Fundamentals of Signals and Systems

Chapter 3 - Fourier Analysis of Signals

Outline of Topics

Main objective

- To learn different representations of signals - signal transformations
- To understand why a signal should be transformed?
- Our objective in this chapter is to study different transforms and more importantly to learn how to analyze problems.

It is the biggest chapter - try to grab the things in common.

About Fourier analysis

Though termed after the French math. & phys. J.B.J. Fourier (1768 - 1830), the Fourier analysis (FA) has a long history involving *many great names* such as L. Euler, D. Bernoulli, and many others,[1] and initiated in *the investigation* of vibrations, physical motions, astronomical events, and optics.

Fourier analysis, as the *corner-stone* for modern signal processing and system theory, has been developed all the time.

[1] Fourier claimed the Fourier series (FS) in 1807 without a proof given later by Poisson, Cauchy and Dirichlet; His 1807 paper, though in favor by three reviewers including Laplace, was rejected due to Lagrange; FA for DT signals, as the discipline of numerical approximation, was traced back to Newton's time in 1600s; The fast Fourier transform, claimed by Cooley and Tukey in 1965, was surprisingly founded in Gauss's notebooks.

Periodic signals

Look at

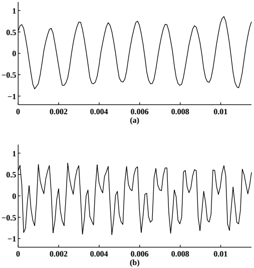


Fig. 3.1: Waveforms for the music note *DO*. (a) a keyboard; (b) a recorder.

Both are for *Do* and have almost the same period. What is the most significant feature that can distinguish the notes produced by the two instruments? The *components*!

As defined before, a signal $s(t)$ is said to be *periodic* if there exists some positive constant T such that

$$s(t) = s(t+T), \ \forall\, t \in \mathcal{R}$$

The smallest T, denoted as T_0, is the (fundamental) *period* of the signal.

Let $x_1(t)$ and $x_2(t)$ be periodic with period T_1 and T_2, respectively. Then, it is easy to show that the sum $x(t) \triangleq x_1(t) + x_2(t)$ is periodic if the ratio T_1/T_2 is rational, i.e.,

$$\frac{T_1}{T_2} = \frac{p}{q}, \quad p,\ q\ \in \mathcal{Z}^+ \triangleq \{1, 2, 3, \cdots\} \tag{1}$$

and that $T_0 = qT_1$ is the period if p and q are *co-prime*.

Now, let us consider how to synthesize a periodic signal.

Let $T_0 > 0$ be constant and denote $\omega_0 \triangleq \frac{2\pi}{T_0}$. Generally speaking, a signal of form

$$x_N(t) = \sum_{m=-N}^{N} c[m] e^{jm\omega_0 t} \tag{2}$$

satisfies [2]

$$x_N(t + T_0) = x_N(t), \ \forall\, t$$

and hence is periodic for any nonnegative integer N and complex constant sequence $c[m]$:

$$c[m] = \alpha[m] + j\beta[m], \ \forall m \tag{3}$$

with $\alpha[m], \beta[m]$ all real.

[2] Here, T_0 may be a multiple of the period of $x_N(t)$.

Let $x(t)$: $x(t+T_0) = x(t)$. What is the $x_N(t)$ that best approximates $x(t)$?

Consider three sets of $\{c[m]\}$ given below

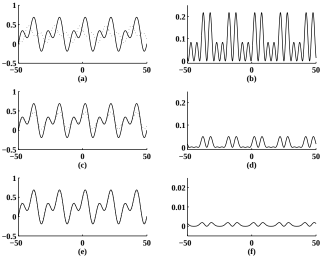


Fig. 3.2: Waveforms for a periodical signal $x(t)$ (solid line) and its approximations $x_2(t)$ (dotted line) with three different sets of $\{c[m]\}$.

As observed, $x_N(t)$ is determined by the values of $\{c[m]\}$. A question we may ask is how to find the $x_N(t)$ that is closest to $x(t)$?

Measure for *best*: Error signal $e(t) \triangleq x(t) - x_N(t)$ is measured with

$$\sigma \triangleq \int_{-T_0/2}^{T_0/2} |e(t)|^2 dt = \int_{T_0} |e(t)|^2 dt \tag{4}$$

where $\int_{T_0}$ means an integration for one period T_0, starting from any point.[3]

The best $x_N(t)$ can then be found from the following minimization

$$\min_{c[k],\ \forall k} \sigma \Longleftrightarrow \min_{\alpha[k],\ \beta[k],\ \forall k} \sigma \tag{5}$$

Any solution to (5) should satisfy

$$\frac{\partial \sigma}{\partial \alpha[k]} = 0, \quad \frac{\partial \sigma}{\partial \beta[k]} = 0, \quad \forall k \tag{6}$$

[3] As the error $e(t)$ is periodic, the integral interval can be taken for one period starting from any point other than $t = -T_0/2$. In fact, $\int_{\tau}^{T_0+\tau} |e(t)|^2 dt = \int_{\tau}^{0} |e(t)|^2 dt + \int_{0}^{T_0} |e(t)|^2 dt + \int_{T_0}^{T_0+\tau} |e(t)|^2 dt = \int_{0}^{T_0} |e(t)|^2 dt$.

It can be shown (see **Appendix C**) that (6) is equivalent to

$$\int_{T_0} e(t)\, e^{-j\omega_0 kt} dt = 0, \;\; \forall\, k \tag{7}$$

This is the so-called orthogonality principle.

Note $e(t) = x(t) - x_N(t) = x(t) - \sum_{m=-N}^{N} c[m] e^{jm\omega_0 t}$ and

$$\int_{T_0} e^{j(m-k)\omega_0 t} dt = \begin{cases} T_0 & , \quad m = k \\ 0 & , \quad m \neq k \end{cases} \tag{8}$$

It then follows from (7) that the optimal coefficients are given by

$$c[k] = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt, \;\; \forall\, k \tag{9}$$

We note that:

- It turns out of $e^*(t) = x^*(t) - \sum_{m=-N}^{N} c^*[m]e^{-jm\omega_0 t}$ and (7) that

$$\begin{aligned} \min_{\{c[k]\}} \sigma &= \int_{T_0} e(t)x^*(t)dt = \int_{T_0} |x(t)|^2 dt - T_0 \sum_{k=-N}^{N} |c[k]|^2 \\ &= E_x - E_{x_N} \triangleq \epsilon_N \end{aligned} \tag{10}$$

where E_{x_N}, see **Problem 2.8**, is the energy of $x_N(t)$ in one period.

- The optimal $c[k]$ by (9) are independent of N and hence $\epsilon_N \geq \epsilon_{N+1}$, indicating that the approximation is improved with increase of N;
- It follows from $\epsilon_N \geq 0$ that $\int_{T_0} |x(t)|^2 dt \geq T_0 \sum_{k=-N}^{N} |c[k]|^2$. This suggests that $c[k]$ be finite if so is $\int_{T_0} |x(t)|^2 \, dt$.

Note $x^*(t)$ is also periodic and hence its best optimal coefficients are

$$\tilde{c}[k] = \frac{1}{T_0}\int_{T_0} x^*(t)e^{-jk\omega_0 t}dt = (\frac{1}{T_0}\int_{T_0} x(t)e^{-j(-k)\omega_0 t}dt)^* = c^*[-k], \ \ \forall\, k$$

Furthermore, if the signal $x(t)$ is real-valued, then $\tilde{c}[k] = c[k]$, i.e.,

$$c[k] = c^*[-k] \ \ \forall k$$

which implies that if $c[k]$ is available, so is $c[-k]$. Let $c[k] = \frac{a[k]}{2}e^{j\phi[k]}$, then

$$a[k] = a[-k], \ \ \phi[-k] = -\phi[k], \ \ \forall\, k$$

Interestingly,

$$\begin{aligned} x_N(t) &= c[0] + \sum_{m=1}^{N}\{c[m]e^{j\omega_0 mt} + c[-m]e^{-j\omega_0 mt}\} \\ &= c[0] + \sum^{N} a[m]cos(m\omega_0 t + \phi[m]) \end{aligned} \tag{11}$$

$\boxed{\textbf{Example 3.1}}$: Consider signals $p_1(t), p_2(t), x(t)$ which are periodic with a period of one. Given

$$p_1(t) = cos(2\pi t)w_{\frac{1}{2}}(t), \ -0.5 < t < 0.5, \ \ p_2(t) = t. \ \ 0 < t < 1$$

and $x(t) = p_1(t) + p_2(t)$. See Fig. 3.3. Compute the optimal coefficients for each of the signals.

