classmate
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Solution of Algebraic and Transcedental Equations

Algebraic Equation: An expression of the form $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ is called a polynomial in x of degree n.

- The Polynomial $f(x)=0$ is called an algebraic Equation of degree n.

- If $f(x)$ contains some other functions such as trigonometric, logrithmic, exponential etc then $f(x)=0$ is called a transcedental equation.

for example

$x^3 - 2x - 5 = 0$ Algebraic Equation

$xe^x - 1 = 0$ Transcedental Equation

- The value α of x which satisfies $f(x)=0$ is called a root of $f(x)=0$.

- Geometrically a root of $f(x)=0$ is the value of x where the graph of $y=f(x)$ crosses the x axis

- The Process of finding the roots of an equation is known as the solution of that equation

Need of numerical methods to solve algebraic or transcedental Equations.

If $f(x)$ is a quadratic Equation, say

$$ax^2+bx+c=0$$

we can solve it by shridharacharya method. or by factorizing it.

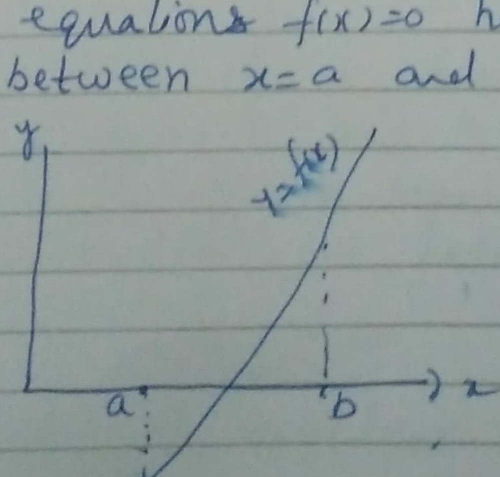
similarly if we have cubic or a biquadratic equation then there are limited methods available.

But if we have to solve higher degree or transcedental equations for which no direct methods exist.

such equations can be solved by approximate or numerical methods.

Intermediate value Property.

If $f(x)$ is continous in the interval $[a,b]$ and $f(a), f(b)$ have different signs, then the equations $f(x)=0$ has at least one root between $x=a$ and $x=b$.



Let us discuss some numerical methods for the solution of algebraic or Transcedental Equat-

We will discuss following methods.

1. Bisection method
2. False Position method
3. secan Method
4. Newton's Raphson Method
5. Iteraton Method

In all the above method we approximate given function by a straight line

firstly we discuss Bisection Method.

BISECTION METHOD

This method is based on intermediate value property.

⇒ which states that if a function $f(x)$ is continous between a and b and $f(a)$ and $f(b)$ are of opposite signs then there exists atleast one root between a and b.

⇒ Let Equation is $f(x)=0$
⇒ and let $f(a)$ be negative and $f(b)$ be positive.

⇒ Then the root lies between a and b and let approximate value be given by

$$x_0 = \frac{a+b}{2}$$

⇒ Now if $f(x_0) = 0$

then we conclude that x_0 is a root of the equation $f(x) = 0$

⇒ otherwise the root lies either between x_0 and b, or between x_0 and a depending on whether $f(x_0)$ is negative or positive.

⇒ we write this new interval as $[a_1 b_1]$ whose length is $\frac{|b-a|}{2}$

⇒ Now as before this is bisected at x_1 and new interval will be exactly half ~~of~~ the length of the Previous one

⇒ we repeat this process until the latest interval (which contains the root) is as small as desired, say ϵ.

