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1.22. LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

A differential equation of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ is known as linear differential equation of second order, where P, Q and R are functions of x alone.

Note. Coefficient of $\frac{d^2y}{dx^2}$ must always be 1.

A linear differential equation of second order can be solved by changing dependent and independent variables as well as by the method of variation of parameters.

Above methods are illustrated below with some suitable and important examples.

1.22.1. Method 1. To Find the Complete Solution of $y'' + Py' + Qy = R$ when Part of Complementary Function is Known (*Method of reduction of order*)

Let $y = u$ be a part of the complementary function of the given differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

where u is a function of x. Then, we have

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad ...(2)$$

Let $y = uv$ be the complete solution of equation (1), where v is a function of x.

Differentiating y w.r.t. x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}\cdot v$$

Again,
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Q(uv) = R$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R \qquad | \text{ Using (2)}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u} \qquad ...(3)$$

Put $\frac{dv}{dx} = p$ then, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

Now (3) becomes, $$\frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)p = \frac{R}{u} \qquad ...(4)$$

Equation (4) is a linear differential equation of I order in p and x.

$$\text{I.F.} = e^{\int\left(\frac{2}{u}\frac{du}{dx} + P\right)dx} = e^{\left(\int \frac{2}{u}du + \int P\,dx\right)} = u^2 e^{\int P\,dx}$$

Solution of (4) is given by

$$pu^2 e^{\int P\,dx} = \int \frac{R}{u} u^2 e^{\int P\,dx}\,dx + c_1$$

where c_1 is an arbitrary constant of integration.

$$\Rightarrow \qquad p = \frac{1}{u^2} e^{-\int P\,dx} \left[\int Ru\, e^{\int P\,dx}\, dx + c_1\right]$$

$$\therefore \qquad \frac{dv}{dx} = \frac{1}{u^2} e^{-\int P\,dx} \left[\int Ru\, e^{\int P\,dx}\, dx + c_1\right]$$

Integration yields, $v = \int \frac{1}{u^2} e^{-\int P\,dx} \left[\int Ru\, e^{\int P\,dx}\, dx + c_1\right] dx + c_2$

where c_2 is an arbitrary constant of integration.

Hence the complete solution of (1) is given by,

$$y = uv$$

$$\Rightarrow \qquad y = u \int \frac{1}{u^2} e^{-\int P\,dx} \left[\int Ru\, e^{\int P\,dx}\, dx + c_1\right] dx + c_2 u$$

To find out the part of C.F. of the linear differential equation of II order given by

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Remember:

Condition	*Part of C.F.*
(*i*) $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$	e^{ax}
(*ii*) $1 + P + Q = 0$	e^{x}
(*iii*) $1 - P + Q = 0$	e^{-x}
(*iv*) $m(m-1) + P\,mx + Qx^2 = 0$	x^m
(*v*) $P + Qx = 0$	x
(*vi*) $2 + 2Px + Qx^2 = 0$	x^2

Proof. (*i*) Let $y = e^{ax}$ be a part of C.F. of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$.

Then, $\qquad a^2e^{ax} + P\,ae^{ax} + Qe^{ax} = 0$

$$\Rightarrow \qquad a^2 + Pa + Q = 0$$

$$\Rightarrow \qquad 1 + \frac{P}{a} + \frac{Q}{a^2} = 0.$$

(*iv*) Let $y = x^m$ be a part of C.F. of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ then,

$$m(m-1)\,x^{m-2} + P\,mx^{m-1} + Qx^m = 0$$

$$\Rightarrow \qquad m(m-1) + P\,mx + Qx^2 = 0$$

Rest all the parts are the deductions from proofs (*i*) and (*iv*).

Steps for solution:

1. Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.

2. Compare the given differential equation with the standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ and hence find P, Q and R which are functions of x only.

