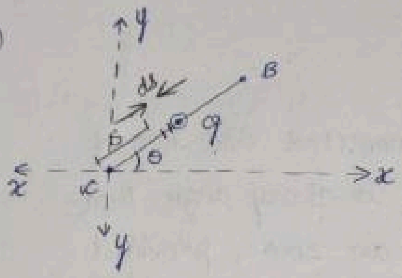
UNIT-3

centroid of line

① Lines.

①



$g \rightarrow$ centroid

$\bar{x} = \frac{l}{2}\cos\theta$

$\bar{y} = \frac{l}{2}\sin\theta$

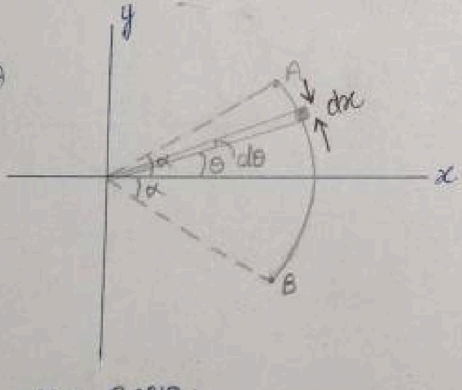
$$\bar{x} = \frac{\int x \cdot ds}{L}$$

$x = s\cos\theta$

$$= \frac{\int_0^L s\cos\theta \, ds}{L} \Rightarrow \frac{\cos\theta \left[\frac{s^2}{2}\right]_0^L}{L} = \frac{l\cos\theta}{2}$$

similary :- $\bar{y} = \frac{l}{2}\sin\theta$

②



$$\bar{x} = \frac{\int x \cdot dL}{\int dL}$$

$$\bar{x} = \frac{\int x \cdot dx}{\int dx}$$

$x = R\cos\theta$

$Rd\theta = dx$

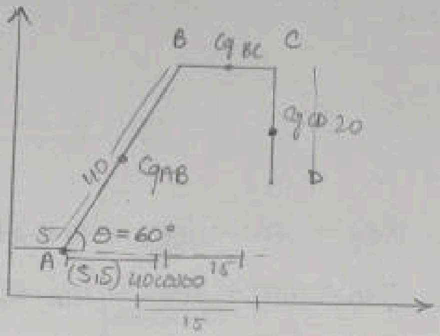
$$\bar{x} = \frac{\int_{-\alpha}^{\alpha} R\cos\theta R d\theta}{\int_{-\alpha}^{\alpha} R d\theta}$$

$\bar{y} = 0$

$$\bar{x} = \frac{R^2 [\sin\theta]_{-\alpha}^{\alpha}}{R(\alpha + \alpha)}$$

$\bar{x} = \frac{R\sin\alpha}{\alpha}$

Ques)



AB = 40 mm

BC = 15 mm

CD = 20 mm

$$\bar{x} = \frac{\Sigma x_i l_i}{\Sigma l_i}$$

i	x_i	l_i	$x_i l_i$
AB	15	40	600
BC	32.5	15	487.5
CD	40	20	800
		+	1887.5

$x_{AB} = \frac{40}{2}\cos60° + 5$
$= 15$

$x_{BC} = 40\cos60° + 5 + \frac{15}{2}$
$= 32.5$

$x_{CD} = 40\cos60° + 5 + 15$
$= 40$

$$\bar{x} = \frac{1887.5}{40+15+20} = 25.16$$

$$\bar{y} = \frac{\Sigma y_i l_i}{\Sigma l_i}$$

i	y_i	l_i	$y_i l_i$
AB	22.32	40	892.8
BC	39.6	15	594
CD	29.6	15	592
		+	2078.8

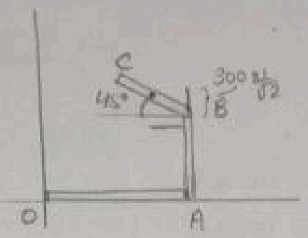
$y_{AB} = \frac{40}{2}\sin60° + 5$

$y_{BC} = 40\sin60° + 5$

$y_{CD} = (40\sin60 + 5) - \frac{20}{2}$

$$\bar{y} = \frac{2078.8}{75} = 27.7$$

Q(iii)



