Signals and Systems (Sem. I / 2014-2015)

Tutorial Solutions

Solutions for Chapter 1

Answer to P 1.1:

Key: $c = x + jy = \rho e^{j\theta}$ with

$$\rho = \sqrt{x^2 + y^2}, \quad tan\theta = \frac{y}{x};$$
$$x = \rho\ cos\theta, \quad y = \rho\ sin\theta$$

- θ is not unique as $e^{j\theta} = e^{j(\theta + 2\pi m)}$ for any integer m. Usually, $|\theta| \leq \pi$ is assumed.

- $-5 = 5e^{j\pm\pi} \;\Rightarrow\; \theta = \pm\pi$, $5 = 5e^{j0} \;\Rightarrow\; \theta = 0$; Also,

$$1 + j = \sqrt{2}e^{j\pi/4}, \; 1 - j = \sqrt{2}e^{-j\pi/4}$$
$$1 + j = -(1 - j) = e^{j\pi}\sqrt{2}e^{-j\pi/4} = 1 + j = \sqrt{2}e^{j3\pi/4}$$

End

Answer to P 1.2: Keep the definitions in mind !

$$\qquad\qquad\qquad\qquad\qquad\qquad \textit{Energy} \qquad\qquad\qquad\qquad \textit{Power}:$$

$$E_x(T) \stackrel{\triangle}{=} \int_{-T}^{T} |x(t)|^2 dt \;\Rightarrow\; E_x = \lim_{T\to\ +\infty} E_x(T); \;\; P_x = \lim_{T\to+\infty} \frac{E_x(T)}{2T}$$

$$E_x[N] \stackrel{\triangle}{=} \sum_{n=-N}^{N} |x[n]|^2 \;\Rightarrow\; E_x = \lim_{N\to\ +\infty} E_x[N]; \;\; P_x = \lim_{N\to+\infty} \frac{E_x[N]}{2N+1}$$

- (a) For $x_1(t) = e^{-2t}$, one has

$$E_{x_1}(T) \stackrel{\triangle}{=} \int_{-T}^{T} |x_1(t)|^2 dt = \int_0^T e^{-4t} dt = \frac{1}{-4} e^{-4t}|_0^T = \frac{1}{4}[1 - e^{-4T}]$$

So, $E_x = \lim_{T\to+\infty} = 1/4$, and hence it is an energy signal. Clearly, $P_x = 0$.

For $x_2[n] = (0.75e^{j\theta})^{|n|}$, one has

$$E_{x_2}[N] \triangleq \sum_{n=-N}^{N} |x_2[n]|^2 = \sum_{n=-N}^{N} |(0.75e^{j\theta})^{|n|}|^2 = \sum_{n=-N}^{N} 0.75^{2|n|}$$

$$= \sum_{n=-N}^{-1} 0.75^{-2n} + \sum_{n=0}^{N} 0.75^{2n} = \sum_{m=0}^{N} 0.75^{2m} - 1 + \sum_{n=0}^{N} 0.75^{2n}$$

$$= 2 \times \frac{1 - 0.75^{2(N+1)}}{1 - 0.75^2} - 1$$

So, $E_x = \lim_{N\to+\infty} = \frac{2}{1-0.75^2} - 1 = \frac{25}{7}$ - an energy signal. Clearly, $P_x = 0$.

- (b) Let $x(t) = x(t + T_0), \ \forall \ t$, then with $T = MT_0$

$$E_x(T) \triangleq \int_{-T}^{T} |x_1(t)|^2 dt = \int_{-MT_0}^{-(M-1)T_0} |x(t)|^2 dt + \cdots$$

$$+ \int_{-T_0}^{0} |x(t)|^2 dt + \int_{0}^{T_0} |x(t)|^2 dt + \cdots + \int_{(M-1)T_0}^{MT_0} |x(t)|^2 dt$$

