Fundamentals of Signals and Systems

Chapter 4 - Frequency-domain Approach to LTI Systems

Outline of Topics

Main objective

- In Chapter 2, we have set up some results in LTI systems in time-domain;
- In Chapter 3, we have learnt to transform a signal from the time domain representation to the frequency domain one;
- Our objective in this chapter is to analyze the effects of LTI systems on inputs from frequency-domain !

Response to a sinusoid

Recall: for an LTI $x[n] \rightarrow y[n]$ one has $y[n] = h[n] * x[n]$.

Now, consider a complex sinusoidal input $x[n] = \rho e^{j\phi_0} e^{j\Omega_0 n}$

$$\begin{aligned} x[n] \rightarrow y[n] &= \sum_{m=-\infty}^{+\infty} h[m]x[n-m] = \sum_{m=-\infty}^{+\infty} h[m]Ae^{j\Omega_0(n-m)} \\ &= [A \sum_{m=-\infty}^{+\infty} h[m]e^{-j\Omega_0 m}]e^{j\Omega_0 n} \triangleq B \; e^{j\Omega_0 n} \end{aligned}$$

What does this tell ?

The output is also a sinusoid with the *same frequency* and an *different amplitude* that is given by

$$B = A \sum_{m=-\infty}^{+\infty} h[m]e^{-j\Omega_0 m}$$

Note that the 2nd factor on the right is equal to $H(e^{j\Omega_0})$, where

$$H(e^{j\Omega}) = \sum_{m=-\infty}^{+\infty} h(m)e^{-j\Omega m} \triangleq |H(e^{j\Omega})|e^{j\phi_h(\Omega)} \tag{1}$$

is usually referred to as the *frequency response* of the (LTI) system with

- $|H(e^{j\Omega})|$ the *magnitude response* and
- and $\phi_h(\Omega)$ the *phase response* of the system.

Therefore,

$$x[n] = Ae^{j\Omega_0 n} \;\; \rightarrow \;\; y[n] = AH(e^{j\Omega_0})e^{j\Omega_0 n} \tag{2}$$

What does it signify? $\frac{B}{A} = H(e^{j\Omega_0})$ – *amplitude gain* !

Now, consider $x[n] = \rho_x \, cos(\Omega_0 n + \phi_x) \;\rightarrow\; y[n] = ?$ As well known,

$$x[n] \;=\; A_1 e^{j\Omega_1 n} + A_2 e^{j\Omega_2 n} \triangleq x_1[n] + x_2[n]$$

where $A_1 = \frac{\rho_x}{2} e^{j\phi_x}, \Omega_1 = \Omega_0$ and $A_2 = \frac{\rho_x}{2} e^{-j\phi_x}, \Omega_2 = -\Omega_0$.

According to (2), we have

$$x_k[n] = A_k e^{j\Omega_k n} \;\rightarrow\; y_k[n] = A_k H(e^{j\Omega_k}) e^{j\Omega_k n}, \;\; k = 1, 2 \qquad (3)$$

$$Linearity \;\Rightarrow\; y[n] = y_1[n] + y_2[n] = A_1 H(e^{j\Omega_1}) e^{j\Omega_1 n} + A_2 H(e^{j\Omega_2}) e^{j\Omega_2 n}$$

Noting that $A_1 = \frac{\rho_x}{2} e^{j\phi_x} = A_2^*, \Omega_1 = \Omega_0 = -\Omega_2$ and $H(e^{j(-\Omega)}) = H^*(e^{j\Omega}) = |H(e^{j\Omega})| e^{-j\phi(\Omega)}$ (as $h[n]$ is assumed real-valued), one finally reaches

$$x[n] = \rho_x \, cos(\Omega_0 n + \phi_x) \;\rightarrow\; y[n] = \rho_y \, cos(\Omega_0 n + \phi_y) \qquad (4)$$

where $\rho_y = \rho_x \, |H(e^{j\Omega_0})|, \quad \phi_y = \phi_x + \phi_h(\Omega_0)$.