OA = 600
AB = 200
BC = 300

$x_{OA} = 300$ $\quad x_{AB} = 600$ $\quad x_{CB} = 600 - \frac{300}{2\sqrt{2}} = \cancel{387.23}\ 493.61$

$y_{OA} = 0$ $\quad y_{AB} = 100$ $\quad y_{CB} = 200 + \frac{300}{2\sqrt{2}} = \cancel{412.76}\ 306.38$

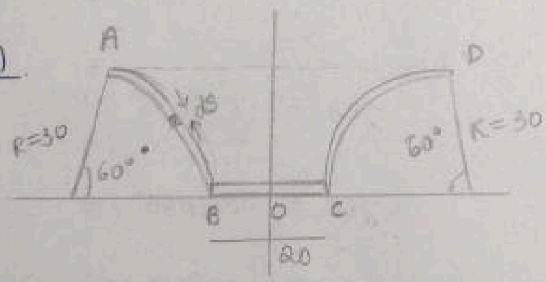
$$\bar{x} = \frac{300 \times 600 + 600 + 200 + \overset{493.61}{\cancel{387.23}} \times 300}{1100}$$

$$= \frac{1800 + 1200 + \overset{1480.82}{\cancel{1161.69}}}{11} = \cancel{378.33}\ 407.34$$

$$\bar{y} = \frac{0 \times 600 + 100 \times 200 + 306.38 \times 300}{1100}$$

$$= \frac{200 + 919.14}{11} = 101.74$$

Q(iv)



$\bar{x} = 0$

$ds = R\,d\theta$

$y_{BOC} = 0$

$y_{AB} = \frac{R\sqrt{3} \times 30}{2 \times \pi}$ $\quad \frac{3R}{2\pi} = \frac{1.5 \times 30}{\pi} = \frac{45}{\pi}$

$= \frac{9\sqrt{3} \times 10}{2 \times 3.41} = 2.28 \times 10$

$$\frac{\int R \sin\theta R\,d\theta + R^2 [\cos\theta]_0^{60}}{R\int_0^{60} d\theta} \Rightarrow \frac{3\left(\frac{1}{2}+1\right)R}{\pi}$$

$y_{CD} = \cancel{2.28 \times 10}$

$$\bar{y} = \frac{0 \times 20 + 2.28 \times \cancel{30}^{10} \times \frac{5\pi}{\cancel{3}} \times 2}{20 + 2 \times \frac{\cancel{30}^{10} \times \pi}{\cancel{3}}}$$

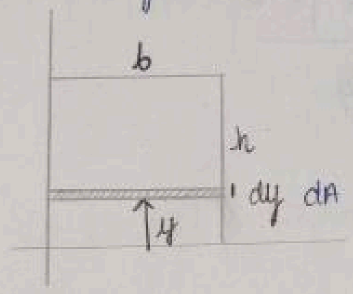
$$\frac{1554.96}{88.2} = 17.62$$

$$\bar{y} = \frac{2 \times \frac{45}{\cancel{\pi}} \times \overset{10}{\cancel{30}} \times \frac{\cancel{\pi}}{\cancel{3}}}{20 + \frac{\overset{10}{\cancel{30}} \times \pi}{\cancel{3}} \times 2} = \frac{900}{88.2} = \underline{10.204}$$

<u>Centroid of Area</u>

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A y\,dA}{\int_A dA}$$

① <u>Rectangle</u>

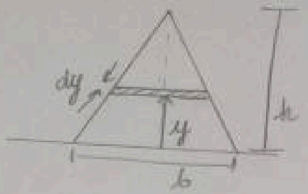


$$\bar{x} = b/2$$
$$\bar{y} = \frac{h}{2}$$
$$dA = b\,dy$$

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^h y\,b\,dy}{\int_0^h b\,dy} = \frac{bh^2}{2 \times bh} = \frac{h}{2}$$

$$\bar{x} = \frac{\int x\,dA}{\int dA} \quad \frac{\int_0^b x\,h\,dx}{\int_0^b h\,dx} = \frac{hb^2}{2 \times bh} = \frac{b}{2} \qquad dA = h\,dx$$

