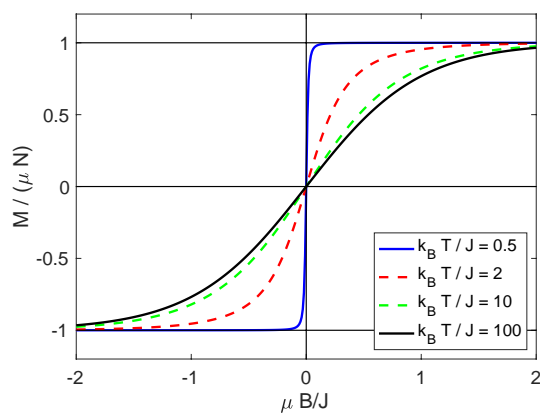
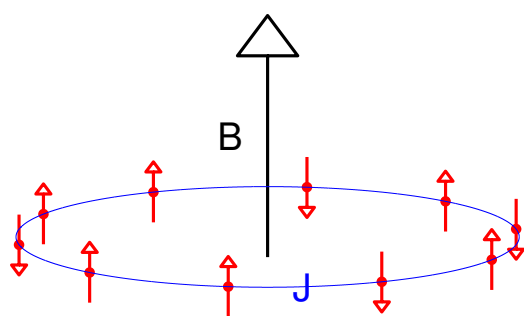


# Advanced Statistical Physics

## (preliminary draft)

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$$M(T, N, B) = N\mu \frac{\sinh(\mu\beta B)}{\sqrt{\sinh^2(\mu\beta B) + e^{-4J\beta}}}$$

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# Contents

<b>1</b>	<b>Phase transitions</b>	<b>4</b>
1.1	Phases . . . . .	4
1.1.1	Isolated systems . . . . .	4
1.1.2	Closed system with $p = \textit{fixed}$ , $T = \textit{fixed}$ . . . . .	7
1.1.3	Number of degrees of freedom . . . . .	8
1.1.4	Clausius-Clapeyron . . . . .	12
1.1.5	Real gases (van der Waals equation) . . . . .	14
1.1.6	Maxwell-Construction . . . . .	18
1.2	Phasetransitions . . . . .	23
1.2.1	Free energy versus $V$ , for $T$ and $N$ fixed . . . . .	24
1.2.2	$F$ and $G$ as function of $T$ . . . . .	25
1.2.3	Ehrenfest classification . . . . .	26
1.3	Critical exponents . . . . .	27
1.3.1	Important critical exponents . . . . .	28
1.3.2	Scaling laws . . . . .	32
<b>2</b>	<b>Ising Model</b>	<b>35</b>
2.1	Exact solution in 1D . . . . .	36
2.1.1	Paramagnet . . . . .	38
2.1.2	Limits . . . . .	39
2.1.3	Magnetization . . . . .	40
2.2	Mean field approximation for any dimension . . . . .	40
2.2.1	Magnetization . . . . .	42
2.2.2	Magnetic susceptibility . . . . .	44
2.2.3	Free energy . . . . .	45
2.2.4	Entropy . . . . .	48
2.2.5	Internal Energy . . . . .	49
2.2.6	Specific heat . . . . .	50
2.3	Exact solution of the 2d Ising model . . . . .	52
2.3.1	Transfermatrix approach . . . . .	52
2.3.2	Graphical approach ( <a href="#">Bachelor thesis J. Pomper</a> ) . . . . .	54
2.3.3	Making the graphs unique . . . . .	56

2.3.4	Decomposing graphs into loops . . . . .	57
2.3.5	Curie-temperature . . . . .	66
2.3.6	Internal energy . . . . .	68
2.3.7	Entropy . . . . .	69
2.3.8	Specific heat . . . . .	70
2.3.9	Spontaneous magnetization . . . . .	75
2.3.10	Critical exponent . . . . .	76
<b>3</b>	<b>Renormalization Group</b>	<b>78</b>
3.1	1D Ising . . . . .	78
3.2	2D Ising . . . . .	88
3.3	Scaling and critical exponents . . . . .	96
<b>4</b>	<b>Magnetism</b>	<b>101</b>
4.1	Hamiltonian . . . . .	101
4.1.1	Diamagnetism of atoms . . . . .	104
4.2	Density matrix and thermodynamic relations . . . . .	105
4.2.1	Magnetic response functions . . . . .	107
4.2.2	Internal energy . . . . .	107
4.3	Paramagnetism of independent moments . . . . .	108
4.3.1	Weak spin-orbit coupling . . . . .	108
4.3.2	Strong spin-orbit coupling . . . . .	112
4.3.3	Plot of $S$ and specific heat . . . . .	117
4.4	Magnetism of the free electron model . . . . .	117
4.4.1	Pauli paramagnetism . . . . .	118
4.4.2	Landau diamagnetism . . . . .	123
4.5	Collective magnetism . . . . .	127
4.5.1	Heisenberg hamiltonian . . . . .	127
4.5.2	Mermin-Wagner Theorem . . . . .	129
4.5.3	Exact ground-state of the ferromagnetic Heisenberg model . . . . .	130
4.5.4	Spin waves in the ferromagnetic Heisenberg model . . .	132
4.5.5	Mean field approximation of the isotropic Heisenberg model . . . . .	137
4.5.6	Curie temperature . . . . .	140
4.5.7	Internal energy . . . . .	141
4.5.8	Magnetic susceptibility . . . . .	141
<b>5</b>	<b>Appendix</b>	<b>142</b>
5.1	Sommerfeld expansion . . . . .	142
5.2	Euler Mac-Laurin formula . . . . .	144
5.3	One-particle density of states . . . . .	145

5.4	Eigenvalues of the free electron gas in a homogeneous magnetic field . . . . .	146
5.5	Proof of the Mermin-Wagner Theorem . . . . .	148
5.5.1	Bogoliubov inequality . . . . .	154
5.5.2	Proof used for the Mermin-Wagner theorem . . . . .	157
<b>6</b>	<b>Bibliography</b>	<b>161</b>

# Chapter 1

## Phase transitions

At the beginning of this lecture, it will be our aim to get a fundamental understanding of different phases and the physics that happens at the boundary of transition between them. This will give some insight into thermodynamic properties of statistical systems, e.g. for the case of real gases where the isotherms that result from van der Waal's equation give us a so-called coexistence region: For certain pressure and volume, multiple phases can coexist in a real gas.

We will then further generalize the definition of a phase transition and have a closer look at how physical quantities behave near a phase transition by introducing critical exponents. The latter will be useful in understanding the physics of the following chapters of this lecture covering the Ising model and magnetism.

### 1.1 Phases

We can generalise the formalism introduced in statistical physics to incorporate different coexisting phases.

#### 1.1.1 Isolated systems

We begin with isolated systems, where the total volume, total energy, and the particle number per component is conserved. Let's assume that there are  $n_p$  coexisting phases, which we enumerate by  $(\nu = 1, \dots, n_p)$  and  $\alpha$  components, enumerated by  $(j = 1, \dots, \alpha)$ . The conditions for isolated systems are

therefore

$$\sum_{\nu=1}^{n_p} V_\nu = V \quad (1.1a)$$

$$\sum_{\nu=1}^{n_p} U_\nu = U \quad (1.1b)$$

$$\sum_{\nu=1}^{n_p} N_{j\nu} = N_j, \text{ for } j = 1, \dots, \alpha. \quad (1.1c)$$

Here  $V_\nu$  ( $U_\nu$ ) is the total volume (energy) of the particles in phase  $\nu$ .  $N_j$  is the total number of particles of component  $j$  and  $N_{j\nu}$  the number of component  $j$  that is in phase  $\nu$ . We also introduce the following vectors

$$\mathbf{N} = (N_{j=1}, N_{j=2}, \dots, N_\alpha)^T \quad (1.2)$$

$$\mathbf{N}_\nu = (N_{j=1,\nu}, N_{j=2,\nu}, \dots, N_{\alpha,\nu})^T. \quad (1.3)$$

The entropy as extensive quantity is

$$S(U, V, \mathbf{N}) = \sum_{\nu=1}^{n_p} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu). \quad (1.4)$$

For an isolated system (micro-canonical ensemble) the entropy represents the thermodynamic potential and in equilibrium it has to be maximal. So we have to maximize the entropy based on the constraints in equation (1.1). This is best achieved by Lagrange multipliers, i.e. we define

$$\mathcal{L} = \sum_{\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) - \lambda^V \sum_{\nu} V_\nu - \lambda^U \sum_{\nu} U_\nu - \sum_j \lambda_j^N \sum_{\nu} N_{j\nu} \quad (1.5)$$

and form the differential

$$d\mathcal{L} = \sum_{\nu=1}^{n_p} \left[ \frac{\partial}{\partial U_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{V_\nu, \mathbf{N}_\nu} - \lambda^U \right] dU_\nu \quad (1.6a)$$

$$+ \sum_{\nu=1}^{n_p} \left[ \frac{\partial}{\partial V_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{U_\nu, \mathbf{N}_\nu} - \lambda^V \right] dV_\nu \quad (1.6b)$$

$$+ \sum_{\nu=1}^{n_p} \sum_{j=1}^{\alpha} \left[ \frac{\partial}{\partial N_{j\nu}} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{U_\nu, V_\nu, N_{i\nu}, i \neq j} - \lambda_j^N \right] dN_{j\nu} \stackrel{!}{=} 0. \quad (1.6c)$$

First of all, the variations in volume, energy, and particle number are independent from each other and, therefore, we have the individual conditions

$$\sum_{\nu=1}^{n_p} \left[ \frac{\partial}{\partial U_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{U_\nu, \mathbf{N}_\nu} - \lambda^U \right] dU_\nu = 0 \quad (1.7a)$$

$$\sum_{\nu=1}^{n_p} \left[ \frac{\partial}{\partial V_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{V_\nu, \mathbf{N}_\nu} - \lambda^V \right] dV_\nu = 0 \quad (1.7b)$$

$$\sum_{\nu=1}^{n_p} \sum_{j=1}^{\alpha} \left[ \frac{\partial}{\partial N_{j\nu}} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{U_\nu, V_\nu, N_{i\nu}, i \neq j} - \lambda_j^N \right] dN_{j\nu} = 0 . \quad (1.7c)$$

According to the constraints equation (1.1) [\[previous page\]](#) , there is one condition for the variables  $U_\nu$ , i.e. we can choose  $U_\nu$  for  $\nu = 1, 2, \dots, n_p - 1$  as we like and  $U_{n_p}$  follows from the constraint. Then we can always choose

$$\lambda^U = \frac{\partial}{\partial U_{n_p}} S_{n_p}(U_{n_p}, V_{n_p}, \mathbf{N}_{n_p}) \Big|_{U_{n_p}, \mathbf{N}_{n_p}}, \quad (1.8)$$

as long as we are eventually able to fulfil the constraint equation (1.7a) . Then the remaining constraint reads

$$\sum_{\nu=1}^{n_p-1} \left[ \frac{\partial}{\partial U_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{V_\nu, \mathbf{N}_\nu} - \lambda^U \right] dU_\nu = 0; .$$

We have thus removed the dependence of the remaining  $dU_\nu$  from each other by making use of our constraints from equation (1.1) [\[previous page\]](#) . Now, all  $dU_\nu$  are independent and the square brackets have to vanish individually. Then we have for all  $\nu$  (the last comes from equation (1.8) )

$$\lambda^U = \frac{\partial}{\partial U_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{V_\nu, \mathbf{N}_\nu} = \frac{1}{T_\nu} . \quad (1.9a)$$

Similarly, we obtain from the other constraints

$$\lambda^V = \frac{\partial}{\partial V_\nu} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{U_\nu, \mathbf{N}_\nu} = \frac{p_\nu}{T_\nu} . \quad (1.9b)$$

and

$$\lambda_j^N = \frac{\partial}{\partial N_{j\nu}} S_\nu(U_\nu, V_\nu, \mathbf{N}_\nu) \Big|_{U_\nu, V_\nu, N_{i\nu}, i \neq j} = -\frac{\mu_{j\nu}}{T_\nu} , \quad \text{for } j = 1, \dots, \alpha . \quad (1.9c)$$

Since the Laplace multipliers on the lhs of these equations are independent of  $\nu$ , we find that in an isolated system in equilibrium all phases have the same temperature  $T$ , the same pressure  $p$  and the same chemical potential  $\mu_j$  for component  $j$ . But in general, we still have  $\mu_j \neq \mu_{j'}$ : The chemical potential in the different phases of one component is equal, but the chemical potential of the different components must not be equal.

STATE VARIABLES IN DIFFERENT PHASES			
$T_\nu = T$	$\forall \nu$		(1.10a)
$p_\nu = p$	$\forall \nu$		(1.10b)
$\mu_{j,\nu} = \mu_j$	$\forall \nu$	.	(1.10c)

### 1.1.2 Closed system with $p = \text{fixed}$ , $T = \text{fixed}$

Another important system is that where pressure and temperature are fixed experimentally. In this case the Free Enthalpy  $G(p, T, \mathbf{N})$  is the relevant thermodynamic potential. The equilibrium condition is  $G$  has to be minimal, or rather  $dG = 0$ . Again  $G$  is extensive, i.e. additive w.r.t. the phases

$$G = \sum_{\nu=1}^{n_p} G_\nu(T, p, \mathbf{N}_\nu) \quad (1.11)$$

Here, we have already exploited that all phases have the same temperature and pressure, which is given at the outset. So the only degree of freedom that can be varied are the particle numbers, still under the constrain used before. A similar derivation now yields

$$\lambda_j^N = \left. \frac{\partial}{\partial N_{j\nu}} G_\nu(T, p, \mathbf{N}_\nu) \right|_{T, p, N_{i,\nu}, i \neq j} = \mu_{j,\nu} . \quad (1.12)$$

and we find that in a closed system, where  $T$  and  $p$  is fixed, the chemical potential of component  $j$  is the same in all phases.



### 1.1.3 Number of degrees of freedom

The chemical potentials will depend on  $T$ ,  $p$ , and in principle on the particle numbers

$$\mu_{j\nu} = \mu_{j\nu}(T, p, \{N_{j\nu}\}) \quad (1.13)$$

Now, the chemical potentials are intensive quantities, i.e. scaling the extensive quantities in its argument, by a common factor does not change the chemical potential. Let's consider an intensive function  $f$  that shall depend on  $T$ ,  $p$ , and some particle numbers  $N_l$ . Scaling the total size by  $\lambda$  results in particle numbers  $\lambda N_l$ . As  $T$  and  $p$  are intensive, we have

$$f(T, p, \{\lambda N_l\}) = f(T, p, \{N_l\}) .$$

This is the case if  $f$  only depends on the concentrations  $c_j = N_j / \sum_i N_i$ , since scaling all  $N_j$  by a common factor does not change the concentration.

1. By phase we mean a spatial area within which no abrupt changes of any physical quantity occur, but at the boundary of which such changes can be observed.
2. The system consists of one or more component/ components. Component means the minimum number of independent chemical substances that we need to produce the phase.
3. The state of the system is described by a number of state variables depending on the type of system.
4. By degrees of freedom  $F$  we mean the number of state variables that we can vary independently of each other without any of the phases disappearing.
5. First of all, we want to assume that no chemical reactions take place in the system. The state of each phase is clearly defined when we specify  $T$ ,  $p$ , and the mole fraction of component  $j$  in phase  $\nu$ .

$$c_j^{(\nu)} := \frac{N_{j\nu}}{N_\nu} \quad (1.14)$$

i.e. the fraction of particles in phase  $\nu$  that belong to component  $j$ .

Then

$$\sum_j c_j^{(\nu)} = 1 , \quad \forall \nu . \quad (1.15)$$

Scaling the total size of the system by a common factor  $\lambda$ , i.e. in particular  $N_{j\nu} \rightarrow \lambda N_{j\nu}$ , does not change the phases. So along with the intensivity of the chemical potentials, we know by now that the latter depend on the mole fractions and not the absolute particle numbers

$$\mu_{j\nu} = \mu_{j\nu}(T, p, \{c_j^{(\nu)}\}) . \quad (1.16)$$

The number of state variables is  $Z_v = 2 + \alpha n_p$ , but they are not independent. According to equation (1.12) [p. 7] we have the equilibrium conditions, which include the conditions for the particle numbers in equation (1.1c) [p. 5]

$$\mu_{j\nu}(T, p, \{N_{j\nu}\}) = \mu_{j\nu'}(T, p, \{N_{j\nu}\}) , \forall \nu \neq \nu' \text{ and } \forall j .$$

These are  $n_p - 1$  conditions for each component, i.e. in total  $\alpha(n_p - 1)$  constraints. E.g. for  $n_p = 2$  we have one condition per component  $\mu_{j,1} = \mu_{j,2}$ . In addition we have the  $n_p$  normalization constraints in equation (1.15) [previous page] . So the total number of constraints is

$$Z_c = \alpha n_p - \alpha + n_p ,$$

and therefore the number of d.o.f. is the number of state variables minus the number of constraints resulting in

$$f = (\alpha n_p + 2) - (\alpha n_p - \alpha + n_p)$$

GIBBS' PHASE RULE	
$f = 2 + \alpha - n_p .$	$(1.17)$

### Example: $H_2O$ Phase diagram

In Figure 1.1 the phase diagram of water is shown. The lines in this diagram indicate the conditions of pressure and temperature, where a phase transition can happen - in the areas between the lines, water exists only in the corresponding phase.

- triple point:  $T_{tp} = 273.16000 \text{ K}$  ( $0.01000 \text{ }^\circ\text{C}$ ),  $p_{tp} = 611.657 \text{ Pa}$ .  
All three phases can coexist at these conditions

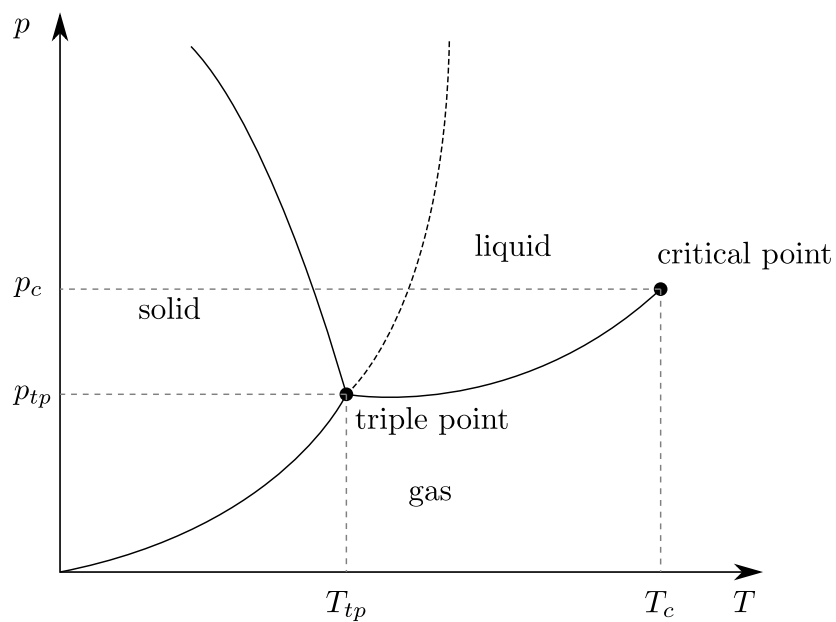


Figure 1.1: Schematic phase diagram for water. Note that the slope of the transition line from solid to liquid has a negative slope. The dashed line corresponds to the transition from solid to liquid for most other substances and shows a positive slope.

- critical point:  $T_c = 647.096$  K,  $p_c = 22.064$  MPa and  $\rho_c = 356$  kg/m<sup>3</sup>. Past this point, the phase transition between gaseous and liquid water vanishes.
- transition solid  $\leftrightarrow$  liquid: melting, freezing
- transition liquid  $\leftrightarrow$  gas: vaporization, condensation
- transition solid  $\leftrightarrow$  gas: sublimation, deposition

What is interesting about water is the phase transition between solid and liquid phase. Contrary to other chemical substances, the line corresponding to this transition in the phase diagram has a negative slope. When we increase the temperature, the pressure, at which the transition happens decreases or phrased in a more intuitive way: The density of ice is smaller than the density of water. What we would normally expect would be a decrease of density when the temperature rises but apparently, water behaves differently. This effect is called the anomaly of water and it is the reason why ice floats on water and why creatures living in water survive in winter.

Water is a one component liquid, i.e.  $\alpha = 1$ , in the following we will consider the different areas of the phase diagram where different phases coexist:

**a) Pure phases (solid,liquid,gas),  $n_p = 1$**

The number of degrees of freedom is, according to Gibb's phase rule,

$$f = 2 + \alpha - n_p = 2 + 1 - 1 = 2 ,$$

which means we can vary to state variables, here  $T$  and  $p$ , independently.

**b) Two coexisting phases,  $n_p = 2$**

- boiling curve (liquid to gas)
- condensation curve (gas to liquid)
- melting curve (solid to liquid)
- solidification curve (liquid to solid)
- sublimation curve (solid to gas)
- resublimation curve (gas to solid)

The number of degrees of freedom is

$$f = 2 + 1 - 2 = 1 ,$$

which means, we have to vary  $T$  and  $p$  on a curve,  $p(T)$ , e.g. melting-point line.

**c) Three coexisting phases,  $n_p = 3$ , triple point**

Here  $f = 0$ . There is no degree of freedom.

It also follows that for a one-component substance there are at most 3 phases, otherwise the number of degrees of freedom would become negative.

**1.1.4 Clausius-Clapeyron**

As a further application of the previous discussion, we consider the coexistence line between liquid and gas (Condensation curve) of a one-component system, such as water.

If we choose  $T$  and  $P$  as the two independent d.o.f. then according to equation (1.12) [p. 7] we have

$$\mu_{fl}(T, p) = \mu_g(T, p) . \quad (1.18)$$

This relation allows to determine the condensation curve  $p(T)$  on which the fluid and the gas phase coexist. Away from this curve, only one phase exists, namely the one where the chemical potential is smaller. This is due to the Gibbs-Duhem relation for the free Enthalpy

$$G(T, p, N) = \mu N .$$

In the case of a single phase, the system will minimize the Free Enthalpy  $G$ . Since  $N$  is fixed, the phase with the smaller chemical potential has a smaller Free Enthalpy.

The free Enthalpy is related to the free energy  $F(T, V, N)$  via a Legendre transformation, where  $V$  is replaced by  $p$ . We start out with

$$\left. \frac{\partial}{\partial V} F(T, V, N) \right|_{T, N} = -p$$

This relation can be inverted for given  $T, N$  and  $p$  we obtain

$$V = V(p, T, N)$$

$$G(T, p, N) = F(T, V(p, T, N), N) + pV . \quad (1.19)$$

Then

$$\begin{aligned} dG &= dF + pdV + Vdp \\ &= -SdT - pdV + \mu dN + pdV + Vdp . \end{aligned}$$

Hence we have

$$dG = \mu dN - SdT + Vdp ,$$

or rather

$$dG - \mu dN = -SdT + Vdp .$$

On the other hand the Gibbs-Duhem relation yields

$$dG = \mu dN + Nd\mu \Rightarrow dG - \mu dN = Nd\mu .$$

Combining the two relation yields

$$-SdT + Vdp = Nd\mu .$$

This is generally valid and it also applies individually to different phases, which here shall have the same  $T$  and  $p$ , i.e.

$$\begin{aligned} -S_\nu dT + V_\nu dp &= N_\nu d\mu_\nu \\ -\frac{S_\nu}{N_\nu} dT + \frac{V_\nu}{N_\nu} dp &= d\mu_\nu . \end{aligned}$$

We introduce entropy and volume per particle

$$s_\nu = \frac{S_\nu}{N_\nu} \tag{1.20}$$

$$v_\nu = \frac{V_\nu}{N_\nu} , \tag{1.21}$$

and obtain

$$-s_\nu dT + v_\nu dp = d\mu_\nu . \tag{1.22}$$

Now we consider changes  $dp, dT$  along the coexistence line. There we have due to equation (1.18) [\[previous page\]](#)

$$d\mu_{fl}(T, p) = d\mu_g(T, p) .$$

Along with equation (1.22) we obtain

$$\begin{aligned} -s_{fl}dT + v_{fl}dp &= -s_gdT + v_gdp \\ \frac{dp}{dT} &= \frac{s_g - s_{fl}}{v_g - v_{fl}} = \frac{\Delta s}{\Delta v} . \end{aligned}$$

Finally, we define the molar vaporation enthalpy

$$q = T(s_g - s_{fl})$$

and find the

CLAUSIUS-CLAPEYRON RELATION	
$\frac{dp}{dT} = \frac{q}{T(v_g - v_{fl})} .$	(1.23)

In the derivation of the Clausius-Clapeyron relation we have assumed that  $S_g \neq S_{fl}$  and  $V_g \neq V_{fl}$ , which means, although

$$\mu_g(T, p) = \mu_{fl}(T, p)$$

that

$$\left. \frac{\partial \mu_g}{\partial T} \right|_p \neq \left. \frac{\partial \mu_{fl}}{\partial T} \right|_p$$

$$\left. \frac{\partial \mu_g}{\partial p} \right|_T \neq \left. \frac{\partial \mu_{fl}}{\partial p} \right|_T .$$

So apparently, the chemical potential  $\mu(T)$  is continuous at the temperature of phase transition, but its first derivative is not (i.e.  $\mu(T)$  is continuous, but not differentiable). Such a phase transition is called a first order phase transition.

Generally, if we have similar behaviour where the the  $n$ -th derivative of  $\mu(T)$  is not continuous, but up until the  $(n-1)$ -th derivative, they are all continuous, we call this a phase transition of  $n$ -th order. We will further look at this in subsection 1.2.3. This means that the Clausius-Clapeyron relation is only valid for first order phase transitions.

### 1.1.5 Real gases (van der Waals equation)

In statistical physics we have discussed the ideal gas and its equation of state

$$pV = Nk_B T .$$

We also introduced finite eigen-volumes of the molecules and obtained

$$p(V - V_0) = Nk_B T .$$

Finally, we also want to include the intermolecular forces. For molecules in the bulk of the material, these forces vanish on average, as they act with equal strength in opposite directions. This is not the case in the surface layer, where the partners outside the surface are missing. This results in an effective force, pulling the molecules in the surface layer into the bulk. I.e. the intermolecular forces act at the surface like an additional pressure (internal pressure). It has the form

$$a \frac{N^2}{V^2}$$

resulting in

VAN DER WAALS EQUATION ( <i>extensive form</i> )
$\left( p + a' \frac{N^2}{V^2} \right) \left( V - \overbrace{Nb'}^{V_0} \right) = Nk_B T . \quad (1.24)$

It can also be expressed in terms of molar volume  $v$ , pressure and temperature, i.e. the intensive form

VAN DER WAALS EQUATION ( <i>intensive representation</i> )
$\left( p + \frac{a'}{v^2} \right) (v - b') = k_B T . \quad (1.25)$

Here,  $b'$  is the eigen-volume of one molecule.

We multiply equation (1.25) by  $v^2$  and find that it becomes a cubic equation



in  $v$

$$\begin{aligned} \left(pv^2 + a'\right)\left(v - b'\right) - k_B T v^2 &= 0 \\ pv^3 + a'v - b'pv^2 - a'b' - k_B T v^2 &= 0 . \end{aligned}$$

Next we introduce suitable units  $p_{cr}, v_{cr}, T_{cr}$  and the corresponding dimensionless quantities  $\pi, \nu, t$  via

$$\begin{aligned} p &= \pi p_{cr} \\ v &= \nu v_{cr} \\ T &= t T_{cr} , \end{aligned}$$

resulting in

$$\begin{aligned} \pi \nu^3 + \frac{a' v_{cr}}{p_{cr} v_{cr}^3} \nu - \frac{b' p_{cr} v_{cr}^2}{p_{cr} v_{cr}^3} \pi \nu^2 - \frac{a' b'}{p_{cr} v_{cr}^3} - \frac{k_B T_{cr} v_{cr}^2}{p_{cr} v_{cr}^3} t \nu^2 &= 0 \\ \pi \nu^3 + \underbrace{\left(\frac{a'}{p_{cr} v_{cr}^2}\right)}_{=T_1} \nu - \underbrace{\left(\frac{b'}{v_{cr}}\right)}_{=T_2} \pi \nu^2 - \underbrace{\left(\frac{a' b'}{p_{cr} v_{cr}^3}\right)}_{=T_3} - \underbrace{\left(\frac{k_B T_{cr}}{p_{cr} v_{cr}}\right)}_{=T_4} t \nu^2 &= 0 \\ \pi \nu^3 + T_1 \nu - T_2 \pi \nu^2 - T_3 - T_4 t \nu^2 &= 0 . \end{aligned}$$

Now we have the freedom to choose the parameters  $p_{cr}, v_{cr}$ , and  $T_{cr}$  suitably. We use

$$T_2 = \left(\frac{b'}{v_{cr}}\right) = \frac{1}{3} \tag{1.26}$$

$$T_3 = \left(\frac{a' b'}{p_{cr} v_{cr}^3}\right) = 1 \tag{1.27}$$

$$T_4 = \left(\frac{k_B T_{cr}}{p_{cr} v_{cr}}\right) = \frac{8}{3} \tag{1.28}$$

then

CRITICAL VALUES ( <i>van der Waals</i> )	
$v_{cr} = 3b'$	(1.29)
$\Rightarrow p_{cr} = \frac{a'b'}{v_{cr}^3} = \frac{a'}{27b'^2}$	(1.30)
$\Rightarrow T_1 = \frac{a'}{p_{cr}v_{cr}^2} = \frac{a'}{\frac{a'}{27b'^2}9b'^2} = 3$	(1.31)
$k_B T_{cr} = \frac{8}{3}p_{cr}v_{cr} = \frac{8}{3}\frac{a'}{27b'^2}3b' = \frac{8a'}{27b'}$	(1.32)

So we have

$$\pi\nu^3 + 3\nu - \frac{\pi\nu^2}{3} - 1 = \frac{8}{3}t\nu^2 \quad (1.33)$$

$$\left(\pi + \frac{3}{\nu^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}t. \quad (1.34)$$

Solving for  $\pi$  yields

$$\pi = \frac{8t}{3\nu - 1} - \frac{3}{\nu^2}$$

The term  $(3\nu - 1)$  originated from  $V - V_0$ , and therefore it is always positive, as the total volume has to be greater than the eigenvolume.

If we plot  $\pi(\nu)$  for fixed  $t$  (isotherm) then we find different behaviour for  $t < 1$ ,  $t = 1$  and  $t > 1$ .

- For  $t < 1$  the curve has a minimum and a maximum in the physical interval  $\nu > 1/3$ .
- For  $t = 1$  minimum and maximum coincide at  $\nu = 1$ , with  $\pi(\nu = 1) = 1$ .
- For  $t > 1$  the curve is monotonically decreasing in the physical interval.

We will see soon that there is a phase transition if the curve has a maximum and a minimum. If it is monotonically decreasing, there is no phase transition. Hence, the parameters  $p_{cr}$ ,  $v_{cr}$ , and  $T_{cr}$  represent the critical point.

Inserting the critical values into the van-der-Waals equation of state in form of the following ratio then we obtain

$$Z := \frac{p_{cr}v_{cr}}{k_B T_{cr}} = \frac{\frac{a'}{27b'^2}3b'}{\frac{8a'}{27b'}} = \frac{3}{8}. \quad (1.35)$$

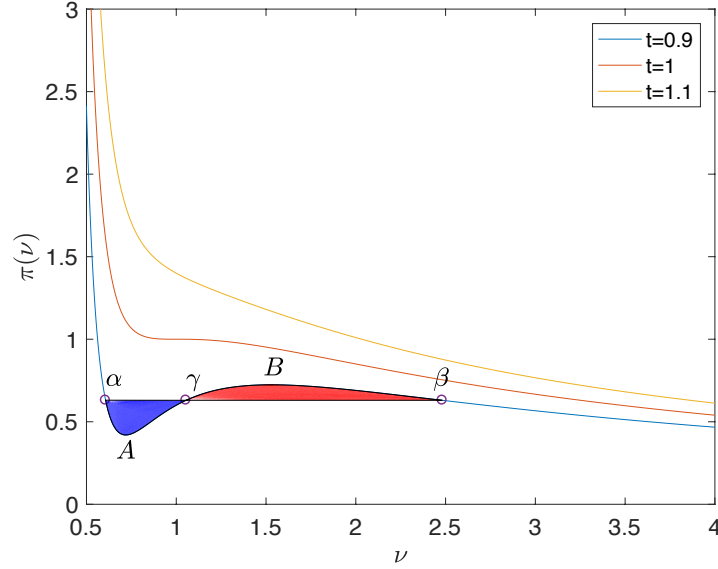


Figure 1.2: Different types of isotherms in the  $\pi$ - $\nu$ -diagram and Maxwell construction

Experimentally one finds for all real gases at the critical point  $Z < 3/8$ , while the ideal gas yields  $Z = 1$ . In this respect the van der Waals model is clearly better. Alternatively, equation (1.35) can be written with  $\rho_c = 1/v_{cr}$  as

$$p_{cr} = \frac{3}{8} k_B T_{cr} \rho_{cr} . \quad (1.36)$$

### 1.1.6 Maxwell-Construction

We have just seen that the equation of state for the van-der-Waals model reads in dimensionless units

$$\left( \pi - \frac{3}{\nu^2} \right) \left( \nu - \frac{1}{3} \right) = \frac{8}{3} t ,$$

and that  $t < 1$  there are regions in the  $\pi - \nu$ -diagram where the isotherm compressibility

$$\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \Big|_T \right) = -\frac{1}{p_{cr} \nu} \left( \frac{\partial \nu}{\partial \pi} \Big|_T \right) < 0 \quad (1.37)$$

becomes negative. This implies that the system is mechanically unstable, it would shrink by itself.

The reason is that the van der Waals model describes a single phase of a one-component system. It is applicable in the pure gas or liquid phase (grey shaded areas in the figure), but in between there is a phase transition, where two phases coexist, which cannot be described by the van-der-Waals equation.

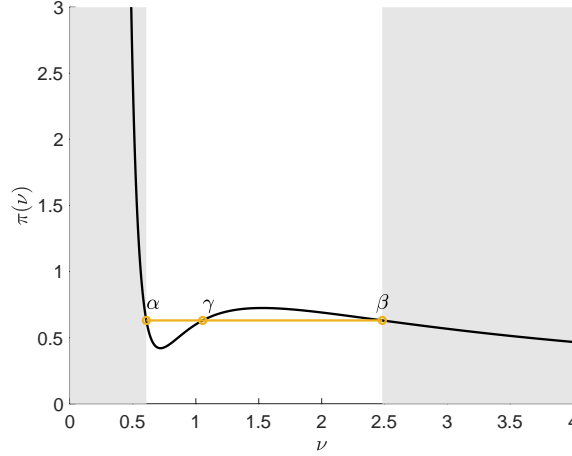


Figure 1.3: **Maxwell construction.**

What happens in reality is the following. We have seen before that generally on the coexistence line between liquid and gas, the number of degrees of freedom is merely 1 and it is described by  $p = p(T)$ . On isotherms,  $T$  is also fixed and therefore  $p$  is fixed as well and takes a constant value.

All isotherms for ( $T < T_c$ ) in the two-phase region have to be horizontal lines in the  $pV$ -diagram. (See golden line in the Figure 1.3)

We denote the pressure on the coexistence line between the points  $\alpha$  and  $\beta$  in the  $\pi$ - $\nu$ -diagram by  $p_{\alpha\beta}$ .

In the pure phases of a one-component system we have only two degrees of freedom, e.g.  $p$  and  $T$ . Since the van-der-Waals model is still valid in the pure phase regions (grey shaded), we have the condition (equation (1.18) [p. 12])

$$\mu_{fl}(T, p) = \mu_g(T, p) .$$

Hence for the isotherm compression, discussed before, the pressure ( $p_{\alpha\beta}$ ) when reducing the volume in the two-phase region, does not change and we have

$$\mu_{fl}(T, p_{\alpha\beta}) = \mu_g(T, p_{\alpha\beta}) .$$

In addition, at the points  $\alpha$  and  $\beta$ , we have pure phases and therefore  $N_\alpha = N$  and  $N_\beta = N$  and consequently at the end points of the horizontal line we have

$$\mu_{fl}(T, P_{\alpha\beta})N_\alpha = \mu_g(T, P_{\alpha\beta})N_\beta$$

Then the Gibbs-Duhem relation ( $G = \mu N$ ) yields for these points

$$\begin{aligned} G_\alpha &= G_\beta \\ F_\alpha + p_\alpha V_\alpha &= F_\beta + p_\beta V_\beta \end{aligned}$$

from which we conclude

$$F_\alpha - F_\beta = -p_{\alpha\beta}(V_\alpha - V_\beta) . \quad (1.38)$$

This is the result for the end points of the coexistence line, where we have exploited some of the features of the two phase region in between, for which we have used  $\pi(\nu) = p_{\alpha\beta}$ , independent of volume  $\nu$ . Alternatively, if we stick to one phase then the van-der-Waals equation determines the pressure curve, the one given in figure 1.2 with the wavy look. In that case, the differential of the free energy is, since  $N$  is fixed and  $T$  is fixed (isotherm),

$$dF = SdT - pdV + \mu dN = -pdV ,$$

and the integral yields

$$\begin{aligned} F_\alpha - F_\beta &= \int_\beta^\alpha dF = - \int_{V_\beta}^{V_\alpha} p(T, V', N) dV' . \\ F_\alpha - F_\beta &= \int_{V_\alpha}^{V_\beta} p(T, V', N) dV' \end{aligned} \quad (1.39)$$

By combining equation (1.38) and equation (1.39) we find

$$\int_{V_\alpha}^{V_\beta} p(T, V', N) dV' = p_{\alpha\beta} (V_\beta - V_\alpha) \quad (1.40)$$

I.e., the area under the vdW curve  $p(t, V, N)$  as function of volume in the interval  $(V_\alpha, V_\beta)$  has to be the same as the area under the constant  $p(V) = p_{\alpha\beta}$  curve in the two-phase region. Consequently, the subareas  $A$  and  $B$  have to be same. This tells us how to determine the points  $\alpha$  and  $\beta$ .

Finally, we will show that the pure phase is unstable w.r.t. the mixed phase. To this end we compare two states with the same  $T$  and  $V$  but different

pressures  $p_{\alpha\beta}$  and the vdW pressure  $p(T, V, N)$ . Since  $T$  and  $V$  is fixed, we have to compare the free energies.

In the pure phase we compute the free energy for a given  $V$  in the interval  $(V_\alpha, V_\beta)$ . To this end we use  $dF = -pdV$  and integrate from  $V_\alpha$  to  $V$

$$F_{vdW}(V, T) = F_\alpha - \int_{V_\alpha}^V p(V', T) dV' . \quad (1.41)$$

In the two-phase region the free energy is the linear combination of the free energy of the two phases according to their relative size, (extensive quantities!) i.e.

$$F_{mp} = c_{fl} F_{fl} + c_g F_g \quad (1.42)$$

$$= c_{fl} F_\alpha + c_g F_\beta , \quad (1.43)$$

with  $c_{fl} = N_{fl}/N$  and  $c_g = N_g/N$ , hence  $c_{fl} + c_g = 1$  . Then

$$\begin{aligned} F_{mp} &= F_\alpha + (c_{fl} - 1)F_\alpha + c_g F_\beta \\ &= F_\alpha - c_g F_\alpha + c_g F_\beta \\ F_{mp} &= F_\alpha + c_g (F_\beta - F_\alpha) . \end{aligned} \quad (1.44)$$

Finally, we want to express  $c_g$  in terms of volume. The total volume is split accordingly,

$$\begin{aligned} V &= c_{fl} V_{fl} + c_g V_g = (1 - c_g) V_\alpha + c_g V_\beta \\ V - V_\alpha &= c_g (V_\beta - V_\alpha) \end{aligned}$$

and we obtain the relative portions of the phases

$$c_g = \frac{V - V_\alpha}{V_\beta - V_\alpha} \quad (1.45a)$$

$$c_{fl} = \frac{V_\beta - V}{V_\beta - V_\alpha} . \quad (1.45b)$$

We then have

$$F_{mp} = F_\alpha + \frac{V - V_\alpha}{V_\beta - V_\alpha} (F_\beta - F_\alpha) . \quad (1.46)$$

Here we can also use equation (1.38) [\[previous page\]](#) resulting in

$$F_{mp} = F_\alpha - p_{\alpha\beta} \frac{V - V_\alpha}{V_\beta - V_\alpha} (V_\beta - V_\alpha) = F_\alpha - p_{\alpha\beta} (V - V_\alpha) \quad (1.47)$$

Since, according to equation (1.41) ,

$$F_\alpha = F_{vdW}(V, T) + \int_{V_\alpha}^V p(V', T) dV'$$

we can rewrite equation (1.47) As

$$F_{mp} = F_{vdW}(V, T) + \int_{V_\alpha}^V p(V', T) dV' - p_{\alpha\beta}(V - V_\alpha)$$

and obtain for the difference of the free energies:

$$F_{vdW}(V, T) - F_{mp}(V, T) = p_{\alpha\beta}(V - V_\alpha) - \int_{V_\alpha}^V p(V', T) dV' \geq 0 .$$

The reason can be seen in figure 1.3. As long as  $V \in (V_\alpha, V_\gamma)$  it is obvious that the integral over  $p(V', T)$  is smaller than the area of the rectangle formed obtained if  $p$  is replaced by  $p_{\alpha\beta}$ . For  $V > V_\gamma$  we can modify the equation as follows

$$\begin{aligned} F_{vdW}(V, T) - F_{mp}(V, T) &= p_{\alpha\beta}(V_\gamma - V_\alpha) + p_{\alpha\beta}(V - V_\gamma) - \left( \int_{V_\alpha}^{V_\gamma} p(V', T) dV' + \int_{V_\gamma}^V p(V', T) dV' \right) \\ &= p_{\alpha\beta}(V_\gamma - V_\alpha) - \underbrace{\int_{V_\alpha}^{V_\gamma} p(V', T) dV'}_{=A} - \int_{V_\gamma}^V (p(V', T) - p_{\alpha\beta}) dV' \end{aligned}$$

For  $V \in (V_\gamma, V_\beta)$  the integrand is positive and

$$\int_{V_\gamma}^V (p(V', T) - p_{\alpha\beta}) dV' \leq B$$

since  $B$  is the value of the integral over the entire interval  $(V_\gamma, V_\beta)$ . According to the Maxwell construction  $B = A$ , hence the integral is less then  $A$  so in total we find

$$F_{vdW}(V, T) - F_{mp}(V, T) \geq 0 ,$$

as we have claimed before.

## 1.2 Phasetransitions

In case of the liquid-gas mixture we have seen that the transition occurs for given  $T$  at a fixed pressure  $p$ , the latter is

$$p_{\alpha,\beta}(T) : \quad \text{vapor pressure}$$

In the transition region we have a mixture of the liquid and the gas state. The percentage portion is given by equation (1.45) [\[previous page\]](#) .

The supplied heat leads to transformation of a fraction of the fluid into gas. This process is isothermal, since heat is only used to overcome the binding energy present in the fluid. Only if the entire fluid is vaporized, further heat transfer results in a temperature rise. Heat transfer that occurs at a constant system temperature but changes the state variable is called latent heat with respect to the variable.

Such phase transitions, that involve *latent heat* are *first order phase transitions*. *Latent heat* in contrast to *sensible heat* does not lead to a change in temperature.

Only for first order phase transitions, the Clausius-Clapeyron equation is applicable, as it requires that entropy and volume per particle are different in the two phases.

We recall that

$$S = -\left.\frac{\partial G}{\partial T}\right|_p \quad (1.48)$$

$$V = -\left.\frac{\partial G}{\partial p}\right|_T . \quad (1.49)$$

A characteristic feature of first order phase transition is therefore the [discontinuity of the first derivative of the thermodynamic potential](#) when crossing the coexistence line in the phase diagram. Along the line it is continuous.

Next we exploit in addition the following relations

$$S = -\left.\frac{\partial F}{\partial T}\right|_V , \quad (1.50)$$

$$p = -\left.\frac{\partial F}{\partial V}\right|_T . \quad (1.51)$$

which allow to compute specific heat and compressibility in term of the free



enthalpy or free energy

$$0 \leq C_p = T \frac{\partial S}{\partial T} \Big|_p = -T \frac{\partial^2 G}{\partial T^2} \Big|_p \Rightarrow \frac{\partial^2 G}{\partial T^2} \Big|_p \leq 0 \quad (1.52)$$

$$0 \leq \kappa_T = -\frac{1}{V} \frac{\partial V}{\partial p} \Big|_T = -\frac{1}{V} \frac{\partial^2 G}{\partial p^2} \Big|_T \Rightarrow \frac{\partial^2 G}{\partial p^2} \Big|_T \leq 0 \quad (1.53)$$

$$0 \leq C_V = T \frac{\partial S}{\partial T} \Big|_V = -T \frac{\partial^2 F}{\partial T^2} \Big|_V \Rightarrow \frac{\partial^2 F}{\partial T^2} \Big|_V \leq 0 \quad (1.54)$$

$$0 \leq \frac{1}{\kappa_T} = -V \frac{\partial p}{\partial V} \Big|_T = V \frac{\partial^2 F}{\partial V^2} \Big|_T \Rightarrow \frac{\partial^2 F}{\partial V^2} \Big|_T \geq 0. \quad (1.55)$$

Therefore we have

$\frac{\partial^2 G}{\partial T^2} \Big _p \leq 0, \quad \frac{\partial^2 G}{\partial p^2} \Big _T \leq 0 \quad (1.56)$	
$\frac{\partial^2 F}{\partial T^2} \Big _V \leq 0, \quad \frac{\partial^2 F}{\partial V^2} \Big _T \geq 0. \quad (1.57)$	

I.e.  $G$  is concave in both variables  $p$  and  $T$ . Due to  $F = G - pV$ , it follows that  $F$  is also concave in  $T$ , but convex in  $p$ .

