Chapter 4
SAMPLING OF CT SIGNALS

Main Topics

1 Periodic sampling

2 Discrete-time processing of continuous-time signals

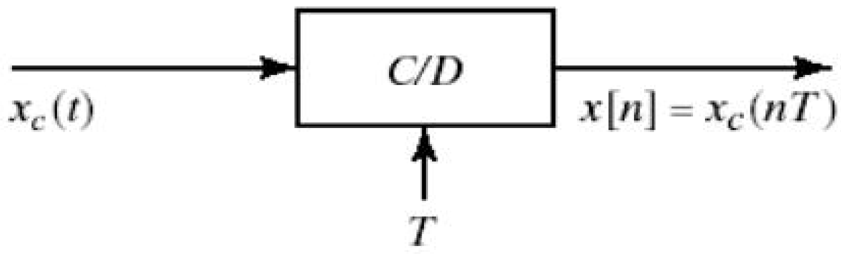
3 Continuous-time processing of discrete-time signal

4 Digital processing of analog signals

5 Changing the sampling rate using discrete-time processing

4.1 Periodic Sampling

Ideal sample

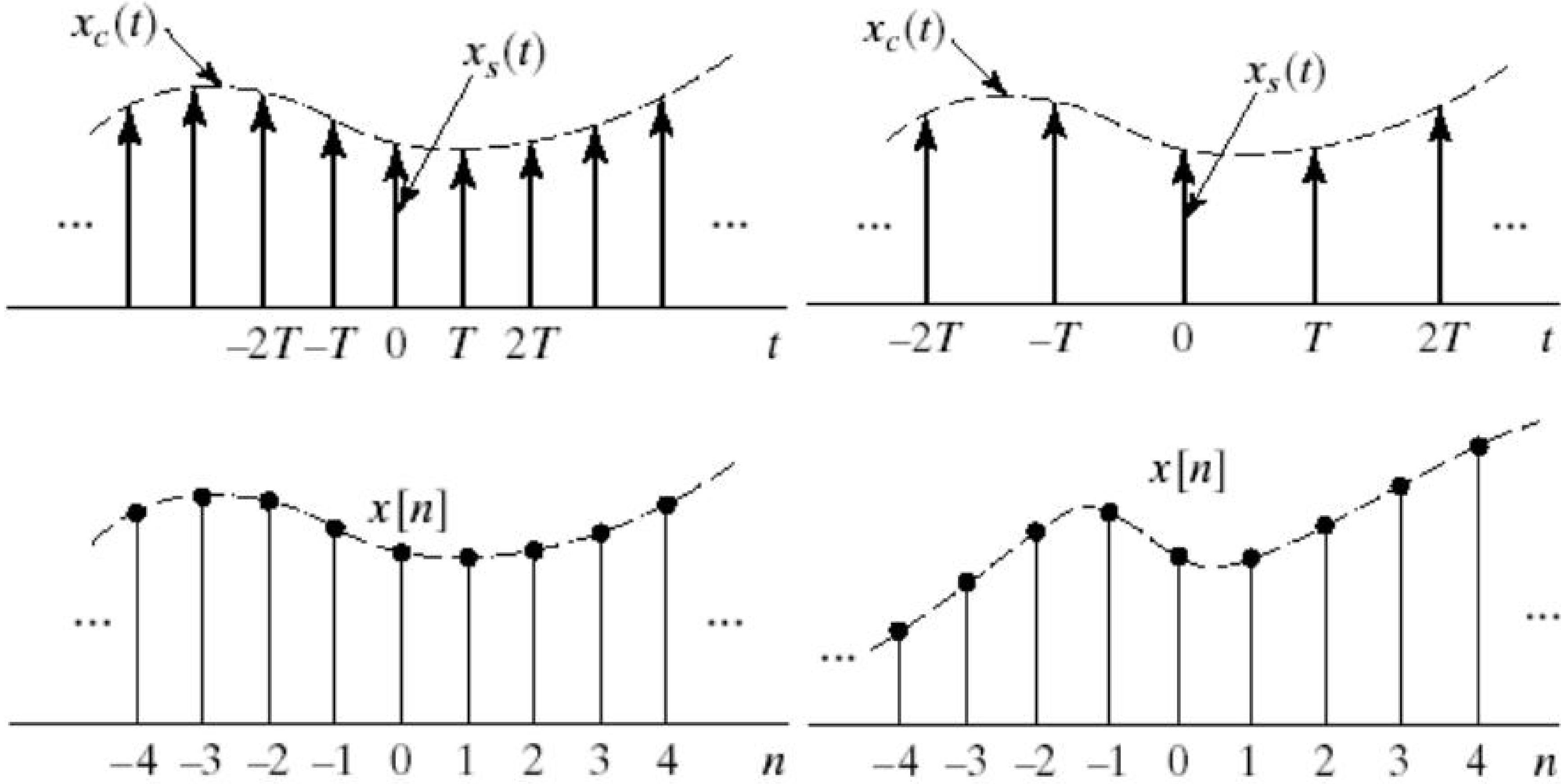


$$x[n] = x_c(t)_{t=nT} = x_c(nT)$$

T : sample period

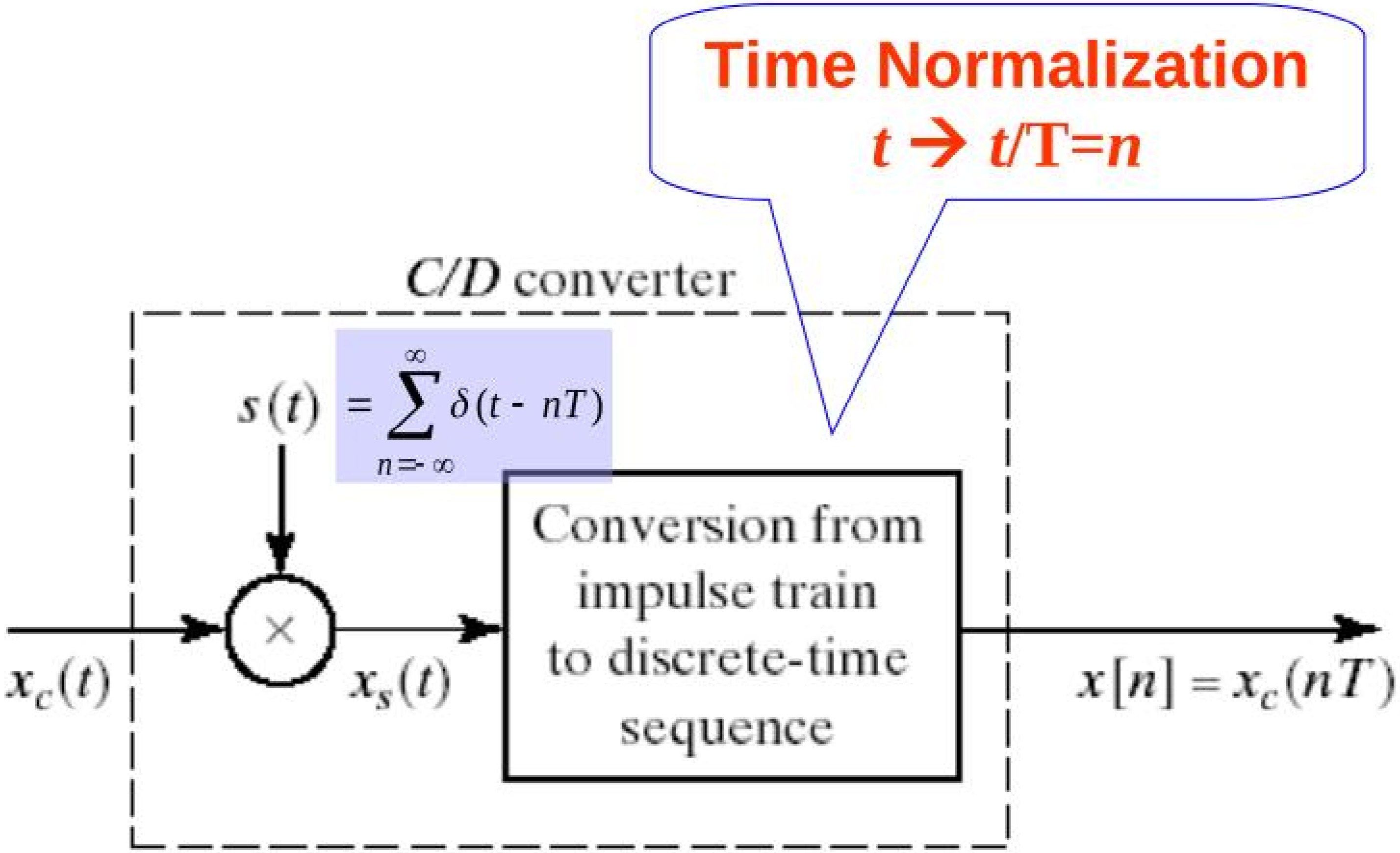
$f_s=1/T$: sample frequency

$\Omega_s=2\pi/T$: sample rate



Time Normalization

