UNIT 1

Differential Equations

1.1. DEFINITIONS

1.1.1. Differential Equation

A **differential equation** is an equation involving differentials or differential coefficients.

Or

An equation involving the dependent variable, independent variable and the differential coefficient (or coefficients) of the dependent variable with respect to the independent variable (or variables) is known as a differential equation. For example:

$$\frac{dy}{dx} = \cot x \qquad \ldots(1) \qquad\qquad \frac{d^2y}{dx^2} + y = 0 \qquad \ldots(2)$$

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 \qquad \ldots(3) \qquad\qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \qquad \ldots(4)$$

$$\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0 \qquad \ldots(5) \qquad\qquad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} \qquad \ldots(6)$$

are all differential equations.

1.1.2. Ordinary Differential Equation

A differential equation which involves only one independent variable is called an ordinary differential equation *e.g.,* All equations excluding (4) given above are ordinary differential equations.

1.1.3. Partial Differential Equation

A differential equation which involves two or more independent variables and partial derivatives with respect to them is called partial differential equation.

e.g., Equation (4) given above is a partial differential equation.

1.1.4. Order of a Differential Equation

The **order** of a differential equation is the order of the highest ordered derivative occurring in the differential equation. *e.g.,* Eqns. (1), (2) and (5) are of orders 1, 2 and 2 respectively.

1.1.5. Degree of a Differential Equation

The **degree** of a differential equation is the degree of the highest ordered derivative present in the differential equation when it is made free from radical signs and fractional powers. *e.g.*, Degree of equation (1) is 1.

Now consider equation (5)

$$\frac{d^2y}{dx^2}+\sqrt{1+\left(\frac{dy}{dx}\right)^3}=0$$

It involves radical sign. So to find the degree, we shall remove the radical sign. To achieve the purpose, squaring, we get

$$\left(\frac{d^2y}{dx^2}\right)^2=1+\left(\frac{dy}{dx}\right)^3$$

Clearly its degree is 2.

Again, consider equation (6)

$$\rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}.$$

It involves fractional powers. So to find the degree, we shall remove the fractional power. To achieve the purpose, squaring, we get

$$\rho^2=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3}{\left(\frac{d^2y}{dx^2}\right)^2}$$

Clearly its degree is 2.

Note. Equations of degree higher than one are called non-linear equations.

1.1.6. Solution of a Differential Equation

The solution is one which satisfies. A solution (or integral) of a differential equation is a relation, free from derivatives, between the variables which satisfies the given equation. It is also called **primitive** because the differential equation can be regarded as a relation derived from it.

Thus if $y=f(x)$ is the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

e.g., $y=c_1\cos x+c_2\sin x$ is the solution of the differential equation $\frac{d^2y}{dx^2}+y=0$.

1.1.7. General Solution

The **general** (or **complete**) **solution** of a differential equation is the solution in which the number of arbitrary constants is equal to the order of the differential equation.

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

1.1.8. Particular Solution

A **particular solution** of a differential equation is the solution which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

The solution of a differential equation of n^{th} order is its particular solution if it contains less than n arbitrary constants.

1.3. FORMATION OF A DIFFERENTIAL EQUATION

Differential equations are formed by elimination of arbitrary constants. To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order. Elimination of n arbitrary constants leads us to n^{th} order derivatives and hence a differential equation of the n^{th} order.

Example. *Eliminate the arbitrary constants A and B from the equation* $y = e^x$ *(A cos x + B sin x) and obtain the differential equation.*

Sol. We have the relation.

$$y = e^x (A \cos x + B \sin x) \quad \text{...(1)}$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x (A \cos x + B \sin x) + e^x (-A \sin x + B \cos x)$$

$$= y + e^x (-A \sin x + B \cos x) \quad \text{...(2)}$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x (-A \sin x + B \cos x) + e^x (-A \cos x - B \sin x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y \quad \text{[Using (1) and (2)]}$$

or

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation.