### 1.2.1 Free energy versus $V$ , for $T$ and $N$ fixed

Since  $F > 0$  is convex in  $V$  and  $\frac{\partial F}{\partial V} = -p < 0$ ,  $F(V)$  is a decreasing, left curving line that is always positive and it is strictly monotonically decreasing, as a zero slope would mean  $p = 0$ .

The Legendre transform of the free energy  $F(V, T, N)$  (in the variable  $V$ ) is the free enthalpy

$$\begin{aligned} G(p, T, N) &= F(V(p), T, N) + pV \\ &= F(V(p), T, N) - \frac{\partial F}{\partial V} \Big|_T V \end{aligned}$$

For  $T > T_c$  the second derivative  $\frac{\partial^2 F}{\partial V^2} = -\frac{\partial p}{\partial V}$  is greater than zero (because  $\frac{\partial p}{\partial V} < 0$ ) for all  $V$ . For  $T < T_c$ , however, there is an interval  $(V_\alpha, V_\beta)$  in which

$\frac{\partial P}{\partial V} = 0$  and hence the second derivative is zero and therefore  $F(V)$  is a linear function in  $V$ . Hence,

$$\left. \frac{\partial F}{\partial V} \right|_T = -p = \text{const} = -p_{\alpha\beta}.$$

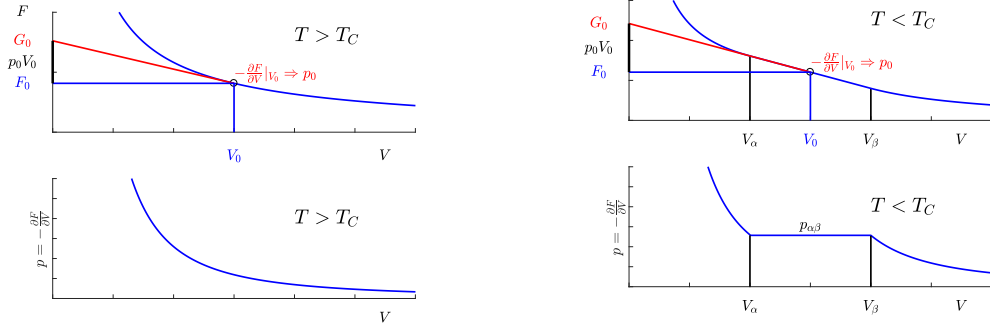


Figure 1.4: **Schematic plot of free energy for  $T > T_C$  (left) and  $T < T_C$  (right).**

### 1.2.2 $F$ and $G$ as function of $T$

As function of temperature, both  $F$  and  $G$  behave similarly, since

$$-S = \left. \frac{\partial G}{\partial T} \right|_p = \left. \frac{\partial F}{\partial T} \right|_V.$$

At a first order phase transition the entropy  $S$  shows a finite jump at  $T_c$ , which is then associated with

LATENT HEAT
$\Delta Q = T_{\alpha\beta} \Delta S. \quad (1.58)$

$\Delta Q$  is, however not a material constant, it also depends on the state variables e.g. for the liquid-gas system,  $\Delta Q$  depends on pressure. On approaching the [critical point](#),  $\Delta Q$  vanishes. Therefore, the definition of the order of the phase transition has to be refined.

### 1.2.3 Ehrenfest classification

We will now define our phase transitions solely by looking at the derivatives of the thermodynamic potential  $G$ . In fact, the order of a phase transition will be defined by the first occurring derivative, that shows a discontinuity. In other words, if the first  $n - 1$  derivatives of  $G$  are continuous, but the  $n$ -th derivative is not, then we are dealing with a phase transition of order  $n$ .

N-TH ORDER PHASE TRANSITION

$$\left. \frac{\partial^m G_\alpha}{\partial T^m} \right|_p = \left. \frac{\partial^m G_\beta}{\partial T^m} \right|_p$$

$\forall m = 1, \dots, n - 1,$

(1.59a)

$$\left. \frac{\partial^m G_\alpha}{\partial p^m} \right|_T = \left. \frac{\partial^m G_\beta}{\partial p^m} \right|_T$$

$\forall m = 1, \dots, n - 1.$

(1.59b)

But

$$\left. \frac{\partial^n G_\alpha}{\partial T^n} \right|_p \neq \left. \frac{\partial^n G_\beta}{\partial T^n} \right|_p$$

(1.59c)

$$\left. \frac{\partial^n G_\alpha}{\partial p^n} \right|_T \neq \left. \frac{\partial^n G_\beta}{\partial p^n} \right|_T.$$

(1.59d)

Of practical importance are first and second order phase transitions. First order phase transitions have already been discussed in detail. In the case of a second order phase transition, we have

SECOND ORDER PHASE TRANSITION			
<hr/>			
$G(T, p)$	continuous	(1.60a)	
$\left. \frac{\partial G}{\partial T} \right _p, \left. \frac{\partial G}{\partial p} \right _T, (\text{i.e. } S(T, p), V(T, p))$	continuous	(1.60b)	
$C_p = -T \left. \frac{\partial^2 G}{\partial T^2} \right _p$	$\kappa_T = -\frac{1}{V} \left. \frac{\partial^2 G}{\partial p^2} \right _T$	discontinuous	(1.60c)

## 1.3 Critical exponents

[Second order phase transitions](#) are of particular interest, as they exhibit universal behaviour close to the phase transition. Completely different systems with second order phase transition show a very similar power law behaviour, which shall be discussed in this section. We define

$$\varepsilon = \frac{T - T_C}{T_C} .$$

Very often it is observed that in a close vicinity of  $T_c$ , typically  $|\varepsilon| < 10^{-2}$ , physical observables  $f(T)$  show the following behaviour

$$f(\varepsilon) = a\varepsilon^\varphi(1 + b\varepsilon^\psi + \dots) \quad , \quad \psi > 0 ,$$

here  $\varphi$  and  $\psi$  are real valued exponents. This behaviour is abbreviated as

$$f(\varepsilon) \simeq \varepsilon^\varphi , \tag{1.61}$$

One says:  $f$  behaves like  $\varepsilon^\varphi$  and  $\varphi$  is called the *critical exponent*. The critical exponent is more generally defined by

$$\varphi = \lim_{\varepsilon \rightarrow 0} \frac{\ln |f(\varepsilon)|}{\ln |\varepsilon|} \tag{1.62}$$

In general, the critical exponent depends on whether we compute it below or above  $T_c$ . For the [order parameter](#) it actually makes only sense below  $T_c$ , as the order parameter by definition is zero above  $T_c$  (consider e.g. the spontaneous magnetization from the Ising model, that vanished above the critical temperature).

Of course, different observables may have different critical exponents. The critical exponent for one observable, however, is almost universal, it only depends on

- spatial dimension
- range of the interaction
- spin dimensionality.

This is the so-called *universality hypothesis* of Griffiths. The range of the particle interaction is grouped into three classes, depending on the value of  $\alpha$  in the following definitions:

- short-range, if the interaction decreases like

$$r^{-(d+2+\alpha)} ; \quad \alpha > 0 .$$

Where the details of the interaction are unimportant. One finds a really universal behaviour.

- long-range, if

$$r^{-(d+2+\alpha)} ; \quad \alpha < \frac{d}{2} - 2 \quad (\text{always negative}).$$

In this case, the *classical theories* apply (Landau Theory, van der Waals model, Weißferromagnet). In this case, however, the critical exponents are independent of the spatial dimension.

- Intermediate range, if

$$r^{-(d+2+\alpha)} ; \quad \frac{d}{2} - 2 < \alpha < 0 .$$

The critical exponents depend on  $\alpha$

Magnetic systems are typically discussed as interacting spin-systems. As spin-dimension ( $n$ ) we understand the relevant components of the spin vector. Ising model /Potts model has  $n = 1$  and is considered as one-dimensional vector, the  $x - y$ -model has  $n = 2$  and the vectors are two-dimensional, and finally the Heisenberg model  $n = 3$  is three-dimensional.

Possible behaviour:

- **Power law decay:**  $f(\varepsilon) \simeq \varepsilon^{-\varphi}$  with  $\varphi > 0$
- **Power law divergence:**  $f(\varepsilon) \simeq \varepsilon^{-\varphi}$  with  $\varphi < 0$
- **logarithmic divergence:**  $f(\varepsilon) = a + b \ln(|\varepsilon|)$  we obtain by the definition of the critical exponent in equation (1.62) [\[previous page\]](#)  $\varphi = 0$

In general, the critical exponent for  $\varepsilon > 0$  (denoted by  $\varphi$ ) can differ from that for  $\varepsilon < 0$  (denoted by  $\varphi'$ ).

### 1.3.1 Important critical exponents

In the following, a few examples for the most important critical exponents that are associated to specific physical quantities often relevant in statistical physics will be given. This to be understood as an outlook for the next chapters: The actual theory behind the models mentioned in the examples will be presented in the following chapters, some of which in more detail than others.

1. Heat capacity:  $(\alpha_{\pm})$  for real gases

$$C_V \simeq A_{\pm} |\varepsilon|^{-\alpha_{\pm}} : \quad \begin{cases} + & \text{for path I, i.e. } T \searrow T_c, \text{ with } \rho = \rho_c \\ - & \text{for path II, i.e. } T \nearrow T_c, \text{ with } \rho = \rho_{g,fl} \end{cases} . \quad (1.63)$$

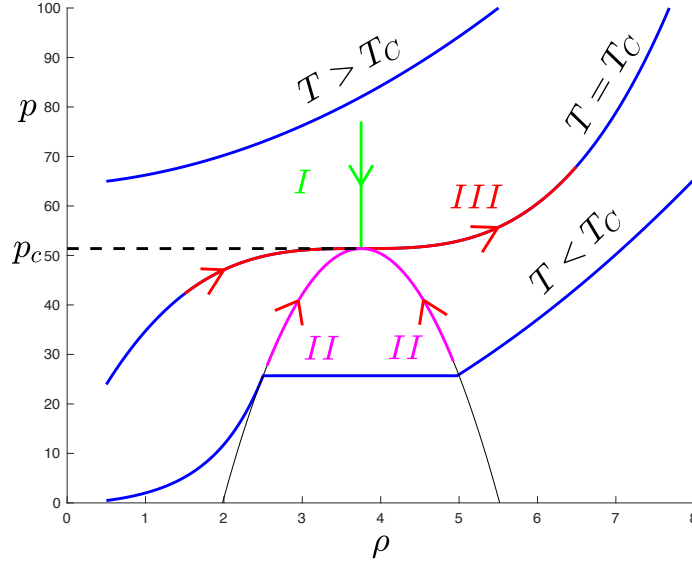


Figure 1.5: **Pressure versus density. Paths for the definition of the critical exponents.**

Heat capacity ( $\alpha'$ ,  $\alpha$ ) for magnets

$$C_B \simeq A_{\pm} |\varepsilon|^{-\alpha_{\pm}} : \quad \begin{cases} + & \text{for path I, i.e. } T \searrow T_c, \text{ with } B = 0 \\ - & T \nearrow T_c, \text{ with } B = 0 \end{cases} . \quad (1.64)$$

The experiment yields  $\alpha_{\pm} \approx 0$ . For the 2D Ising model the exact solution yields a logarithmic divergence, i.e.  $\alpha_{\pm} = 0$ . (see section 2.3.8). Classic theories (i.e. Weißferro-magnet, vdW gas) yield a discontinuity, which is equivalent to  $\alpha_{\pm} = 0$ .

2. Order parameter  $\beta$  (not inverse temperature!). This parameter exists only below  $T_c$  and characterises the order of the system. Two examples are the magnetization  $M$  or for real gases the density difference  $\Delta\rho = \rho_{fl} - \rho_g$ , or rather  $\Delta\rho = \rho_{fl,g} - \rho_c$ , in the two-phase region

$$\frac{\Delta\rho(T)}{2\rho_c} \simeq B |\varepsilon|^{\beta} , \text{ along path II} \quad (1.65)$$

$$\frac{M(T)}{M(0)} \simeq B |\varepsilon|^{\beta} , \text{ for zero magnetic field .} \quad (1.66)$$

The normalizations are introduced to make sure that  $B$  is  $O(1)$ . In principle,  $\beta$  should be  $\beta'$ , as we are below  $T_c$ , but since the order parameter is only defined below  $T_c$ , it is common to use  $\beta$ . Typical experimental values are  $0.35 \pm 0.02$ . classical theories yield  $\beta = 1/2$ . The exact solution of the 2D Ising model yields  $\beta = 1/8$ . (See section 2.3.10). For the 3D Ising model one finds  $\beta = 0.325 \pm 0.001$ , while 3D Heisenberg gives  $\beta = 0.3445 \pm 0.002$ .

3. Compressibilities and susceptibilities ( $\gamma_{\pm}$ ):

$$\kappa_T = -\frac{1}{V} \frac{\partial V}{\partial p} \Big|_T = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \Big|_T ,$$

$$\chi_T = \frac{\partial M}{\partial B} \Big|_T$$

$$\frac{\kappa_T}{\kappa_{T_c}^0} \simeq C_{\pm} |\varepsilon|^{\gamma_{\pm}} : \quad \begin{cases} - & \text{for path II, i.e. } T \nearrow T_c, \text{ with } \rho = \rho_{g,fl} \\ + & \text{for path I, i.e. } T \searrow T_c, \text{ with } \rho = \rho_c \end{cases} . \quad (1.67)$$

Here  $\kappa_c^0$  is the compressibility of the ideal gas at  $T = T_c$ , which follows (due to  $\rho = \frac{p}{k_B T}$ ) from

$$\kappa = \frac{1}{\rho} \frac{1}{k_B T} = \frac{1}{p} .$$

Similarly, one uses for the normalization in the magnetic case, the result of an ideal paramagnet

$$M = \frac{C^*}{T} ,$$

with  $C^*$  being the Curie constant.

$$\frac{\chi_T}{\chi_{T_c}^0} \simeq C_{\pm} |\varepsilon|^{-\gamma_{\pm}} : \quad \begin{cases} - & \text{for path II, i.e. } T \nearrow T_c, \text{ with } H = 0 \\ + & \text{for path I, i.e. } T \searrow T_c, \text{ with } H = 0 \end{cases} . \quad (1.68)$$

Typical experimental values differ somewhat about  $\gamma \approx \gamma' \approx 1.3$ . Model calculations all yield  $\gamma = \gamma'$ . Classical models yield  $\gamma = 1$ , the exact solution of the 2D Ising model yields  $\gamma = 7/4$ , the 3D Ising yields  $\gamma \approx 1.24$  and 3D Heisenberg yields  $\gamma = 1.39$ .

4. Critical isotherm ( $\delta$ ):

We define the critical pressure of the ideal gas as the pressure of the ideal gas at  $T_c$  and  $\rho_c$  via the ideal gas law

$$p_c^{(0)} = k_B T_c \rho_c$$

which clearly differs from the relation of the vdW model (see equation (1.36) [p. 18]). For the real gas we define the isothermal critical exponent as

$$\frac{p - p_c}{p_c^0} \simeq D \left| \frac{\rho - \rho_c}{\rho_c} \right|^\delta \text{sign}(\rho - \rho_c); \quad (\text{along the path III, } T = T_c) . \quad (1.69)$$

If we define

$$B_C^{(0)} = \frac{k_B T_c}{\mu_0 m} m$$

with  $m$  being the magnetic moment per particle, then the corresponding relation for magnets reads

$$\frac{B}{B_C^{(0)}} \simeq D \left| \frac{M(T = T_c, B)}{M(T = 0, B = 0)} \right|^\delta \text{sign}(M) .$$

Experimental values are in the range  $\delta \in (4, 5)$ . The 2D Ising model yields  $\delta = 15$ , which is apparently far off. The 3D Ising and Heisenberg model yields  $\delta = 4.9$  (much better). Classical theories yield  $\delta = 3$ .

5. Correlation lengths ( $\nu, \nu', \eta$ )

We define a pair-correlation function, e.g. density-density or spin-spin,

$$g(\mathbf{r}, \mathbf{r}') = \langle \Delta \rho(\mathbf{r}) \Delta \rho(\mathbf{r}') \rangle \quad (1.70)$$

$$g_{ij} = \langle \Delta \mathbf{S}_i \Delta \mathbf{S}_j \rangle . \quad (1.71)$$

In the critical regions they behave approximately like

## ORNSTEIN-ZERNIKE FUNCTION

$$g(\mathbf{r}, \mathbf{r}') = c_0 \frac{e^{-\frac{|\mathbf{r}-\mathbf{r}'|}{\xi(T)}}}{|\mathbf{r} - \mathbf{r}'|} \quad (1.72)$$

$$g_{ij} = g(\mathbf{r}_i, \mathbf{r}_j) , \quad (1.73)$$



with  $\xi(T)$  is the correlation length. It diverges on approaching the critical point. For real gases one defines

$$\xi \simeq D_{\pm} |\varepsilon|^{-\nu_{\pm}} : \quad \begin{cases} - & \text{on path II ,} \\ + & \text{on path I .} \end{cases} \quad (1.74)$$

For magnets

$$\xi \simeq D_{\pm} |\varepsilon|^{-\nu_{\pm}} : \quad \begin{cases} - & \text{for } T \nearrow T > T_C, B = 0 , \\ + & \text{for } T \searrow T > T_C, B = 0 . \end{cases} \quad (1.75)$$

Finally, we also introduce the pair-correlation function at the critical temperature  $T = T_c$ . According to the Ornstein-Zernike formula in equation (1.72) [\[previous page\]](#), at  $T = T_c$  the correlation length is infinite and, therefore, the pair-correlation would decrease like  $1/|\mathbf{r} - \mathbf{r}'|$ . This is, however, not the case for real systems. The behaviour is slightly different and expressed by

$$g(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'|^{-(d-2+\eta)} \begin{cases} p = p_C & , \text{ real gases ,} \\ B = 0 & , \text{ magnets .} \end{cases}$$

The Ornstein-Zernike formula would imply  $\eta = 3 - d$ . For the other critical exponents  $\nu, \nu'$  the deviation from the  $1/r$  dependence is negligible, as the the correlation length is obtained from the slope of a fit of

$$\ln(g(r)) = c_0 - \frac{r}{\xi(T)} - (d - 2 + \eta) \ln(r)$$

versus  $r$ . Here the last term enters only logarithmically in  $r$ .

### 1.3.2 Scaling laws

**Generalized homogeneous functions**  $f(x, y, \dots)$  in several variables have the following property for arbitrary real  $\lambda$

$$f(\lambda^{a_x} x, \lambda^{a_y} y, \dots) = \lambda f(x, y, \dots) , \quad (1.76)$$

where  $a_x, a_y, \dots$  can be any real numbers. An example would be

$$f(x, y) = 4x^3 + 7y^8 ,$$

with  $a_x = 1/3$  and  $a_y = 1/8$ , while

$$f(x, y) = x + 6x^2 + xy + y^5$$

is no generalized homogeneous function. Next we will define the [scaling hypothesis](#) in the case of the free energy  $F(T, B)$  of a magnetic system. We are only interested in the non-analytic parts of  $F$  near  $T_C$  which we denote by  $F(\varepsilon, B)$ .

SCALING HYPOTHESIS	
$F(\lambda^{a_\varepsilon} \varepsilon, \lambda^{a_B} B) = \lambda F(\varepsilon, B) .$	(1.77)

This is not yet strictly proven in general, but there are many important cases where the scaling hypothesis is fulfilled.

Now we can use the scaling hypothesis to express the critical exponents in terms of the scaling parameters  $a_\varepsilon$  and  $a_B$ . Differentiation w.r.t.  $B$  yields

$$\left. \frac{\partial}{\partial b} F(\lambda^{a_\varepsilon} \varepsilon, b) \right|_{b=\lambda^{a_B} B} \lambda^{a_B} \stackrel{!}{=} \lambda \frac{\partial}{\partial B} F(\varepsilon, B) .$$

Now we have

$$M = -\frac{\partial F}{\partial B}$$

and therefore we have

$$\lambda^{a_B} M(\lambda^{a_\varepsilon} \varepsilon, \lambda^{a_B} B) = \lambda M(\varepsilon, B) . \quad (1.78)$$

a) [Exponent  \$\beta\$](#) : We use  $B = 0$  and obtain

$$\lambda^{a_B} M(\lambda^{a_\varepsilon} \varepsilon, 0) = \lambda M(\varepsilon, 0) ,$$

which is valid for any  $\lambda$ , also for

$$\lambda = (-\varepsilon)^{-1/a_\varepsilon} .$$

This value for  $\lambda$  was chosen in such a way, that, if plugged in, the  $\varepsilon$ -dependency of  $M(\lambda^{a_\varepsilon} \varepsilon, 0)$  vanishes and we are only left with a constant expression  $M(O(1), 0)$ . Only then, the  $\varepsilon$ -dependency of the entire expression lies only in the factor  $\varepsilon^\beta$ , which is the desired form for the critical exponent  $\beta$ . We obtain then:

$$\begin{aligned} \lambda^{a_\varepsilon} &= (-\varepsilon)^{-1} = -\frac{1}{\varepsilon} , \\ \lambda^{a_B} &= -(\varepsilon)^{-\frac{a_B}{a_\varepsilon}} , \\ \lambda^{a_B-1} &= -(\varepsilon)^{-\frac{a_B-1}{a_\varepsilon}} \end{aligned}$$

Plugging this into the above equation yields

$$M(\varepsilon, 0) = \lambda^{a_B-1} M(\lambda^{a_\varepsilon} \varepsilon, 0) = (-\varepsilon)^{\frac{1-a_B}{a_\varepsilon}} M(-1, 0) \simeq (\varepsilon)^{\frac{1-a_B}{a_\varepsilon}} .$$

Hence we obtain the following result for the order parameter:

$$\beta = \frac{1-a_B}{a_\varepsilon} .$$

b) **Exponent  $\delta$  (critical isotherm):** Next we use  $\varepsilon = 0$  and have

$$M(0, B) = \lambda^{a_B-1} M(0, \lambda^{a_B} B) .$$

Here we choose  $\lambda = B^{-1/a_B}$  for similar reasons as above and obtain

$$\begin{aligned} M(0, B) &\simeq B^{\frac{1-a_B}{a_B}} \\ B &\simeq M^{\frac{a_B}{1-a_B}} , \end{aligned}$$

and hence  $\delta = \frac{a_B}{1-a_B}$ . In summary, we have

$$a_B = \frac{\delta}{1+\delta} \tag{1.79}$$

$$a_\varepsilon = \frac{1}{\beta} \frac{1}{1+\delta} \tag{1.80}$$

$$\tag{1.81}$$

We can express further critical exponents in terms of the scaling parameters, which then in turn allows us to find relations among the critical exponents: From the scaling laws one finds

$$\alpha = \alpha' ; \quad \gamma = \gamma' ; \quad \nu = \nu' . \tag{1.82}$$

and also

$$\alpha + 2\beta + \gamma = 2 \tag{1.83}$$

$$\alpha + \beta(1+\delta) = 2 \tag{1.84}$$

$$\beta = \frac{\gamma}{\delta-1} \tag{1.85}$$

$$\nu = \frac{\gamma}{2-\eta} . \tag{1.86}$$

# Chapter 2

## Ising Model

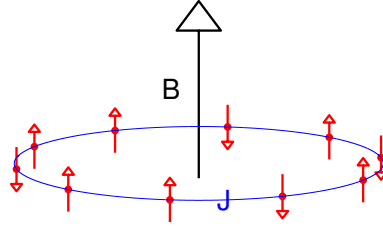


Figure 2.1: Arrangement of Ising-spins in a 1d chain with pbc.

We will now examine collective magnetism. The simplest model for this purpose is the Ising model, which is described by the following Hamilton function

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - B \hat{M} \quad (2.1)$$

$$\text{with} \quad \hat{M} = \mu \sum_i S_i, \quad \left( \mu := -\frac{\mu_B g_e}{2} \right) \quad (2.2)$$

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - \mu B \sum_i S_i, \quad (2.3)$$

where the spins  $S_i$  can only take the values  $S_i = \pm 1$ .  $\mu_B$  is the Bohr magneton and  $g_e$  the electronic Landé factor.  $J$  is the exchange coupling and the sum over sites is restricted to nearest neighbour sites, **such that each neighbouring pair is only counted once!**  $B$  stands for the magnetic flux density. The Ising model is also used to describe binary alloys. However, the parameters then have a different meaning. The Ising model can be solved exactly in 1d and 2d and in 2d it even has a phase transition. We consider periodic boundary

conditions <sup>1</sup>

## 2.1 Exact solution in 1D

The Hamilton function of the Ising model in 1D reads

$$H = -J \sum_{i=1}^N S_i S_{i+1} - \mu B \frac{1}{2} \sum_i (S_i + S_{i+1})$$

(pbc):  $S_{i+N} = S_i$  .

### Canonical partition function

The evaluation is particularly simply in the canonical ensemble:

$$\begin{aligned} Z(T, N, B) &= \sum_{\{S_i\}=\pm 1} e^{\beta J \sum_i S_i S_{i+1} + \frac{\mu \beta B}{2} \sum_i (S_i + S_{i+1})} \\ &= \sum_{\{S_i\}=\pm 1} \prod_{i=1}^N e^{j S_i S_{i+1} + \frac{b}{2} (S_i + S_{i+1})} . \end{aligned}$$

We introduced the abbreviations  $j = \beta J$  for the exchange coupling and  $b = \mu B \beta$  for the magnetic flux density. We also define the **transfer matrix**

$$\mathcal{T}_{s,s'} := e^{j s s' + \frac{b}{2} (s + s')} \quad (2.4a)$$

which has the matrix elements

$$\mathcal{T}_{s,s'} = \begin{array}{c|cc} & +1 & -1 \\ \hline +1 & e^{j+b} & e^{-j} \\ -1 & e^{-j} & e^{j-b} \end{array} \quad (2.4b)$$

then:

$$Z(T, N, B) = \sum_{\{S_i\}=\pm 1} \prod_{i=1}^N \mathcal{T}_{S_i, S_{i+1}} . \quad (2.4c)$$

We first consider the case  $N = 2$ , where the partition function reads

$$Z(T, N = 2, B) = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \mathcal{T}_{s_1, s_2} \mathcal{T}_{s_2, s_1} = \sum_{s_1} (\mathcal{T}^2)_{s_1, s_1} = \text{tr}\{\mathcal{T}^2\}$$

The generalization to  $N > 2$  leads obviously to

$$Z(T, N, B) = \text{tr}\{\mathcal{T}^N\} .$$

---

<sup>1</sup>In most cases, boundary conditions are irrelevant in the thermodynamic limits.

The transfer matrix is real-symmetric and can be expressed in the spectral representation as follows

$$\mathcal{T} = UDU^\dagger ,$$

here  $U$  is the unitary matrix of eigenvectors and  $D$  the diagonal matrix of eigenvalues  $(d_1, d_2)$ . Hence we have

$$\begin{aligned} Z(T, N, B) &= \text{tr}\{\mathcal{T}^N\} = \text{tr}\{(UDU^\dagger)^N\} = \text{tr}\{UD^N U^\dagger\} = \text{tr}\{D^N\} \\ &= d_1^N + d_2^N . \end{aligned}$$

The eigenvalues of the transfer matrix are

$$\begin{aligned} d_{1/2} &= \frac{e^{j+b} + e^{j-b}}{2} \pm \sqrt{\left(\frac{e^{j+b} - e^{j-b}}{2}\right)^2 + e^{-2j}} \\ &= e^j \left( \cosh(b) \pm \sqrt{\sinh^2(b) + e^{-4j}} \right) \end{aligned}$$

We can use  $d_1 > d_2$  in the calculation of the partition function in thermodynamic limits

$$Z(T, N, B) = d_1^N \left[ 1 + \left( \frac{d_2}{d_1} \right)^N \right] .$$

### Free energy

The free energy reads

$$\begin{aligned} F(T, N, B) &= -k_B T \ln [Z(T, B)] = -k_B T N \ln(d_1) - k_B T \ln \left[ 1 + \underbrace{\left( \frac{d_2}{d_1} \right)^N}_{\rightarrow 0} \right] \\ &= -N k_B T \ln(d_1) . \end{aligned}$$

### Magnetization

The mean magnetization is

$$M = \mu \left\langle \sum_i S_i \right\rangle .$$

The comparison with the partition function immediately shows

$$\begin{aligned}
M &= \frac{1}{\beta} \left( \frac{\partial \ln(Z)}{\partial B} \right) \Big|_{T,N} = - \left( \frac{\partial F}{\partial B} \right) \Big|_{T,N} \\
&= -\mu\beta \left( \frac{\partial F}{\partial b} \right) \Big|_{T,N} \\
&= -\mu\beta(-k_B T N) \left( \frac{\partial \ln(d_1)}{\partial b} \right) \Big|_{T,N} \\
&= N\mu \frac{\frac{\partial d_1}{\partial b}}{d_1} = N\mu \frac{\sinh(b) + \frac{\sinh(b) \cosh(b)}{\sqrt{\sinh^2(b) + e^{-4j}}}}{\cosh(b) + \sqrt{\sinh^2(b) + e^{-4j}}} \\
&= N\mu \sinh(b) \frac{1 + \frac{\cosh(b)}{\sqrt{\sinh^2(b) + e^{-4j}}}}{\cosh(b) + \sqrt{\sinh^2(b) + e^{-4j}}} \\
&= N\mu \frac{\sinh(b)}{\sqrt{\sinh^2(b) + e^{-4j}}} \frac{\sqrt{\sinh^2(b) + e^{-4j}} + \cosh(b)}{\cosh(b) + \sqrt{\sinh^2(b) + e^{-4j}}} .
\end{aligned}$$

MAGNETIZATION OF THE 1D ISING MODEL
-------------------------------------

$M(T, N, B) = N\mu \frac{\sinh(\mu\beta B)}{\sqrt{\sinh^2(\mu\beta B) + e^{-4J\beta}}} .$
---

### 2.1.1 Paramagnet

Without interaction of the magnetic moments ( $J = 0$ ) we obtain

$$M(T, N, B) = N\mu \frac{\sinh(\mu\beta B)}{\sqrt{\sinh^2(\mu\beta B) + 1}} = N\mu \tanh(\mu\beta B) ,$$

the well-known result for paramagnetism.

### 2.1.2 Limits

It depends on the order of the limits. If we first set  $B = 0$ , we obtain for any finite temperature

$$M(T, N, B = 0) = 0$$

a vanishing magnetization, since  $\sinh(\mu\beta B)$  always yields 0 independent of  $\beta$ . In the opposite order, i.e. keeping  $B > 0$  finite and let  $T$  go to zero, i.e.  $\beta \rightarrow \infty$  we obtain

$$\begin{aligned} M(T, B \neq 0) &= N\mu \frac{\sinh(\mu\beta B)}{\sqrt{\underbrace{\sinh^2(\mu\beta B)}_{\gg 1} + \underbrace{e^{-4J\beta}}_{\ll 1}}} \\ &\xrightarrow{T \rightarrow 0} N\mu \frac{\sinh(\mu\beta B)}{\pm \sinh(\mu\beta B)} \\ &= N\mu \operatorname{sign}(B) . \end{aligned}$$

In this limiting case we get perfect alignment of all spins, even if we now let  $B$  go to zero.

The Ising model in one dimension has a "phase transition" at  $T = 0$ .



### 2.1.3 Magnetization

We use the exchange coupling  $J$  as energy unit. Then two independent parameters remain,  $k_B T$  and  $\tilde{B} := \mu B$ . We plot magnetization as a function of  $\tilde{B}$

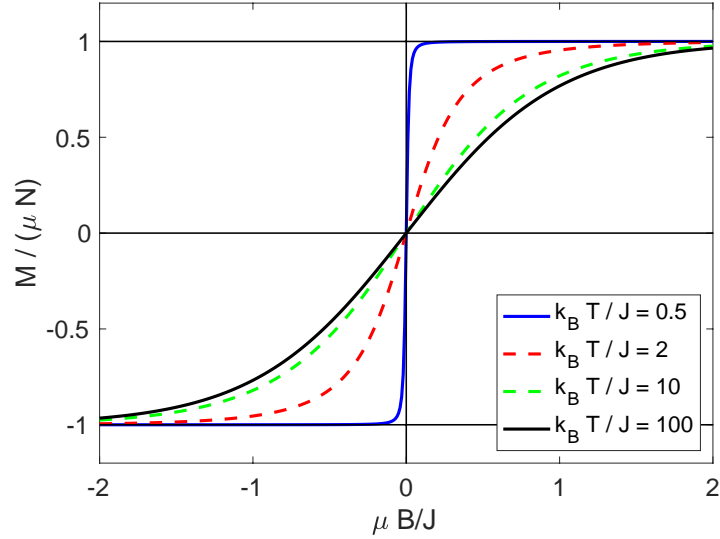


Figure 2.2: Magnetization curve of the 1d Ising model.

We can see from the comparison with the result for the paramagnet that the amount of magnetization increases everywhere due to the influence of the interaction. For low temperatures, the magnetization abruptly enters the fully polarized state.

## 2.2 Mean field approximation for any dimension

(See *Ising Model for Magnetism (Springer)*)

We will study the concept of the mean-field approximation in terms of the Ising model. We start out from the Ising model expressed as

$$H = -J \sum_{\langle ij \rangle} S_i S_j - B \cdot \hat{M}$$

We recall that the Ising model describes spin-1/2 objects. The total magnetisation  $\mathbf{M}$  is related to the spins via

$$\hat{M} = \mu \sum_i S_i .$$

Now the Ising model takes only the component along the quantization axis into account and  $S_i^z$  takes the values  $\frac{\hbar}{2}\sigma_i$ , with  $\sigma_i \in \{-1, +1\}$ . As before, we introduce the variable

$$h = \mu B .$$

and obtain

$$H = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i$$

Which is precisely the Hamiltonian of the Ising model. The mean field approximation is defined for any two dynamical variables  $A$  and  $B$  by

$$AB \approx \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle .$$

In our case, assuming translational invariance, we have

$$S_i S_j \approx \langle S \rangle S_j + S_i \langle S \rangle - \langle S \rangle^2 .$$

This leads to the mean-field Hamilton function

$$\begin{aligned} H_{\text{MFA}} &= -J \sum_{\langle ij \rangle} \langle S \rangle S_i - J \sum_{\langle ij \rangle} \langle S \rangle S_j + J \sum_{\langle ij \rangle} \langle S \rangle^2 - h \sum_i S_i \\ &= -2J \langle S \rangle \sum_{\langle ij \rangle} S_i + J \sum_{\langle ij \rangle} \langle S \rangle^2 - h \sum_i S_i \end{aligned}$$

Since we have used the convention that the sum over nearest neighbours avoids double counting, each site  $i$  has  $z/2$  nearest neighbours, where for simple cubic lattices  $z = 2d$  is defined as the total number of nearest neighbours. Hence:

$$H_{\text{MFA}} = -Jz \langle S \rangle \sum_i S_i + \frac{JNz}{2} \langle S \rangle^2 - h \sum_i S_i .$$

This can also be written as

$$H_{\text{MFA}} = -h' \sum_i S_i + \frac{JNz}{2} \langle S \rangle^2 \quad (2.5)$$

$$\text{with} \quad h' = h + Jz \langle S \rangle . \quad (2.6)$$

The canonical partition function is then given by

$$Z(T, N, B) = e^{-\frac{\beta N J z}{2} \langle S \rangle^2} \sum_{\{S_i\}} e^{\beta h' \sum_i S_i} , \quad (2.7)$$

The partition function can easily be computed

$$\begin{aligned} Z(T, N, B) &= e^{-\frac{\beta N J z}{2} \langle S \rangle^2} \prod_i \sum_{S_i = \pm 1} e^{h' S_i} \\ &= \left( e^{-\frac{\beta J z}{2} \langle S \rangle^2} 2 \cosh(\beta h') \right)^N \end{aligned} \quad (2.8)$$

$$\ln(Z(T, N, B)) = N \left( -\frac{\beta J z}{2} \langle S \rangle^2 + \ln(2) + \ln(\cosh(\beta h')) \right) \quad (2.9)$$

### 2.2.1 Magnetization

Next we compute the mean total spin  $\mathcal{S} := \sum_i S$

$$\langle \mathcal{S} \rangle = \frac{1}{Z} \sum_{\{S_i\}} \left\{ \mathcal{S} e^{-\frac{\beta N J z}{2} \langle S \rangle^2} e^{\beta h' \mathcal{S}} \right\} = \frac{k_B T}{Z} \frac{\partial}{\partial h'} Z .$$

Hence

$$\langle \mathcal{S} \rangle = k_B T \frac{\partial \ln(Z)}{\partial h'} = N \tanh(\beta h') . \quad (2.10)$$

We introduce as  $m$  the magnetization per site, divided by  $\mu$

$$m = \frac{M}{\mu N} = \frac{1}{N} \langle \sum_i S_i \rangle = \langle S \rangle . \quad (2.11)$$

and obtain the equation of state by using  $h' = h + Jz\langle S \rangle$  and dividing by  $N$ :

MAGNETIZATION <i>(of the Ising model)</i>
$m = \tanh \left( \beta (Jz m + h) \right) . \quad (2.12)$

We are particularly interested in the spontaneous magnetization, i.e. the magnetization for  $B \rightarrow 0 \Rightarrow h \rightarrow 0$ . So we seek the solution of

$$m = \tanh(\beta J z m) . \quad (2.13)$$

We introduce an auxiliary temperature  $T^*$ , defined by

$$k_B T^* = J z . \quad (2.14)$$

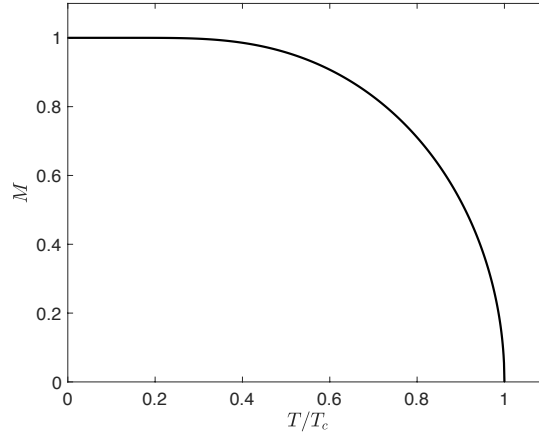


Figure 2.3: *Mean-field result for the Magnetization for arbitrary spatial dimension  $d$*

The condition for the spontaneous magnetization can therefore be expressed in a form independent of the dimension

$$m = \tanh\left(\frac{T^*}{T}m\right) \quad (2.15)$$

or rather with  $x = \frac{T^*}{T}m$

$$x = \frac{T^*}{T} \tanh(x) . \quad (2.16)$$

The function  $\tanh(x)$  starts at  $x = 0$  with the value 0 and slope 1, then it increases monotonically with decreasing slope and approaches 1 for  $x \rightarrow \infty$ . equation (2.16) has always the trivial solution  $x = 0 \Leftrightarrow M = 0$ . For the non-trivial solution, we need  $\frac{T^*}{T} > 1$ , or rather  $T < T^*$ . According to the behaviour of  $\tanh(x)$ , for  $\frac{T^*}{T} > 1$  there is precisely one non-trivial solution. So we see that  $T^*$  is indeed the critical/ Curie temperature, i.e

$$k_B T_C = Jz . \quad (2.17)$$

In 2D, we have  $(\beta_c J) = .25$ , as compared to the exact value  $\beta_c J = 0.4407$ . This not so bad, however, for 1D the mean-field result predicts a phase-transition at  $(\beta_c J) = .5$ , while the exact result is  $\infty$ .

In figure 2.3, the magnetization, obtained from equation (2.13) [\[previous page\]](#), is depicted as function of  $T$ . Interestingly, the curve is independent of the spatial dimension. The latter only enters in the value of  $T_c$

### Critical exponent

For temperatures  $T$  slightly below  $T_c$ , the magnetization is very small and we can use the Taylor expansion with  $\tau = T/T_C$

$$\begin{aligned} m &= \tanh\left(\frac{m}{\tau}\right) = \left(\frac{m}{\tau} - \frac{(m/\tau)^3}{3}\right) = \frac{m}{\tau} \left(1 - \frac{(m/\tau)^2}{3}\right) \\ \tau &= 1 - \frac{(m/\tau)^2}{3} \\ \left(\frac{m}{\tau}\right)^2 &= 3(1 - \tau). \end{aligned}$$

Up to this order we have for  $T < T_C$

$$m = 3\tau\sqrt{1 - \tau} \propto \varepsilon^{\frac{1}{2}} \quad (2.18)$$

$$\varepsilon := \frac{T_C - T}{T_C} \quad (2.19)$$

The critical exponent is therefore  $\beta = 1/2$ .

### 2.2.2 Magnetic susceptibility

Starting from the definition of the susceptibility via

$$\chi = \left. \frac{\partial M}{\partial B} \right|_{T, h=0}$$

we obtain with  $M = mN\mu$  and  $B = h/\mu$  the relation

$$\chi = \left[ \frac{\partial M}{\partial m} \frac{\partial h}{\partial B} \frac{\partial m}{\partial h} \right]_{T, h=0} = \mu^2 N \left. \frac{\partial m}{\partial h} \right|_{T, h=0}$$

We start out from

$$m = \tanh\left(\beta h + \frac{T_C}{T} m\right)$$

The derivative w.r.t.  $h$  is

$$\begin{aligned} \xi &= \left. \frac{\partial m}{\partial h} \right|_{T, h=0} = \frac{\beta + \frac{T_C}{T} \xi}{\cosh^2\left(\beta h + \frac{T_C}{T} m\right)} \Big|_{h=0} = \frac{\beta + \frac{T_C}{T} \xi}{\cosh^2\left(\frac{T_C}{T} m\right)} \\ \xi \cosh^2\left(\frac{T_C}{T} m\right) &= \beta + \frac{T_C}{T} \xi \\ \xi \left( \cosh^2\left(\frac{T_C}{T} m\right) - \frac{T_C}{T} \right) &= \beta. \end{aligned}$$

Close to  $T_C$  the magnetisation is very small and we can Taylor expand the magnetisation resulting in

$$\begin{aligned}\xi &= \frac{\beta}{\cosh^2\left(\frac{T_C}{T}m\right) - \frac{T_C}{T}} \\ &= \frac{\beta}{1 - \frac{T_C}{T} + \left(\frac{T_C}{T}m\right)^2} \\ &= \frac{1}{k_B} \frac{1}{T - T_C + T_C \frac{T_C}{T} m^2} .\end{aligned}$$

We can replace  $T$  by  $T_C$  in the last term in the denominator, as it already contains  $O(m^2)$

$$\xi = \frac{1}{k_B} \frac{1}{T - T_C + T_C m^2} .$$

**For**  $T \searrow T_C$  the magnetisation is zero and we have

$$\xi = \frac{1}{k_B T_C} (\varepsilon)^{-1}$$

with  $\varepsilon = (T - T_C)/T_C$ .

**For**  $T \nearrow T_C$  we use equation (2.18) [\[previous page\]](#) for  $m$  and obtain

$$m^2 = 3\tau^2(1 - \tau) = \frac{3}{T_C}(T_C - T) + \mathcal{O}(\Delta T/T_C)^2$$

and have

$$\xi = \frac{1}{k_B} \frac{1}{T - T_C + 3(T_C - T)} = \frac{1}{2k_B T_C} |\varepsilon|^{-1}$$

According to the definition of the critical exponent of the susceptibility

$$\chi \simeq A_{\pm} |\varepsilon|^{-\gamma_{\pm}}$$

we have  $A_+ = 2A_-$  and  $\gamma_+ = \gamma_- = 1$ .

### 2.2.3 Free energy

Next we compute the free energy based equation (2.9) [\[p. 42\]](#)

$$\begin{aligned}f &:= \frac{F}{k_B N} = -\frac{k_B T}{k_B N} \ln(Z) \\ &= -T \left( -\frac{\beta J z}{2} \langle S \rangle^2 + \ln(2) + \ln(\cosh(\beta h')) \right) \\ &= \frac{J z}{2 k_B} m^2 - T \ln(2) - T \ln\left(\cosh\left(\beta h + \frac{J z}{k_B T} m\right)\right) .\end{aligned}$$

The natural variables are  $T$ ,  $N$ , and  $h$ . However, we will see that the magnetization  $m$  is more suitable than  $h$  and we will introduce a corresponding Legendre transformation later.