Measuring frequency response:

$$|H(e^{j\Omega_0})| = \frac{\rho_y}{\rho_x}, \;\; \phi_h(\Omega_0) = \phi_y - \phi_x \tag{5}$$

Sweeping Ω_0 from 0 to π, we can then find the complete frequency response of the system.

Now, let us consider two examples that will help us have a better understanding of frequency response.

$\boxed{\textbf{Example 4.1}}$: We have an LTI system

$$x[n] = \kappa + (-1)^n \;\; \rightarrow \; y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

where κ is constant but unknown. See Fig. 4.1(a).

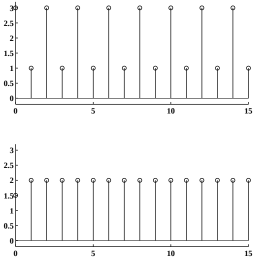

Fig. 4.1: WTime-domain waveforms for Example 4.1. (a) $x[n]$ with $\kappa = 2$; (b) $y[n]$.

With the input signal $x[n]$ given in Fig. 4.1(a), the output can be obtained directly from the difference eqn. and is shown in Fig. 4.1(b). It seems that the unknown constant κ is 2.

Why is that? The answer can be obtained from the concept of frequency response with the analysis below.

First of all, $h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$ leads to $h[n] = 0,\ \forall n \neq 0, 1$ and $h[0] = h[1] = \frac{1}{2}$. So,

$$H(e^{j\Omega}) = \frac{1}{2} + \frac{1}{2}e^{-j\Omega} = cos(\Omega/2)e^{-j\Omega/2}$$

Therefore, the magnitude response is $|H(e^{j\Omega})| = |cos(\Omega/2)|$ and the phase response is of form $\phi_h(\Omega) = -\Omega/2,\ \ |\Omega| \leq \pi$. See Fig. 4.2 for $-\pi \leq \omega \leq \pi$.

Analysis:

With $x[n] = \kappa\, cos(0n + 0) + cos(\pi n + 0)$, applying (4) yields

$$y[n] = \kappa \times |H(e^{j0})|cos[0n + 0 + \phi_h(0)] + 1 \times |H(e^{j\pi})|cos[\pi n + 0 + \phi_h(\pi)]$$

Note

$$H(e^{j0}) = 1,\ H(e^{j\pi}) = 0 \quad \Rightarrow \quad y[n] = \kappa \ \ \Rightarrow \ \ \kappa = 2$$

which is confirmed by the actual computation result in Fig. 4.1(b).

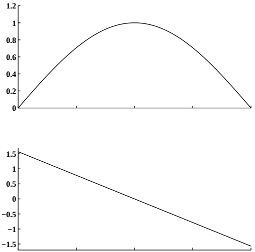

Fig. 4.2: Frequency response for Example 4.1, where the x-axis denotes angular frequency $[-\pi, \pi]$. (a) $|H(e^{j\Omega})|$ - the magnitude response; (b) $\phi_h(\Omega)$ - the phase response.

This system blocks the high frequency component $x_2[n] = (-1)^n$ but lets $x_1[n] = \kappa$ pass - a typical *low-pass* filtering operation.

Example 4.2 : A causal LTI system is given by the following

$$y[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-1]$$

Compute $H(e^{j\Omega})$ and determine $y[n]$ to the same $x[n]$ as in Example 4.1.

Solution: Applying the same procedure, we have $h[n] = 0, \; \forall \, n \neq 0, 1$ and $h[0] = \frac{1}{2}, h[1] = -\frac{1}{2}$. Noting $j = e^{j\pi/2}$, we have

$$H(e^{j\Omega}) = \frac{1}{2} - \frac{1}{2}e^{-j\Omega} = jsin(\Omega/2)e^{-j\Omega/2} = sin(\Omega/2)e^{-j(\frac{\Omega}{2}-\frac{\pi}{2})}$$

With $x[n] = K + (-1)^n$, the output is in the *same* form

$$y[n] = K \times |H(e^{j0})|cos[0n + 0 + \phi(0)] + 1 \times |H(e^{j\pi})|cos[\pi n + 0 + \phi(\pi)]$$

but this time, $H(e^{j0}) = 0, \; H(e^{j\pi}) = 1$, which leads to

$$y[n] = (-1)^n = x_2[n]$$

This system blocks $x_1[n] = K$ and lets $x_2[n] = (-1)^n$ pass - a typical *high-pass* filtering operation.