It follows from $x(t) = x(t + kT_0)$ and then $\int_{(k-1)T_0}^{kT_0} |x(t)|^2 dt = \int_0^{T_0} |x(t)|^2 dt$

that $E_x = \lim_{T\to+\infty} E_x(T) = \lim_{M\to+\infty} 2M\int_0^{T_0}|x(t)|^2dt \quad\to\quad +\infty$, and $P_x = \lim_{T\to+\infty}\frac{E_x(T)}{2T} = \lim_{M\to+\infty}\frac{2M\int_0^{T_0}|x(t)|^2dt}{2MT_0} = \frac{1}{T_0}\int_0^{T_0}|x(t)|^2dt$ - a power signal.

- (c) Let $E_x = \int_{-\infty}^{+\infty}|x(t)|^2dt$. Now, with $y(t) = x(\kappa t)$

$$E_y = \int_{-\infty}^{+\infty}|y(t)|^2dt = \int_{-\infty}^{+\infty}|x(\kappa t)|^2dt$$

Define $\tau = \kappa t$, then $dt = \kappa^{-1}d\tau$ and then

$$E_y = \kappa^{-1}\int_{-\infty}^{+\infty}|x(\tau)|^2d\tau = \kappa^{-1}E_x$$

As required, $E_y = 1$. This means

$$\kappa = E_x$$

End

Answer to P 1.3: As given, the system I/O is $y[n] = nx[n] + \alpha$.

- *Linearity*

  Let $x_k[n] \rightarrow y_k[n] = nx_k[n] + \alpha$, $k = 1, 2$. Clearly, $x[n] = \beta_1 x_1[n] + \beta_2 x_2[n] \rightarrow y[n] = nx[n] + \alpha$, i.e., $y[n] = n\{\beta_1 x_1[n] + \beta_2 x_2[n]\} + \alpha$. So,

  $$\begin{aligned} y[n] &= \beta_1\{nx_1[n] + \alpha\} + \beta_2\{nx_2[n] + \alpha\} + \alpha[1 - (\beta_1 + \beta_2)] \\ &= \beta_1 y_1[n] + \beta_2 y_2[n] + \alpha + \alpha[1 - (\beta_1 + \beta_2)] \end{aligned}$$

  Generally, $y[n] \neq \beta_1 y_1[n] + \beta_2 y_2[n]$ as β_1, β_2 are all arbitrary. So, the system is NOT linear unless $\alpha = 0$.

- *Time-invariance*

  Let $x[n] \rightarrow y[n] = nx[n] + \alpha$. So, $\hat{x}[n] = x[n - n_0] \rightarrow \hat{y}[n] = n\hat{x}[n] + \alpha = nx[n - n_0] + \alpha$ for any n_0 given. Knowing $y[n - n_0] = (n - n_0)x[n - n_0] + \alpha$, we realize that $\hat{y}[n] \neq y[n - n_0]$, which implies that the system is NOT time-invariant.

- *Causality*

  Clearly, the system IS causal as $y[n_0]$ has nothing to do with $x[n]$ for $n > n_0$, where n_0 is any given integer.

- *Stability*

  Let $x[n] = 1$, which is bounded. Note $|y[n]| = |x[n] + \alpha| \geq |n| - |\alpha|$ can be bigger than any given number. This means that $y[n]$ is unbounded and hence the system is UNSTABLE.

End

Answer to P 1.4: Given that $x[n] = cos(\omega_0 n)$, if it is periodic, there should exists N such that

$$x[n] = x[n+N],\ \forall\ n \quad \Leftrightarrow \quad cos(\omega_0 n) = cos(\omega_0 n + \omega_0 N), \quad \forall\ n$$

which implies that there should exist some integer K such that

$$\omega_0 N = 2\pi K \quad \Rightarrow \quad N = \frac{2\pi}{\omega_0} K$$

Keeping that N is integer in mind, the above is impossible for the case when $\frac{\pi}{\omega_0}$ is NOT rational!