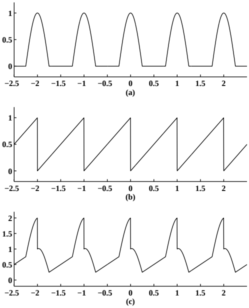


Fig. 3.3: Three periodic signals.

Solution: First of all, the fundamental frequency is $\omega_0 = \frac{2\pi}{T_0} = 2\pi$.

According to (9), the optimal coefficients for $p_1(t)$ are given[4]

$$\begin{aligned} c_1[k] &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} p_1(t)e^{-jk\omega_0 t}dt = \int_{-1/4}^{1/4} cos(2\pi t)e^{-jk\omega_0 t}dt = \cdots \\ &= \begin{cases} \frac{1}{4}[sinc(\frac{k-1}{2}) + sinc(\frac{k+1}{2})], & k \neq 1 \\ \frac{1}{4}, & k = 1 \end{cases}, \ \forall\, k \geq 0 \end{aligned}$$

where $sinc(t) \triangleq \frac{sin(t\pi)}{t\pi}$.

Similarly, for $p_2(t)$ using the technique of *integration by parts* we have

$$\begin{aligned} c_2[k] &= \frac{1}{T_0}\int_{0}^{T_0} p_2(t)e^{-jk\omega_0 t}dt = \int_{0}^{1} te^{-jk\omega_0 t}dt = \cdots \\ &= \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{j}{2k\pi}, & k \neq 0 \end{cases}, \ \forall\, k \geq 0 \end{aligned}$$

[4]Hint: Use Euler's formula for the integration.

As to the third signal $x(t) = p_1(t) + p_2(t)$, the corresponding optimal coefficients are

$$\begin{aligned} c[k] &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{T_0} [p_1(t) + p_2(t)]e^{-jk\omega_0 t}dt \\ &= c_1[k] + c_2[k],\ \forall\, k \end{aligned}$$

In general, the coefficients of $x(t) = \sum_m \alpha_m p_m(t)$ are $c[k] = \sum_m \alpha_m c_m[k]$, where $p_m(t) = p_m(t + T_0)$, $\forall\, t, m$ and $c_m[k]$ is the set of the optimal coefficients of $p_m(t)$.

For a given N, we can compute $c[k] = c_1[k] + c_2[k]$ first and then the best approximate $x_N(t)$ can be obtained with (2).

For example, for $N = 4$, calculations show that $c[0] = 0.8183$ and

$$\begin{aligned} c[1] &= 0.5927e^{j0.5669},\ c[2] = 0.2653e^{j0.6435} \\ c[3] &= 0.1061e^{j1.5708},\ c[4] = 0.0902^{j2.0608} \end{aligned}$$

and hence $c[-k] = c^*[k]$ can be obtained for $k = 1, 2, 3, 4$.

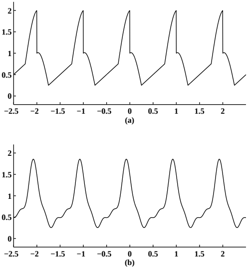


Fig. 3.4: (a) $x(t)$. (b) $x_4(t)$.

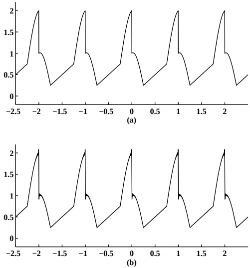


Fig. 3.5: (a) $x(t)$. (b) $x_{115}(t)$.

As seen, there is no significant difference between $x(t)$ and $x_{115}(t)$. So, it *seems* that the optimized $x_N(t)$ goes to some signal that is very close to $x(t)$.

$\boxed{\textit{Fourier series}}$: Let $x(t)$: $x(t+T_0) = x(t)$ and $c[k]$ computed using (9), the following

$$x_\infty(t) \triangleq \sum_{k=-\infty}^{+\infty} c[k] e^{jk\frac{2\pi}{T_0}t} \tag{12}$$

is called the *Fourier series* (FS) of $x(t)$, expanded in $\{e^{jk\frac{2\pi}{T_0}t}\}$, while $c[k]$, $\forall\, k$ are named as the corresponding FS *coefficients*.

One fundamental question to be asked is what the relationship between $x(t)$ and its FS $x_\infty(t)$ is. In what follows, two somewhat different results are presented without proof.

Theorem

Let $x_\infty(t) = \sum_{k=-\infty}^{+\infty} c[k] e^{jk\frac{2\pi}{T_0}t}$ *be the FS of a periodic signal* $x(t)$*. If*

$$\int_{T_0} |x(t)|^2 dt < +\infty \tag{13}$$

then $c[k]$ *are all* finite *and*

$$\int_{T_0} |x(t) - x_\infty(t)|^2 dt = 0 \tag{14}$$

The result (14) implies that the energy σ of $e(t) = x(t) - x_\infty(t)$ over one period is nil. It then follows from (10) that

$$\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c[k]|^2 \tag{15}$$

which is referred to as Parseval's relation.

The second result is due to P.L. Dirichlet.

Theorem

Let $x_\infty(t) = \sum_{k=-\infty}^{+\infty} c[k] e^{jk\frac{2\pi}{T_0}t}$ *be the FS of a periodic signal* $x(t)$. *If the signal* $x(t)$

- *is absolutely integrable over one period:* $\int_{-T_0/2}^{T_0/2} |x(t)|dt < \infty$
- *has a* finite *number of maxima and minima during a single period, and*
- *has only a finite number of discontinuities in a single period and furthermore, each of these discontinuities is finite,*

then all the $c[k]$ *are* finite *and*

$$x_\infty(t) = \frac{x(t_-) + x(t_+)}{2}, \ \forall\, t$$

where t_-, t_+ *are the left and right limits of the point* t, *respectively.*

It should be noted that

- The conditions in the two theorems are just SCs, not NC.
- In general, the FS $x_\infty(t)$ is not equal to the signal $x(t)$.
- If $x(t)$ is continuous everywhere, then $x(t) = x_\infty(t)$, $\forall\, t$ and hence

$$\left\{ \begin{array}{rcl} x(t) & = & \sum_{k=-\infty}^{+\infty} c[k] e^{jk\omega_0 t} \\ c[k] & = & \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \end{array} \right. \tag{16}$$

which, as an one-to-one mapping, is usually denoted as

$$x(t) \leftrightarrow c[k]$$

For all practical purposes, it is assumed in the sequel that (16) holds.

Example 3.2 : Consider the square wave $x(t)$ of period T_0 and $x(t) = \frac{1}{\tau} w_\tau(t)$ for $0 < \tau < T_0/2$. Work out the FS coefficients of $x(t)$.

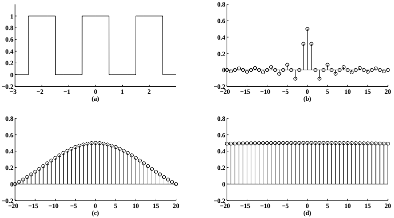


Fig. 3.6: The square wave $x(t)$ and its FS coefficients. (a) $x(t)$ with $T_0 = 2, \tau = 1$; (b) $c[k]$ with $\tau = 1$; (c) $c[k]$ with $\tau = 0.1$; (d) $c[k]$ with $\tau = 0.01$.