② Triangle



$$\frac{h}{h-y} = \frac{b}{b'}$$

$$b' = \frac{b(h-y)}{h}$$

$dA = b'dy$

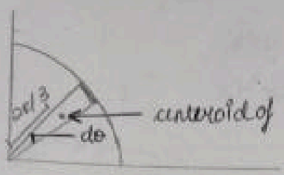
$$\bar{y} = \frac{\int_A y\, dA}{\int_A dA}$$

$$\bar{y} = \frac{\int y \times \frac{b(h-y)}{h} dy}{\int \frac{b(h-y)}{h} dy} = \frac{\frac{bh}{h}\int y dy - \frac{b}{h}\int y^2 dy}{b\int dy - \frac{b}{h}\int y dy}$$

$$= \frac{\frac{bh}{h}\left(\frac{y^2}{2}\right)_0^h - \frac{b}{h}\left(\frac{y^3}{3}\right)_0^h}{\frac{hbh}{h} - \frac{b}{h}\frac{h^2}{2}}$$

$$= \frac{\frac{bh^3}{2h} - \frac{bh^3}{3h}}{\frac{bh^2}{h} - \frac{bh^2}{2h}} = \frac{\frac{bh^{\cancel{3}2}}{6\cancel{h}}}{\frac{bh^{\cancel{2}}}{2\cancel{h}}} = \frac{\cancel{b}h^{\cancel{2}} \times \cancel{2}}{\cancel{3}6 \times \cancel{bh}} = \boxed{\frac{h}{3}}$$

③ Circular sector



centroid of triangle

$$\bar{x} = \frac{\int \bar{x} \cdot dA}{\int dA}$$

$$dA = \frac{Rd\theta \times R}{2} = \frac{R^2 d\theta}{2}$$

$$\int \frac{2R}{3}\sin\theta \times \frac{R^2 d\theta}{2} \Big/ \int \frac{R^2}{2} d\theta = \frac{\int_0^{\pi/2} \frac{2R}{3}\cos\theta \times \frac{R^2 d\theta}{\cancel{2}}}{\int \frac{R^2}{2}d\theta} = \frac{\frac{R^3}{3}\int\cos\theta \, d\theta}{\frac{R^2}{2}\left(\frac{\pi}{2}-0\right)}$$

$$\frac{4R}{3\pi} \; -\cos\theta\Big]_0^{\pi}$$

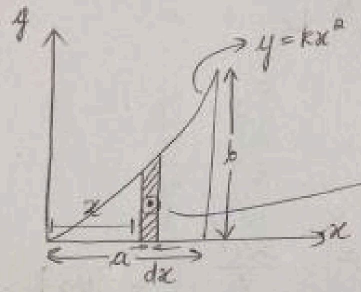
$$(0+1)$$

$$= \frac{4R^{\cancel{3}}}{\pi 3 \cancel{R^2}}\left[\sin\theta\right]_0^{\pi/2}$$

$$= \frac{4R}{3\pi}$$

similarly $\bar{y} = \frac{4R}{3\pi}$

④ Spandral → 1st method



centroid of rectangular strip

$$(x', y') = \left(x, \frac{y}{2}\right) = \left(x, \frac{kx^2}{2}\right)$$

$$\bar{x} = \frac{\int_A x' dA}{\int_A dA}$$

- $x' = x$
- $dA = y\,dx$
- $dA = kx^2 dx$

$$= \frac{\int_0^a x k x^2 dx}{\int_0^a kx^2 dx} = \frac{\frac{x^4}{4}\Big]_0^a}{\frac{x^3}{3}\Big]_0^a} = \frac{a^{\cancel{8}4} \times 3}{4 \times a^3} = \boxed{\frac{3a}{4}}$$

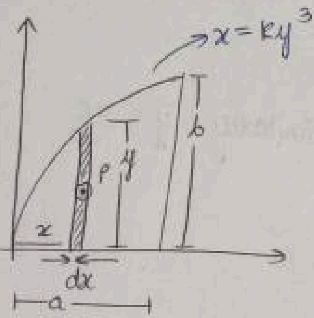
$$\bar{y} = \frac{\int_A y' dA}{\int_A dA} \qquad \bar{y}' = \frac{y}{2} \qquad y' = \frac{kx^2}{2}$$

$$dA = y\,dx = kx^2 dx$$

$$k = \frac{b}{a^2}$$

$$= \frac{\int_0^a \frac{kx^2}{2} kx^2 dx}{\int_0^a kx^2 dx} = \frac{k \left.\frac{x^5}{10}\right]_0^a}{\left.\frac{x^3}{3}\right]_0^a} = \frac{b}{a^2} \times \frac{a^2 \times 3}{10} = \boxed{\frac{3b}{10}}$$