We have seen before that  $k_B T_C = Jz$  so we can express the free energy as

$$\frac{F}{Nk_B} = \frac{1}{2}T_c m^2 - T \ln(2) - T \ln \cosh\left(\beta h + \frac{T_c}{T}m\right).$$

Along with

$$\cosh(x) = (1 - \tanh^2(x))^{-1/2}$$

and the self consistency equation (2.15) [p. 43]

$$m = \tanh\left(\beta h + \frac{T_c}{T}m\right) \quad (2.20)$$

we can express the free energy as

$$\frac{F}{Nk_B} = \frac{T_c m^2}{2} - T \ln(2) + \frac{T}{2} \ln(1 - m^2). \quad (2.21)$$

Next we compute the Helmholtz free energy, by the following Legendre transform, where we introducing  $M$  as natural variable instead of  $h$ :

$$A(T, M) = F(T, h) + Mh. \quad (2.22)$$

Then the total differential reads

$$\begin{aligned} dA &= dF(T, h) + h dM + M dh \\ &= -S dT - M dh + h dM + M dh \\ &= -S dT + h dM. \end{aligned}$$

exercise: proof  $\frac{\partial F}{\partial h} = -M$  by using equation (2.21) and equation (2.20).  
Hence

$$\left. \frac{\partial A}{\partial M} \right|_T = h \quad (2.23)$$

$$\left. \frac{\partial A}{\partial T} \right|_M = -S. \quad (2.24)$$

If we want to have spontaneous magnetisation, i.e. a finite magnetisation  $M$  without external field, then according to equation (2.23) we are looking for a finite value of the magnetization  $M$  for which

$$\left. \frac{\partial A(M, T)}{\partial M} \right|_T = 0. \quad (2.25)$$

Before we can exploit this equation, we need to express  $h$  in terms of  $M$  (or rather  $m$ ). To this end we invert

$$m = \tanh \left( \beta h + \frac{T_C}{T} m \right)$$

leading to

$$\beta h + \frac{T_C}{T} m = \tanh^{-1}(m) .$$

Along with

$$\tanh^{-1}(b) = \frac{1}{2} \ln \left( \frac{1+b}{1-b} \right)$$

we obtain

$$\begin{aligned} \beta h + \frac{T_c}{T} m &= \frac{1}{2} \ln \left( \frac{1+m}{1-m} \right) \\ h &= \frac{k_B T}{2} \ln \left( \frac{1+m}{1-m} \right) - k_B T_c m \end{aligned}$$

Then we obtain for the Helmholtz free energy of equation (2.21) [\[previous page\]](#)

$$\begin{aligned} \frac{A}{Nk_B} &= \frac{T_C m^2}{2} - T \ln(2) + \frac{T}{2} \ln(1-m^2) + \frac{mh}{k_B} \\ &= \frac{1}{2} T_c m^2 - T \ln(2) + \frac{T}{2} \ln(1-m^2) + \frac{m}{k_B} \left( \frac{k_B T}{2} \ln \left( \frac{1+m}{1-m} \right) - k_B T_c m \right) \\ &= \frac{1}{2} T_c m^2 - T \ln(2) + \frac{T}{2} \ln(1-m^2) + \frac{mT}{2} \ln \left( \frac{1+m}{1-m} \right) - T_c m^2 . \end{aligned}$$

or rather

$$\begin{aligned} \frac{A}{Nk_B T_c} &= \frac{A}{NJz} = -\frac{1}{2} m^2 - \tau \ln(2) + \frac{\tau}{2} \left[ \ln((1-m)(1+m)) + m \ln \left( \frac{1+m}{1-m} \right) \right] \\ &= -\frac{1}{2} m^2 - \tau \ln(2) + \frac{\tau}{2} \left( (1+m) \ln(1+m) + (1-m) \ln(1-m) \right) , \end{aligned} \tag{2.26}$$

with  $\tau = T/T_c$ . According to equation (2.25) [\[previous page\]](#) the magnetization is given by the points where the slope as function of  $m$  vanishes (see figure 2.4).



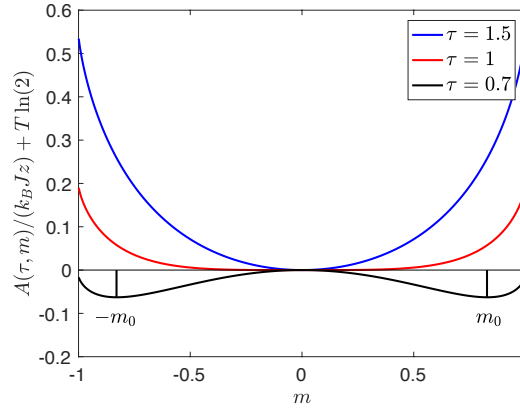


Figure 2.4: *Helmholtz free energy for the Ising model in mean-field approximation versus order parameter without external field.*

## 2.2.4 Entropy

According to equation (2.24) [p. 46] , which was

$$S = -\left. \frac{\partial A}{\partial T} \right|_M$$

along with equation (2.26) [previous page] the entropy is

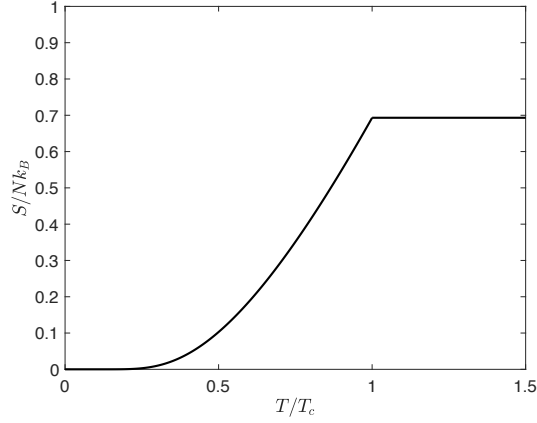
$$-\frac{S}{Nk_B T_C} = \frac{\partial}{\partial \tau} \left( -\frac{1}{2} m^2 - \tau \ln(2) + \frac{\tau}{2} \left( (1+m) \ln(1+m) + (1-m) \ln(1-m) \right) \right) \bigg|_m \frac{d\tau}{dT} .$$

Hence

<p>ENTROPY OF THE ISING MODEL (in MFA for zero field)</p> <hr style="border: 0.5px solid black;"/> $\frac{S}{Nk_B} = \ln(2) - \frac{1}{2} \left( (1+m) \ln(1+m) + (1-m) \ln(1-m) \right) . \quad (2.27)$
--

It has the correct limiting behaviour: For  $T \rightarrow 0$ , i.e.  $m \rightarrow 1$  we obtain

$$\frac{S}{Nk_B} = \ln(2) - \frac{1}{2} \left( 2 \left( \ln(2) + 0 \right) \right) = 0 ,$$

Figure 2.5: *Entropy of the Ising model in MFA.*

and for  $T \rightarrow \infty$ , i.e.  $m \rightarrow 0$  the entropy becomes

$$\frac{S}{Nk_B} = \ln(2) - \frac{1}{2} \left( \ln(1) + \ln(1) \right) = \ln(2) .$$

Recall that

$$S = k_B \ln(\text{number of micro states}) .$$

For  $T \rightarrow \infty$  all states can be reached with the same probability. Hence the number of micro states is  $2^N$  and

$$S = Nk_B \ln(2) ,$$

in agreement with the above result.

For the entire  $T$  dependence of  $S$  we have to insert the self-consistent solution for  $m(T)$  in equation (2.15) [p. 43].

### 2.2.5 Internal Energy

The internal energy is defined as the expectation value of the hamiltonian. Using the mean field expression of equation (2.5) [p. 41] we have

$$U = N \left( -h'm + \frac{Jz}{2} m^2 \right)$$

with  $h' = Jzm + h$  we obtain:

$$\begin{aligned} \frac{U}{NJ} &= -\frac{hm}{J} - zm^2 + \frac{z}{2} m^2 \\ &= -\frac{z}{2} m^2 - \frac{hm}{J} . \end{aligned}$$

For zero external field it simplifies to (we also use  $z = 2d$ )

<p>INTERNAL ENERGY OF THE ISING MODEL (in MFA for zero field)</p> <hr style="border: 0.5px solid black; margin: 10px 0;"/> $\frac{U}{NJ} = -d m^2 . \quad (2.28)$
---

### 2.2.6 Specific heat

For the specific heat we need

$$\begin{aligned} C_{h=0} &= \left. \frac{\partial U}{\partial T} \right|_{h=0} = -dNJ \frac{d m^2}{dT} \\ \frac{C_{h=0}}{NJ} &= -d \frac{d}{dT} m^2 . \end{aligned}$$

Above  $T_C$  the magnetization is zero and hence  $C = 0$ . Slightly below  $T_C$  we can replace  $m$  by equation (2.18) [p. 44], which gives

$$\begin{aligned} \frac{C_{h=0}}{NJ} &= -d \frac{d}{dT} 3\tau^2 (1 - \tau) \\ &= -\frac{3d}{T_C^3} \frac{d}{dT} (T^2 T_C - T^3) \\ &= -\frac{3d}{T_C^3} T (2T_C - 3T) . \end{aligned}$$

Hence, approaching  $T_C$  from below we obtain

$$\frac{C(T_C)_{h=0}}{NJ} = \frac{3d}{T_C} = \frac{3dk_B}{Jz} = \frac{3dk_B}{J2d} \quad (2.29)$$

$$\frac{C(T_C)_{h=0}}{N} = \frac{3k_B}{2} . \quad (2.30)$$

Therefore, in MFA, the specific heat has no power-law behaviour close at  $T_C$ , but rather a discontinuity from  $\frac{3}{2}k_B$  below  $T_C$  to 0 above  $T_C$ . For the entire  $T$ -dependence below  $T_C$  we continue with

$$\frac{C_{h=0}}{NJ} = -2d m \frac{dm}{dT} . \quad (2.31)$$

First we consider

$$\xi := \frac{dm}{dT}.$$

Exploiting equation (2.20) [p. 46] yields

$$\xi = \frac{d}{dT} \tanh(\beta h + \beta k_B T_C m) \Big|_{h=0} \quad (2.32)$$

$$\xi = \frac{(h + k_B T_C m) \frac{d\beta}{dT} + \beta k_B T_C \xi}{\cosh^2(\beta(h + k_B T_C m))} \Big|_{h=0} \quad (2.33)$$

$$= \frac{-\frac{T_c}{T^2} m + \frac{T_C}{T} \xi}{\cosh^2\left(\frac{T_C}{T} m\right)}. \quad (2.34)$$

Then

$$\begin{aligned} \xi \cosh^2\left(\frac{T_C}{T} m\right) &= -\frac{T_c}{T^2} m + \frac{T_C}{T} \xi \\ \xi \left( \cosh^2\left(\frac{T_C}{T} m\right) - \frac{T_C}{T} \right) &= -\frac{T_C}{T^2} m \\ \xi &= -\frac{1}{T_C} \frac{\frac{m}{\tau}}{\tau \cosh^2\left(\frac{m}{\tau}\right) - 1} \end{aligned}$$

That leads to

$$\begin{aligned} \frac{C_{h=0}}{NJ} &= -z m \xi \\ &= \frac{z}{T_C} \frac{\frac{m^2}{\tau}}{\tau \cosh^2\left(\frac{m}{\tau}\right) - 1} \\ &= \frac{z}{zJ/k_B} \frac{\frac{m^2}{\tau}}{\tau \cosh^2\left(\frac{m}{\tau}\right) - 1}. \end{aligned}$$

Finally, we have

$$\frac{C_{h=0}}{N} = k_B \frac{\left(\frac{m}{\tau}\right)^2}{\cosh^2\left(\frac{m}{\tau}\right) - 1/\tau}. \quad (2.35)$$

Together with the self consistent equation for  $m(T)$ :

$$m(T) = \tanh\left(\frac{T_C}{T} m(T)\right),$$

which have solved numerically before, we can plot the specific heat.

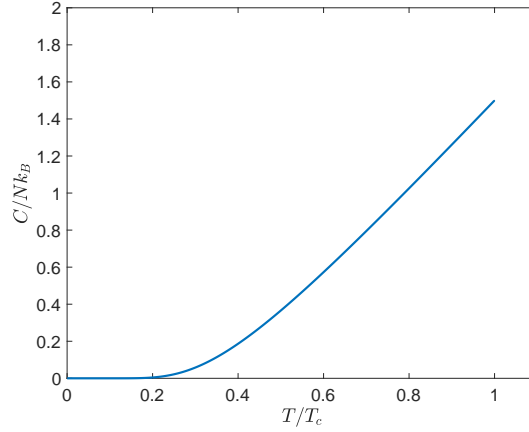


Figure 2.6: *Specific heat of the Ising model in MFA.*

We have seen in statistical physics that the heat capacity is connected to  $\langle(\Delta H)^2\rangle$  in the following

$$C = \frac{1}{k_B T^2} \left\langle (\Delta H)^2 \right\rangle ,$$

i.e.  $C$  is large when there are pronounced energy fluctuations, which is the case in the vicinity of  $T_C$ . In MFA, above  $T_C$  there are no fluctuations because  $H \propto m$  and therefore  $U = 0$  above  $T_C$ . This is an artefact of MFA.

## 2.3 Exact solution of the 2d Ising model

So far we have derived the exact solution of the Ising model in 1d and the mean field approximation as a way to make statements about the Ising model in arbitrary dimensions. However, it is also possible to obtain an exact result in the 2-dimensional case. There are different methods to approach this and some of them will be presented on the next pages.

### 2.3.1 Transfermatrix approach

We will first briefly explain how the Transfer matrix approach works in the 2d case. To this end we will represent the hamiltonian of the 2d Ising model ( $L_x \times L_y$ ) by writing the two cartesian indices explicitly in the form  $S_{ij}$

$$-\beta H = j \sum_{l=1}^{L_y} \sum_{i=1}^{L_x} (S_{i,l} S_{i+1,l} + S_{i,l} S_{i,l+1}) + h \sum_l \sum_i S_{il} .$$

Now we combine the spins of column  $l$  (i.e.  $S_{il}$ ) in a vector

$$\mathcal{S}^{(l)} = \begin{pmatrix} S_{1,l} \\ S_{2,l} \\ \dots \\ S_{L_x,l} \end{pmatrix}, \text{ i.e. } (\mathcal{S}^{(l)})_i = S_{il}$$

Then the hamiltonian can be written as

$$-\beta H = \sum_l A(\mathcal{S}^{(l)}, \mathcal{S}^{(l+1)}) \quad (2.36)$$

with the definitions

$$A(\mathcal{S}^{(l)}, \mathcal{S}^{(k)}) = \frac{1}{2} \left( \tilde{A}(\mathcal{S}^{(l)}, \mathcal{S}^{(k)}) + \tilde{A}(\mathcal{S}^{(k)}, \mathcal{S}^{(l)}) \right) \quad (2.37)$$

$$\tilde{A}(\mathcal{S}^{(l)}, \mathcal{S}^{(k)}) = j \sum_i \left( (\mathcal{S}^{(l)})_i (\mathcal{S}^{(l)})_{i+1} + (\mathcal{S}^{(l)})_i (\mathcal{S}^{(k)})_i \right) + h \left( \sum_i \mathcal{S}^{(l)}_i \right). \quad (2.38)$$

$A$  is by construction a real symmetric matrix. Inserting indices yields

$$\begin{aligned} \sum_l \tilde{A}(\mathcal{S}^{(l)}, \mathcal{S}^{(l+1)}) &= j \sum_{l,i} (S_{i,l} S_{i+1,l} + S_{i,l} S_{i,l+1}) + h \sum_{i,l} S_{i,l} = -\beta H \\ \sum_l \tilde{A}(\mathcal{S}^{(l+1)}, \mathcal{S}^{(l)}) &= j \sum_{l,i} (S_{i,l+1} S_{i+1,l+1} + S_{i,l+1} S_{i,l}) + h \sum_{i,l} S_{i,l+1} \\ (l+1 \rightarrow l' \text{ plus pbc } \Rightarrow) &= j \sum_{l',i} (S_{i,l'} S_{i+1,l'} + S_{i,l'} S_{i,l+1}) + h \sum_{i,l} S_{i,l'} = -\beta H. \end{aligned}$$

Hence equation (2.36) is correct. Then

$$Z = \sum_{\{S_{ij}\}} e^{\sum_{l=1}^L A(\mathcal{S}^{(l)}, \mathcal{S}^{(l+1)})} = \prod_{l=1}^L \sum_{\mathcal{S}^{(l)}} e^{A(\mathcal{S}^{(l)}, \mathcal{S}^{(l+1)})}.$$

Now, we introduce the transfer matrix  $\mathcal{T}$  with matrix elements

$$\mathcal{T}_{\mathcal{S}^{(l)}, \mathcal{S}^{(l')}} = e^{A(\mathcal{S}^{(l)}, \mathcal{S}^{(l')})}$$

Then the partition function can be written as (remember that we use pbc)

$$\begin{aligned} Z &= \sum_{\mathcal{S}^{(1)}} \sum_{\mathcal{S}^{(2)}} \cdots \sum_{\mathcal{S}^{(L_y)}} \prod_l \mathcal{T}_{\mathcal{S}^{(l)}, \mathcal{S}^{(l+1)}} \\ &= \sum_{\mathcal{S}^{(1)}} \sum_{\mathcal{S}^{(2)}} \cdots \sum_{\mathcal{S}^{(L_y)}} \mathcal{T}_{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}} \mathcal{T}_{\mathcal{S}^{(2)}, \mathcal{S}^{(3)}} \cdots \mathcal{T}_{\mathcal{S}^{(L_y-1)}, \mathcal{S}^{(L_y)}} \mathcal{T}_{\mathcal{S}^{(L_y)}, \mathcal{S}^{(1)}} \\ &= \text{tr} \{ \mathcal{T}^{L_y} \} \\ &= \lambda_{\max}^{L_y}. \end{aligned}$$

This is the straight-forward generalization of equation (2.4) [p. 36] and is to be understood as follows:

As before, the dominant eigenvector ( $\lambda_{\max}$ ) predominates in the thermodynamic limit ( $L_y \rightarrow \infty$ ). The eigenvalue problem of the ( $2^{L_x} \times 2^{L_x}$ )-dimensional transfer matrix for the 2d case, is much more complicated than in the 1d case. It can be found in the book of K. Huang (*Kerson Huang, Statistical Mechanics, Wiley and Sons (1963)*).

### 2.3.2 Graphical approach (Bachelor thesis J. Pomper)

Here we will present instead the exact solution of the 2d Ising model based on a graphical representation, however, without external field. The ideas go back to M. Lawrence Glasser, American Journal of Physics 38, 1033 (1970), and were didactically improved by W. Nolting: Quantum Theory of Magnetism (Springer Verlag).

Starting point is the partition function which, without an external field, is

$$Z = \sum_{\{S_i\}} \prod_{\langle ij \rangle} e^{j S_i S_j} . \quad (2.39)$$

Next we expand the exponential in a Taylor series

$$\begin{aligned} e^{j S_i S_j} &= \sum_{n=0}^{\infty} \frac{j^n}{n!} (S_i S_j)^n \\ &= \sum_{n=0}^{\infty} \frac{j^{2n}}{(2n)!} \underbrace{(S_i S_j)^{2n}}_{=1} + \sum_{n=0}^{\infty} \frac{j^{2n+1}}{(2n+1)!} \underbrace{(S_i S_j)^{2n+1}}_{=S_i S_j} \\ &= \cosh(j) (1 + \tanh(j) S_i S_j) \end{aligned}$$

Inserted in equation (2.39) results in

$$Z = \cosh^{2N}(j) \sum_{\{S_i\}} \prod_{\langle ij \rangle} (1 + t S_i S_j) ,$$

where  $t = \tanh(j)$ . The power  $2N$  arises as there are  $2N$  terms in the product in equation (2.39). For each factor we have the choice to use the term 1 or  $t S_i S_j$ . In total there are  $2^{Nz}$  such terms, for which we still have to sum over all spin configurations. Graphically, we represent the terms  $t S_i S_j$  as lines on a square lattice connecting site  $i$  and  $j$ . Such a line can be considered as *edge of a graph* and the sites that are connected by edges are denoted as *vertices*. In graph theory the number of edges connected to a vertex is called *order of the vertex*. In the present context, it can only be an integer  $n \in \{0, 1, 2, 3, 4\}$ .

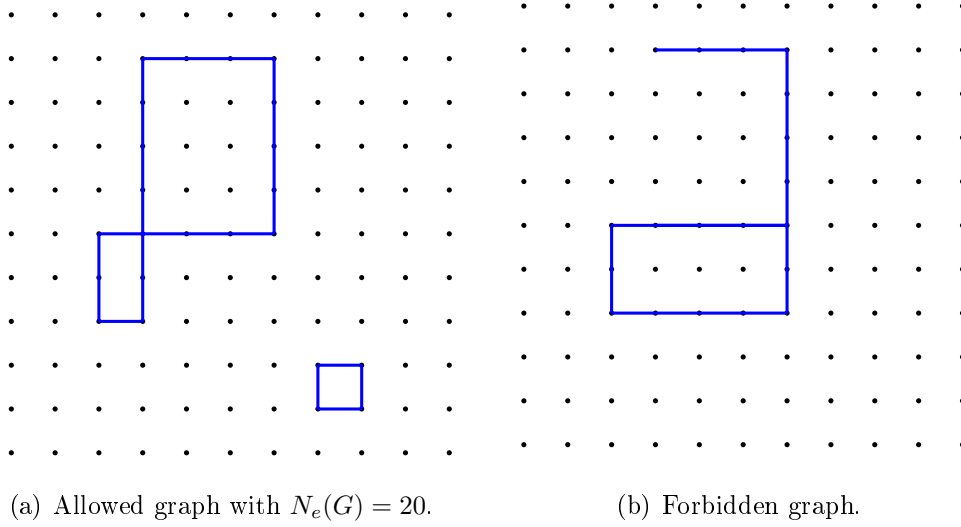


Figure 2.7: *Graphical representation of the partition function of the Ising model.*

Each edge, that reaches site (vertex)  $i$  carries a factor  $S_i$ . Therefore, we obtain a term  $S_i^n$ , where  $n$  is the order of the vertex.  $n \in \{0, 1, 2, 3, 4\}$ . If the order is odd, the sum over  $S_i$  vanishes, otherwise it gives 2. Hence, only graphs where all vertices have an even order (either 0, 2 or 4) are allowed. By now we have

$$Z = 2^N \cosh^{2N}(j) \sum_G t^{N_e(G)},$$

The sum runs over all possible graphs on a square lattice of given size, with periodic boundary conditions, with vertices of even order.  $N_e(G)$  is the total number of edges of the graph. The elements of the individual graphs need not to be connected. In figure 2.7 two examples are given, one for an allowed and one for a forbidden graph. We can rewrite the sum over allowed graphs also in the form

$$Z = 2^N \cosh^{2N}(j) \left( 1 + \sum_{n=4}^{\infty} g_n t^n \right), \quad (2.40)$$

where  $n$  is the number of edges and  $g_n$  is the number of allowed graphs with  $n$  edges. We have already exploited that the smallest graph (besides the empty graph) has 4 edges. Since each vertex has even order, the allowed graphs contain closed paths, as can be seen in figure 2.7.



### 2.3.3 Making the graphs unique

The figure also contains a *node*, i.e. a vertex of order 4. Let's try to draw the graph with a pencil on a piece of squared paper. We start at an arbitrary vertex of order 2 and draw a line to one of the two connected vertices. If the next vertex also has order 2, it is obvious how to continue the drawing. When we reach a node, however, we have 3 options to continue: *turn left*, *go straight*, *turn right*. It will turn out to be advantageous, to replace the graph by objects that allow to draw them in a unique way. Therefore, we will introduce graphical objects, that make a node unique. To this end a node is split into the three graphical objects, shown in figure 2.8. This splitting

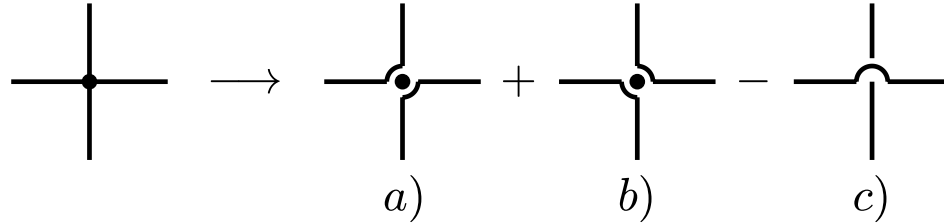


Figure 2.8: *Splitting of a node with corresponding weights.*

results in  $3^{N_n}$  different new graphs, obtained from a graph with  $N_n$  nodes. This would increase  $g_n$  in a complicated way. To avoid this complication, each new graphical element obtains a weight factor  $\pm 1$ , as shown in figure 2.8. Now the three contributions of each node add up to 1 again and the total count does not change. The new graphs are denoted by  $\tilde{G}$ . They consist of one or more loops. As a remainder, a *loop* (with e.g.  $n$  edges) is a simply connected set of edges, i.e. an object that can be drawn by starting at an arbitrary vortex on the loop, following the edges, and returning at the starting point after  $n$  steps. The shortest loop has 4 edges and the number of edges is always even. In the new graphs  $\tilde{G}$ , the number of edges has not changed, as compared to the original graph, but it has an extra weight factor

$$w(\tilde{G}) = (-1)^{N_c(\tilde{G})} ,$$

where  $N_c(\tilde{G})$  stands for the number of **crossings** (decomposition c in figure 2.8) of a graph  $\tilde{G}$  of the new type. By now  $g_n$  is

$$g_n = \sum_{\tilde{G}}^{N_e(\tilde{G})=n} (-1)^{N_c(\tilde{G})} .$$

### 2.3.4 Decomposing graphs into loops

We are still not able to calculate the sum of all graphs analytically. To this end we need to transform the graphs further. Each graph  $\tilde{G}$  with  $n$  edges consists of one or more loops, which in total have  $n$  edges. If a particular graph  $\tilde{G}$  consists of  $l$  loops, we can divide the total weight between the loops. A loop  $L$  contributes a factor  $(-1)^{N_c(L)}$ , where  $N_c(L)$  is the number of crossings in loop  $L$ . The total weight is the product of the weights of the loops of which the graph is formed. Clearly, there is a one-to-one correspondence between all graphs  $\tilde{G}$  with  $n$  edges and all sets of loops with a total number  $n$  of edges. We define

$$D_l = \begin{cases} \sum_L^{\text{loops with } l \text{ edges}} (-1)^{N_c(L)} & \text{if } l \text{ is even} \\ 0 & \text{otherwise} \end{cases}. \quad (2.41)$$

Hence for  $n \geq 4$  we may think that we can decompose  $g_n$  into the contribution of the loop decomposition:

$$g_n = \sum_{\tilde{G}} \delta_{N_n(\tilde{G})=n} (-1)^{N_c(\tilde{G})} = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{l_1 \dots l_m=4} \delta_{\sum_{\nu=1}^m l_\nu=n} \prod_{\nu=1}^m D_{l_\nu} \quad (2.42)$$

Here  $m$  is the number of loops and  $l_\nu$  stands for the number of edges of loop  $\nu$  and  $D_{l_\nu}$  is the corresponding weight obtained by summing over all realizations of a loop with  $l_\nu$  edges. The factor  $1/m!$  is required as the right hand side creates a particular set of  $m$  loops in  $m!$  different permutations. We can insert equation (2.42) in equation (2.40) [p. 55] and obtain

$$\begin{aligned} \sum_{n=4}^{\infty} g_n t^n &= \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{l_1 \dots l_m=4} \underbrace{\sum_{n=4}^{\infty} \delta_{\sum_{\nu=1}^m l_\nu=n}}_{=1} \prod_{\nu=1}^m D_{l_\nu} t^{l_\nu} \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \prod_{\nu=1}^m \left( \sum_{l=4} D_l t^l \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \left( \sum_{l=4} D_l t^l \right)^m \\ &= \exp \left( \sum_{l=4} D_l t^l \right) - 1 \end{aligned}$$

Hence according to equation (2.40) [p. 55] we have

$$\ln(Z) = N \ln(2) + 2N \ln(\cosh(j)) + \sum_{m=1}^{\infty} D_m t^m. \quad (2.43)$$

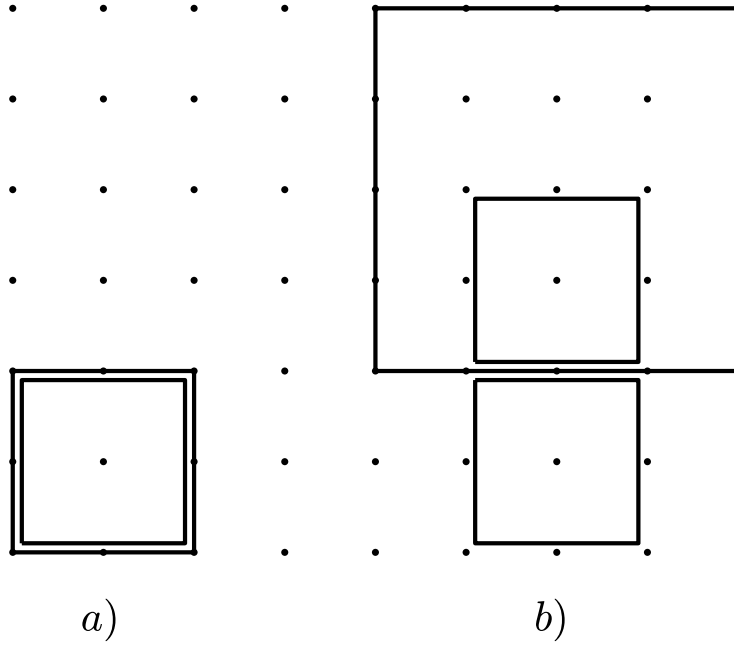
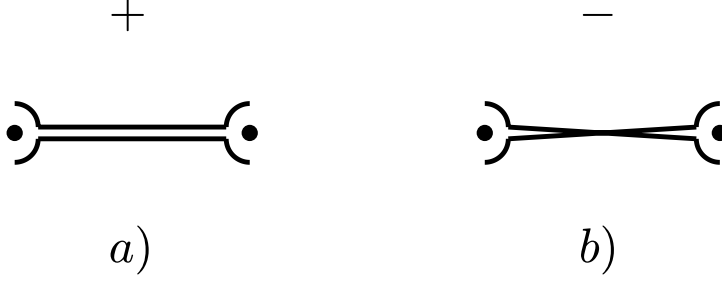
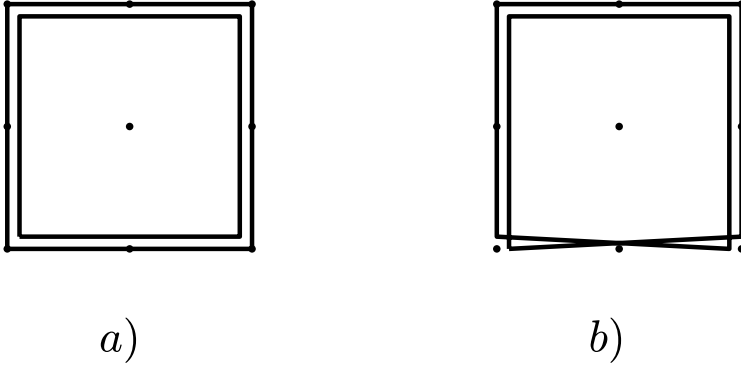


Figure 2.9: *Examples of configurations that occur if loops are placed independently. Such configurations are no valid graphs  $\tilde{G}$ .*

The sum can start at  $m = 1$  (instead of  $m = 4$ ) since equation (2.41) counts loops and the shortest loop has length 4 anyways. However, equation (2.42) is not really correct. On the rhs the sum runs over all configurations of independently arranged loops. I.e. there will be configurations where edges occur more than once, like in figure 2.9. Such a configuration is not included in  $\tilde{G}$ . However, there is a very simple cure. We allow such configurations also in  $\tilde{G}$ , but in addition to double bonds, say, as in figure 2.9 a) or schematically depicted in figure 2.10 a) we also include configurations that occur if we cross the lines as shown in figure 2.10 b) and give them a weight  $-1$ . The sum of these configurations adds up to zero. The same holds true, if there are threefold edges as in figure 2.9 b) or even  $m$ -fold edges. Generally, in the case of  $m$ -fold edges there are  $m!$  possible connections (permutations) of the incoming and outgoing lines. The number of crossings is even/odd if the permutation is even/odd. As the number of even permutations is always equal to the number of odd permutations, the signs add up to zero. In the case of figure 2.9 a) the new configurations are depicted in figure 2.11. Now a) represents a double loop generated by placing loops independently, while b) shows a single loop, which is passed twice. Including b) in the allowed single loops ensures the cancelation of forbidden configurations generated by

Figure 2.10: *Double occupied edges.*Figure 2.11: *Double occupied edges.*

placing loops independently. The same holds true for even more complex structures, such as triple loops.

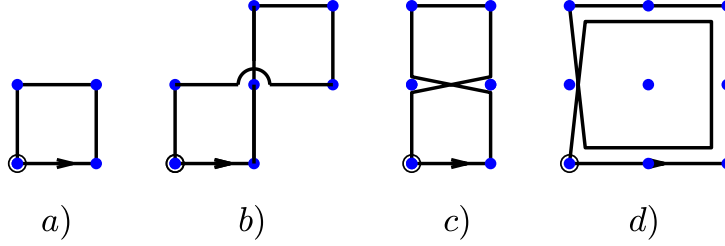
If we include the new type of crossings in the loops, then the previous formulas (equation (2.41) [p. 57] and equation (2.53) [p. 65]) remain unchanged, with the exception that these crossings also contribute to the number of crossings.

### Counting then number of loops and corresponding crossing

The remaining task is the determination of  $D_l$ , which includes the task of counting the crossings. To this end we introduce directed paths. The  $n$ -th step of the path is encoded in

$$S^{(n)} = (\mathbf{x}^{(n)}, \mathbf{d}^{(n)}) \quad (2.44)$$

where  $\mathbf{x}^{(n)}$  represents the initial site of the  $n$ -th step, and  $\mathbf{d}^{(n)} \in \{\pm \mathbf{e}_x, \pm \mathbf{e}_y\}$  the direction of the  $n$ -th step. A *path* from  $S^{(0)}$  to  $S^{(m)}$  is defined by the

Figure 2.12: *Representative paths and weights.*

sequence

$$\mathcal{P} = S^{(0)}, S^{(1)}, \dots, S^{(m)}$$

Clearly, we have the condition

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \mathbf{d}^{(n)} . \quad (2.45)$$

As the allowed loops do not contain elements that occur when  $\mathbf{d}^{(n+1)} = -\mathbf{d}^{(n)}$ , we omit such steps. We define the weight of such a path as

$$w(\mathcal{P}) = \prod_{l=1}^m e^{i\frac{1}{2}\Phi(\mathbf{d}^{(l-1)}, \mathbf{d}^{(l)})} , \quad (2.46)$$

where  $\Phi(\mathbf{d}, \mathbf{d}')$  is the angle between the vectors  $\mathbf{d}$  and  $\mathbf{d}'$ , defined as follows.

$$\Phi(\mathbf{d}, \mathbf{d}') = \begin{cases} 0 & \text{if } \mathbf{d}, \mathbf{d}' \text{ are parallel} \\ \frac{\pi}{2} & \text{if } \mathbf{d}' \text{ is anti-clockwise rotated from } \mathbf{d} \\ -\frac{\pi}{2} & \text{if } \mathbf{d}' \text{ is clockwise rotated from } \mathbf{d} \\ \text{forbidden} & \text{if } \mathbf{d}, \mathbf{d}' \text{ are anti-parallel} \end{cases} \quad (2.47)$$

In figure 2.12 some examples of allowed loops (paths) are given, for which we want to compute the weight. We start at the lower left corner and follow the arrow. We recall that turning left at a vertex gives  $+\frac{\pi}{2}$ , a right turn  $-\frac{\pi}{2}$  and crossing straight adds zero. In subfigure a) the 4 angles  $\Phi$  encountered during the path are all  $\pi/2$ . The four angle add up to  $2\pi$ . In other words we have performed four left turns. The weight is then

$$w = e^{i\frac{1}{2}2\pi} = -1 .$$

For example b) the angles are in units of  $\frac{\pi}{2}$ : 1, 0, -1, -1, -1, 0, 1, 1, with a sum of 0 and a weight  $w = +1$ . In other words the number of left and right

turns is equal. That is also the case in subfigure c). In example d) a loop is twice passed, so the total angle is  $4\pi$  and a phase factor  $+1$ . We find in all cases

$$w(\mathcal{P}) = -(-1)^{N_c(L)} ,$$

which is the required weight in equation (2.41) [p. 57] apart from a global minus sign.

Now we define a matrix  $\mathcal{M}$  with matrix elements

$$\langle S' | M_m | S \rangle = \text{sum of the weights of all paths from } S \text{ to } S' \text{ in } m \text{ steps} .$$

The value of the matrix element is zero, if there is no path connecting  $S$  and  $S'$  in  $m$  steps. For  $m = m_1 + m_2$  we have by definition

$$\langle S' | M_m | S \rangle = \sum_{S''} \langle S' | M_{m_1} | S'' \rangle \langle S'' | M_{m_2} | S \rangle ,$$

which is the common matrix product. Consequently, we have

$$\mathcal{M}_m = (\mathcal{M}_1)^m .$$

$\mathcal{M}_1$  has the dimension  $4N \times 4N$ , as each step  $S = (\mathbf{x}, \mathbf{d})$  has  $N$  possible sites  $\mathbf{x}$  and 4 possible directions  $\mathbf{d}$ . Most of the matrix elements, however, are zero. The fact that a path cannot be retraced is also accounted for in  $\mathcal{M}_1$ .

We can now easily express the sought-for weight  $D_m$  as

$$D_m = -\frac{1}{2m} \sum_S \langle S | \mathcal{M}_m | S \rangle = -\frac{1}{2m} \text{tr}\{\mathcal{M}_1^m\}$$

---

*proof*

1. The trace is required to sum over all initial vertices of the loop and to make sure that the loop is really closed
2. Since each vertex of a loop occurs as initial point in the trace, we have to divide by  $m$
3. Each loop can be traversed in two direction, which explains the factor  $1/2$ .
4. The minus sign has been explained before.

q.e.d. ✓

---

In total we therefore have according to equation (2.53) [p. 65]

$$\begin{aligned}
 \ln(Z) &= N \ln(2) + 2N \ln(\cosh(j)) + \sum_{m=1} D_m t^m \\
 &= N \ln(2) + 2N \ln(\cosh(j)) - \frac{1}{2} \text{tr} \left\{ \sum_{m=1} \frac{(t\mathcal{M}_1)^m}{m} \right\} \\
 &= N \ln(2) + 2N \ln(\cosh(j)) + \frac{1}{2} \text{tr} \{ \ln(\mathbf{1} - t\mathcal{M}_1) \} \\
 &= N \ln(2) + 2N \ln(\cosh(j)) + \frac{1}{2} \ln \det[(\mathbf{1} - t\mathcal{M}_1)] .
 \end{aligned}$$

As said before, the size of the matrix  $\mathcal{M}_1$  is  $4N \times 4N$ . The matrix elements are defined via

$$\langle \mathbf{x}, \mathbf{d} | \mathcal{M}_1 | \mathbf{x}', \mathbf{d}' \rangle .$$

Since we use pbc, it is advantageous to perform a Fourier transform with respect to  $\mathbf{x}$  and  $\mathbf{x}'$ . The determinant is invariant against such a unitary transformation. Moreover, due to translational invariance of the problem, we obtain

$$\begin{aligned}
 \langle \mathbf{q}, \mathbf{d} | \mathcal{M}_1 | \mathbf{q}', \mathbf{d}' \rangle &= \frac{1}{N} \sum_{\mathbf{x}, \mathbf{x}'} e^{-i(\mathbf{x} \cdot \mathbf{q} - \mathbf{x}' \cdot \mathbf{q}')} \langle \mathbf{0}, \mathbf{d} | \mathcal{M}_1 \left| \underbrace{\mathbf{x}' - \mathbf{x}}_{\Delta \mathbf{x}}, \mathbf{d}' \right\rangle \\
 &= \frac{1}{N} \underbrace{\sum_{\mathbf{x}} e^{-i\mathbf{x} \cdot (\mathbf{q} - \mathbf{q}')} }_{=\delta_{\mathbf{q}\mathbf{q}'}} \sum_{\Delta \mathbf{x}} e^{i\mathbf{q}' \cdot \Delta \mathbf{x}} \langle \mathbf{0}, \mathbf{d} | \mathcal{M}_1 | \Delta \mathbf{x}, \mathbf{d}' \rangle \\
 &= \delta_{\mathbf{q}\mathbf{q}'} \sum_{\Delta \mathbf{x}} e^{i\mathbf{q} \cdot \Delta \mathbf{x}} \langle \mathbf{0}, \mathbf{d} | \mathcal{M}_1 | \Delta \mathbf{x}, \mathbf{d}' \rangle .
 \end{aligned}$$

According to equation (2.45) [p. 60] we have the condition

$$\begin{aligned}
 \mathbf{x}' &= \mathbf{x} + \mathbf{d} \\
 \Rightarrow \quad \Delta \mathbf{x} &= \mathbf{x}' - \mathbf{x} = \mathbf{d} .
 \end{aligned}$$

Hence the  $4 \times 4$  matrix  $M(\mathbf{q})$  has the matrix elements

$$\begin{aligned}
 \langle \mathbf{q}, \mathbf{d} | \mathcal{M}_1 | \mathbf{q}', \mathbf{d}' \rangle &= \delta_{\mathbf{q}\mathbf{q}'} e^{i\mathbf{q} \cdot \mathbf{d}} \langle \mathbf{0}, \mathbf{d} | \mathcal{M}_1 | \mathbf{d}, \mathbf{d}' \rangle \\
 &= \delta_{\mathbf{q}\mathbf{q}'} \underbrace{e^{i\mathbf{q} \cdot \mathbf{d}} e^{i\frac{1}{2}\Phi(\mathbf{d}, \mathbf{d}')}}_{=M_{\mathbf{d}, \mathbf{d}'}}
 \end{aligned}$$

The inverse transformation is

$$\langle \mathbf{x}, \mathbf{d} | \mathcal{M}_1 | \mathbf{x}', \mathbf{d}' \rangle = \frac{1}{N} \sum_{\mathbf{q}} e^{i(\mathbf{x}-\mathbf{x}') \cdot \mathbf{q}} \langle \mathbf{q}, \mathbf{d} | \mathcal{M}_1 | \mathbf{q}, \mathbf{d}' \rangle ,$$

and, therefore,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{d} | \mathcal{M}_1^n | \mathbf{x}', \mathbf{d}' \rangle &= \frac{1}{N} \sum_{\mathbf{q}} e^{i(\mathbf{x}-\mathbf{x}') \cdot \mathbf{q}} \langle \mathbf{q}, \mathbf{d} | (\mathcal{M}_1)^n | \mathbf{q}, \mathbf{d}' \rangle \\ &= \frac{1}{N} \sum_{\mathbf{q}} e^{i(\mathbf{x}-\mathbf{x}') \cdot \mathbf{q}} \left( (M(\mathbf{q}))^n \right)_{\mathbf{d}, \mathbf{d}'} . \end{aligned}$$

The remaining  $4 \times 4$  matrix  $M(\mathbf{q})$  has the matrix elements

$\mathbf{d} \backslash \mathbf{d}'$	$\mathbf{e}_x$	$-\mathbf{e}_x$	$\mathbf{e}_y$	$-\mathbf{e}_y$
$\mathbf{e}_x$	$e^{iq_1}$	$0$	$\lambda e^{iq_1}$	$\lambda^* e^{iq_1}$
$-\mathbf{e}_x$	$0$	$e^{-iq_1}$	$\lambda^* e^{-iq_1}$	$\lambda e^{-iq_1}$
$\mathbf{e}_y$	$\lambda^* e^{iq_2}$	$\lambda e^{iq_2}$	$e^{iq_2}$	$0$
$-\mathbf{e}_y$	$\lambda e^{-iq_2}$	$\lambda^* e^{-iq_2}$	$0$	$e^{-iq_2}$

with the definition  $\lambda = e^{i\frac{\pi}{4}}$ . This can also be written as

$$M(\mathbf{q}) = D(\mathbf{q}) \tilde{M} \quad (2.48a)$$

$$D(\mathbf{q}) = \text{diag}[e^{iq_1}, e^{-iq_1}, e^{iq_2}, e^{-iq_2}] \quad (2.48b)$$

$$\tilde{M} = \begin{pmatrix} 1 & 0 & \lambda, \lambda^* \\ 0 & 1 & \lambda^*, \lambda \\ \lambda^*, \lambda & 1 & 0 \\ \lambda, \lambda^* & 0 & 1 \end{pmatrix} \quad (2.48c)$$

Here  $\tilde{M}$  is a hermitean matrix but  $M$  is not. As the matrix is blockdiagonal in  $\mathbf{q}, \mathbf{q}'$  we have

$$\det(\mathbf{1} - t\mathcal{M}_1) = \det(t\mathcal{M}_1 - \mathbf{1}) = \prod_{\mathbf{q}}^{1bc} \det(tM(\mathbf{q}) - \mathbf{1})$$