Solution: With $\omega_0 = 2\pi/T_0$, we have

$$c[k] = \frac{1}{\tau T_0} \int_{-\tau/2}^{\tau/2} e^{-j\omega_0 kt} dt = \begin{cases} \frac{1}{T_0}, & k = 0 \\ \frac{1}{T_0} \frac{sin(k\omega_0 \tau/2)}{k\omega_0 \tau/2}, & k \neq 0 \end{cases}$$

As observed, $\lim_{\tau \to 0} c[k] = \frac{1}{T_0}$.

Note that $x(t)$ can be represented as

$$x(t) = \sum_{m=-\infty}^{+\infty} \frac{1}{\tau} w_\tau(t - mT_0) \;\Rightarrow\; p(t) \triangleq \lim_{\tau \to 0} x(t) = \sum_{m=-\infty}^{+\infty} \delta(t - mT_0)$$

The FS coefficients $c_p[k]$ of $p(t)$ [5] are expected to be equal to $\lim_{\tau \to 0} c[k] = \frac{1}{T_0}$. In fact, this is the case:

$$c_p[k] = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p(t) e^{-j\omega_0 kt} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 kt} dt = \frac{1}{T_0}$$

Therefore, we have the following equivalence to be used in Chapter 5

$$\sum_{m=-\infty}^{+\infty} \delta(t - mT_0) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} e^{j\omega_0 kt} \tag{17}$$

[5] This signal does not satisfy any set of conditions in Theorems 1 and 2.

Example 3.3 : Determine the signal $x(t)$ based on the following facts:

- $x(t)$ is periodic with period $T_0 = 4$ and has FS coefficients $c[k]$.
- $x(t)$ is real-valued.
- $c[k] = 0$ for $|k| > 1$.
- The signal $y(t)$, whose FS coefficients are $d[k] = e^{-j\pi k/2}c[k]$, is odd.[6]
- $\frac{1}{4}\int_{-2}^{2} |x(t)|^2 dt = 1/2$.

Solution: The first condition simply tells us that

$$x(t) = \sum_k c[k]e^{j\omega_0 kt}$$

$\omega_0 = 2\pi/T_0 = \pi/2$. With $c[k] = r_k e^{j\phi_k}$, the 2nd condition implies

$$c[-k] = c^*[k] = r_k e^{-j\phi_k}$$

[6] A function $f(t)$ is said to be *odd* if $f(-t) = -f(t)$, $\forall\, t$ and *even* if $f(-t) = f(t)$, $\forall\, t$.

It follows from the 3rd condition that $x(t)$ can be further specified as

$$x(t) = c[-1]e^{-j\omega_0 t} + c[0] + c[1]e^{j\omega_0 t} = c[0] + 2r_1 cos(\omega_0 t + \phi_1)$$

The 4th condition suggests

$$y(t) = d[-1]e^{-j\tilde{\omega}_0 t} + d[0] + d[1]e^{j\tilde{\omega}_0 t} = c[0] + 2r_1 sin(\tilde{\omega}_0 t + \phi_1)$$

where $\tilde{\omega}_0$ may be different from ω_0. As given, $y(-t) = -y(t)$ leading to

$$c[0] = -2r_1 sin\phi_1 cos(\tilde{\omega}_0 t)$$

which suggests $r_1 sin\phi_1 = 0$ and hence $c[0] = 0$ and $r_1 = 0$ or $\phi_1 = m\pi$.

The last condition, according to the Parseval's relation (15), is equivalent to say $1/2 = \sum_k |c[k]|^2 = |c[-1]|^2 + |c[1]|^2 = 2r_1^2$, giving $r_1 = 1/2$. Therefore,

$$x(t) = 2r_1 cos(\omega_0 t + \phi_1) = cos(\omega_0 t + m\pi) = (-1)^m cos(\omega_0 t)$$

Let $x[n]$ be: $x[n] = x[n+N_0]$ for $N_0 > 0$. Then

Basis signals: $\{e^{jk\Omega_0 n},\ k = 0, 1, \cdots, N_0 - 1\}$ with $\Omega_0 \triangleq \frac{2\pi}{N_0}$

$\boxed{\textit{Discrete-time Fourier series}}$:

$$x[n] = \sum_{k=n_0}^{N_0-1+n_0} X_p[k] e^{jk\Omega_0 n} \tag{18}$$

where n_0 is arbitrary and

$$X_p[k] = \frac{1}{N_0} \sum_{n=n_0}^{N_0-1+n_0} x[n] e^{-jk\Omega_0 n},\ \forall\ k \tag{19}$$

Please study this section yourself !

$\boxed{\textbf{Example 3.5}}$: Let $x[n] = cos(\pi n/8 - \pi/6) + 2cos(\pi n/4 + \pi/4)$. See Fig. 3.7. Find out the DTFS coefficients of $x[n]$.

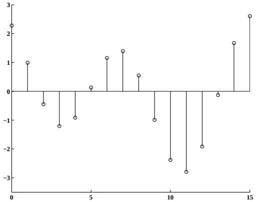


Fig. 3.7: Waveform of the periodic $x[n]$ for one period in Example 3.5.

Solution: First of all, $N_1 = 16, \;\; N_2 = 8 \;\; \Rightarrow \;\; N_0 = 16$.

The DTFS coefficients $X_p[k]$ can be obtained by either (19) or the following

$$\begin{aligned} x[n] &= cos(\pi n/8 - \pi/6) + 2cos(\pi n/4 + \pi/4) \\ &= \frac{1}{2}[e^{j(\pi n/8 - \pi/6)} + e^{-j(\pi n/8 - \pi/6)}] + [e^{j(\pi n/4 + \pi/4)} + e^{-j(\pi n/4 + \pi/4)}] \end{aligned}$$

As $\Omega_0 = \frac{2\pi}{N_0} = \frac{\pi}{8}$, we have

$$x[n] = \frac{e^{-j\pi/6}}{2} e^{j\Omega_0 n} + \frac{e^{j\pi/6}}{2} e^{j(N_0-1)\Omega_0 n} + e^{j\pi/4} e^{j2\Omega_0 n} + e^{-j\pi/4} e^{j(N_0-2)\Omega_0 n}$$

Comparing the above with $x[n] = \sum_{k=0}^{N_0-1} X_p[k] e^{jk\Omega_0 n}$ leads to $X_p[1] = \frac{e^{-j\pi/6}}{2}$, $X_p[15] = \frac{e^{j\pi/6}}{2}$, $X_p[2] = e^{j\pi/4}$, $X_p[14] = e^{-j\pi/4}$, and $X_p[k] = 0,\ k = 0, 3, 4, \cdots, 13.$

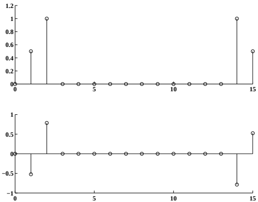


Fig. 3.8: $X_p[m]$ of the periodic $x[n]$ in Example 3.5. Top: $|X_p[m]|$; Bottom: $\phi[m]$.

$\boxed{\textbf{Example 3.6}}$: Let $X_p[m] = cos(2m\frac{2\pi}{11}) + 2jsin(3m\frac{2\pi}{11})$ be the DTFS coefficients of a signal $x[n]$. What is the corresponding signal $x[n]$?

Solution: Note $X_p[m+11] = X_p[m]$, we take $N_0 = 11$ and hence $\Omega_0 = \frac{2\pi}{11}$. With the help of Euler's formula

$$\begin{aligned} X_p[m] &= cos(2m\Omega_0) + 2jsin(3m\Omega_0) \\ &= \frac{1}{2}[e^{-jm\Omega_0 2} + e^{-jm\Omega_0(11-2)}] + [-e^{-jm\Omega_0 3} + e^{-jm\Omega_0(11-3)}] \end{aligned}$$

Comparing it with (19), we have

$$x[2] = x[9] = 11/2,\ x[8] = 11,\ x[3] = -11$$

and $x[n] = 0$ for other n.

