# Chapter 3
# THE Z-TRANSFORM

# Main Topics

3.1 Definition

3.2 Properties of ROC

3.3 Inverse z-transform

3.4 Z-transform properties

# 3.1 Definition

The z-transform of a general discrete-time signal $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] \xleftrightarrow{Z} X(z)$$

$z$ is a complex variable $z = re^{j\omega}$

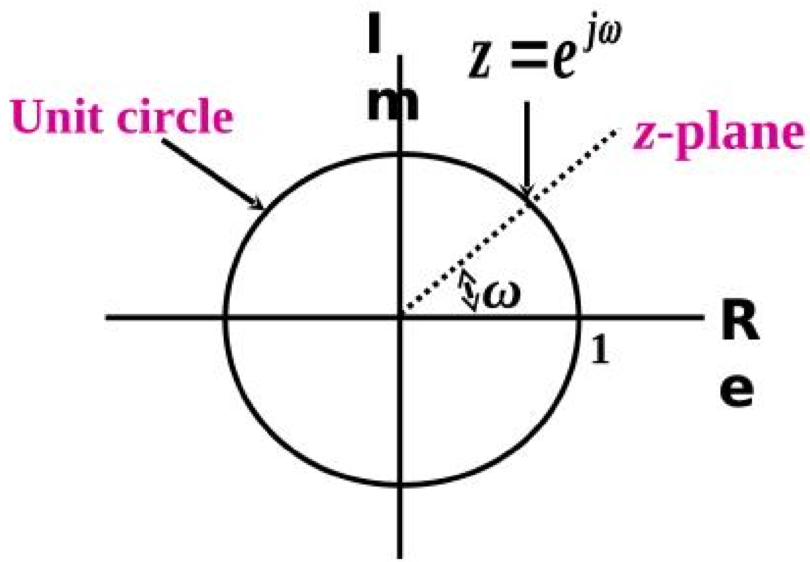
$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\}e^{-j\omega n}$$

$$X(z) = F\{x[n]r^{-n}\}$$

$$X(z) = F\{x[n]r^{-n}\}$$

For $r = 1$, or equivalently, $|z| = 1$,
z-transform equation reduces to the Fourier transform (on the unit circle in z-plane).

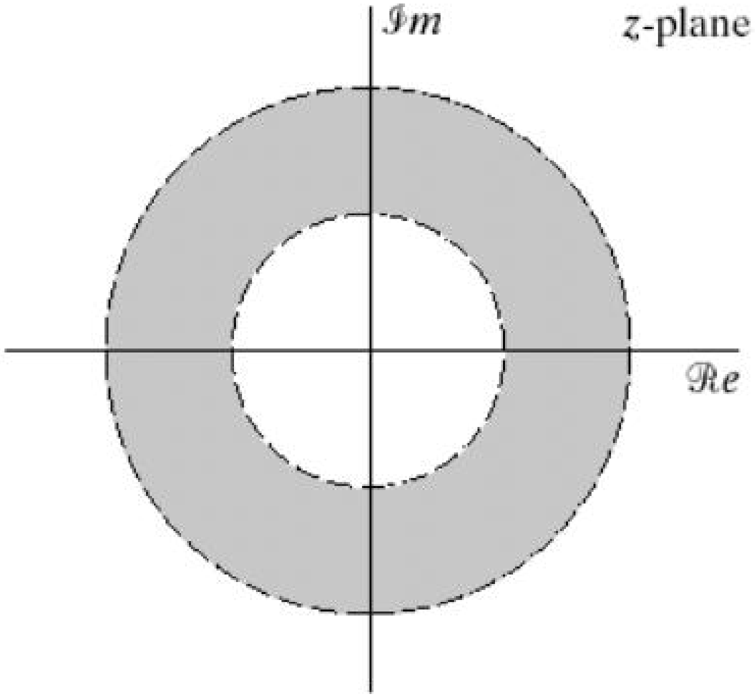
$$X(z)\Big|_{z=e^{j\omega}} = F\{x[n]\} = X(e^{j\omega})$$