Ques)



$$k = \frac{a}{b^3}$$

$$P(x', y') = \left(x, \frac{y}{2}\right) = \left(x, \frac{1}{2}\left(\frac{x}{k}\right)^{1/3}\right)$$

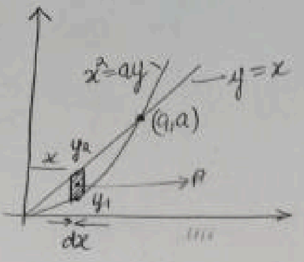
$$dA = \left(\frac{x}{k}\right)^{1/3} dx$$

$$\bar{x} = \frac{\int_A x' dA}{\int_A dA} = \frac{\left(\frac{1}{k}\right)^{1/3} \int_0^a x \cdot x^{1/3} dx}{\left(\frac{1}{k}\right)^{1/3} \int_0^a x^{1/3} dx} = \frac{\frac{3a^{7/3}}{7}}{\frac{3a^{4/3}}{4}} = \boxed{\frac{4a}{7}}$$

$$\bar{y} = \frac{\int_A y' dA}{\int_A dA} = \frac{\frac{1}{2(k)^{1/3}} \int_0^a x^{1/3} \cdot x^{1/3} dx}{\left(\frac{1}{k}\right)^{1/3} \int_0^a x^{1/3} dx} = \frac{\frac{\cancel{3}a^{5/3}}{\cancel{2}\,5}}{\frac{\cancel{3}a^{4/3}}{\cancel{4}\,2}} \times \left(\frac{1}{k}\right)^{1/3}$$

$$= \frac{2}{5} \frac{a^{1/3} b}{a^{1/3}} = \boxed{\frac{2b}{5}}$$

Q(ii)



$$P\left(x, \frac{y_2-y_1}{2}\right) \Rightarrow \left(x, \frac{x}{2}-\frac{x^2}{2a}\right)$$

$$y_2 = x \quad y_1 = \frac{x^2}{a}$$

$$dA = dx(y_2 - y_1)$$

$$dA = dx\left(x - \frac{x^2}{a}\right)$$

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA}$$

$$\bar{x} = \frac{\int_0^a x\left(x-\frac{x^2}{a}\right)dx}{\int_0^a \left(x-\frac{x^2}{a}\right)dx} = \frac{\int_0^a \left(x^2-\frac{x^3}{a}\right)dx}{\int_0^a \left(x-\frac{x^2}{a}\right)dx}$$

$$\bar{x} = \frac{\frac{a^3}{3}-\frac{a^4}{4a}}{\frac{a^2}{2}-\frac{a^3}{3a}} = \frac{a^3 \times 6}{12a^2} = \boxed{\frac{a}{2}}$$

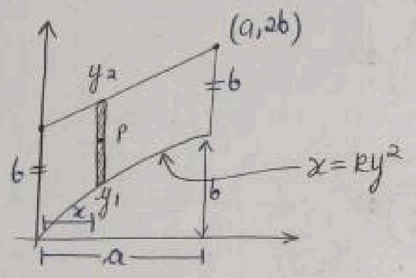
$$\bar{y} = \frac{\int_A y'\,dA}{\int_A dA} = \frac{\int_0^a \left(\frac{x}{2}-\frac{x^2}{2a}\right)\left(x-\frac{x^2}{a}\right)dx}{\int_0^a \left(x-\frac{x^2}{a}\right)dx}$$

$$\bar{y} = \frac{\int_0^a \left(\frac{x^2}{2} - \frac{x^3}{2a} - \frac{x^3}{2a} + \frac{x^4}{2a^2}\right) dx}{\int_0^a \left(x - \frac{x^2}{a}\right) dx}$$