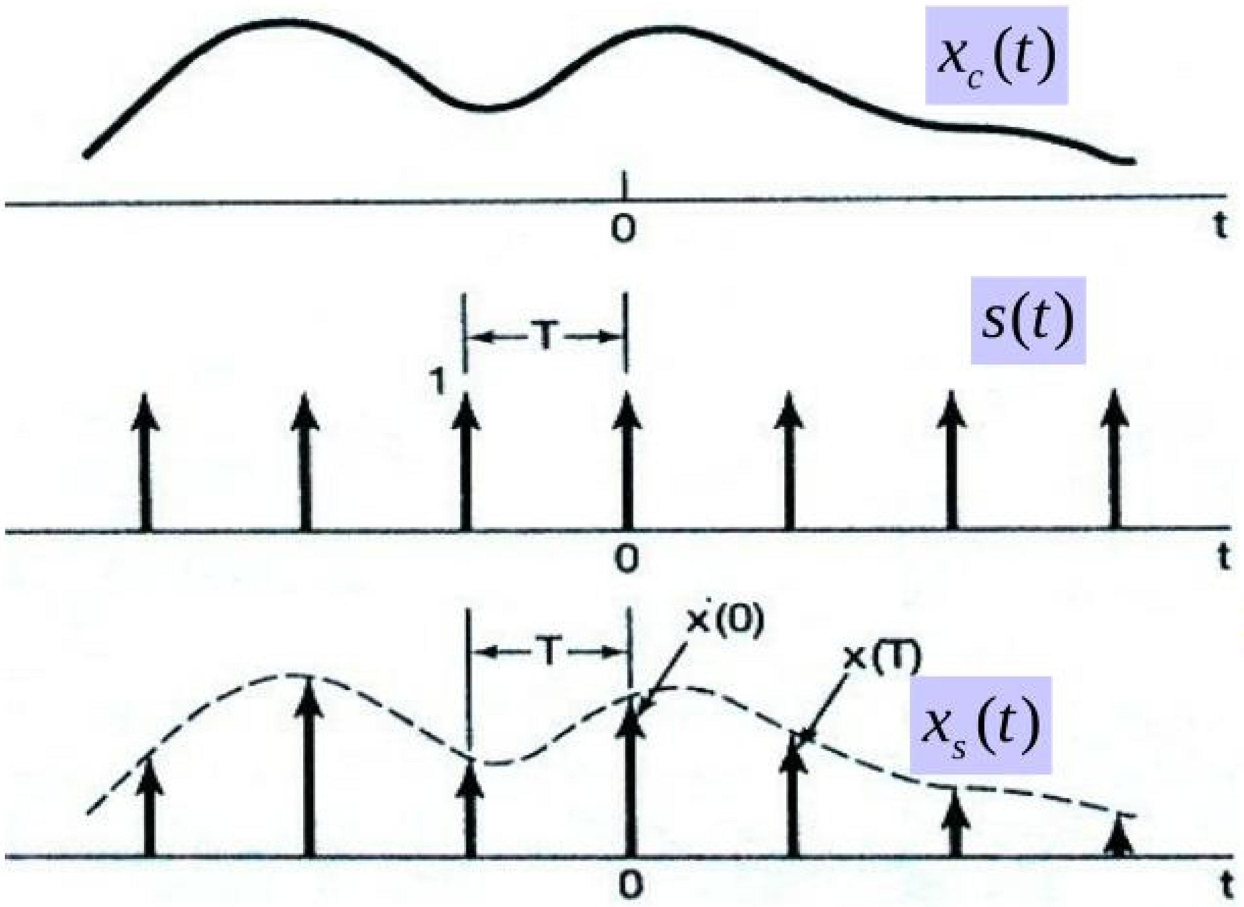
$t \rightarrow t/\mathrm{T} = n$



mathematic model for ideal C/D

4.2 F-DOMAIN REPRESENTATION OF SAMPLING

Time domain:



Sampling period: T

Sampling frequency:

Frequency domain:

$$x_c(t) \xleftrightarrow{CTFT} X_c(j\Omega)$$