Examples: 1) $\omega_0 = 0.3\pi \quad \Rightarrow \quad \frac{\pi}{\omega_0} = 10/3 \quad \Rightarrow \quad N = 20K/3$. So, $K = 3, 6, ..., 3k, ..$ and hence the signal is periodic; 2) when $\omega_0 = 0.3$, there exists no integer K and hence no N, meaning the signal is NOT periodic.

End

Answer to P 1.5 and **Answer tp P 1.8**: To be given on the class on board.

Answer to P 1.6:

- (a) Let $y_1(t) \stackrel{\triangle}{=} x(4 - t/2)$. With $x(t)$ given, one notes

  $y_1(t) = x(-\frac{1}{2}(t-8))$. Let $\tilde{y}_1(t) = x(-\frac{1}{2}t)$, then $y_1(t) = \tilde{y}_1(t-8)$. Therefore, $\tilde{y}_1(t)$ is obtained by applying a time reversal to $x(t)$ then followed by a time scaling of $1/2$. Time-shifting $\tilde{y}_1(t)$ by 8 yields $y_1(t)$.

- $y_2(t) = [x(t) + x(-t)]u(t)$ to be sketch on board.

- Noting

$$\begin{aligned} y_3(t) &= x(t)[\delta(t+3/2) - \delta(t-3/2)] \\ &= x(-3/2)\delta(t+3/2) - x(3/2)\delta(t-3/2) \end{aligned}$$

  one has $y_2(t) = -0.5\delta(t+3/2) - 0.5\delta(t-3/2)$.

  Sketch it on board. **End**

Answer to P 1.7:

- Let $y_1[n] \stackrel{\triangle}{=} x[3n+1]$. Observing $x[n]$, one can see $y_1[n] = 0$ for those integer valued n: $3n+1 < -4 \;\Leftrightarrow\; n < -1 \quad and \quad 3n+1 > 3 \;\Leftrightarrow\; n \geq 1$

  So, there are two points: $n = -1, 0$, for which $y_1[n]$ may not be zero.

  Knowing $y_1[-1] = x[-2] = 1/2, y_1[0] = x[1] = 1$, we then have

$$y_1[n] = 1/2\delta[n+1] + \delta[n]$$

- Let $y_2[n] \stackrel{\triangle}{=} x[n]u[3-n]$. Denote $w[n] = u[1-n] = u[-(n-1)]$, obtained by time reversal on $u[n]$, then time-shifting by 1.

  $y_2[n]$ is the signal obtained by the product of $x[n]$ and $w[n]$ point by point.

$$y_2[n] = -\delta[n+4] - 0.5\delta[n+3] + 0.5\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1]$$

End

Answer to P 1.10:

Noting $w_\tau(t)$ is an even function, i.e., $w_\tau(t) = w_\tau(-t)$, one has $w_\tau(\alpha t) = w_\tau(|\alpha| t)$.

Sketching $w_\tau(|\alpha| t)$, one realizes $w_\tau(|\alpha| t) = w_{\tau/|\alpha|}(t)$.

According to the definition,

$$\delta(\alpha t) = \lim_{\tau \to 0} \frac{w_\tau(\alpha t)}{\tau} = \lim_{\tau \to 0} \frac{w_{\tau/|\alpha|}(t)}{\tau} = \lim_{\tau \to 0} |\alpha|^{-1} \frac{w_{\tau/|\alpha|}(t)}{\tau/|\alpha|}$$

that is

$$\delta(\alpha t) = |\alpha|^{-1} \lim_{\tilde{\tau} \to 0} \frac{w_{\tilde{\tau}}(t)}{\tilde{\tau}} = |\alpha|^{-1} \delta(t)$$

End

Answer to P 1.12:

Key: To memorize and understand the definitions.