Both magnitude and phase responses are plotted in Fig. 4.3 for $-\pi \leq \Omega \leq \pi$.

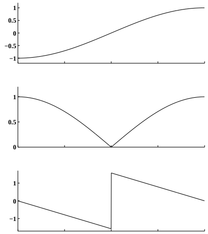

Fig. 4.3: Frequency response for Example 4.2, where the x-axis denotes angular frequency $[-\pi, \pi]$, and (a) $sin(\Omega/2)$; (b) $|H(e^{j\Omega})|$ - the magnitude response; (c) $\phi_h(\Omega)$ - the phase response.

For an CT LTI system of $h(t)$, the frequency response is defined as

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau = |H(j\omega)| e^{j\phi_h(\omega)} \tag{6}$$

where $|H(j\omega)|$ and $\phi_h(\omega)$ are the magnitude and phase responses, respectively.

It can be shown that

$$x(t) = Ae^{j\omega_x t} \;\leftrightarrow\; y(t) = AH(j\omega_x)\, e^{j\omega_x t} \tag{7}$$

and furthermore,

$$x(t) = \rho_x\, cos(\omega_x t + \phi_x) \;\rightarrow\; y(t) = \rho_y\, cos(\omega_x t + \phi_y) \tag{8}$$

with

$$\phi_y \triangleq \phi_x + \phi_h(\omega_x), \;\; \rho_y \triangleq \rho_x\, |H(j\omega_x)|$$

as long as the LTI system is real-valued.

Example 4.3 : The differentiator $y(t) = \frac{dx(t)}{dt}$ is a causal LTI system. Determine its frequency response.

Solution: As $h(t) = \frac{\delta(t)}{dt}$ has an FT $j\omega$, the frequency response is

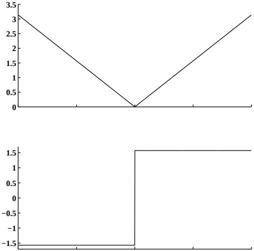

Fig. 4.4: (a) Magnitude response; (b) Phase response.

This system blacks the low frequency components and amplifies the high ones.

Such a system can be used for edge detection. An image is represented by a grey scale function $G = f(x, y)$ in such a way that $G = 0$ means that the pixel (x, y) is *black*, while a large value of G corresponds to a *white* pixel.

$$G(x,y) \;\; \rightarrow \;\; \tilde{G}(x,y) \triangleq \sqrt{\{\frac{\partial f(x,y)}{\partial x}\}^2 + \{\frac{\partial f(x,y)}{\partial y}\}^2}$$

Fig. 4.5: Effects of a differentiator on the image. Left - the original picture; (b) Right - the one obtained using differentiating.

Properties of FR

For discrete-time LTI systems, we have

- $H(e^{j\Omega}) = |H(e^{j\Omega})|e^{j\phi_h(\Omega)}$ and hence both $|H(e^{j\Omega})|$ and $\phi_h(\Omega)$ are all periodic in Ω with a period of 2π.
- For *real-valued* $h[n]$, $|H(e^{j\Omega})|$ is an even function, while $\phi_h(\Omega)$ is an odd function.

It is due to these properties that the magnitude and phase responses $|H(e^{j\Omega})|$ (or $20log_{10}|H(e^{j\Omega})|$) and $\phi_h(\Omega)$ are usually plotted with Ω just for

$$0 \leq \Omega < \pi$$

For a real-valued CT LTI system, the frequency response is given for $\omega \geq 0$ because

- $|H(j\omega)|$ is an even function, while $\phi_h(\omega)$ is an odd function.
- Very often the *bode plot* is used, in which both $20log_{10}|H(j\omega)|$ and $\phi_h(\omega)$ are presented with a *logarithmic scaled* frequency $log_{10}\omega$ for $\omega > 0$.

