Chebyshev Method

We assume for the function $f(x)$, a polynomial of degree two and write as

$$f(x) = a_0 x^2 + a_1 x + a_2 = 0 \qquad a_0 \neq 0 \qquad (1)$$

where a_0, a_1, a_2 are arbitrary parameters to be determined by prescribing three appropriate conditions on $f(x)$ and/or its derivatives

We determine a_0, a_1 and a_2 in 1 using the conditions

$$\left.\begin{aligned} f_k &= a_0 x_k^2 + a_1 x_k + a_2 \\ f_k' &= 2a_0 x_k + a_1 \\ f_k'' &= 2a_0 \end{aligned}\right\} \qquad (2)$$

On eliminating a_i's from (1) and (2) we get

$$f_k + (x - x_k)^2 f_k' + \frac{1}{2}(x - x_k)^2 f_k'' = 0 \qquad (3)$$

which is the Taylor's expansion of $f(x)$ about $x = x_k$ such that the terms of order $(x - x_k)^3$ and higher Powers are neglected.

Equation ③ is a quadratic equation and can be solved easily.

only one of the two roots converges to the correct root. In order to get the next approximation to the correct root we write ③ as

$$f_K + (x_{K+1} - x_K) f'_K + \frac{1}{2} (x_{K+1} - x_K)^2 f''_K = \quad \text{--- ④}$$

or

$$(x_{K+1} - x_K) f'_K = - f_K - \frac{1}{2} (x_{K+1} - x_K)^2 f''_K \quad \text{--- ⑤}$$

or

$$(x_{K+1} - x_K) = - \frac{f_K}{f'_K} - \frac{1}{2} \frac{(x_{K+1} - x_K)^2 f''_K}{f'_K} \quad \text{--- ⑥}$$

we substitute value of $x_{K+1} - x_K$ from Newton Raphson's Method

$$x_{K+1} = x_K - \frac{f(x_K)}{f'(x_K)}$$

or
$$x_{K+1} - x_K = - \frac{f(x_K)}{f'(x_K)}$$

$$x_{K+1} - x_K = - \frac{f_K}{f'_K} \quad \text{--- ⑦}$$

putting this value in ⑥ we get

$$x_{k+1} - x_k = -\frac{f_k}{f'_k} - \frac{1}{2}\left(-\frac{f_k}{f'_k}\right)^2 \cdot \frac{f''_k}{f'_k}$$

or

$$x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{1}{2}\frac{(f_k)^2 \, f''_k}{(f'_k)^3}$$

which is called chebyshev Method. This method requires three evaluation for each iteration

Now if we write eq. (4) as below

$$f_k + (x_{k+1} - x_k)\left\{f'_k + \frac{1}{2}(x_{k+1} - x_k)^2 f''_k\right\} = 0$$

i.e taking $(x_{k+1} - x_k)$ common from 2nd and 3rd term.

or

$$(x_{k+1} - x_1)\left\{f'_k + \frac{1}{2}(x_{k+1} - x_k)^2 f''_k\right\} = -f_k$$

or $$x_{k+1} - x_k = -\frac{f_k}{f'_k + \frac{1}{2}(x_{k+1} - x_k)^2 f''_k} \quad -(8)$$

Now putting value of $x_{k+1} - x_k$ from (7) in (8) we get

$$= -\frac{f_k}{f'_k + \frac{1}{2}(x_{k+1} - x_k) f''_k} \quad - (9)$$

$$\boxed{x_{k+1} = x_k - \frac{f_k}{f'\left(x_k + \frac{1}{2}(x_{k+1} - x_k)\right)}} \quad - (10)$$

$\because$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

Similarly we can write

$$f'(x+h) = f'(x) + h f''(x) + \cancel{\frac{h^2}{2!} f'''(x) + \cdots}$$

then

$$f'(x+h) = f'(x) + h f''(x)$$

here if we take $h = \frac{1}{2}(x_{k+1} - x_k)$
then

$$f'\left(x + \frac{1}{2}(x_{k+1} - x_k)\right) = f'(x) + \frac{1}{2}(x_{k+1} - x_k) f''(x)$$