- Memoryless: $y(t)$ $(y[n])$ depends only on $x(t)$ $(x[n])$ for all t (n).
- Time-invariant: Let $x(t) \rightarrow y(t)$. Compute the output $\tilde{y}(t)$ when the input is $\tilde{x}(t) = x(t - t_0)$. Calculate $y(t - t_0)$ and check if $\tilde{y}(t)$ is equal to $y(t - t_0)$.
- Linear: Let $x_k \rightarrow y_k,\ k = 1, 2$. Compute the output y when the input is $x = \alpha_1 x_1 + \alpha_2 x_2$. Check if y is equal to $\alpha_1 y_1 + \alpha_2 y_2$.
- Causal: Check if $y[n_0]$ depends *ONLY* on the values of $x[n]$ for $n \leq n_0$ for all n_0 and $x[n]$. If yes, it is causal.
- Stable: For any $|x| < M_x$, check if y is bounded.

See **Problem 1.3**.

Examples

- $x(t) \;\rightarrow\; y_1(t) = x(t-2) + x(2-t)$.

  It is easy to see that it is NOT memoryless, Non-causal, stable, and linear, but is it TI?

  Let $\tilde{x}(t) = x(t-t_0)$, then $\tilde{y}(t) = \tilde{x}(t-2) + \tilde{x}(2-t)$. Noting that

  $$\tilde{x}(t) = x(t-t_0) \;\Rightarrow\; \tilde{x}(t-2) = x((t-2)-t_0), \;\; \tilde{x}(2-t) = x((2-t)-t_0)$$

  we have

  $$\tilde{y}(t) = x((t-2)-t_0) + x((2-t)-t_0) = x((t-t_0)-2) + x(2-(t+t_0))$$

  Since $y_1(t) = x(t-2) + x(2-t)$,

  $$y_1(t-t_0) = x((t-t_0)-2) + x(2-(t-t_0))$$

  Clearly, $\tilde{y}_1(t) \neq y_1(t-t_0)$. Therefore, it is NOT time-invariant!

- $x(t) \;\rightarrow\; y_3(t) = \frac{dx(t)}{dt}$.

It is easy to see that it is time-invariant, unstable ($u(t) \;\rightarrow\; \delta(t)$) and linear, but is it causal and memoryless?

$$y_3(t) = \lim_{\Delta t \;\rightarrow\; 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

So, it is Not memoryless because it depends on $x(t+\Delta t)$ when $|\Delta t| \neq 0$ very small, and it is NOT causal since it depends on $x(t+\Delta t)$, where Δt can be positive.

Both cases are *Marginal*!

End

Answer to P 1.13: As seen, $y_1(t)$ is the output of the LTI system excited by $x_1(t) = u(t) - u(t-2)$:

$$x_1(t) \quad \rightarrow \quad y_1(t)$$

- Note that $x_2(t)$ can be expressed in terms of $x_1(t)$:

$$x_2(t) = x_1(t) - x_1(t-2) \quad \rightarrow \quad y_2(t) = \quad ??$$

If $q(t) \stackrel{\triangle}{=} x_1(t-2) \quad \rightarrow \quad p(t)$, *linearity* implies

$$y_2(t) = y_1(t) - p(t)$$

$y_1(t)$ has been given, what about $p(t)$?

Time invariance suggests $p(t) = y_1(t-2)$ and consequently,

$$y_2(t) = y_1(t) - y_1(t-2)$$

- Observing carefully, one can see that

$$x_3(t) = x_1(t+1) + x_1(t)$$

and immediately

$$y_3(t) = y_1(t+1) + y_1(t)$$

With $y_1(t)$ given, one can sketch $y_2(t)$ and $y_3(t)$ obtained above with little difficulty.