1.4. LINEARLY DEPENDENT AND INDEPENDENT SOLUTIONS

Consider a second order differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad \text{...(1)}$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous on an interval (a, b) and $a_0(x) \neq 0\ \forall\ x \in (a, b)$. Then two solutions $y_1(x)$ and $y_2(x)$ of equation (1) are said to be linearly dependent if $\exists$ two constants c_1 and c_2, not both zero, such that

$$c_1y_1 + c_2y_2 = 0 \quad \forall\ x \in (a, b)$$

Two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if they are not linearly dependent.

In other words, two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if

$$c_1y_1 + c_2y_2 = 0$$

$$\Rightarrow \qquad c_1 = 0 \quad \text{and} \quad c_2 = 0\ ;\ x \in (a, b).$$

1.5. THE WRONSKIAN OR WRONSKI DETERMINANT

I.M. Hone (1778–1853) was a polish mathematician who changed his name to Wronski.

Let $y_1(x)$ and $y_2(x)$ be two solutions of the second order differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0.$$

Then the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(y_1, y_2) \quad \text{or} \quad W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Again,

Let $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ be n solutions of

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + P_ny = 0$$

Then, Wronskian of $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ is denoted by $W(y_1, y_2, \ldots\ldots, y_n)$ or $W(x)$ and defined by the determinant

$$W(x) = \begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_n \\ y_1' & y_2' & \cdots\cdots & y_n' \\ y_1'' & y_2'' & \cdots\cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots\cdots & y_n^{(n-1)} \end{vmatrix}.$$

1.6. SOME IMPORTANT THEOREMS

(*i*) If $y_1(x)$ and $y_2(x)$ are any two solutions of $a_0(x) \dfrac{d^2y}{dx^2} + a_1(x) \dfrac{dy}{dx} + a_2(x)\, y = 0$, then the linear combination $c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are constants is also a solution of the given equation.

(*ii*) Two solutions $y_1(x)$ and $y_2(x)$ of the equation

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = 0 \; ; \; a_0(x) \neq 0, \; x \in (a, b)$$

are linearly dependent iff their Wronskian is identically zero.

Corollary. Solutions are linearly independent iff their Wronskian is not zero at some point $x_0 \in (a, b)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants c_1 and c_2 so that*

$$\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x).$$

Sol. Given equation is $\quad y'' + y = 0 \quad$ ...(1)

Here, $\quad y_1(x) = \sin x$

$\therefore \quad y_1'(x) = \cos x \quad$ and $\quad y_1''(x) = -\sin x$

Since, $y_1''(x) + y_1(x) = -\sin x + \sin x = 0$ hence $y_1(x)$ is a solution of (1).

Similarly, we can show that $y_2(x)$ is also a solution of (1).

Now, the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x\,(\cos x + \sin x) - \cos x(\sin x - \cos x) = 1 \neq 0$$

which shows that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

Given that $\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$

$$\equiv c_1 \sin x + c_2(\sin x - \cos x)$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$c_1 + c_2 = 1 \quad \text{and} \quad -c_2 = 3$$

so that, $$c_1 = 4, \; c_2 = -3.$$

Example 2. *Prove that the functions 1, x, x^2 are linearly independent. Hence form the differential equation whose roots are 1, x, x^2.*

Sol. Let $y_1(x) = 1, \; y_2(x) = x$ and $y_3(x) = x^2$.

Then the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

or $$W(x) = 2 \neq 0 \quad \forall\, x \in (-\infty, \infty)$$

$\therefore$ y_1, y_2 and y_3 are linearly independent.

The general solution of the required differential equation may be written as

$$y = c_1y_1 + c_2y_2 + c_3y_3 = c_1 + c_2x + c_3x^2 \quad ...(1)$$

where c_1, c_2, c_3 are arbitrary constants.

