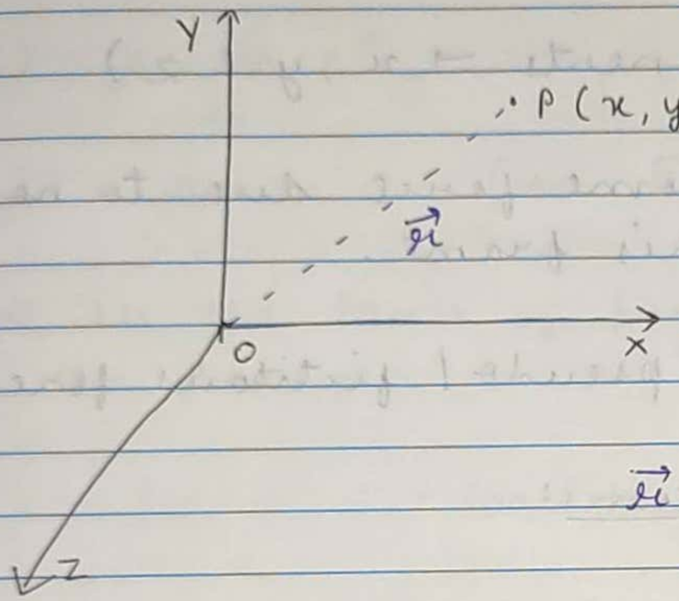
Unit - 1
Introductory Mechanics & Theory of Relativity

Frame of reference - It is nothing but a simple geometrical framework which is used to describe the occurence of event in space.



$\cdot P(x, y, z, t) \rightarrow$ Position, time description of event P.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{at 't' instant}$$

- $$\vec{v}_P = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

- $$\vec{a}_P = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

$$= a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Classification of frame of reference:

i) Inertial frame of reference - This frame follows law of inertia. The special theory of relativity is based on this frame.
$\rightarrow a = 0, \ v =$ constant (uniform motion)

ii) Non-Inertial frame of reference - This frame doesn't follows law of inertia. General theory of relativity is based on this frame

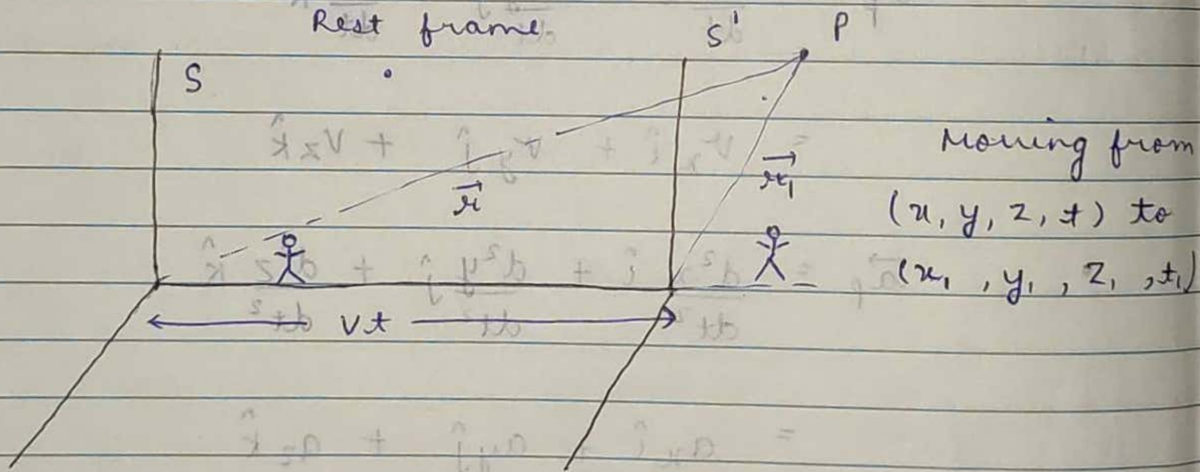
→ $a = \frac{d^2r}{dt^2} \neq 0$, accelerated frame (which is true for all components → x, y & z)

→ particle experiences some force due to non-zero acceleration of this frame.

→ The force is called pseudo / fictitious force.