Note: The solution is dependent of the choice for N_0. For this example, we can take $N_0 = 22$ and a different solution can be obtained in the same way.

What does the DTFS (18) tell us?

It says that a signal $x[n]$ (in time domain) can be completely recovered from its DTFS coefficients $X_p[m]$ as long as Ω_0 is given. So, $X_p[m]$ is an alternative representation of the signal $x[n]$.

A *signal transformation* refers to the procedure for representing signals in an alternative way. For example, (19) defines a signal transformation that changes the signal representation from $x[n]$ to $X_p[k]$. The procedure for recovering $x[n]$ from the alternative representation is called the *inverse transformation*.

What kind of advantages can we gain from a transformation? Let us consider two application examples, which can give some insights on the question.

$\boxed{\textit{Case I Data compression}}$: Look at Example 3.5. Consider the following two ways to store the 16 samples digitally.

- *Direct storage*: each sample is represented with 16 bits, leading to a memory size of

$$16 \times 16 = 256 \ \ bits$$

- *Indirect storage*: This periodic signal can be represented using its DTFS $X_p[m]$. As obtained before, there are 4 non-zero coefficients $X_p[15] = X_p^*[1], \ \ X_p[14] = X_p^*[2]$, i.e., 4 real numbers requiring

$$16 \times 4 = 64 \, bits$$

A music is generated as a sequence of symbols such as **Do Re Mi Fa Sol La Xi**. As observed from Fig. 3.1, a symbol, when represented as a signal, is periodic with a fixed fundamental frequency. See Fig. 3.9.

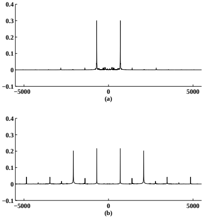


Fig. 3.9: Plot of $|c[m]|$ for the two signals $k(t)$ and $r(t)$ in Fig. 3.1.

As seen, there are only a few significant terms of non-zero FS coefficients that need to be coded for this symbol. Therefore, we can code the signals for all the notes efficiently using the transform domain coding that is explained above.

$\boxed{\textit{Case II Frequency estimation}}$: to estimate the sinusoidal components underlying a signal - a very important area.

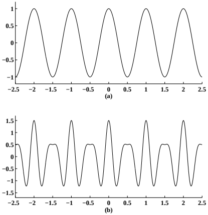


Fig. 3.10: (a) $x_1(t)$; (b) $x_2(t)$.

As seen, both are periodical with $T_0 = 1$. Applying the FS would yield

For $x_1(t)$: $c(1) = c(-1) = 1/2$ and $c(k) = 0$, $\forall k \neq \pm 1$, leading to

$$x_1(t) = \frac{1}{2}e^{-j2\pi t} + \frac{1}{2}e^{j2\pi t} = cos(2\pi t)$$

For $x_2(t)$: $c(\pm 2) = 1/2$, $c(\pm 3) = 1/4$, and $c(k) = 0$, $\forall k \neq \pm 2, \pm 3$, implying that there are two (real-valued) sinusoidal components in $x_2(t)$ and that the corresponding frequencies are $2\ Hz$ and $3\ Hz$, respectively:

$$x_2(t) = cos(2\pi \times 2t) + \frac{1}{2} cos(2\pi \times 3t)$$

So, with the FS transform one can identify the frequency components underlying a *periodic signal* as long as the the period is known.

But what to do if $x(t)$ is aperiodic, say $x(t) = cos(2\pi \times \sqrt{2}t) + \frac{1}{2} cos(2\pi \times 2t)$ as the FS transform is not applicable to such signals? More general transform is needed.

Let $x(t)$ be an *arbitrary* signal, as depicted in Fig. 3.11.

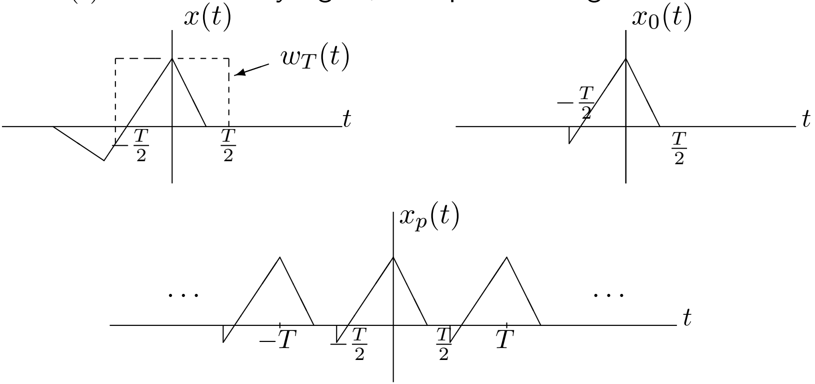


Fig. 3.11: Waveforms of $x(t)$, $x_0(t)$ and $x_p(t)$.

Denote $x_0(t) \triangleq x(t)w_T(t)$. We then construct a periodic signal $x_p(t)$:

$$x_p(t) \triangleq \sum_{k=-\infty}^{+\infty} x_0(t - kT)$$

It follows from the FS pair that

$$x_p(t) = \sum_{m=-\infty}^{+\infty} c[m] e^{jm\frac{2\pi}{T}t}, \ \forall\, t$$

where

$$c[m] = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jm\frac{2\pi}{T}t} dt, \ \forall\, m$$

Denote

$$X_T(j\omega) \triangleq \int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt$$

where ω is a continuous variable $-\infty < \omega < +\infty$. Then

$$c[m] = \frac{X_T(j\omega_m)}{T}, \ \ \omega_m \triangleq \frac{2\pi}{T} m, \ \ \forall\, m \in \mathcal{Z}$$

Define

$$X(j\omega) \triangleq \lim_{T\to+\infty} X_T(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \tag{20}$$

We note $x(t) = x_p(t),\ -T/2 < t < T/2$ and hence

$$x(t) = \lim_{T\to\ +\infty} x_p(t) = \lim_{T\to\ +\infty} \sum_{m=-\infty}^{+\infty} c[m]e^{jm\frac{2\pi}{T}t}$$

Noting $c[m] = \frac{X_T(j\omega_m)}{T}$, we obtain

$$x(t) = \frac{1}{2\pi} \lim_{T\to\ +\infty} \sum_{m=-\infty}^{+\infty} X_T(j\omega_m)e^{jm\frac{2\pi}{T}t}\frac{2\pi}{T}$$

As $\omega_m = \frac{2\pi m}{T}$, $\Delta\omega_m \triangleq \omega_m - \omega_{m-1} = \frac{2\pi}{T}$. So,
$x(t) = \frac{1}{2\pi}\lim_{T\to+\infty}\sum_{m=-\infty}^{+\infty} X_T(j\omega_m)e^{j\omega_m t}\Delta\omega_m$, that is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \tag{21}$$

The two form an FT pair, denoted as

$$x(t) \;\leftrightarrow\; X(j\omega)$$

Convergence issue: See Theorems 1 and 2:

- $$\int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty \Rightarrow \; |X(j\omega)| < +\infty \tag{22}$$

  and that the signal defined by (21) is equivalent to $x(t)$ in the sense that the energy of the difference between the two is nil.

- If

  $$\int_{-\infty}^{+\infty} |x(t)| dt < +\infty \tag{23}$$

  hold, then $X(j\omega)$ by (20) is finite and that the integration of (21) is equal to the mean of the left and right limits of $x(t)$ at any point t.

Example 3.7 : Compute the FT of the signal $x(t) = e^{-\gamma t}u(t)$, where γ is a constant with $\mathcal{R}_e(\gamma) > 0$.

Solution: By definition,

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} e^{-\gamma t}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{+\infty} e^{-(\gamma+j\omega)t}dt = \frac{1}{j\omega+\gamma} \end{aligned}$$

due to $\mathcal{R}_e(\gamma) > 0$. Therefore,

$$e^{-\gamma t}u(t),\ \mathcal{R}_e(\gamma) > 0 \quad \leftrightarrow \quad \frac{1}{j\omega+\gamma} \tag{24}$$

This is a very important FT pair.