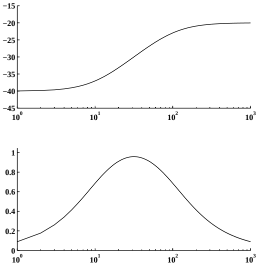

Fig. 4.6: Bode plot for $H(j\omega) = \kappa\,\frac{1+j\omega/\omega_1}{1+j\omega/\omega_2}$ with $\kappa = 0.01, \omega_1 = 10$ and $\omega_2 = 100$. (a) $20log_{10}|H(j\omega)|$; (b) $\phi_h(\omega)$, where the x-axis is $log_{10}\omega$.

The use of logarithmic scale allows details to be displayed over a wider dynamic range. E.g., the detailed variations around a value of 10^{-5} and a value of 10^{5} on the same graph, the logarithmic scaling proves a powerful tool.

The same argument applies to the logarithmic frequency scale used in a bode plot, where the frequency varies from 0 Hz to infinity, while it is not used in frequency response plot of discrete-time systems as the frequency range is just from 0 to π.

Now, let us consider the frequency response of the following system

$$H(j\omega) = \kappa \, \frac{1 + j\omega/\omega_1}{1 + j\omega/\omega_2} \tag{9}$$

with $\kappa, \omega_1, \omega_2$ all constant. So,

$$20log_{10}|H(j\omega)| = 20log_{10}|\kappa| + 20log_{10}|1 + j\omega/\omega_1| - 20log_{10}|1 + j\omega/\omega_2|$$

The *straight-line approximation* of Bode magnitude plot is the graph

obtained with the following approximation rule:

$$20log_{10}|1 + j\omega/\omega_k| \approx \begin{cases} 0, & 0 \leq \omega < |\omega_k| \\ 20log_{10}\,\omega - 20log_{10}\,|\omega_k|, & \omega \geq |\omega_k| \end{cases} \tag{10}$$

The *straight-line approximation* of the Bode magnitude plot for the $H(j\omega)$ given by (9) is shown in Fig. 4.7.

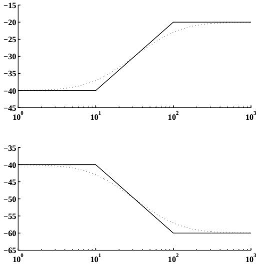

Fig. 4.7: The straight-line approximation (solid-line) for $20log_{10}|H(j\omega)| = 20log_{10}|\kappa\,\frac{1+j\omega/\omega_1}{1+j\omega/\omega_2}|$ (dotted-line) with $\kappa = 0.01$. (a) $\omega_1 = 10$ and $\omega_2 = 100$; (b) $\omega_1 = 100$ and $\omega_2 = 10$.

Since both $|1 + j\frac{\omega}{\omega_p}|$ and $|1 - j\frac{\omega}{\omega_p}|$ have the same straight-line approximation, a straight-line corresponds to two possible frequency responses.

Example 4.4 : A straight-line approximation of a *causal* LTI system is given by Fig. 4.8. Determine the frequency response of this system.

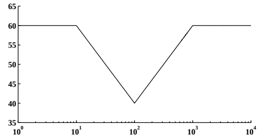

Fig. 4.8: Straight-line approximation for Example 4.4.

Solution: Denote $\omega_1 = 10, \omega_2 = 100$ and $\omega_3 = 1000$. Observing the plot given, we know that the frequency response of the system is of form

$$H(j\omega) = \kappa \frac{1}{1 \pm j\frac{\omega}{\omega_1}} (1 \pm j\frac{\omega}{\omega_2})^2 \frac{1}{1 \pm j\frac{\omega}{\omega_3}}$$

where $|\kappa| = 10^{60/20} = 10^3$.

Since

$$\omega_p e^{-\omega_p t} u(t) \ \leftrightarrow \ \frac{1}{1 + j\frac{\omega}{\omega_p}}, \quad \omega_p e^{\omega_p t} u(-t) \ \leftrightarrow \ \frac{1}{1 - j\frac{\omega}{\omega_p}}$$

and the system is *causal*, we have

$$H(j\omega) = \pm 10^3 \, \frac{(j\omega \pm \omega_2)^2}{(j\omega + \omega_1)(j\omega + \omega_3)}$$

which yields four possible frequency responses.