3. Apply the mentioned conditions on P and Q. If any one of the conditions is satisfied, write down the corresponding part of C.F. as u.
4. Let $y = uv$ be the complete solution of the given differential equation, where v is another function of x, to be determined.
5. Find y' and y'' from y and substitute the required values in the given equation.
6. Arrange the expression in decreasing order of the derivatives of v w.r.t. x.
7. Check whether the coefficient of v is zero ? If not, we are somewhere wrong. It is the time to recheck.
8. Put $\frac{dv}{dx} = p$ so that $\frac{d^2v}{dx^2} = \frac{dp}{dx}$. Then we get a I order differential equation in p and x, which we solve by the methods discussed before.
9. Integrating $\frac{dv}{dx}$ w.r.t. x, we get v in terms of x.
10. At last $y = uv$ will be the complete solution of the given differential equation having two arbitrary constants.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $\sin^2 x \frac{d^2y}{dx^2} = 2y$, *given that* $y = \cot x$ *is a solution of it.*

Sol.
$$\frac{d^2y}{dx^2} - 2(\text{cosec}^2 x)y = 0 \qquad ...(1)$$

Here, a part of C.F. $= \cot x$

Let $y = v \cot x$ be the complete solution of (1)

$$\therefore \quad \frac{dy}{dx} = v(-\text{cosec}^2 x) + \cot x \frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v(2\,\text{cosec}^2 x \cot x) - 2\,\text{cosec}^2 x \frac{dv}{dx} + \cot x \frac{d^2v}{dx^2}$$

Substituting the values of y, $\frac{d^2y}{dx^2}$, we get

$$\cot x \frac{d^2v}{dx^2} - 2\,\text{cosec}^2 x \frac{dv}{dx} = 0$$

$$\Rightarrow \quad \cot x \frac{dp}{dx} - 2\,\text{cosec}^2 x \,.\, p = 0, \quad \text{where } p = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dp}{p} = 2\,\frac{\text{cosec}^2 x}{\cot x}\, dx$$

Integration yields,

$$\log p = -2 \log \cot x + \log c_1$$

$$\Rightarrow \quad \log p + \log \cot^2 x = \log c_1 \qquad | \; c_1 \text{ is arbitrary constant}$$

$$\Rightarrow \qquad p = c_1 \tan^2 x$$

$$\therefore \qquad \frac{dv}{dx} = c_1 (\sec^2 x - 1)$$

Integrating with respect to x, we get

$$v = c_1 (\tan x - x) + c_2$$

where c_2 is an arbitrary constant of integration.

Hence the complete solution is given by

$$y = v \cot x = [c_1 (\tan x - x) + c_2] \cot x$$

$$y = c_1(1 - x \cot x) + c_2 \cot x.$$

Example 2. *Solve:* $\dfrac{d^2 y}{dx^2} - \cot x \dfrac{dy}{dx} - (1 - \cot x)y = e^x \sin x.$

Sol. Comparing with the standard form, we get

$$P = -\cot x,\ Q = -(1 - \cot x),\ R = e^x \sin x$$

$$1 + P + Q = 1 - 1 + \cot x - \cot x = 0$$

$$\therefore \qquad \text{A part of C.F.} = e^x$$

Let $y = v\, e^x$ be the complete solution of given equation, then

$$\frac{dy}{dx} = v\, e^x + e^x \frac{dv}{dx}$$

$$\frac{d^2 y}{dx^2} = v\, e^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2 v}{dx^2}$$

Substituting for y, $\dfrac{dy}{dx}$ and $\dfrac{d^2 y}{dx^2}$ in given equation, we get

$$\frac{d^2 v}{dx^2} + (2 - \cot x) \frac{dv}{dx} = \sin x$$

$$\Rightarrow \qquad \frac{dp}{dx} + (2 - \cot x)p = \sin x \qquad \ldots(1) \quad \text{where } p = \frac{dv}{dx}.$$

This is a linear differential equation of I order in p and x.

$$\text{I.F.} = e^{\int (2 - \cot x)\, dx} = \frac{e^{2x}}{\sin x}$$

Solution of (1) is, $\qquad p \dfrac{e^{2x}}{\sin x} = \displaystyle\int \sin x \cdot \frac{e^{2x}}{\sin x}\, dx + c_1 = \frac{e^{2x}}{2} + c_1$

where c_1 is an arbitrary constant of integration.

$$p = \frac{1}{2} \sin x + c_1\, e^{-2x} \sin x$$

$$\frac{dv}{dx} = \frac{1}{2} \sin x + c_1\, e^{-2x} \sin x$$

Integrating, we get $\qquad v = -\dfrac{1}{2} \cos x - \dfrac{1}{5} c_1\, e^{-2x} (\cos x + 2 \sin x) + c_2$

Hence the complete solution is given by,

$$y = v\,e^x = \left[-\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(\cos x + 2\sin x) + c_2\right]e^x.$$

Example 3. *Solve:* $(1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x(1-x^2)^{3/2}.$