⇒ It is clear that the interval width is reduced by a factor of one-half at each step and at the end ~~othe~~ of the nth step the new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$.

we then have $\frac{|b-a|}{2^n} \le \epsilon$

which gives on simplification

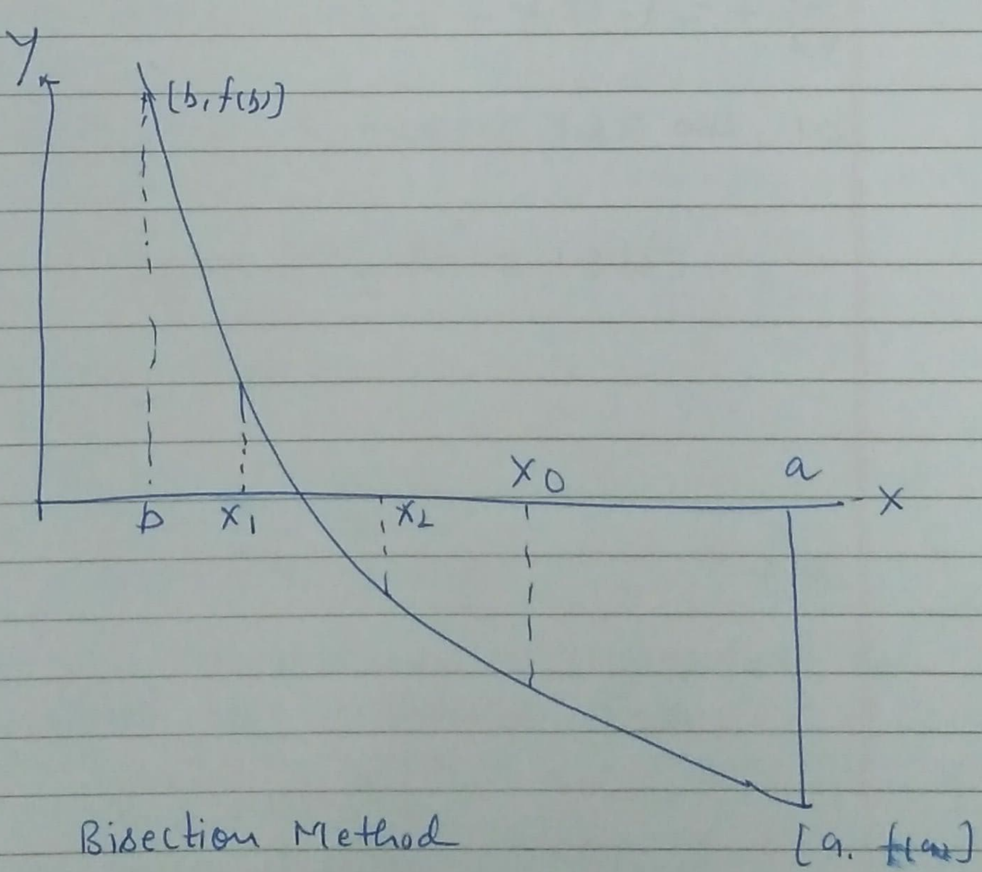
$$n \geq \frac{\log_e \left(\frac{|b-a|}{\epsilon} \right)}{\log_e 2} \quad \text{— ①}$$

Inequality ① gives the number of iterations required to achieve an accuracy ϵ

for example if $|b-a| = 1$ and $\epsilon = .001$, then it can be seen that

$$n \geq 10$$

Method is shown graphically in figure 1



Bisection Method

Example: Find a real root of the equation

$$f(x) = x^3 - x - 1 = 0$$

Sol.

$f(0) = 0 - 0 - 1 = -1$, $f(1) = 1 - 1 - 1 = -1$, $f(2) = 5$

Since $f(1)$ is negative and $f(2)$ is Positive

$\therefore$ root lies between 1 and 2 and therefore we take $x_0 = \frac{3}{2}$. Then $f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$ which is positive.