The FT $X(j\omega)$ of $x(t)$ is usually referred to as *spectrum* of the signal:

$$X(j\omega) = |X(j\omega)|e^{j\phi_x(\omega)}$$

For the example above, we have

$$|X(j\omega)| = \frac{1}{\sqrt{[\mathcal{R}_e(\gamma)]^2 + [\omega + \mathcal{I}_m(\gamma)]^2}}, \quad \phi_x(\omega) = -atan\frac{\omega + \mathcal{I}_m(\gamma)}{\mathcal{R}_e(\gamma)}$$

which are sketched in Fig. 3.12 for $\gamma = 1$.

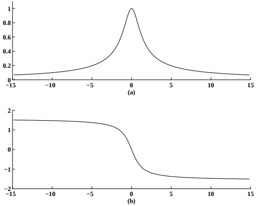


Fig. 3.12: Signal spectrum for Example 3.7 with $\gamma = 1$. (a) $|X(j\omega)|$; (b) $\phi_x(\omega)$.

Using the same procedure, we can obtain the following FT pair

$$-e^{-\beta t}u(-t),\ \mathcal{R}_e(\beta) < 0 \quad \leftrightarrow \quad \frac{1}{j\omega + \beta} \tag{25}$$

which is another important FT pair.

$\boxed{\textbf{Example 3.8}}$: Compute the FT of the window function $x(t) = \kappa w_\tau(t)$, where κ is constant.

Solution: Based on the definition, the FT of the signal is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} \kappa w_\tau(t)e^{-j\omega t}dt \\ &= \kappa \int_{-\tau/2}^{\tau/2} e^{-j\omega t}dt \end{aligned}$$

which leads to

$$\kappa w_\tau(t) \;\leftrightarrow\; \kappa\tau\frac{sin(\omega\tau/2)}{\omega\tau/2} = \kappa\tau s_\tau(\omega) \tag{26}$$

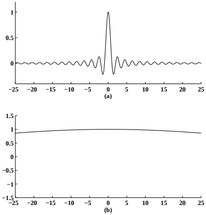


Fig. 3.13: Signal spectrum for Example 3.8 with $\kappa = \tau^{-1}$. (a) $X(j\omega)$ with $\tau = 1$; (b) $X(j\omega)$ with $\tau = 0.01$.

As seen, when τ gets bigger (i.e., the window gets wider), the main lobe of signal spectrum becomes narrower and vice versa.

With $\kappa = \tau^{-1}$ in (26), we have the FT pair $\lim_{\tau \to 0} \kappa w_\tau(t) \;\leftrightarrow\; \lim_{\tau \to 0} \frac{sin(\omega\tau/2)}{\omega\tau/2}$, which leads to the following important FT pair

$$\delta(t) \;\leftrightarrow\; 1 \tag{27}$$

What is the signal $x(t)$ whose FT is $X(j\omega) = 2\pi\delta(\omega)$? Using the IFT equation (21), one has

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = e^{-j0t} = 1$$

which implies

$$1 \;\leftrightarrow\; 2\pi\delta(\omega) \tag{28}$$

Properties of FT

As the proofs of a common property shared by FS and FT as well as the DTFS & DTFT to be studied later are very similar, we just focus on discussing these properties for the FT.

$\boxed{\textit{P.1 Linearity}}$: If $x_k(t) \leftrightarrow X_k(j\omega)$, $\forall k \in \mathcal{Z}$, then for any constants $\{\alpha_k\}$

$$x(t) \triangleq \sum_k \alpha_k x_k(t) \ \leftrightarrow \ X(j\omega) = \sum_k \alpha_k X_k(j\omega)$$

For example, according to (24) and (25) that for any constant α with $\mathcal{R}_e(\alpha) > 0$,

$$e^{-\alpha|t|} = e^{\alpha t}u(-t) + e^{-\alpha t}u(t) \ \leftrightarrow \ -\frac{1}{j\omega - \alpha} + \frac{1}{j\omega + \alpha} = \frac{2\alpha}{\omega^2 + \alpha^2}$$

Let us consider the FT of $u(t)$. First of all, define $\tilde{u}(t) \triangleq \frac{1}{2} + \frac{1}{2}[x_1(t) - x_2(t)]$, where

$$x_1(t) = e^{-\alpha t}u(t), \;\; x_2(t) = e^{\alpha t}u(-t), \; \alpha > 0$$

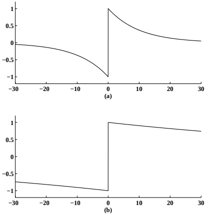


Fig. 3.14: Wave-form of $\tilde{u}(t)$. (a) $\alpha = 0.1$; (b) $\alpha = 0.01$.

It follows from the FT pairs (24) and (25) that

$$x_1(t) \; \leftrightarrow \; X_1(j\omega) = \frac{1}{\alpha + j\omega}, \;\; x_2(t) \; \leftrightarrow \; X_2(j\omega) = \frac{1}{\alpha - j\omega}$$

and hence

$$\begin{aligned}\tilde{u}(t) \leftrightarrow \tilde{U}(j\omega) &= \pi\delta(\omega) + \frac{1}{2}[\frac{1}{\alpha + j\omega} - \frac{1}{\alpha - j\omega}] \\ &= \pi\delta(\omega) + \frac{1}{2} \times \begin{cases} 0 & , \quad \omega = 0 \\ \frac{1}{\alpha+j\omega} - \frac{1}{\alpha-j\omega} & , \quad \omega \neq 0 \end{cases}\end{aligned}$$

Noting $\lim_{\alpha\to 0} \tilde{u}(t) = u(t)$, one concludes

$$u(t) \leftrightarrow U(j\omega) = \lim_{\alpha\to 0} \tilde{U}(j\omega) = \begin{cases} \pi\delta(\omega) & , \quad \omega = 0 \\ \frac{1}{j\omega} & , \quad \omega \neq 0 \end{cases}$$

which can be simply denoted as

$$u(t) \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega) \tag{29}$$

$\boxed{\textit{P.2 Time shift}}$: If $x(t) \leftrightarrow X(j\omega)$, then for any constant τ

$$y(t) \triangleq x(t-\tau) \;\leftrightarrow\; Y(j\omega) = X(j\omega)e^{-j\omega\tau}$$

As given by (27), $\delta(t) \leftrightarrow 1$. The *time shift* implies

$$\delta(t-\tau) \leftrightarrow e^{-j\omega\tau}$$

which can be verified with a direct computation with (20).

What is the FT of $x(t) = e^{-\alpha t}u(t-t_0),\ \alpha > 0$?

$\boxed{\textit{P.3 Frequency shift}}$: If $x(t) \leftrightarrow X(j\omega)$, then for any constant ω_0

$$y(t) \triangleq x(t)e^{j\omega_0 t} \;\leftrightarrow\; Y(j\omega) = X(j(\omega-\omega_0))$$

As known from (28), $1 \leftrightarrow 2\pi\delta(\omega)$, it turns out of the *frequency shift* that

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0) \tag{30}$$

Basically, most of the FT properties can be proved easily just based on the definition specified by (20) and this is particular true for the three properties listed above. We leave the proofs of the three properties to the readers.

Example 3.9 : Consider the signal $x(t) = cos(2\pi \times \sqrt{2}t) + \frac{1}{2}cos(4\pi t)$. What is the FT of this signal?

