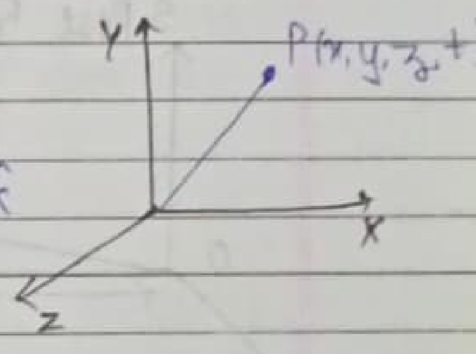
UNIT - 1

RELATIVISTIC MECHANICS

* **Frame Of Reference**

A geometrical framework which is used to describe the occurence of event in the space is frame of reference.
The description of the event can be told by - 3 - space coordinates
1 - time coordinate

- **Position vector of P at instant 't'**



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Velocity - $\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

Acceleration - $\vec{a} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$

* Classification Of Frame of Reference

Inertial Frame [Non accelerated frames]
(It follows the law of inertia)

$a = \frac{d^2r}{dt^2} = 0$ | In component form -

$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$

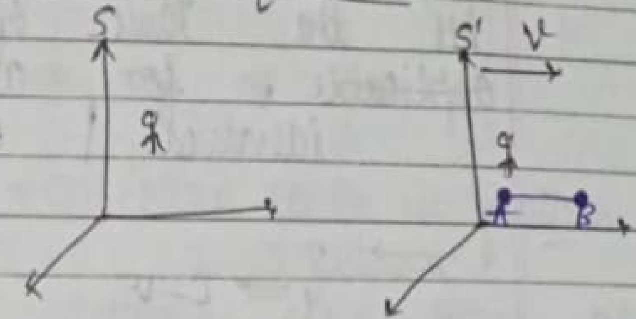
Non- inertial frame
(Does not follow law of inertia)

$a = \frac{d^2r}{dt^2} \neq 0$

[Accelerated frames of Reference]

* Consequences of Transformation Equation

→ Length of an object is absolute (invariant)



S' Frame: $A(x_1', y_1', z_1')$ $B(x_2', y_2', z_2')$

S Frame: $A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

$$L' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2}$$

Let us transform x_1', x_2' etc. using G.T

$$= \sqrt{(x_2 - vt - x_1 + vt)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= L$$

→ Velocity of moving object :-

$$x' = x - vt \quad (1)$$

$$\frac{dx'}{dt} = \frac{dx}{dt} - v$$

$$\boxed{V' = V - v} \quad \text{or} \quad V' = \quad \boxed{V = V' + v}$$

$\cancel{v' = v}$

Galilean Addition of velocity formula

→ Acceleration

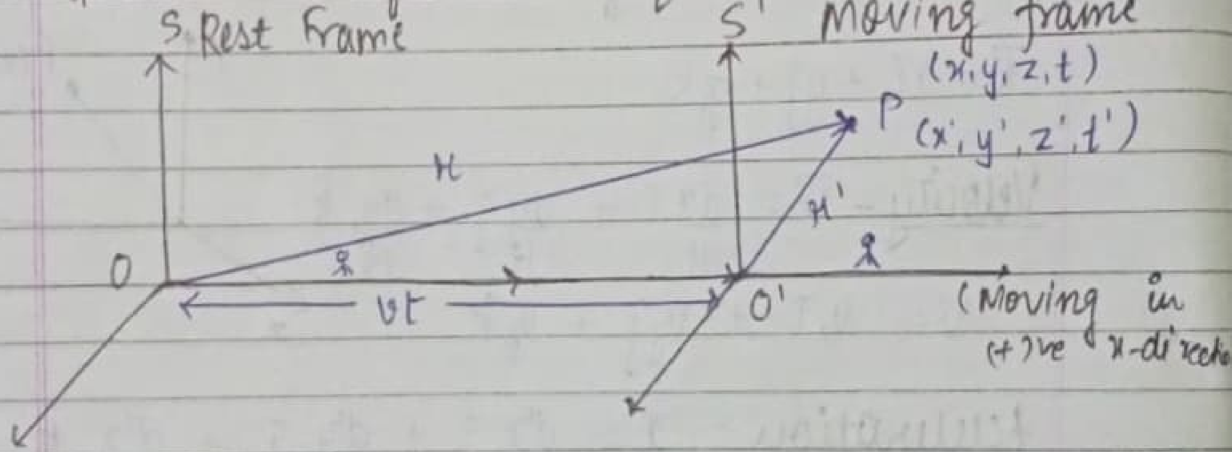
$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} - 0$$