$$s(t) \xleftrightarrow{F.S.} a_k = \frac{1}{T} \quad (Periodic \quad signal)$$

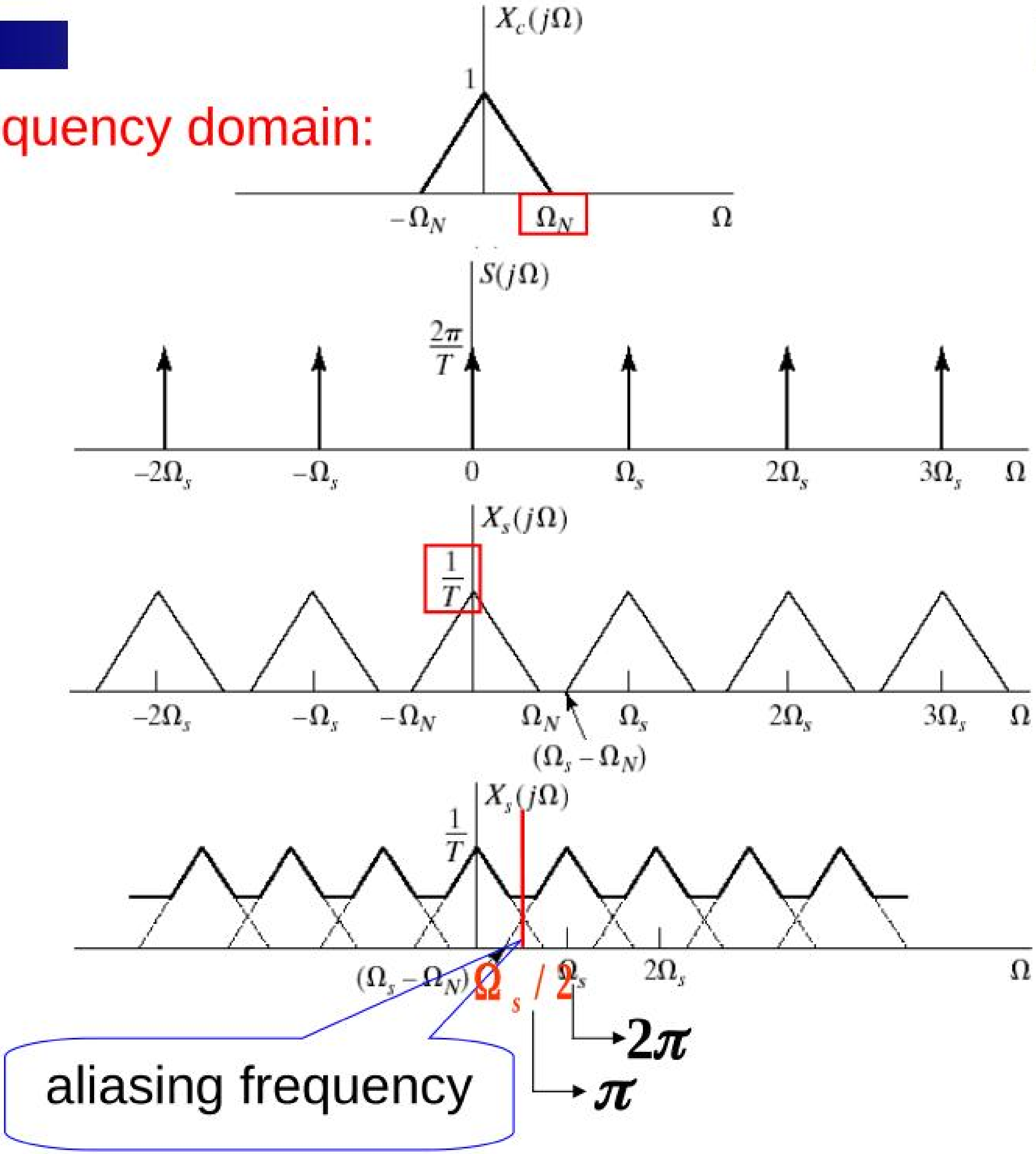
$$s(t) \xleftrightarrow{CTFT} S(j\Omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\Omega - k\Omega_s) = \sum_{k=-\infty}^{+\infty} \Omega_s \delta(\Omega - k\Omega_s)$$