Consider the class of LTI systems that are constrained with

$$x[n] \rightarrow y[n]: \; y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \qquad (11)$$

The existence of such LTI systems has been proved in Chapter 2. What is the $H(e^{j\Omega})$ for such a system?

It can be shown (see the textbook) that

$$H(e^{j\Omega}) = \frac{\sum_{m=0}^{M} b_m e^{-j\Omega m}}{1 + \sum_{k=1}^{N} a_k e^{-j\Omega k}} \qquad (12)$$

With such an expression, the response response can be evaluated much easily once the coefficients a_k, b_m are given.

Example 4.5 : Determine the frequency response of the LTI system given by $y[n] - \frac{1}{4}y[n-2] = 2x[n]$.

Solution: Based on (12), the frequency response is given directly

$$H(e^{j\Omega}) \quad = \quad \frac{2}{1 - \frac{1}{4}e^{-j2\Omega}}$$

This procedure avoids computing the unit impulse response of the system which is given by the IDTFT of $H(e^{j\omega})$

$$h[n] = 0.5^n u[n] + (-0.5)^n u[n]$$

Is the system stable?

For an CT LTI system constrained with

$$\frac{d^N y(t)}{dt^N} + \sum_{k=1}^{N} \alpha_k \frac{d^{N-k} y(t)}{dt^{N-k}} = \sum_{k=0}^{M} \beta_k \frac{d^{M-k} x(t)}{dt^{M-k}} \tag{13}$$

if its frequency response exists, then

$$H(j\omega) = \frac{\sum_{k=0}^{M} \beta_k (j\omega)^{M-k}}{(j\omega)^N + \sum_{k=1}^{N} \alpha_k (j\omega)^{N-k}} \tag{14}$$

Example 4.6 : Consider the rectifier shown in Fig. 1.3, where $R >> r$ and $x(t) = cos(2\pi F_0 t),\ F_0 = 1\ Hz$. The design problem is to choose R and C such that the output $y(t)$ is close to a constant.

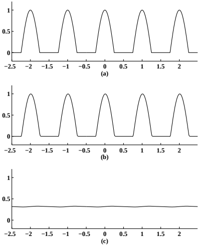

Fig. 4.9: Signals for Example 4.6. (a) $p(t)$; (b) $y(t)$ with $RC = 0.01$; (c) $y(t)$ with $RC = 10$.

Fig.s 4.9(b) and 4.9(c) show the output $y(t)$ for $RC = 0.01$ and $C = 10$, respectively. Try to explain why the difference is so big by evaluating the voltage $y(t)$ across the capacitor C.

Solution: Ideally, $p(t)$ shown in Fig. 1.3 is a periodic with $T_0 = 1/F_0 = 1$ second. See 4.9(a).

Denote

$$x_0(t) \triangleq x(t) w_{T_0/2}(t) \;\Rightarrow\; p(t) = \sum_k x_0(t - kT_0)$$

As $p(t)$ is periodic, $p(t) = \sum_m c[m] e^{j\omega_0 mt}$, where

$$c[m] = X_0(j2\pi m) = \frac{1}{4}\,[\frac{sin((m-1)\pi/2)}{(m-1)\pi/2} + \frac{sin((m+1)\pi/2)}{(m+1)\pi/2}]$$

Particularly,

$$c[0] = \frac{1}{\pi},\;\; c[1] = \frac{1}{4}$$

Since the RC circuit is an LTI system constrained with

$$y(t) + RC\,\frac{dy(t)}{dt} = p(t) \quad \Leftrightarrow \quad H(j\omega) = \frac{1}{1 + j\omega RC}$$

Therefore, the linearity suggests

$$\begin{aligned} y(t) &= \sum_m c[m]H(jm\omega_0)e^{jm\omega_0 t} = \sum_m \frac{c[m]}{1+jm\omega_0 RC}\, e^{jm\omega_0 t} \\ &\triangleq \sum_m d[m]\, e^{jm\omega_0 t} = \frac{1}{\pi} + \sum_{m\neq 0} d[m]\, e^{jm\omega_0 t} \end{aligned}$$

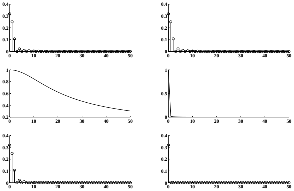

Fig. 4.10: Spectral relationship for Example 4.6. (a) $RC = 0.01$; (b) $RC = 10$.