$$\Rightarrow \boxed{a' = a} \quad \Rightarrow \boxed{F' = F}$$

Non-inertial frames—
Particle will experience some force due to acceleration of reference known as pseudo force.

Theory of Relativity
→ which is applicable to inertial frame is known as special theory of relativity
→ which is applicable to non-inertial frame is known as general theory of relativity.

* Galilean Transformation Equations



→ At $t = t' = 0$, origins of both frames coincide.

$$\boxed{r = r' + vt}$$

In component form.

$$x = x' + vt$$
or
$$x' = x - vt$$
$$y' = y, \quad z' = z, \quad t' = t$$

Galilean Transformation.
valid for smaller velocities

- Gallilean Hypothesis of Invariance :-
All the laws of physics are ~~applicable~~ identical for all the observers of inertial frame.

$$\left.\begin{array}{l}\xrightarrow{c}\\ \xrightarrow{v}\end{array}\right\} \Rightarrow c-v \qquad \left.\begin{array}{l}\xrightarrow{c}\\ \xleftarrow{v}\end{array}\right\} \Rightarrow c+v$$

Michaelson Morley Experiment (1920)

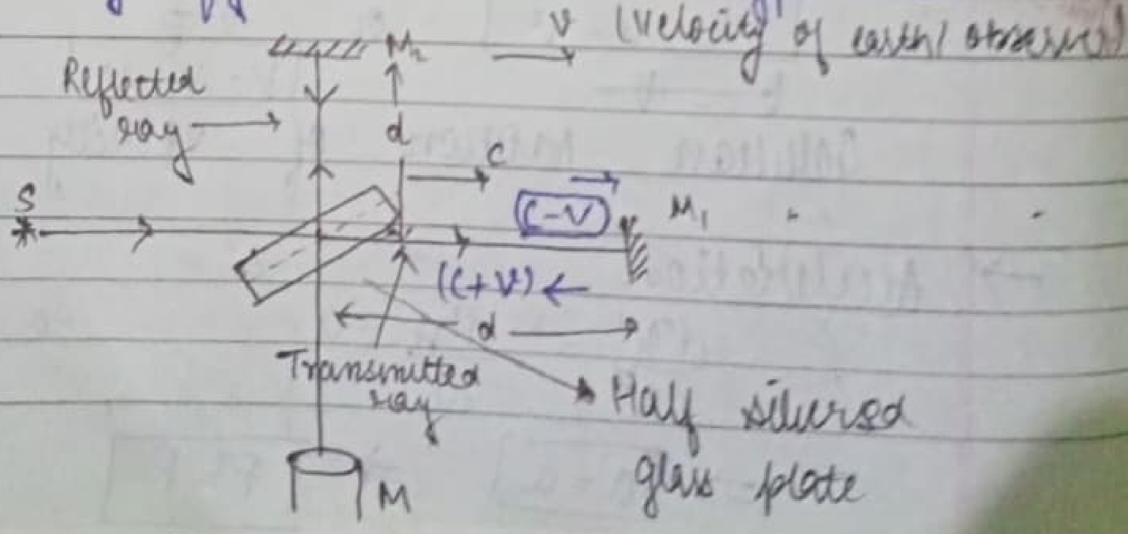
Objective → Whether speed of light c gets modified in accordance with G.T.

→ Aether - Material medium which was supposed to be present throughout the universe

→ Perfectly elastic
→ Highly transparent
→ Negligible density

→ Existence of aether was assumed as absolute frame of reference relative to which the motion of bodies can be detected.