$$x_s(t) \xleftrightarrow{CTFT} X_s(j\Omega) = \frac{1}{2\pi}[X_c(j\Omega) * S(j\Omega)]$$

$$= \frac{\Omega_s}{2\pi} \sum_{k=-\infty}^{+\infty} X(j(\Omega - k\Omega_s))$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j(\Omega - k\Omega_s))$$

Frequency domain:



$\Omega_s - \Omega_N \geq \Omega_N$

No aliasing

$\Omega_s - \Omega_N < \Omega_N$

Aliasing

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

以 **Ωs** 为周期

$$\omega = \Omega T = \frac{\Omega}{f_s}$$

Frequency Normalization

以 **2π** 为周期

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k2\pi)/T)$$

1. Nyquist sampling theorems

let $x_c(t)$ be a bandlimited signal with $X_c(j\Omega) = 0, |\Omega| \geq \Omega_N$

then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), n = 0, \pm 1, \pm 2, \cdots$$

if $\Omega_s - \Omega_N \geq \Omega_N$, that is

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N \qquad \left(\text{or } f_s = \frac{1}{T} \geq 2 f_N \right)$$

$\Omega_s / 2$: *Nyquist frequency*

$2\Omega_N$: *Nyquist rate*

$\Omega_s > 2\Omega_N$: *oversampling*

$\Omega_s < 2\Omega_N$: *undersampling*

e.g.