With  $Q_\alpha = te^{iq_\alpha}$  the matrix argument of the determinant reads

$$tM(\mathbf{q}) - \mathbf{1} = - \begin{pmatrix} Q_1 - 1 & 0 & \lambda Q_1 & \lambda^* Q_1 \\ 0 & Q_1^* - 1 & \lambda^* Q_1^* & \lambda Q_1^* \\ \lambda^* Q_2 & \lambda Q_2 & Q_2 - 1 & 0 \\ \lambda Q_2^* & \lambda^* Q_2^* & 0 & Q_2^* - 1 \end{pmatrix} .$$



The determinant yields (according to MATHEMATICA)

$$\ln \left[ \det (\mathbf{1} - t\mathcal{M}_1) \right] = \sum_{\mathbf{q}}^{1bz} \ln \left[ (1 + t^2)^2 - 2t(1 - t^2)(\cos(q_1) + \cos(q_2)) \right] .$$

Then the free energy per site is

$$-\beta \frac{F}{N} = \ln(2) + 2 \ln(\cosh(j)) + \frac{1}{2N} \sum_{\mathbf{q}}^{1bc} \ln \left[ (1 + t^2)^2 - 2t(1 - t^2)(\cos(q_1) + \cos(q_2)) \right] \quad (2.49)$$

We introduce the normalized density of states  $\rho(\varepsilon)$  corresponding to the dispersion

$$\varepsilon(\mathbf{q}) = -(\cos(q_1) + \cos(q_2)) \quad (2.50)$$

$$\rho(\varepsilon) = \frac{1}{N} \sum_{\mathbf{q}}^{1bc} \delta(\varepsilon(\mathbf{q}) - \varepsilon) \quad (2.51)$$

Then the free energies reads

$$-\beta \frac{F}{N} = \ln(2) + 2 \ln(\cosh(j)) + \frac{1}{2} \int d\varepsilon \rho(\varepsilon) \ln \left[ (1 + t^2)^2 + 2t(1 - t^2) \varepsilon \right] . \quad (2.52)$$

Moreover, we use

$$\begin{aligned} \cosh(j) &= \frac{1}{\sqrt{1 - t^2}} , \quad (t := \tanh(j)) \\ 2 \ln(\cosh(j)) &= \frac{1}{2} \ln((1 - t^2)^{-2}) = \frac{1}{2} \int d\varepsilon \rho(\varepsilon) \ln((1 - t^2)^{-2}) \end{aligned}$$

to obtain

$$\begin{aligned} -\beta \frac{F}{N} &= \ln(2) + \frac{1}{2} \int d\varepsilon \rho(\varepsilon) \ln \left[ \left( \frac{1 + t^2}{1 - t^2} \right)^2 + \left( \frac{2t}{1 - t^2} \right) \varepsilon \right] \\ &= \ln(2) + \frac{1}{2} \int d\varepsilon \rho(\varepsilon) \ln \left[ \underbrace{\left( \frac{1 + t^2}{1 - t^2} \right)^2}_{= (1 - \sinh(2j))^2} + \underbrace{\left( \frac{2t}{1 - t^2} \right)}_{= \sinh(2j)} (2 + \varepsilon) \right] . \end{aligned}$$

---

*proof*

$$\begin{aligned}
\frac{2t}{1-t^2} &= \frac{2 \tanh(j)}{1 - \tanh^2(j)} \\
&= \frac{2sh/ch}{\underbrace{(ch^2 - sh^2)}_{=1}/ch^2} \\
&= 2 \sinh(j) \cosh(j) = \sinh(2j) \\
\left(\frac{1+t^2}{1-t^2}\right)^2 - \left(\frac{4t}{1-t^2}\right) &= \underbrace{\left(\frac{\overbrace{ch^2 + sh^2}^{=\cosh(2j)}}{ch^2 - sh^2}\right)^2}_{=1} - \underbrace{\left(2\frac{2t}{1-t^2}\right)}_{=2\sinh(2j)} \\
&= \cosh^2(2j) - 2\sinh(2j) \\
&= 1 + \sinh^2(2j) - 2\sinh(2j) \\
&= (1 - \sinh(2j))^2.
\end{aligned}$$

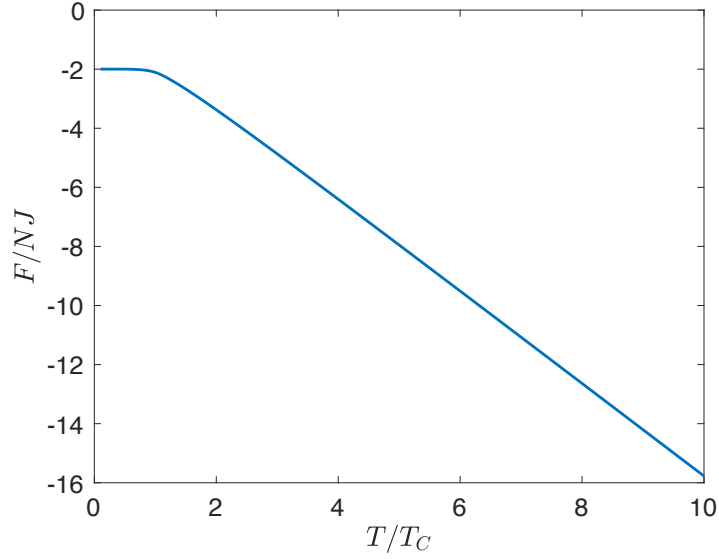
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q.e.d. ✓

The final result for the free energy reads thereore

<p>FREE ENERGY OF THE 2D ISING MODEL (without external field)</p> <hr style="border: 0.5px solid black;"/> $ \frac{\ln(Z)}{N} = -\beta \frac{F}{N} = \ln(2) + \frac{1}{2} \int d\varepsilon \rho(\varepsilon) \ln \left[ (1 - \sinh(2j))^2 + \sinh(2j) (2 + \varepsilon) \right]. $ <p style="text-align: right;">(2.53)</p>
--

It is to be remembered that  $j = J\beta > 0$  for the ferrromagnetic model. Therefore,  $\sinh(2j) > 0$ . According to equation (2.50) [\[previous page\]](#) , the energies  $\varepsilon$  are restricted to the interval  $[-2, 2]$ . The density of states  $\rho(\varepsilon)$  is

Figure 2.13: *Free energy of the 2D Ising model.*

that of the 2d tight binding model, which is given by

$$\rho(\varepsilon) = \frac{\theta(|\varepsilon| \leq 2)}{2\pi^2} \mathcal{K}\left(1 - \left(\frac{\varepsilon}{2}\right)^2\right), \quad (2.54)$$

where  $\mathcal{K}(x)$  is the elliptic integral of the first kind

$$\mathcal{K}(x) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - x^2(\sin \varphi)^2}}. \quad (2.55)$$

The free energy is related to the entropy and internal energy through

$$F = U - TS.$$

As we will easily see later on, for  $T \rightarrow 0$  we have  $U \rightarrow -2JN$  and  $S \rightarrow 0$  (3. law of thermodynamics), hence  $F \rightarrow -2JN$ ; while for  $k_B T/J \gg 1$  we have  $U \rightarrow 0$  and  $S \rightarrow Nk_B \ln(2)$ , hence  $F \rightarrow -Nk_B T \ln(2)$ .

### 2.3.5 Curie-temperature

If we would have computed the free energy in the presence of an external magnetic field, we could have computed the magnetization and from that directly the transition temperature. In the absence of an external field, the magnetization is always zero, due to symmetry. But in order to allow for a

phase transition the free energy has to have an irregularity. (The argument will be given later.) The argument of the logarithm is non-negative, but it can be zero which yields the required irregularity. For the argument to become zero, both terms  $(1 - \sinh(2j))^2$  and  $\sinh(2j)(2 + \varepsilon)$  have to be zero. The first condition yields

$$\sinh(2j_C) = 1 \quad (2.56)$$

$$\begin{aligned} 1 - \sinh(2j) &= 1 - \frac{e^{2j_C} - e^{-2j_C}}{2} = 0 \\ \Rightarrow \quad e^{2j_C} - e^{-2j_C} - 2 &= 0 \\ e^{4j_C} - 2e^{2j_C} - 1 &= 0 . \end{aligned}$$

This quadratic equation has the solutions

$$e^{2j_C} = 1 \pm \sqrt{2} .$$

Since  $e^{2j} > 0$  the only solution is

$$j_C = J\beta_C = \frac{1}{2} \ln(1 + \sqrt{2}) = 0.4407 . \quad (2.57)$$

The mean field result was  $J\beta_C = 0.25$ . The irregularity of the free energy is a necessary prerequisite for phase transition, but not sufficient for a rigorous prove. Nor does it tell us the order of the phase transition. The rigorous prove requires the dependence of the free energy on the an external field.

### 2.3.6 Internal energy

$$U = \langle H \rangle = -J \sum_{\langle ij \rangle} \langle S_i S_j \rangle .$$

Due to the translational invariance all n.n. correlations are the same

$$\frac{U}{NJ} = -2\langle S_1 S_2 \rangle .$$

For high temperatures the spins are uncorrelated resulting in

$$\frac{U}{NJ} = -2\langle S_1 \rangle \langle S_2 \rangle = 0 ,$$

while for  $T = 0$ , the lowest energy is obtained if all spins are parallel and

$$\frac{U}{NJ} = -2 .$$

Now we compute  $U$  for arbitrary temperature. To this end, we write  $\ln(Z)$  in a more compact form

$$\ln(Z) = \ln(2) + \frac{1}{2} \left\langle \underbrace{\ln \left( (1 - \kappa)^2 + \kappa(2 + \varepsilon) \right)}_{=g(\kappa)} \right\rangle ,$$

the average here is taken w.r.t.  $\rho(\varepsilon)$ , and  $\kappa = \sinh(2\beta J)$ . Then

$$\begin{aligned} \frac{U}{N} &= \frac{\langle H \rangle}{N} = -\frac{\partial \ln(Z)}{\partial \beta} \\ &= -\frac{1}{2} \left( \frac{\partial}{\partial \kappa} g(\kappa) \right) \frac{d\kappa}{d\beta} . \end{aligned}$$

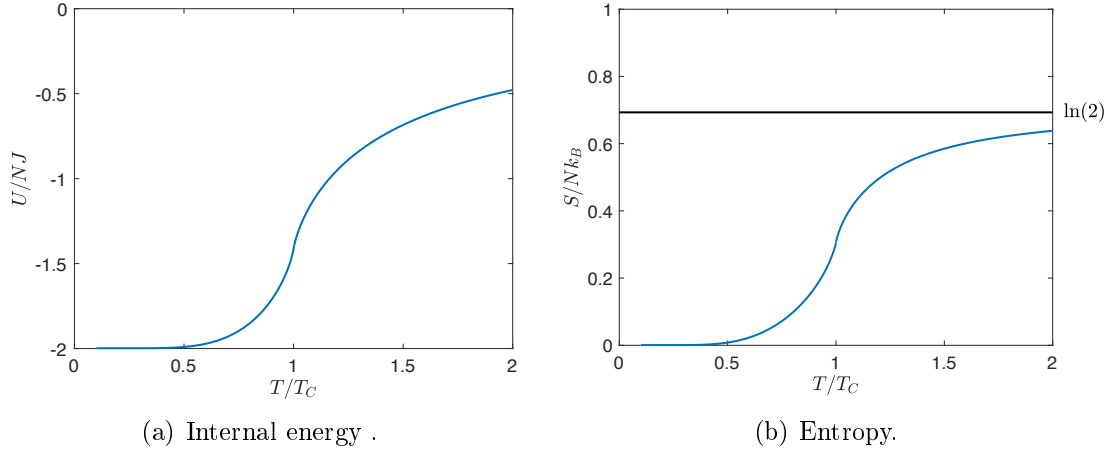
with

$$\frac{\partial \kappa}{\partial \beta} = \frac{\partial}{\partial \beta} \sinh(2J\beta) = 2J \cosh(2J\beta) = 2J \sqrt{1 + \sinh^2(2J\beta)}$$

we have

$$\begin{aligned} \frac{U}{N} &= -J g'(\kappa) \sqrt{1 + \kappa^2} \\ &= -J \left\langle \frac{2\kappa + \varepsilon}{(1 - \kappa)^2 + \kappa(2 + \varepsilon)} \right\rangle \sqrt{1 + \kappa^2} . \end{aligned}$$

In summary we have

Figure 2.14: *2d Ising model for  $B = 0$ .*

INTERNAL ENERGY	
$\frac{U}{NJ} = - \left\langle \frac{2\kappa + \varepsilon}{(1 - \kappa)^2 + \kappa(2 + \varepsilon)} \right\rangle \sqrt{1 + \kappa^2} .$	(2.58)

### 2.3.7 Entropy

From  $F = U - TS$  we get

$$\frac{S}{Nk_B} = j \frac{U}{NJ} - \beta \frac{F}{N} = j \left( \frac{U}{NJ} \right) + \left( \frac{\ln(Z)}{N} \right) .$$

Hence, we can express the entropy in terms of equation (2.58) and equation (2.53) [p. 65].

We see no critical behaviour in the entropy, which is the first derivative of the free energy w.r.t.  $T$ . The latter is continuous at  $T_C$ . According to [section \(1.2.3\)](#) [Seite 26] it is therefore not a first order phase transition.

### 2.3.8 Specific heat

The order of the phase transition can also be inferred from the specific heat at  $B = 0$ .

$$\begin{aligned} \frac{C}{N} &= \frac{1}{N} \frac{\partial U}{\partial T} \Big|_N = \frac{1}{N} \frac{\partial U}{\partial \kappa} \underbrace{\frac{\partial \kappa}{\partial \beta}}_{=2J\sqrt{1+\kappa^2}=-\frac{1}{k_B T^2}} \underbrace{\frac{\partial \beta}{\partial T}} \\ &= \left[ \frac{\partial}{\partial \kappa} \left( -Jg'(\kappa)\sqrt{1+\kappa^2} \right) \right] \left( -k_B \beta^2 (2J\sqrt{1+\kappa^2}) \right) \end{aligned}$$

The final result reads

$$\frac{C(T)}{Nk_B} = 2j^2 \left[ (1+\kappa^2)g''(\kappa) + \kappa g'(\kappa) \right]. \quad (2.59a)$$

with

$$g'(\kappa) = \left\langle \frac{2\kappa + \varepsilon}{(1-\kappa)^2 + \kappa(2+\varepsilon)} \right\rangle \quad (2.59b)$$

$$g''(\kappa) = \left\langle \frac{2}{(1-\kappa)^2 + \kappa(2+\varepsilon)} \right\rangle - \left\langle \left( \frac{2\kappa + \varepsilon}{(1-\kappa)^2 + \kappa(2+\varepsilon)} \right)^2 \right\rangle \quad (2.59c)$$

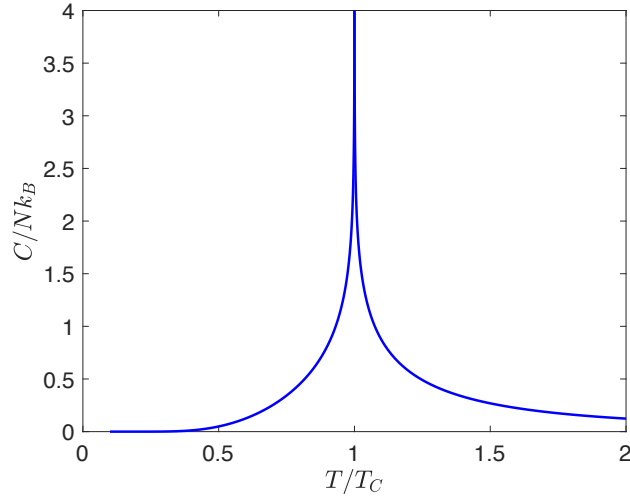
We see in figure 2.15 that the specific heat diverges at  $T_C$ . In order to unravel the type of divergency we perform a Taylor expansion in  $T$  about  $T_C$ , i.e.  $T = T_C + \Delta T$ . This corresponds to

$$\begin{aligned} j &= j_C + \Delta j \\ \kappa &= \sinh(2J_C) + \Delta\kappa = 1 + \Delta\kappa \end{aligned}$$

In the last step we have used equation (2.56) [p. 67]. The only divergent behaviour can come from  $g'$  or  $g''$ . The other terms can, therefore, be replaced by  $T = T_C$ . I.e.

$$\frac{C(T_C + \Delta T)}{Nk_B} = 2j_C^2 \left[ 2g''(\kappa_c + \Delta\kappa) + g'(\kappa_c + \Delta\kappa) \right].$$

There is a critical behaviour in the second derivative of  $F$ , i.e. the specific heat. Therefore, the Ising model has a [second order phase transition](#).

Figure 2.15: *Specific heat of the 2d Ising model.*

### Type of divergency

The divergency originates from the term  $\frac{1}{(1-\kappa)^2 + \kappa(\varepsilon+2)}$  at the vicinity of  $\varepsilon = -2$ . Since, near  $T_C$ , we obtain  $\kappa \rightarrow 1$ , this term becomes  $\frac{1}{\kappa(\varepsilon+2)}$  and the divergency is entirely determined by the value of  $\varepsilon$ . The dos  $\rho(\varepsilon)$  in principle also has a divergency at  $\varepsilon = 0$  but it is integrable and does not result in a divergence in the specific heat. To study the divergency at  $\varepsilon = -2$  we split

$$\rho(\varepsilon) = \underbrace{\rho(-2)}_{=\rho_0} + \Delta\rho(\varepsilon) .$$

Then the expectation values become

$$\langle F(\varepsilon) \rangle = \langle F(\varepsilon) \rangle_{\rho_0} + \langle F(\varepsilon) \rangle_{\Delta\rho} .$$

The second term cause no divergency, hence we merely need to consider terms of the form

$$\langle F(\varepsilon) \rangle_{\rho_0} . \tag{2.60}$$

---

*Proof*



$$\begin{aligned}
& \lim_{\varepsilon \rightarrow -2} \frac{\Delta\rho(\varepsilon)(2 + \varepsilon + 2\Delta\kappa)}{(\Delta\kappa)^2 + (1 + \Delta\kappa)(\varepsilon + 2)} \\
&= \lim_{\eta \rightarrow 0} \frac{\Delta\rho(-2 + \eta)(\eta + 2\Delta\kappa)}{(\Delta\kappa)^2 + (1 + \Delta\kappa)\eta} \\
&\stackrel{L'Hospital}{=} \lim_{\eta \rightarrow 0} \frac{\Delta\rho'(-2 + \eta)(\eta + 2\Delta\kappa) + \Delta\rho(-2 + \eta)}{(\Delta\kappa)^2 + (1 + \Delta\kappa)\eta} \\
&= \lim_{\eta \rightarrow 0} \frac{(2\Delta\kappa\Delta\rho'(-2 + \eta) + \Delta\rho(-2 + \eta))}{(\Delta\kappa)^2} \\
&= \frac{2}{\Delta\kappa} \lim_{\eta \rightarrow 0} \underbrace{\Delta\rho'(-2 + \eta)}_{=0} + \frac{1}{(\Delta\kappa)^2} \lim_{\eta \rightarrow 0} \underbrace{\Delta\rho(-2 + \eta)}_{=0} \\
&= 0 .
\end{aligned}$$

or for  $g''$  we need

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow -2} \Delta\rho(\varepsilon) \left( \frac{2 + \varepsilon + \Delta\kappa}{(\Delta\kappa)^2 + (1 + \Delta\kappa)(\varepsilon + 2)} \right)^2 \\
&= \lim_{\eta \rightarrow 0} \frac{\Delta\rho(-2 + \eta)(\eta + \Delta\kappa)^2}{((\Delta\kappa)^2 + (1 + \Delta\kappa)\eta)^2} \\
&\stackrel{L'Hospital}{=} \lim_{\eta \rightarrow 0} \frac{\Delta\rho'(-2 + \eta)(\eta + \Delta\kappa)^2 + 2\Delta\rho(-2 + \eta)(\eta + \Delta\kappa)}{2((\Delta\kappa)^2 + (1 + \Delta\kappa)\eta)(1 + \Delta\kappa)} \\
&\stackrel{L'Hospital}{=} \lim_{\eta \rightarrow 0} \frac{\Delta\rho'(-2 + \eta)(\Delta\kappa)^2 + 2\Delta\kappa\Delta\rho(-2 + \eta)}{2((\Delta\kappa)^2)(1 + \Delta\kappa)} \\
&= \frac{(\Delta\kappa)^2}{2((\Delta\kappa)^2)(1 + \Delta\kappa)} \lim_{\eta \rightarrow 0} \underbrace{\Delta\rho'(-2 + \eta)}_{=0} + \frac{2\Delta\kappa}{2((\Delta\kappa)^2)(1 + \Delta\kappa)} \lim_{\eta \rightarrow 0} \underbrace{\Delta\rho(-2 + \eta)}_{=0} \\
&= 0 .
\end{aligned}$$

q.e.d. ✓

For the first term in equation (2.59) [p. 70] we need

$$g'(1 + \Delta\kappa) = \left\langle \frac{\varepsilon + 2 + \Delta\kappa}{(2 + \varepsilon)(1 + \Delta\kappa) + (\Delta\kappa)^2} \right\rangle_{\rho_0} ,$$

for which we find

$$g'(1 + \Delta\kappa) = \frac{1}{1 + \Delta\kappa} \cdot \{4\rho_0 + \Delta\kappa\rho_0 \ln(4) - 2\rho_0\Delta\kappa \ln(\Delta\kappa) + O(\Delta\kappa)\}$$

$$\xrightarrow{\Delta\kappa \rightarrow 0} 4\rho_0 = O(1) .$$

---

*Proof*

First we will transform the denominator

$$(2 + \varepsilon)(1 + \Delta\kappa) + (\Delta\kappa)^2 = (1 + \Delta\kappa) \left( 2 + \varepsilon + \frac{(\Delta\kappa)^2}{1 + \Delta\kappa} \right)$$

$$= (1 + \Delta\kappa) \left( 2 + \varepsilon + (\Delta\kappa)^2 + O((\Delta\kappa)^3) \right)$$

Then

$$\frac{\varepsilon + 2 + \Delta\kappa}{(2 + \varepsilon)(1 + \Delta\kappa) + (\Delta\kappa)^2}$$

$$= \frac{1}{1 + \Delta\kappa} \cdot \frac{\varepsilon + 2 + \Delta\kappa}{2 + \varepsilon + (\Delta\kappa)^2 + O((\Delta\kappa)^3)}$$

$$= \frac{1}{1 + \Delta\kappa} \cdot \frac{\varepsilon + 2 + (\Delta\kappa)^2 + O((\Delta\kappa)^3) + \Delta\kappa(1 - \Delta\kappa) + O((\Delta\kappa)^3)}{2 + \varepsilon + (\Delta\kappa)^2 + O((\Delta\kappa)^3)} .$$

I.e.

$$\frac{\varepsilon + 2 + \Delta\kappa}{(2 + \varepsilon)(1 + \Delta\kappa) + (\Delta\kappa)^2} = \frac{1}{1 + \Delta\kappa} \cdot \left\{ 1 + \frac{\Delta\kappa(1 - O((\Delta\kappa)^2))}{2 + \varepsilon + (\Delta\kappa)^2(1 + O(\Delta\kappa))} \right\} .$$

(2.61)

So we have

$$g'(1 + \Delta\kappa) = \left\langle \frac{\varepsilon + 2 + \Delta\kappa}{(2 + \varepsilon)(1 + \Delta\kappa) + (\Delta\kappa)^2} \right\rangle_{\rho_0}$$

$$= \frac{1}{1 + \Delta\kappa} \cdot \left\{ \langle 1 \rangle_{\rho_0} + \Delta\kappa(1 - O((\Delta\kappa)^2)) \left\langle \frac{1}{2 + \varepsilon + (\Delta\kappa)^2(1 + O(\Delta\kappa))} \right\rangle_{\rho_0} \right\}$$

The definition of the expectation value is

$$\langle f(\varepsilon) \rangle_{\rho_0} := \int_{-2}^2 f(\varepsilon) \rho_0 d\varepsilon = \rho_0 \int_{-2}^2 f(\varepsilon) d\varepsilon .$$

Therefore

$$\langle 1 \rangle_{\rho_0} = 4\rho_0 .$$

$$\begin{aligned} & \left\langle \frac{1}{2 + \varepsilon + (\Delta\kappa)^2(1 + O(\Delta\kappa))} \right\rangle_{\rho_0} \\ &= \rho_0 \int_{-2}^2 \frac{1}{2 + \varepsilon + (\Delta\kappa)^2(1 + O(\Delta\kappa))} d\varepsilon \\ &= \rho_0 \ln \left( 2 + 2 + (\Delta\kappa)^2(1 + O(\Delta\kappa)) \right) - \rho_0 \ln \left( 2 - 2 + (\Delta\kappa)^2(1 + O(\Delta\kappa)) \right) \\ &= \rho_0 \ln(4) - \rho_0 \ln((\Delta\kappa)^2) + O(\Delta\kappa) \end{aligned}$$

In total we have

$$\begin{aligned} g'(1 + \Delta\kappa) &= \left\langle \frac{\varepsilon + 2 + \Delta\kappa}{(2 + \varepsilon)(1 + \Delta\kappa) + (\Delta\kappa)^2} \right\rangle_{\rho_0} \\ &= \frac{1}{1 + \Delta\kappa} \cdot \{ 4\rho_0 + \Delta\kappa\rho_0 \ln(4) - 2\rho_0\Delta\kappa \ln((\Delta\kappa)^2) + O(\Delta\kappa) \} \\ &\xrightarrow{\Delta\kappa \rightarrow 0} 0 . \end{aligned}$$

q.e.d. ✓

Similarly, for the second term of  $g''$  we find with equation (2.61) [\[previous page\]](#)

$$\begin{aligned} & \left\langle \left( \frac{2\kappa + \varepsilon}{(1 - \kappa)^2 + \kappa(2 + \varepsilon)} \right)^2 \right\rangle_{\rho_0} \\ &= \left( \frac{1}{1 + \Delta\kappa} \right)^2 \cdot \left\langle \left( 1 + \frac{\Delta\kappa(1 - O((\Delta\kappa)^2))}{2 + \varepsilon + (\Delta\kappa)^2(1 + O(\Delta\kappa))} \right)^2 \right\rangle_{\rho_0} \\ &= O(1) + O(\Delta\kappa \ln(\Delta\kappa)) + O((\Delta\kappa)^2) \int_{-2}^2 \frac{1}{2 + \varepsilon + (\Delta\kappa)^2} d\varepsilon \\ &= O(1) + O(\Delta\kappa \ln(\Delta\kappa)) + O((\Delta\kappa)^2) \left( O(1) + O\left(\frac{1}{(\Delta\kappa)^2}\right) \right) \\ &= O(\Delta\kappa \ln(\Delta\kappa)) + O(1) + O((\Delta\kappa)^2) \end{aligned}$$

Again no divergency. The only divergent term stems from the first part of  $g''$ , which yields

$$\begin{aligned}
 \left\langle \frac{2}{(\Delta\kappa)^2 + (1 + \Delta\kappa)(2 + \varepsilon)} \right\rangle_{\rho_0} &= \int_{-2}^2 \frac{\rho_0}{(\Delta\kappa)^2 + (1 + \Delta\kappa)(2 + \varepsilon)} d\varepsilon \\
 &= \rho_0 \ln [(\Delta\kappa)^2 + (1 + \Delta\kappa)(2 + 2)] - \rho_0 \ln [(\Delta\kappa)^2 + (1 + \Delta\kappa)(2 - 2)] \\
 &= \rho_0 \ln [4 + O(\Delta\kappa)] - 2\rho_0 \ln(|\Delta\kappa|); \\
 &= O(1) - 2 \ln(|\Delta\kappa|);
 \end{aligned}$$

The specific heat obviously has a logarithmic divergency, which corresponds to the critical exponent

$$\alpha = \alpha' = 0 . \quad (2.62)$$

For the definition of the critical exponents see [section \(1.3\)](#) [\[Seite 27\]](#).

### 2.3.9 Spontaneous magnetization

So far we have only treated the 2d Ising model in the case where there is no external field and it has not been possible to obtain a solution including such a field. This is only doable for infinitesimal  $B$  values, which is enough to determine the spontaneous magnetization. It amounts to computing the spin-spin correlation  $\langle S_i S_j \rangle$  and use

$$\lim_{l \rightarrow \infty} \langle S_i S_{i+l} \rangle \longrightarrow \left( \langle S_i \rangle \right)^2 = \left( \frac{M}{N} \right)^2 .$$

The graphical solution amounts to count all graphs as before but with the peculiarity that the vertices  $i$  and  $j$  have an odd order. The result reads

SPONTANEOUS MAGNETIZATION OF THE 2D ISING MODEL	
$\frac{M}{N} = \begin{cases} (1 - \sinh^{-4}(2j))^{1/8} & \text{for } T \leq T_C \\ 0 & \text{otherwise} \end{cases}$	(2.63)

The order parameter, the magnetization, is continuous, which corroborates the statement, that it is a second order phase transition.

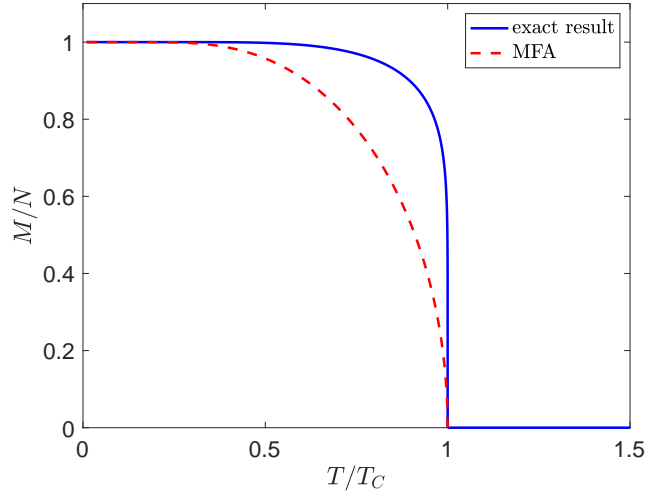


Figure 2.16: *Spontaneous magnetization of Ising spins.*

### 2.3.10 Critical exponent

For the corresponding critical exponent we start our considerations from the before obtained solution Equation 2.63. In order to evaluate the magnetization at  $T \searrow T_C$  we let  $j = j_C + \Delta j = j_C + J \Delta\beta$  and compute the magnetization:

$$\begin{aligned} \frac{M}{N} &= \left( 1 - [\sinh(2j_C + 2J\Delta\beta)]^{-4} \right)^{\frac{1}{8}} \\ &= \left( 1 - \underbrace{[\sinh(2j_C) + \cosh(2j_C)J\Delta\beta]^{-4}}_{\stackrel{(2.56)}{=} 1} \right)^{\frac{1}{8}}. \end{aligned}$$

Where a series expansion of  $\sinh(2j_C + 2J\Delta\beta)$  around  $\Delta\beta = 0$  was performed. In addition, we have

$$\cosh(2j_C) = \sqrt{1 + \sinh^2(2j_C)} = \sqrt{2}.$$

Hence, using  $(1 + \sqrt{2}J\Delta\beta)^{-4} \approx 1 - 4\sqrt{2}J\Delta\beta$ , we obtain for the magnetization:

$$\begin{aligned}
 \frac{M}{N} &= \left(1 - [1 + \sqrt{2}J\Delta\beta]^{-4}\right)^{\frac{1}{8}} \\
 &= \left(1 - 1 + 4\sqrt{2}J\Delta\beta\right)^{\frac{1}{8}} \\
 &= \left(4\sqrt{2}J\Delta\beta\right)^{\frac{1}{8}} \\
 &\propto \left(\frac{1}{T} - \frac{1}{T_C}\right)^{\frac{1}{8}} \\
 \Rightarrow \frac{M}{N} &\propto \varepsilon^{\frac{1}{8}}
 \end{aligned}$$

And the desired critical exponent for the magnetization in the 2d Ising model is  $\beta = 1/8$  in contrast to the 1d solution where we obtained  $\beta = 1/2$ .

## Chapter 3

# Renormalization Group

In this chapter we will have a look at the Ising model from a different point of view. After having presented some ways to analytically approach the solution of the Ising model in both 1d and 2d cases, it will now be our aim to make statements about how the spins in such a system correlate to each other, specifically over longer distances than just nearest-neighbour correlation. To obtain these solutions we will make use of a certain iterative renormalization scheme for the partition function.

The general idea for this scheme will be to trace out the effect of designated sites in our Ising-chain, namely, in the 1d case, by marginalizing the probability density function over all odd-labeled sites in the chain. Repeating this process (and thus, further tracing out the effect of some spin sites) will allow us to gain insight into the behaviour of the Ising model in long-range cases.

### 3.1 1D Ising

We discuss and motivate the basic ideas of the renormalization group (RNG) approach in terms of the 1d Ising model

$$H = -J \sum_{i=1}^N s_i s_{i+1} - h' \sum_i s_i , \quad (3.1)$$

as every step can be performed analytically. We assume  $N = 2^L$ , which allows us to thin out every other spin repeatedly. In addition we assume periodic boundary conditions (pbc), i.e.  $s_{N+1} = s_1$ . For the partition function we need

$$\tilde{H} := -\beta H := K \sum_i s_i s_{i+1} + h \sum_i s_i + N \cdot C , \quad (3.2)$$

where we have added a constant energy per site  $C$ , as it will become relevant in the renormalization scheme. The pdf for a spin configuration  $\{s\}$  is then given by

$$\rho_N(s) = \prod_{i=1}^N e^{K s_i s_{i+1} + \frac{h}{2}(s_i + s_{i+1}) + C} . \quad (3.3)$$

In the B-field term we have expanded the sum over two different spins  $s_i$  and  $s_{i+1}$ . To avoid the double counting of these spins, the factor  $\frac{1}{2}$  is introduced. As we will see, writing the sum like this turns out to be useful later. Next we want to determine the reduced pdf for "even sites only", which is obtained by tracing out the spins on the odd sites. To this end we rewrite  $\rho_N$  as follows (starting again from the form of Equation 3.2)

$$\rho_N(s) = \prod_{i=1}^N e^{K s_i s_{i+1} + h s_i + C} \quad (3.4)$$

$$= \prod_{i=1}^{\text{even}} e^{h s_i + 2C} \prod_{i=1}^{\text{odd}} e^{K(s_{i-1} s_i + s_i s_{i+1}) + h s_i} \quad (3.5)$$

$$= \prod_{i=1}^{\text{even}} e^{h s_i + 2C} \prod_{i=1}^{\text{odd}} e^{(K(s_{i-1} + s_{i+1}) + h) s_i} \quad (3.6)$$

The product sum over the  $h s_i$  is simply split up into the odd and even components. The product sum over  $s_i s_{i+1}$  is carried out only for odd values of  $i$  - in order not to lose any factors, both directions  $s_{i+1}$  and  $s_{i-1}$  are considered in this term. Then we obtain by the marginalization of  $\rho_N$

$$\rho_{\text{even}} = \left( \prod_i^{\text{odd}} \sum_{s_i = \pm 1} \right) \rho_N(s) \quad (3.7)$$

$$= \left( \prod_j^{\text{even}} e^{h s_j + 2C} \right) \left( \prod_i^{\text{odd}} \sum_{s_i = \pm 1} \right) e^{(K(s_{i-1} + s_{i+1}) + h) s_i} . \quad (3.8)$$

This marginal pdf is required, if we want to calculate the spin-spin correlation for spins on even sites only. In particular, it allows to compute the spin-spin correlation

$$\langle s_{2i} s_{2(i+1)} \rangle . \quad (3.9)$$

If we repeat the marginalization once more, we obtain a marginal pdf that allows to compute

$$\langle s_{4i} s_{4(i+1)} \rangle . \quad (3.10)$$



Hence, the repeated marginalization of half of the sites allows to determine how spin spin-correlations behave on different length scales. Now we want to really compute equation (3.7) .

$$\rho_{\text{even}} = \left( \prod_i^{\text{even}} e^{hs_i+2C} \right) \left( \prod_i^{\text{odd}} \sum_{s_i=\pm 1} e^{s_i(K(s_{i-1}+s_{i+1})+h)} \right) \quad (3.11)$$

$$= \left( \prod_i^{\text{even}} e^{hs_i+2C} \right) \left( \prod_i^{\text{odd}} 2 \cosh(K(s_{i-1} + s_{i+1}) + h) \right) \quad (3.12)$$

$$= \prod_i^{\text{even}} e^{hs_i+2C+\ln(2)} \cosh(K(s_i + s_{i+2}) + h) \quad (3.13)$$

$$= \prod_i^{N/2} e^{\frac{h}{2}(s_{2i}+s_{2(i+1)})+2C+\ln(2)} \cosh(K(s_{2i} + s_{2(i+1)}) + h) . \quad (3.14)$$

Next we define  $s_i^{(b)} = s_{bi}$  and find in particular for  $b = 2$

$$\rho_{\text{even}} = \left( \prod_i^{N/2} e^{\frac{h}{2}(s_i^{(2)}+s_{i+1}^{(2)})+2C+\ln(2)} \cosh(K(s_i^{(2)} + s_{i+1}^{(2)}) + h) \right) . \quad (3.15)$$

Finally, we want to express the marginal density formally identically to equation (3.3)

$$\rho_{\text{even}} := \rho_{N/2} = \prod_{i=1}^{N/2} e^{K' s_i^{(2)} s_{i+1}^{(2)} + \frac{h'}{2}(s_i^{(2)}+s_{i+1}^{(2)})+C'} , \quad (3.16)$$

which is possible, since each factor

$$e^{K' s_i^{(2)} s_{i+1}^{(2)} + \frac{h'}{2}(s_i^{(2)}+s_{i+1}^{(2)})+C'} = e^{\frac{h}{2}(s_i^{(2)}+s_{i+1}^{(2)})+2C+\ln(2)} \cosh(K(s_i^{(2)} + s_{i+1}^{(2)}) + h)$$

only depends on  $s_i^{(2)} + s_{i+1}^{(2)}$ , and  $s_i^{(2)} s_{i+1}^{(2)}$ , for which 3 different values are possible each, which defines the 3 parameters  $K', h', C'$ . The corresponding conditions are

$$s_i^{(2)} = s_{i+1}^{(2)} = 1 : \quad e^{K'+h'+C'} = e^{h+2C+\ln(2)} \cosh(h + 2K) \quad (3.17a)$$

$$s_i^{(2)} = s_{i+1}^{(2)} = -1 : \quad e^{K'-h'+C'} = e^{-h+2C+\ln(2)} \cosh(h - 2K) \quad (3.17b)$$

$$s_i^{(2)} = -s_{i+1}^{(2)} : \quad e^{-K'+C'} = e^{2C+\ln(2)} \cosh(h) \quad (3.17c)$$

From equation (3.17c) we obtain

$$e^{C'} = e^{K'+2C+\ln(2)} \cosh(h) \quad (3.18)$$

Insertion in the first two equations yields

$$e^{2K'+h'} \cosh(h) = e^h \cosh(h+2K) \quad (3.19a)$$

$$e^{2K'-h'} \cosh(h) = e^{-h} \cosh(h-2K) . \quad (3.19b)$$

Multiplication of these equations yields

$$e^{4K'} = \frac{\cosh(h+2K) \cosh(h-2K)}{\cosh^2(h)} , \quad (3.20)$$

and division of these equations yields

$$e^{2h'} = e^{2h} \frac{\cosh(h+2K)}{\cosh(h-2K)} . \quad (3.21)$$

equation (3.18) [\[previous page\]](#) can alternatively be written as

$$e^{4C'} = e^{4K'} e^{8C+4\ln(2)} \cosh^4(h) \quad (3.22)$$

Along with equation (3.20) we obtain

$$e^{4C'} = \cosh(h+2K) \cosh(h-2K) e^{8C+4\ln(2)} \cosh^2(h) . \quad (3.23)$$

The equations 3.20, 3.21, and 3.23 uniquely define the values of  $h', K', C'$ , which we now denote as  $h^{(2)}, K^{(2)}, C^{(2)}$ . The key finding is that the reduced density matrix is formally identical to the original one with modified parameters and due to the translational invariance it is actually the same for the even and odd sites. Therefore we denote it simply by  $\rho^{(2)}$ . Now we can repeat this procedure and obtain  $\rho^{(3)}$ , which is the reduced density if only every fourth site is retained. The corresponding parameters  $h^{(3)}, K^{(3)}, C^{(3)}$  are related to the parameters  $h^{(2)}, K^{(2)}, C^{(2)}$  of the previous iteration via equation (3.20), equation (3.21), and equation (3.23). In general we obtain the iteration scheme

$$e^{4K^{(n+1)}} = \frac{\cosh(h^{(n)} + 2K^{(n)}) \cosh(h^{(n)} - 2K^{(n)})}{\cosh^2(h^{(n)})} \quad (3.24a)$$

$$e^{2h^{(n+1)}} = e^{2h^{(n)}} \frac{\cosh(h^{(n)} + 2K^{(n)})}{\cosh(h^{(n)} - 2K^{(n)})} \quad (3.24b)$$

$$e^{4C^{(n+1)}} = \cosh(h^{(n)} + 2K^{(n)}) \cosh(h^{(n)} - 2K^{(n)}) e^{8C^{(n)}+4\ln(2)} \cosh^2(h^{(n)}) \quad (3.24c)$$

The iteration starts with  $K^{(1)} = K, h^{(1)} = h$ , and  $C^{(1)} = C$ . For a first discussion, we consider the case  $h = 0$ , i.e. no external magnetic field. Then the first iteration yields for  $h$

$$e^{2h^{(2)}} = \frac{\cosh(2K^{(1)})}{\cosh(2K^{(1)})} = 1 , \quad (3.25a)$$

i.e.  $h^{(2)} = 0$ . Hence,  $h^{(n)} = 0$  for all iteration steps. For the parameter  $K$  we then obtain the recursion relation

$$e^{4K^{(n+1)}} = \cosh^2(2K^{(n)}) = \frac{1}{4} \left( e^{2K^{(n)}} + e^{-2K^{(n)}} \right)^2 \quad (3.26)$$

$$e^{-4K^{(n+1)}} = \frac{4}{\left( e^{2K^{(n)}} + e^{-2K^{(n)}} \right)^2} = \frac{4e^{-4K^{(n)}}}{\left( 1 + e^{-4K^{(n)}} \right)^2} \quad (3.27)$$

We introduce the definition  $x^{(n)} = e^{-4K^{(n)}}$  for which we obtain

$$x^{(n+1)} = f(x^{(n)})$$

$$f(x) := \frac{4x}{(1+x)^2}$$

The figure illustrates that if we start the recursion with any value  $0 < x^{(1)} \leq$

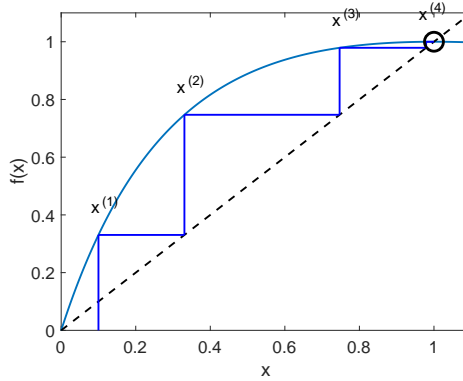


Figure 3.1: **Recursion relation for  $x$  in the 1d Ising model. Starting with an arbitrary  $x^{(1)} \in (0, 1]$  the fixed point is  $x^{(\infty)} = 1$ . Only for  $x^{(1)} = 0$  the other fixed point  $x^{(\infty)} = 0$  is relevant.**

1, i.e.  $0 \leq K < \infty$ , which corresponds to the parameter of the original physical system, the iteration ends at the fixed point  $x^{(\infty)} = 1$ . This means that the physical features (e.g. spin-spin correlation) for very long distances is equivalent to that of an Ising model with  $x = 1$ , which corresponds to  $K = 0$ . This fixed point is called *high temperature fixed point*, as  $x = 1$ , or rather  $K = 0$  is also obtained for  $T \rightarrow \infty$  ( $\beta \rightarrow 0$ ). The 1d Ising model considered at very long length scales looks like an infinite temperature or non-interacting solution, which means it is disordered (no long range order). This fixed point is stable or attractive.

The other fixed point  $x^* = 0$  is the non-trivial *critical fixed point*. It corresponds to  $K = \infty$ , i.e.  $T = 0$ . In this case, starting with the physical parameter  $x = 0$ , i.e.  $T = 0$ , the system stays at  $x = 0$  and is therefore ordered even over long range. This is the trivial case: At  $T \rightarrow 0$  there are no thermal fluctuations, since there are no quantum fluctuations. Accordingly, the spins of such a 1-dimensional Ising will be ordered no matter how far the considered spins lie apart from each other. This result is in contrast to the case of the spin-1/2 Heisenberg model.

### Energy Parameter

Next we study the energy parameter  $C$  for the case  $h = 0$ . Then we have

$$C^{(n+1)} = 2C^{(n)} + \frac{1}{2} \ln \left( \cosh(2K^{(n)}) \right) + \ln(2) . \quad (3.28)$$

As the recursion for  $K^{(n)}$  is independent of that for  $C^{(n)}$ , we can insert the previous result for  $K^{(n)}$ . Hence for  $n \rightarrow \infty$  (very long length scale), we have  $K^{(n)} \rightarrow 0$  (or  $x \rightarrow 1$  in Figure 3.1) and we find

$$C^{(n+1)} = 2C^{(n)} + \ln(2) .$$

The factor 2 is obvious due to the decimation of the number of spins by the factor 2.