→ To justify the aether hypothesis.



$$\Delta t = t_2 - t_1 = \frac{2d}{c} - \frac{2d}{c} = 0$$

$\Rightarrow$ Earth is at rest

* $\Delta t = t_2 - t_1 \neq 0$ (When earth in motion is considered)

Some path diff. will definitely occur either b/w transmitted and reflected ray.

Path diff $= \frac{dv^2}{c^2}$ Fringe pattern should appear and visible

$\boxed{\text{No fringe shift was experimentally observed}}$

$\Rightarrow v = 0$ → Motion of earth could not be detected related to aether.

→ Earth is absolutely at rest in aether.

$\boxed{\text{Negative Result}}$

Aether's hypothesis was rejected.

* G.T. are not valid for c. The speed of light is universal constant ~~in~~ identical for all the observers of inertial frame.

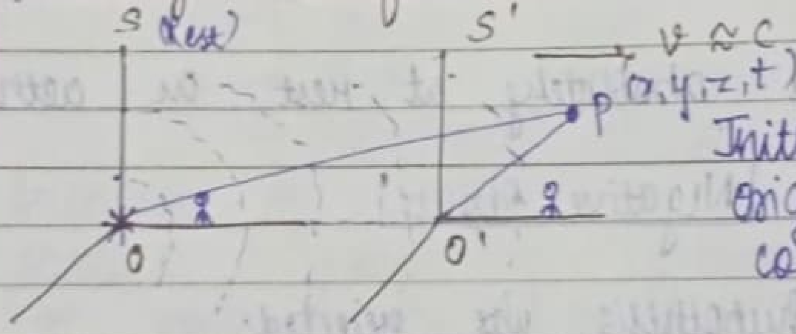
23/12/2020

* Basic Postulates of Relativity:-

i) All the laws of physics are identical for all the observers of the inertial frame that move with a constant velocity relative to one another.

* ii) The velocity of light is a universal constant and is identical for all the observers of the inertial frame. This is also known as constancy of velocity of light.

• Lorentz Transformations



Initially at $t = t' = 0$ origins are coinciding

When S & S' are coinciding a light signal emits from O.
When light signal reaches at P, the coordinates of incident wavefront observed by rest observer is (x, y, z, t)
Similarly, that by the moving frame is (x', y', z', t')

Time taken by light signal to reach P for S- frame observer :-

$$t = \frac{OP}{c} = \frac{(x^2+y^2+z^2)^{1/2}}{c}$$

$$\Rightarrow x^2+y^2+z^2 = c^2t^2 \quad \text{—— (1)}$$

In S' frame observer —

$$t' = \frac{O'P}{c} = \frac{(x'^2+y'^2+z'^2)^{1/2}}{c}$$

$$\Rightarrow x'^2+y'^2+z'^2 = c^2t'^2 \quad \text{—— (2)}$$

Since c is identical for both the observers

The new transformations will be such that eq (2) transform eq (1).

Let us apply G.T. again,

$x' = x - vt$, $y' = y$, $z' = z$, $t' = t$

$$(x-vt)^2 + y^2 + z^2 = c^2t^2$$

$$x^2 + v^2t^2 - 2vxt + y^2 + z^2 = c^2t^2 \quad \text{—— (3)}$$

Comparing eq (1) & eq (3)

Extra term $\longrightarrow$ $v^2t^2 - 2vxt$

$\Rightarrow$ G.T. fail or need some modification.

$\therefore$ new transformations b/w (x, y, z, t) & (x', y', z', t') should be such that eq^n (3) transform eq^n (1) & extra term $v^2t^2 - 2vxt$ should cancel.

Also new transformation should lead to G.T. for smaller velocities

Let the new / modified eq^{n}s can be written as :-

$$x' = \alpha(x - vt) \; ; \; t' = \alpha'(t + f x)$$

where α, α' & f are const.