## Region of Convergence ( ROC ):

For convergence of the z-transform, we require:

$$\sum_{n=-\infty}^{\infty} | x[n] r^{-n} | < \infty$$



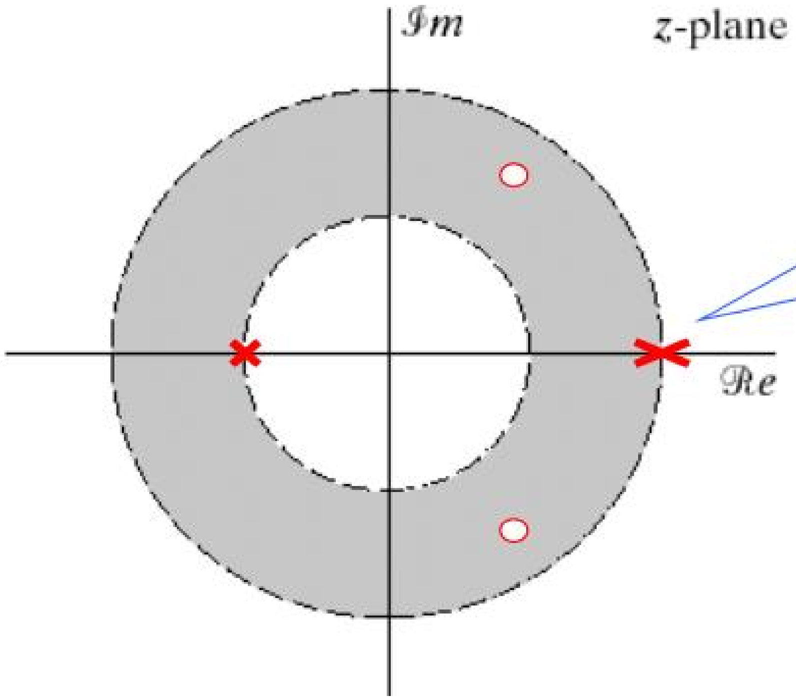

$$Rx- < | z | < Rx+$$

## Example 3.1-3.2

$$x_1[n] = a^n u[n], x_2[n] = -a^n u[-n-1]$$

$$X(z) = \frac{1}{1 - az^{-1}},$$

$$ROC \quad is \quad |z| > |a| \quad and \quad |z| < |a| \quad seperately$$




ROC takes the poles as its boundary

# 3.2 Properties of ROC

**Property 1** The ROC is a ring or disk in the $z$-plane centered at the origin.

$$\sum_{n=-\infty}^{+\infty} \left| x[n] r^{-n} \right| < \infty$$

Convergence is dependent only on $r = |z|$ and not on $\omega$.

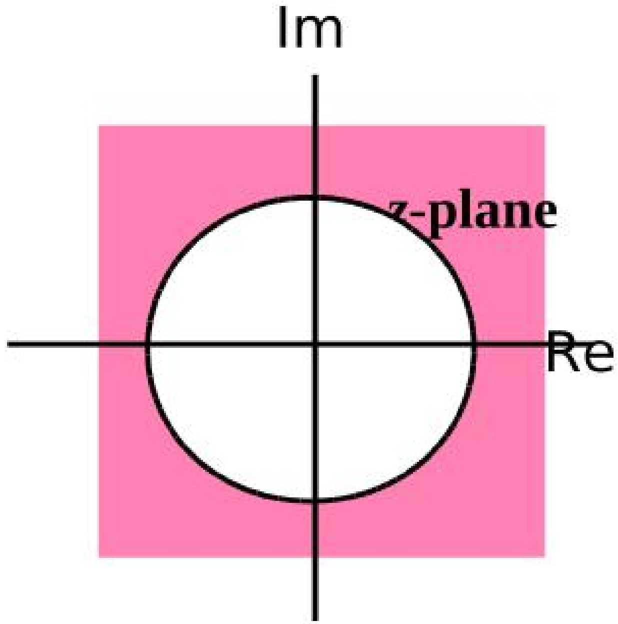
**Property 2** The FT of $x[n]$ converges absolutely if and only if the ROC of the zT of $x[n]$ includes the unit circle.

