Method of Variation of parameter

$$\boxed{\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.}$$

1). first find C.F

2). let C.F = $c_1u + c_2v$

3). consider complete solution is $y = Au + Bv$

4). A and B are determined by.

$$A = -\int \frac{Rv}{uv_1 - u_1v}\,dx + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + c_2$$

(1). $$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

$$(D^2 + a^2)y = \sec ax$$

$m^2 + a^2 = 0$ $\qquad m = \pm ia$

C.F = $c_1 \cos ax + c_2 \sin ax$

$u = \cos ax$ $\qquad v = \sin ax$

let $y = Au + Bv$

$$A = -\int \frac{\sec ax \cdot \sin ax}{a\cos^2 ax - (-a\sin ax)\cdot \sin ax}\,dx + c_1$$

$$= -\int \frac{\tan ax}{a(\cos^2 ax + \sin^2 ax)}\,dx + c_1$$

$$= -\frac{1}{a}\int \tan ax\,dx + c_1 \qquad = -\frac{1}{a^2}\log \sec ax + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1 v}\, dx + C_2$$

$$= \int \frac{\sec ax \cdot \cos ax}{a(\cos^2 ax + \sin^2 ax)}\, dx + C_2$$

$$= \frac{1}{a} \int dx + C_2 = \frac{1}{a} x + C_2 .$$

hence complete solution $y = Au + Bv$

$$y = \left(\frac{1}{a^2} \log \cos ax + C_1 \right) \cos ax + \left(\frac{x}{a} + C_2 \right) \sin ax$$

(2). Use the variation of parameter to solve the differential equation.

$$x^2 y'' + xy' - y = x^2 e^x$$

Solving (2), (3) and (4), we get A_1, B_1 and C_1 which by integration will give A, B and C.

As the solution is obtained by varying the arbitrary constants of the complementary function, the above method is known as that of **Variation of Parameters.**

ILLUSTRATIVE EXAMPLES

Example 1. *Solve by the method of variation of parameters:*

$$\frac{d^2y}{dx^2} + a^2y = \sec ax.$$ [U.P.T.U. (SUM) 2008; U.K.T.U. 2012]

Sol. Here, $u = \cos ax$, $v = \sin ax$ are two parts of C.F.

Also, $R = \sec ax$.

Let the complete solution be

$$y = A\cos ax + B\sin ax \quad ...(1)$$

where A and B are suitable functions of x.

To determine the values of A and B, we have

$$A = \int \frac{-Rv}{uv_1 - u_1v}\,dx + c_1$$

$$= \int \frac{-\sec ax \, . \sin ax}{\{\cos ax \, . \, a\cos ax - (-a\sin ax)\sin ax\}}\,dx + c_1$$

$$= -\int \frac{\tan ax}{a}\,dx + c_1$$

$$\Rightarrow \quad A = \frac{1}{a^2}\log\cos ax + c_1$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + c_2$$

$$= \int \frac{\sec ax \, . \cos ax}{\{\cos ax \, . \, a\cos ax - (-a\sin ax)\sin ax\}}\,dx + c_2$$

$$= \frac{1}{a}\int dx + c_2 = \frac{x}{a} + c_2$$

where c_2 is an arbitrary constant of integration.

Hence the complete solution is given by

$$y = A\cos ax + B\sin ax$$

$$= \left(\frac{\log\cos ax}{a^2} + c_1\right)\cos ax + \left(\frac{x}{a} + c_2\right)\sin ax.$$

Example 2. *Solve by method of variation of parameters:*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$$

Sol. Parts of C.F. are 1 and e^{2x}.

Let $u = 1, v = e^{2x}$ Also, $R = e^x \sin x$

Let $y = A + Be^{2x}$ be the complete solution

where A and B are suitable functions of x determined by

$$A = \int \frac{-Rv}{uv_1 - u_1 v} dx + c_1 = -\int \frac{e^x \sin x \,.\, e^{2x}}{1.\,2\,e^{2x}} dx + c_1$$

$$= -\frac{1}{2}\int e^x \sin x \, dx + c_1 = -\frac{1}{2}\left[\frac{e^x}{1+1}(\sin x - \cos x)\right] + c_1$$

$$= -\frac{e^x}{4}(\sin x - \cos x) + c_1$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{Ru}{uv_1 - u_1 v} dx + c_2 = \int \frac{e^x \sin x \,.\, 1}{1.\,2e^{2x}} dx + c_2$$

$$= \frac{1}{2}\int e^{-x} \sin x \, dx + c_2 = \frac{1}{2}\left[\frac{e^{-x}}{1+1}(-\sin x - \cos x)\right] + c_2$$

$$= -\frac{e^{-x}}{4}(\sin x + \cos x) + c_2$$

where c_2 is an arbitrary constant of integration.