End

Answer to P 1.14:

- For the system: $x(t) \ \rightarrow \ y(t) = x(\alpha t)$, we have

$$\tilde{x}(t) \triangleq x(t-t_0) \ \rightarrow \ \tilde{y}(t) = \tilde{x}(\alpha t)$$

  On the one hand, $\tilde{x}(t) \triangleq x(t-t_0) \ \Rightarrow \ \tilde{x}(\alpha t) = x(\alpha t - t_0)$, $\tilde{y}(t) = x(\alpha t - t_0)$. On the other hand, $y(t-t_0) = x(\alpha t - \alpha t_0)$. Clearly, $\tilde{y}(t) \neq y(t-t_0)$, which implies that the system is NOT time-invariant, unless $\alpha = 1$.

  Let $x_k(t) \ \rightarrow \ y_k(t) = x_k(\alpha t), \ k = 1, 2$. Noting that

$$\begin{aligned} x(t) = \beta_1 x_1(t) + \beta_2 x_2(t) \quad \rightarrow \ y(t) &= x(\alpha t) = \beta_1 x_1(\alpha t) + \beta_2 x_2(\alpha t) \\ &= \beta_1 y_1(t) + \beta_2 y_2(t) \end{aligned}$$

  we conclude that the system is LINEAR.

- As given, the system $x(t) \;\to\; y(t)$ is TI, that is $x(t+t_0) \;\to\; y(t+t_0)$ for ANY t_0.

  Particularly, for the period T_0 we have

$$x(t+T_0) \;\to\; y(t+T_0)$$

  Knowing $x(t) = x(t+T_0) \;\to\; y(t)$, one concludes

$$y(t) = y(t+T_0) \quad \Rightarrow \quad y(t) \;\; is \;\; periodic$$

  and the period, denoted as $\tilde{T}_0$, according to definition, should not be larger than T_0.

End

Answer to P 1.16: Given that

$$x_p(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_p), \quad T_p > 0$$

Since

$$x_p(t + T_p) = \sum_{k=-\infty}^{+\infty} x_0((t + T_p) - kT_p) = \sum_{k=-\infty}^{+\infty} x_0(t - (k - 1)T_p)$$

With $m = k - 1$, one has

$$x_p(t + T_p) = \sum_{m=-\infty}^{+\infty} x_0(t - mT_p) = x_p(t), \quad \forall \;\; t$$

This tells us that $x_p(t)$ is periodic and the period is not bigger than T_p.

With $x_0(t)$ given, sketch $x_p(t)$ on board for different T_p.

Answer to P 1.17: Based on the diagram given, $e[n] = x[n] - y[n]$.

As $y[n] = 0.75e[n-1]$, one finally has

$$y[n] = 0.75x[n-1] - 0.75y[n-1], \quad y[n] = 0, \ \ \forall\, n < 0$$

- When $x[n] = \delta[n]$, $y[0] = 0.75\delta[-1] - 0.75y[-1] = 0, y[1] = 0.75\delta[0] - 0.75y[0] = 0.75$, and $y[n] = -0.75y[n-1], \quad \forall\, n \geq 2$. Clearly,

$$y[n] = 0.75(-0.75)^{n-1}u[n-1]$$

- When $x[n] = u[n]$, $y[0] = 0.75u[-1] - 0.75y[-1] = 0$, and

$$y[n] = 0.75 - 0.75y[n-1]), \quad \forall\, n \geq 1$$

Clearly, $y[1] = 0.75, \ y[2] = 0.75 - 0.75^2, \cdots ,.$ A closed-form expression

$$y[n] = \frac{0.75}{1.75}\,[0.75(-0.75)^{n-1} + 1]u[n-1]$$

can be obtained easily using the technique in Chapter 2. **End**

Solutions for Chapter 2

Answer to P 2.1: By a graphical approach, one can see that

$$\begin{cases} 0, & n < 0 \\ n+1, & 0 \geq n, n-N \geq 0 \;\Rightarrow\; 0 \leq n \leq N \\ N+1, & n-N > 0, n \leq 9 \;\Rightarrow\; N < n \leq 9 \\ 9-(n-N)+1, & 9 < n, n-N \leq 9 \;\Rightarrow\; 9 < n \leq 9+N \\ 0, & n-N > 9 \;\Rightarrow\; n > 9+N \end{cases}$$