Gallilean transformations:

Law of inertia is also known as gallilean law



i) at $t = t_1$; origin of both frames coincide

$$r = r' + vt$$ (also in component form)

ii) velocity of moving object

$$r' = r - vt$$

$$\frac{dr'}{dt} = \frac{dr}{dt} - V\frac{dt}{dt}$$

$$v_1 = v - V$$

iii) for acc^n;

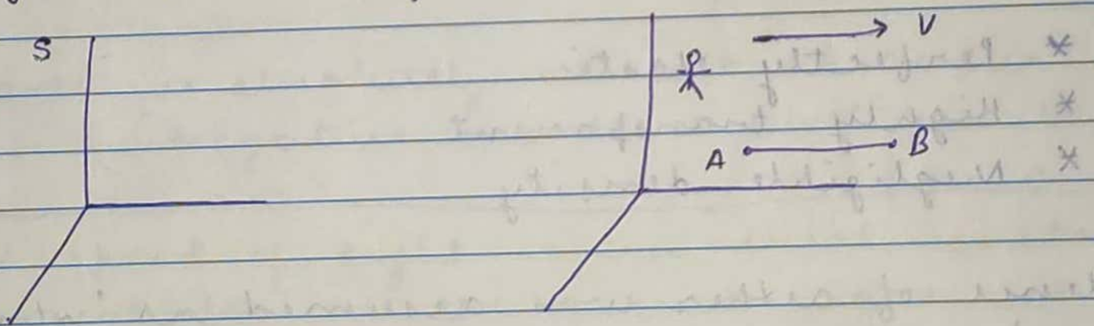
$$a_1 = a \quad (\text{so } F_1 = F \text{ also})$$

$\Rightarrow$ All the laws of physics are identical for the observer in all inertial frames.

– Gallilean hypothesis of invariance

Consequences of G.T.

Length of an object is absolute (invariant)



S' frame : $A(x_1', y_1', z_1')$ $B(x_2', y_2', z_2')$

S frame : $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

S' frame observer

$$L' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}$$

Let us perform x_1', x_2' etc. by G.T.

$$= \sqrt{[(x_2 - vt) - (x_1 - vt)]^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

= same as previous one.

$\Rightarrow$ so, length is independent of frame.

→ Michelson - Morley Experiment (1927)

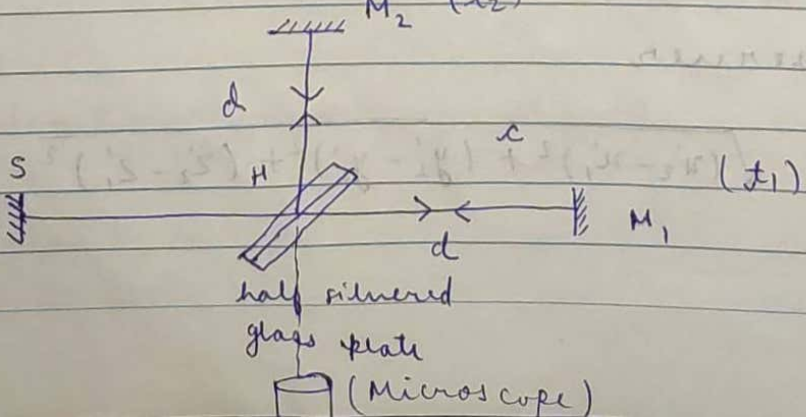
- Whether speed of light c gets modified in accordance with G.T.

• Aether - Material medium which was supposed (to be present throughout the universe.

* Perfectly elastic
* Highly transparent
* Negligible density

Existence of aether was assumed as 'absolute frame' relative to which the motion of the bodies can be detected.

To justify aether hypothesis:



i) $\Delta t = t_2 - t_1 = \frac{2d}{c} - \frac{2d}{c} = 0$ (Earth is at rest)

ii) $\Delta t = t_2 - t_1 \neq 0$ (when earth is in motion)

$$\boxed{\text{path difference} = \frac{dv^2}{c^2}}$$

v – velocity of earth , c → velocity of light

Fringe patterns appears in this (ii) case.

After performing this exp. again & again,
(no fringe shift was experimentally observed)
↳ outcome.

⇒ Motion of Earth could not be detected relative to aether.

⇒ Earth is absolutely at rest in aether.
(Negative result – null result).