The complete solution is

$$y = A + B\,e^{2x}$$

$$= \frac{e^x}{4}(\cos x - \sin x) + c_1 + \left[-\frac{e^{-x}}{4}(\sin x + \cos x) + c_2\right]e^{2x}$$

$$= \frac{e^x}{4}(\cos x - \sin x) + c_1 - \frac{e^x}{4}(\sin x + \cos x) + c_2\,e^{2x}$$

$$\Rightarrow \qquad y = c_1 + c_2\,e^{2x} - \frac{e^x}{2}\sin x$$

Example 3. *Apply the method of variation of parameters to solve the ordinary differential equations:*

(*i*) $\dfrac{d^2y}{dx^2} + y = \tan x$ [U.P.T.U. 2009; U.K.T.U. 2011]

(*ii*) $(D^2 - 1)y = 2(1 - e^{-2x})^{-1/2}$. [M.T.U. 2011; U.P.T.U. (SUM) 2009]

Sol. (*i*) Parts of C.F. are $u = \cos x$ and $v = \sin x$

Let $y = A\cos x + B\sin x$ be the complete solution of the given equation where A and B are determined as:

Hence the complete solution is

$$y = [\log (e^{-x} + 1) + c_1]\, e^x + [\log (1 + e^{-x}) - (1 + e^{-x}) + c_2]\, e^{2x}$$

Example 6. *Solve by method of variation of parameters:*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} \log x.$$ (U.P.T.U. 2008)

Sol. Parts of C.F. are $u = e^{-x}$, $v = xe^{-x}$ and $R = e^{-x} \log x$

Let $y = Ae^{-x} + Bxe^{-x}$ be the complete solution where A and B are some suitable functions of x. To determine A and B, we have

$$A = -\int \frac{Rv}{uv_1 - u_1v}\,dx + c_1 = -\int \frac{e^{-x} \log x \,.\, xe^{-x}}{e^{-x}(e^{-x} - xe^{-x}) + xe^{-2x}}\,dx + c_1$$

$$= -\int x \log x \,dx + c_1 = -\frac{x^2}{2}\log x + \frac{x^2}{4} + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + c_2 = \int \frac{e^{-x} \log x \,.\, e^{-x}}{e^{-2x}}\,dx + c_2$$

$$= \int \log x \,dx + c_2 = x \log x - x + c_2$$

Hence the complete solution is

$$y = Ae^{-x} + Bxe^{-x} = \left(-\frac{x^2}{2}\log x + \frac{x^2}{4} + c_1\right)e^{-x} + (x \log x - x + c_2)\,xe^{-x}$$

Example 7. *Use the variation of parameter method to solve the differential equation* $x^2y'' + xy' - y = x^2e^x$. (U.P.T.U. 2006)

Sol. The given equation is

$$x^2y'' + xy' - y = x^2e^x \quad ...(1)$$

$$\Rightarrow \quad y'' + \frac{y'}{x} - \frac{y}{x^2} = e^x \quad ...(2)$$

Here, $R = e^x$

Consider the equation $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$ for finding parts of C.F.

Put $x = e^z$ so that $z = \log x$ and Let $D \equiv \frac{d}{dz}$ then the above equation reduces to

$$[D(D-1) + D - 1]\, y = 0$$

$$\Rightarrow \quad (D^2 - 1)\, y = 0 \qquad \ldots(3)$$

Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$\therefore \quad \text{C.F.} = c_1 e^z + c_2 e^{-z} = c_1 x + c_2 \cdot \frac{1}{x}$$

Hence parts of C.F. are x and $\frac{1}{x}$

Let $\quad u = x$ and $v = \frac{1}{x}$

Let $y = Ax + \frac{B}{x}$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows:

$$A = -\int \frac{Rv}{uv_1 - u_1 v}\, dx + c_1 = -\int \frac{e^x \cdot \frac{1}{x}}{x \cdot \left(\frac{-1}{x^2}\right) - 1 \cdot \left(\frac{1}{x}\right)}\, dx + c_1$$

$$= -\int \frac{e^x \cdot \frac{1}{x}}{\left(\frac{-2}{x}\right)}\, dx + c_1 = \frac{1}{2} e^x + c_1$$

and

$$B = \int \frac{Ru}{uv_1 - u_1 v}\, dx + c_2 = \int \frac{e^x \cdot x}{x\left(\frac{-1}{x^2}\right) - 1\left(\frac{1}{x}\right)}\, dx + c_2$$

$$= \int \frac{e^x \cdot x}{\left(\frac{-2}{x}\right)}\, dx + c_2 = -\frac{1}{2}\int x^2 e^x\, dx + c_2$$