With $RC = 10$, (say $C = 10\mu F$ and $R = 100\ k\Omega$) the mission of generating a dc can be accomplished.

The important conclusions:

$$y = x * h \;\; \Leftrightarrow \;\; Y = XH \tag{15}$$

Here, just provide the proof for CT case.

First of all, we note

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

and

$$h(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)e^{j\omega(t-\tau)}d\omega$$

Substituting the latter into the former, we obtain

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} x(\tau)[\frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)e^{j\omega(t-\tau)}d\omega]d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)[\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau]e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega
\end{aligned}
$$

This implies that the FT of $y(t)$ is $H(j\omega)X(j\omega)$ and hence completes the proof.

The output y (in time-domain) can then be evaluated by the inverse transform, namely

$$x,\ h\ \Rightarrow\ X,\ H\ \Rightarrow\ Y = XH\ \Rightarrow\ y = Inverse\,transform\,of\ Y$$

Example 4.8 : Consider a causal LTI system described with

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Derive a closed-form expression for the output $y(t)$ in response to $x(t) = e^{-t}u(t)$.

Solution: For this LTI system, we have

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$$

Time domain approach: $h(t)$ is required:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3}$$

where, by comparing the coefficients of the numerator, $A = B = 1/2$, that the unit impulse response is $h(t) = [\frac{1}{2}\, e^{-t} + e^{-3t}]u(t)$ and hence the output can be evaluated with $y(t) = x(t) * h(t)$.

Frequency domain approach: Note $X(j\omega) = \frac{1}{1+j\omega}$. Then

$$\begin{aligned} Y(j\omega) = H(j\omega)X(j\omega) &= \frac{j\omega + 2}{(j\omega+1)(j\omega+3)} \frac{1}{1+j\omega} \\ &= \frac{C_1}{1+j\omega} + \frac{C_2}{(1+j\omega)^2} + \frac{C_3}{3+j\omega} \end{aligned}$$

By comparing the coefficients, one has

$$C_1 = 1/4, \;\; C_2 = 1/2, \;\; C_3 = -1/4$$

Finally,

$$y(t) = \frac{1}{4}\,[e^{-t} + 2te^{-t} - e^{-t}]u(t)$$

This approach seems simpler than directly computing the convolution.

The most important advantage of the frequency-domain approach over the time-domain one is due to the fact that the design of systems can be made much easier in frequency-domain.

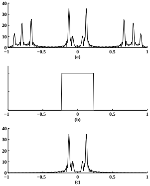

Fig. 4.11: Spectra of $x[n] = x_0[n] + e[n]$, $h[n]$ and $y[n]$.

A priori information: $\Omega_0 < \Omega_e$

Look at the filter $H(e^{j\Omega}) = \sum_{k=0}^{7} \frac{1}{8} e^{-j\Omega k}$, used in Chapter 2 for processing $x[n]$ shown in Fig. 2.3.

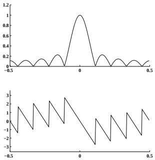

Fig. 4.12: Frequency response of $H(e^{j\Omega}) = \sum_{k=0}^{8} \frac{1}{9} e^{-j\Omega k}$: (a) Magnitude response; (b) Phase response.

As observed, $|H(e^{j\Omega})| \approx 1$ around $\Omega = 0$, much larger than that for the higher frequencies.

To introduce several classes of well-known LTI systems.

For LTI systems,

$$Y(\cdot) = H(\cdot)X(\cdot) \;\Rightarrow\; \begin{cases} |Y(\cdot)| &= |H(\cdot)X(\cdot)| \\ \phi_y(\cdot) &= \phi_h(\cdot) + \phi_x(\cdot) \end{cases} \tag{16}$$

Generally speaking,

- both $|H(\cdot)X(\cdot)|$ and $\phi_h(\cdot)$ affect the spectrum of the input signal though in different manners.
- Besides, as suggested by *Parseval* Theorem, $\phi_h(\cdot)$ has not effect on the signal energy distribution of the input signal.