### Correlation length

Next we want to discuss the correlation length, For that we first introduce a different recursion relation for  $K^{(n)}$  (still for the case  $h = 0$ ). From equation (3.26) [\[previous page\]](#) we obtain

$$K^{(n+1)} = g(K^n) \quad (3.29)$$

$$g(K) = \frac{1}{2} \ln \left( \cosh(2K) \right) , \quad (3.30)$$

which for large  $K^{(n)}$  can be rewritten as follows

$$\begin{aligned} K^{(n+1)} &= \frac{1}{2} \ln \left( \frac{e^{2K^{(n)}}(1 + e^{-4K^{(n)}})}{2} \right) \\ &= K^{(n)} - \frac{1}{2} \ln(2) + \frac{1}{2} \ln(1 + e^{-4K^{(n)}}) . \end{aligned}$$

Hence we have for large  $K^{(n)}$

$$K^{(n+1)} \approx K^{(n)} - \frac{\ln(2)}{2} . \quad (3.31)$$

If this relation applies, then

$$\begin{aligned}
K^{(n)} &\approx K^{(n-1)} - \frac{\ln(2)}{2} \\
&\approx K^{(n-2)} - \frac{\ln(2)}{2} - \frac{\ln(2)}{2} \\
&\dots \\
K^{(n)} &\approx K^{(0)} - \frac{n}{2} \cdot \ln(2) .
\end{aligned} \tag{3.32}$$

Note that here we begin the iteration counter with index 0. Now we want to use the renormalization group approach to study how the correlation length depends on  $\beta$  or rather  $K$ . If there is no long range order, the spin-spin correlation decreases exponentially as

$$\langle s_0 s_m \rangle \propto e^{-m/\xi(K)} .$$

The correlation length  $\xi(K)$  will depend on  $\beta J = K$ . In each renormalization step, the unit cell increases by a factor of 2. In the original system we have

$$\langle s_0 s_{\textcolor{red}{2}m} \rangle_{K^{(0)}} \propto e^{-(\textcolor{red}{2}m)/\xi(K^{(0)})} .$$

The same correlation function is equal to that of the renormalized system as follows

$$\langle s_0 s_{2m} \rangle_{K^{(0)}} = \langle s_0^{(2)} s_m^{(2)} \rangle_{K^{(1)}} \propto e^{-m/\xi(K^{(1)})} .$$

Comparing the exponentials yields

$$\xi(K^{(0)}) = 2\xi(K^{(1)}) = 2\xi(g(K^{(0)})) .$$

Repeating the renormalization  $m$  times we obtain the relation

$$\xi(K^{(0)}) = 2^m \xi(g^{(m)}(K^{(0)})) . \tag{3.33}$$

The left hand side is the quantity we are interested in for low temperature or rather  $K \gg 1$ . In this case, according to equation (3.32), after  $m$  RNG steps we have

$$g^{(m)}(K^{(0)}) = K^{(m)} \approx K^{(0)} - \frac{m}{2} \ln(2) .$$

Remember we are interested in  $K \gg 1$ . Each iteration reduces  $K$  by  $\ln(2)/2 \approx 0.35$ . Then if we choose  $m$  sufficiently large, eventually, we will

reach an iteration  $m^*$  with  $K^{(m^*)} = g^{(m^*)}(K^{(0)}) \sim O(1)$ . The required number of steps is

$$\begin{aligned} K^{(0)} - \frac{m^*}{2} \ln(2) &= \mathcal{O}(1) \\ m^* &= \frac{2(K^{(0)} - \mathcal{O}(1))}{\ln(2)} \\ m^* &\approx \frac{2K^{(0)}}{\ln(2)} . \end{aligned}$$

Along with equation (3.33) [\[previous page\]](#) we obtain for  $m = m^*$  RNG iterations (from now on we omit the upper index 0)

$$\xi(K) = 2^{m^*} \underbrace{\xi(g^{(m^*)}(K))}_{=\mathcal{O}(1)} .$$

Now  $g(K = \mathcal{O}(1))$  is some unimportant constant  $C$  and we finally have

$$\xi(K) \sim 2^{m^*} = 2^{\frac{2K}{\ln(2)}} = e^{\ln\left(2^{\frac{2K}{\ln(2)}}\right)} = e^{2K} .$$

CORRELATION LENGTH	
$\xi(K) \sim e^{2K} .$	(3.34)

So the bottom line is that the correlation length increases exponentially with decreasing temperature and becomes infinite at  $T = 0$ , but it is finite for finite  $T$  in agreement to the previous finding that there is no long range order for  $T > 0$ .

### Free Energy

Finally, we compute the free energy for the case  $h = 0$ . We start out from the partition function for the original system and integrate out the spins on the odd sites and use equation (3.16) [\[p. 80\]](#) .

$$Z_N(K, C) = \text{tr}\{e^{-\beta H}\} = \text{tr}\left\{e^{K \sum_{\langle ij \rangle} s_i s_j + NC}\right\} = Z_{N/2}(K', C') ,$$

with  $K'$  given in equation (3.29) [p. 83] and  $C'$  given in equation (3.28) [p. 83]

$$K^{(1)} = \frac{1}{2} \ln \left( \cosh(2K^{(0)}) \right) \quad (3.35)$$

$$C^{(1)} = 2C^{(0)} + K^{(1)} + \ln(2) . \quad (3.36)$$

Next, we express the partition function slightly differently by prepending  $C$

$$Z_N(K, C) = e^{NC^{(0)}} Z_N^{(0)}(K^{(0)}) = e^{\frac{N}{2}C^{(1)}} Z_{N/2}^{(0)}(K^{(1)}) . \quad (3.37)$$

Taking the logarithm per lattice site (apart from a factor  $(-k_B T)$  this is the free energy per lattice site)

$$\begin{aligned} f(K^{(0)}, C^{(0)}) &= \frac{\ln(Z_N(K^{(0)}, C^{(0)}))}{N} = C^{(0)} + \underbrace{\frac{\ln(Z_N^{(0)}(K^{(0)}))}{N}}_{:=\tilde{f}(K^{(0)})} \\ f(K^{(0)}, C^{(0)}) &= C^{(0)} + \tilde{f}(K^{(0)}) . \end{aligned} \quad (3.38)$$

On the other hand, based on the second equation in equation (3.37) we obtain

$$\begin{aligned} f(K^{(0)}, C^{(0)}) &= \frac{1}{N} \left( \frac{N}{2} C^{(1)} + \ln(Z_{N/2}^{(0)}(K^{(1)})) \right) \\ &= \frac{1}{2} \left( C^{(1)} + \frac{\ln(Z_{N/2}^{(0)}(K^{(1)}))}{N/2} \right) \\ f(K^{(0)}, C^{(0)}) &= \frac{1}{2} \left( C^{(1)} + \tilde{f}(K^{(1)}) \right) . \end{aligned}$$

Comparison with equation (3.38) gives

$$\tilde{f}(K^{(0)}) = \frac{1}{2} \left( C^{(1)} - 2C^{(0)} + \tilde{f}(K^{(1)}) \right) .$$

Inserting equation (3.36) finally yields

$$\tilde{f}(K^{(0)}) = \frac{1}{2} \left( K^{(1)} + \ln(2) + \tilde{f}(K^{(1)}) \right) \quad (3.39)$$

From a second iteration we get

$$\tilde{f}(K^{(1)}) = \frac{1}{2} \left( K^{(2)} + \ln(2) + \tilde{f}(K^{(2)}) \right) .$$

Inserting into equation (3.39) yields

$$\begin{aligned}\tilde{f}(K^{(0)}) &= \frac{1}{2} \left( K^{(1)} + \ln(2) + \frac{1}{2} \left[ K^{(2)} + \ln(2) + \tilde{f}(K^{(2)}) \right] \right) \\ &= \frac{K^{(1)} + \ln(2)}{2^1} + \frac{K^{(2)} + \ln(2)}{2^2} + \frac{\tilde{f}(K^{(2)})}{2^2} .\end{aligned}$$

Clearly, this leads after  $m$  iterations to

$$\begin{aligned}\tilde{f}(K^{(0)}) &= \sum_{\nu=1}^m \frac{\ln(2)}{2^\nu} + \sum_{\nu=1}^m \frac{K^{(\nu)}}{2^\nu} + \frac{\tilde{f}(K^{(m)})}{2^m} \\ &= \ln(2) \sum_{\nu=1}^m \frac{1}{2^\nu} + \sum_{\nu=1}^m \frac{K^{(\nu)}}{2^\nu} + \frac{\tilde{f}(K^{(m)})}{2^m} \\ &= \ln(2) \sum_{\nu=1}^m \frac{1}{2^\nu} + \sum_{\nu=1}^m \frac{K^{(\nu)}}{2^\nu} + \frac{\tilde{f}(K^{(m)})}{2^m} .\end{aligned}$$

We have seen, that  $K^{(n)} \rightarrow 0$  for  $n \rightarrow \infty$ . The partition function for  $K \rightarrow 0$  can be obtained analytically as

$$Z_N(K \rightarrow 0) = 2^N ,$$

hence, according to the definition  $\tilde{f} = \ln(Z)/N$ , we obtain

$$\tilde{f}(K \rightarrow 0) = \frac{\ln(Z_N)(K \rightarrow 0)}{N} = \ln(2)$$

So, if we perform an infinite number of renormalization steps, we obtain

$$\begin{aligned}\tilde{f}(K^{(0)}) &= \ln(2) \underbrace{\left( \sum_{\nu=0}^{\infty} \left( \frac{1}{2} \right)^\nu - 1 \right)}_{=1} + \sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{2^\nu} + \underbrace{\lim_{L \rightarrow \infty} \frac{\ln(2)}{2^L}}_{=0} \\ &= \ln(2) + \sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{2^\nu} .\end{aligned}$$

We have seen in chapter 1 that the exact result is given by

$$\begin{aligned}Z_N &= d_1^N \\ \tilde{f}(K) &= \ln(d_1) \\ d_1 &= 2 \cosh(K^0) ,\end{aligned}$$

Hence, since  $C^{(0)} = 0$

$$f(K^{(0)}, C^{(0)}) = \tilde{f}(K^{(0)}) = \ln(2) + \ln(\cosh(K^{(0)})) .$$



Numerical comparison shows that both results agree, i.e.

$$\sum_{\nu=1}^{\infty} \frac{K^{(\nu)}}{2^{\nu}} = \ln(\cosh(K^{(0)}))$$

with

$$K^{(n)} = \frac{1}{2} \ln \left( \cosh(2K^{(n-1)}) \right).$$

In the 1D case, no approximations were necessary for the RNG procedure.

## 3.2 2D Ising

We decompose the sc-lattice into an A-B-lattice, i.e. sites belong either to sub-lattice A (blue circles with even indices) or B (red crossed with odd indices). All nearest neighbors of a point of sub-lattice A belong to sub-lattice B and vice versa.

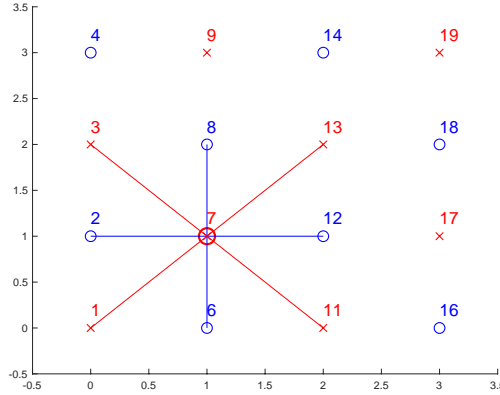


Figure 3.2: **Illustration of the RG scheme for 2D Ising.**

The goal is again to decimate the lattice by integrating out sublattice B, i.e. to compute the reduced density matrix for sub-lattice A. The reduced density matrix on sub-lattice A is

$$\rho_A = \sum_{\{S_i\} \in B} e^{-\beta H}$$

To get an idea of one renormalization step, we consider all terms of the hamiltonian that contain the spin on site 7, which is part of sub-lattice B.

$$\begin{aligned} -\beta H_7 &= C + hS_7 + K\mathcal{S}_7^{\text{nn}}S_7 \\ \mathcal{S}_7^{\text{nn}} &= S_2 + S_6 + S_8 + S_{12} . \end{aligned}$$

Next we separate the contribution of this spin from the rest

$$\begin{aligned} \rho_A &= \sum_{\{S_i\} \in B/S_7} e^{-\beta H'} \cdot Z_7 \\ Z_7 &= \sum_{S_7} e^{-\beta H_7} \end{aligned}$$

with  $H' = H - H_7$  being the hamiltonian without site 7. The trace over  $S_7$  yields

$$Z_7 := \sum_{S_7} e^{C+S_7(K\mathcal{S}_7^{\text{nn}}+h)} = e^C \cdot 2 \cosh(K\mathcal{S}_7^{\text{nn}} + h) . \quad (3.40)$$

We want to express this function again as

$$e^{-\beta \tilde{H}} .$$

The most general exponential form for this expression, since  $S_i^2 = 1$ , is given by

$$Z_7 = \exp \left( C' + h'\mathcal{S}_7^{\text{nn}} + \overline{K} \sum'_{ij} S_i S_j + b \sum'_{ijk} S_i S_j S_k + c S_2 S_6 S_8 S_{12} \right) . \quad (3.41)$$

The indices in the sums are from the set  $I_7^{\text{nn}} = \{2, 6, 8, 12\}$ , and the prime symbols over the summation sign indicate that no indices must occur twice and all the indices shall be ordered, e.g.  $S_i < S_j < S_k$ . We have used the symmetry that equation (3.40) is invariant against interchange of any two indices  $i \leftrightarrow j$ . Otherwise, each product of spins would have its own prefactor.

In the double sum there are also the products  $S_6 S_8$  and  $S_2 S_{12}$ , which belong to next-nearest neighbour sites on the decimated lattice, which only consists of sub-lattice A (blue circles). I.e. starting from nn coupling the decimation also introduces nnn coupling and also terms with three and four spins. The latter will turn out negligible but the nnn coupling is relevant. To obtain

the parameter mapping in an RG step, we include the nnn terms also in the original hamiltonian, i.e.

$$\begin{aligned} -\beta H_7 &= C + S_7(h + K\mathcal{S}_7^{\text{nn}} + L\mathcal{S}_7^{\text{nnn}}) \\ \mathcal{S}_7^{\text{nn}} &= S_2 + S_6 + S_8 + S_{12} \\ \mathcal{S}_7^{\text{nnn}} &= S_1 + S_3 + S_{11} + S_{13} . \end{aligned}$$

Again, we separate the contribution of this spin from the rest

$$\begin{aligned} \rho_A &= \sum_{\{S_i\} \in B/S_7} e^{-\beta H'} \cdot Z_7 \\ Z_7 &= \sum_{S_7} e^{-\beta H_7} \end{aligned}$$

The trace over  $S_7$  now yields

$$Z_7 = e^C \cdot 2 \cosh(K\mathcal{S}_7^{\text{nn}} + L\mathcal{S}_7^{\text{nnn}} + h) . \quad (3.42)$$

Here we have a problem, since  $\mathcal{S}_7^{\text{nnn}}$  contains spins of sublattice B (e.g.  $S_{13}$ ) that still need to be integrated out and that are also contained in the residual hamiltonian  $H'$ . In this combination the sum over  $S_{13}$  is not as trivial as that over  $S_7$ , which we had before. Moreover, the complexity increases, as  $Z_7$  also contains other spins of sub-lattice  $B$  ( $S_1$  say) and after the trace over  $S_{13}$  the expression containing  $S_1$  is getting even more complicated, etc.. Hence, to keep things manageable, we replace  $\mathcal{S}_7^{\text{nnn}}$  at this point by its mean value, which is zero. Then we are left with

$$Z_7 = e^C \cdot 2 \cosh(K\mathcal{S}_7^{\text{nn}} + h) . \quad (3.43)$$

As we will see soon, that does not mean that the original nnn coupling has no influence at all. We can use the general exponential from of equation (3.41) [previous page] again

$$Z_7 = \exp \left( C' + h'\mathcal{S} + \overline{K} \sum_{ij} S_i S_j + b \sum_{ijk} S_i S_j S_k + c S_2 S_6 S_8 S_{12} \right) .$$

As before, the indices in the sums are from the set  $I_7^{\text{nn}} = \{2, 6, 8, 12\}$ . The term with 3 spins can equivalently be written as

$$\begin{aligned} \sum_{ij} S_i S_j S_k &= \mathcal{S} \cdot S_2 S_6 S_8 S_{12} \\ \mathcal{S} &= (S_{12} + S_8 + S_6 + S_2) \end{aligned}$$

---

*proof*

$$\begin{aligned}
\sum_{ij}^I S_i S_j S_k &= S_2 S_6 S_8 + S_2 S_6 S_{12} + S_2 S_8 S_{12} + S_6 S_8 S_{12} \\
S_i^2 = 1 &\Rightarrow S_2 S_6 S_8 S_{12}^2 + S_2 S_6 S_{12} S_8^2 + S_2 S_8 S_{12} S_6^2 + S_6 S_8 S_{12} S_2^2 \\
&= S_2 S_6 S_8 S_{12} (S_{12} + S_8 + S_6 + S_2)
\end{aligned}$$

---

q.e.d. ✓

So we have the constraint

$$C + \ln \left( 2 \cosh [K\mathcal{S} + h] \right) = C' + h'\mathcal{S} + \bar{K} \sum_{ij}^I S_i S_j + (b\mathcal{S} + c) S_1 S_2 S_3 S_4. \quad (3.44)$$

This allows us to establish a connection between the parameters  $h, h', K$  and  $\bar{K}$  and thus, we know how the parameters in our renormalization scheme are related to each other. It is now our goal to obtain the recursion formula for the parameters  $h'$  and  $\bar{K}$  by specifically looking at all possible spin configurations and hence, find a suitable renormalization scheme:

$$\begin{aligned}
1) \quad (+ + + +) : \quad & C + \ln \left( 2 \cosh [4K + h] \right) = C' + 4h' + 6\bar{K} + (4b + c) \cdot 1 \\
2) \quad (+ + + -) : \quad & C + \ln \left( 2 \cosh [2K + h] \right) = C' + 2h' + 0\bar{K} + (2b + c) \cdot (-1) \\
3) \quad (+ + - -) : \quad & C + \ln \left( 2 \cosh [0K + h] \right) = C' + 0h' - 2\bar{K} + (0b + c) \cdot 1
\end{aligned}$$

Since the geometric position of the spins does not enter in equation (3.44) we get the same equations if we permute the spins.

The remaining spin configurations are obtained by the transformation  $S_i \rightarrow -S_i$ , which changes the sign in the terms with an odd number of spins. I.e.

3) is invariant, and for the other two we obtain

$$\begin{aligned} 1') \quad (- - - -) : \quad C + \ln \left( 2 \cosh [-4K + h] \right) &= C' - 4h' + 6\bar{K} + (-4b + c) \cdot 1 \\ 2') \quad (- - - +) : \quad C + \ln \left( 2 \cosh [-2K + h] \right) &= C' - 2h' + 0\bar{K} + (-2b + c) \cdot (-1) \end{aligned}$$

We simplify these equations a bit further

$$\begin{aligned} 1) \quad (+ + + +) : \quad \ln \left( 2 \cosh [4K + h] \right) &= \Delta C + 4h' + 6\bar{K} + 4b + c \\ 1') \quad (- - - -) : \quad \ln \left( 2 \cosh [-4K + h] \right) &= \Delta C - 4h' + 6\bar{K} - 4b + c \\ 2) \quad (+ + + -) : \quad \ln \left( 2 \cosh [2K + h] \right) &= \Delta C + 2h' - 2b - c \\ 2') \quad (- - - +) : \quad \ln \left( 2 \cosh [-2K + h] \right) &= \Delta C - 2h' + 2b - c \\ 3) \quad (+ + - -) : \quad \ln \left( 2 \cosh [h] \right) &= \Delta C - 2\bar{K} + c, \end{aligned}$$

with  $\Delta C = C' - C$ .

When we add or subtract equation 1 and 1' and likewise for 2 and 2', we obtain

$$\begin{aligned} a) : \quad \ln \left( 4 \cosh [4K + h] \cosh [4K - h] \right) &= 2 \left( \Delta C + 6\bar{K} + c \right) \\ a') : \quad \ln \left( \cosh [4K + h] / \cosh [4K - h] \right) &= 8(h' + b) \\ b) : \quad 2 \ln \left( 4 \cosh [2K + h] \cosh [2K - h] \right) &= 4 \left( \Delta C - c \right) \\ b') : \quad 2 \ln \left( \cosh [2K + h] / \cosh [2K - h] \right) &= 8(h' - b). \end{aligned}$$

Equation 3) yields

$$c) : \quad \Delta C + c = \ln \left( 2 \cosh [h] \right) + 2\bar{K}$$

Inserting c) into a) gives

$$\begin{aligned} \tilde{a}) : \quad \ln \left( 4 \cosh [4K + h] \cosh [4K - h] \right) &= 2 \ln \left( 2 \cosh [h] \right) + 16\bar{K} \\ 16\bar{K} &= \ln \left( \frac{\cosh [4K + h] \cosh [4K - h]}{\cosh^2 [h]} \right) \end{aligned}$$

We see that  $\overline{K}$  only depends on  $K$  and  $h$ . If we add a') and b') we obtain

$$16h' = \ln \left( \frac{\cosh [4K + h] \cosh^2 [2K + h]}{\cosh [4K - h] \cosh^2 [2K - h]} \right).$$

Also  $h'$  only depends on  $h$  and  $K$ . Especially for  $h = 0$ , we find  $h' = 0$ . So the renormalization does not introduce a B-field, when we start with  $B = 0$ . In the following we will consider only the case  $h = 0$ . Then we obtain

$$\overline{K} = \frac{1}{16} \ln(\cosh^2[4K]) = \frac{1}{8} \ln(\cosh[4K]). \quad (3.45)$$

For the determination for the other parameters for  $h = 0$  we have to solve

$$b) \Rightarrow \bar{b} \quad \ln \left( 2 \cosh [2K] \right) = \Delta C - c \quad (\bar{b})$$

$$b') \Rightarrow \quad 0 = 8(h' - b) \quad (\bar{b}')$$

$$c) \Rightarrow \bar{c} \quad \ln(2) + 2\overline{K} = \Delta C + c \quad (\bar{c})$$

We know already that  $h = 0$  also means  $h' = 0$  and according to  $\bar{b}'$  this implies  $b = 0$ . I.e. the three-spin coupling vanishes, which is reasonable, because without an external B-field no spin-direction is favoured, i.e. the hamiltonian is invariant under the reversion of all spins. This symmetry would be violated by the three-spin coupling. Finally,  $\bar{c} - \bar{b}$  yields

$$c = \overline{K} - \frac{1}{2} \ln(\cosh(2K)) \quad (3.46)$$

In addition, from  $\bar{c} + \bar{b}$  we obtain

$$\Delta C = \frac{1}{2} (2 \ln(2) + \ln(\cosh(2K)) + 2\overline{K}).$$

Along with equation (3.45) we obtain

$$\Delta C = \ln(2) + \frac{1}{2} \ln(\cosh(2K)) + \frac{1}{8} \ln(\cosh(4K)) \quad (3.47)$$

$$\Delta C = \ln \left( 2 \cosh^{\frac{1}{2}}(2K) \cosh^{\frac{1}{8}}[4K] \right). \quad (3.48)$$

**First approximation, ignoring nnn interaction**

In equation (3.44) [p. 91] we have the new two-spins interaction term in the decimated lattice, which is given by

$$E_7(S) := \overline{K} \sum'_{ij} S_i S_j = \overline{K} \underbrace{(S_2 S_6 + S_6 S_{12} + S_{12} S_8 + S_8 S_2)}_{=\text{nn terms in dec. lattice}} + \dots \quad (3.49)$$

$$+ \overline{K} \underbrace{(S_2 S_{12} + S_6 S_8)}_{=\text{nnn terms in dec. lattice}} \quad (3.50)$$

This is the interaction that comes from the sum over  $S_7$ . Among others, there is a coupling between  $S_8$  and  $S_{12}$ . Integrating out  $S_{13}$  also mediates a coupling between these spins. These contributions add up. Therefore, we have for the new nn coupling between site 8 and 12

$$K' = 2\overline{K} .$$

Moreover, in addition to the nn coupling we also have a nnn coupling in the decimated lattice. Both couplings support the same (ferromagnetic) orientation. As a first approximation, we want to omit these terms and use a modified nn coupling instead.

We consider the isolated cluster with sites 2, 6, 8, 12 with  $S_7$  being integrated out. For a perfect ferromagnetic alignment (all spins in the cluster equal +1) we have for the energy of that cluster if we include the nnn term

$$E_7(S) = 6\overline{K} .$$

Omitting the nnn term and using  $\tilde{K}$  for the nn coupling instead, we find

$$\tilde{E}_7(S) = 4\tilde{K} .$$

These energies shall be the same and we therefore use

$$\tilde{K} = \frac{3}{2}\overline{K} .$$

As mentioned before, this coupling gets a factor two by the integration of the neighbouring spins in the B sub-lattice. Hence in this approximation we have

$$K' = 3\overline{K} = \frac{3}{8} \ln \left( \cosh(4K) \right) . \quad (3.51)$$

In figure 3.3 the mapping of the coupling parameter  $K$  during the renormalization steps is depicted. We find three fixed points,  $K_1^* = 0$ ,  $K_2^* = \infty$ , and

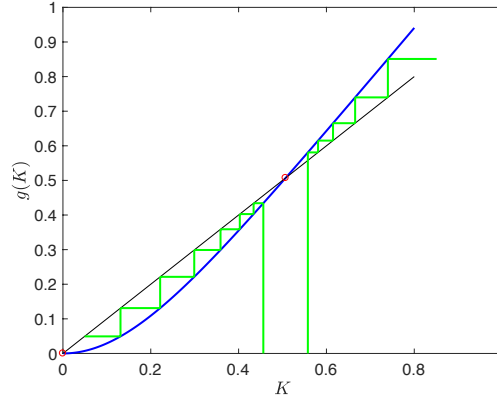


Figure 3.3: **Flux of the renormalization iterations in the first approximation described by equation (3.51).**

$K_3^* = 0.507$ . The first two are the same as in the 1D case, **but with the important difference that now both fixed points are stable, while in the 1D case the low-temperature fixed point  $K^* = \infty$  was unstable.** The  $K^* = 0$  fixed point is trivial and corresponds to  $T = \infty$  where the correlations between neighboring spins vanish. The other fixed point ( $K^* = \infty$ ) is a so-called critical fixed point as it describes the situation that the correlation length diverges for  $T \rightarrow 0$  but there are nonetheless no fluctuations since all spins are perfectly aligned. Only the third fixed point describes a true **critical point**. This one is unstable. If we start with a physical parameter  $K > K_3^*$  the interaction leads to  $K = \infty$ , while if we start with  $K < K_3^*$  the iterations lead to  $K = 0$ . Only starting precisely with  $K = K_3^*$ , the value of  $K$  does not change. So  $K_3^* = \frac{J}{k_B T^*}$  corresponds to the phase transition. For  $T < T^*$  ( $K > K_3^*$ ) the fixed point is  $K \rightarrow \infty$ , i.e.  $T = 0$  and hence long range order and in the other case the system behaves like  $T = \infty$ .

The value  $K_3^* = 0.507$  has to be compared with the exact result

$$\beta^* J = 0.4407 .$$

We have ignored so far the four-spin term. It has the coupling parameter  $c$ , which is given in equation (3.52) . According to equation (3.51) [\[previous page\]](#) we have

$$\overline{K} = \frac{K'}{3}$$

As the recursion equation for  $K$  is independent of the other parameters, we can replace  $K$  by its fixed point values  $K^*$  and we obtain

$$c = \frac{K^*}{3} - \frac{1}{2} \ln(\cosh(2K^*)) = -0.053. \quad (3.52)$$



The result confirms that the four-spin coupling is much smaller than the two-spin terms and can, therefore, be either ignored or taken into account perturbationally.

### 3.3 Scaling and critical exponents

We originally introduced the renormalization group formalism to inspect the physics that is happening at long-range distances. So now we will now try and obtain both the correlation length of the spins and a corresponding critical exponent like we did before for the 1-dimensional Ising model RNG.

To illustrate this idea we consider again the first approximation (ignoring the nnn iterations). The iteration equation was given in equation (3.51) [p. 94] by

$$K' = \frac{3}{8} \ln(\cosh(4K)) := R(K) :$$

We consider recursion steps in the vicinity of the fixed point. We start out from the recursion relation for  $K = K^* + \delta K$

$$\begin{aligned} K' = R(K) &= R(K^*) + \underbrace{\frac{d}{dK} R(K)}_{=\lambda} \Big|_{K=K^*} \delta K \\ \Rightarrow \quad \delta K' &= \lambda \delta K . \end{aligned}$$

The derivative yields

$$\lambda = \frac{3}{2} \tanh(4K^*) = 1.4489 . \quad (3.53)$$

In the present decimation scheme the length is scaled by  $b = \sqrt{2}$  since, if we remove all sites from sublattice B, the remaining sites from sublattice A lie diagonally in the original lattice. Obviously, the length scaling for the diagonal is  $b = \sqrt{2}$ . If the further trace out spin sites, this length scaling paramter  $b$  changes again. Accordingly, also  $\lambda$  will change, i.e.

$$\lambda = \lambda(b) .$$

When we perform 2 successive decimation steps we have

$$\delta K'' = \lambda(b) \delta K' = (\lambda(b))^2 \delta K .$$

On the other hand, we could have performed the two decimation steps as one combined step of a new decimation scheme with  $b' = b \cdot b$ , then

$$\begin{aligned} \delta K'' &= \lambda(b') \delta K = \lambda(b^2) \delta K , \\ \Rightarrow \quad \lambda(b^2) &= \lambda^2(b) \end{aligned} \quad (3.54)$$

The last equation requires

$$\lambda(b) = b^y ,$$

where  $y$  is a parameter that depends on the underlying physical system but not on  $b$ ; otherwise equation (3.54) [\[previous page\]](#) would not hold. For the present decimation scheme  $b = \sqrt{2}$  and  $\lambda = 1.4489$  (see equation (3.53) [\[previous page\]](#)), resulting in

$$y = \frac{\ln(\lambda)}{\ln(\sqrt{2})} = 1.070 . \quad (3.55)$$

### Correlation length

Since we now know how the scaling factor  $\lambda$  in our series expansion behaves depending on  $b$ , we can use this to gain insight about the correlation length. We have:

$$\delta K = K - K_c = \frac{J}{k_B} \left( \frac{1}{T} - \frac{1}{T_c} \right) = \frac{J}{k_B T} \left( \frac{T_c - T}{T_c} \right) = K \varepsilon \approx K^* \varepsilon .$$

Where in the last step, the  $K^*$  can be used for  $K$  since we performed an expansion of  $K$  around the fixed point. From the relation

$$\delta K' = b^y \delta K$$

we obtain (again for  $K \approx K^*$ )

$$K^* \varepsilon' = b^y K^* \varepsilon \Rightarrow \varepsilon' = b^y \varepsilon . \quad (3.56)$$

Moreover, close to  $T_C$  we have generally

$$\frac{\xi}{a} \sim \varepsilon^{-\nu}$$

and in the decimated system similarly

$$\frac{\xi'}{a'} \sim \varepsilon'^{-\nu} ,$$

where  $a$  and  $a'$  is the lattice constant in the corresponding system. Then equation (3.56) yields

$$\begin{aligned} \frac{\xi'}{a'} &\sim \varepsilon'^{-\nu} = b^{-\nu y} \varepsilon^{-\nu} \sim b^{-\nu y} \frac{\xi}{a} \\ \Rightarrow \quad \xi' &= \underbrace{\frac{a'}{a}}_{=b(\text{def.})} b^{-\nu y} \xi \\ \xi' &= b^{1-\nu y} \xi . \end{aligned}$$

Where the definition of the length-scaling factor  $b$  that was discussed before was used:  $b$  gives the changing of the lattice constant throughout one step of lattice decimation.

Again approaching the fixed point, the correlation length is converged and does not change. I.e.

$$\xi^* = b^{1-\nu y} \xi^* .$$

Hence,

$$\begin{aligned} \nu y &= 1 \\ \nu &= \frac{1}{y} = 0.9346 . \end{aligned}$$

Compared to the exact result ( $\nu = 1$ ) for the critical exponent the RG result is not so bad.

### Free energy and specific heat (a bit tricky, will be discussed later)

Next we want to compute the critical exponent of the specific heat from the free energy

$$C_V = -T \left. \frac{\partial F}{\partial T} \right|_V . \quad (3.57)$$

For the critical exponent which is introduced in equation (1.63) [\[p. 28\]](#) as

$$C_V \simeq |\varepsilon|^{-\alpha}$$

we only need the close neighbourhood of the phase transition. Hence the factor  $T$  in equation (3.57) can be replaced by  $T_C$ . Moreover, we have

$$F = -k_B T N \underbrace{\frac{\ln(Z(K, N))}{N}}_{:=f(K, N)} .$$

Also here the prefactor  $T$  can be replaced by  $T_C$  as it does not modify the critical behaviour. So we have

$$C_V \sim \frac{\partial^2 f}{\partial T^2} \sim |\varepsilon|^{-\alpha} .$$

We recall

$$\varepsilon = \frac{T - T_C}{T_C} ,$$

which yields  $f(\varepsilon, N)$  has to have the following behaviour near the phase transition

$$\begin{aligned} \frac{\partial^2}{\partial f^2}(\varepsilon, N) &\sim |\varepsilon|^{-\alpha} \\ \Rightarrow f(\varepsilon, N) &\sim |\varepsilon|^{2-\alpha} . \end{aligned} \quad (3.58)$$

The integration would also yield a contribution  $c + d \varepsilon$ , which is, however, irrelevant for the critical behaviour.

Next we need to compute  $f$ . We start out with the partition function in the physical parameters

$$Z(K, N) = \sum_{\{S_i\}_{i=1}^N} e^{K \sum_{\langle ij \rangle} S_i S_j} .$$

After one decimation we have

$$Z(K, N) = e^{\Delta a N} \sum_{\{S_i\}_{i=1}^{N/2}} e^{K' \sum_{\langle ij \rangle} S_i S_j} = e^{\Delta a N} Z(K', N/2) .$$

For the quantity  $f(K) := \ln(Z(K, N))/N$  we obtain

$$\begin{aligned} f(K, N) &= \frac{\ln(Z(K, N))}{N} \\ &= \frac{\Delta a N + \ln(Z(K', N/2))}{N} \\ &= \Delta a + \frac{1}{2} \underbrace{\frac{\ln(Z(K', N/2))}{N/2}}_{=f(K', N/2)} \end{aligned}$$

Since  $F = -k_B T \ln(Z)$  is an extensive quantity,  $f(K, N)$  is independent of  $N$ . So we have

$$\begin{aligned} f(K) &= \Delta a + \frac{1}{2} f(K') \\ f(K') &= 2f(K) - \Delta a \end{aligned}$$

Inserting equation (3.47) [p. 93] yields

$$f(K') = 2f(K) - \ln\left(2 \cosh^{\frac{1}{2}}(2K) \cosh^{\frac{1}{8}}[4K]\right) . \quad (3.59)$$

The second term is analytic and cannot lead to a critical behaviour. The only relevant, singular part  $f_s$  behaves like

$$f_s(K') = 2f_s(K) = b^d f_s(K) .$$

The scaling factor  $b = \sqrt{2}$  and the spatial dimension is  $d = 2$ . Or rather

$$f_s(K) = b^{-d} f_s(K') .$$

Close to the phase transition we have

$$\begin{aligned} K &= K^* + \delta K \\ K' &= R(K^*) + \underbrace{\left. \frac{dR(K)}{dK} \right|_{K=K^*}}_{=\lambda} \delta K \\ \delta K' &= \lambda \delta K \end{aligned}$$

As discussed before we have  $\lambda = b^{-y}$  and hence

$$\delta K' = b^y \delta K = b^y K^* \varepsilon$$

So we have

$$\begin{aligned} f_s(K) &= b^{-d} f_s(K^* + b^y K^* \varepsilon) \\ &= b^{-d} f_s(K^* [1 + b^y \varepsilon]) . \end{aligned}$$

This relation has to be valid for any decimation scheme, or rather  $b$ ; so we can choose

$$b = |\varepsilon|^{-1/y}$$

resulting in

$$\begin{aligned} f_s(K) &= b^{-d} f_s(K^* + b^y K^* \varepsilon) \\ &= |\varepsilon|^{d/y} \underbrace{f_s(K^* [1 + \frac{\varepsilon}{|\varepsilon|}])}_{=const} . \end{aligned}$$

So we have

$$f(\varepsilon, N) \sim |\varepsilon|^{d/y} .$$

Comparison with equation (3.58) [\[previous page\]](#) yields

$$\begin{aligned} 2 - \alpha &= \frac{d}{y} \\ \alpha &= 2 - \frac{d}{y} . \end{aligned}$$

Inserting the numerical value from equation (3.55) [\[p. 97\]](#) gives

$$\alpha = 0.131 .$$

The exact value is  $\alpha = 0$  corresponding to a logarithmic divergency. While the mean field result of equation (2.29) [\[p. 50\]](#) gave constant specific heat when approach from below  $T_C$  and  $C_V = 0$  above  $T_C$ .

# Chapter 4

## Magnetism

In this chapter we will study further theories on magnetism, including first the general Hamiltonian for a magnetic system and then looking at different approaches to obtain an expression for magnetization in different cases - some of which more involved than others.

Lastly, we will revisit collective magnetism by introducing the Heisenberg Hamiltonian as a generalization of the by now well-known Ising model.

### 4.1 Hamiltonian

The hamiltonian reads

$$H = \sum_j \left( \mathbf{p}_j + e\mathbf{A}(\mathbf{r}_j) \right)^2 + H_{\text{Coul}} - \sum_j m_j^s B(\mathbf{r}_j) + H_{\text{rel}} \quad (4.1)$$

So far, this hamiltonian describes any arbitrary system, from isolated atoms up to any crystal. The first term describes the coupling of the electrons to the electromagnetic field in combination with the kinetic energy of the electrons, the second contains Coulomb interaction of the electrons to the atomic core and the electron-electron interaction and last term covers the interaction of the magnetic moment of the electronic spins to the external field. The spin of electron  $j$ , denoted by  $\mathbf{S}_j$  corresponds to a magnetic moment

$$\mathbf{m}_j^s = -\frac{g_e e}{2m} \mathbf{S}_j = -\frac{g_e \mu_B}{\hbar} \mathbf{S}_j \quad (4.2)$$

with  $g_e = 2.0023$  being the Landé Factor for the electron, and the Bohr magneton

$$\mu_B = \frac{e\hbar}{2m} . \quad (4.3)$$

The last term in equation (4.1) covers relativistic effects, such as the spin orbit coupling, which we will briefly discuss later on. In the hamiltonian the dipole-dipole interaction of the spins is neglected, as it is of minor importance. First, we transform the first part of the hamiltonian

$$\left( \mathbf{p}_j + e\mathbf{A}(\mathbf{r}_j) \right)^2 = \mathbf{p}_j^2 + e\mathbf{p}_j\mathbf{A}(\mathbf{r}_j) + e\mathbf{A}(\mathbf{r}_j)\mathbf{p}_j + e^2\mathbf{A}^2(\mathbf{r}_j)$$

We use the Coulomb gauge, in which

$$\text{div}\mathbf{A}(\mathbf{r}) = 0$$

holds. Moreover, we will only consider a homogeneous magnetic field  $\mathbf{B}$ , for which in the Coulomb gauge we can write

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r} .$$

In this case

$$\mathbf{p}_j\mathbf{A}(\mathbf{r}_j) = \frac{1}{2}\mathbf{p}_j \cdot (\mathbf{B} \times \mathbf{r}_j) .$$

We have to take into account that  $\mathbf{P}$  and  $\mathbf{r}$  are operators, while  $\mathbf{B}$  is a classical vector. Then the spin-term simplifies to

$$\sum_j m_j^s B(\mathbf{r}_j) = \mathbf{B} \cdot \sum_j m_j^s .$$

Moreover, we need

$$\begin{aligned} \mathbf{p} \cdot (\mathbf{B} \times \mathbf{r}) &= p_i \varepsilon_{ijk} B_j r_k \\ &= -B_j \varepsilon_{jik} p_i r_k \\ &= B_j \varepsilon_{jki} r_k p_i . \end{aligned}$$

In the last step we have used that  $r_k$  and  $p_j$  commute for different indices. If the indices of  $r$  and  $p$  would be equal, then the Levi-Civita symbol would vanish. Hence,

$$\mathbf{p}_j \cdot (\mathbf{B} \times \mathbf{r}_j) = \mathbf{B} \cdot (\mathbf{r}_j \times \mathbf{p}_j) = \mathbf{B} \cdot \mathbf{L}_j . \quad (4.4)$$

Similarly, we find

$$\begin{aligned} (\mathbf{B} \times \mathbf{r}) \cdot \mathbf{p} &= \varepsilon_{ijk} B_j r_k p_i \\ &= B_j \varepsilon_{jki} r_k p_i . \end{aligned}$$

We still need

$$\begin{aligned}
 \mathbf{A}^2(r) &= \frac{1}{4} \left( \mathbf{B} \times \mathbf{r} \right)^2 \\
 &= \frac{1}{4} \varepsilon_{ijk} \varepsilon_{imn} B_i r_j B_m r_n \\
 &= \frac{1}{4} \left( B_i r_j B_i r_j - B_i r_i B_j r_j \right) \\
 &= \frac{1}{4} \left( \mathbf{B}^2 \mathbf{r}^2 - (\mathbf{B} \cdot \mathbf{r})^2 \right) .
 \end{aligned}$$

In total we therefore find

$$H = \frac{1}{2m} \sum_j \mathbf{p}_j^2 + \frac{e}{2m} \mathbf{B} \sum_j \cdot \mathbf{L}_j + \frac{e^2 B^2}{8m} \sum_j \left( \mathbf{B}^2 \mathbf{r}_j^2 - (\mathbf{B} \cdot \mathbf{r}_j)^2 \right) + H_{\text{Coul}} - \mathbf{m}^s \cdot \mathbf{B} ,$$

where  $\mathbf{m}^s$  is the total magnetic moment of all spins. We also relate the angular momentum with the magnetic moment

$$\begin{aligned}
 \mathbf{m}_j^L &= -\frac{e}{2m} \mathbf{L}_j = -\frac{\mu_B}{\hbar} \mathbf{L}_j \\
 \mathbf{m}^L &= -\frac{e}{2m} \sum_j \mathbf{L}_j = -\frac{\mu_B}{\hbar} \sum_j \mathbf{L}_j .
 \end{aligned}$$

Then the hamiltonian simplifies to

$$\begin{aligned}
 H &= \frac{1}{2m} \sum_j \mathbf{p}_j^2 + H_{dia} + H_{\text{Coul}} - \mathbf{m}^{perm} \cdot \mathbf{B} , \\
 H_{dia} &= \frac{e^2 B^2}{8m} \sum_j \left( \mathbf{B}^2 \mathbf{r}_j^2 - (\mathbf{B} \cdot \mathbf{r}_j)^2 \right) ,
 \end{aligned}$$

with the total *permanent* magnetic moment defined by

$$\mathbf{m}^{perm} = \mathbf{m}^s + \mathbf{m}^L = -\frac{\mu_B}{\hbar} (\mathbf{L} + g_e \mathbf{S}) . \quad (4.5)$$

It results in *paramagnetism*.

The permanent moments tend to align in the magnetic field, while thermal fluctuations have the opposite effect. There is yet another term in the hamiltonian equation (4.1) [p. 101] that contains magnetic moments resulting from the electronic motion: the diamagnetic term  $H_{dia}$ . One can show that the total magnetic moment, defined as

$$\mathbf{m} = \mathbf{m}^{perm} + \mathbf{m}^{ind} ,$$



can be obtained by

$$\mathbf{m} = -\nabla_{\mathbf{B}} H .$$

Then we see that the induced magnetic moment is given by

$$\mathbf{m}^{ind} = -\frac{e^2}{4m} \sum_j (\mathbf{r}_j^2 \mathbb{1} - \mathbf{r}_j \mathbf{r}_j^T) \mathbf{B} .$$

W.l.o.g. we can choose  $\mathbf{B} = B\mathbf{e}_z$  then

$$\mathbf{m}^{ind} = -B \frac{e^2}{4m} \sum_j (\mathbf{r}_j^2 \mathbf{e}_z - z_j \mathbf{r}_j)$$

### 4.1.1 Diamagnetism of atoms

We consider atoms or ions with closed shells, e.g. helium or other noble gases. in this case  $S = 0$ ,  $L = 0$ . Then the only term contribution to a magnetic moment is the diamagnetic term. We will treat this term in first order perturbation theory, i.e. we first determine the ground state of the system for  $B = 0$  and use the corresponding eigenvectors to determine the first order energy correction

$$\Delta E_1 = \langle H_{dia} \rangle ,$$

from which we obtain the resulting moment

$$\langle \mathbf{m}^{ind} \rangle = -B \frac{e^2}{4m} \sum_j (\langle \mathbf{r}_j^2 \rangle \mathbf{e}_z - \langle z_j \mathbf{r}_j \rangle) .$$

where  $r_{j,\nu}$  is the  $\nu$ -th cartesian component of the vector  $\mathbf{r}_j$ . We consider

$$\langle z_j \mathbf{r}_j \rangle = \sum_{\nu} \mathbf{e}_{\nu} \langle z_j r_{j,\nu} \rangle ,$$

For closed atomic shells, the ground state wavefunction is rotationally invariant, therefore

$$\langle z_j r_{j,\nu} \rangle = \delta_{\nu,z} \langle z_j^2 \rangle = \delta_{\nu,z} \frac{1}{3} \langle r_j^2 \rangle$$

So in total , the atomic induced magnetic moment is

$$\mathbf{m}_{ind} = -\mathbf{B} \frac{e^2}{6m} \sum_j \langle r_j^2 \rangle$$

	He	Ne	Ar	Kr	Xe
$\chi$ per mole in $10^{-6} \text{ cm}^3/\text{mole}$	-1.9	-7.2	-15.4	-28.0	-43.0

The magnetic moment is always in the opposite direction of the applied magnetic field, i.e. *diamagnetic reaction*, it reduced the total magnetic field. It is called *Langevin diamagnetism*.