**Property 3** The ROC can not contain any poles.

**Property 4** If $x[n]$ is a finite duration sequence, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

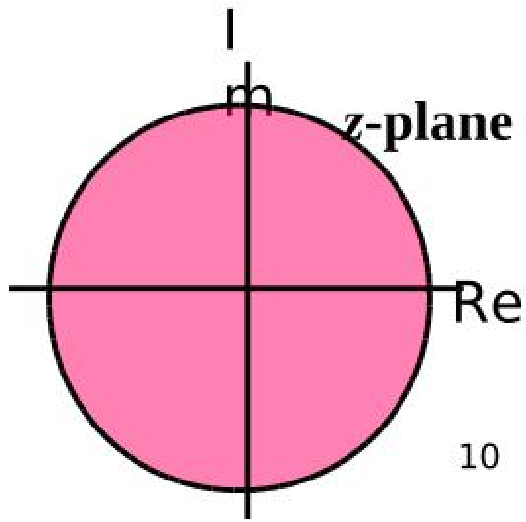
**Property 5**

If $x[n]$ is a right-sided sequence, the ROC is outside the outmost finite pole.



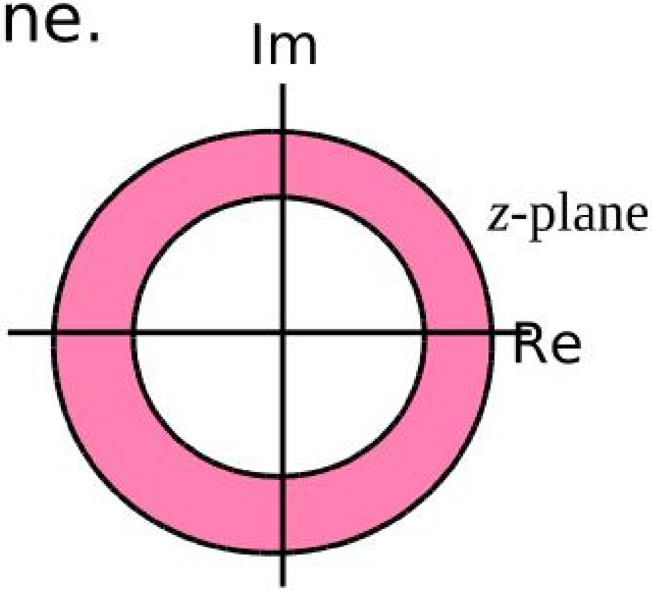

**Property 6**

If $x[n]$ is a left-sided sequence, the ROC is inside a circle in the z-plane.



## Property 7

If $x[n]$ is two sided, the ROC will consist of a ring in the z-plane.




## Property 8

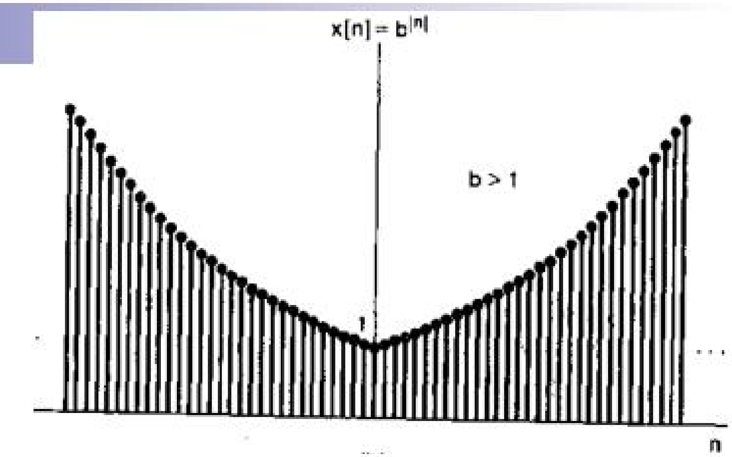
If the $z$-transform $X(z)$ is rational, then its ROC is bounded by poles or extends to infinity.

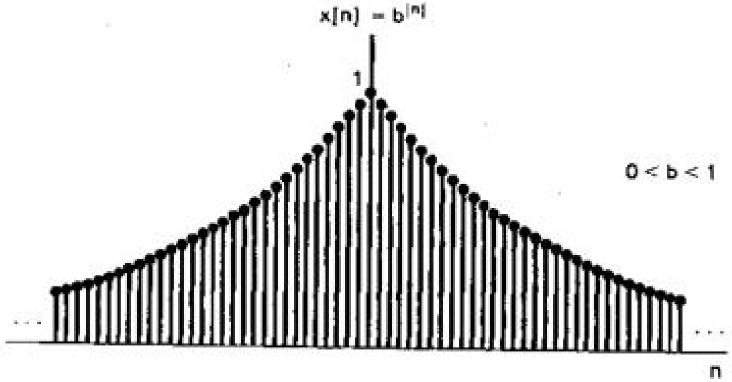
## Example $x[n]=b^{|n|}, \quad b>0$

$$x[n]=b^{n}u[n]+b^{-n}u[-n-1]$$

$$b^{n}u[n]\xleftrightarrow{Z}\frac{1}{1-bz^{-1}} \qquad |z|>b$$

$$b^{-n}u[-n-1]\xleftrightarrow{Z}\frac{-1}{1-b^{-1}z^{-1}} \qquad |z|<\frac{1}{b}$$