On substituting these in eq^n (2).

$$\alpha^2(x-vt)^2 + y^2 + z^2 = c^2\alpha'^2(t+fx)^2$$

$$\alpha^2(x^2 + v^2t^2 - 2vxt) + y^2 + z^2 = c^2\alpha'^2(t^2 + f^2x^2 + 2fxt)$$

On equating coefficients of x^2, x and constant we get

$$x^2[\alpha^2 - f^2\alpha'^2c^2] - 2xt[\alpha^2 v - fc^2\alpha'^2] + y^2 + z^2 = c^2t^2\left[\alpha'^2 - \frac{\alpha^2 v^2}{c^2}\right]$$

On comparing with eq^n (2)

$$\alpha^2 - f^2\alpha'^2c^2 = 1$$

$$\alpha^2 v + fc^2\alpha'^2 = 0$$

$$\alpha^2 - \frac{\alpha^2 v^2}{c^2} = 1$$

On solving above equations -

$$\alpha = \alpha' = \frac{1}{\sqrt{1 - v^2/c^2}} \; ; \; f = -\frac{v}{c^2}$$

On substituting these in eq^n

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \; y' = y, \; z' = z, \; t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

These are known as direct Lorentz transformations.

For small velocities, $\Rightarrow \frac{v}{c^2} \to 0$

$x' = x - vt$, $y' = y$, $z' = z$, $t' = t$
These are Galilean transformation.

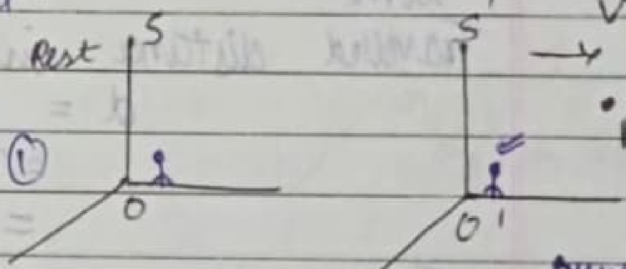
$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Inverse Lorentz Transformation

$$\begin{cases} x = \alpha(x' + vt') & t = \alpha'(t' + \beta x') \\ & \alpha = \alpha' = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \beta = v/c^2 \end{cases}$$

* Consequences of Lorentz Transformations:-

i) Time dilation: slowing down of clock relative to stationery object.



Proper time interval
$(\Delta t')_{rest} = t_2' - t_1'$ — (1)

Observed time interval
(improper) $(\Delta t)_{motion} = t_2 - t_1$ — (2)

Event 1 t_1', t_2'
Event 2 t_1, t_2

On applying Inverse Lorentz transformation

$$\cancel{= \frac{t_2' - \frac{v}{c^2}x'}{\sqrt{1 - \frac{c^2}{c^2}}} - \frac{t_1' + vx'}{c^2}}$$

$$(\Delta t)_{mot} = \frac{t_2' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}} = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}} = \frac{(\Delta t')_{rest}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{(\Delta t)_{motion} > (\Delta t')_{rest}}$$

Time Dilation is a real effect -

→ Example from high energy physics (Elementary particle)

μ- mesons are elementary particles which are produced in the upper atmosphere at high altitudes by the action of cosmic ray showers on π- mesons. These are highly unstable and their lifetime in own frame of reference is 2.2×10^{-6} seconds. So the distance traversed by μ- mesons in this life time -

Travelled distance in own frame.

$$d = \text{life time} \times \text{velocity}$$
$$= 2.2 \times 10^{-6} s \times 0.998c$$
$$= 660 \text{ meter}$$

Therefore, there is no expectation of finding these particles near to the surface of the earth. But experimentally these particles had been detected near to the surface of the earth. This can be justified considering the time dilation effect.