From the magnetic moment, we determine the magnetic susceptibility

$$\chi_{\nu\mu} = \mu_0 \frac{\partial \langle \mathbf{m}_\nu \rangle}{\partial B_\mu} = -\delta_{\mu\nu} \frac{\mu_0 e^2}{6m} \langle \sum_j r_j^2 \rangle$$

To estimate the order of magnitude, we replace  $\langle r_j^2 \rangle$  by the square of the Bohr radius. One obtains

$$\chi \text{ per mole} \approx -3 \times 10^{-6} \frac{\text{cm}^3}{\text{mol}} .$$

For noble gases the values of  $\chi$  are listed in the table below. This diamagnetic contribution is also present in open shell systems, but then it is orders of magnitude smaller than the contribution of the other term.

## 4.2 Density matrix and thermodynamic relations

We consider the canonical ensemble with fixed particle number  $N$ , which we will not explicitly mention as natural variable in the arguments. We only write  $T$  and  $\mathbf{B}$

$$\rho(T, \mathbf{B}) = \frac{1}{Z(T, \mathbf{B})} e^{-\beta H}$$

$$Z = \text{tr} \{ e^{-\beta H} \} .$$

The free energy reads

$$F(T, \mathbf{B}) = -k_B T \ln(Z(T, \mathbf{B})) . \quad (4.6)$$

The internal energy follows from

$$U = \langle H \rangle = -\frac{\partial}{\partial \beta} \ln(Z)$$

$$= \frac{\partial}{\partial \beta} (\beta F)$$

$$= F - \underbrace{\beta(k_B T^2)}_{=T} \underbrace{\frac{\partial}{\partial T} F}_{=-S}$$

$$U = F - TS .$$

This is equivalent to

$$\begin{aligned} S &= \frac{1}{T}(U - F) \\ &= k_B \operatorname{tr}\{\rho(\beta H)\} + k_B \ln(Z) \\ &= k_B \left( \operatorname{tr}\{\rho(\beta H)\} + \ln(Z) \right). \end{aligned}$$

Then we use

$$\begin{aligned} \ln(\rho) &= \ln\left(\frac{e^{-\beta H}}{Z}\right) = \ln\left(e^{-\beta H}\right) - \ln(Z) = -\beta H - \ln(Z) \\ \beta H &= -\left(\ln(\rho) + \ln(Z)\right). \end{aligned}$$

Inserting above yields

$$\begin{aligned} S &= k_B \left( -\operatorname{tr}\{\rho(\ln(\rho) + \ln(Z))\} + \ln(Z) \right) \\ &= k_B \left( -\operatorname{tr}\{\rho \ln(\rho)\} - \underbrace{\operatorname{tr}\{\rho\} \ln(Z)}_{\substack{=1 \\ =0}} + \ln(Z) \right) \\ &= \operatorname{tr}\{-k_B \rho \ln(\rho)\} \\ &= -k_B \langle \ln(\rho) \rangle. \end{aligned}$$

Hence the entropy is the trace of the quantum density operator.

For the total differential of  $F$  we have

$$\begin{aligned} dF &= \left. \frac{\partial F(T, \mathbf{B})}{\partial T} \right|_{\mathbf{B}} dT + \left. \nabla_{\mathbf{B}} F(T, \mathbf{B}) \right|_T \cdot d\mathbf{B} \\ &= -SdT - \mathbf{M} \cdot d\mathbf{B} \end{aligned}$$

Proof of the last step:

$$\begin{aligned} \left. \nabla_{\mathbf{B}} F(T, \mathbf{B}) \right|_T &= -k_B T \nabla_{\mathbf{B}} \ln(Z) \\ &= -\frac{k_B T}{Z} \nabla_{\mathbf{B}} Z \\ &= -\frac{k_B T}{Z} \operatorname{tr} \left( \left. \nabla_{\mathbf{B}} e^{-\beta H} \right|_T \right) \\ &= \underbrace{-\frac{k_B T}{Z} (-\beta)}_{=\frac{1}{Z}} \operatorname{tr} \left( e^{-\beta H} \underbrace{\left. \nabla_{\mathbf{B}} H \right|_T}_{=-\mathbf{m}} \right) \\ &= -\operatorname{tr} \left( \frac{e^{-\beta H}}{Z} \mathbf{m} \right) = -\mathbf{M} \end{aligned}$$

Hence

$$\left. \frac{\partial F}{\partial \mathbf{B}} \right|_T = -\mathbf{M} . \quad (4.7)$$

Oftentimes the magnetization is defined as total mean magnetic moment per volume.

### 4.2.1 Magnetic response functions

We define the specific heat in analogy to the case of gases that was treated before:

$$C_{\mathbf{B}} = \left. \frac{\partial U}{\partial T} \right|_{\mathbf{B}} = T \left. \frac{\partial S}{\partial T} \right|_{\mathbf{B}} = -T \left. \frac{\partial^2 F}{\partial T^2} \right|_{\mathbf{B}} . \quad (4.8)$$

The derivation of the following result is conducted in a similar way to before without a magnetic field. For the magnetic susceptibility we obtain:

$$\chi_{T,\nu\mu} = \mu_0 \left. \frac{\partial M_\nu}{\partial B_\mu} \right|_T = -\mu_0 \left. \frac{\partial^2 F}{\partial B_\nu \partial B_\mu} \right|_T \quad (4.9)$$

In many times, the susceptibility is also expressed per volume.

### 4.2.2 Internal energy

We have defined the internal energy as  $U = \langle H \rangle$ , which is the energy of the material including the interaction of the electromagnetic field. It was derived from the purely electronic part. However, the field energy itself is not included. It is common to define a second internal energy  $U'$  that also includes the field energy

$$U' = U + \mathbf{M} \cdot \mathbf{B} . \quad (4.10)$$

resulting in

$$dU' = TdS + \mathbf{B} \cdot d\mathbf{M} .$$

It is the Legendre transformation in the pair  $(\mathbf{M}, \mathbf{B})$ .

With

$$\begin{aligned} \left. \frac{\partial U'}{\partial S} \right|_{\mathbf{M}} &= T \\ \left. \frac{\partial U'}{\partial \mathbf{M}} \right|_S &= \mathbf{B} . \end{aligned}$$

This results in the Maxwell relation

$$\left. \frac{\partial \mathbf{B}}{\partial S} \right|_{\mathbf{M}} = \left. \frac{\partial T}{\partial \mathbf{M}} \right|_S .$$

### 4.3 Paramagnetism of independent moments

We consider the situation, where the permanent magnetic moments of the atoms do not interact. Then the hamiltonian has the structure

$$H = \sum_i H^{(i)} .$$

where  $i$  enumerates the atoms. All atoms shall be the same, so all  $H^{(i)}$  are the same. In the following we suppress the index  $i$ . The hamiltonian  $H^{(i)}$  has the form

$$H^{(i)} = H_0 + \underbrace{\gamma(\mathbf{LS})}_{\text{spin-orbit coupling}} + \underbrace{\frac{\mu_B B}{\hbar} (J^z + S^z)}_{\text{Zeeman term}} . \quad (4.11)$$

Here  $J^z = L^z + S^z$ . w.l.o.g. we have assumed that  $\mathbf{B} = B\mathbf{e}_z$ . In addition we have used  $g_e \approx 2$ .

$H_0$  commutes with  $L^2, S^2, L^z, S^z$ , which is also true for the total hamiltonian without spin-orbit coupling. For the calculation of the partition function we have, since there is no interaction between the atoms

$$Z = \prod_{i=1}^N Z^{(i)} = (Z^{(1)})^N ,$$

i.e. it suffices to consider a single atom.

#### 4.3.1 Weak spin-orbit coupling

In this case we ignore the spin-orbit term and the eigenvectors are given by

$$|\kappa, L, S, L_z, S_z\rangle ,$$

where  $\kappa$  covers all other quantum numbers. The corresponding eigenvalues are based on equation (4.11) with  $J^z$  replaced by  $L^z + S^z$ .

$$E(\kappa, L, S, L_z, S_z) = E_0(\kappa, L, S) + \frac{\mu_B B}{\hbar} (L_z + 2S_z) ,$$

The canonical partition function reads

$$Z = \prod_{i=1}^N Z^{(i)} = \left( Z^{(1)} \right)^N \quad (4.12)$$

$$Z^{(1)} = \sum_{\kappa, L, S, L_z, S_z} e^{-\beta E_0(\kappa, L, S) - \frac{\beta \mu_B B}{\hbar} (L_z + 2S_z)} \quad (4.13)$$

$$Z^{(1)} = \sum_{\kappa, L, S} e^{-\beta E_0(\kappa, L, S)} \sum_{L_z, S_z} e^{-\frac{\beta \mu_B B}{\hbar} (L_z + 2S_z)} . \quad (4.14)$$

Typically, the Zeeman splitting is small compared to the other energy differences, when changing  $\kappa, L, S$ . For moderate temperature we only need to consider the ground state quantum numbers  $(\kappa_0, L_0, S_0)$  (Hund's rule) and obtain (we use  $L_z = \hbar l_z$  and  $S_z = \hbar s_z$ , with  $l_z, s_z \in \mathbb{N}$ )

$$Z^{(1)} = e^{-E_0(\kappa_0, L_0, S_0)} \underbrace{\sum_{l_z=-L_0}^{L_0} e^{-bL_z} \sum_{s_z=-S_0}^{S_0} e^{-2bs_z}}_{=\mathcal{Z}(L_0, S_0)}$$

Here  $b = \beta B \mu_B$ . If there is a gap between the ground state multiplet and the first excited one of  $\Delta E$ , then the next to leading order contribution  $((\kappa_1, L_1, S_1))$  is

$$\begin{aligned} Z^{(1)} &= e^{-\beta E_0(\kappa_0, L_0, S_0)} \mathcal{Z}(L_0, S_0) + e^{-E_0(\kappa_1, L_1, S_1)} \mathcal{Z}(L_1, S_1) \\ &= e^{-\beta E_0(\kappa_0, L_0, S_0)} \left( \mathcal{Z}(L_0, S_0) + e^{-\beta \Delta E} \mathcal{Z}(L_1, S_1) \right) \end{aligned}$$

If  $|\Delta E| \gg k_B T$  then we can neglect the higher order terms and

$$Z^{(1)} = e^{-\beta E_0(\kappa_0, L_0, S_0)} \mathcal{Z}(L_0, S_0) \quad (4.15)$$

The key elements are sums of the form

$$\begin{aligned} z_M(\eta) &= \sum_{m=-M}^M e^{-\eta m} = e^{\eta M} \sum_{m=0}^{2M} e^{-\eta m} = e^{\eta M} \frac{1 - e^{-\eta(2M+1)}}{1 - e^{-\eta}} \\ &= \frac{e^{\eta(M+1/2)} - e^{-\eta(M+1/2)}}{e^{\eta/2} - e^{-\eta/2}} \\ &= \frac{\sinh(\eta(M+1/2))}{\sinh(\eta/2)} \end{aligned}$$

Then we obtain immediately

$$\begin{aligned} \frac{F}{N} &= -k_B T \ln(Z^1) \\ &= E_0(\kappa_0, L_0, S_0) - k_B T \ln \left( z_{L_0}(b) \right) - k_B T \ln \left( z_{S_0}(2b) \right). \end{aligned}$$

Next we compute the magnetisation

$$\mathbf{M} = - \left. \frac{\partial F}{\partial \mathbf{B}} \right|_T$$

Since  $\mathbf{B} = \mathbf{e}_z$  we obtain

$$\mathbf{M} = -\mathbf{e}_z \left. \frac{\partial F}{\partial B} \right|_T$$

$E_0$  is independent of the magnetic field. We need

$$\frac{\partial}{\partial B} z_M(\eta) = \frac{z'_M(\eta)}{z_M(\eta)} \frac{d\eta}{dB}$$

For the latter derivative we need

$$\begin{aligned} \frac{db}{dB} &= \beta \mu_B \\ \frac{d(2b)}{dB} &= 2\beta \mu_B . \end{aligned}$$

Next we compute

$$\begin{aligned} z'_M(\eta) &= \frac{d}{d\eta} \frac{\sinh(\eta(M + 1/2))}{\sinh(\eta/2)} \\ &= \left(M + \frac{1}{2}\right) \frac{\cosh(\eta(M + 1/2))}{\sinh(\eta/2)} - \frac{1}{2} \frac{\sinh(\eta(M + 1/2)) \cosh(\eta/2)}{\sinh^2(\eta/2)} . \end{aligned}$$

Then

$$\begin{aligned} \frac{z'_M(\eta)}{z_M(\eta)} &= \left(M + \frac{1}{2}\right) \coth(\eta(M + 1/2)) - \frac{1}{2} \coth(\eta/2) \\ &= M \left( \frac{2M + 1}{2M} \coth\left(\frac{M\eta(2M + 1)}{2M}\right) - \frac{1}{2M} \coth\left(\frac{M\eta}{2M}\right) \right) . \end{aligned}$$

Finally we have

$$\frac{z'_M(\eta)}{z_M(\eta)} = M \mathcal{B}_M(M\eta) , \quad (4.16)$$

with

BRILLOUIN FUNCTION	
$\mathcal{B}_M(x) = \frac{2M + 1}{2M} \coth\left(\frac{(2M + 1)}{2M} x\right) - \frac{1}{2M} \coth\left(\frac{x}{2M}\right)$	(4.17)
	(4.18)

Then the final result reads

$$\frac{\mathbf{M}}{N} = \mathbf{e}_z \mu_B B \left( L_0 \mathcal{B}_{L_0}(bL_0) + 2S_0 \mathcal{B}_{S_0}(2bS_0) \right). \quad (4.19)$$

### Properties of the Brillouin function

For small or large arguments,  $\coth(x)$  behaves like

$$\begin{aligned} x \ll 1 : \quad & \coth(x) = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} O(x^5) \\ x \rightarrow \infty : \quad & \coth(x) \rightarrow 1. \end{aligned}$$

Hence, we have for  $x \ll 1$  (MATHEMATICA)

$$\mathcal{B}_M(x) = \frac{M+1}{3M} x - \frac{2M^3 + 4M^2 + 3M + 1}{90M^3} x^3. \quad (4.20)$$

And for  $x \rightarrow \infty$

$$\mathcal{B}_M(x) \rightarrow \frac{2M+1}{2M} - \frac{1}{2M} = 1.$$

In summary

$$\begin{aligned} \eta \ll 1 : \quad & \mathcal{B}_M(\eta) = \frac{M+1}{3M} \eta \\ \eta \rightarrow \pm\infty : \quad & \mathcal{B}_M(\eta) \rightarrow \pm 1. \end{aligned}$$

This yields for the magnetization in equation (4.19)

$$\begin{aligned} b \ll 1 : \quad \frac{\mathbf{M}}{N} &= \mathbf{e}_z \mu_B \left( L_0 \frac{L_0+1}{3L_0} (bL_0) + 2S_0 \frac{S_0+1}{3S_0} (2bS_0) \right) \\ &= \mathbf{e}_z \frac{\mu_B^2 B}{3k_B T} \left( L_0(L_0+1) + 4S_0(S_0+1) \right) \end{aligned}$$

The susceptibility  $\chi_{\mu\nu} = \mu_0 \frac{\partial M_\mu}{\partial B_\nu}$  then reads

CURIE LAW	
$\chi_{\mu\nu} = \delta_{\mu\nu} \frac{C}{T}, \quad (4.21)$	



with

$$C = \frac{\mu_0 \mu_B^2}{3k_B} \left( L_0(L_0 + 1) + 4S_0(S_0 + 1) \right)$$

We still need to consider the large  $b$  limit. Here we have:

$$b \rightarrow \infty : \quad \frac{\mathbf{M}}{N} \rightarrow \pm \mathbf{e}_z \mu_B (L_0 + 2S_0) .$$

### Brillouin function for $M = 1/2$ and $M = \infty$

An important special case is given for spin-1/2, then

$$\begin{aligned} \mathcal{B}_{1/2}(x) &= 2 \coth(2x) - \coth(x) \\ &= \frac{2(e^{2x} + e^{-2x})}{e^{2x} - e^{-2x}} - \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{e^x + e^{-x}}{e^x + e^{-x}} \\ &= \frac{2e^{2x} + 2e^{-2x} - e^{2x} - e^{-2x} - 2}{(e^x - e^{-x})(e^x + e^{-x})} \\ &= \frac{e^{2x} + e^{-2x} - 2}{(e^x - e^{-x})(e^x + e^{-x})} . \end{aligned}$$

So we finally have

$$\mathcal{B}_{1/2}(x) = \tanh(x) \tag{4.22}$$

For large spin ( $M \gg 1$ ) with  $\varepsilon = 1/2M \ll 1$  find

$$\begin{aligned} \mathcal{B}_M(x) &= (1 + \varepsilon) \coth(x + \varepsilon x) - \varepsilon \coth(\varepsilon x) \\ &= \coth(x) + \mathcal{O}(\varepsilon) - \frac{\varepsilon}{\varepsilon x} + \mathcal{O}(\varepsilon) \end{aligned}$$

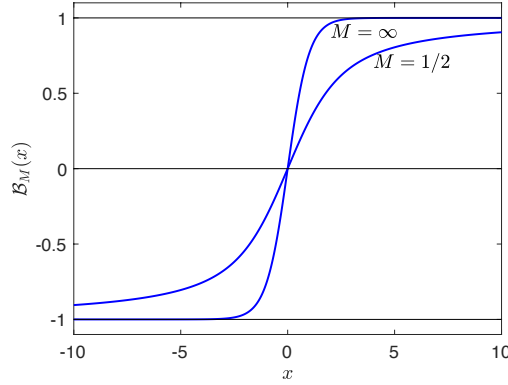
Hence

$$\mathcal{B}_{M \rightarrow \infty}(x) = \coth(x) - \frac{1}{x} : \tag{4.23a}$$

This is the classical limit.

### 4.3.2 Strong spin-orbit coupling

To avoid confusion we mark operators by hats. If the spin-orbit coupling is strong compared to the magnetic field, then we cannot ignore the spin-orbit

Figure 4.1: Brillouin function for  $M = 1/2$  and  $M = \infty$ .

term. In this case,  $H$  still commutes with  $L$  and  $S$ , but no longer with  $L^z$  and  $S^z$ . Instead, based on the relations

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}} \quad (4.24)$$

$$\hat{\mathbf{L}}\hat{\mathbf{S}} = \frac{1}{2} \left( \hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2 \right), \quad (4.25)$$

$H$  now commutes with  $L^2, S^2, J^2, J_z$ . The eigenvectors are therefore

$$|\kappa, L, S, J, J_z\rangle.$$

The corresponding eigenvalues are given by equation (4.11) [p. 108] as

$$E(\kappa, L, S, J, J_z) = E_0(\kappa, L, S) + \frac{B\mu_B}{\hbar} \left( J_z - \langle \hat{S}^z \rangle \right).$$

Here  $\langle \hat{S}^z \rangle$  is the expectation value of  $\hat{S}^z$  in one of the eigenstates  $|\kappa, L, S, J, J_z\rangle$ , while  $J_z$  is already the eigenvalue of the operator  $\hat{J}^z$ . The computation of the expectation value  $\langle \hat{S}^z \rangle$  yields eventually

$$\begin{aligned} E(\kappa, L, S, J, J_z) &= E_0(\kappa, L, S) + \frac{\mu_B B}{\hbar} \underbrace{\left( 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \right)}_{=g_J} J_z \\ &= E_0(\kappa, L, S) + \mu_B B g_J m, \end{aligned}$$

with  $J_z = \hbar m$  and  $m \in \{-J, -J+1, \dots, J\}$

---

*proof*

To compute the expectation value we proceed as follows. We start out with the identity

$$\hat{\mathbf{S}} (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) - (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) \hat{\mathbf{S}} = -i\hbar \hat{\mathbf{S}} \times \hat{\mathbf{L}} . \quad (4.26)$$

which follows from

$$\begin{aligned} \hat{S}_\nu \hat{L}_\mu S_\mu - \hat{L}_\mu \hat{S}_\mu \hat{S}_\nu &= \hat{S}_\nu \hat{S}_\mu \hat{L}_\mu - \hat{S}_\mu \hat{S}_\nu \hat{L}_\mu = [\hat{S}_\nu, \hat{S}_\mu] \hat{L}_\mu \\ &= i\hbar \varepsilon_{\nu\mu\rho} \hat{S}_\rho \hat{L}_\mu = -i\hbar \varepsilon_{\nu\rho\mu} \hat{S}_\rho \hat{L}_\mu \\ &= -i\hbar (\hat{\mathbf{S}} \times \hat{\mathbf{L}})_\nu \end{aligned}$$

. Next we multiply equation (4.26) with  $\times \hat{\mathbf{J}}$  and obtain

$$\hat{\mathbf{S}} (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) \times \hat{\mathbf{J}} - (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) \hat{\mathbf{S}} \times \hat{\mathbf{J}} = -i\hbar (\hat{\mathbf{S}} \times \hat{\mathbf{L}}) \times \hat{\mathbf{J}} \quad (4.27)$$

Now,  $(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}})$  commutes with all components of the vector operator  $\hat{\mathbf{J}}$ , since according to equation (4.27)  $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$  is given by  $\frac{1}{2}(\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2)$  which can be seen as follows

$$\begin{aligned} [\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}, J^\alpha] &= \frac{1}{2} \left( \underbrace{[J^2, J^\alpha]}_{=0} - [L^2, J^\alpha] - [S^2, J^\alpha] \right) \\ &= -\frac{1}{2} \left( [L^2, L^\alpha + S^\alpha] + [S^2, L^\alpha + S^\alpha] \right) \\ &= -\frac{1}{2} \left( [L^2, L^\alpha] + [S^2, S^\alpha] \right) \\ &= 0 . \end{aligned}$$

Then we can move  $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$  in the first term of equation (4.27) to the right of  $\hat{\mathbf{J}}$ , resulting in

$$(\hat{\mathbf{S}} \times \hat{\mathbf{J}}) (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) - (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) (\hat{\mathbf{S}} \times \hat{\mathbf{J}}) = -i\hbar (\hat{\mathbf{S}} \times \hat{\mathbf{L}}) \times \hat{\mathbf{J}}$$

If we now compute expectation values in the eigenvectors  $|\kappa, L, S, J, J_z\rangle$ , we can replace the operators  $(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}})$  by the eigenvalues  $\frac{1}{2}(J(J+1) - L(L+1) - S(S+1))$  and the remaining expectation value on the left hand side is zero. Hence, we obtain from the right hand side

$$\langle (\hat{\mathbf{S}} \times \hat{\mathbf{L}}) \times \hat{\mathbf{J}} \rangle = 0 , \quad (4.28)$$

which is valid in the eigenstates. From

$$\begin{aligned}
 [(\hat{\mathbf{S}} \times \hat{\mathbf{L}}) \times \hat{\mathbf{J}}]_i &= \varepsilon_{ijk} (\hat{\mathbf{S}} \times \hat{\mathbf{L}})_j \hat{J}_k \\
 &= \varepsilon_{ijk} \varepsilon_{jmn} \hat{S}_m \hat{L}_n \hat{J}_k \\
 &= \varepsilon_{jki} \varepsilon_{jmn} \hat{S}_m \hat{L}_n \hat{J}_k \\
 &= (\delta_{km} \delta_{in} - \delta_{kn} \delta_{im}) \hat{S}_m \hat{L}_n \hat{J}_k \\
 &= \hat{S}_k \hat{L}_i \hat{J}_k - \hat{S}_i \hat{L}_k \hat{J}_k \\
 &= \hat{L}_i \hat{S}_k \hat{J}_k - \hat{S}_i \hat{L}_k \hat{J}_k .
 \end{aligned}$$

we obtain for the double vector product

$$\begin{aligned}
 (\hat{\mathbf{S}} \times \hat{\mathbf{L}}) \times \hat{\mathbf{J}} &= \hat{\mathbf{L}} (\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}) - \hat{\mathbf{S}} (\hat{\mathbf{L}} \cdot \hat{\mathbf{J}}) \\
 &= (\hat{\mathbf{J}} - \hat{\mathbf{S}}) (\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}) - \hat{\mathbf{S}} (\hat{\mathbf{L}} \cdot \hat{\mathbf{J}}) \\
 &= \hat{\mathbf{J}} (\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}) - \hat{\mathbf{S}} ((\hat{\mathbf{L}} + \hat{\mathbf{S}}) \cdot \hat{\mathbf{J}}) \\
 &= \hat{\mathbf{J}} (\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}) - \hat{\mathbf{S}} \hat{\mathbf{J}}^2 .
 \end{aligned}$$

From equation (4.28) [\[previous page\]](#) we therefore obtain

$$(\hat{\mathbf{S}} \times \hat{\mathbf{L}}) \times \hat{\mathbf{J}} = \hat{\mathbf{J}} (\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}) - \hat{\mathbf{S}} \hat{\mathbf{J}}^2 = 0 .$$

and for the expectation value in the eigenstates  $|\kappa, L, S, J, J^z\rangle$  we have

$$\langle \hat{J}_z (\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}) \rangle - \langle \hat{S}_z J^2 \rangle = 0 .$$

Next we use

$$\begin{aligned}
 \hat{\mathbf{S}} \cdot \hat{\mathbf{J}} &= \hat{\mathbf{S}} \cdot \hat{\mathbf{L}} + \hat{\mathbf{S}}^2 \\
 &= \frac{1}{2} (\hat{\mathbf{J}}^2 - \hat{\mathbf{S}}^2 - \hat{\mathbf{L}}^2) + \hat{\mathbf{S}}^2 \\
 &= \frac{1}{2} (\hat{\mathbf{J}}^2 + \hat{\mathbf{S}}^2 - \hat{\mathbf{L}}^2) .
 \end{aligned}$$

and obtain

$$\begin{aligned}
 J_z \frac{1}{2} \langle \hat{\mathbf{J}}^2 + \hat{\mathbf{S}}^2 - \hat{\mathbf{L}}^2 \rangle - \langle \hat{S}_z J(J+1) \rangle &= 0 \\
 J_z \frac{1}{2} (J(J+1) + S(S+1) - L(L+1)) - \langle \hat{S}_z \rangle J(J+1) &= 0 .
 \end{aligned}$$

The expectation value  $\langle S_z \rangle$  in the eigenstates  $|\kappa, L, S, J, J^z\rangle$  is therefore:

$$\langle S_z \rangle = J_z \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} .$$

Hence, the eigenvalues are

$$E(\kappa, L, S, J, J_z) = E_0(\kappa, L, S) + \frac{\mu_B B}{\hbar} \underbrace{\left(1 + \frac{1}{2}[J(J+1) + S(S+1) - L(L+1)]\right)}_{=g_J} J_z .$$

q.e.d. ✓

---

The remaining calculation is similar to the previous one and one obtains

$$\frac{F}{N} = E_0 - k_B T \ln \left( z_J(b) \right)$$

with  $b = g_J \mu_B \beta B$ . Then

$$\hat{\mathbf{M}} = \mathbf{e}_z N J g_J \mu_B B \mathcal{B}_J(bJ) .$$

For high  $T$  and small  $B$ , the susceptibility yields

CURIE LAW	
$\chi_{\mu\nu} = \delta_{\mu\nu} \frac{C}{T} .$	(4.29)

with

$$C = \frac{N \mu_0 \mu_B^2 g_J^2 J(J+1)}{3k_B} ,$$

and for large  $b$  we have

$$\frac{\mathbf{M}}{N} \xrightarrow{b \rightarrow \infty} \pm \mathbf{e}_z g_J \mu_B J .$$

## Entropy

For the entropy we obtain

$$\frac{S}{N} = - \frac{\partial F/N}{\partial T} \Big|_{\mathbf{B}} = k_B \ln(z_J(b)) + k_B T \frac{z'_J(b)}{z_J(b)} \frac{db}{dT}$$

$$\frac{db}{dT} = -g_J \mu_B B \frac{1}{k_B T^2} = -\beta^2 g_J \mu_B B = -\beta b .$$

$$S = k_B \left( \ln(z_J(b)) - b \frac{z_J'(b)}{z_J(b)} \right)$$

Along with equation (4.16) [p. 110] we obtain

$$\frac{S}{N} = k_B \left( \ln \left( \frac{\sinh(b(J + 1/2))}{\sinh(b/2)} \right) - bJ \mathcal{B}_J(bJ) \right) .$$

For  $b \ll 1$  (i.e.  $T \rightarrow \infty$ ) we find

$$\begin{aligned} \frac{S}{N} &= k_B \left( \ln \left( \frac{b(J + 1/2)}{b/2} \right) - (bJ)^2 \frac{J + 1}{3J} \right) \\ &= k_B \ln(2J + 1) . \end{aligned}$$

The argument is the number of eigenvalues of  $J_z$ .

### 4.3.3 Plot of $S$ and specific heat

In Figure 4.2 the before obtained entropy per particle  $\frac{S}{N}$  is depicted. Note also that for  $b \ll 1$  we can now check visually that  $\frac{S}{N} \rightarrow k_B \ln(2J + 1)$ . Additionally, the heat capacity is shown as it is connected to the entropy by:

$$\begin{aligned} C_V &= T \frac{\partial S}{\partial T} = T \frac{\partial S}{\partial b} \frac{\partial b}{\partial T} = T \frac{\partial S}{\partial b} \frac{\partial}{\partial T} \frac{B \mu_B}{k_B T} \\ C_V &= T \frac{\partial S}{\partial b} \left( -\frac{B \mu_B}{k_B T^2} \right) = \frac{\partial S}{\partial b} \left( -\frac{B \mu_B}{k_B T} \right) \\ C_V &= -b \frac{\partial S}{\partial b} \end{aligned}$$

## 4.4 Magnetism of the free electron model

The free electron gas is based on the following assumptions: no electron-electron interactions, the electrons experience no potential due to the crystal, they are confined to a box. In addition, we apply a constant homogeneous external magnetic field.

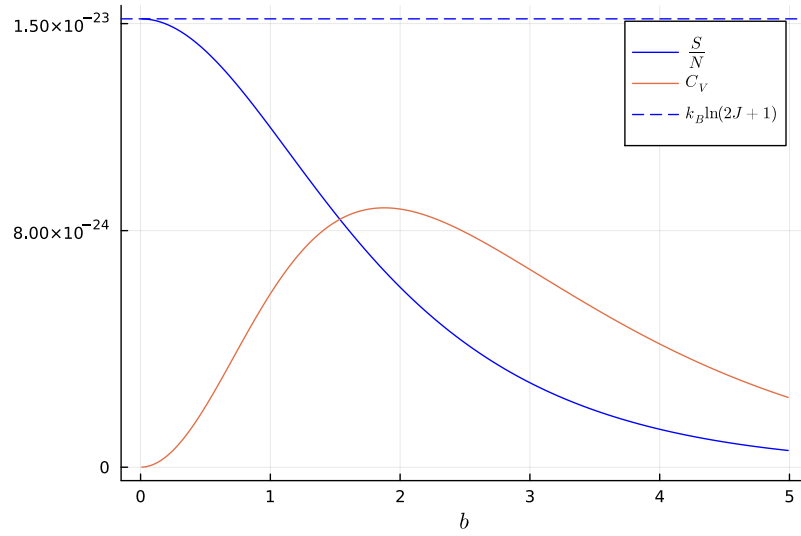


Figure 4.2: Entropy and specific heat for strong spin-orbit coupling and  $J = 1$ .

#### 4.4.1 Pauli paramagnetism

First, we restrict the discussion to the coupling of the electronic spin to the magnetic field, i.e. we ignore the angular momentum. The energy one-particle eigenvalues are

$$\begin{aligned}\varepsilon_\sigma(\mathbf{k}) &= \frac{\hbar^2 \mathbf{k}^2}{2m} + \sigma b, \\ b &= \frac{\mu_B g_e}{2} B.\end{aligned}$$

with the quantized wave vectors  $\mathbf{k}$ . The mean occupation of the one-particle orbitals is given by the Fermi-Dirac distribution

$$n_F(\varepsilon_\sigma(\mathbf{k})|T, \mu)$$

The mean total number of electrons in the Zeeman level represented by  $\sigma$  is

$$N_\sigma = \sum_{\mathbf{k}} n_F(\varepsilon_\sigma(\mathbf{k})|T, \mu) = \int d\varepsilon \rho(\varepsilon) n_F(\varepsilon + \sigma b|T, \mu).$$

As derived in appendix (5.3) the 3D dos is

$$\rho(\varepsilon) = D\sqrt{\varepsilon}, \quad (4.30a)$$

$$\text{with } D = \frac{Vm^{3/2}}{\hbar^3 \pi^2 \sqrt{2}}. \quad (4.30b)$$

Then

$$N_\sigma = D \int_0^\infty d\varepsilon \sqrt{\varepsilon} n_F(\varepsilon + \sigma b | T, \mu) .$$

For small magnetic field we can use a Taylor expansion in  $b$  around  $b = 0$

$$\begin{aligned} N_\sigma &= D \int_0^\infty d\varepsilon \sqrt{\varepsilon} \left( n_F(\varepsilon | T, \mu) + \sigma b \frac{\partial}{\partial \varepsilon} n_F(\varepsilon | T, \mu) + \mathcal{O}(b^2) \right) \\ &= D \int_0^\infty d\varepsilon \sqrt{\varepsilon} n_F(\varepsilon | T, \mu) + \sigma b D \int_0^\infty d\varepsilon \sqrt{\varepsilon} n'_F(\varepsilon | T, \mu) + \mathcal{O}(b^2) \end{aligned}$$

By now plugging in  $\sigma = \pm 1$  and adding  $N_{+1}$  and  $N_{-1}$  together, the term depending on  $\sigma$  cancels out and only the first term is left. Hence, the total particle number follows as:

$$\begin{aligned} N &= N_+ + N_- = 2D \int_0^\infty d\varepsilon \sqrt{\varepsilon} n_F(\varepsilon | T, \mu) + \mathcal{O}(b^2) \\ &= 2D \int_0^\infty d\varepsilon \sqrt{\varepsilon} n_F(\varepsilon | T, \mu) + \mathcal{O}(b^2) . \end{aligned} \quad (4.31)$$

For small  $b$  and low temperature we can use the Sommerfeld expansion, outlined in appendix (5.1),

SOMMERFELD EXPANSION	
$\begin{aligned} I &= \int_0^\infty f(\varepsilon) n_F(\varepsilon   \mu, T) d\varepsilon \\ &= \int_0^\mu f(\varepsilon) d\varepsilon + 2 \sum_{n=1}^{\text{odd}} \left( 1 - \frac{1}{2^n} \right) \zeta(n+1) (k_B T)^{n+1} f^{(n)}(\mu) , \end{aligned} \quad (4.32)$	$(4.32)$

which reads to leading order using  $\zeta(2) = \frac{\pi^2}{6}$ :

$$I = \int_0^\mu f(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 f'(\mu) + \mathcal{O}\left(\left(\frac{k_B T}{\mu}\right)^4\right) .$$

For the total particle number we obtain

$$\begin{aligned} N &= 2D \left( \int_0^\mu \sqrt{\varepsilon} d\varepsilon + \frac{\pi^2}{6} \frac{d}{d\mu} \sqrt{\mu} (k_B T)^2 + \dots \right) \\ &= 2D \left( \frac{2}{3} \mu^{3/2} + \frac{\pi^2}{12} \mu^{-1/2} (k_B T)^2 + \dots \right) . \end{aligned}$$



$$\frac{3N}{4D} = \mu^{3/2} \left( 1 + \frac{\pi^2}{8} \left( \frac{k_B T}{\mu} \right)^2 + \dots \right) \quad (4.33)$$

For  $T = 0$  the chemical potential is equivalent to the Fermi energy  $\varepsilon_F$  and we have

$$\frac{3N}{4D} = \varepsilon_F^{3/2}, \quad (4.34)$$

or rather

$$\varepsilon_F = \left( \frac{3N\hbar^3(2\pi)^2}{4V(2m)^{3/2}} \right)^{2/3} = \left( \frac{3N\pi^2}{V} \right)^{2/3} \frac{\hbar^2}{2m}.$$

FERMI ENERGY IN THE FREE ELECTRON GAS
$\varepsilon_F = \left( 3\pi^2 n \right)^{2/3} \frac{\hbar^2}{2m}. \quad (4.35)$

This defines the Fermi wave number  $k_F$ , through

$$\begin{aligned} \varepsilon_F &= \left( 3\pi^2 n \right)^{2/3} \frac{\hbar^2}{2m} = \frac{\hbar^2 k_F^2}{2m} \\ k_F &= \left( 3\pi^2 n \right)^{1/3} = \left( \frac{3\pi^2}{v} \right)^{1/3}, \end{aligned}$$

where  $v$  is the average volume per electron. Hence

$$k_F \propto \frac{1}{r},$$

where  $r$  is the mean distance between the electrons. Inserting equation (4.34) in equation (4.33) yields apart from higher order terms

$$\begin{aligned} \varepsilon_F &= \mu \left( 1 + \frac{\pi^2}{8} \left( \frac{k_B T}{\mu} \right)^2 \right)^{2/3} \\ \mu &= \varepsilon_F \left( 1 + \frac{\pi^2}{8} \left( \frac{k_B T}{\mu} \right)^2 \right)^{-2/3}. \end{aligned}$$

We can solve this equation iteratively, starting with  $\mu = \varepsilon_F$ . The first iteration yields

$$\mu = \varepsilon_F \left( 1 + \frac{\pi^2}{8} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right)^{-2/3}.$$

For low temperatures,  $\frac{k_B T}{\varepsilon_F} < 1$  we can as well write

$$\mu = \varepsilon_F \left( 1 - \frac{\pi^2}{8} \frac{2}{3} \left( \frac{k_B T}{\varepsilon_F} \right)^2 + \mathcal{O} \left( \left( \frac{k_B T}{\varepsilon_F} \right)^4 \right) \right).$$

Further iterations do not change the second order term and we generally have

CHEMICAL POTENTIAL (free electron gas)	
$\mu = \varepsilon_F \left( 1 - \frac{\pi^2}{12} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right) + \mathcal{O} \left( \left( \frac{k_B T}{\varepsilon_F} \right)^4 \right)$	(4.36)

The partition function for non-interacting particles has already been derived previously. For one-particle energies  $\varepsilon_\nu$  it was

$$\ln(Z) = \sum_{\nu} \ln \left( 1 + e^{-\beta(\varepsilon_\nu - \mu)} \right).$$

In the present case the index  $\nu$  stands for the wave vector  $\mathbf{k}$  and the spin direction. Hence

$$\ln(Z) = \sum_{\sigma} \int_0^{\infty} \rho(\varepsilon) \ln \left( 1 + e^{-\beta(\varepsilon + \sigma b - \mu)} \right). \quad (4.37)$$

The corresponding grand potential reads

$$\Omega(T, \mathbf{B}) = -k_B T \ln(Z) = -k_B T \sum_{\sigma} \int_0^{\infty} \rho(\varepsilon) \ln \left( 1 + e^{-\beta(\varepsilon + \sigma b - \mu)} \right).$$

The magnetization in z-direction is obtain via

$$\begin{aligned}
 M &= -\frac{\partial \Omega}{\partial B} \Big|_T \\
 &= k_B T \sum_{\sigma} \left( -\frac{\beta \sigma \mu_B g_e}{2} \right) \int_0^{\infty} d\varepsilon \rho(\varepsilon) \frac{e^{-\beta(\varepsilon + \sigma b - \mu)}}{1 + e^{-\beta(\varepsilon + \sigma b - \mu)}} \\
 &= -\frac{\mu_B g_e}{2} \sum_{\sigma} \sigma \int_0^{\infty} d\varepsilon \rho(\varepsilon) n_F(\varepsilon + \sigma b | T, \mu) \\
 &= -\frac{\mu_B g_e}{\hbar} \underbrace{\frac{\hbar(N_+ - N_-)}{2}}_{=\langle S_{\text{total}}^z \rangle}
 \end{aligned}$$

In agreement with equation (4.2) [p. 101]. Again we assume that  $b$  is small and employ a Taylor expansion.

$$\begin{aligned}
 M &= -\frac{\mu_B g_e}{2} \sum_{\sigma} \sigma \int_0^{\infty} d\varepsilon \left( \rho(\varepsilon) n_F(\varepsilon | T, \mu) + \sigma b n'_F(\varepsilon | T, \mu) + O(b^2) \right) \\
 &= -\frac{\mu_B g_e}{2} 2b \int_0^{\infty} d\varepsilon \rho(\varepsilon) n'_F(\varepsilon | T, \mu) + O(b^2) \\
 &= -\frac{(\mu_B g_e)^2}{2} B \int_0^{\infty} d\varepsilon \rho(\varepsilon) n'_F(\varepsilon | T, \mu) + O(b^2) .
 \end{aligned}$$

Then the susceptibility reads

$$\begin{aligned}
 \chi_T &= \mu_0 \frac{\partial M}{\partial B} \Big|_T \Big|_{B=0} \\
 &= -\mu_0 \frac{(\mu_B g_e)^2}{2} \int_0^{\infty} d\varepsilon \rho(\varepsilon) n'_F(\varepsilon | T, \mu) .
 \end{aligned}$$

This can also be written as

$$\begin{aligned}
 \chi_T &= -\mu_0 \mu_B^2 \left( \frac{g_e}{2} \right)^2 2 \int d\varepsilon \rho(\varepsilon) n'_F(\varepsilon | T, \mu) \\
 &= \mu_0 \mu_B^2 \left( \frac{g_e}{2} \right)^2 2 \frac{\partial}{\partial \mu} \int d\varepsilon \rho(\varepsilon) n_F(\varepsilon | T, \mu) .
 \end{aligned}$$

Comparison with equation (4.31) [p. 119] yields

$$\chi_T = \mu_0 \mu_B^2 \left( \frac{g_e}{2} \right)^2 \frac{\partial N}{\partial \mu} \Big|_{T, B=0} \quad (4.38)$$

We use the Sommerfeld expansion, derived in appendix (5.1) to expand the integral in powers of  $k_B T/\mu$ .

$$\begin{aligned}
\chi &= -\mu_0 \frac{(\mu_B g_e)^2}{2} \int_0^\infty d\varepsilon \rho(\varepsilon) n'_F(\varepsilon|T, \mu) \\
&= \mu_0 \frac{(\mu_B g_e)^2}{2} \int_0^\infty d\varepsilon \rho'(\varepsilon) n_F(\varepsilon|T, \mu) \\
&= \mu_0 \frac{(\mu_B g_e)^2}{2} \left( \int_0^\mu d\varepsilon \rho'(\varepsilon) + \frac{\pi^2}{6} (k_B T)^2 \rho''(\mu) + \mathcal{O}\left(\left(\frac{k_B T}{\mu}\right)^4\right) \right) \\
&= \mu_0 \frac{(\mu_B g_e)^2}{2} \left( \rho(\mu) + \frac{\pi^2}{6} (k_B T)^2 \rho''(\mu) + \mathcal{O}\left(\left(\frac{k_B T}{\mu}\right)^4\right) \right).
\end{aligned}$$

The final result reads

$$\chi = \mu_0 \frac{(\mu_B g_e)^2}{2} \rho(\mu) \left( 1 + \frac{\pi^2}{6} (k_B T)^2 \frac{\rho''(\mu)}{\rho(\mu)} + \dots \right) \quad (4.39)$$

Now we recall, that  $\rho(\varepsilon) = D\sqrt{\varepsilon}$ , resulting in  $\rho''(\mu)/\rho(\mu) = -1/(4\mu^2)$ .

<p>MAGNETIC SUSCEPTIBILITY (free electron gas)</p> <hr style="border: 0.5px solid black; margin: 10px 0;"/> $\chi_P = \mu_0 \frac{(\mu_B g_e)^2}{2} \rho(\mu) \left( 1 - \frac{\pi^2}{24} \left( \frac{k_B T}{\mu} \right)^2 + \dots \right). \quad (4.40)$
---

We see that indeed  $k_B T/\mu$  is the relevant small parameter. This describes the **Pauli-spin-paramagnetism**, which is almost temperature independent for low  $T$ ; in strong contrast to the Curie  $1/T$  behaviour. The reason for the discrepancy lies in the fermi-statistics. Only the spin of the electrons in the vicinity of  $\mu$  can contribute. The number of thermally excited electrons is  $k_B T \rho(\mu)$  which compensates the  $1/T$  behaviour.