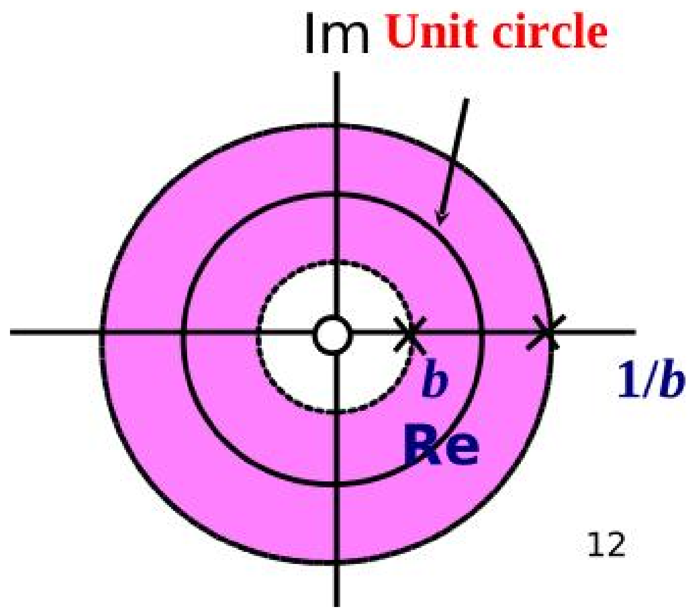



For $b<1$,

$$X(z)=\frac{1}{1-bz^{-1}}-\frac{1}{1-b^{-1}z^{-1}}$$

$$=\frac{b^{2}-1}{b}\frac{z}{(z-b)(z-b^{-1})}, \quad b<|z|<\frac{1}{b}$$



**Property 9**

If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outermost pole. Furthermore, if $x[n]$ is causal, then the ROC also includes $z = \infty$.

**Property 10**

If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole.

In particular, if $x[n]$ is anticausal, then the ROC also includes $z = 0$.

## ROC and LTI system properties

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} : system \quad function$$

$$causal : R_{x-} < |z| \leq \infty, include \quad \infty$$

$$stable : include\, |z| = 1$$

$$causal\ \&\ stable : poles \quad are \quad all \quad in \quad unit \quad circle$$

# 3.3 The Inverse z-Transform

# 3.3 The Inverse z-Transform

1. Inspection method
2. Partial fraction expansion
3. Power series expansion

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > a$$

$$-a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| < a$$

## Example

$$X(z) = \frac{1}{(1-2z^{-1})(1-0.5z^{-1})}, 0.5 < |z| < 2$$

$$= \frac{B_1}{1-2z^{-1}} + \frac{B_2}{1-0.5z^{-1}}$$

$$B_1 = (1-2z^{-1})X(z)|_{z=2} = 4/3,$$

$$B_2 = (1-0.5z^{-1})X(z)|_{z=0.5} = -1/3$$

$$x[n] = -\frac{4}{3}2^n u[-n-1]) - \frac{1}{3}0.5^n u[n]$$

## Partial-fraction expansion（负幂）

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$X(z)=\frac{A}{(1-\frac{1}{4}z^{-1})}+\frac{B}{(1-\frac{1}{3}z^{-1})}$$

$$A=X(z)(1-\frac{1}{4}z^{-1})\Big|_{z=\frac{1}{4}}=\frac{3-\frac{5}{6}z^{-1}}{1-\frac{1}{3}z^{-1}}\Bigg|_{z=\frac{1}{4}}=1$$

$$X(z)=\frac{1}{(1-\frac{1}{4}z^{-1})}+\frac{2}{(1-\frac{1}{3}z^{-1})}$$

$$B=X(z)(1-\frac{1}{3}z^{-1})\Big|_{z=\frac{1}{3}}=\frac{3-\frac{5}{6}z^{-1}}{1-\frac{1}{4}z^{-1}}\Bigg|_{z=\frac{1}{3}}=2$$