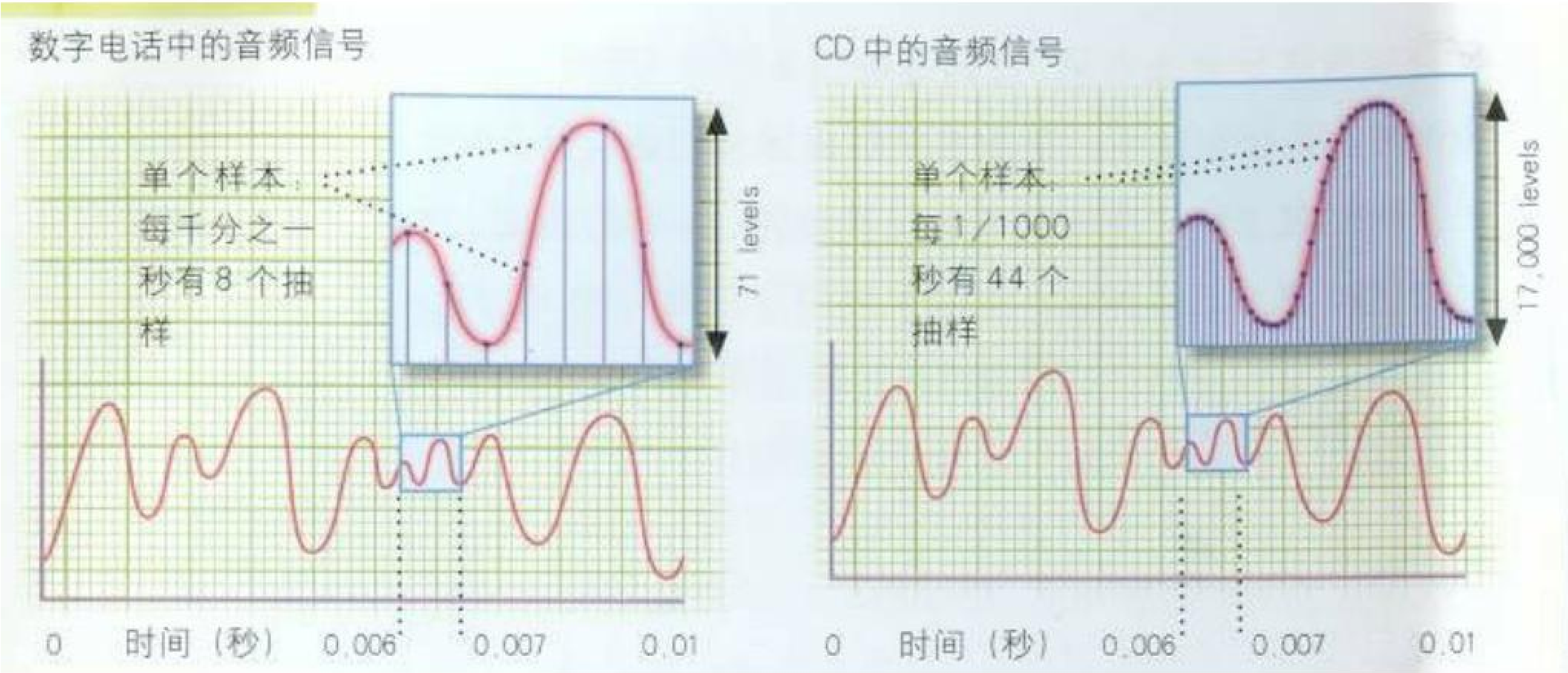
1. The highest frequency of analog signal ,which wav file with sampling rate 16kHz can show ， is：