Sol. Comparing with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = \frac{x}{1-x^2},\ Q = -\frac{1}{1-x^2},\ R = x\sqrt{1-x^2}$$

$\because$ $\quad P + Qx = 0$

$\therefore$ $\quad$ A part of C.F. $= x$

Let $y = vx$ be the complete solution of the given differential equation

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting the values, we get

$$\frac{d^2v}{dx^2} + \frac{2-x^2}{x(1-x^2)}\frac{dv}{dx} = (1-x^2)^{1/2}$$

$$\Rightarrow \quad \frac{dp}{dx} + \frac{(2-x^2)}{x(1-x^2)}p = (1-x^2)^{1/2} \qquad \ldots(1) \qquad \text{where } p = \frac{dv}{dx}.$$

$$\text{I.F.} = e^{\int \frac{(2-x^2)}{x(1-x^2)}dx} = e^{\int\left(\frac{2}{x}+\frac{x}{1-x^2}\right)dx} = \frac{x^2}{\sqrt{1-x^2}}$$

Solution of (1) is,
$$p.\frac{x^2}{\sqrt{1-x^2}} = \int \sqrt{1-x^2}\cdot\frac{x^2}{\sqrt{1-x^2}}\,dx + c_1$$

$$\Rightarrow \quad p\frac{x^2}{\sqrt{1-x^2}} = \frac{x^3}{3} + c_1 \qquad \mid c_1 \text{ is arbitrary constant}$$

$$\Rightarrow \quad p = \frac{x}{3}\sqrt{1-x^2} + c_1\frac{\sqrt{1-x^2}}{x^2}$$

$$\frac{dv}{dx} = \frac{x}{3}\sqrt{1-x^2} + c_1\frac{\sqrt{1-x^2}}{x^2}$$

Integrating, we get
$$v = \frac{1}{3}\int x\sqrt{1-x^2}\,dx + c_1\int\frac{\sqrt{1-x^2}}{x^2}\,dx + c_2$$

$$= -\frac{1}{9}(1-x^2)^{3/2} + c_1\left[\frac{-\sqrt{1-x^2}}{x} - \sin^{-1}x\right] + c_2$$

where c_2 is also an arbitrary constant of integration.

Hence the complete solution is given by,

$$y = vx$$

$$\Rightarrow \quad y = \frac{-x(1-x^2)^{3/2}}{9} - c_1\left[\sqrt{1-x^2} + x\sin^{-1}x\right] + c_2 x.$$

Example 4. *Solve:* $(x \sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = 0$ *of which* $y = x$ *is a solution.*

Sol.
$$\frac{d^2y}{dx^2} - \left(\frac{x\cos x}{x\sin x + \cos x}\right)\frac{dy}{dx} + \left(\frac{\cos x}{x\sin x + \cos x}\right)y = 0 \qquad \ldots(1)$$

Here a part of C.F. $= x$ | Given

Let $y = vx$ be the complete solution of equation (1)

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting the values of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ in (1), we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{dp}{dx} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)p = 0, \qquad \text{where } p = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dp}{p} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)dx = 0$$

Integration yields, $px^2 = c_1(x\sin x + \cos x)$

or
$$\frac{dv}{dx} = c_1\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)$$

Again integration yields,

$$v = -c_1\frac{\cos x}{x} + c_2$$

Hence the complete solution is given by

$$y = vx$$

$$\Rightarrow \quad y = -c_1\cos x + c_2 x$$

where c_1 and c_2 are the arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0$

2. $x\dfrac{d^2y}{dx^2} - (3+x)\dfrac{dy}{dx} + 3y = 0$

3. $x^2\dfrac{d^2y}{dx^2} - (x^2+2x)\dfrac{dy}{dx} + (x+2)y = x^3e^x$ of which $y = x$ is a solution.

4. $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x$

5. $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is a solution.

6. Solve : $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, given that $x + \frac{1}{x}$ is one integral.

Answers

1. $y = (c_1 \log x + c_2)e^x$
2. $y = -c_1(x^3 + 3x^2 + 6x + 6) + c_2 e^x$
3. $y = x(x-1)e^x + c_1 x e^x + c_2 x$
4. $y = -e^x - \frac{1}{4}c_1(2x+5) + c_2 e^{2x}$
5. $y = e^{x^2}(c_1 x + c_2)$
6. $y = \frac{A}{x} + c_2\left(x + \frac{1}{x}\right)$.