$$= -\frac{1}{2}\left[x^2 e^x - \int 2x\, e^x\, dx\right] + c_2 = -\frac{1}{2}[x^2 e^x - 2(x-1)\, e^x] + c_2$$

$$= -\frac{1}{2} x^2 e^x + (x-1)\, e^x + c_2$$

Hence the complete solution is given by

$$y = Ax + \frac{B}{x} = \left(\frac{1}{2} e^x + c_1\right) x + \left[-\frac{1}{2} x^2 e^x + (x-1)\, e^x + c_2\right] \cdot \frac{1}{x}$$

$$\Rightarrow \quad y = c_1 x + \frac{c_2}{x} + \left(1 - \frac{1}{x}\right) e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Using variation of parameters method, solve:*

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x.$$

Sol. Consider the equation

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = 0 \text{ for finding parts of C.F.}$$

Put $x = e^z$ so that $z = \log x$ and Let $D \equiv \frac{d}{dz}$ then the given equation reduces to

$$[D(D-1) + 2D - 12]\, y = 0$$

$\Rightarrow$

$$(D^2 + D - 12)\, y = 0$$

Auxiliary equation is

$$m^2 + m - 12 = 0 \quad \Rightarrow \quad m = 3, -4$$

$\therefore$

$$\text{C.F.} = c_1 e^{3z} + c_2 e^{-4z} = c_1 x^3 + c_2 x^{-4}$$

Hence, parts of C.F. are x^3 and x^{-4}

Let

$$u = x^3 \quad \text{and} \quad v = x^{-4}. \quad \text{Also,} \quad R = x \log x$$

Let $y = Au + Bv$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows:

$$A = -\int \frac{Rv}{uv_1 - u_1 v}\, dx + c_1 = -\int \frac{x \log x \,.\, x^{-4}}{x^3 \,.\, (-4x^{-5}) - 3x^2 (x^{-4})}\, dx + c_1$$

$$= -\int \frac{x^{-3} \log x}{-7x^{-2}}\, dx + c_1 = \frac{1}{7} \int \frac{\log x}{x}\, dx + c_1 = \frac{1}{14} (\log x)^2 + c_1$$

and

$$B = \int \frac{Ru}{uv_1 - u_1 v}\, dx + c_2 = \int \frac{x \log x \,.\, x^3}{-7x^{-2}}\, dx + c_2$$

$$= -\frac{1}{7} \int x^6 \log x \; dx + c_2 = -\frac{1}{7} \left[\log x \,.\, \frac{x^7}{7} - \int \frac{1}{x} \,.\, \frac{x^7}{7}\, dx \right] + c_2$$

$$= -\frac{1}{7} \left[\frac{x^7 \log x}{7} - \frac{1}{7} \left(\frac{x^7}{7} \right) \right] + c_2 = \frac{x^7}{49} \left(\frac{1}{7} - \log x \right) + c_2$$

Hence the complete solution is given by

$$y = Ax^3 + Bx^{-4} = \left[\frac{1}{14} (\log x)^2 + c_1 \right] x^3 + \left[\frac{x^7}{49} \left(\frac{1}{7} - \log x \right) + c_2 \right] x^{-4}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 9. *By the method of variation of parameters, solve the differential equation*

$$\frac{d^2y}{dx^2} + (1 - \cot x) \frac{dy}{dx} - y \cot x = \sin^2 x.$$

Sol. Take $y'' + (1 - \cot x)\, y' - y \cot x = 0$ ...(i)

Obviously, $y = e^{-x}$ is a part of C.F.

Example ...

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{2x}.$$

Sol. Here, $u = e^x$, $v = e^{2x}$, $w = e^{3x}$ which are parts of C.F.

Let $y = Ae^x + Be^{2x} + Ce^{3x}$ be the complete solution of the given equation, where A, B and C are the suitable functions of x.

To determine the values of A, B and C, we have the equations

$$A_1(e^x) + B_1(e^{2x}) + C_1(e^{3x}) = 0 \quad \text{...(1)}$$

$$A_1(e^x) + B_1(2e^{2x}) + C_1(3e^{3x}) = 0 \quad \text{...(2)}$$

$$A_1(e^x) + B_1(4e^{2x}) + C_1(9e^{3x}) = e^{2x} \quad \text{...(3)}$$

From (1) and (2), $\dfrac{A_1}{e^{5x}} = \dfrac{B_1}{-2e^{4x}} = \dfrac{C_1}{e^{3x}} = \lambda$ (say)