Now,

$$y[4] = 5 \quad \Rightarrow \quad n = 4 \; belongs \; to \quad 0 \leq n \leq N \quad \Rightarrow \quad 4 \leq N$$

$$y[14] = 0 \quad \Rightarrow \quad n = 14 \; belongs \; to \quad n > 9+N \quad \Rightarrow \quad 14 > 9+N$$

Therefore,

$$N = 4$$

Answer to P 2.2: Given

$$x(t) = (t+1)w_1(t-1/2) + (2-t)w_1(t-3/2), \quad h(t) = \delta(t+2) + 2\delta(t+1)$$

Consider $x(t)$ as the unit impulse response of an LTI system: $\delta(t) \quad \rightarrow \quad x(t)$ and $h(t)$ is the input. So, one has

$$y(t) = x(t) * h(t) = h(t) * x(t) = x(t+2) + 2x(t+1)$$

Sketch $x(t)$, then $2x(t+1), x(t+2)$. $y(t)$ can be obtained by adding the two one-by-point.

End

Answer to P 2.3: Given $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$.

- Direct approach

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{3}^{5} 1\ e^{-3(t-\tau)}u(t-\tau)d\tau$$

$$= \begin{cases} 0, & t < 3 \\ \int_3^t 1\ e^{-3(t-\tau)}d\tau, & 3 < t < 5 \\ \int_3^5 1\ e^{-3(t-\tau)}d\tau, & 5 < t \end{cases}$$

Therefore,

$$y(t) = \frac{1}{3}[1 - e^{-3(t-3)}]\ u(t-3) - \frac{1}{3}[1 - e^{-3(t-5)}]\ u(t-5)$$

- As $\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$,

$$\begin{aligned} g(t) &\stackrel{\triangle}{=} (\frac{dx(t)}{dt}) * h(t) = h(t-3) - h(t-5) \\ &= e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5) \end{aligned}$$

- Clearly, $\frac{dy(t)}{dt} = g(t)$. So,

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} g(\tau)d\tau + y(-\infty) \\ &= \int_{-\infty}^{t} e^{-3(\tau-3)}u(\tau-3)d\tau - \int_{-\infty}^{t} e^{-3(\tau-5)}u(\tau-5)d\tau \\ &= \int_{3}^{t} e^{-3(\tau-3)}d\tau u(t-3) - \int_{5}^{t} e^{-3(\tau-5)}d\tau u(t-5) = ... \end{aligned}$$

End

Answer to P 2.6 Given three sub LTI systems with UIRs

$$h_1(t) = tu(t), \quad h_2(t) = \delta(t-1), \quad h_3(t) = -\delta(t)$$

- How to design a system with the three s.t the system has an UIR below?

$$h(t) = -(t-1)u(t-1)$$

By observation, $h(t) = -h_1(t-1) = h_1(t) * h_2(t) * h_3(t)$, which is a cascade of the three sub-systems.

- How to design a system with the three s.t the system has an UIR below?

$$h(t) = tu(t) - \delta(t-1) - (t-1)u(t-1)$$

By observation, $h(t) = h_1(t) - h_2(t) + h_1(t) * h_2(t) * h_3(t)$.

Draw the block-diagrams on board.

Answer to P 2.7: Note

$$x(t) = u(t-1) - u(t-2), \quad y(t) = (t-2)[u(t-2) - u(t-3)] + u(t-3)$$

and hence

$$\frac{dx(t)}{dt} = \delta(t-1) - \delta(t-2), \quad \frac{dy(t)}{dt} = u(t-2) - u(t-3)$$

Since

$$y(t) = x(t) * h(t) \quad \Rightarrow \quad \frac{dy(t)}{dt} = (\frac{dx(t)}{dt}) * h(t)$$

one has

$$u(t-2) - u(t-3) = h(t-1) - h(t-2)$$

By observation,

$$h(t) = u(t-1)$$

Answer to P 2.10: Given that $y[n] = \sum_{k=-\infty}^{n} x[k]$,

- Obviously,

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n]$$

- Note

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{+\infty} x[k]u[n-k] = x[n] * u[n]$$

According to the theorem, the system is LTI with $h[n] = u[n]$.