1.22.2. Method 2. To Find the Complete Solution of $y'' + Py' + Qy = R$ when it is Reduced to Normal Form (*Removal of first derivative*)

When the part of C.F. can not be determined by the previous method, we reduce the given differential equation in **normal form** by eliminating the term in which there exists first derivative of the dependent variable.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

Let $y = uv$ be the complete solution of eqn. (1), where u and v are the functions of x.

$$\therefore \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

and
$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eqn. (1), we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + v\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u} \qquad ...(2)$$

Let us choose u such that $\frac{2}{u}\frac{du}{dx} + P = 0$ ...(3)

which on solving gives,

$$u = e^{-\int \frac{P}{2}dx} \qquad ...(4)$$

From (3), $$\frac{du}{dx} = -\frac{Pu}{2}$$

Differentiating, we get $$\frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\left(\frac{du}{dx}\right) + \frac{dP}{dx}(u)\right]$$

$$= -\frac{1}{2}\left[P\left(\frac{-Pu}{2}\right) + u\frac{dP}{dx}\right] = \frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx}$$

$$\text{Coefficient of } v = \frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q = \frac{1}{u}\left[\frac{P^2u}{4} - \frac{u}{2}\frac{dP}{dx}\right] + \frac{P}{u}\left(\frac{-Pu}{2}\right) + Q$$

$$= Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = I \text{ (say)}$$

$$\text{RHS} = \frac{R}{u} = Re^{\frac{1}{2}\int P\,dx} = S \text{ (say)}$$

Then (2) becomes, $\dfrac{d^2v}{dx^2} + Iv = S$ ...(5)

This is known as the normal form of equation (1).

Solving (5), we get v in terms of x. Ultimately, $y = uv$ is the complete solution.

Steps for solution:

1. Make the coefficient of $\dfrac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare with standard form $y'' + Py' + Qy = R$ to get P, Q and R.
3. Let $y = uv$ be the complete solution of the given equation.
4. Find $u = e^{-\frac{1}{2}\int P\,dx}$, $I = Q - \dfrac{1}{2}\dfrac{dP}{dx} - \dfrac{P^2}{4}$ and $S = \dfrac{R}{u}$.
5. Check whether I is a constant or $\dfrac{\text{constant}}{x^2}$? If I is not any one of them, the method is not applicable.
6. If I is a constant, we get a linear differential equation of II order with constant coefficients while if I is $\dfrac{\text{constant}}{x^2}$, we get a homogeneous linear differential equation with variable coefficients.
7. Normal form is given by $\dfrac{d^2v}{dx^2} + Iv = S$ which we solve for v.
8. $y = uv$ will be the complete solution of the given differential equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x.$

Sol. Here, $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2}\sin 2x$

Let $y = uv$ be the complete solution.

Now, $$u = e^{-\frac{1}{2}\int(-4x)\,dx} = e^{x^2}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1.$$

Also, $$S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

Hence normal form is,

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

Auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

where c_1 and c_2 are arbitrary constants of integration.

$$\text{P.I.} = \frac{1}{D^2+1}(-3\sin 2x) = \frac{-3}{(-4+1)}\sin 2x = \sin 2x$$

∴ Solution is, $$v = c_1 \cos x + c_2 \sin x + \sin 2x$$

Hence the complete solution of given differential equation is

$$y = uv = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x).$$

Example 2. *Solve:* $\dfrac{d^2y}{dx^2} + \dfrac{1}{x^{1/3}}\dfrac{dy}{dx} + \left(\dfrac{1}{4x^{2/3}} - \dfrac{1}{6x^{4/3}} - \dfrac{6}{x^2}\right)y = 0$.

Sol. Comparing with the standard form, we get

$$P = x^{-1/3},\ Q = \frac{1}{4}x^{-2/3} - \frac{1}{6}x^{-4/3} - 6x^{-2},\ R = 0$$

Let $y = uv$ be the complete solution of given equation.