8kHz

2. According to what you know about the sampling rate of MP3 file ， judge the sound we can feel frequency rangeB（ ）

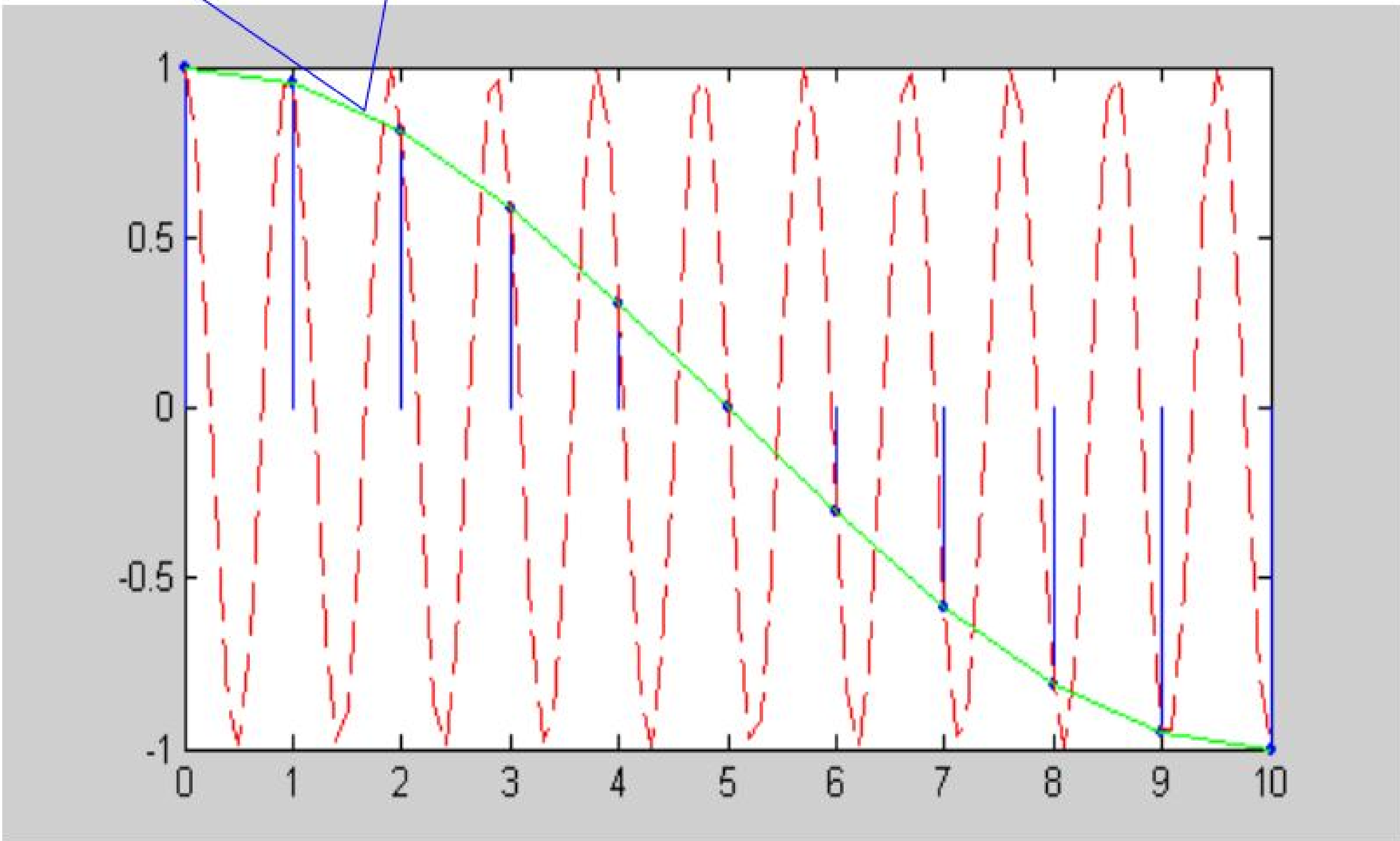
（A） 20~44.1kHz （B） 20~20kHz
（C） 20~4kHz （D） 20~8kHz

The higher sampling rate of audio files, the better fidelity.