Answer to P 2.11: Recall the relationship between causality and stability of an LTI system with its unit impulse response h.

For the LTI with $h[n] = 5^n u[3-n]$, it is

- not causal because of $h[-3] = 5^{-3} \neq 0$ - against the condition of initial rest;
- stable because of $\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} 5^n u[3-n]$, equal to

$$\sum_{n=-\infty}^{3} 5^{(n-3)} 5^3 = 5^3 \sum_{k=0}^{+\infty} 5^{-k} = \frac{5^3}{1-1/5} < +\infty$$

For the LTI with $h(t) = e^{-6|t|}$, it is

- not causal because of $h(t) \neq 0$ for all $t < 0$.
- stable because of $\int_{-\infty}^{+\infty} |h(t)| dt = 2 \int_0^{+\infty} e^{-6t} dt = 1/3 < +\infty$.

End

Answer to P 2.12: Consider the signal $x[n] = \alpha^n \; u[n]$.

- Clearly, one has

$$g[n] \stackrel{\triangle}{=} x[n] - \alpha x[n-1] = \alpha^n \; u[n] - \alpha\alpha^{(n-1)} \; u[n-1]$$

$$= \alpha^n \; u[n] - \alpha^n \; u[n-1] = \alpha^n \{u[n] - u[n-1]\} = \delta[n]$$

- As given, $x[n] * h[n] = (\frac{1}{2})^n \{u[n+2] - u[n-2]\} \stackrel{\triangle}{=} p[n]$. On the one hand,

$$g[n] * h[n] = (x[n] - \alpha x[n-1]) * h[n]$$

$$= x[n] * h[n] - \alpha x[n-1] * h[n] = p[n] - \alpha p[n-1]$$

On the other hand, the fact that $g[n] = \delta[n]$ implies $g[n] * h[n] = h[n]$. So,

$$h[n] = p[n] - \alpha p[n-1]$$

Note: The solution depends on the value of α.

End

Answer to P 2.13: Given an LTI system: $x(t) \;\rightarrow\; y(t)$ and

$$e(t) = e^{\alpha t}u(t) \;\rightarrow\; r(t), \quad \frac{de(t)}{dt} \;\rightarrow\; \beta r(t) + e^{-2t}u(t)$$

It follow from $\frac{de(t)}{dt} = \alpha e^{\alpha t}u(t) + e^{\alpha t}\delta(t) = \alpha e^{\alpha t}u(t) + \delta(t)$ that the output in response of such an input should be

$$\alpha r(t) + h(t)$$

and as given,

$$\alpha r(t) + h(t) = \beta r(t) + e^{-2t}u(t) \;\Rightarrow\; \quad h(t) = (\beta - \alpha)r(t) + e^{-2t}u(t)$$

End

Answer to P 2.15: This exercise is used to strengthen your understanding of the concepts.

- *If $h(t)$ is the impulse response of of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable* - **true**!

  Justification: As the sufficient and necessary condition for LTI systems to be stable if

  $$\int_{-\infty}^{+\infty} |h(t)|dt < +\infty$$

  If $h(t+T) = h(t)$ with T the period, then

  $$\int_{-\infty}^{+\infty} |h(t)|dt = \lim_{N \to +\infty} \sum_{k=-N}^{N} \int_{(k-\frac{1}{2})T}^{(k+\frac{1}{2})T} |h(t)|dt = +\infty$$

  because of

  $$\int_{(k-\frac{1}{2})T}^{(k+\frac{1}{2})T} |h(t)|dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} |h(t)|dt \neq 0$$

- *The inverse of a causal LTI system is always causal* - **false**!