Now, $$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^{-1/3}\,dx} = e^{-\frac{3}{4}x^{2/3}}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{2}\left(-\frac{1}{3}x^{-4/3}\right) - \frac{1}{4}\cdot\frac{1}{x^{2/3}}$$

⇒ $$I = -\frac{6}{x^2}$$

Also, $$S = \frac{R}{u} = 0$$

∴ Normal form is,

$$\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0$$

⇒ $$x^2\frac{d^2v}{dx^2} - 6v = 0 \qquad \text{...(1)}$$

Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then eqn. (1) becomes,

$$[D(D-1)-6]\, v = 0$$

$$\Rightarrow \qquad (D^2 - D - 6)\, v = 0 \qquad \ldots(2)$$

Solution of eqn. (2) is, $\qquad v = c_1 e^{3z} + c_2 e^{-2z} = c_1 x^3 + c_2 x^{-2}$

where c_1 and c_2 are arbitrary constants of integration.

$\therefore$ Complete solution of given differential equation is

$$y = e^{-\frac{3}{4}x^{2/3}}\left(c_1 x^3 + \frac{c_2}{x^2}\right).$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $x\frac{d}{dx}\left(x\frac{dy}{dx} - y\right) - 2x\frac{dy}{dx} + 2y + x^2 y = 0$

2. $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right) + y\cos^2 x = 0$

3. $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = n^2 y$

4. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x \sec x$

5. $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x$

6. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$

7. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y = e^{\frac{1}{2}(x^2 + 2x)}$ *[G.B.T.U. (C.O.) 2011]*

8. $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - 8)\, y = x^2 e^{-x^2/2}$. *[G.B.T.U. 2012, 2013]*

Answers

1. $y = x(c_1 \cos x + c_2 \sin x)$

2. $y = \sec x\,(c_1 \cos \sqrt{2}\,x + c_2 \sin \sqrt{2}\,x)$

3. $y = \frac{1}{x}(c_1 e^{nx} + c_2 e^{-nx})$

4. $y = \sec x\left[c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x + \frac{e^x}{7}\right]$

5. $y = \frac{1}{2}(\sin x - x\cos x) + c_1 x\cos x + c_2 \cos x$

6. $y = e^{x^2}(c_1 e^x + c_2 e^{-x} - 1)$

7. $y = e^{x^2/2}(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x) + \frac{1}{4}e^{\frac{1}{2}(x^2 + 2x)}$

8. $y = e^{-\frac{x^2}{2}}\left[c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)\right]$.

1.22.3. Method 3. To Find the Complete Solution of $y'' + Py' + Qy = R$ by Changing the Independent Variable

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad ...(1)$$

Let us relate x and z by the relation,

$$z = f(x) \quad ...(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \quad ...(3)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right)$$

$$= \frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + \frac{dz}{dx} \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} \quad ...(4)$$

Substituting in (1), we get

$$\frac{dy}{dz} \cdot \frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + P\frac{dy}{dz} \cdot \frac{dz}{dx} + Qy = R$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad ...(5)$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$, $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Here P_1, Q_1, R_1 are functions of x which can be transformed into functions of z using the relation $z = f(x)$.

Choose z such that Q_1 = constant = a^2 (say)

$$\Rightarrow \quad \frac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2 \quad \Rightarrow \quad a\frac{dz}{dx} = \sqrt{Q}$$

$$\Rightarrow \quad dz = \frac{\sqrt{Q}}{a}\, dx$$

Integration yields, $\quad z = \displaystyle\int \frac{\sqrt{Q}}{a}\, dx$

If this value of z makes P_1 as constant then equation (5) can be solved.

Steps for solution:

1. Make the coefficient of $\dfrac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare the equation with standard form $y'' + Py' + Qy = R$ and get P, Q and R.

3. Choose z such that $\left(\frac{dz}{dx}\right)^2 = Q$

Here Q is taken in such a way that it remains the whole square of a function without surd and its negative sign is ignored.

4. Find $\frac{dz}{dx}$ hence obtain z (on integration) and $\frac{d^2z}{dx^2}$ (on differentiation).

5. Find P_1, Q_1 and R_1 by the formulae

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}.$$

6. Reduced equation is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ which we solve for y in terms of z.

7. We write the complete solution as y in terms of x by replacing the value of z in terms of x.

ILLUSTRATIVE EXAMPLES

Example 1. *By changing the independent variable, solve the differential equation*

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4.$$

Sol.
$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4 \qquad ...(1)$$

Here, $$P = -\frac{1}{x},\ Q = 4x^2,\ R = x^4$$

Choose z such that $$\left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \qquad \frac{dz}{dx} = 2x \qquad ...(2)$$

$$z = x^2 \text{ (on integrating)} \qquad ...(3)$$

From (2), $$\frac{d^2z}{dx^2} = 2 \qquad | \text{ Differentiating (2) w.r.t. } x$$

$$\therefore \qquad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}(2x)}{4x^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4}$$

