Q find a Positive root of the equation $xe^x = 1$ which lies between 0 and 1.

sol. Eq is $xe^x = 1 \implies xe^x - 1 = 0$

Let $f(x) = xe^x - 1$

Now let us find $f(0) = -1$

$f(1) = 1.718$

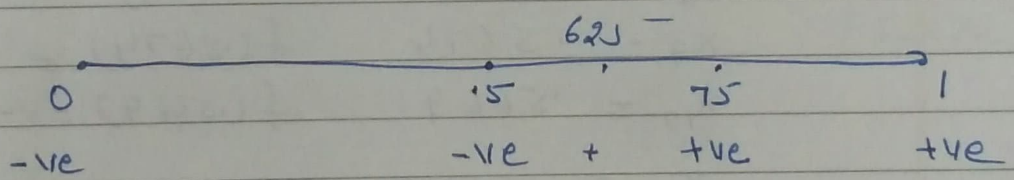
i.e. root lies between 0 and 1.

Now first approximate value of the root $x_0 = \frac{0+1}{2}$

$= 0.5$

$\therefore \quad f(x_0) = f(0.5) = .5\, e^{.5} - 1$

$= -ve$



$\therefore$ Root lies between .5 and 1.

So next approximate value of the root $= x_1 = \frac{.5+1}{2}$

$= .75$

$f(x_1) = f(.75) = .75\, e^{.75} - 1$

+ve

$\therefore$ Root lies between .5 and .75

$\therefore \quad x_2 = \frac{.5 + .75}{2} = .625$

$f(x_2) = f(.625) = .625\, e^{.625} - 1 =$

$= +ve$

$\therefore$ Root lies between $.5$ and $.625$

Repeating this Procedure we get following different approximations of the root

$x_3 = .5625 \qquad f(.5625) = -ve$

$x_4 = .59375 \qquad f(.59375) = +ve$

$x_5 = .5781 \qquad f(.5781) = +ve$

$x_6 = .5703 \qquad f(.5703) = +ve$

$x_7 = .5664 \qquad f(.5664) = -ve$

$x_8 = .5684 \qquad f(.5684) = +ve$

$x_9 = .5674 \qquad f(.5674) = +ve$

$x_{10} = .5669 \qquad f(.5669) = -ve$

$x_{11} = .5671 \qquad f(.5671) = -ve$

$\therefore$ Required root is $.567$ correct to three decimal places.

Rate of convergence:

Let $x_0, x_1, x_2 \ldots\ldots$ be the values of a root (α) of an equation at 0th, 1st, 2nd iterations while its actual value is 3.5567.

The values of this root calculated by three different methods, are given below

Root	Ist Method	2nd Method	3rd Method
x_0	5	5	5
x_1	5.6	3.8527	3.8327
x_2	6.4	3.5693	3.56834
x_3	8.3	3.55798	3.55743
x_4	9.7	3.55687	3.55672
x_5	10.6	3.55676	
x_6	11.9	3.55671	

- The values in the Ist Method do not converge towards the root 3.5567.
- In the 2nd and 3rd Methods, the value converge to the root after 6th and 4th iterations respectively.
- clearly 3rd Method converges faster than the 2nd Method.

This fastness of convergence in any method, is represented by its **rate of convergence**.

- If e be the error then $e_i = \alpha - x_i$
  $= x_{i+1} - x_i$

- If $\frac{e_{i+1}}{e_i}$ is almost constant, then convergence is said to be linear i.e slow

- If $\frac{e_{i+1}}{e_i^P}$ is nearly constant, convergence is said to be of order P i.e faster.

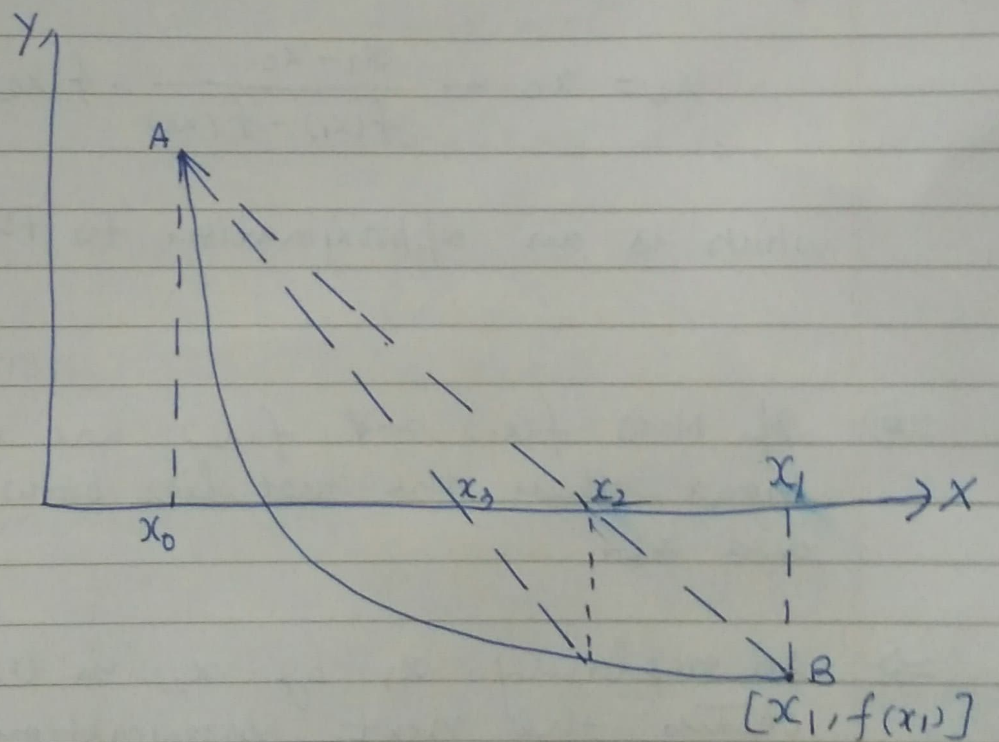
Note: As the error decreases with each step by a factor of $\frac{1}{2}$ (i.e $\frac{e_{n+1}}{e_n} = \frac{1}{2}$), the convergence in the bisection method is linear.