#### 4.4.2 Landau diamagnetism

So far we have only considered the spin degrees of freedom of the free electron gas. The orbital moments also contribute to the magnetization, which results in the Landau diamagnetism.

The single electron energies are now

$$\begin{aligned}\varepsilon_{n,\sigma}(k_z) &= \varepsilon(k_z) + \hbar\omega_c(n + \tfrac{1}{2}) + \sigma b \\ \varepsilon(k_z) &= \frac{\hbar^2 k_z^2}{2m}\end{aligned}$$

with  $b = \frac{\mu_B g_e}{2}$  and the cyclotron frequency  $\omega_c = \frac{eB}{m}$ , and  $\mu_B = \frac{e\hbar}{2m}$ . The degeneracy of these levels is

$$N_{\text{deg}} = \frac{L_x L_y e B}{2\pi\hbar} = BC_{\text{deg}}.$$

We use again the formula for the grand canonical partition function of non-interacting particles

$$\ln(Z) = \sum_{\nu} \ln(1 + e^{-\beta(\varepsilon_{\nu} - \mu)})$$

Inserting the one particle energies results in

$$\ln(Z) = BC_{\text{deg}} \sum_{\sigma} \sum_{n=0}^{\infty} \underbrace{\int d\varepsilon \rho^{(1d)}(\varepsilon) \ln\left(1 + e^{-\beta(\varepsilon + \hbar\omega_c(n + \frac{1}{2}) + b\sigma - \mu)}\right)}_{\substack{:= F(\mu - \hbar\omega_c(n + \frac{1}{2}) - b\sigma) \\ := f(n + \frac{1}{2})}}. \quad (4.41)$$

Now we invoke the leading Euler-Mac-Laurin formula (see section 5.2) in equation (5.2) [p. 144] to leading order and use  $f'(\infty) = 0$ .

$$\begin{aligned}\sum_{n=0}^{\infty} f(n + \tfrac{1}{2}) &= \int_0^{\infty} f(x) dx + \frac{1}{24} f'(0) + \dots \\ &= \int_0^{\infty} F(\mu - \hbar\omega_c x - b\sigma) dx + \frac{1}{24} \frac{\partial}{\partial x} F(\mu - \hbar\omega_c x - b\sigma) \Big|_{x=0} + \dots \\ &= \frac{1}{\hbar\omega_c} \int_0^{\infty} F(\mu - y - b\sigma) dy - \frac{\hbar\omega_c}{24} \frac{\partial}{\partial \mu} F(\mu - b\sigma) + \dots\end{aligned}$$

Next we expand this expression in terms of  $B$ , or rather  $b$ , up to second order. We also include already the sum over the electronic spin, which allows

to omit term proportional to  $\sigma^m$  for odd  $m$ .

$$\begin{aligned}
\sum_{\sigma} \sum_n f(n + \tfrac{1}{2}) &= \sum_{\sigma} \frac{1}{\hbar\omega_c} \left( \int_0^{\infty} F(\mu - y) dy - \cancel{b\sigma} \int_0^{\infty} \frac{\partial}{\partial \mu} F(\mu - y) dy + \right. \\
&\quad \left. + \frac{b^2}{2} \int_0^{\infty} \frac{\partial^2}{\partial \mu^2} F(\mu - y) dy - \frac{(\hbar\omega_c)^2}{24} \frac{\partial}{\partial \mu} F(\mu) + \mathcal{O}(B^4) \right) \\
&= \frac{2}{\hbar\omega_c} \left( \int_{-\infty}^{\mu} F(z) dz + \right. \\
&\quad \left. + \frac{b^2}{2} \int_0^{\infty} \frac{\partial^2}{\partial y^2} F(\mu - y) dy - \frac{(\hbar\omega_c)^2}{24} \frac{\partial}{\partial \mu} F(\mu) + \mathcal{O}(B^4) \right) \\
&= \frac{2}{\hbar\omega_c} \left( \int_{-\infty}^{\mu} F(z) dz + \right. \\
&\quad \left. + \frac{b^2}{2} \left( \frac{\partial}{\partial \mu} F(\mu - y) \right) \Big|_{y=0}^{y=\infty} - \frac{(\hbar\omega_c)^2}{24} \frac{\partial}{\partial \mu} F(\mu) + \mathcal{O}(B^4) \right) \\
&= \frac{2}{\hbar\omega_c} \left( \int_{-\infty}^{\mu} F(z) dz + \frac{1}{2} \left( b^2 - \frac{(\hbar\omega_c)^2}{12} \right) \frac{\partial}{\partial \mu} F(\mu) + \mathcal{O}(B^4) \right).
\end{aligned}$$

Inserting the definition of  $\omega_c$  and  $b$  yields

$$\begin{aligned}
b^2 - \frac{(\hbar\omega_c)^2}{12} &= B^2 \left( \mu_B^2 \left( \frac{g_e}{2} \right)^2 - \frac{1}{12} \left( \frac{\hbar e}{m} \right)^2 \right) \\
&= B^2 \mu_B^2 \left( \left( \frac{g_e}{2} \right)^2 - \frac{4}{12} \right)
\end{aligned}$$

The grand potential up to order  $B^2$  reads

$$\begin{aligned}
\Omega &= -k_B T \ln(Z) \\
&= - \underbrace{\frac{2k_B T C_{deg} B}{\hbar\omega_c}}_{=D} \left( \int_0^{\infty} F(\mu - y) dy + \frac{\mu_B^2 B^2}{2} \left( \left( \frac{g_e}{2} \right)^2 - \frac{4}{12} \right) \frac{\partial}{\partial \mu} F(\mu) \right) \\
&= D \int_0^{\infty} F(\mu - y) dy + \frac{1}{2} \left( \left( \frac{g_e}{2} \right)^2 - \frac{4}{12} \right) D \frac{\partial}{\partial \mu} F(\mu).
\end{aligned}$$

For  $B = 0$  we obtain

$$\Omega(T, \mu, B = 0) = D \int_0^{\infty} F(\mu - y) dy$$

We can exploit

$$\begin{aligned} \frac{\partial}{\partial \mu} \int_{-\infty}^{\mu} F(z) dz &= F(\mu) \\ \Rightarrow \quad \frac{\partial}{\partial \mu} F(\mu) &= \frac{\partial^2}{\partial \mu^2} \int_{-\infty}^{\mu} F(z) dz \end{aligned}$$

to rewrite the grand potential as

$$\Omega(T, \mu, B) = \Omega(T, \mu, B = 0) + \frac{\mu_B^2 B^2}{2} \left( \left( \frac{g_e}{2} \right)^2 - \frac{1}{3} \right) \frac{\partial^2}{\partial \mu^2} \Omega(T, \mu, B = 0) .$$

The susceptibility is defined as

$$\chi = - \frac{\partial \Omega}{\partial B^2} \Big|_{\mu, T} = - \mu_B^2 \left( \left( \frac{g_e}{2} \right)^2 - \frac{1}{3} \right) \frac{\partial^2}{\partial \mu^2} \Omega(T, \mu, B = 0) ,$$

Finally, we exploit the equation  $\frac{\partial \Omega}{\partial \mu} \Big|_{T, B} = N(T, \mu, B)$  resulting in

$$\chi = - \frac{\partial \Omega}{\partial B^2} \Big|_{\mu, T} = \left( 1 - \frac{1}{3} \left( \frac{2}{g_e} \right)^2 \right) \left( \mu_B^2 \left( \frac{g_e}{2} \right)^2 \frac{\partial}{\partial \mu} N(T, \mu, B = 0) \right) ,$$

According to equation (4.38) [p. 122] the last bracket is equal to the Pauli-spin susceptibility, i.e.

$$\chi = \chi_P - \frac{1}{3} \left( \frac{2}{g_e} \right)^2 \chi_P$$

The additional term is called *Landau diamagnetism* and for  $g_e \approx 2$  is

$$\chi_L = -\frac{1}{3} \chi_P$$

The total susceptibility is still positive

$$\chi \approx \frac{2}{3} \chi_P .$$

LANDAU DIAMAGNETISM  
(free electron gas)

$$\chi_L = -\frac{1}{3} \chi_P \tag{4.42}$$

## 4.5 Collective magnetism

### 4.5.1 Heisenberg hamiltonian

In many crystalline systems, the electrons can be split into those that are localized at the atomic sites and others which can move freely through the crystal. The former result in localized magnetic moments or spins  $\mathbf{S}_i$ , if they do not belong to closed shells. We have discussed these local spins  $\mathbf{S}_i$  before, but there we have focussed on systems where these spins do not interact. The Coulomb interaction between the electrons can lead to various types of exchange interactions between these spins. The exchange can either result from exchange processes of the electrons that form the local spins on neighboring sites *direct exchange* or it can be mediated by additional electrons *indirect exchange*. In the latter case, the interaction can be long ranged, like in the case of the so-called *RKKY* (Ruderman-Kittel-Kasuya-Yosida). In RKKY the interaction decays like  $1/|\mathbf{R}_i - \mathbf{R}_j|^3$  with the distance between the spins at site  $\mathbf{R}_i$  and  $\mathbf{R}_j$ . In addition, the exchange interaction changes sign as function of distance, i.e. depending on the distance between the localized spins it can be ferromagnetic and anti-ferromagnetic. In all cases, the interaction can be described by a prominent model for collective magnetism, the

#### HEISENBERG MODEL

$$H = - \sum_{jj'} \left( J_{jj'}^z S_j^z S_{j'}^z + J_{jj'}^{xy} (S_j^x S_{j'}^x + S_j^y S_{j'}^y) \right) + b \sum_j S_j^z. \quad (4.43)$$

Here  $J_{jj'} = J(|\mathbf{R}_j - \mathbf{R}_{j'}|)$ , with  $J_{jj} = 0$ .  $J_{jj'}$  can always be chosen symmetric, since by substituting  $j \leftrightarrow j'$  we find

$$H_H = - \sum_{\alpha} \sum_{jj'} J_{jj'} S_j^{\alpha} S_{j'}^{\alpha} = - \sum_{\alpha} \sum_{jj'} J_{j'j} S_{j'}^{\alpha} S_j^{\alpha} = - \sum_{\alpha} \sum_{jj'} J_{j'j} S_j^{\alpha} S_{j'}^{\alpha}$$

Hence

$$H = \frac{1}{2} \left( - \sum_{\alpha} \sum_{jj'} J_{jj'} S_j^{\alpha} S_{j'}^{\alpha} - \sum_{\alpha} \sum_{jj'} J_{j'j} S_{j'}^{\alpha} S_j^{\alpha} \right) = - \sum_{\alpha} \sum_{jj'} \left( \frac{J_{jj'} + J_{j'j}}{2} \right) S_j^{\alpha} S_{j'}^{\alpha}.$$

We also introduce the ladder operators

$$S_j^{\pm} = S_j^x \pm i S_j^y. \quad (4.44)$$



In these operators, the hamiltonian reads

$$H = - \sum_{jj'} \left( J_{jj'}^z S_j^z S_{j'}^z + J_{jj'}^{xy} S_j^+ S_{j'}^- \right) + b \sum_j S_j^z .$$

The order of the operators  $S_j^+ S_{j'}^-$  is irrelevant, since  $j \neq j'$  and in that case they commute. There are three important limiting cases depending on the anisotropy of the crystal in spin-space.

- ( $|J^z| \ll |J^{xy}|$ ) results in the so-called *xy- model*

XY-MODEL	
$H = - \sum_{jj'} J_{jj'}^{xy} S_j^+ S_{j'}^- + b \sum_j S_j^z .$	(4.45)

- ( $|J^z| \gg |J^{xy}|$ ) results in the *Ising model*, which is discussed in great detail in chapter (2) [\[page 35\]](#)
- ( $J^z = J^{xy}$ ) represents the isotropic Heisenberg model

HEISENBERG MODEL ( <i>isotropic case</i> )	
$H = - \sum_{jj'} J_{jj'} \mathbf{S}_j \mathbf{S}_{j'} + b \sum_j S_j^z .$	(4.46)

Very often the exchange coupling  $J_{jj'} = J(|\mathbf{R}_j - \mathbf{R}_{j'}|)$  decreases very rapidly with distance and it suffices to take only nearest neighbour (nn) interactions into account, i.e.

$$- \sum_{jj'} J_{jj'} \approx -J \sum_{\langle jj' \rangle} . \quad (4.47)$$

In general,  $J$  can be positive or negative. In the first (second) case, ferromagnetic (anti-ferromagnetic) spin alignment is favoured. If the exchange coupling between localized spins is mediated by itinerant electrons, the so-called *RKKY-interaction* (Ruderman-Kittel-Kasuya-Yosida) results, which is long ranged and oscillating as function of distance, i.e. depending on the distance between the localized spins it can be ferromagnetic and anti-ferromagnetic.

### 4.5.2 Mermin-Wagner Theorem

There are only a few exact results available for the Heisenberg model. Presumably the most important one is due to Mermin and Wagner and states: *In one and two dimensions, continuous symmetries cannot be broken spontaneously at finite temperature in systems with sufficiently short-ranged interactions.*

We add an external field

$$\mathbf{B}_j = b \mathbf{e}_z e^{i\mathbf{Q}\mathbf{R}_j} ,$$

to the isotrop ferromagnetic Heisenberg model. This B-field couples to different types of magnetic order (e.g. ferro, anti-ferro), depending on the choice of the wave vector  $\mathbf{Q}$ . We consider the order parameter

$$\mathcal{S}_{\mathbf{Q}}(T, b) = \frac{1}{\hbar N} \langle \mathbf{S}_{\mathbf{Q}}^z \rangle_{T, b} . \quad (4.48)$$

The operator  $\mathcal{S}_{\mathbf{Q}}^z$  stands for the Fourier transform of  $S_j^z$ , defined in equation (5.11) [p. 149].

After a straightforward but tedious calculation, which can be found in appendix 5.5, the result reads

$$1D) \quad |\mathcal{S}_{\mathbf{Q}}(T, b)| \leq \left( \frac{|b|}{T^2} \right)^{1/3} \xrightarrow{b \rightarrow 0} 0 \quad (4.49a)$$

$$2D) \quad |\mathcal{S}_{\mathbf{Q}}(T, b)| \leq \frac{const}{\sqrt{T |\ln(|b|)|}} \xrightarrow{b \rightarrow 0} 0 \quad (4.49b)$$

$$3D) \quad |\mathcal{S}_{\mathbf{Q}}(T, b)| \leq \frac{const}{\sqrt{T}} \quad (4.49c)$$

For finite  $T$  the rhs in 1D and 2D goes to zero for vanishing field ( $b \rightarrow 0$ ). Hence there is no finite order parameter possible, irrespective of the wave vector  $\mathbf{Q}$  (order). For 3D, however, the Mermin-Wagner theorem does not rule out spontaneous magnetization for finite  $T$ .

### 4.5.3 Exact ground-state of the ferromagnetic Heisenberg model

Here we consider only the **homogeneous** ferromagnetic ( $J_{jj'} > 0$ ) Heisenberg model with a homogenous magnetic field pointing in the  $z$ -direction. We readily see that the ground state is given by the state, where all spins are maximally aligned in the negative  $z$ -direction

$$|0\rangle = \otimes_{j=1}^N |-S\rangle_j \quad (4.50)$$

with

$$S_j^z |m\rangle_j = \hbar m |m\rangle_j \quad (4.51a)$$

$$S_j^2 |m\rangle_j = \hbar^2 S(S+1) |m\rangle_j . \quad (4.51b)$$

Applying the hamiltonian to  $|0\rangle$  yields

$$\begin{aligned} H|0\rangle &= - \sum_{jj'} J_{jj'} S_j^z S_{j'}^z |0\rangle - \sum_{jj'} J_{jj'} S_j^+ \underbrace{S_{j'}^-}_{=0} |0\rangle + b \sum_j S_j^z |0\rangle \\ &= - \sum_{jj'} J_{jj'} \hbar^2 S^2 |0\rangle - b \sum_j \hbar S |0\rangle \\ &= -\hbar \left( N \sum_l J(|\mathbf{R}_l|) \hbar S^2 + b \sum_j S \right) |0\rangle \\ &= -\hbar N \left( \hbar \tilde{J}(0) S^2 + b S \right) |0\rangle \\ \tilde{J}(0) &= \sum_l J(|\mathbf{R}_l|) . \end{aligned}$$

We have introduced the Fourier transform of the exchange coupling.

$$\tilde{J}(\mathbf{k}) = \sum_l J(|\mathbf{R}_l|) e^{i\mathbf{k}\mathbf{R}_l} . \quad (4.52)$$

Obviously,  $|0\rangle$  is an eigenvector. We consider

$$\begin{aligned} (\mathbf{S}_j + \mathbf{S}_{j'})^2 &= \mathbf{S}_j^2 + \mathbf{S}_{j'}^2 + 2\mathbf{S}_j \mathbf{S}_{j'} \\ -\mathbf{S}_j \mathbf{S}_{j'} &= \frac{1}{2} (\mathbf{S}_j^2 + \mathbf{S}_{j'}^2) - \frac{1}{2} (\mathbf{S}_j + \mathbf{S}_{j'})^2 \end{aligned}$$

All spins have magnitude  $S$ . We consider an arbitrary vector  $|\psi\rangle$ , which is an eigenstate of all  $\mathbf{S}_j^2$  with eigenvalue  $S(S+1)$  but otherwise arbitrary. Then

$$-\langle \mathbf{S}_j \mathbf{S}_{j'} \rangle = S(S+1) - \frac{1}{2} \langle (\mathbf{S}_j + \mathbf{S}_{j'})^2 \rangle$$

Now  $\mathbf{S}' := \mathbf{S}_j + \mathbf{S}_{j'}$  is an effective spin where the eigenvalues of  $\mathbf{S}'^2$  are given by  $S'(S' + 1)$  with  $S' \in \hbar\{0, 1, \dots, 2S\}$ . Consequently,

$$\left\langle (\mathbf{S}_j + \mathbf{S}_{j'})^2 \right\rangle = \langle \mathbf{S}'^2 \rangle \leq 2S(2S + 1) .$$

Then by plugging this inequality into the equation above we obtain

$$-\langle \mathbf{S}_j \mathbf{S}_{j'} \rangle \geq S(S + 1) - S(2S + 1) = -S^2$$

Finally that means if  $J_{jj'} \geq 0 \forall j, j'$

$$\begin{aligned} \langle H_0 \rangle &= \sum_{jj'} J_{jj'} \left( -\langle \mathbf{S}_j \mathbf{S}_{j'} \rangle \right) \\ &\geq -S^2 \hbar^2 \sum_{jj'} J_{jj'} \\ \langle H_0 \rangle &\geq -N \hbar^2 \tilde{J}(0) . \end{aligned}$$

For the Zeeman term we see readily

$$\langle H_Z \rangle \geq -bS\hbar$$

Hence in total we have for the energy

$$E \geq -N \hbar^2 \tilde{J}(0) - bS\hbar = E_0$$

Since there is no energy lower than the ferromagnetic state, it is the ground-state. Moreover, we see that

$$S_{total}^z = \sum_j \mathbf{S}_j^z |0\rangle = \hbar NS |0\rangle .$$

The groundstate is an eigenstate of  $S_{total}^z$  and also of

$$\begin{aligned} \mathbf{S}_{total}^2 &= \sum_{j \neq j'} \mathbf{S}_j \mathbf{S}_{j'} + \sum_j \mathbf{S}_j^2 \\ \mathbf{S}_{total}^2 |0\rangle &= \sum_{j \neq j'} \mathbf{S}_j \mathbf{S}_{j'} |0\rangle + N \hbar^2 S(S + 1) |0\rangle \\ \mathbf{S}_{total}^2 |0\rangle &= \sum_{j \neq j'} \hbar^2 S^2 |0\rangle + \underbrace{S_j^+ S_{j'}^-}_{=0} |0\rangle + NS(S + 1) \hbar^2 |0\rangle \\ &= \left( N(N - 1) \hbar^2 S^2 + NS(S + 1) \hbar^2 \right) |0\rangle \\ &= \hbar^2 \left( N^2 S^2 - \cancel{NS^2} + \cancel{NS^2} + NS \right) |0\rangle \\ &= \hbar^2 NS(NS + 1) |0\rangle \end{aligned}$$

So it  $|0\rangle$  an eigenstate of  $\mathbf{S}_{total}$  with maximum magnitude  $NS\hbar$ .

#### 4.5.4 Spin waves in the ferromagnetic Heisenberg model

Next we study the low-lying excited states. To this end we first consider the Hamiltonian in Fourier space. The details of the Fourier transform as introduced in equation (5.11) [p. 149] are

$$S_k^\alpha = \sum_j S_j^\alpha e^{i\mathbf{k}\mathbf{R}_j}, \quad \alpha \in \{x, y, z\} \quad (4.53a)$$

$$S_j^\alpha = \frac{1}{N} \sum_k S_k^\alpha e^{-i\mathbf{k}\mathbf{R}_j} \quad (4.53b)$$

We recall the commutation relations

$$[S_j^\alpha S_{j'}^\beta] = \delta_{jj'} i\hbar \varepsilon_{\alpha\beta\gamma} S_j^\gamma, \quad (4.54a)$$

$$[S_j^z S_{j'}^\pm] = \pm \delta_{jj'} \hbar S_j^\pm, \quad (4.54b)$$

$$[S_j^-, S_{j'}^+] = -2\hbar \delta_{jj'} S_j^z, \quad (4.54c)$$

$$[S_j^+, S_{j'}^+] = 0. \quad (4.54d)$$

Then

$$[S_{\mathbf{k}}^z S_{\mathbf{k}'}^\pm] = \pm \hbar S_{\mathbf{k}+\mathbf{k}'}^\pm, \quad (4.55a)$$

$$[S_{\mathbf{k}}^-, S_{\mathbf{k}'}^+] = -2\hbar S_{\mathbf{k}+\mathbf{k}'}^z, \quad (4.55b)$$

$$[S_j^+, S_{j'}^+] = 0. \quad (4.55c)$$

These commutators can be proved by employing equation (4.53) in the commutator and then making use of the commutation relations that are given above. In the following equations, first the necessary commutators will be calculated and then used in the ansatz using the Fourier transformed version. Lastly, by identifying again the Fourier transformed operators, the compact form of the commutation relations is obtained.

We need to know the following commutators:

$$\begin{aligned} [S_j^z S_{j'}^\pm] &= [S_j^z S_{j'}^x] \pm i [S_j^z S_{j'}^y] \\ &= \delta_{jj'} i\hbar \left( \varepsilon_{zxy} S_j^y \pm i \varepsilon_{zyx} S_j^x \right) \\ &= \delta_{jj'} \hbar \left( i \underbrace{\varepsilon_{zxy}}_{=+1} S_j^y \mp \underbrace{\varepsilon_{zyx}}_{=-1} S_j^x \right) \\ &= \delta_{jj'} \hbar \left( i S_j^y \pm S_j^x \right) \\ &= \pm \delta_{jj'} \hbar \left( S_j^x \pm i S_j^y \right). \end{aligned}$$

$$\begin{aligned}
[S_{\mathbf{k}}^z S_{\mathbf{k}'}^\pm] &= \sum_{jj'} e^{i(\mathbf{k}\mathbf{x}_j + \mathbf{k}'\mathbf{x}_{j'})} [S_j^z S_{j'}^\pm] \\
&= \pm \hbar \sum_{jj'} e^{i(\mathbf{k}\mathbf{x}_j + \mathbf{k}'\mathbf{x}_{j'})} \delta_{jj'} S_j^\pm \\
&= \pm \hbar \sum_j e^{i(\mathbf{k} + \mathbf{k}')\mathbf{x}_j} S_j^\pm \\
&= \pm \hbar S_{\mathbf{k} + \mathbf{k}'}^\pm .
\end{aligned}$$

$$\begin{aligned}
[S_j^-, S_{j'}^+] &= [(S_j^x - iS_j^y), (S_{j'}^x + iS_{j'}^y)] \\
&= i[S_j^x, S_{j'}^y] - [S_j^y, S_{j'}^x] \\
&= 2i\delta_{jj'} (i\hbar S_j^z) \\
&= -2\hbar\delta_{jj'} S_j^z .
\end{aligned}$$

Now we look at the commutators of the operators as given above:

$$\begin{aligned}
[S_{\mathbf{k}}^-, S_{\mathbf{k}'}^+] &= \sum_{jj'} e^{i(\mathbf{k}\mathbf{j} + \mathbf{k}'\mathbf{j}')} \underbrace{[S_j^-, S_{j'}^+]}_{=-2\hbar\delta_{jj'} S_j^z} \\
&= -2\hbar \sum_j e^{i(\mathbf{k} + \mathbf{k}')\mathbf{j}} S_j^z \\
&= -2\hbar S_{\mathbf{k} + \mathbf{k}'}^z
\end{aligned}$$

and obtain by employing the Fourier transform in the known Hamiltonian

$$\begin{aligned}
H &= - \sum_{jj'} J_{jj'} \left( S_j^z S_{j'}^z + S_j^+ S_{j'}^- \right) + b \sum_j S_j^z \\
&= - \frac{1}{N^2} \sum_{kk'} \sum_{jj'} J(\underbrace{\mathbf{R}_{j'} - \mathbf{R}_j}_{=\mathbf{R}_l}) e^{-i(\mathbf{R}_j \mathbf{k} + \mathbf{R}_{j'} \mathbf{k}')} \left( S_k^z S_{k'}^z + S_k^+ S_{k'}^- \right) \\
&\quad + b \frac{1}{N} \sum_k \underbrace{\sum_j e^{i\mathbf{R}_j \mathbf{k}} S_k^z}_{=N\delta_{\mathbf{k},0}} \\
&= - \frac{1}{N^2} \sum_{kk'} \sum_{jl} J(|\mathbf{R}_l|) e^{-i(\mathbf{R}_j \mathbf{k} + (\mathbf{R}_j + \mathbf{R}_l) \mathbf{k}')} \left( S_k^z S_{k'}^z + S_k^+ S_{k'}^- \right) + b S_{\mathbf{k}=0}^z \\
&= - \frac{1}{N} \sum_{kk'} \sum_l J(|\mathbf{R}_l|) e^{-i\mathbf{k}' \mathbf{R}_l} \underbrace{\sum_j e^{-i\mathbf{R}_j (\mathbf{k} + \mathbf{k}')}}_{=N\delta_{\mathbf{k}', -\mathbf{k}}} \left( S_k^z S_{k'}^z + S_k^+ S_{k'}^- \right) + b S_{\mathbf{k}=0}^z .
\end{aligned}$$

Eventually, we have with equation (4.52) [p. 130] (Fourier transform of the exchange coupling  $J$ )

$$H = -\frac{1}{N} \sum_k \tilde{J}(k) \left( S_k^z S_{-k}^z + S_k^+ S_{-k}^- \right) + b S_{\mathbf{k}=0}^z . \quad (4.56)$$

There exist exact eigenstates for the low-lying excitations given by

$$\left| \tilde{\psi}(q) \right\rangle = S_q^+ |0\rangle . \quad (4.57)$$

First we will prove that these states are indeed eigenstates

$$\begin{aligned} H \left| \tilde{\psi}(q) \right\rangle &= H S_q^+ |0\rangle \\ &= S_q^+ \underbrace{H |0\rangle}_{=E_0|0\rangle} + [H, S_q^+] |0\rangle \\ &= E_0 \left| \tilde{\psi}(q) \right\rangle + [H, S_q^+] |0\rangle . \end{aligned}$$

For the commutator we first evaluate the Zeeman part of the hamiltonian

$$[H_Z, S_q^+] |0\rangle = b [S_{q=0}^z, S_q^+] |0\rangle = b \hbar S_{q+0}^+ |0\rangle = \hbar b \left| \tilde{\psi}(q) \right\rangle .$$

For the Heisenberg part of the hamiltonian we obtain

$$\begin{aligned} [H_H, S_q^+] &= -\frac{1}{N} \sum_k \tilde{J}(k) \left( [S_k^z S_{-k}^z, S_q^+] + [S_k^+ S_{-k}^-, S_q^+] \right) \\ &= -\frac{1}{N} \sum_k \tilde{J}(k) \left( S_k^z [S_{-k}^z, S_q^+] + [S_k^z, S_q^+] S_{-k}^z + S_k^+ [S_{-k}^-, S_q^+] \right) \\ &= -\frac{\hbar}{N} \sum_k \tilde{J}(k) \left( \underbrace{S_k^z S_{q-k}^+}_{=S_{q-k}^+ S_k^z + [S_k^z, S_{q-k}^+]} + S_{q+k}^+ S_{-k}^z - 2 S_k^+ S_{q-k}^z \right) \\ &\quad \underbrace{[S_k^z, S_{q-k}^+]}_{=\hbar S_q^+} \\ &= -\frac{\hbar}{N} \sum_k \tilde{J}(k) \left( S_{q-k}^+ S_k^z + S_{q+k}^+ S_{-k}^z - 2 S_k^+ S_{q-k}^z \right) - \frac{\hbar^2}{N} \left( \sum_k \tilde{J}(k) \right) S_q^+ . \end{aligned}$$

The last term vanishes because

$$\sum_k \tilde{J}(\mathbf{k}) = \sum_l J(|\mathbf{R}_l|) \underbrace{\sum_k e^{i\mathbf{k}\mathbf{R}_l}}_{=N\delta_{l,0}} = N J_l = 0 .$$

For the first terms we need to consider how  $S_z^k$  acts on the state  $|0\rangle$ :

$$S_k^z |0\rangle = \sum_l e^{i\mathbf{k}\mathbf{R}_l} \underbrace{S_l^z |0\rangle}_{=-\hbar S |0\rangle} = -\hbar S \left( \sum_l e^{i\mathbf{k}\mathbf{R}_l} \right) |0\rangle = -\hbar S N \delta_{\mathbf{k},0} |0\rangle .$$

Hence,

$$\begin{aligned} [H_H, S_q^+] |0\rangle &= \frac{\hbar^2 S N}{N} \sum_{\mathbf{k}} \tilde{J}(\mathbf{k}) \left( S_{\mathbf{q}-\mathbf{k}}^+ \delta_{\mathbf{k},0} + S_{\mathbf{q}+\mathbf{k}}^+ \delta_{\mathbf{k},0} - 2S_{\mathbf{k}}^+ \delta_{\mathbf{k}-\mathbf{q}} \right) |0\rangle \\ &= \hbar^2 S \left( 2\tilde{J}(0) - 2\tilde{J}(q) \right) S_q^+ |0\rangle \\ &= 2\hbar^2 S (\tilde{J}(0) - \tilde{J}(q)) \left| \tilde{\Psi}(q) \right\rangle \end{aligned}$$

Putting all terms together proves that  $\left| \tilde{\psi}(q) \right\rangle$  is an eigen vector

$$H \left| \tilde{\Psi}(q) \right\rangle = (E_0 + \hbar\omega_q) \left| \tilde{\Psi}(q) \right\rangle , \quad (4.58)$$

with eigenvalue

$$\omega_q = b + 2\hbar S (\tilde{J}(0) - \tilde{J}(q)) . \quad (4.59)$$

For nearest neighbour exchange interactions,

$$\tilde{J}(q) = J \sum_{\delta} e^{i\delta\mathbf{q}} = 2J \sum_{\nu=1}^d \cos(q_{\nu}) , .$$

The sum runs over the cartesian coordinates. Hence

$$\tilde{J}(0) - \tilde{J}(q) = 2J \sum_{\nu} (1 - \cos(q_{\nu})) .$$

For  $b = 0$  and small values of  $q_{\nu}$  the leading order of the Taylor expansion yields

$$\begin{aligned} \hbar\omega_q &= 2\hbar^2 S (\tilde{J}(0) - \tilde{J}(q)) \\ &= 2\hbar^2 S \cdot 2J \sum_{\nu} \left( 1 - \left( 1 - \frac{q_{\nu}^2}{2} \right) \right) \\ &= 4\hbar^2 S J \sum_{\nu} \frac{q_{\nu}^2}{2} \\ &= 2\hbar^2 S J \mathbf{q}^2 \\ &= D \mathbf{q}^2 \end{aligned}$$

The excitation has a quadratic dispersion with a *spin wave stiffness*  $D = 2\hbar^2 S J$ . The excitation energy tends continuously to zero for small  $q$ . This is a typical feature of a *Goldstone mode*, which tries to restore the broken symmetry.



### Properties of the single magnon state

We will study further properties of the one-magnon excitation. We start out with the scalar product of one-magnon states for different wavevectors

$$\begin{aligned}
 \langle \tilde{\Psi}(q) | \tilde{\Psi}(q') \rangle &= \langle 0 | S_{-q}^- S_{q'}^+ | 0 \rangle \\
 &= \langle 0 | S_{q'}^+ \underbrace{S_{-q}^-}_{=0} | 0 \rangle + \langle 0 | [S_{-q}^-, S_{q'}^+] | 0 \rangle \\
 &= -2\hbar \langle 0 | \underbrace{S_{q'-q}^z}_{= \delta_{q,q'}(-SN\hbar)} | 0 \rangle \\
 &= \delta_{q,q'} 2SN\hbar^2 .
 \end{aligned}$$

The one-magnon states are orthogonal and the correctly normalized vectors are

ONE-MAGNON STATES		
eigenvector:	$ \psi(\mathbf{q})\rangle = \frac{1}{\hbar\sqrt{2NS}} S_q^+  0\rangle$	(4.60a)
excitation energy:	$\omega_q = b + 2S(\tilde{J}(0) - \tilde{J}(q))$	(4.60b)

The one-magnon vectors are also eigenvectors of

$$S_{total}^z = \sum_l S_l^z = S_{q=0}^z .$$

The proof is as follows

$$\begin{aligned}
 S_{q=0}^z S_q^+ |0\rangle &= S_q^+ \underbrace{S_{q=0}^z}_{=-SN\hbar} |0\rangle + \underbrace{[S_{q=0}^z, S_q^+]}_{=\hbar S_{q+0}^+} |0\rangle \\
 &= \hbar(-NS + 1) S_q^+ |0\rangle
 \end{aligned}$$

The eigenvalue of  $S_{total}^z$  has been changed by  $+\hbar$ . Now it is interesting to compare that with the expectation value of  $S_j^z$  of the spin at site  $\mathbf{R}_j$

$$\langle \psi(\mathbf{q}) | S_j^z | \psi(\mathbf{q}) \rangle = \frac{1}{N} \sum_k e^{-i\mathbf{k}\mathbf{R}_j} \langle \psi(\mathbf{q}) | S_k^z | \psi(\mathbf{q}) \rangle . \quad (4.61)$$

We need

$$\begin{aligned}
\langle 0 | S_{-q}^- S_k^z S_q^+ | 0 \rangle &= \langle 0 | S_{-q}^- S_q^+ \underbrace{S_k^z | 0 \rangle}_{=-NSh\delta_{k,0}|0\rangle} + \langle 0 | S_{-q}^- [S_k^z S_q^+] | 0 \rangle \\
a &= -SN\hbar\delta_{k,0} \underbrace{\langle 0 | S_{-q}^- S_q^+ | 0 \rangle}_{=\langle\psi(\mathbf{q})|\psi(\mathbf{q})\rangle} + \hbar \underbrace{\langle 0 | S_{-q}^- S_{k+q}^+ | 0 \rangle}_{=\langle\psi(\mathbf{q})|\psi(\mathbf{q}+\mathbf{k})\rangle} \\
&= \langle\psi(\mathbf{q})|\psi(\mathbf{q})\rangle \delta_{\mathbf{k},0} \hbar (-SN + 1) .
\end{aligned}$$

Hence

$$\langle S_{\mathbf{k}}^z \rangle = \frac{\langle 0 | S_{-\mathbf{q}}^- S_{\mathbf{k}}^z S_{\mathbf{q}}^+ | 0 \rangle}{\langle 0 | S_{-q}^- S_q^+ | 0 \rangle} = \delta_{k,0} \hbar (-SN + 1) .$$

Inserting in equation (4.61) [\[previous page\]](#) yields

$$\langle\psi(\mathbf{q})| S_j^z |\psi(\mathbf{q})\rangle = -\hbar\left(S - \frac{1}{N}\right) \sum_k e^{-\mathbf{k}\mathbf{R}_j} \delta_{k,0} = -\hbar S + \frac{\hbar}{N} .$$

The value of  $S_j^z$  is increased by  $\hbar/N$  at each site, irrespective of the index  $j$ .

#### 4.5.5 Mean field approximation of the isotropic Heisenberg model

The derivation of the MFA results is very similar to that in the Ising model. Remember that the approach for the mean field approximation for the Ising model was the following relation for two dynamical variables A and B:

$$AB \approx \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle$$

We employ this approximation and the hamiltonian in MFA for an homogeneous magnetic field is

$$H^{\text{MFA}} = -\frac{1}{2} \sum_{jj'} J_{jj'} (\langle \mathbf{S}_j \rangle \mathbf{S}_{j'} + \mathbf{S}_j \langle \mathbf{S}_{j'} \rangle) + \frac{1}{2} \sum_{jj'} J_{jj'} \langle \mathbf{S}_j \rangle \langle \mathbf{S}_{j'} \rangle - \mathbf{b} \sum_j \mathbf{S}_j . \quad (4.62)$$

We consider only the restricted MFA, where we retain the translational symmetry of the hamiltonian, i.e.

$$\langle \mathbf{S}_j \rangle = \mathbf{S} \quad \forall j$$

It is obvious in an isotropic model to assume that the mean field will point into the same direction as the external field. Without loss of generality,

we assume that the external field defines the  $z$ -direction ( $\mathbf{b} = b\mathbf{e}_z$ ). With  $\mathcal{S} = \langle S_j^z \rangle$  Then

$$\begin{aligned} H^{\text{MFA}} &= - \underbrace{\left( \sum_{j'} J_{jj'} \right)}_{=\tilde{J}(0)} \mathcal{S} \sum_j S_j^z + \mathcal{S}^2 \frac{1}{2} \underbrace{\sum_{jj'} J_{jj'}}_{=N\tilde{J}(0)} - b \sum_j S_j^z \\ &= - \underbrace{(\tilde{J}(0) \mathcal{S} + b)}_{=b'} \sum_j S_j^z + \frac{N\tilde{J}(0)}{2} \mathcal{S}^2. \end{aligned}$$

We have employed equation (4.52) [p. 130] and emphasize that  $S_j^z$  is an operator.

The partition function is readily computed in the eigen basis of the  $S_i^z$  operators. Since the spins are no longer interacting and the effective field  $b'$  is translational invariant, the partition function of  $N$  spins  $Z_N$  is simply the  $N$ -th power of  $Z_1$ , or rather (see also equation (2.7) [p. 41] of the Ising model)

$$\begin{aligned} \ln(Z_N) &= N \ln(Z_1) \\ Z_1 &= e^{-\frac{\beta\tilde{J}(0)}{2}\mathcal{S}^2} \sum_{\sigma=-S}^S e^{h'\sigma} \\ &= e^{-\frac{\beta\tilde{J}(0)}{2}\mathcal{S}^2} e^{-Sh'} \sum_{\sigma=0}^{2S} e^{h'\sigma} \\ &= e^{-\frac{\beta\tilde{J}(0)}{2}\mathcal{S}^2} e^{-Sh'} \frac{e^{h'(2S+1)} - 1}{e^{h'} - 1} \\ &= e^{-\frac{\beta\tilde{J}(0)}{2}\mathcal{S}^2} \frac{\sinh(h'(2S+1)/2)}{\sinh(h'/2)} \end{aligned}$$

with  $h' = \beta b'$ . Hence

$$\frac{\ln(Z_N)}{N} = -\beta \frac{\tilde{J}(0)}{2} \mathcal{S}^2 + \ln [\sinh(h'(2S+1)/2)] - \ln [\sinh(h'/2)]$$

and for the free energy  $\frac{F}{N} = -k_B T \frac{\ln(Z_N)}{N}$  we obtain

FREE ENERGY OF THE ISOTROPIC HEISENBERG MODEL  
(in mean-field approximation)

$$\frac{F}{N} = \frac{\tilde{J}(0)\mathcal{S}^2}{2} - k_B T \left( \ln [\sinh (h'(2S+1)/2)] - \ln [\sinh (h'/2)] \right) \quad (4.63)$$

Like in the Ising case, the easiest way to compute the order parameter  $\mathcal{S}$  is via

$$\begin{aligned} \mathcal{S} &= \frac{1}{N} \frac{d}{dh'} \ln(Z) = \frac{d}{dh'} \left( \ln [\sinh (h'(2S+1)/2)] - \ln [\sinh (h'/2)] \right) \\ &= \frac{2S+1}{2} \coth (h'(2S+1)/2) - \frac{1}{2} \coth (h'/2) . \end{aligned}$$

In terms of the Brillouin function, defined in equation (4.17) [p. 110], we have

ORDER PARAMETER OF THE ISOTROPIC HEISENBERG MODEL  
(in mean-field approximation)

$$\frac{\mathcal{S}}{S} = \mathcal{B}_S(Sh') \quad (4.64a)$$

$$h' = \beta(\tilde{J}(0)\mathcal{S} + b) . \quad (4.64b)$$

For  $S = 1/2$  we obtain according to equation (4.22) [p. 112]

$$2\mathcal{S} = \mathcal{B}_{1/2}\left(\frac{h'}{2}\right) = \tanh\left(\frac{h'}{2}\right) . \quad (4.65)$$

This result agrees with that of the MFA result in equation (2.12) [p. 42] for the nn Ising model, if we take into account

$$\tilde{J}(0) = \sum_l J(|\mathbf{R}_l|) = zJ .$$

Then equation (4.65) becomes

$$2\mathcal{S} = \tanh\left(\beta\left(\frac{zJ}{4}2\mathcal{S} + \frac{b}{2}\right)\right) . \quad (4.66)$$

We also have to take into account for the spin-1/2 case

$$H_H^{\text{MFA}} = -\tilde{J}_H(0)\langle S_1 \rangle \sum_i S_i^z - b_H \sum_j S_j.$$

If the spin eigenvalues are  $S_i = \frac{1}{2}\sigma_i$ , with  $\sigma_i = \pm 1$  then

$$H_H^{\text{MFA}} = -\frac{\tilde{J}_H(0)}{4}\langle \sigma_1 \rangle \sum_i \sigma_i^z - \frac{b_H}{2} \sum_j \sigma_j$$

So we have the following relations between the Heisenberg- and the Ising parameters

$$J_I = \frac{J_H}{4}, \quad b_I = \frac{b_H}{2}.$$

With these parameters equation (4.66) [\[previous page\]](#) turns into

$$\langle \sigma_1 \rangle = \tanh \left( \beta (z J_I \langle \sigma_1 \rangle + b_I) \right).$$

which is identical to that of the Ising model in equation (2.12) [\[p. 42\]](#).

### 4.5.6 Curie temperature

According to equation (4.20) [\[p. 111\]](#) the equation for the order parameter without external field reads for  $T \leq T_C$

$$\begin{aligned} \frac{S}{S} &= \mathcal{B}_S \left( \underbrace{\beta \tilde{J}(0) S^2}_{=\alpha} \underbrace{\frac{S}{S}}_{=x} \right) \\ x &= \frac{S+1}{3S} \alpha x - \underbrace{\frac{2S^3 + 4S^2 + 3S + 1}{90S^3}}_{=D} \alpha^3 x^3. \end{aligned}$$

Removing the trivial solution ( $x = 0$ ) we are left with

$$\begin{aligned} Dx^2 &= \left( \frac{S+1}{3S} \alpha - 1 \right) \\ x &\propto \left( \frac{S(S+1)}{3} \tilde{J}(0) \beta - 1 \right)^{1/2}. \end{aligned}$$

There is a temperature (Curie temperature), at which  $x$  goes to zero. The condition for it yields

$$k_B T_C = \tilde{J}(0) \frac{S(S+1)}{2}. \quad (4.67)$$

Then the order parameter slightly below  $T_C$  can also be written as

$$x \propto \left( \frac{T_C}{T} - 1 \right) \simeq \varepsilon^{1/2} .$$

Hence the critical exponent for the order parameter is again  $\beta = 1/2$ .

#### 4.5.7 Internal energy

Like in the Ising case we obtain for the internal energy

$$U = \langle H^{MFA} \rangle = -\frac{N\tilde{J}(0)}{2} \mathcal{S}^2 , \quad (4.68)$$

which is zero above  $T_C$  and slightly below  $T_C$  it has the form

$$\frac{U}{N} = \frac{\tilde{J}(0)}{2D} (T - T_C) ,$$

resulting in a specific heat close to  $T_C$

$$\frac{C}{N} = \begin{cases} \frac{\tilde{J}(0)}{2D} > 0 & \text{for } T < T_C \\ 0 & \text{for } T > T_C . \end{cases}$$

Like in the Ising case there is a discontinuous jump the the specific heat, but no power law behaviour

#### 4.5.8 Magnetic susceptibility

$$\chi = \mu_0 \left. \frac{\partial M}{\partial h} \right|_T \Big|_{h=0}$$

Since the equation has the same structure as in the Ising case, we again find

$$\chi \simeq A_{\pm} |\varepsilon|^{-\gamma_{\pm}}$$

with  $A_+ = 2A_-$  and  $\gamma_+ = \gamma_- = 1$ .