$$X(z)=\frac{1}{(1-\frac{1}{4}z^{-1})}+\frac{2}{(1-\frac{1}{3}z^{-1})}$$

$|z|>\frac{1}{3}$

$$x[n]=(\frac{1}{4})^n u[n]+2(\frac{1}{3})^n u[n]$$

$\frac{1}{4}<|z|<\frac{1}{3}$

$$x[n]=(\frac{1}{4})^n u[n]-2(\frac{1}{3})^n u[-n-1]$$

$|z|<\frac{1}{4}$

$$x[n]=-(\frac{1}{4})^n u[-n-1]-2(\frac{1}{3})^n u[-n-1]$$

## Partial-fraction expansion（正幂）

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}=\frac{3z^2-\frac{5}{6}z}{(z-\frac{1}{4})(z-\frac{1}{3})}$$

先除以 z， 进行部分分式展开

$$\frac{X(z)}{z}=\frac{3z-\frac{5}{6}}{(z-\frac{1}{4})(z-\frac{1}{3})}=\frac{A}{z-\frac{1}{4}}+\frac{B}{z-\frac{1}{3}}=\frac{1}{z-\frac{1}{4}}+\frac{2}{z-\frac{1}{3}}$$

再乘以 z

$$X(z)=\frac{z}{z-\frac{1}{4}}+\frac{2z}{z-\frac{1}{3}}$$

$$|z|>\frac{1}{3}$$

$$x[n]=(\frac{1}{4})^n u[n]+2(\frac{1}{3})^n u[n]$$

## Example

$$X(z) = 1 + 0.5z^{-1} + 0.25z^{-2}$$

$$x[n] = \begin{cases} 1 & n = 0 \\ 0.5 & n = 1 \\ 0.25 & n = 2 \\ 0 & other \end{cases}$$

Example $X(z)=\frac{1}{1-az^{-1}},\quad |z|>|a|$

Sequence $x[n]$ is right-sided, arrange the numerator polynomial and the denominator polynomial with an order of the power of $z$ decreasing （分子分母多项式按降幂排列）

long division

$$
\begin{array}{r|l}
 & 1+az^{-1}+a^2z^{-2}+\cdots \\
\hline
1-az^{-1} & 1 \\
 & \underline{1-az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1}-a^2z^{-2}} \\
 & \qquad\qquad a^2z^{-2} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$X(z)=1+az^{-1}+a^2z^{-2}+\cdots$$

$$=\sum_{n=0}^{\infty}a^nz^{-n}$$

$$x[n]=a^nu[n]$$

$$X(z)=\frac{1}{1-az^{-1}}, \quad |z|<|a|$$

Sequence $x[n]$ is left-sided, arrange the numerator polynomial and the denominator polynomial with an order of the power of $z$ increasing （分子分母多项式按升幂排列）

$$\begin{array}{r|l} & -a^{-1}z-a^{-2}z^2-\cdots \\ \hline -az^{-1}+1 & 1 \\ & \underline{1-a^{-1}z} \\ & \quad a^{-1}z \\ & \quad \cdots \end{array}$$

$$X(z)=-a^{-1}z-a^{-2}z^2-\cdots$$

$$x[n]=-a^n u[-n-1]$$

# 3.4 z-Transform Properties

# 1. Linearity

If $$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), \quad with\ ROC = R_1$$

$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), \quad with\ ROC = R_2$$

then

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z), \quad with\ ROC\ containing\ R_1 \cap R_2$$

Note: ROC is at least the intersection of R1 and R2, which could be empty, also can be larger than the intersection.

## 2. Time Shifting

If $\quad x[n] \xleftrightarrow{Z} X(z), \quad with\ ROC = R$

then $\quad x[n - n_0] \xleftrightarrow{Z} z^{-n_0} X(z), \quad with\ ROC = R$

Except for the possible addition or deletion of the origin or infinity.