$$\bar{y} = \frac{\frac{a^3}{6} - \frac{a^4}{4a} - \frac{a^5}{5\times 2a^2}}{\frac{a^2}{2} - \frac{a^3}{3a}}$$

$$\bar{y} = \frac{\frac{a^3}{6} - \frac{a^3}{4} + \frac{a^3}{10}}{\frac{a^2}{6}} = \frac{a^3 \times 6}{2\times 30 a^2} = \boxed{\frac{a}{10}}$$

ous)



$$y = \frac{bx}{a} + b$$

$$P(x', y')$$

$$P\left(x, \frac{y_2 - y_1}{2}\right)$$

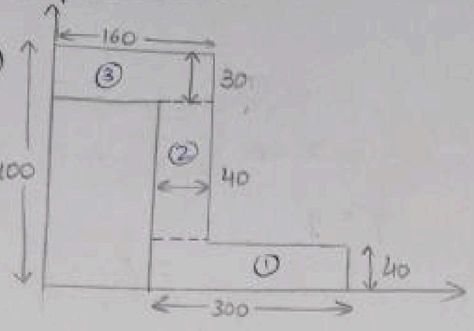
$$P\left(x, \frac{bx}{2a} + \frac{b}{2} - \frac{1}{2}\left(\frac{x}{k}\right)^{1/2}\right)$$

$$\bar{x} = \frac{\int_A x' dA}{\int_A dA} = \frac{\int_0^a (x)\left(\frac{bx}{a} + b - \left(\frac{x}{k}\right)^{1/2}\right) dx}{\int_0^a \left(\frac{bx}{a} + b - \left(\frac{x}{k}\right)^{1/2}\right) dx}$$

$$\bar{x} = \frac{\frac{ba^{\cancel{3}2}}{\cancel{a}3} + \frac{ba^2}{2} - \frac{1}{\sqrt{k}}\frac{a^{5/2}\times 2}{5}}{\frac{bx^2}{a\times 2} + bx - \frac{1}{\sqrt{k}}\frac{x^{3/2}\times 2}{3}} = \frac{\frac{a^2 b \times 13}{30}}{\frac{5ab}{6}} = \frac{a \times 13 \times 6}{30\times 5}$$

$$= \boxed{\frac{13a}{25}}$$

composite area

(1) 

💡 If reference axis is not given then choose it in a way that figure remains in I^{st} quadrant.

$A_1 = 300 \times 40$, $A_2 = 40 \times (400-70)$, $A_3 = 30 \times 160$

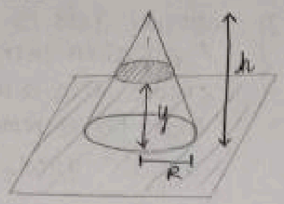
				yA
A_1	$x_i = 120+150 = 270$	$y_i = 20$	$x_iA_i = 324 \times 10^4$	24×10^4
A_2	$x_i = 140$	$y_i = \frac{330}{2} + 40$	$x_iA_i = 184.8 \times 10^4$	270.6×10^4
A_3	$x_i = 80$	$y_i = 400-15$	$x_iA_i = 38.4 \times 10^4$	184.8×10^4

$$\bar{x} = \frac{547.2 \times 10^4}{3 \times 10^4} = 182.4$$

$$\bar{y} = \frac{457.4 \times 10^4}{3 \times 10^4} = 152.4$$

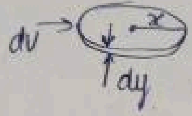
Centroid of Volume

① cone



$$\bar{y} = \frac{\int y' \, dv}{\int dv} \qquad \bar{x} = 0$$

$$y' = y$$



$$dv = \pi x^2 dy$$

$$\bar{y} = \frac{\int y \pi x^2 dy}{\int \pi x^2 dy}$$

$$\frac{h}{h-y} = \frac{R}{x}$$

$$hx = Rh - Ry$$

$$y = \frac{Rh - hx}{R}$$

$$\cancel{dy} = h - \frac{h}{R}x$$

$$dy = -\frac{h}{R}dx$$

$$= \frac{\int \frac{h(R-x)}{R} \pi x^2 \times -\frac{h}{R} dx}{\int -\pi x^2 \times \frac{h}{R} dx}$$

$$= \frac{\frac{h}{R} \int_0^R (R\pi x^2 - \pi x^3) dx}{\pi \int_0^R x^2 dx}$$

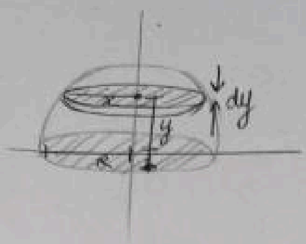
$$= \frac{\frac{h}{R} \left(\frac{Rx^3}{3} - \frac{x^4}{4} \right]_0^R}{\frac{x^3}{3} \Big]_0^R}$$