Method of False Position Method:
(Regula-falsi Method)

Note: This method is the oldest method of finding the real root of an equation $f(x)=0$ and closely resembles the bisection Method.

Here we choose two Points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs

i.e graph of $y=f(x)$ crosses the x axis between these points



This indicates that root lies between x_0 and x_1 and consequently $f(x_0) \cdot f(x_1) < 0$

Now Equation of the chord joining the Points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

⇒ The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x axis as an approximation to the root.

So the abscissa of the point where the chord cuts the x axis $(y=0)$ is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0) \quad -(1)$$

which is an approximation to the root

⇒ If Now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2.

⇒ So replacing x_1 by x_2 in (1) we obtain the next approximation x_3

⇒ This Procedure is repeated till the ~~round is~~ root is found to the desired accuracy.

- The iteration process based on (1) is known as the method of false position
- Its rate of convergence is faster than that of the bisection method.

Ex. Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.

sol.

$$f(x) = x^3 - 2x - 5$$

$$f(0) = -5$$
$$f(1) = 1 - 2 - 5 = -6$$
$$f(2) = 2^3 - 2\times2 - 5 = -1$$
$$f(3) = 3^3 - 2\times3 - 5 = 16$$

i.e root lies between 2 and 3.

$\therefore$ Taking $x_0 = 2$, $x_1 = 3$
$f(x_0) = -1$, $f(x_1) = 16$ in the method of false position, we get

(i)

$$\boxed{x_2 = x_0 - \frac{(x_1 - x_0) f(x_0)}{f(x_1) - f(x_0)}} = 2 - \frac{(3-2)(-1)}{16-(-1)}$$

$$= 2 + \frac{1}{17} = 2.0588$$

$$f(x_2) = f(2.0588) = -0.3908$$

i.e the root lies between 2.0588 and 3

Taking $x_0 = 2.0588$ $f(x_0) = -.3908$

$x_1 = 3$ $f(x_1) = 16$

in (i) we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-.3908)$$

$$= 2.0813$$

Repeating this process, the successive approximation are

$x_4 = 2.0862$

$x_5 = 2.0915$

$x_6 = 2.0934$

$x_7 = 2.0941$

$x_8 = 2.0943$

Hence the root is 2.094 correct to three decimal places.

Q Find the root of the equation $xe^x = \cos x$ using the ~~f~~ regula falsi Method.

Sol
$$f(x) = \cos x - x e^x = 0$$

$$f(0) = 1$$
$$f(1) = \cos 1 - 1 e^1 = -2.17798$$

i.e the root lies between 0 and 1

$\therefore$ Taking $x_0 = 0$ $\quad f(x_0) = 1$
$x_1 = 1$ $\quad f(x_1) = -2.17798$

in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0)$$

$$= 0 + \frac{(1-0)}{-2.17798 - 1} \cdot 1$$

$$= .31467$$

Now $f(x_2) = f(.31467) = 0.51987$

$\therefore$ root lies between 0.31467 and 1

$\therefore$ Taking $x_0 = .31467$ $\quad f(x_0) = .31467$
$x_1 = 1$ $\quad f(x_1) = -2.17798$

we get $\quad x_3 = .31467 + \frac{.68533}{2.69785} \times .51987$

$$= 0.44673$$

$f(.44673) = 0.20356$

$\therefore$ root lies between 0.44673 and 1.

Taking $x_0 = 0.44673$ $\quad f(.44673) = .20356$

$x_1 = 1$ $\quad f(x_1) = -2.17798$

$$x_4 = .44673 + \frac{0.55327}{2.38154} \times 0.20356$$

$$= 0.49402$$

Repeating this Process, the successive approximations are

$x_5 = 0.50995$

$x_6 = 0.51520$

$x_7 = 0.51692$

$x_8 = 0.51748$

$x_9 = 0.51767$

$x_{10} = 0.51775$ etc

Hence the root is 0.5177 correct to 4 decimal places.

Solve following question using False Position Method. (correct to three decimal places).

(i) $x^3 + x - 1 = 0$

(ii) $x^3 - 4x - 9 = 0$

(iii) $x e^x = 2$

(iv) $\cos x = 3x - 1$