## 3. Multiplication by Exponential Sequence (Scaling in the z-Domain)

If $\quad x[n] \xleftrightarrow{Z} X(z), \quad with\ ROC = R$

then $\quad z_0^{\,n} x[n] \xleftrightarrow{Z} X\left(\dfrac{z}{z_0}\right), \quad with \quad ROC = |z_0| R$

## 4. Differentiation in the z-Domain

$$nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz} \qquad \text{with } ROC = R$$

$$n^m x[n] \leftrightarrow \left[-z\frac{d}{dz}\right]^m X(z)$$

Example

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

$$na^n u[n] \xleftrightarrow{Z} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z| > |a|$$

$$na^n u(n) \xleftrightarrow{Z} -z\frac{d\left(\frac{z}{z-a}\right)}{dz} = \frac{az}{(z-a)^2} \quad , |z| > |a|$$

## 5. Conjugation

$$x^*[n] \xleftrightarrow{Z} X^*(z^*), \quad with\ ROC = R.$$

If $\mathbf{x[n]}$ is real, $\mathbf{X(z) = X^*(z^*)}$

Thus, if $\mathbf{X(z)}$ has a pole (or zero) at $\mathbf{z = z_0}$, it must also have a pole (or zero) at the complex conjugate point $\mathbf{z = z_0^*}$.

## 6. Time Reversal

If $x[n] \xleftrightarrow{Z} X(z), \quad with\ ROC = R$

then $x[-n] \xleftrightarrow{Z} X\left(\frac{1}{z}\right), \quad with\ ROC = \frac{1}{R}$

# 7. The Convolution Property

If $\quad x_1[n] \xleftrightarrow{Z} X_1(z), \quad with\ ROC = R_1$

and $\quad x_2[n] \xleftrightarrow{Z} X_2(z), \quad with\ ROC = R_2$

then

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z), \quad with\ ROC\ containing\ R_1 \cap R_2$$

Example: let $w[n]$ be the running sum of $x[n]$:

$$w[n] = \sum_{k=-\infty}^{n} x[k].$$

Consider its z-transform. (Suppose $X(z)$ is given)

Since

$$w[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{\infty} x[k]u[n-k] = u[n] * x[n]$$

From the convolution property,

$$W(z) = \mathrm{ZT}\{u[n]\} \cdot X(z) = \frac{z}{z-1} X(z)$$

If the ROC of $X(z)$ is R, then the ROC of $W(z)$ must includes at least the interconnection of R with $|z| > 1$.

## 8. The Initial- and Final-Value Theorems

If $x[n]$ is a causal sequence, i.e., $x[n] = 0$, for $n < 0$,

Initial-value theorem :

$$x[0] = \lim_{z \to \infty} X(z)$$

Final -value theorem :

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z - 1) X(z)$$

## *. Time Expansion

$$x_{(k)}[n] \xleftrightarrow{Z} X(z^k), \quad with\ ROC = R^{1/k}$$

# 3.5* Geometric Evaluation of The Fourier Transform From The Pole-Zero Plot

When unit circle ($|z| = 1$) is in the ROC , the discrete-time Fourier transform exists.

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

$$h[n] \xleftrightarrow{Z} H(z)$$

For example: First-order causal discrete-time system

$$h[n] = a^n u[n] \qquad H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

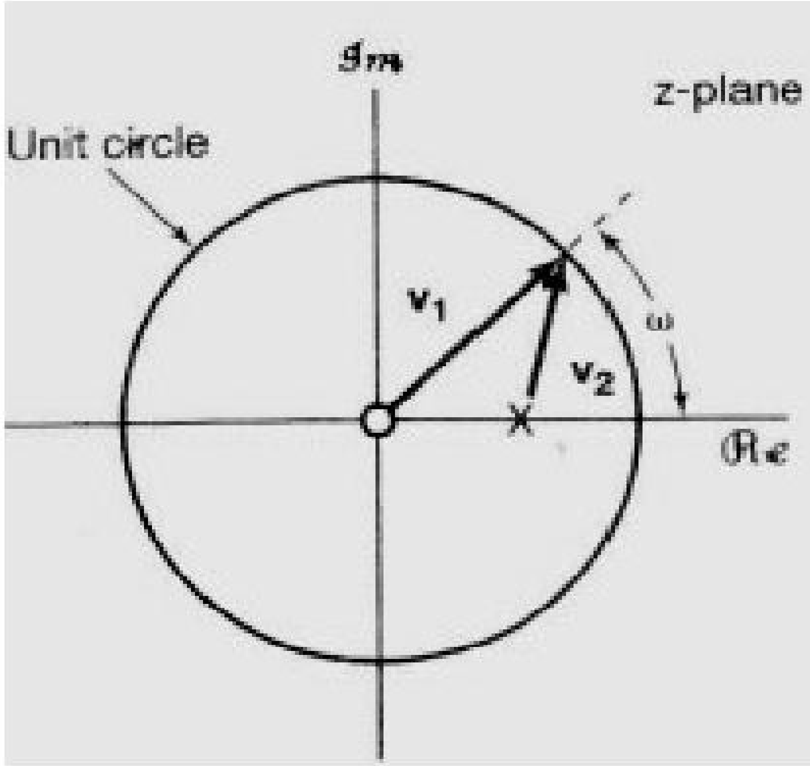
For $|a| < 1$, the ROC includes the unit circle, and the Fourier transform of $h[n]$ converges.