Solution: As $cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{j(-\omega_0)t}]$ and the pair $e^{j\omega_c t} \;\leftrightarrow\; 2\pi\delta(\omega - \omega_c)$ for any constant ω_c, the linearity suggests that

$$cos(\omega_0 t) \;\leftrightarrow\; \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

and hence the frequency components can be detected form $X(j\omega)$

$$\begin{aligned} x(t) \leftrightarrow X(j\omega) &= \pi[\delta(\omega - 2\sqrt{2}\pi) + \delta(\omega + 2\sqrt{2}\pi)] \\ &+ \frac{\pi}{2}[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)] \end{aligned}$$

$\boxed{\textit{P.4 Time scaling}}$: If $x(t) \leftrightarrow X(j\omega)$, then

$$x(\alpha t) \leftrightarrow \frac{1}{|\alpha|}X(j\frac{\omega}{\alpha}), \quad \alpha \neq 0$$

Particularly, the time reversal signal $x(-t)$ has an FT $X(-j\omega)$.

Proof: It follows from

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

that

$$x(\alpha t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega\alpha t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{|\alpha|}X(j\frac{\xi}{\alpha})e^{j\xi t}d\xi$$

Comparing with (21), we realize

$$x(\alpha t) \leftrightarrow \frac{1}{|\alpha|}X(j\frac{\omega}{\alpha})$$

This completes the proof.

As to be seen, we have the following FT pair

$$x(t) = \frac{2sin(0.5t)}{\pi t} cos(1.5t) \ \leftrightarrow \ \ X(j\omega) = w_1(\omega + 1.5) + w_1(\omega - 1.5)$$

Fig. 3.15 shows the waveform of $x(\alpha t)$ and its FT for different values of α.

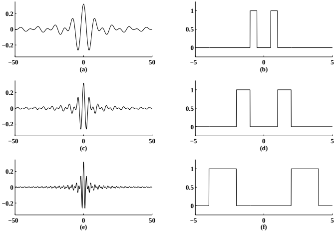


Fig. 3.15: Plots of $x(\alpha t)$ and $\alpha X(j\omega/\alpha)$. (a)-(b) $\alpha = 0.5$; (c)-(d) $\alpha = 1$; (e)-(f) $\alpha = 2$.

This explains the demonstration of the effects of α on $x(\alpha t)$ in the 3rd session.

$\boxed{\textit{P.5 Duality}}$: If $x(t) \leftrightarrow X(j\omega)$, then

$$X(-jt) \leftrightarrow 2\pi x(\omega) \Leftrightarrow \frac{1}{2\pi} X(jt) \leftrightarrow x(-\omega)$$

Before giving the proof, let us specify what this property signifies with the window signal. As indicated by (26), we have

$$x(t) \triangleq w_{2\omega_0}(t) \leftrightarrow X(j\omega) = 2\omega_0 \frac{sin(\omega\omega_0)}{\omega\omega_0}$$

The *duality* tells us that the following is true

$$\frac{sin(\omega_0 t)}{\pi t} \leftrightarrow w_{2\omega_0}(\omega) \tag{31}$$

Fig. 3.16 shows graphically the two pairs of FT.

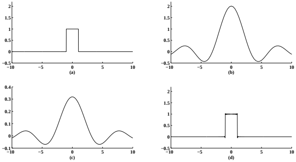


Fig. 3.16: Demonstration of the duality property. (a)-(b) $w_{2\omega_0}(t)$ and its FT; (c)-(d) $\frac{sin(\omega_0 t)}{\pi t}$ and its FT.

Proof: First of all,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\xi)e^{j\xi t}d\xi$$

By multiplying both sides with 2π and changing $t \rightarrow \omega$, we have

$$\begin{aligned} 2\pi x(\omega) &= \int_{-\infty}^{+\infty} X(j\xi)e^{j\xi\omega}d\xi = \int_{-\infty}^{+\infty} X(-jt)e^{-j\omega t}dt \\ \Rightarrow & \quad X(-jt) \leftrightarrow 2\pi x(\omega) \end{aligned}$$

$\boxed{\textit{P.6 Conjugate symmetry}}$: If $x(t) \leftrightarrow X(j\omega) = |X(j\omega)|e^{j\phi_x(\omega)}$, then

$$x^*(t) \leftrightarrow X^*(-j\omega)$$

Furthermore, if $x(t)$ is real, then $X(-j\omega) = X^*(j\omega)$. This signifies that

- $|X(j\omega)|$ is an even function of ω: $|X(j(-\omega))| = |X(j\omega)|$
- $\phi_x(\omega)$ is an odd function of ω: $\phi_x(-\omega) = -\phi_x(\omega)$.

Proof: From the definition of FT, the FT of $x^*(t)$ is

$$\int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}dt = (\int_{-\infty}^{+\infty} x(t)e^{j\omega t}dt)^* = X^*(-j\omega)$$

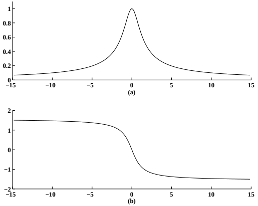
which is actually the first part of the property.

If $x(t)$ is real-valued, then $x^*(t) = x(t)$ and hence its FT is $X(j\omega)$, namely, $X^*(-j\omega) = X(j\omega)$ or $X(-j\omega) = X^*(j\omega)$. With $X(j\omega) = |X(j\omega)|e^{j\phi_x(\omega)}$, the latter yields

$$|X(-j\omega)|e^{j\phi_x(-\omega)} = |X(j\omega)|e^{-j\phi_x(\omega)}$$

Noting $-\pi < \phi_x(\omega) \leq \pi$, we have $|X(j(-\omega))| = |X(j\omega)|$ and $\phi_x(-\omega) = -\phi_x(\omega)$, which ends the proof.

As seen before, $e^{-t}u(t) \quad \leftrightarrow \quad \frac{1}{j\omega+1}$.



$\boxed{\textit{P.7 Differentiation in time}}$: If $x(t) \leftrightarrow X(j\omega)$, then

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$$

Proof: Differentiating both sides of (21) *w.r.t* time variable t will do.

$\boxed{\textit{P.8 Differentiation in frequency}}$: If $x(t) \leftrightarrow X(j\omega)$, then

$$tx(t) \leftrightarrow j\frac{dX(j\omega)}{d\omega}$$

Proof: Differentiating both sides of (20) *w.r.t.* ω yields

$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} x(t)(-jt)e^{-j\omega t}dt,$$

namely,

$$j\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} tx(t)e^{-j\omega t}dt$$

which tells us that the FT of $tx(t)$ is $j\frac{dX(j\omega)}{d\omega}$.

$\boxed{\textit{P.9 Convolution in time}}$: If $v(t) \leftrightarrow V(j\omega)$ and $w(t) \leftrightarrow W(j\omega)$, then

$$v(t) * w(t) \leftrightarrow V(j\omega)W(j\omega)$$

The proof will be given in the next chapter.

Property	Given FT pair	:	$x(t) \leftrightarrow X(j\omega)$	Remarks
Linearity	$\sum_k \alpha_k x_k(t)$	$\leftrightarrow$	$\sum_k \alpha_k X_k(j\omega)$	
Time shift	$x(t-\tau)$	$\leftrightarrow$	$X(j\omega)e^{-j\omega\tau}$	
Time scaling	$x(\alpha t)$	$\leftrightarrow$	$\frac{1}{\lvert\alpha\rvert}X(\frac{j\omega}{\alpha})$	$\alpha \neq 0$ real
Frequency shift	$x(t)e^{j\omega_0 t}$	$\leftrightarrow$	$X(j(\omega-\omega_0))$	ω_0 real
Derivative in time $\frac{d}{dt}$	$\frac{dx(t)}{dt}$	$\leftrightarrow$	$j\omega X(j\omega)$	
Conjugate symmetry	$x^*(t)$	$\leftrightarrow$	$X^*(-j\omega)$	
Derivative in frequency	$tx(t)$	$\leftrightarrow$	$j\frac{dX(j\omega)}{d\omega}$	
Convolution in time	$x(t) * y(t)$	$\leftrightarrow$	$X(j\omega)Y(j\omega)$	
Duality	$X(jt)$	$\leftrightarrow$	$2\pi x(-\omega)$	
Multiplication in time	$x(t)y(t)$	$\leftrightarrow$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)\ w.r.t.\ \omega$	
Integration in time	$\int_{-\infty}^{t} x(\tau)d\tau$	$\leftrightarrow$	$\frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$	
Parseval Theorem	$\int_{-\infty}^{+\infty} x(t)y(t)dt$	$=$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\xi)Y(-j\xi)d\xi$	

Table: Properties of the Fourier transform

The last three can be derived with the properties listed above them.