All-pass systems

A system is said to be *all-pass* if $|H(\cdot)| = c$ (say, $c = 1$) for all frequencies.

E.g.,

- $H(j\omega) = e^{-j\alpha\omega}$: $|H(j\omega)| = 1$, $\phi(\omega) = -\alpha\omega$, yielding $y(t) = x(t-\alpha)$.
- $H(e^{j\Omega}) = \frac{e^{-j\Omega}-\beta}{1-\beta e^{-j\Omega}} = \frac{e^{-j\Omega}(1-\beta e^{j\Omega})}{1-\beta e^{-j\Omega}}$ with β real-valued is all-pass.

Show if $A(e^{j\Omega}) = 1 + a_1 e^{-j\Omega} + \cdots + a_p e^{-jp\Omega}$ with $\{a_k\}$ all real-valued, then $A^*(e^{j\Omega}) = A(e^{-j\Omega})$ and hence the following is all-pass

$$H(e^{j\Omega}) = \frac{e^{-jp\Omega} A(e^{-j\Omega})}{A(e^{j\Omega})}$$

Will an all-pass LTI system affect the energy density of the input signal?

Linear phase response systems

A discrete-time system is said of linear phase response if its phase response $\phi_h(\Omega)$ is linear in frequency variable Ω.

The *group delay* is a measure used in study of this topic:

$$g(\Omega) \triangleq -\frac{d\phi_h(\Omega)}{d\Omega} \tag{17}$$

So, an LTI system $h[n]$ of linear phase response actually implies that the group delay $g(\Omega)$ is constant.

All these apply to continuous-time LTI systems.

Example 4.9 : Let $y[n]$ be the output of the system $H(e^{j\Omega}) = e^{-j\alpha\Omega}$ in response to $x[n]$, where α is not necessarily an integer. What is the time-domain relationship between $x[n]$ and $y[n]$?

Solution : First of all, $y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\Omega})e^{jn\Omega}d\Omega$, i.e.,

$$\begin{aligned} y[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j(n-\alpha)\Omega}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_m x[m]e^{-jm\Omega}e^{j(n-\alpha)\Omega}d\Omega \\ &= \sum_m x[m]\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n-\alpha-m)\Omega}d\Omega = \sum_m x[m]\frac{sin[(n-\alpha-m)\pi]}{\pi(n-\alpha-m)} \end{aligned}$$

As to be seen in next chapter, if $x[n]$ is obtained by sampling $x(t)$ with sampling period T small enough, then $y[n] = x(nT - \alpha T)$.

Is this system causal?

Typical frequency responses of ideal filters

Ideal low-pass:

$$\begin{aligned} |C(\omega)| &= w_{\omega_l}(\omega - \omega_l/2), \;\; \omega \geq 0 \\ |D(\Omega)| &= w_{\Omega_l}(\Omega - \Omega_l/2), \;\; 0 \leq \Omega \leq \pi \end{aligned}$$

Ideal high-pass:

$$\begin{aligned} |C(\omega)| &= u(\omega - \omega_h), & \omega \geq 0 \\ |D(\Omega)| &= u(\Omega - \Omega_h) - u(\Omega - \pi), & 0 \leq \Omega \leq \pi \end{aligned}$$

Ideal band-pass:

$$\begin{aligned} |C(\omega)| &= u(\omega - \omega_l) - u(\omega - \omega_h), & \omega \geq 0 \\ |D(\Omega)| &= u(\Omega - \Omega_l) - u(\Omega - \Omega_h), & 0 \leq \Omega \leq \pi \end{aligned}$$

Ideal band-stop:

$$
\begin{aligned}
|C(\omega)| &= 1 - [u(\omega-\omega_l) - u(\omega-\omega_h)], \qquad \omega \geq 0 \\
|D(\Omega)| &= 1 - [u(\Omega-\Omega_l) - u(\Omega-\Omega_h)], \quad 0 \leq \Omega \leq \pi
\end{aligned}
$$