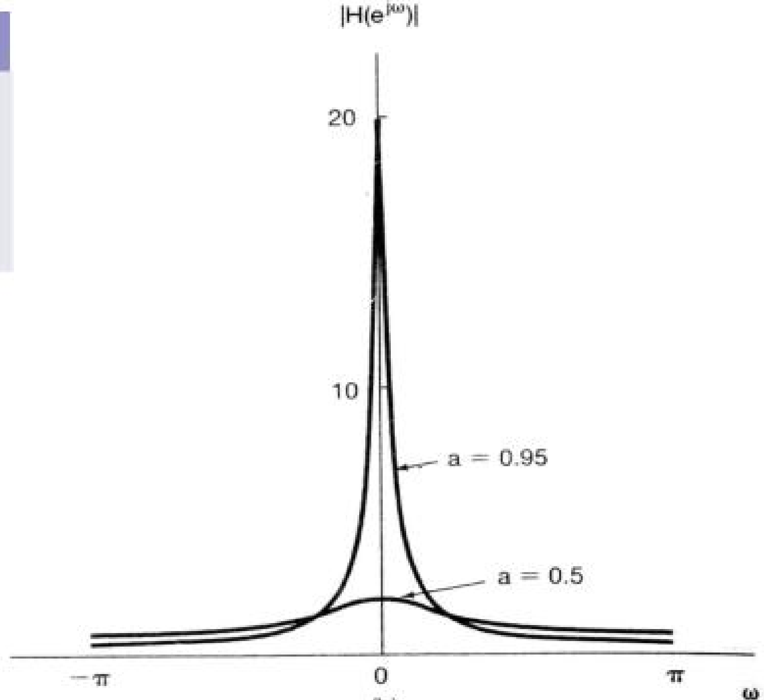
$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}$$

$$H(e^{j\omega}) = \frac{e^{j\omega} - 0}{e^{j\omega} - a} = \left|H(e^{j\omega})\right| e^{\arg H(e^{j\omega})}$$

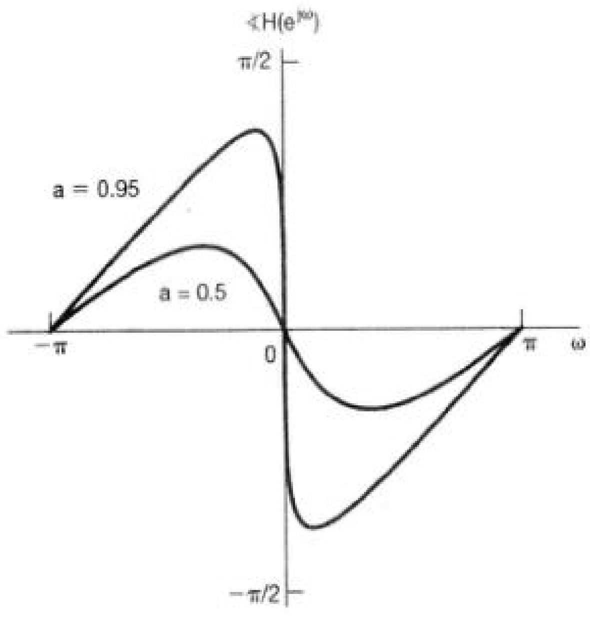
pole: $z = a$

zero: $z = 0$



(b)



(c)