(11)

so putting this value in (9) from (11)
we get formula (10) which is multipoint iteration method
where $k = 0, 1, 2 \ldots$

x_{k+1} =

For computation purpose we may write (10) as the two stage method

$$\boxed{\begin{aligned} x^*_{k+1} &= x_k - \frac{1}{2}\frac{f_k}{f'_k} \\ \text{and} \quad x_{k+1} &= x_k - \frac{f_k}{f'(x^*_{k+1})} \end{aligned}}$$

Q Perform two iterations of the chebyshev method to find the smallest positive root of the equation

$$x^3 - 5x + 1 = 0$$

Sol

Let $f(0) = 0 - 5\times 0 + 1 = 1$

$f(1) = 1 - 5 + 1 = -3$

Root lies between 0 and 1

Let initial approximation $= \frac{0+1}{2} = 0.5$

we have $f(x) = x^3 - 5x + 1$ $\qquad f(.5) = -1.375$

$f'(x) = 3x^2 - 5$ $\qquad f'(.5) = -4.25$

$f''(x) = 6x$ $\qquad f''(.5) = 3$

$$x_{k+1} = x_k - \frac{f_k}{f'_k} - \frac{1}{2}\left(\frac{f_k}{f'_k}\right)^2 \left(\frac{f''_k}{f'_k}\right)$$

put $k=0$

$$x_1 = x_0 - \frac{f_0}{f'_0} - \frac{1}{2}\left(\frac{f_0}{f'_0}\right)^2 \left(\frac{f''_0}{f'_0}\right)$$

$$= 0.5 - 0.323529 - 0.5(0.104671)(-0.70580)$$

$$= 0.213414$$

$f(x_1) = -.057350$

$f'(x_1) = -4.863363$

$f''(x_1) = 1.280484$

$$x_2 = x_1 - \frac{f_1}{f'_1} - \frac{1}{2}\left(\frac{f_1}{f'_1}\right)^2 \left(\frac{f''_1}{f'_1}\right)$$

$$= 0.213414 - 0.011792 - 0.5(0.000139)(-0.263294)$$

$$= 0.201640$$

Ans

Q. Perform three iterations of the multipoint iteration method to find the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

initial approximation $x_0 = 0.5$

we have $f(x) = x^3 - 5x + 1$

$$f'(x) = 3x^2 - 5$$

for multipoint iteration method

$$\left.\begin{aligned} x^*_{k+1} &= x_k - \frac{1}{2}\frac{f_k}{f'_k} \\ x_{k+1} &= x_k - \frac{f_k}{f'^*_{k+1}} \end{aligned}\right\} \quad (1) \quad k = 0, 1, 2 \ldots$$

for $x_0 = 0.5$ $\quad f(x) = -1.375$, $\quad f'(x) = -4.25$

put $k = 0$ in 1 we get

$$x^*_1 = x_0 - \frac{1}{2}\frac{f_0}{f'_0} \quad \text{— (a)}$$

$$x_1 = x_0 - \frac{f_0}{f'^*_1} \quad \text{— (b)}$$

using (a) we get

$$x_1^* = x_0 - \frac{1}{2}\frac{f_0}{f_0'}$$

$$= .338235$$

$$f_1'(x^*) = f_1'^* = -4.656791$$

Now using (b) we get

$$\therefore x_1 = x_0 - \frac{f_0}{f_1'^*}$$

$$= .204732$$

$$f_1 = -.015079$$

$$f_1'^* = -4.874254$$

$$x_2 = x_1 - \frac{1}{2}\frac{f_1}{f_1'^*} = .203185$$

$$f_2'^* = 4.876148$$

$$x_2 = x_1 - \frac{f_1}{f_2'^*} = .201640$$

Ans

Q.1. Perform three iterations of the multipoint iteration method, to find the root of the equation $f(x) = \cos x - x e^{x} = 0$

Q.2. Solve the above equation using chebyshev method also (find three iterations)