"The speed of light is universal constant & is identical for all the observers of inertial frames"

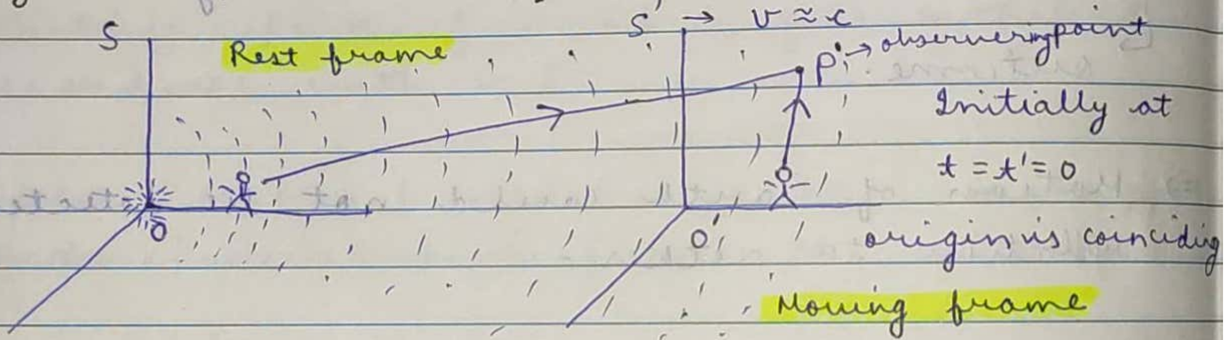
Presence of aether was discarded. G.T. is applica-ble for small velocities only (not for c)

Basic postulates of relativity

1.) All the laws of physics are identical for all the observers of the inertial frame that move with a constant velocity relative to one another.

2.) The velocity of light is a universal constant, and is identical for all the observers of the inertial frames. This is also known as constancy of velocity of light.

Lorentz Transformation equation → needed to deal with velocities like 'c'.



When S and S' are coinciding, light signal emits from O.

Coordinates of P by S frame observer is
$(x, y, z)(t)$

" " " S' frame " "
$(x', y', z')(t')$

→ Time taken by light signal to reach P for S frame observer.

$$t = \frac{OP}{c} = \frac{(x^2+y^2+z^2)^{1/2}}{c}$$

$$x^2+y^2+z^2 = c^2t^2 \quad \cdots (i)$$

Similarly for S' frame observer,

$$t' = \frac{O'P}{c}$$

$$= \frac{(x'^2 + y'^2 + z'^2)^{1/2}}{c}.$$

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \cdots (ii)$$

- The transformation will be such that eqn ① transform eqn②.

Let's apply Gallilean transformation,

$$\boxed{x' = x - vt \ , \ y' = y \ , \ z' = z \ , \ t' = t}$$

so, (ii) can be written as

$$(x-vt)^2 + y^2 + z^2 = c^2t^2$$

$$x^2 + v^2t^2 - 2xvt + y^2 + z^2 = c^2t^2 \quad \cdots (iii)$$

(Gallilean transformation fails) (needs modificat

New transformations to be included only in order to transform (i) to (iii)

Let the modified eqn's be written as.

$$x' = \alpha(x - v.t) \quad , \quad t' = \alpha'(t + fx)$$

where α, α', f are constant.

Substituting these in eq^n (2)

$$\alpha^2(x-vt)^2 + y^2 + z^2 = c^2\alpha'^2(t+fx)^2$$

$$\alpha^2(x^2+v^2t^2-2vxt) + y^2+z^2 = c^2\alpha'^2(t^2+f^2x^2+2fxt)$$

on equating co-efficients of x^2, x and const

we get,

$$x^2[\alpha^2 - f^2\alpha'^2c^2] - 2xt(\alpha^2 v + fc^2\alpha'^2) + y^2+z^2 =$$

$$c^2 t^2\left[\alpha'^2 - \frac{\alpha^2v^2}{c^2}\right]$$

comparing with eq (2)

so,

$$\boxed{\alpha^2 - f^2\alpha'^2c^2 = 1,\quad \alpha^2 v + fc^2\alpha'^2 = 0,\quad \alpha'^2 - \frac{\alpha^2v^2}{c^2} = 1}$$

on solving these eq^n we get,

$$\boxed{\alpha = \alpha' = \frac{1}{\sqrt{1-v^2/c^2}}\ ;\ f = -\frac{v}{c^2}}$$

Substituting these values in eq^n above

$$x' = \frac{x-vt}{\sqrt{1-v^2/c^2}},\quad y' = y,\quad z' = z,\quad t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}$$

Direct Lorentz transformations

And for small velocities, $\frac{v}{c^2} \rightarrow 0$

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t \quad \left.\right\} \text{Gallilean Transformation}$$

2) In case of light source lieing at S' frame then, we feel S frame is moving in '-ve' x-axis direction.