For laboratory frame observer on the ground

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6} s}{\sqrt{1 - \frac{(0.998c)^2}{c^2}}} = 3.17 \times 10^{-5} s$$

New distance (observed by laboratory frame observer) :-

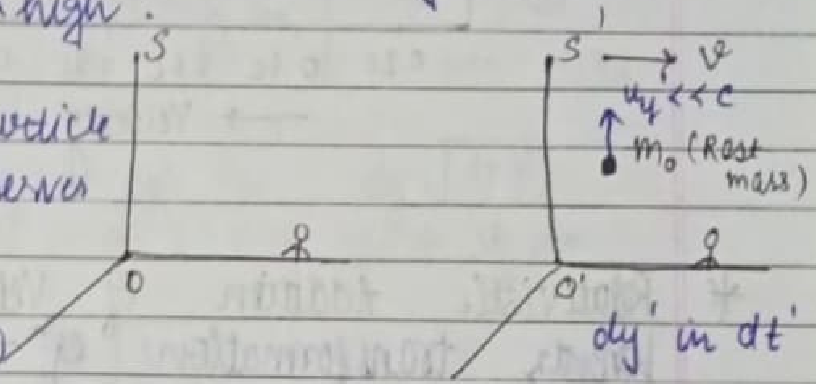
$$d' = 3.17 \times 10^{-5} \times 3.0 \times 10^{8}$$

$d' = 9500\ m$ (approx.)

⇒ The presence of μ-mesons near to the surface of the earth is justified.
⇒ Time dilation is a real effect.

ii) <u>Mass of Particle</u> :- Mass undergoes variation with velocity when velocity become extremely high.

Momentum of particle for S' frame observer



$P_y' = m_0 u_y'$

$P_y' = m_0 \dfrac{dy'}{dt'}$ —①

Momentum of this same particle for S frame observer

$dy' = dy$, $dt' = \dfrac{dt'}{\sqrt{1-\frac{v^2}{c^2}}}$

Using these in eqⁿ ①

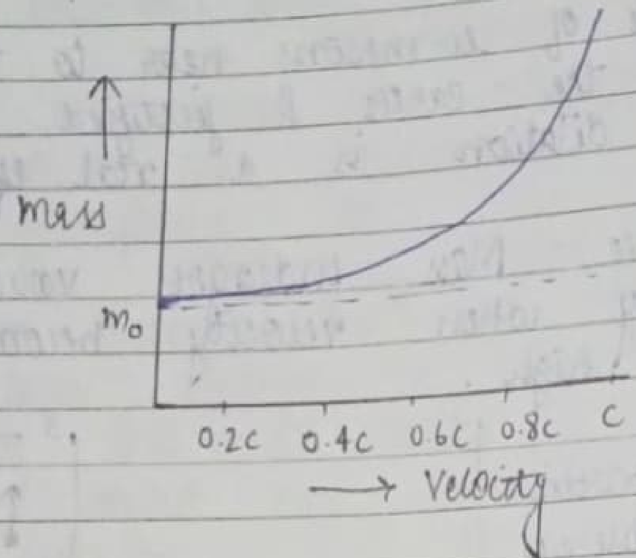
$$P_y = \frac{m_0\, dy}{dt\sqrt{1-\frac{v^2}{c^2}}}$$

$$P_y = \frac{m_0 u_y}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$P_y = \underline{m} u_y$$

$$\boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}}$$

* Graphical Representation



* Relativistic Addition of Velocities :-

Lorentz transformation eq^n :-

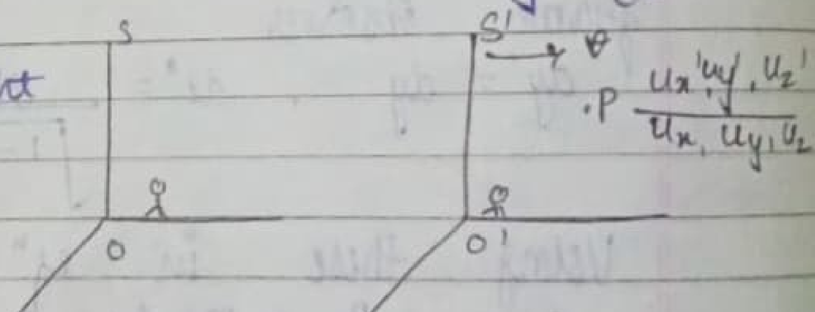
$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$; $y' = y$; $z' = z$; $t' = \frac{t - \frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$