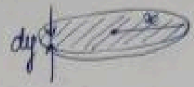
$$= \frac{\frac{h}{R} \left(\frac{R^4}{3} - \frac{R^4}{4} \right)}{\frac{R^3}{3}}$$

$$= \frac{h}{\cancel{R}} \; \frac{R^{\cancel{4}} \times \cancel{3}}{_4 \cancel{12} \times \cancel{R^3}}$$

$$= \boxed{\frac{h}{4}}$$

② Hemisphere





$dv = \pi x^2 dy$

$\bar{x} = 0$

$$\bar{y} = \frac{\int y' \, dv}{\int dv} \qquad (y' = y)$$

$$x^2 + y^2 = R^2$$
$$x^2 = R^2 - y^2$$

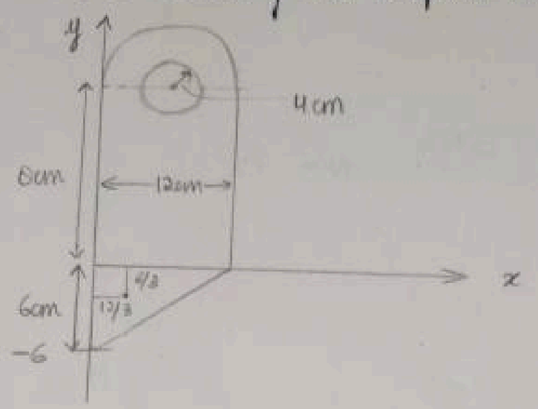
$$\bar{y} = \frac{\int y \pi x^2 dy}{\frac{2}{3}\pi R^3}$$

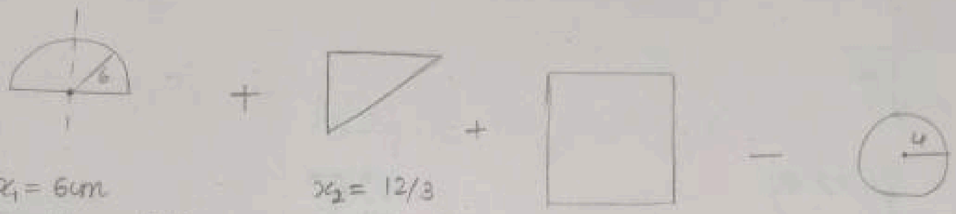
$$\bar{y} = \frac{\pi \int_0^R y(R^2 - y^2) dy}{\frac{2}{3}\pi R^3}$$

$$\bar{y} = \frac{\pi \left(\frac{R^4}{2} - \frac{R^4}{4}\right)}{\frac{2}{3}\pi R^3}$$

$$\bar{y} = \boxed{\frac{3R}{8}}$$

Ques) Find centroid of this composite area.