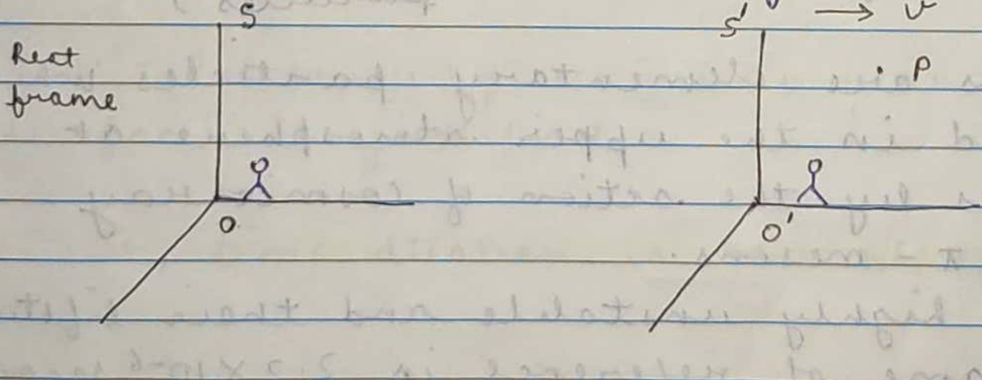
So,

$$\underbrace{x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Inverse Lorentz transformations

Consequences of Lorentz Transformations:

i) Time dilation - slowing down of clock relative to stationary observer.



Taking case of a simple pendulum which is at mean position at t_1' & t_2' seconds (S' frame)

so, time duration $(\Delta t')_{rest} = t_2' - t_1'$

for S frame person
observed time interval $(\Delta t)_{motion} = t_2 - t_1$
↳ improper

We apply lorentz transformation for S frame, as observatn was not appropriate

Applying L.T,

$$(\Delta t)_{motion} = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{(\Delta t')_{rest}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{(\Delta t)_{motion} > (\Delta t')_{rest}}$$

Time dilation is a real effect

* Example - high energy physics (elementary particles)

- μ-mesons are elementary particles which are produced in the upper atmosphere at high altitudes by the action of cosmic ray showers on π-mesons.
  These are highly unstable and their lifetime in own frame of reference is 2.2×10^{-6} seconds

So, the distance traversed by μ-mesons in this life time - (own frame)

$$d = \text{life time} \times \text{velocity}$$
$$= 2.2 \times 10^{-6}\ s \times 0.998c\ m/s$$
$$= 660\ m.$$

$\therefore$ There is no expectation of finding these particles near to the surface of earth. But experimentally these particles have been detected near to the surface of earth.

This can be justified considering time-dilation effect.

own frame

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.2 \times 10^{-6}}{\sqrt{1 - \frac{(0.998c)^2}{c^2}}}$$

laboratory frame observer

$$= 3.17 \times 10^{-5}\ s$$

Now, new $d' = 3.17 \times 10^{-5} \times 0.998c$ (we can use 'c' for ease)

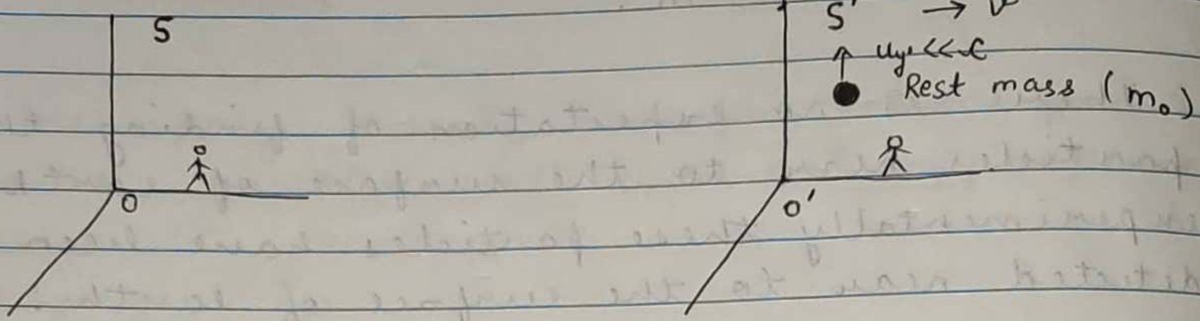
$\simeq 9500\ m$ (approx.)

as observed by laboratory frame

$\therefore$ Presence of μ-mesons near to the earth's surface is justified.

So, time dilation is a real effect.