Velocity component

$u_x' = \frac{dx'}{dt'}$

$u_y' = \frac{dy'}{dt'}$

$u_z' = \frac{dz'}{dt'}$



On diff. Lorentz transformation eq^n -

$dx' = \frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}}$, $dy' = dy$, $dz' = dz$

$dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1 - \frac{v^2}{c^2}}}$

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$$u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{\sqrt{1-\frac{v^2}{c^2}}} \times \frac{\sqrt{1-v^2/c^2}}{dt - \frac{v}{c^2}dx}$$

$$= \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

Therefore, $\boxed{u_x' = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}}$ — (i)

Similarly $u_y' = \frac{dy'}{dt'} = \frac{dy\sqrt{1-\frac{v^2}{c^2}}}{dt - \frac{v}{c^2}dx}$

$$= \frac{\frac{dy}{dt}\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$\boxed{u_y' = \frac{u_y\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{v}{c^2}u_x}}$$ — (2)

$u_z' = \frac{dz'}{dt'} \Rightarrow \boxed{u_z' = \frac{u_z\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{v}{c^2}u_x}}$ — (3)

Let the moving object moves with velocity c –

$$u_x' = \frac{c - v}{1 - \frac{v.c}{c^2}} = \left(\frac{c-v}{c-v}\right)c$$

$$= c$$

Confirms second basic postulate

* <u>Einstein's Mass- Energy Relation</u> $E = mc^2$

Consider a body moves with velocity under the action of force F, therefore.

$$F = \frac{dP}{dt} = \frac{d}{dt}(mv) \quad \text{— (1)}$$

In relativistic mechanics, both the mass & velocity are variable.

$$F = \frac{m dv}{dt} + \frac{v dm}{dt} \quad \text{— (2)}$$

Work done on the body $dW = dK = F.ds$

$$dK = \frac{m dv}{dt} . ds + \frac{v dm}{dt} dx$$

$$dK = mv dv + v^2 dm \quad \text{— (3)}$$

Mass variation can be expressed as—

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow m^2\left(1-\frac{v^2}{c^2}\right) = m_0^2$$

$$\Rightarrow m^2[c^2-v^2] = m_0^2c^2$$

$$\Rightarrow m^2c^2 - m^2v^2 = m_0^2c^2 \quad \text{— (4)}$$

On differentiating eq^n (4)

$$\cancel{2}\cancel{m} dm c^2 - [m^{\cancel{2}} \cancel{2} v dv + v^2 . \cancel{2} \cancel{m} dm] = 0$$

$$\Rightarrow mv dv + v^2 dm = c^2 dm$$

On substituting this in eq^n (3)

$$\therefore dK = c^2 dm \quad \text{— (5)}$$

Let K be the K.E. of the body &

during movement, its mass changes from m_0 to m.

Total K.E. , $K = \int_{m_0}^{m} c^2 \, dm = (m - m_0)c^2$ — (6)

Total Energy, E = Rest mass energy + K.E.

$= m_0c^2 + (m - m_0)c^2$

$$\Rightarrow \boxed{E = mc^2}$$

* <u>Relativistic Kinetic Energy</u>

$$K = (m - m_0)c^2 \quad \text{— (1)}$$

Classical value of Kinetic Energy $= \frac{1}{2}mv^2$

$$= \left(\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0\right)c^2$$

$$= m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

$$= m_0c^2\left[1 + \frac{v^2}{2c^2} + \dots - 1\right]$$

For ~~higher~~ smaller velocities, higher powers can be neglected.

$$\boxed{KE = \frac{1}{2}mv^2}$$

* ~~Rlat~~ <u>Relativistic Momentum & Energy</u>

$$E^2 = p^2c^2 + m_0^2c^4$$

$$E^2 - p^2c^2 = m_0^2c^4$$

$$m^2c^4 - m^2v^2c^2$$

$$\frac{m_0^2c^4}{\left(1-v^2/c^2\right)} - \frac{m_0^2c^2v^2}{\left(1-v^2/c^2\right)}$$

On simplifying

$$= m_0^2c^4 \quad (RHS)$$