$x_1 = 6cm$

$y_1 = \frac{4R}{3\pi} = \frac{4 \times 6}{3\pi}$

$A_1 = \pi(6)^2$

$x_2 = 12/3$

$y_2 = -6/3$

$A_2 = \frac{1}{2} \times 6 \times 12$

$x_3 = 6cm$

$y_3 = 4cm$

$A_3 = 8 \times 12$

$x_4 = 6cm$

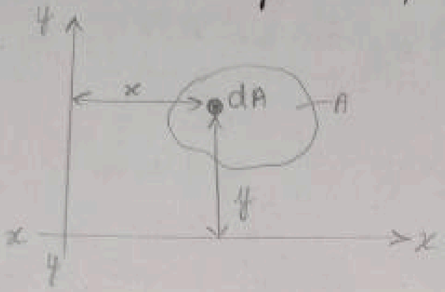
$y_4 = 8cm$

$A_4 = \pi 4^2$

$$\bar{x} = \frac{A_1x_1 + A_2x_2 + A_3x_3 - A_4x_4}{A_2 + A_1 + A_3 - A_4}$$

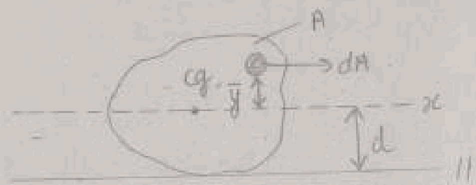
Moment of Inertia

Area moment of inertia / Moment of inertia



$$I_{yy} = \int_A x^2 dA$$

Parallel Axis Theorem



$$I_{||} = I_{cg} + Ad^2$$

$$I_{cg} = \int_A \bar{y}^2 dA$$

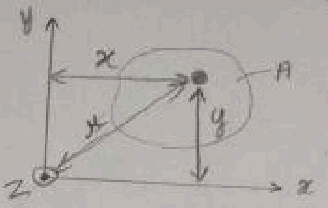
$$I_{||} = \int_A (d+\bar{y})^2 dA$$

$$I_{||} = \int_A (d^2 + \bar{y}^2 + 2dy) dA$$

$$I_{||} = \int_A d^2 dA + \int_A \bar{y}^2 dA + \cancelto{0}{\int_A 2dy\,dA}$$

$$I_{||} = d^2 A + I_{cg}$$

Perpendicular Axis Theorem



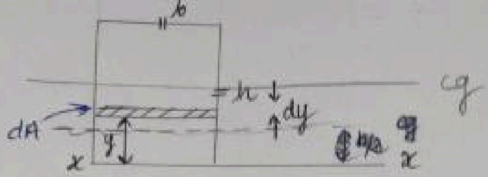
r = distance from z axis

$$I_{zz} = I_{xx} + I_{yy} \quad \Rightarrow \quad I_{zz} = \int r^2 \, dA$$

$$\Rightarrow \quad I_{zz} = \int x^2 \, dA + \int y^2 \, dA$$

$$\Rightarrow \quad I_{zz} = I_{xx} + I_{yy}$$

① Rectangle



$$dA = b\,dy \qquad y' = y$$

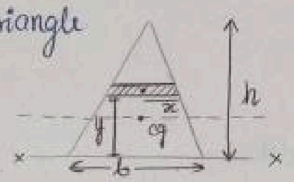
$$I_{xx} = \int_0^h y^2 b\,dy$$

$$= \boxed{\frac{bh^3}{3}}$$

$$I_{cg} = \frac{bh^3}{3} - bh \times \frac{h^2}{4}$$

$$= \frac{bh^3}{12}$$

② Triangle



$$\frac{h}{h-y} = \frac{b}{2x}$$

$$x = \frac{b(h-y)}{2h}$$

$$dA = dy \times 2x$$

$$dA = \frac{b(h-y)}{h}dy$$

$$I_{xx} = \int_0^h \frac{b}{h} y^2 (h-y)\,dy$$

$$= \frac{b}{h}\int_0^h (y^2 h - y^3)\,dy$$

$$= \frac{b}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right)$$

$$= \frac{bh^3}{12}$$

$$I_{cg} = I_{xx} - Ad^2$$

$$I_{cg} = \frac{bh^3}{12} - \frac{bh}{2} \times \left(\frac{h}{3}\right)^2 = \boxed{\frac{bh^3}{36}}$$

01

$$x = (-ky^2 + \cancel{3}4)$$

$$I_{xx} = \int y^2 \, dA$$

$$= \int y^2 (-ky^2 + 4) \, dy$$

$$= -\int k y^4 dy + \int 4y^2 dy$$

$$= -\frac{ky^5}{5} + \frac{4y^3}{3}$$

$$= -\frac{4}{\cancel{9}} \times \frac{(3)^{\cancel{5}\,3}}{5} + \frac{4(3)^{\cancel{3}\,2}}{\cancel{3}}$$

$$= -\frac{27 \times 4}{5} + 36$$

$$= \cancel{\frac{108 - 36}{5}} = \cancel{\frac{72}{5}} \quad \frac{180 - 108}{5} = \frac{72}{5}$$