  Inverse: If $x(t) \rightarrow y(t) \rightarrow x(t)$, then the system: $y(t) \rightarrow x(t)$ is said the inverse of the system: $x(t) \rightarrow y(t)$.

  Justification: As $y(t) = x(t - t_0)$ with $t_0 > 0$ is causal and LTI, and $w(t) = y(t + t_0)$ is LTI, the second system is the inverse of the first one (because of $w(t) = x(t)$) but it is non-causal.

- *If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with $h[n]$ as unit impulse response is stable* - *false*!

  Justification:

$$|u[n]| \leq 1 \ \Rightarrow \ \sum_n |h[n]| = +\infty$$

  which implies that the system is unstable.

- *If an LTI system has an impulse response $h[n]$ of finite duration, the system is stable* - **true** - as long as $|h[n]| < +\infty$!

  Justification: Such a $h[n]$ is of form

$$h[n] = \sum_{k=N_1}^{N_2} h[k]\delta[n-k]$$

  where both N_1, N_2 are finite with $N_2 \geq N_1$. Clearly, $\sum_n |h[n]| < +\infty$ is always met.

- *If an LTI system is causal, then it is stable* - **false**!

  Justification: $\sum_n |h[n]| < +\infty$ may not be met. Say the causal system $h[n] = 2^n u[n]$, which is unstable.

- The cascade of a non-causal LTI system with a causal one is necessarily non-causal - **false**!

Justification: Assume $h_1[n] = \delta[n-3]$ - causal and $h_2[n] = u[n+1]$ - non-causal. Clearly, the cascade of the two has a unit impulse response $h[n] = h_1[n] * h_2[n] = u[n-2]$ - causal.

- *An LTI system with $u(t) \rightarrow s(t)$ – unit step response stable if and only if $\int_{-\infty}^{+\infty} |s(t)| \, dt < +\infty$* - **false**!

  Justification: The system $h(t) = e^{-t}u(t)$ is stable but $s(t) = h(t) * u(t) = [1 - e^{-t}]u(t)$ is not absolutely integrable.

- *An LTI system with $u[n] \rightarrow s[n]$ is causal if and only if $s[n] = 0, \ \forall \ n < 0$* - **true**!

  Justification: *Sufficient condition:* Assume $s[n] = 0, \ \forall \ n < 0$. As $[n]$ is the output of the system when the input is $u[n]$ and $h[n]$ is the output when the input is $\delta[n] = u[n] - u[n-1]$, the system being LTI means

$h[n] = s[n] - s[n-1]$. Clearly,

$$h[n] = 0,\ \forall\ n < 0 \quad \Leftrightarrow \quad \textit{system causal}$$

Necessary condition: Assume the system is causal, then $h[n] = 0,\ \forall\ n < 0$, that is $h[n] = h_0[n]\ u[n]$. Note

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{+\infty} h_0[k]u[k]u[n-k] = \sum_{k=0}^{n} h_0[k]\ u[n]$$

Therefore,

$$s[n] = 0,\ \forall\ n < 0$$

End

Answer to P 2.16:

With the information given, one has

$$y[n]/y[n-1] = 1/2 \;\Rightarrow\; y[n]-1/2\,y[n-1] = 0, \quad y[0] = h \;\Rightarrow\; y[n] = (1/2)^n\,h$$

With $h = 3$, $y[n] < 0.1$ implies

$$(1/2)^n\,3 < 0.1 \qquad \Rightarrow \quad 30 \geq 2^n \quad \Rightarrow \quad n \geq 5$$

Answer to P 2.17: By analysis,

$$y[n] = x[n] + (1 + r)y[n - 1]$$

with $y[0] = 100, \quad r = 0.25\%, \; x[n] = 1000 + 100n.$