Mass of particle : (# Mass variation with very extremely very high velocity)



Suppose m_0 has moved dy' distance in dt' time in y - direction

Momentum of particle for S' frame observer

$$P_y' = m_0 u_y' = m_0 \frac{dy'}{dt'} \quad \text{--- (1)}$$

Momentum of same particle for S frame observer

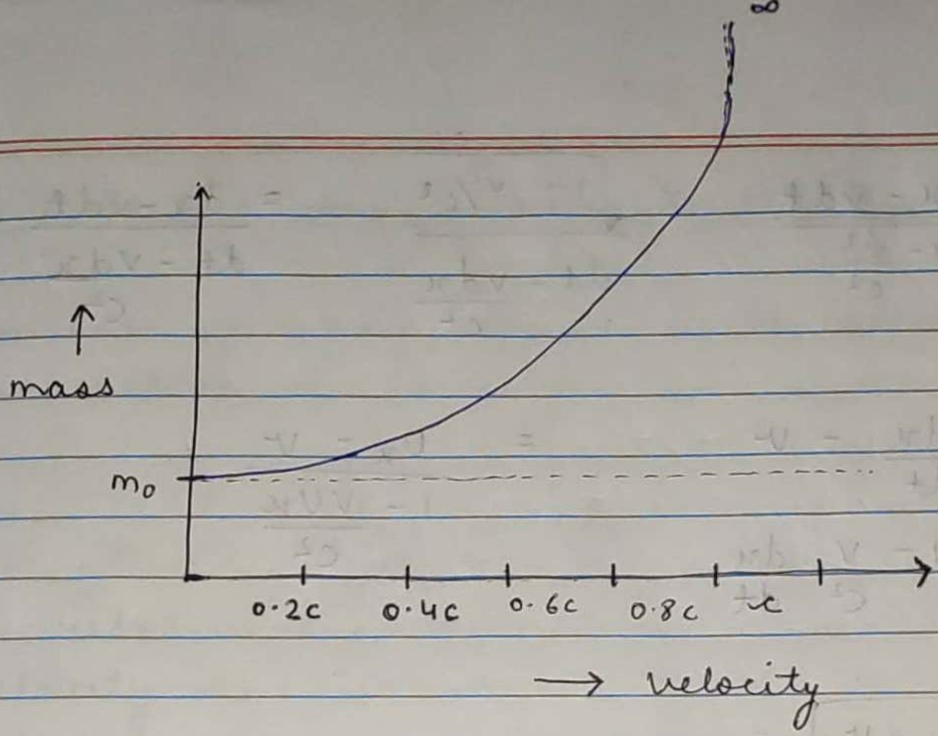
$$dy' = dy, \quad dt = \frac{dt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

putting values of dy & dt (1)

$$P_y = m_0 \frac{dy}{dt\sqrt{1-v^2/c^2}} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \cdot u_y = m \cdot u_y$$

so ; $\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$

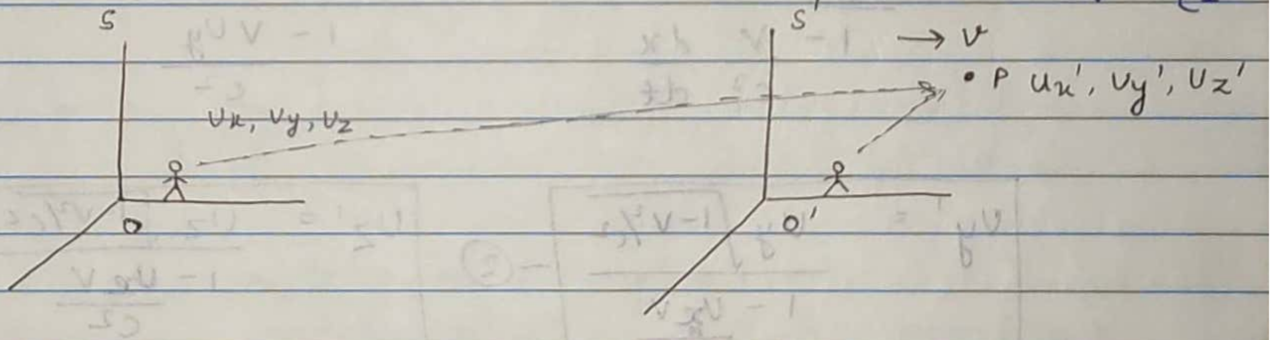
⇒ Graphical Representation



Relativistic addition of velocities

Lorentz transformations are –

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$



Velocity component $u_x' = \frac{dx'}{dt'}$, $u_y' = \frac{dy'}{dt'}$

$$u_z' = \frac{dz'}{dt'}$$

on differentiating Lorentz transformation,

$$dx' = \frac{dx - v\,dt}{\sqrt{1-v^2/c^2}}, \quad dy' = dy, \quad dz' = dz, \quad dt' = \frac{dt - \frac{v\,dx}{c^2}}{\sqrt{1-v^2/c^2}}$$

