Newton's Raphson Method:

Let x_0 be an approximate root of the equation $f(x)=0$

If x_0+h be the exact root, then $f(x_0+h)=0$

h is the difference between approximate and exact value of the root

$\therefore$ Expanding $f(x_0+h)$ by Taylor's series

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots$$

$$0 = f(x_0) + h f'(x_0)$$

($\because$ h is small $\therefore$ neglecting h^2 and higher powers of h)

$$\therefore h = -\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

similarly next approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad n=0,1,2 \cdots$$

It is known as Newton-Raphson formula

Ex. Let us find the smallest Positive root of $x^3-5x+3=0$

sol

$$f(x) = x^3 - 5x + 3$$
$$f(0) = 0 - 0 + 3 = 3$$
$$f(1) = 1 - 5 + 3 = -1$$

root lies between 0 and 1

If we take $x_0 = 1$, we get using N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)}$$

$\because$ $f'(x) = 3x^2 - 5$
$f'(1) = 3\times1^2 - 5 = -2$

$\therefore$ $x_1 = 1 - \frac{(-1)}{(-2)} = 1 - \frac{1}{2} = 0.5$

similarly next iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= .5 + \frac{.5}{.34} = 0.64$$

$$x_3 = x_2 + \frac{f(x_2)}{f'(x_2)}$$

$$= .64 + \frac{.062144}{3.7712} = .6565$$

$$x_4 = .6565 + \frac{.0004464}{3.70702325} = .656620$$

$$x_5 = .656620 + \frac{.00000115976}{3.70655053} = .65662043$$

we observe that convergence is very rapid.

~~as~~ Approximate value of the root correct ~~to~~ to three decimal places is .656 Ans

Note: ① This method is useful in cases of large values of $f'(x)$ i.e when the graph of $f(x)$ while crossing the x axis is nearly vertical

② If $f'(x)$ is zero or nearly 0, the method fails

③ Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root.

④ This method is also used to find the complex roots.

another Example :

using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.

$$f(x) = x \log_{10} x - 1.2$$

$f(1) = -1.2$

$f(2) = 2 \log_{10} 2 - 1.2 = \cdot 59794 = -ve$

$f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = \cdot 23136$ +ve

so a root of $f(x) = 0$ lies between 2 and 3

let us take $x_0 = 2$

Also $f'(x) = \log_{10} x + x \times \frac{1}{x} \log_{10} e$

$= \log_{10} x + \cdot 43429$

Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= \frac{\cdot 43429\, x_n + 1.2}{\log_{10} x_n + 0.43429}$$

putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + \cdot 43429} = 2.81$$

putting $n = 1, 2, 3, 4$ we get

$$x_2 = \frac{.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74064$$

$$x_4 = \frac{.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + .43429} = 2.74065$$

$$x_5 = \frac{.43429 \times 2.7465 + 1.2}{\log_{10} 2.74065 + .43429} = 2.74065$$

$\because x_4 = x_5$

Hence required roots is 2.74065 Correct to five decimal places, Ans

Note:⇒ Newton's Method is generally used to improve the result obtained by other methods.

⇒

Convergence of Newton's Raphson Method

Suppose x_n differs from the root α by a small quantity ϵ_n so that

$$x_n = \alpha + \epsilon_n$$
$$x_{n+1} = \alpha + \epsilon_{n+1}$$

then using the Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\cancel{\alpha} + \epsilon_{n+1} = \cancel{\alpha} + \epsilon_n - \frac{f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$= \epsilon_n - \frac{\overset{\to 0}{f(\alpha)} + \epsilon_n f'(\alpha) + \frac{1}{2!}\epsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots}$$

$$= \frac{\{\cancel{\epsilon_n f'(\alpha)} + \epsilon_n^2 f''(\alpha) + \cdots\} - \{\cancel{\epsilon_n f'(\alpha)} + \frac{\epsilon_n^2}{2!} f''(\alpha) \cdots\}}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots}$$

$$= \frac{\frac{\epsilon_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + \epsilon_n f''(\alpha) + \cdots}$$

$$\epsilon_{n+1} \simeq \frac{\epsilon_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \Rightarrow \frac{\epsilon_{n+1}}{\epsilon_n^2} = \frac{f''(\alpha)}{2 f'(\alpha)}$$

This shows that N.R Method has quadratic convergence.

Q1. using Newton's iterative method, find a root of the following equations correct to 4 decimal places.

(i) $x^4 - x - 10 = 0$

(ii) $x^5 - 5x^2 + 3 = 0$

Q.2. using Newton's Raphson method to find a root of the following equations correct to 3 decimal places

(i) $x \sin x + \cos x = 0$ which is near $x = \pi$

(ii) $e^x = x^3 + \cos 25x$ which is near 4.5

Q.3 describe the geometrical interpretation of N-R. Method.

Q4 find the convergence criteria of N-R. Method.