Reduced equation is,

$$\frac{d^2y}{dz^2} + y = \frac{z}{4} \qquad | \because \quad z = x^2 \text{ from (3)}$$

Auxiliary equation is,

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\left(\frac{z}{4}\right) = (1 + D^2)^{-1}\left(\frac{z}{4}\right)$$

$$= (1 - D^2 + \ldots\ldots)\left(\frac{z}{4}\right) = \frac{z}{4} \qquad | \text{ Leaving higher powers}$$

$\therefore$ Solution is $\quad y = c_1 \cos z + c_2 \sin z + \frac{z}{4}$

Complete solution is given by

$$y = c_1 \cos (x^2) + c_2 \sin (x^2) + \frac{x^2}{4}.$$

Example 2. *Solve:* $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4 \cos \log (1+x).$

Sol.

$$\frac{d^2y}{dx^2} + \frac{1}{1+x}\frac{dy}{dx} + \frac{y}{(1+x)^2} = \frac{4}{(1+x)^2} \cos \log (1+x) \qquad \ldots(1)$$

Choose z such that,

$$\left(\frac{dz}{dx}\right)^2 = \frac{1}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{1+x} \qquad \ldots(2)$$

Integration yields, $\quad z = \log (1+x) \qquad \ldots(3)$

From (2),

$$\frac{d^2z}{dx^2} = -\frac{1}{(1+x)^2}$$

$\therefore$

$$P_1 = \frac{-\frac{1}{(1+x)^2} + \frac{1}{1+x} \cdot \frac{1}{1+x}}{\frac{1}{(1+x)^2}} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 4 \cos \log (1 + x) = 4 \cos z \qquad | \text{ From (3)}$$

Reduced equation is

$$\frac{d^2y}{dz^2} + y = 4 \cos z$$

Auxiliary equation is $m^2 + 1 = 0 \qquad \Rightarrow \qquad m = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2 + 1} (4 \cos z) = 4 \cdot \frac{z}{2} \sin z = 2z \sin z$$

Complete solution is

$$y = c_1 \cos z + c_2 \sin z + 2z \sin z$$

$$y = c_1 \cos \log (1 + x) + c_2 \sin \log (1 + x) + 2 \log (1 + x) \sin \log (1 + x).$$

Example 3. *Solve by changing the independent variable :*

$$\frac{d^2y}{dx^2} + (3 \sin x - \cot x) \frac{dy}{dx} + 2y \sin^2 x = e^{-\cos x} \sin^2 x.$$

Sol. Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \quad \frac{dz}{dx} = \sin x \qquad \ldots(1)$$

Integration yields, $\quad z = -\cos x \qquad \ldots(2)$

From (1), $\quad \frac{d^2z}{dx^2} = \cos x$

$$\therefore \qquad P_1 = \frac{\cos x + (3 \sin x - \cot x) \sin x}{\sin^2 x} = 3$$

$$Q_1 = \frac{2 \sin^2 x}{\sin^2 x} = 2$$

$$R_1 = \frac{e^{-\cos x} \cdot \sin^2 x}{\sin^2 x} = e^{-\cos x} = e^z$$

Reduced equation is, $\quad \frac{d^2y}{dz^2} + 3 \frac{dy}{dz} + 2y = e^z$

Auxiliary equation is $\quad m^2 + 3m + 2 = 0 \quad \Rightarrow \quad m = -1, -2$

$$\therefore \qquad \text{C.F.} = c_1 e^{-z} + c_2 e^{-2z}$$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2} (e^z) = \frac{e^z}{6}$$

Complete solution is, $\quad y = c_1 e^{-z} + c_2 e^{-2z} + \frac{e^z}{6}$

$$\Rightarrow \qquad y = c_1 e^{\cos x} + c_2 e^{2 \cos x} + \frac{e^{-\cos x}}{6}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve by changing the independent variable*

$$(1 + x^2)^2 \frac{d^2 y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} + 4y = 0.$$

Sol. The given equation is

$$\frac{d^2 y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0 \quad ...(1)$$

Here, $P = \frac{2x}{1+x^2}, \quad Q = \frac{4}{(1+x^2)^2}, \quad R = 0$

Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \frac{4}{(1+x^2)^2}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{2}{1+x^2} \quad ...(2)$$