$\boxed{\textit{Multiplication in time}}$: As the properties of duality and convolution lead to

$$X(jt) * Y(jt) \; \leftrightarrow \; 4\pi^2 x(-\omega)y(-\omega)$$

applying duality again yields $4\pi^2 x(t)y(t) \; \leftrightarrow \; 2\pi X(j\omega) * Y(j\omega)$, which is equivalent to

$$x(t)y(t) \; \leftrightarrow \; \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

$\boxed{\textit{Integration in time}}$: It can be proved with the *convolution in time* by noting $\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$ and (29).

$\boxed{\textit{Parseval Theorem}}$: It follows directly from *multiplication in time* which implies

$$\frac{1}{2\pi}X(j\omega) * Y(j\omega) = \int_{-\infty}^{+\infty} x(t)y(t)e^{-j\omega t}dt$$

Particularly, let $y(t) = x^*(t)$ and hence $Y(j\omega) = X^*(-j\omega)$ (due to the conjugate symmetry property), we have

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\xi)|^2 d\xi$$

Usually, the function $P(\omega)$ defined below

$$P(\omega) \triangleq \frac{1}{2\pi}|X(j\omega)|^2$$

is referred to as the *energy spectrum density function* of $x(t)$.

Example 3.10 : In this example, two signal are considered.

- What is the FT of $x_1(t) = \frac{d}{dt}\{te^{-\alpha t}u(t) * w_\tau(t)\}$ with $\mathcal{R}_e(\alpha) > 0$?

  First of all, denote $g_1(t) \triangleq t\, g_2(t)$ with

  $g_2(t) = e^{-\alpha t}u(t) \ \leftrightarrow \ G_2(j\omega) = \frac{1}{j\omega+\alpha}$. With the property of derivative in frequency, we have

$$g_1(t) = te^{-\alpha t}u(t) \ \leftrightarrow \ G_1(j\omega) = j\frac{dG_2(j\omega)}{d\omega} = \frac{1}{(j\omega+\alpha)^2}$$

  It then follows from the properties of derivative and convolution both in time domain that

$$X_1(j\omega) = j\omega G_1(j\omega)W_\tau(j\omega) = \frac{j\omega}{(j\omega+\alpha)^2}\frac{\tau sin(\omega\tau/2)}{\omega\tau/2}$$

- Let us consider the FT of $x_2(t) = tw_2(t)$.

  Denote $g(t) \triangleq \frac{dx_2(t)}{dt}$, that is

$$g(t) = w_2(t) + t[\delta(t+1) - \delta(t-1)] = w_2(t) - \delta(t+1) - \delta(t-1)$$

Note $w_2(t) \leftrightarrow \frac{2sin\omega}{\omega}$ and $\delta(t-t_0) \leftrightarrow e^{-j\omega t_0}$. So,

$$g(t) \leftrightarrow G(j\omega) = \frac{2sin\omega}{\omega} - e^{j\omega} - e^{-j\omega} = \frac{2sin\omega}{\omega} - 2cos\omega$$

Since $x_2(-\infty) = 0$,

$$x_2(t) - x_2(-\infty) = \int_{-\infty}^{t} g(\tau)d\tau \Rightarrow x_2(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

With the *integration in time* property and $G(j0) = 0$, we finally obtain

$$X_2(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(j0)\delta(\omega) = \frac{G(j\omega)}{j\omega}$$

This problem can also be attacked using the property of *derivative in frequency* as there is a factor t in the signal.

Inverse Fourier transform

Given an FT $X(j\omega)$, find the corresponding $x(t)$

- Direct way:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

  This integration can be evaluated with the *residue theorem*. See **Appendix D**.

- Indirect way: using the *partial-fraction expansion* techniques. See **Appendix E**. The procedure of this approach is demonstrated with the following example.

Example 3.11 : Let $X(j\omega) = \frac{4+j3\omega}{(2+j\omega)(j\omega)} + \delta(\omega - \omega_0) + \frac{e^{-j\omega}}{j\omega - 3}$. Find out $x(t)$.

Solution: First of all, using partial fraction expansions (see **Appendix E**) we have

$$\frac{4+j3\omega}{(2+j\omega)(j\omega)} = \frac{A}{j\omega+2} + \frac{B}{j\omega}, \quad A = 1, \quad B = 2$$

and hence

$$X(j\omega) \;=\; \frac{A}{2+j\omega} + B[\frac{1}{j\omega} + \pi\delta(\omega)] - B\pi\delta(\omega) + \delta(\omega-\omega_0) + \frac{e^{-j\omega}}{j\omega - 3}$$

The FT pairs specified by (24), (29), (30) and the property of time shift suggest that

$$\begin{aligned} x(t) &= Ae^{-2t}u(t) + Bu(t) + \frac{1}{2\pi}[-B\pi e^{j0t} + e^{j\omega_0 t}] - e^{3(t-1)}u(-t+1) \\ &= [e^{-2t}+2]u(t) - 1 + \frac{e^{j\omega_0 t}}{2\pi} - e^{3(t-1)}u(-t+1) \end{aligned}$$

By considering $x[n]$ as the FS coefficients of a periodic signal repeating itself every 2π, we can show that

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{jn\Omega}d\Omega \leftrightarrow X(e^{j\Omega}) \triangleq \sum_{n=-\infty}^{+\infty} x[n]e^{-jn\Omega} \qquad (32)$$

called the *discrete-time Fourier transform* (DTFT) pair. [7]

Remarks:

- By nature, the pair of DTFT is equivalent to that of FS;
- Inherently, $X(e^{j\Omega})$ is periodic in Ω, repeating itself every 2π !

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[7] The section is briefed as it is for self-study. Another reason for that is DTFT is usually studied in the more advanced courses such as *Digital Signal Processing*.

Example 3.12 : Determine the DTFT of $x[n] = \gamma^n u[n]$ with $|\gamma| < 1$.

Solution: By definition, the DTFT of $x[n]$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jn\Omega} = \sum_{n=0}^{+\infty} \gamma^n e^{-jn\Omega} = \frac{1}{1-\gamma e^{-j\Omega}}$$

due to $|\gamma e^{-j\Omega}| = |\gamma| < 1$. So, we have the following DTFT pair:

$$\gamma^n u[n] \; \leftrightarrow \; \frac{1}{1-\gamma e^{-j\Omega}}, \; |\gamma| < 1 \tag{33}$$

which is one of the basic but important DFFT pairs.

Like the FT, the DTFT is a complex function in Ω and hence

$$X(e^{j\Omega}) = |X(e^{j\Omega})|e^{j\phi_x(\Omega)}$$

where as in the FT case, $|X(e^{j\Omega})|$ and $\phi_x(\Omega)$ are called the *magnitude spectrum* and *phase spectrum* of $x[n]$, respectively.

For the DTFT by (33), assume $\gamma = \rho e^{j\psi}$ and then

$$|X(e^{j\Omega})| = \frac{1}{\sqrt{1 - 2\rho cos(\Omega - \psi) + \rho^2}}$$
$$\phi_x(\Omega) = -atan\frac{-\rho sin(\Omega - \psi)}{1 - \rho cos(\Omega - \psi)}$$

Fig. 3.17 shows the two spectra for $\gamma = 0.75$, i.e., $\rho = 0.75, \psi = 0$.

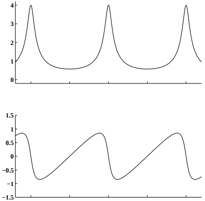

Fig. 3.17: Spectra of $\gamma^n u[n]$ with $\gamma = 0.75$. (a) $|X(e^{j\Omega})|$; (b) $\phi_x(\Omega)$.