# Chapter 5

## Appendix

### 5.1 Sommerfeld expansion

We are interested in integrals of the form

$$I = \int_0^\infty f(\varepsilon) n_F(\varepsilon|\mu, T) d\varepsilon$$

for low temperatures. The function  $f(\varepsilon)$  is assumed to be independent of  $T$  and regular at  $\varepsilon = \mu$ . First we split the integral as follows

$$I = \int_0^\mu f(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} d\varepsilon + \int_\mu^\infty f(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} d\varepsilon ,$$

and modify the first term using  $1/(e^x + 1) = 1 - 1/(e^{-x} + 1)$

$$I = \underbrace{\int_0^\mu f(\varepsilon) d\varepsilon}_{=I_1} - \underbrace{\int_0^\mu f(\varepsilon) \frac{1}{e^{-\beta(\varepsilon-\mu)} + 1} d\varepsilon}_{=I_2} + \underbrace{\int_\mu^\infty f(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} d\varepsilon}_{=I_3}$$

In  $I_2$  we use the transformation

$$\begin{aligned} x &= -\beta(\varepsilon - \mu) , \\ \varepsilon &= \mu - k_B T x , \\ d\varepsilon &= -k_B T dx , \end{aligned}$$

and find

$$I_2 = -k_B T \int_0^{\beta\mu} \frac{f(\mu - k_B T x)}{e^x + 1} .$$

The kernel  $1/(e^x - 1)$  decreases rapidly with  $x$ ; and for small values of  $T$  the upper integration limit  $\beta\mu$  is much greater than 1, so we can as well extend the integral all the way up to infinity

$$I_2 = -k_B T \int_0^\infty \frac{f(\mu - k_B T x)}{e^x + 1} dx .$$

For  $I_3$  we introduce the substitution

$$\begin{aligned} x &= \beta(\varepsilon - \mu) , \\ \varepsilon &= \mu + k_B T x , \\ d\varepsilon &= k_B T dx , \end{aligned}$$

and obtain

$$I_3 = k_B T \int_0^\infty \frac{f(\mu + k_B T x)}{e^x + 1} dx .$$

Since  $1/(e^x + 1)$  decreases rapidly,  $x$  is essentially restricted to  $x \lesssim 1$ . Hence  $k_B T x \lesssim k_B T$ . So if  $k_B T/\mu$  is small, then we can use a Taylor expansion

$$f(\mu + s k_B T x) = \sum_{n=0}^{\infty} s^n \frac{(k_B T x)^n}{n!} f^{(n)}(\mu) .$$

We need

$$\begin{aligned} I_3 - I_2 &= k_B T \int_0^\infty dx \left( f(\mu + k_B T x) - f(\mu - k_B T x) \right) \frac{1}{e^x + 1} \\ &= 2 \sum_{n=1}^{\text{odd}} \frac{(k_B T)^{n+1}}{n!} f^{(n)}(\mu) \int_0^\infty dx \frac{x^n}{e^x + 1} \\ &= 2 \sum_{n=1}^{\text{odd}} \frac{(k_B T)^{n+1}}{n!} f^{(n)}(\mu) \left(1 - \frac{1}{2^n}\right) \Gamma(n+1) \zeta(n+1) . \end{aligned}$$

The final result reads

SOMMERFELD EXPANSION
$\int_0^\infty f(\varepsilon) n_F(\varepsilon \mu, T) d\varepsilon = \int_0^\mu f(\varepsilon) d\varepsilon + 2 \sum_{n=1}^{\text{odd}} \left(1 - \frac{1}{2^n}\right) \zeta(n+1) (k_B T)^{n+1} f^{(n)}(\mu) \quad (5.1)$



So the leading order terms in the Sommerfeld expansion are

$$I = \int_0^\mu f(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 f'(\mu) + \mathcal{O}\left(\left(\frac{k_B T}{\mu}\right)^4\right)$$

## 5.2 Euler Mac-Laurin formula

We are interested in sums of the form  $\sum_{n=0}^\infty f(n + \frac{1}{2})$ . We begin with the integral over  $f(x)$ , which shall be differentiable at least twice and the integral over  $(0, \infty)$  shall exist. Then

$$\begin{aligned} \int_0^\infty f(x) dx &= \sum_{n=0}^\infty \int_n^{n+1} f(x) dx = \sum_{n=0}^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} f(n + \frac{1}{2} + \xi) d\xi \\ &= \sum_{n=0}^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( f(n + \frac{1}{2}) + f'(n + \frac{1}{2})\xi + f''(n + \frac{1}{2})\frac{\xi^2}{2} + \dots \right) d\xi \\ &= \sum_{n=0}^\infty \left( f(n + \frac{1}{2}) \int_{-\frac{1}{2}}^{\frac{1}{2}} d\xi + f'(n + \frac{1}{2}) \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi d\xi + \frac{f''(n + \frac{1}{2})}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \xi^2 d\xi + \dots \right) d\xi \\ &= \sum_{n=0}^\infty \left( f(n + \frac{1}{2}) + f''(n + \frac{1}{2}) \frac{2}{2 \cdot 3 \cdot 2^3} + \dots \right) d\xi. \end{aligned}$$

The leading order terms are

$$\sum_{n=0}^\infty f(n + \frac{1}{2}) = \int_0^\infty f(x) dx - \frac{1}{24} \sum_{n=0}^\infty f''(n + \frac{1}{2}) + \dots$$

We can use this formula again to express the sum on the rhs also by an integral, resulting in

$$\sum_{n=0}^\infty f(n + \frac{1}{2}) = \int_0^\infty f(x) dx - \frac{1}{24} \left( \int_0^\infty f''(x) dx - \frac{1}{24} \sum_{n=0}^\infty f^{(iv)}(n + \frac{1}{2}) \right) + \dots,$$

and we obtain

EULER-MACLAURIN FORMULA  
(leading order)

$$\sum_{n=0}^\infty f(n + \frac{1}{2}) = \int_0^\infty f(x) dx + \frac{1}{24} f'(0) - \frac{1}{24} f'(\infty) + \dots \quad (5.2)$$

These are actually the first terms of the *Euler MacLaurin formula* formula.

### 5.3 One-particle density of states

The one-particle density of states in  $D$  spatial dimensions is

$$\begin{aligned}
 \rho^{(D)}(\varepsilon) &= \sum_{\mathbf{k}} \delta\left(\varepsilon - \frac{\hbar^2 \mathbf{k}^2}{2m}\right) \\
 &= \frac{V^{(D)}}{(2\pi)^D} \int d^D \mathbf{k} \delta\left(\varepsilon - \left(\frac{\hbar \mathbf{k}}{\sqrt{2m}}\right)^2\right) \\
 &= \frac{V^{(D)}}{(2\pi)^D} \left(\frac{\sqrt{2m}}{\hbar}\right)^D \int d^D \mathbf{x} \delta(\varepsilon - \mathbf{x}^2) \\
 &= \frac{V^{(D)}(2m)^{D/2}}{(2\pi)^D \hbar^D} \Omega_D \int_0^\infty dx x^{D-1} \delta(\varepsilon - x^2) \\
 &= \frac{V^{(D)}(2m)^{D/2}}{(2\pi)^D \hbar^D} \Omega_D \theta(\varepsilon \geq 0) \int_0^\infty dx x^{D-1} \frac{\delta(x - \sqrt{\varepsilon})}{2x} \\
 &= \frac{V^{(D)}(2m)^{D/2}}{(2\pi)^D \hbar^D} \frac{\Omega_D}{2} \varepsilon^{\frac{D-2}{2}} \theta(\varepsilon \geq 0)
 \end{aligned}$$

In appendix C of the statistics I script we have derived the surface  $\Omega_D$  of a  $D$ -dimensional hypersphere of unit radius

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} .$$

The  $D$ -dimensional dos of the free electron gas is therefore

DENSITY OF STATES  
(*D*-dimensional free electron gas)

$$\rho^{(D)}(\varepsilon) = \frac{V^{(D)} m^{\frac{D}{2}}}{\hbar^D (2\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \varepsilon^{\frac{D-2}{2}} \theta(\varepsilon \geq 0) . \quad (5.3a)$$

$$\rho^{(1)}(\varepsilon) = \frac{L\sqrt{m}}{\hbar\sqrt{2\pi}} \frac{1}{\sqrt{\varepsilon}} \theta(\varepsilon \geq 0) . \quad (5.3b)$$

$$\rho^{(2)}(\varepsilon) = \frac{V^{(2)} m}{\hbar^2 2\pi} \theta(\varepsilon \geq 0) , \quad (5.3c)$$

$$\rho^{(3)}(\varepsilon) = \frac{V m^{\frac{3}{2}}}{\hbar^3 \sqrt{2\pi^2}} \sqrt{\varepsilon} \theta(\varepsilon \geq 0) . \quad (5.3d)$$

## 5.4 Eigenvalues of the free electron gas in a homogeneous magnetic field

Here we determine the one-particle eigenvalues of the free electron gas in a homogeneous magnetic field. Since the electron do not interact with each other, it suffices to consider the hamiltonian of single electron, which reads

$$H = \frac{(\hat{\mathbf{p}} + e\hat{\mathbf{A}})^2}{2m} + \mu_B g_e \mathbf{B} \hat{\mathbf{S}} ,$$

where  $\mathbf{S}$  is the vector operator of the electronic spin. The direction of the magnetic field defines the  $z$ -direction. Since there is no spin-orbit coupling, the eigenvector is a tensor product of the orbital and spin degrees of freedom. For the latter the vector is the eigenvector of the operator  $S_z$ , i.e.

$$|\psi\rangle = |\Phi\rangle \otimes |\sigma\rangle$$

The eigenvalue problem turns into

$$H |\psi\rangle = |\sigma\rangle \otimes \left( \frac{(\hat{\mathbf{p}} + e\hat{\mathbf{A}})^2}{2m} + \underbrace{\frac{\mu_B \hbar g_e}{2}}_{=b} \sigma \right) |\Phi\rangle = \varepsilon |\sigma\rangle \otimes |\Phi\rangle$$

The orbital part of the eigenvalue problem reads

$$\frac{(\hat{\mathbf{p}} + e\hat{\mathbf{A}})^2}{2m} |\Phi\rangle = \varepsilon' |\Phi\rangle$$

#### 5.4. EIGENVALUES OF THE FREE ELECTRON GAS IN A HOMOGENEOUS MAGNETIC FIELD

and the the eigenvalues of the entire hamitonian are

$$\varepsilon = \varepsilon' + b\sigma .$$

The vector potential for the homogeneous magnetic field in  $z$ -direction can be chosen (Landau gauge) as

$$\mathbf{A} = B(0, x, 0) .$$

We readily see that it gives the correct  $\mathbf{B}$  field

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} \\ &= B \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \end{aligned}$$

Inserting  $\mathbf{A}$  into the orbital eigenvalue problem gives

$$\frac{\hat{p}_x^2 + (\hat{p}_y + eB\hat{x})^2 + \hat{p}_z^2}{2m} |\Phi\rangle = \varepsilon' |\Phi\rangle$$

The  $y$  and  $z$  coordinate only enters via the momenta, therefore, the eigenvector is a tensor product of the form

$$|\Phi\rangle = |\Phi_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle ,$$

where  $|k_x\rangle$  ( $|k_x\rangle$ ) is an eigenvector of the momentum operator  $p_x$  ( $p_y$ ) with the corresponding eigenvalue. Then

$$\begin{aligned} \frac{\hat{p}_x^2 + (\hat{p}_y + eB\hat{x})^2 + \hat{p}_z^2}{2m} |\Phi\rangle &= |p_y\rangle \otimes |p_z\rangle \otimes \frac{\hat{p}_x^2 + (p_y + eB\hat{x})^2 + p_z^2}{2m} |\Phi_x\rangle \\ &= |p_y\rangle \otimes |p_z\rangle \otimes \left( \varepsilon' |\Phi_x\rangle \right) \end{aligned}$$

The remaining 1D problem reads

$$\frac{\hat{p}_x^2 + (p_y + eB\hat{x})^2 + p_z^2}{2m} |\Phi_x\rangle = \varepsilon' |\Phi_x\rangle$$

The electron gas is confined to a box of size  $L_x \times L_y \times L_y$ . The momentum eigenvalues are therefore quantized as

$$p_y = \hbar \frac{2\pi}{L_y} n_y , \quad \text{with } n_y \in \mathbf{N} \quad (5.4)$$

$$p_z = \hbar \frac{2\pi}{L_z} n_z , \quad \text{with } n_z \in \mathbf{N} \quad (5.5)$$

$$(5.6)$$

The 1D hamiltonian can be rewritten in the form

$$\left( \frac{1}{2m} \hat{p}_x^2 + \frac{eB}{2m} (\hat{x} + \underbrace{\frac{p_y}{eB}}_{:= -x_0})^2 + \frac{p_z^2}{2m} \right) |\Phi_x\rangle = \varepsilon' |\Phi_x\rangle$$

This is the hamiltonian of a harmonic oscillator shifted by  $x_0$ . The prefactor of the  $x^2$  term corresponds to  $\omega_c/2$ , hence  $\omega_c = eB/m$ . The eigenvalues of the harmonic oscillator yield

$$\varepsilon' = \frac{p_z^2}{2m} + \hbar\omega_c \left( n + \frac{1}{2} \right) .$$

These eigenvalues are independent of  $p_y$  and, therefore, degenerate.  $p_y$  defines the center  $x_0$  of the harmonic oscillator, which has to be within  $(0, L_x)$ , resulting in the condition

$$\begin{aligned} 0 &\leq \frac{p_y}{eB} \leq L_x \\ 0 &\leq \frac{\hbar 2\pi}{L_y eB} n_y \leq L_x \\ 0 &\leq n_y \leq \frac{L_x L_y eB}{2\pi\hbar} . \end{aligned}$$

The number of allowed  $p_y$  values defines the degeneracy

$$N_{\text{deg}} = \left\lfloor \frac{L_x L_y eB}{2\pi\hbar} \right\rfloor + 1 .$$

This number will be much greater than 1 and we can, therefore, ignore the fact that it has to be an integer and we simply use

$$N_{\text{deg}} = \frac{L_x L_y eB}{2\pi\hbar} . \quad (5.7)$$

## 5.5 Proof of the Mermin-Wagner Theorem

The proof is based on the

### BOGOLIUBOV INEQUALITY

$$\frac{\beta}{2} \langle [A, A^\dagger]_+ \rangle \langle [[C, H]_-, C^\dagger]_- \rangle \geq |[C, A]_-|^2 \quad (5.8)$$

the proof of which is given in appendix (5.5.1).

Now we start with the Mermin-Wagner theorem, which is valid for the following hamiltonian

$$H = - \sum_{jj'} J_{jj'} \mathbf{S}_j \mathbf{S}_{j'} + b S_{\mathbf{Q}}^z . \quad (5.9)$$

The division by  $\hbar$  makes the order parameter dimensionless and the factor  $1/N$  leads to the magnetization per site in the ferromagnetic case. The spin operators obey the common commutator relations of angular momenta

$$[S_j^z S_{j'}^\pm] = \pm \delta_{jj'} \hbar S_j^\pm \quad (5.10a)$$

$$[S_j^+ S_{j'}^-] = \delta_{jj'} 2\hbar S_j^z . \quad (5.10b)$$

It is expedient to introduce Fourier transformed operators

$$S_k^\alpha = \sum_j S_j^\alpha e^{i\mathbf{k}\mathbf{R}_j} , \quad \alpha \in \{x, y, z\} \quad (5.11a)$$

$$S_j^\alpha = \frac{1}{N} \sum_k S_k^\alpha e^{-i\mathbf{k}\mathbf{R}_j} \quad (5.11b)$$

From that we obtain

$$S_{\mathbf{k}}^\pm := \sum_j e^{i\mathbf{k}\mathbf{R}_j} S_j^\pm = \sum_j e^{i\mathbf{k}\mathbf{R}_j} (S_j^x \pm i S_j^y) = S_{\mathbf{k}}^x \pm i S_{\mathbf{k}}^y \quad (5.12)$$

The commutation relations in these operators are

$$[S_k^z S_{k'}^\pm] = \pm \hbar S_{k+k'}^\pm \quad (5.13a)$$

$$[S_k^+ S_{k'}^-] = 2\hbar S_{k+k'}^z . \quad (5.13b)$$

We assume that the exchange coupling decays rapidly enough to ensure

$$X := \frac{S(S+1)\hbar^2}{N} \sum_{jj'} |J_{jj'}| |\mathbf{R}_j - \mathbf{R}_{j'}|^2 < \infty , \quad \forall N . \quad (5.14)$$

For the proof of the Mermin-Wagner theorem the following operators are used

$$A = A_{\mathbf{k}} = S_{\mathbf{Q}-\mathbf{k}}^- , \quad A_{\mathbf{k}}^\dagger = S_{\mathbf{k}-\mathbf{Q}}^+ \quad (5.15a)$$

$$C = C_{\mathbf{k}} = S_{\mathbf{k}}^+ , \quad C_{\mathbf{k}}^\dagger = S_{-\mathbf{k}}^- . \quad (5.15b)$$

The first term in the Bogoliubov inequality that we compute is

$$\langle [C_{\mathbf{k}}, A_{\mathbf{k}}]_- \rangle = \langle [S_{\mathbf{k}}^+, S_{\mathbf{Q}-\mathbf{k}}^-] \rangle = 2\hbar N \mathcal{S}_{\mathbf{Q}}^z(T, b) . \quad (5.16)$$

The second term that we consider is

$$\sum_{\mathbf{k}} \langle [A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger]_+ \rangle = \sum_{\mathbf{k}} \langle S_{\mathbf{Q}-\mathbf{k}}^- S_{\mathbf{k}-\mathbf{Q}}^+ + S_{\mathbf{k}-\mathbf{Q}}^+ S_{\mathbf{Q}-\mathbf{k}}^- \rangle \quad (5.17)$$

$$= \sum_{jj'} \sum_{\mathbf{k}} e^{i(\mathbf{Q}-\mathbf{k})\mathbf{R}_j} e^{i(\mathbf{k}-\mathbf{Q})\mathbf{R}_{j'}} \langle S_j^- S_{j'}^+ + S_{j'}^+ S_j^- \rangle \quad (5.18)$$

$$= \sum_{jj'} e^{i\mathbf{Q}(\mathbf{R}_j - \mathbf{R}_{j'})} \underbrace{\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{R}_j - \mathbf{R}_{j'})}}_{=N\delta_{jj'}} \langle S_j^- S_{j'}^+ + S_{j'}^+ S_j^- \rangle \quad (5.19)$$

$$= N \sum_j \langle S_j^- S_j^+ + S_j^+ S_j^- \rangle \quad (5.20)$$

$$= 2N \sum_j \langle (S_j^x)^2 + (S_j^y)^2 \rangle , \quad (5.21)$$

This term can be estimated by

$$\sum_{\mathbf{k}} \langle [A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger]_+ \rangle \leq 2N \sum_j \underbrace{\langle (\mathbf{S}_j)^2 \rangle}_{=\hbar^2 S(S+1)} = 2N^2 \hbar^2 S(S+1) ,$$

resulting in

$$\beta N^2 \hbar^2 S(S+1) \geq \frac{\beta}{2} \sum_{\mathbf{k}} \langle [A, A^\dagger]_+ \rangle , \quad (5.22)$$

For the remaining term in the Bogoliubov inequality we find (proof in appendix appendix (5.5.2))

$$4\hbar^2 N (|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X) \geq \left\langle \left[ [C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger \right]_- \right\rangle \quad (5.23)$$

with the definition of  $X$  in equation (5.14) [\[previous page\]](#) . The Bogoliubov inequality reads

$$\frac{\beta}{2} \langle [A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger]_+ \rangle \langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger]_- \rangle \geq |\langle [C_{\mathbf{k}}, A_{\mathbf{k}}]_- \rangle|^2$$

According to equation (5.29) [p. 156] the second factor is greater zero, therefore

$$\begin{aligned} \frac{\beta}{2} \langle [A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger]_+ \rangle &\geq \frac{|\langle [C_{\mathbf{k}}, A_{\mathbf{k}}]_- \rangle|^2}{\langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger]_- \rangle} \\ \Rightarrow \quad \frac{\beta}{2} \sum_{\mathbf{k}} \langle [A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger]_+ \rangle &\geq \sum_{\mathbf{k}} \frac{|\langle [C_{\mathbf{k}}, A_{\mathbf{k}}]_- \rangle|^2}{\langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger]_- \rangle}. \end{aligned}$$

Finally, we use the operators in equation (5.15) [p. 149] in the Bogoliubov inequality, sum over  $\mathbf{k}$  and use equation (5.16) [previous page], equation (5.22) [previous page] and equation (5.23) [previous page], resulting in

$$\begin{aligned} \beta N^2 \hbar^2 S(S+1) &\geq \frac{\beta}{2} \sum_{\mathbf{k}} \langle [A_{\mathbf{k}}, A_{\mathbf{k}}^\dagger]_+ \rangle \\ &\geq \sum_{\mathbf{k}} \frac{|[C_{\mathbf{k}}, A_{\mathbf{k}}]_-|^2}{\langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger]_- \rangle} \\ \beta N^2 \hbar^2 S(S+1) &\geq \frac{4\hbar^2 N^2 \mathcal{S}_{\mathbf{Q}}^2(T, b)}{4N} \sum_{\mathbf{k}} \frac{1}{|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X} \\ \beta S(S+1) &\geq \frac{\mathcal{S}_{\mathbf{Q}}(T, b)}{N} \sum_{\mathbf{k}} \frac{1}{|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X} \\ \beta S(S+1) &\geq \mathcal{S}_{\mathbf{Q}}^2(T, b) I, \end{aligned}$$

with

$$\begin{aligned} I &:= \frac{1}{N} \sum_{\mathbf{k}} \frac{1}{|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X} \\ &= \frac{V}{N(2\pi)^D} \int_{V_{1bc}} d^D k \frac{1}{|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X} \end{aligned}$$

The volume  $V_{1bc}$  of integration is the first Brillouin zone of a simple square lattice, i.e. a  $D$ -dimensional cube with edges ranging from  $-\pi$  to  $\pi$ . Since the integrand is positive, the integral becomes smaller, if we replace the volume



by that of a sphere,  $V_{sp}$ , that lies entirely within  $V_{1bc}$ , i.e.,

$$\begin{aligned}
I &\geq I_{sp} \\
I_{sp} &= \frac{V}{N(2\pi)^D} \int_{V_{sp}} d^D k \frac{1}{|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X} \\
&= \frac{V\Omega_D}{N(2\pi)^D} \int_0^{k_0} dk \frac{k^{D-1}}{|b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X} \\
&= \frac{const}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|} \int_0^{k_0} \frac{dk}{k} \frac{k^D}{1 + \underbrace{\mathbf{k}^2 \frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}}_{:=x^2}} \\
&= \frac{const}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|} \left( \frac{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}{X} \right)^{D/2} \int_0^{k_0} \sqrt{\frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} \frac{dx}{x} \frac{x^D}{1+x^2} \\
&= const |b\mathcal{S}_{\mathbf{Q}}(T, b)|^{\frac{D-2}{2}} \int_0^{k_0} \sqrt{\frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} \frac{dx}{x} \frac{x^D}{1+x^2} .
\end{aligned}$$

So finally we have

$$S(S+1) \geq const k_B T \frac{\mathcal{S}_{\mathbf{Q}}^2(T, b)}{\hbar^2} |b\mathcal{S}_{\mathbf{Q}}(T, b)|^{\frac{D-2}{2}} \int_0^{k_0} \sqrt{\frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} \frac{dx}{x} \frac{x^D}{1+x^2} ,$$

or alternative

$$1 \geq const \cdot T \cdot \left| \mathcal{S}_{\mathbf{Q}}(T, b) \right|^{\frac{D+2}{2}} |b|^{\frac{D-2}{2}} \underbrace{\int_0^{k_0} \sqrt{\frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} \frac{dx}{x} \frac{x^D}{1+x^2}}_{=K_D} . \quad (5.24)$$

We are interested in the spontaneous magnetization  $b \rightarrow 0$ , i.e. the upper integration limit goes to  $\infty$ .

## 1D

In the 1D case, the integral for  $b \rightarrow 0$  becomes

$$K_1 = \int_0^{k_0} \sqrt{\frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} dx \frac{1}{1+x^2} \xrightarrow{b \rightarrow 0} \int_0^{\infty} dx \frac{1}{1+x^2} = \frac{\pi}{2}$$

Hence

$$\begin{aligned}
1 &\geq \text{const } T \left( \mathcal{S}_{\mathbf{Q}}(T, b) \right)^{\frac{3}{2}} |b|^{-\frac{1}{2}} \\
\frac{|b|}{T^2} &\geq \text{const } \left( \mathcal{S}_{\mathbf{Q}}(T, b) \right)^3 \\
\mathcal{S}_{\mathbf{Q}}(T, b) &\leq \left( \frac{|b|}{T^2} \right)^{1/3}.
\end{aligned}$$

For finite temperature, the rhs tends to zero for  $b \rightarrow 0$ . Hence for 1D and finite  $T$

$$\lim_{b \rightarrow 0} \mathcal{S}_{\mathbf{Q}}(T, b) = 0, \quad (5.25)$$

i.e. there is no spontaneous magnetization, irrespective of  $\mathbf{Q}$ , for finite  $T$ . **The inequality does not rule out a finite spontaneous magnetization for  $T = 0$ .**

## 2D

In the 2D case, the integral  $K_D$  is a bit more tricky as it diverges for  $b \rightarrow 0$

$$\begin{aligned}
K_2 &= \int_0^{k_0} \sqrt{\frac{X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} dx \frac{x}{1+x^2} \\
&= \frac{1}{2} \int_0^{\frac{k_0^2 X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}} dz \frac{1}{1+z} \\
&= \frac{1}{2} \ln \left( 1 + \frac{k_0^2 X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|} \right).
\end{aligned}$$

For  $b \rightarrow 0$  it behaves like

$$K_2 = \frac{1}{2} \ln \left( \frac{k_0^2 X}{|b\mathcal{S}_{\mathbf{Q}}(T, b)|} \right) = -\frac{1}{2} \ln \left( \frac{|b\mathcal{S}_{\mathbf{Q}}(T, b)|}{k_0^2 X} \right) = \frac{1}{2} \left| \ln \left( \left| \frac{b}{b_0} \right| \right) \right|,$$

where we have defined the positive parameter  $b_0 = k_0^2 X / |\mathcal{S}_{\mathbf{Q}}(T, b)|$  and we have used that  $b \rightarrow 0$  result in a negative logarithm. Then the inequality equation (5.24) [\[previous page\]](#) yields

$$\begin{aligned}
1 &\geq \frac{\text{const}}{2} T \left( \mathcal{S}_{\mathbf{Q}}(T, b) \right)^2 |b|^0 \left| \ln \left( |b/b_0| \right) \right| \\
\left( \mathcal{S}_{\mathbf{Q}}(T, b) \right)^2 &\leq \frac{\text{const}}{T \left| \ln \left( |b/b_0| \right) \right|}.
\end{aligned}$$

The story is the same as in 1D. The rhs goes to zero for  $b \rightarrow 0$  and finite  $T$ . **Nothing can be said about  $T = 0$ .**

### 3D

Eventually in the 3D case, the integral  $K_D$  yields

$$\begin{aligned} K_3 &= \int_0^{k_0} \sqrt{\frac{x}{|b\mathcal{S}_{\mathbf{Q}}(T,b)|}} dx \frac{x^2}{1+x^2} \\ &= \int_0^{k_0} \sqrt{\frac{x}{|b\mathcal{S}_{\mathbf{Q}}(T,b)|}} dx \left(1 - \frac{1}{1+x^2}\right) \\ &= \sqrt{\frac{c}{|b\mathcal{S}_{\mathbf{Q}}(T,b)|}} - \underbrace{K_{1D}}_{=\pi/2} \end{aligned}$$

Inserting in the inequality equation (5.24) [p. 152] yields

$$\begin{aligned} 1 &\geq \text{const } T \left( \mathcal{S}_{\mathbf{Q}}(T,b) \right)^{\frac{5}{2}} |b|^{\frac{1}{2}} \left( \frac{c}{\sqrt{|b\mathcal{S}_{\mathbf{Q}}(T,b)|}} - \frac{\pi}{2} \right) \\ &\geq \text{const } T \left( \mathcal{S}_{\mathbf{Q}}(T,b) \right)^2 \\ \left( \mathcal{S}_{\mathbf{Q}}(T,b) \right)^2 &\leq \frac{\text{const}}{T} \end{aligned}$$

In this case we do not obtain a useful inequality. In other words, the Mermin-Wagner theorem does not rule out spontaneous magnetization for finite  $T$  in 3D.

#### 5.5.1 Bogoliubov inequality

Here we will prove the Bogoliubov inequality. To this end we will exploit the Schwarz inequality

$$|(A|B)|^2 \leq (A|A)(B|B), \quad (5.26)$$

where  $(X|Y)$  are scalar products between any vectors  $A$  and  $B$ . To be able to use this relation, we have to introduce a suitable scalar product, that involves the operators  $A$  and  $B$

$$(A|B) := \sum_{n,m}^{E_m \neq E_n} \langle m| A^\dagger |n\rangle \langle n| B |m\rangle \frac{\rho_m - \rho_n}{E_n - E_m}$$

with

$$\rho_m = \frac{e^{-\beta E_m}}{\text{tr}\{e^{-\beta H}\}}.$$

This product fulfils the four defining properties of a scalar product

- $(A|B) = (B|A)^*$ .

This is obviously the case

$$\begin{aligned}
 (B|A)^* &= \sum_{\substack{E_m \neq E_n \\ n, m}} (\langle m| B^\dagger |n\rangle \langle n| A |m\rangle)^* \frac{\rho_m - \rho_n}{E_n - E_m} \\
 &= \sum_{\substack{E_m \neq E_n \\ n, m}} \langle m| A^\dagger |n\rangle \langle n| B |m\rangle \frac{\rho_m - \rho_n}{E_n - E_m}, \\
 &= (A|B). \quad \checkmark
 \end{aligned}$$

- Linearity  $(A|\alpha B_1 + \beta B_2) = \alpha(A|B_1) + \beta(A|B_2)$ , which follows from the linearity of the matrix elements  $\langle n| B |m\rangle$ .
- $(A|A) \geq 0$ , since

$$(A|A) = \sum_{\substack{E_m \neq E_n \\ n, m}} \underbrace{|\langle n| A^\dagger |m\rangle|}_{\geq 0}^2 \underbrace{\frac{\rho_m - \rho_n}{E_n - E_m}}_{\geq 0}.$$

- From  $A = 0$  follows clearly  $(A|A) = 0$ , but not conversely

So we can conclude that  $(A|B)$  is a semi-definite scalar product, for which the Schwarz inequality applies. In addition, the product has the special feature

$$(H|A) = 0, \quad \forall \text{ operators } A. \quad (5.27)$$

This is easily seen, since  $\langle m| H^\dagger |n\rangle = E_n \delta_{mn}$ ; but  $m = n$  is excluded from the sum.

Next we exploit it for the prove of the Bogoliubov inequality. To this end, we first use  $B = [C^\dagger, H]$  in the Schwarz relation. For the lhs we need  $(A|B)$

$$\begin{aligned}
 (A|B) &= (A|[C^\dagger, H]) \\
 &= \sum_{\substack{E_m \neq E_n \\ mn}} \langle m| A^\dagger |n\rangle \langle n| [C^\dagger, H] |m\rangle \frac{\rho_m - \rho_n}{E_n - E_m} \\
 &= \sum_{\substack{E_m \neq E_n \\ mn}} \langle m| A^\dagger |n\rangle \langle n| C^\dagger |m\rangle (E_m - E_n) \frac{\rho_n - \rho_m}{E_m - E_n} \\
 &= \sum_{\substack{E_m \neq E_n \\ mn}} \langle m| A^\dagger |n\rangle \langle n| C^\dagger |m\rangle (\rho_n - \rho_m).
 \end{aligned}$$

Now we can omit the restriction in the sum, since  $(\rho_n - \rho_m)$  is zero for  $n = m$ , anyways. Then

$$(A|B) = \sum_{mn} \rho_n \langle n| C^\dagger |m\rangle \langle m| A^\dagger |n\rangle - \sum_{mn} \rho_m \langle m| A^\dagger |n\rangle \langle n| C^\dagger |m\rangle .$$

Renaming  $n \leftrightarrow m$  in the last term eventually leads to

$$(A|B) = \left\langle [C^\dagger, A^\dagger]_- \right\rangle \quad (5.28)$$

If we also choose  $A = B = [C^\dagger, H]_-$  in equation (5.28) , we obtain

$$(B|B) = \left\langle [C^\dagger, [H, C]_-]_- \right\rangle = \left\langle [[C, H], C^\dagger]_- \right\rangle \geq 0 . \quad (5.29)$$

For the next steps we use  $(\rho_m = e^{-\beta E_m} / Z)$

$$\rho_m > \rho_n \quad \text{if} \quad E_n > E_m .$$

Hence, for any  $E_n \neq E_m$

$$0 < \frac{\rho_m - \rho_n}{E_n - E_m} = \frac{\rho_m + \rho_n}{E_n - E_m} \cdot \frac{\rho_m - \rho_n}{\rho_m + \rho_n} . \quad (5.30)$$

The second factor can be modified

$$\begin{aligned} \frac{\rho_m - \rho_n}{\rho_m + \rho_n} &= \frac{e^{-\beta E_m} - e^{-\beta E_n}}{e^{-\beta E_m} + e^{-\beta E_n}} \\ &= \frac{e^{-\frac{\beta}{2} E_m} e^{-\frac{\beta}{2} E_n}}{e^{-\frac{\beta}{2} E_m} e^{-\frac{\beta}{2} E_n}} \frac{e^{-\frac{\beta}{2} (E_m - E_n)} - e^{-\frac{\beta}{2} (E_n - E_m)}}{e^{-\frac{\beta}{2} (E_m - E_n)} + e^{-\frac{\beta}{2} (E_n - E_m)}} \\ &= \tanh \left( \frac{\beta}{2} (E_n - E_m) \right) \end{aligned}$$

Then equation (5.30) yields

$$0 < \frac{\rho_m - \rho_n}{E_n - E_m} = \frac{\beta}{2} (\rho_m + \rho_n) \frac{\tanh \left( \frac{\beta}{2} (E_n - E_m) \right)}{\frac{\beta}{2} (E_n - E_m)} .$$

Since  $\tanh(x)/x < 1$ , we find

$$0 < \frac{\rho_m - \rho_n}{E_n - E_m} < \frac{\beta}{2} (\rho_m + \rho_n) .$$

From this relation we obtain

$$(A|A) = \sum_{mn}^{E_m \neq E_n} \langle n| A^\dagger |m\rangle \langle m| A |n\rangle \frac{\rho_m - \rho_n}{E_n - E_m} < \frac{\beta}{2} \sum_{mn}^{E_m \neq E_n} \langle n| A^\dagger |m\rangle \langle m| A |n\rangle (\rho_m + \rho_n) .$$

The omitted terms for  $E_n = E_m$  are positive, so including them does not violate the inequality

$$(A|A) < \frac{\beta}{2} \sum_{mn} \langle n| A^\dagger |m\rangle \langle m| A |n\rangle (\rho_m + \rho_n) = \frac{\beta}{2} \langle [A^\dagger, A]_+ \rangle \quad (5.31)$$

Now we return to the Schwarz inequality in equation (5.26) [\[p. 154\]](#)

$$|(A|B)|^2 \leq (A|A)(B|B) , \quad (5.32)$$

and insert piece by piece the relations we just derived for the various terms. Beginning with equation (5.28) [\[previous page\]](#) and equation (5.29) [\[previous page\]](#) we get

$$|\langle [C, A] \rangle|^2 \leq (A|A) \langle [[C, H], C^\dagger]_- \rangle ,$$

Along with equation (5.31) we finally obtain

$$|\langle [C, A] \rangle|^2 \leq \frac{\beta}{2} \langle [A^\dagger, A]_+ \rangle \langle [[C, H], C^\dagger]_- \rangle , \quad (5.33)$$

which proves the Bogoliubov inequality.

### 5.5.2 Proof used for the Mermin-Wagner theorem

Here we prove the relation

$$\left\langle \left[ [C, H]_-, C^\dagger \right]_- \right\rangle \leq 4\hbar^2 N \left( |b\mathcal{S}_{\mathbf{Q}}(T, b)| + \mathbf{k}^2 X \right) . \quad (5.34)$$

We recall that

$$C = S_{\mathbf{k}}^+ = \sum_l e^{i\mathbf{k}\mathbf{R}_l} S_l^+ .$$

Hence We begin with  $[S_k^+, H]_-$

$$\begin{aligned}
[C, H_H]_- &= [S_{\mathbf{k}}^+, H_H]_- \\
&= - \sum_l e^{i\mathbf{R}_l \mathbf{k}} \sum_{jj'} J_{jj'} \left( [S_l^+, S_j^+ S_{j'}^-] + [S_l^+, S_j^z S_{j'}^z] \right) \\
&= - \sum_l e^{i\mathbf{R}_l \mathbf{k}} \sum_{jj'} J_{jj'} \left( S_j^+ [S_l^+, S_{j'}^-] + S_j^z [S_l^+, S_{j'}^z] + [S_l^+, S_j^z] S_{j'}^z \right) \\
&= -\hbar \sum_l e^{i\mathbf{R}_l \mathbf{k}} \sum_{jj'} J_{jj'} \left( 2S_j^+ \delta_{lj'} S_l^z - S_j^z \delta_{lj'} S_l^+ - \delta_{lj} S_l^+ S_{j'}^z \right) \\
&= -\hbar \sum_l e^{i\mathbf{R}_l \mathbf{k}} \sum_j J_{jl} \left( 2S_j^+ S_l^z - S_j^z S_l^+ - S_l^+ S_j^z \right) \\
&= -2\hbar \sum_l e^{i\mathbf{R}_l \mathbf{k}} \sum_j J_{jl} \left( S_j^+ S_l^z - S_l^+ S_j^z \right).
\end{aligned}$$

We have used the symmetry  $J_{jj'} = J_{j'j}$  and  $S_j^z S_l^+ = S_l^+ S_j^z + [S_j^z, S_l^+]$ . The commutator is proportional to  $\delta_{jl}$  and hence to  $J_{jj}$ , which is zero. Meanwhile we have

$$[C, H_H]_- = -2\hbar \sum_{lj} (e^{i\mathbf{R}_l \mathbf{k}} - e^{i\mathbf{R}_j \mathbf{k}}) J_{jl} S_j^+ S_l^z.$$

We continue with the second commutator

$$[[C, H_H]_-, C^\dagger]_- = -2\hbar \sum_{lj} (e^{i\mathbf{R}_l \mathbf{k}} - e^{i\mathbf{R}_j \mathbf{k}}) J_{jl} \sum_{l'} e^{-i\mathbf{k} \mathbf{R}_{l'}} [S_j^+ S_l^z, S_{l'}^-].$$

With

$$\begin{aligned}
[S_j^+ S_l^z, S_{l'}^-] &= S_j^+ [S_l^z, S_{l'}^-] + [S_j^+, S_{l'}^-] S_l^z \\
&= -\delta_{ll'} \hbar S_j^+ S_l^- + 2\hbar \delta_{l'l} S_j^z S_l^z
\end{aligned}$$

we obtain

$$\begin{aligned}
[[C, H_H]_-, C^\dagger]_- &= -2\hbar^2 \sum_{lj} (e^{i\mathbf{R}_l \mathbf{k}} - e^{i\mathbf{R}_j \mathbf{k}}) J_{jl} \left( -e^{-i\mathbf{k} \mathbf{R}_l} S_j^+ S_l^- + 2e^{-i\mathbf{k} \mathbf{R}_j} S_j^z S_l^z \right) \\
&= 2\hbar^2 \sum_{lj} J_{jl} \left( (1 - e^{i(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k}}) S_j^+ S_l^- - 2(e^{-i(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k}} - 1) S_j^z S_l^z \right) \\
&= 2\hbar^2 \sum_{lj} J_{jl} \left( 1 - e^{i(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k}} \right) \left( S_j^+ S_l^- + 2S_j^z S_l^z \right).
\end{aligned}$$

In the last step we have substituted  $j \leftrightarrow l$  and used the symmetry of  $J_{jl}$ . Finally, we repeat the calculation for the field term

$$\begin{aligned} [C, H_b]_- &= [S_{\mathbf{k}}^+, H_b]_- = b \sum_{l'} e^{i\mathbf{R}_l \mathbf{k}} e^{i\mathbf{R}_{l'} \mathbf{Q}} \underbrace{[S_l^+, S_{l'}^z]}_{=-\delta_{ll'} \hbar S_l^+} \\ &= b \hbar \sum_l e^{i\mathbf{R}_l (\mathbf{k} + \mathbf{Q})} S_l^+ . \end{aligned}$$

Then

$$\begin{aligned} [[C, H_b]_-, C^\dagger] &= b \hbar \sum_l e^{i\mathbf{R}_l (\mathbf{k} + \mathbf{Q})} \sum_{l'} e^{-i\mathbf{k} \mathbf{R}_{l'}} \underbrace{[S_l^+, S_{l'}^-]}_{=\delta_{ll'} 2\hbar S_l^z} \\ &= 2b \hbar^2 \sum_l e^{i\mathbf{R}_l \mathbf{Q}} S_l^z \\ &= 2b \hbar^2 S_{\mathbf{Q}}^z \\ &= 2b N \hbar^2 \mathcal{S}_{\mathbf{Q}}(T, b) . \end{aligned}$$

Hence for  $C = C_{\mathbf{k}}$  we have

$$\begin{aligned} \langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger] \rangle &= 2b N \hbar^2 \mathcal{S}_{\mathbf{Q}}(T, b) \\ &+ 2\hbar^2 \sum_{lj} J_{jl} \left( 1 - e^{i(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k}} \right) \underbrace{\left( \langle S_j^+ S_l^- \rangle + \langle S_j^z S_l^z \rangle + \langle S_j^- S_l^+ \rangle \right)}_{=\langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle} \\ &= 2b N \hbar^2 \mathcal{S}_{\mathbf{Q}}(T, b) \\ &+ 2\hbar^2 \sum_{lj} J_{jl} \left( 1 - e^{i(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k}} \right) \left( \langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle + \langle S_j^z S_l^z \rangle \right) . \end{aligned}$$

From equation (5.29) [p. 156] we know that  $[[C, H_b]_-, C^\dagger]$  originates from the scalar product  $(B|B)$ . It is therefore a real and non-negative function for all  $\mathbf{k}$ . Then

$$\begin{aligned} \langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger] \rangle &\leq \langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger] \rangle + \langle [[C_{-\mathbf{k}}, H]_-, C_{-\mathbf{k}}^\dagger] \rangle \\ \langle [[C_{\mathbf{k}}, H]_-, C_{\mathbf{k}}^\dagger] \rangle &\leq 4b \hbar^2 N \mathcal{S}_{\mathbf{Q}}(T, b) \\ &+ 4\hbar^2 \sum_{lj} J_{jl} \left( 1 - \cos(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k} \right) \left( \langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle + \langle S_j^z S_l^z \rangle \right) \\ &\leq 4N \hbar^2 |b \mathcal{S}_{\mathbf{Q}}(T, b)| \\ &+ 4\hbar^2 \sum_{lj} \underbrace{|J_{jl}| \left( 1 - \cos(\mathbf{R}_j - \mathbf{R}_l) \mathbf{k} \right)}_{\geq 0} \left( |\langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle| + |\langle S_j^z S_l^z \rangle| \right) . \end{aligned}$$



Finally, we can use the Schwarz inequality for the scalar products  $\langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle$

$$\begin{aligned} |\langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle|^2 &\leq \langle \mathbf{S}_j^2 \rangle \langle \mathbf{S}_l^2 \rangle = \left( \hbar^2 S(S+1) \right)^2 \\ |\langle \mathbf{S}_j \cdot \mathbf{S}_l \rangle| &\leq \hbar^2 S(S+1) . \end{aligned}$$

Moreover,  $|\langle S_j^z S_l^z \rangle| \leq |\langle S_j S_l \rangle| \leq \hbar^2 S(S+1)$ . In total we have

$$\langle [[C, H]_-, C^\dagger] \rangle \leq 4N\hbar^2 \left( |b\mathcal{S}_Q(T, b)| + 2S(S+1)\hbar^2 \frac{1}{N} \sum_{jl} |J_{jl}| \left( 1 - \cos(\mathbf{R}_j - \mathbf{R}_l) \cdot \mathbf{k} \right) \right)$$

Moreover we use

$$1 - \cos(x) \leq \frac{x^2}{2} ,$$

and obtain

$$\langle [[C, H]_-, C^\dagger] \rangle \leq 4N\hbar^2 \left( |b\mathcal{S}_Q(T, b)| + \underbrace{\mathbf{k}^2 \frac{S(S+1)\hbar^2}{N} \sum_{jl} |J_{jl}| |\mathbf{R}_j - \mathbf{R}_l|^2}_{=X \text{ with } 0 \leq X < \infty} \right)$$

Which proves equation (5.34) [\[p. 157\]](#).

# Chapter 6

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