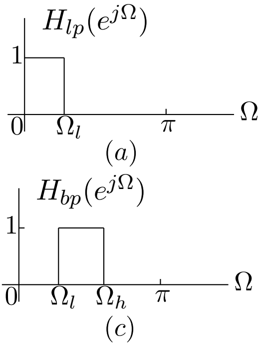

Fig. 4.15: Four types of ideal digital filters with $0 < \Omega_l < \Omega_h < \pi$. (a) LP; (b) HP; (c) BP; (d) BS.

Study **Example 4.10** by yourself - a typical example to be consider in the course *Communications Principles*!

Though having different frequency characteristics, these ideal filters have one thing in common - all non-causal and hence practically not implementable.

Filter design:

$$\min_{H} ||H(e^{j\Omega}) - D(\Omega)||$$

where

$$H(e^{j\Omega}) = \frac{b_0 + b_1 e^{-j\Omega} + \cdots + b_N e^{-jN\Omega}}{1 + a_1 e^{-j\Omega} + \cdots + a_N e^{-jN\Omega}}$$

is a causal LTI system.

For example, the causal $H(e^{j\Omega}) = \sum_{n=0}^{33} b_n e^{-jn\Omega}$

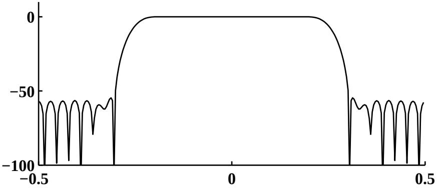

Fig. 4.19: Magnitude response (in dB) of a causal low-pass digital filter with $\Omega_p = 0.4\pi$ and $\Omega_s = 0.6\pi$.

When $\Omega_l = \Omega_h$, the stop-band filter is called an ideal *notch filter* and can be approximated with

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

where $a_1 = -2\rho cos\frac{\pi}{4},\ a_2 = \rho^2,\ b_0 = b_2 = 1,\ b_1 = -2cos\theta_0$.

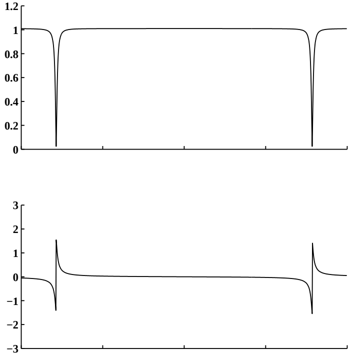

Fig. 4.20: Frequency response of a 2nd order IIR notch system with $\rho = 0.99, \theta_0 = \pi/4$. (a) Magnitude response (b) Phase response.

Ideal channels

Let $x(t)$ be a signal whose spectrum is just spread within (ω_1, ω_2). The ideal channel for transmitting this signal is

$$C_I(j\omega) = \begin{cases} \rho\, e^{-j\tau\omega}, & \omega_1 \leq \omega \leq \omega_2 \\ not\ interested, & otherwise \end{cases} \tag{18}$$

with $\rho > 0,\ \tau$ constant as the received signal $y(t)$ is given by $y(t) = \rho\, x(t-\tau)$ and hence contains exactly the same information as $x(t)$ does.

In practice, due to the multi-path effect in communications systems the signal received is $r(t) = x(t) + \beta x(t-\tau)$. The frequency response of this channel is

$$C(j\omega) = 1 + \beta e^{-j\omega\tau}$$

See Fig. 4.13 for $\beta = 0.25, \tau = 0.005$.

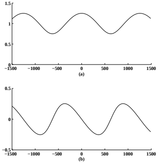

Fig. 4.13: (a) Magnitude response $|C(j\omega)|$; (b) Phase response for $\beta = 0.25$, $\tau = 0.005$.

One way to recover back the transmitted signal $x(t)$ is to feed the received signal $r(t)$ into a well designed system $E(j\omega)$, called *channel equalizer* such that $r(t) \rightarrow x(t-\xi)$. More discussions can be found from the textbooks on communications.