Integration yields, $z = 2\tan^{-1} x$

From (2), $$\frac{d^2 z}{dx^2} = \frac{-4x}{(1+x^2)^2}$$

$$\therefore \quad P_1 = \frac{\frac{-4x}{(1+x^2)^2} + \frac{2x}{1+x^2}\cdot\frac{2}{1+x^2}}{\frac{4}{(1+x^2)^2}} = 0$$

$$Q_1 = \frac{\frac{4}{(1+x^2)^2}}{\frac{4}{(1+x^2)^2}} = 1, \quad R_1 = \frac{0}{\{4/(1+x^2)^2\}} = 0$$

Reduced equation is $$\frac{d^2 y}{dz^2} + y = 0$$

Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = 0$$

$$\therefore \quad y = c_1 \cos z + c_2 \sin z$$

Complete solution is $y = c_1 \cos(2\tan^{-1} x) + c_2 \sin(2\tan^{-1} x)$
where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve by changing the independent variable*

$$\cos x \frac{d^2 y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x.$$

Sol. The given equation is

$$\frac{d^2 y}{dx^2} + \tan x \frac{dy}{dx} - 2y\cos^2 x = 2\cos^4 x \quad ...(1)$$

Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \cos^2 x \Rightarrow \quad \frac{dz}{dx} = \cos x$$

Integration yields, $z = \sin x$ ...(2)

From (1), $\dfrac{d^2z}{dx^2} = -\sin x$

$$\therefore \quad P_1 = \frac{-\sin x + \tan x \cos x}{\cos^2 x} = 0$$

$$Q_1 = \frac{-2\cos^2 x}{\cos^2 x} = -2$$

$$R_1 = \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x = 2(1 - z^2)$$

Reduced equation is

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

Auxiliary equation is

$$m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$$

$$\therefore \quad \text{C.F.} = c_1 \cosh\sqrt{2}\,z + c_2 \sinh\sqrt{2}\,z$$

$$\text{P.I.} = \frac{1}{D^2 - 2}[2(1 - z^2)] = \frac{2}{-2 + D^2}(1 - z^2) = -\left(1 - \frac{D^2}{2}\right)^{-1}(1 - z^2)$$

$$= -\left(1 + \frac{D^2}{2}\right)(1 - z^2) = -\left[1 - z^2 + \frac{1}{2}(-2)\right] = -[1 - z^2 - 1] = z^2$$

$$\therefore \quad y = c_1 \cosh\sqrt{2}\,z + c_2 \sinh\sqrt{2}\,z + z^2$$

Complete solution is

$$y = c_1 \cosh\sqrt{2}\,(\sin x) + c_2 \sinh\sqrt{2}\,(\sin x) + \sin^2 x$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations by changing the independent variable:

1. $\dfrac{d^2y}{dx^2} + \cot x \dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$ (*U.K.T.U. 2012*)
2. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$
3. $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3 y = 2x^3$ (*M.T.U. 2013*)
4. $x^6 \dfrac{d^2y}{dx^2} + 3x^5 \dfrac{dy}{dx} + a^2 y = \dfrac{1}{x^2}$
5. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - y \sin^2 x = 0$
6. $x^4 \dfrac{d^2y}{dx^2} + 2x^3 \dfrac{dy}{dx} + n^2 y = 0$ (*U.K.T.U. 2011*)
7. $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y \cos^2 x = 0$
8. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - y \sin^2 x = \cos x - \cos^3 x$. (*G.B.T.U. 2011*)

Answers

1. $y = c_1 \cos\left(2 \log \tan \dfrac{x}{2}\right) + c_2 \sin\left(2 \log \tan \dfrac{x}{2}\right)$

2. $y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$

3. $y = (c_1 + c_2 x^2)\, e^{-x^2} + \dfrac{1}{2}$

4. $y = c_1 \cos\left(\dfrac{a}{2x^2}\right) + c_2 \sin\left(\dfrac{a}{2x^2}\right) + \dfrac{1}{a^2 x^2}$

5. $y = c_1 e^{-\cos x} + c_2 e^{\cos x}$

6. $y = c_1 \cos\left(-\dfrac{n}{x}\right) + c_2 \sin\left(-\dfrac{n}{x}\right)$

7. $y = c_1 \sin(\sin x) + c_2 \cos(\sin x)$

8. $y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x$