# LTI system properties

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

for FIR: $a_k = 0, \text{ for } k \neq 0$

$$b_k = h[k]$$

# Example Relation between H(z) and frequency response:

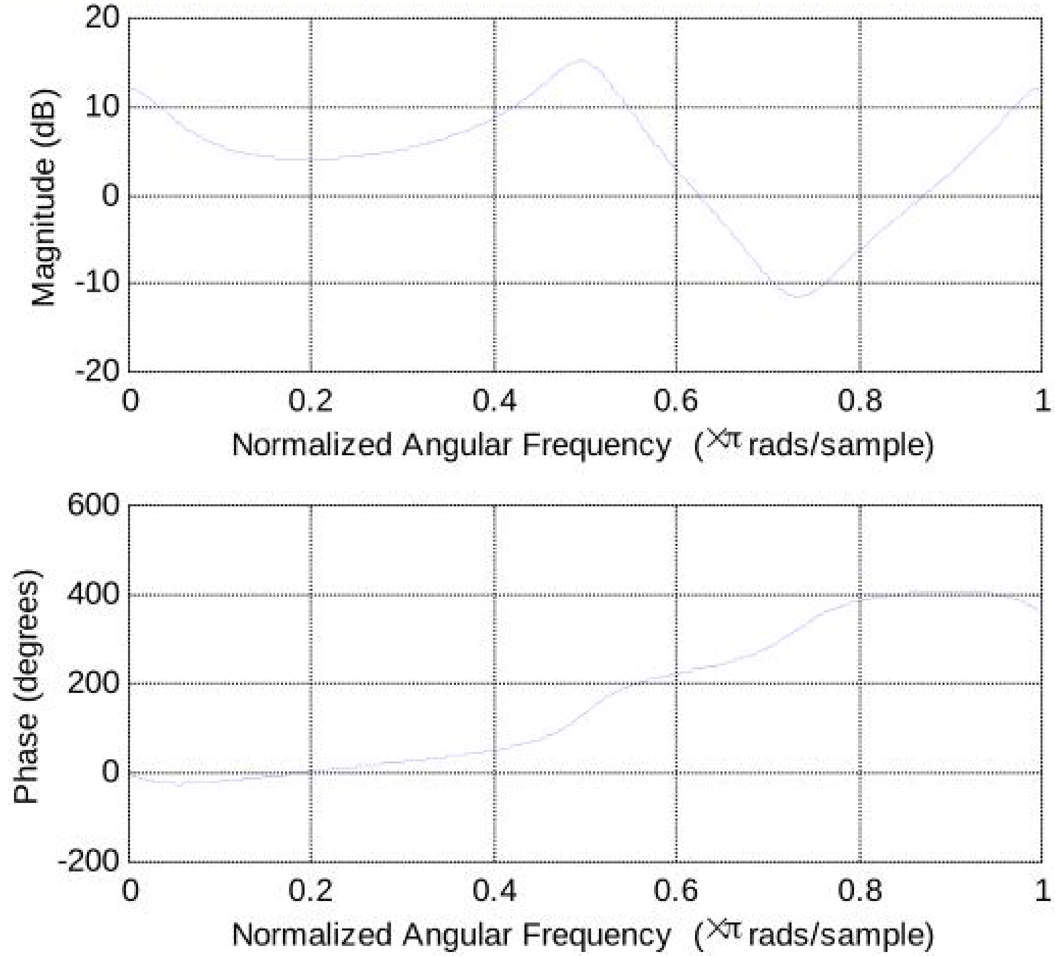
$$H(z) = \frac{2 + z^{-1} - z^{-3}}{1 + 0.5z^{-2} - z^{-4}}$$

$$y[n] = 2x[n] + x[n-1] - x[n-3] - 0.5y[n-2] + y[n-4]$$

**B=[2,1,0,-1]**

**A=[1,0,0.5,0,-1]**

**freqz(B,A)**



# LTI system properties

$$\begin{tikzcd}
H(z) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & h[n] \arrow[r, leftrightarrow] \arrow[dl, leftrightarrow] & \text{convolution(FIR's realization)} \\
H(e^{j\omega}) & \text{difference equation(IIR's realization)} &
\end{tikzcd}$$

## summary:

Keys and difficulties：

ROC

the convolution property

the relationship among

system function

the impulse response

frequency response

difference equation

## Exercises

第二版

- 3.40
- 3.43

第三版

- 3.42
- 3.45