Other common but important DTFT pairs include

$$-\gamma^n u[-n-1] \leftrightarrow \frac{1}{1-\gamma e^{-j\Omega}},\ 1 < |\gamma| \tag{34}$$

$$\delta[n] \leftrightarrow 1 \tag{35}$$

$$e^{j\Omega_0 n},\ \ |\Omega_0| < \pi \ \leftrightarrow \ 2\pi \sum_{k=-\infty}^{+\infty} \delta(\Omega - \Omega_0 + 2\pi k) \tag{36}$$

$$u[n] \leftrightarrow \frac{1}{1-e^{-j\Omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\Omega - 2\pi k) \tag{37}$$

Inverse DTFT

The direct way is demonstrated below.

$\boxed{\textbf{Example 3.16}}$: Compute the IDTFT of $X(e^{j\Omega})$ given below for $|\Omega| \leq \pi$:

$$X(e^{j\Omega}) = \begin{cases} e^{-j\Omega\tau}, & |\Omega| < \Omega_0 < \pi \\ 0, & otherwise \end{cases}$$

where τ is a constant.

Solution: Using (32), we have

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{jn\Omega}d\Omega = \begin{cases} \frac{\Omega_0}{\pi} & , \quad n = \tau \\ \frac{sin[\Omega_0(n-\tau)]}{\pi(n-\tau)} & , \quad n \neq \tau \end{cases}$$

The indirect way is based on partial fraction. See Example 3.17.

Continuous-time periodic signals

For any function $g(t) \leftrightarrow G(j\omega)$ and constant $T_0 > 0$ given,

$$x(t) = \sum_{m=-\infty}^{+\infty} g(t - mT_0) \tag{38}$$

satisfies $x(t + T_0) = x(t)$, $\forall\, t$.

Fig. 3.20 in the textbook show $g(t) = (1 - |t|)w_2(t)$ abd the corresponding $x(t)$ by (38) using $T_0 = 1.5$.

Recall: Since $x(t + T_0) = x(t)$, $\forall\, t$, the FS suggests

$$x(t) = \sum_{m=-\infty}^{+\infty} c[m]e^{jm\omega_0 t} \leftrightarrow c[m] = \frac{1}{T_0}\int_{T_0} x(t)e^{-jm\omega_0 t}dt, \ \ \omega_0 = \frac{2\pi}{T_0}$$

How $c[m]$ and $G(j\omega)$ are related ?

On the one hand,

$$x(t) = \sum_{m=-\infty}^{+\infty} c[m] e^{jm\omega_0 t} \leftrightarrow X(j\omega) = \sum_{m=-\infty}^{+\infty} 2\pi c[m] \delta(\omega - m\omega_0) \quad (39)$$

The FS coefficients $c[m]$ are usually referred to as *line spectrum* of $x(t)$.

On the other hand, applying FT's properties to $x(t)$ yields

$$X(j\omega) = \sum_{m=-\infty}^{+\infty} G(j\omega) e^{-jmT_0\omega} = G(j\omega) \sum_{m=-\infty}^{+\infty} e^{-jmT_0\omega}$$

Noting that the periodic signal $p(\omega) \triangleq \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$ has a period of ω_0 and the corresponding FS coefficients are all equal to $1/\omega_0$, we have

$$p(\omega) = \sum_{m=-\infty}^{+\infty} \omega_0^{-1} e^{jmT_0\omega}$$

and hence

$$\sum_{m=-\infty}^{+\infty} e^{jmT_0\omega} = \frac{2\pi}{T_0} \sum_{m=-\infty}^{+\infty} \delta(\omega - m\omega_0) \tag{40}$$

It turns out from (40) that

$$X(j\omega) = \sum_{m=-\infty}^{+\infty} \frac{2\pi}{T_0} G(jm\omega_0)\delta(\omega - m\omega_0)$$

Comparing it with (39), we conclude

$$c[m] = \frac{1}{T_0} G(jm\omega_0) \tag{41}$$

$\boxed{\textbf{Example 3.18}}$: A periodic signal $x(t)$ is depicted in Figure 3.21. Find out the FS coefficients of this signal.

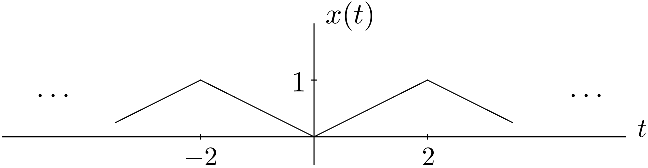

Fig. 3.21: The signal $x(t)$ in Example 3.12.

Solution: Note the period is $T_0 = 4$ and hence $\omega_0 = 2\pi/4 = \pi/2$.

Let $g(t) \triangleq x(t)\, w_4(t) = -\frac{t}{2}\, w_2(t+1) + \frac{t}{2} w_2(t-1)$. Denote $\tilde{g}(t) \triangleq \frac{dg(t)}{dt}$, then

$$\tilde{g}(t) \;=\; -\frac{1}{2}\, w_2(t+1) + \frac{1}{2} w_2(t-1) + \delta(t+2) - \delta(t-2)$$

and hence

$$\begin{aligned}\tilde{G}(j\omega) &= -\frac{1}{2} \times [2\,\frac{sin\omega}{\omega} e^{j\omega} - 2\,\frac{sin\omega}{\omega} e^{-j\omega}] + e^{j2\omega} - e^{-j2\omega} \\ &= j2\,\frac{\omega sin(2\omega) - sin^2\omega}{\omega}\end{aligned}$$

Note $g(t) - g(-\infty) = \int_{-\infty}^{t} \tilde{g}(\tau)d\tau$ and $g(-\infty) = 0$. It then follows from the integration property and $\tilde{G}(j0) = 0$ that

$$G(j\omega) = \frac{\tilde{G}(j\omega)}{j\omega} = 2\,\frac{\omega sin(2\omega) - sin^2\omega}{\omega^2}$$

Since $G(j0) = 2$ and $\omega_0 = \pi/2$, we finally have $c[m] = G(jm\omega_0)/T_0$, namely,

$$c[0] = 1/2, \;\; c[m] = -\frac{2sin^2(m\pi/2)}{(m\pi)^2}, \; \forall\, m \neq 0$$

Discrete-time periodic signals

Like the continuous-time case, a periodic signal $x[n]$ satisfying $x[n+N_0] = x[n]$ can be represented by

$$x[n] = \sum_{m=-\infty}^{+\infty} g[n - mN_0] = \sum_{k=0}^{N_0-1} X_p[k] e^{jk\Omega_0 n}$$

for some sequence $g[n] \leftrightarrow G(e^{j\Omega})$, where $\Omega_0 = 2\pi/N_0$.

It can be shown that

$$X_p[k] = \frac{1}{N_0} G(e^{jk\Omega_0}), \ \forall\, k \tag{42}$$

$\boxed{\textbf{Example 3.19}}$: Let $x_0[n] = \gamma^{|n|}$ with $|\gamma| < 1$. Compute the DTFS coefficients of $x[n] \triangleq \sum_{k=-\infty}^{+\infty} x_0[n-5k]$.

Solution: First of all, $x_0[n] = \gamma^{|n|} = \gamma^{-n}u[-n-1] + \gamma^n u[n]$ and hence

$$x_0[n] \leftrightarrow X_0(e^{j\Omega}) = \frac{(\gamma - \gamma^{-1})e^{-j\Omega}}{(1-\gamma^{-1}e^{-j\Omega})(1-\gamma e^{-j\Omega})}$$

As the signal is periodic with $N_0 = 5$, the DTFS is

$$x[n] = \sum_{k=0}^{4} X_p[k]e^{jk\Omega_0 n}$$

where $X_p[k] = \frac{X_0(e^{jk\Omega_0})}{5}$ with $\Omega_0 = \frac{2\pi}{5}$.