So, $U_x' = \dfrac{dx - v\,dt}{\cancel{\sqrt{1-\frac{v^2}{c^2}}}} \times \dfrac{\cancel{\sqrt{1-v^2/c^2}}}{dt - \frac{v\,dx}{c^2}} = \dfrac{dx - v\,dt}{dt - \frac{v\,dx}{c^2}}$

$$= \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\cdot\frac{dx}{dt}} = \frac{U_x - v}{1 - \frac{v U_x}{c^2}}$$

$$\boxed{U_x' = \frac{U_x - v}{1 - \frac{U_x v}{c^2}}} \quad \text{— (1)}$$

Similarly, $U_y' = \dfrac{dy'}{dt'} = \dfrac{dy\sqrt{1-v^2/c^2}}{dt - \frac{v}{c^2}dx}$

$$= \frac{\frac{dy}{dt}\sqrt{1-v^2/c^2}}{1 - \frac{v}{c^2}\cdot\frac{dx}{dt}} = \frac{U_y\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{v U_y}{c^2}}$$

$$\boxed{U_y' = \frac{U_y\sqrt{1-v^2/c^2}}{1 - \frac{U_x v}{c^2}}} \quad \text{— (2)} \qquad \boxed{U_z' = \frac{U_z\sqrt{1-v^2/c^2}}{1 - \frac{U_x v}{c^2}}} \quad \text{— (3)}$$

ii) Let the moving object move with 'c' velocity

$U_x' = \dfrac{c - v}{1 - \frac{v\cdot c}{c^2}}$; $U_x' = \dfrac{c - v}{1 - \frac{v}{c}} = c$ (invarient)

It confirms 2nd basic postulate

Einstein's mass energy relation

Consider a body moves with velocity v under the action of force F. Therefore -

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt} \quad — (1)$$

In relativistic mechanics, both mass and velocity are variable.

$$F = m\frac{dv}{dt} + v \cdot \frac{dm}{dt} \quad — (2)$$

Work done on the body $dW = dK = F \cdot dS$

$$dK = m\frac{dv \cdot dS}{dt} + v \cdot \frac{dm}{dt} \cdot dS$$

$$= mvdv + v^2 dm \quad — (3)$$

$\because$ $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$; $m_0^2 = m^2\left[1 - \frac{v^2}{c^2}\right]$

$$m_0^2 c^2 = m^2 [c^2 - v^2]$$

$$m^2c^2 - m^2v^2 = m_0^2c^2 \quad — (4)$$

Diff (4), we get,

$$2mdmc^2 - [m^2 \cdot 2vdv + v^2 2m \cdot dm] = 0$$

$$mvdv + v^2 dm = c^2 dm \quad ; \text{ subst. in (3)}$$

$$dK = c^2 dm \quad — (5).$$

Let K be the kinetic energy of the body (during movement), suppose mass changes from $m_0 \rightarrow m$

$$\text{Total KE} = \int_{m_0}^{m} c^2 dm = (m - m_0)c^2 \quad -(6)$$

~~K.E~~ =

Total energy = Rest mass energy + Kinetic energy

$$= m_0c^2 + (m - m_0)c^2$$

$$\boxed{E = mc^2}$$

Relativistic kinetic energy $= (m - m_0)c^2$

$$= \left(\frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 \right) c^2$$

$$= m_0c^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right]$$

$$= m_0c^2 \left[1 + \frac{v^2}{2c^2} + \cdots - 1 \right]$$

for smaller velocities, higher powers can be neglected.

$$= m_0c^2 \cdot \frac{v^2}{2c^2} = \boxed{\frac{1}{2} m_0 v^2}$$

Relativistic momentum & Energy

$$\boxed{E^2 = p^2c^2 + m_0^2c^4}$$

Proof; ~~m^2~~ $E^2 - p^2c^2 = m^2c^4 - m^2v^2c^2$

$$= \frac{m_0^2 c^4}{\left(1 - \frac{v^2}{c^2}\right)} - \frac{m_0^2 c^2 v^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$= m_0^2 c^4$$ (verified)