Differentiating (1), we get $\quad y' = c_2 + 2c_3 x \quad ...(2)$

Differentiating again, we get $\quad y'' = 2c_3 \quad ...(3)$

Differentiating (3), we get $\quad y''' = 0$ or $\dfrac{d^3y}{dx^3} = 0 \quad ...(4)$

Since equation (4) is free from arbitrary constants c_1, c_2 and c_3, hence equation (4) is the

Example 3. *Determine the differential equation whose set of independent solutions is* $\{e^x, xe^x, x^2e^x\}$.

Sol. Let the general solution of the required differential equation be

$$y = c_1e^x + c_2 xe^x + c_3 x^2e^x \quad \text{...(1)}$$

Differentiating (1) w.r.t. x, we get

$$y' = c_1e^x + c_2(x+1)e^x + c_3(x^2+2x)e^x \quad \text{...(2)}$$

From (1) and (2), we get

$$y = y' - c_2e^x - 2c_3 xe^x \quad \text{...(3)}$$

Differentiating (3) w.r.t. x, we get

$$y' = y'' - c_2e^x - 2c_3\,(x+1)e^x \quad \text{...(4)}$$

From (3) and (4), we get

$$y = y' + y' - y'' + 2c_3e^x = 2y' - y'' + 2c_3e^x \quad \text{...(5)}$$

1.7. DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

A differential equation of the form $\frac{dy}{dx} = f(x, y)$ or $Mdx + Ndy = 0$, where M, N are functions of x and y, is called a differential equation of the first order and first degree.

1.9. VARIABLES SEPARABLE FORM

If it is possible to write a differential equation of first order and first degree in the form

$$f_1(x)\,dx = f_2(y)\,dy$$

we say that the variables are separable. Such equations can be solved immediately by integration and the solution is given by

$$\int f_1(x)\,dx = \int f_2(y)\,dy + c$$

where c is an arbitrary constant of integration.

1.9.1. Steps for Solution

1. Separate the variables as $f_1(x)\,dx = f_2(y)\,dy$.
2. Integrate both sides as $\int f_1(x)\,dx = \int f_2(y)\,dy$.
3. Add an arbitrary constant to any of the sides.

1.9.2. Differential Equations of the Form $\dfrac{dy}{dx} = f(ax + by + c)$ ...(1)

can be reduced to a form in which the variables are separable by the substitution $ax+by+c=t$

so that
$$a + b\frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$\therefore$ Equation (1) becomes $\dfrac{1}{b}\left(\dfrac{dt}{dx} - a\right) = f(t)$ or $\dfrac{dt}{dx} = a + bf(t)$

or
$$\frac{dt}{a + bf(t)} = dx$$

After integrating both sides, t is to be replaced by its value.

Example 4. *Solve:* $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$.

Sol. Separating the variables, we get

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Integrating both sides, we get

$$\tan^{-1} y = \tan^{-1} x + \tan^{-1} c$$

$$\Rightarrow \quad \tan^{-1} y - \tan^{-1} x = \tan^{-1} c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1} c$$

$$\Rightarrow \quad \frac{y-x}{1+xy} = c.$$

where c is an arbitrary constant.

1.10. LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

The general form of a linear differential equation of the first order is $\frac{dy}{dx} + Py = Q$ ...(1)

where P and Q are functions of x only or constants.

Equation (1) is also known as *Leibnitz's linear equation**.

To solve it, we multiply both sides by $e^{\int P dx}$ and get

$$\frac{dy}{dx} e^{\int P dx} + y(e^{\int P dx} P) = Qe^{\int P dx}$$

or

$$\frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get

$$ye^{\int P dx} = \int Qe^{\int P dx}\, dx + c$$

Note 1. In the general form of a linear differential equation, the coefficient of $\frac{dy}{dx}$ is unity.

Note 2. The factor $e^{\int P dx}$ on multiplying by which the L.H.S. of (1) becomes the differential coefficient of a single function is called the integrating factor (briefly written as I.F.) of (1).