Hence root lies between 1 and 1.5 so so now we get

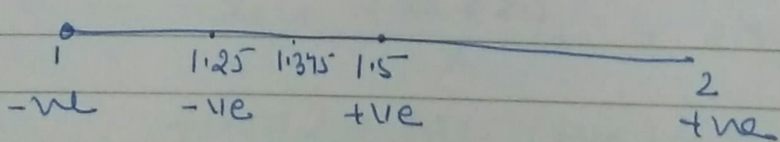
$$x_1 = \frac{1 + 1.5}{2} = 1.25$$

$$\therefore f(x_1) = (1.25)^3 - 1.25 - 1 = -\frac{19}{64} \quad (-ve)$$

so root lies between 1.25 and 1.5

$$\therefore x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

we can show



Now the Procedure is repeated and successive approximations are

$x_3 = 1.3125$

$x_4 = 1.34375$

$x_5 = 1.328125$

Q. Find a root of the equation $x^3 - 3x - 5 = 0$ by bisection Method

sol. Let $f(x) = x^3 - 3x - 5 = 0$ by bisection Method.

Now $f(0) = -5$

$f(1) = 1^3 - 3\times1 - 5 = -7$

$f(2) = 2^3 - 3\times2 - 5 = -3$

$f(3) = 3^3 - 3\times3 - 5 = 13$

Thus a root lies between 2 and 3.

Let ~~$x_0 = 2.5$~~ $x_0 = \frac{2+3}{2} = 2.5$

$f(x_0) = (2.5)^3 - 3\times2.5 - 5 = 3.125$ (+ve)

$\therefore$ Root lies between 2.0 and 2.5

then $x_1 = \frac{2+2.5}{2} = 2.25$

$f(x_1) = -.359375$ (-ve)

$\therefore$ Root lies between 2.25 and 2.5.

$\therefore$ $x_2 = \frac{2.25+2.5}{2} = 2.375$

$f(2.375) = 1.275$ (+ve)

Hence root lies between 2.25 and 2.375

$\therefore$ $x_3 = \frac{2.25+2.375}{2} = 2.3125$

$f(2.3125) = .4289$ (+ve)

root lies between 2.25 and 2.3125

$\vdots$

$$\therefore \quad x_4 = \frac{2.25 + 2.3125}{2}$$

$$= 2.28125$$

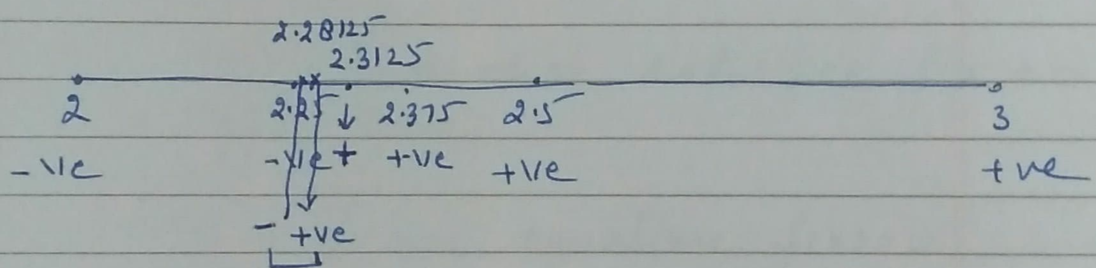
$$f(2.28125) = 0.0281 \quad (+ve)$$

$$x_5 = \frac{2.25 + 2.28125}{2}$$

$$= 2.265625$$

$$f(2.265625) = 11.6295815 - 6.796875 - 5$$

$$= -.1672935$$



Root lies between 2.28125 and 2.3125

$$\therefore \quad x_6 = \frac{2.28125 + 2.3125}{2}$$

$$= 2.296875$$

continuing this Process till the required accuracy we get the desired root

Root $\simeq$ 2.29

Q. Find a root of the following equations correct to three decimal places

(i) $x^3 - x - 11 = 0$

(ii) $x^4 - x - 10 = 0$

(iii) $x - \cos x = 0$

(iv) $x \log_{10} x = 1.2$

Q. The value of x that satisfies $f(x) = 0$ is called the

(A) root of an equation $f(x) = 0$

(B) root of a function $f(x)$

(C) zero of an equation $f(x) = 0$

(D) none of these