Substituting the values of A_1, B_1, C_1 in (3), we get

$$e^{2x} = \lambda(e^{6x} - 8e^{6x} + 9e^{6x}) = 2\lambda e^{6x}$$

$$\Rightarrow \qquad \lambda = \frac{1}{2}e^{-4x}$$

$$\therefore \qquad A_1 = \frac{1}{2}e^x, \; B_1 = -1, \; C_1 = \frac{1}{2}e^{-x}$$

Integrating, $\quad A = \dfrac{1}{2}e^x + a, \quad B = -x + b, \quad C = -\dfrac{1}{2}e^{-x} + c$

$\therefore$ The complete solution is

$$y = \frac{1}{2}e^{2x} + ae^x - xe^{2x} + be^{2x} - \frac{1}{2}e^{2x} + ce^{3x} = ae^x + be^{2x} + ce^{3x} - xe^{2x}$$

where a, b and c are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations by the method of variation of parameters:

1. $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$

2. $y_2 + 4y = 4\tan 2x$

3. $\dfrac{d^2y}{dx^2} + y = x$

4. $\dfrac{d^2y}{dx^2} + y = \sec x \tan x$

5. $y_2 - 3y_1 + 2y = e^{2x} + x^2$

6. $(D^2 + 1)y = \tan^2 x$

7. $(D^2 + 1)y = \operatorname{cosec} x \cot x$

8. (*i*) $\dfrac{d^2y}{dx^2} + y = x\cos x$ (*ii*) $\dfrac{d^2y}{dx^2} + y = x\sin x$

9. (*i*) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = \sin(\log x)$ (*ii*) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x\log x$

10. (*i*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \tan x$ (*ii*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \log x$

11. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \sin e^{-x}$ (M.T.U. 2012)

12. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 3x\, e^{-x}$

13. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{e^{3x}}{x^2}$

14. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$ [G.B.T.U. 2012]

15. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x^2 + \dfrac{1}{x^2}$

16. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 48x^5$. [G.B.T.U. (C.O.) 2010]

17. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 3x = \dfrac{e^t}{1+e^t}$ [G.B.T.U. 2013]

Answers

1. $y = (a - x)\cos x + (b + \log \sin x)\sin x$
2. $y = c_1 \cos 2x + b \sin 2x - \cos 2x \log \tan\left(\dfrac{\pi}{4} + x\right)$
3. $y = c_1 \cos x + c_2 \sin x + x$
4. $y = c_1 \cos x + c_2 \sin x + x\cos x + \sin x \log \sec x - \sin x$
5. $y = c_1 e^x + c_2 e^{2x} + xe^{2x} + \dfrac{3}{2}x + \dfrac{7}{4} + \dfrac{1}{2}x^2 - e^{2x}$
6. $y = c_1 \cos x + c_2 \sin x - \cos x(\sec x + \cos x) + \sin x \log(\sec x + \tan x) + \sin^2 x$.
7. $y = c_1 \cos x + c_2 \sin x - \cos x \log \sin x - x \sin x - \sin x \cot x$
8. (i) $y = c_1 \cos x + \left(c_2 - \dfrac{1}{8}\right)\sin x + \dfrac{x^2}{4}\sin x + \dfrac{x}{4}\cos x$

   (ii) $y = c_1 \cos x + c_2 \sin x + \dfrac{x}{4}\sin x - \dfrac{x^2}{4}\cos x$
9. (i) $y = c_1x^2 + c_2x^3 + \dfrac{1}{10}(\sin\log x + \cos\log x)$ (ii) $y = c_1\, x\log x + c_2x + \dfrac{1}{6}x(\log x)^3$
10. (i) $y = e^x(c_1\cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$

    (ii) $y = (c_1 + c_2x)\, e^x + x^2 e^x\left(\dfrac{1}{2}\log x - \dfrac{3}{4}\right)$
11. $y = c_1e^x + c_2 e^{2x} - e^{2x}\sin e^{-x}$
12. $y = c_1e^{-2x} + c_2x\, e^{-2x} + \dfrac{1}{2}x^3e^{-2x}$
13. $y = (c_1x + c_2)\, e^{3x} - e^{3x}\log x$
14. $y = (e^x + c_1)\dfrac{1}{x} + [(1-x)\,e^x + c_2]\dfrac{1}{x^2}$
15. $y = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{1}{12}x^2 - \dfrac{1}{x^2}\log x$
16. $y = (4x^2 + c_1)\,x^3 + (c_2 - x^8)\,x^{-3}$.
17. $x = \left[\dfrac{1}{2}\log(e^{-t}+1) + c_1\right]e^t + \left[-\dfrac{1}{4}(e^{-t}+1)^2 - \dfrac{1}{2}\log(e^{-t}+1) + (e^{-t}+1) + c_2\right]e^{3t}$