Thus I.F. $= e^{\int P dx}$ and the solution is

$$\boxed{y\text{ (I.F.)} = \int Q\text{ (I.F.) } dx + c.}$$

Note 3. Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. The equation can then be put as $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y only or constants. The integrating factor in this case is $e^{\int P dy}$ and the solution is

$$\boxed{x\text{ (I.F.)} = \int Q\text{ (I.F.) } dy + c.}$$

Example 10. *Solve:* $\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$.

Sol. Comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{3x^2}{1+x^3}, \quad Q = \frac{\sin^2 x}{1+x^3}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{3x^2}{1+x^3} dx} = e^{\log(1+x^3)} = 1 + x^3$$

*Gottfried wilhelm Leibnitz (1646–1716) was a German mathematician.

Hence, solution is given by

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3}(1+x^3)\,dx + c$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx + c = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

$$\Rightarrow \quad y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

where c is an arbitrary constant of integration.

Example 11. *Solve:* $\dfrac{dy}{dx} + \dfrac{3y}{x} = \dfrac{1}{x^4}$.

[U.P.T.U. (C.O.) 2009]

Sol. Comparing with $\dfrac{dy}{dx} + \text{P}y = \text{Q}$, we get $\text{P} = \dfrac{3}{x}$ and $\text{Q} = \dfrac{1}{x^4}$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{3\int \frac{1}{x}\,dx} = e^{3\log x} = x^3$$

$\therefore$ The solution is

$$y(\text{I.F.}) = \int \text{Q(I.F.)}\,dx + c$$

$$\Rightarrow \quad yx^3 = \int \frac{1}{x^4}(x^3)\,dx + c$$

$$\Rightarrow \quad yx^3 = \log x + c$$

where c is an arbitrary constant of integration.

Example 12. *Solve:* $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

Sol. Dividing throughout by $\cos^2 x$, we get

$$\frac{dy}{dx} + \sec^2 x \,.\, y = \tan x \sec^2 x$$

Comparing with $\dfrac{dy}{dx} + \text{P}y = \text{Q}$, we get, $\text{P} = \sec^2 x$, $\quad \text{Q} = \tan x \sec^2 x$

$$\therefore \quad \text{I.F.} = e^{\int \text{P}\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan x}$$

Hence the solution is given by

$$y\,.\,e^{\tan x} = \int e^{\tan x}\,.\,\tan x \sec^2 x\,dx + c \quad | \text{ } c \text{ is an arbitrary constant}$$

$$= \int e^t\,.\,t\;dt + c \qquad \left|\; \begin{array}{l} \text{Put } \tan x = t \\ \therefore \; \sec^2 x\,dx = dt \end{array}\right.$$

$$= t\,e^t - e^t + c$$

$$= \tan x\,.\,e^{\tan x} - e^{\tan x} + c$$

$$\Rightarrow \quad y = \tan x - 1 + c\,e^{-\tan x}.$$

Example 15. *Solve the equation:*

$$(y \log y)\, dx + (x - \log y)\, dy = 0.$$ [U.P.T.U. (B. Pharm.) 2008]

Sol. We have

$$(y \log y)\, dx + (x - \log y)\, dy = 0 \quad ...(1)$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y} \quad ...(2)$$

This is a linear differential equation of the form $\dfrac{dx}{dy} + \mathrm{P}x = \mathrm{Q}$

Here, $$\mathrm{P} = \frac{1}{y \log y}, \quad \mathrm{Q} = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \mathrm{P} dy} = e^{\int \frac{1}{y \log y} dy} = e^{\log (\log y)} = \log y$$

Solution to equation (2) is

$$x \,.\, \log y = \int \frac{1}{y} \,.\, \log y \; dy + c$$

$$\Rightarrow \quad x \log y = \frac{1}{2} (\log y)^2 + c$$

where c is an arbitrary constant of integration.