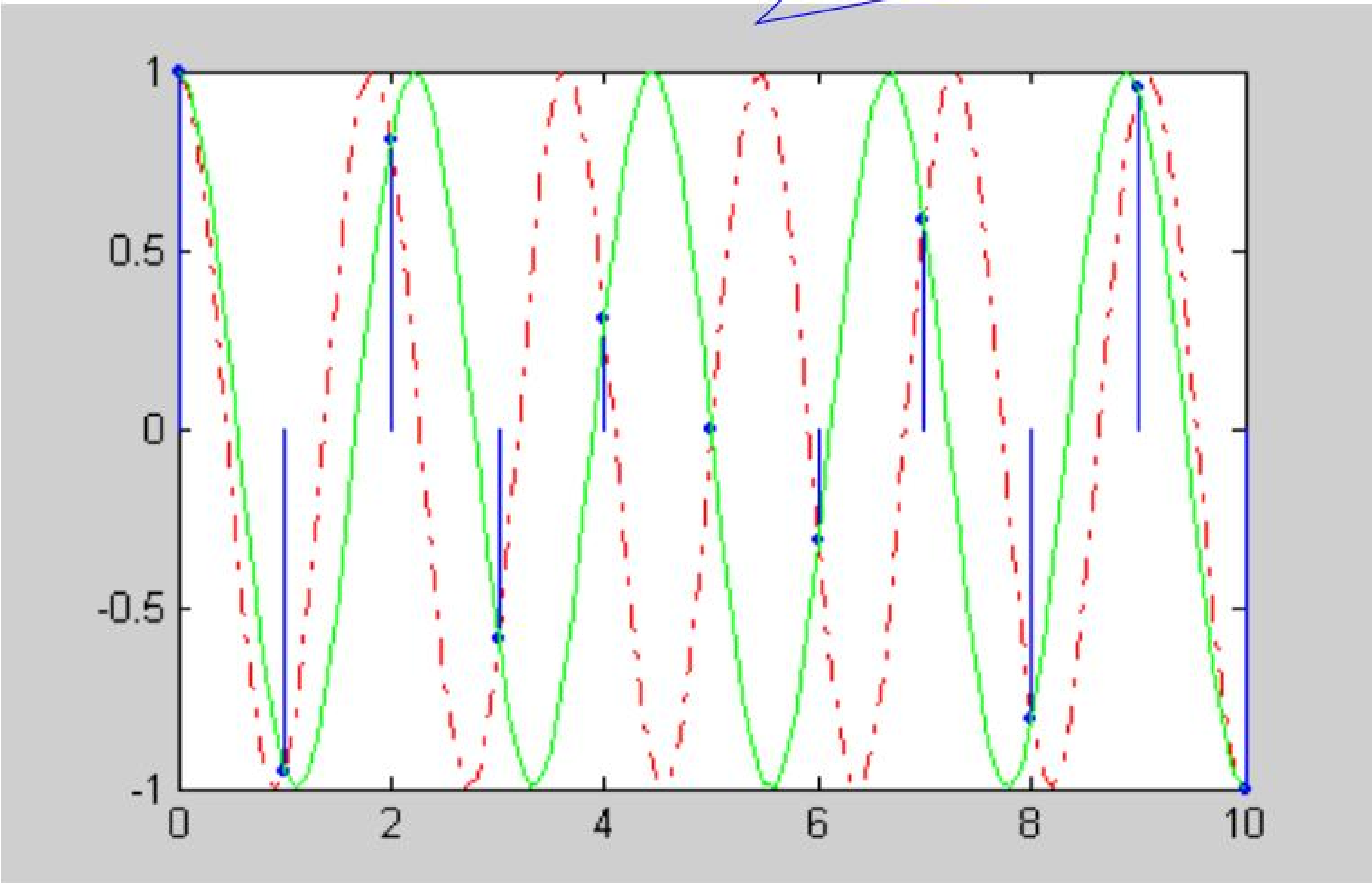
数字电话中的音频信号
单个样本：每千分之一秒有8个抽样
71 levels
0
时间（秒）
0.006
0.007
0.01
CD中的音频信号
单个样本：每1/1000秒有44个抽样
17,000 levels
0
时间（秒）
0.006
0.007
0.01

Period =2π in frequency domain：
w=2.1π and w=0.1π are the same



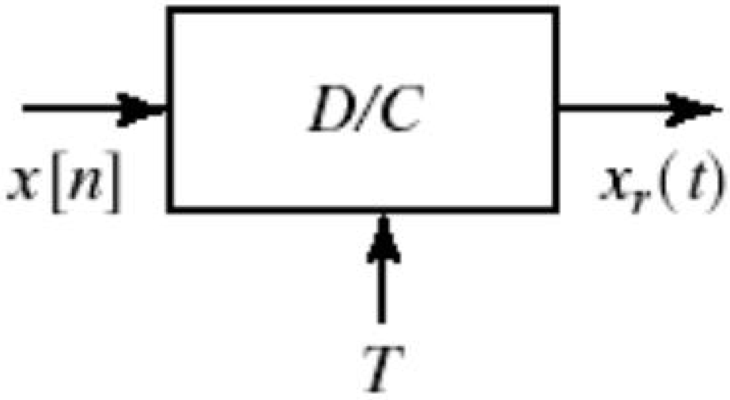
$$\cos(2.1\pi n) = \cos(0.1\pi n)$$

high frequency is changed into low frequency in time domain ： w=1.1π and w=0.9π are the same

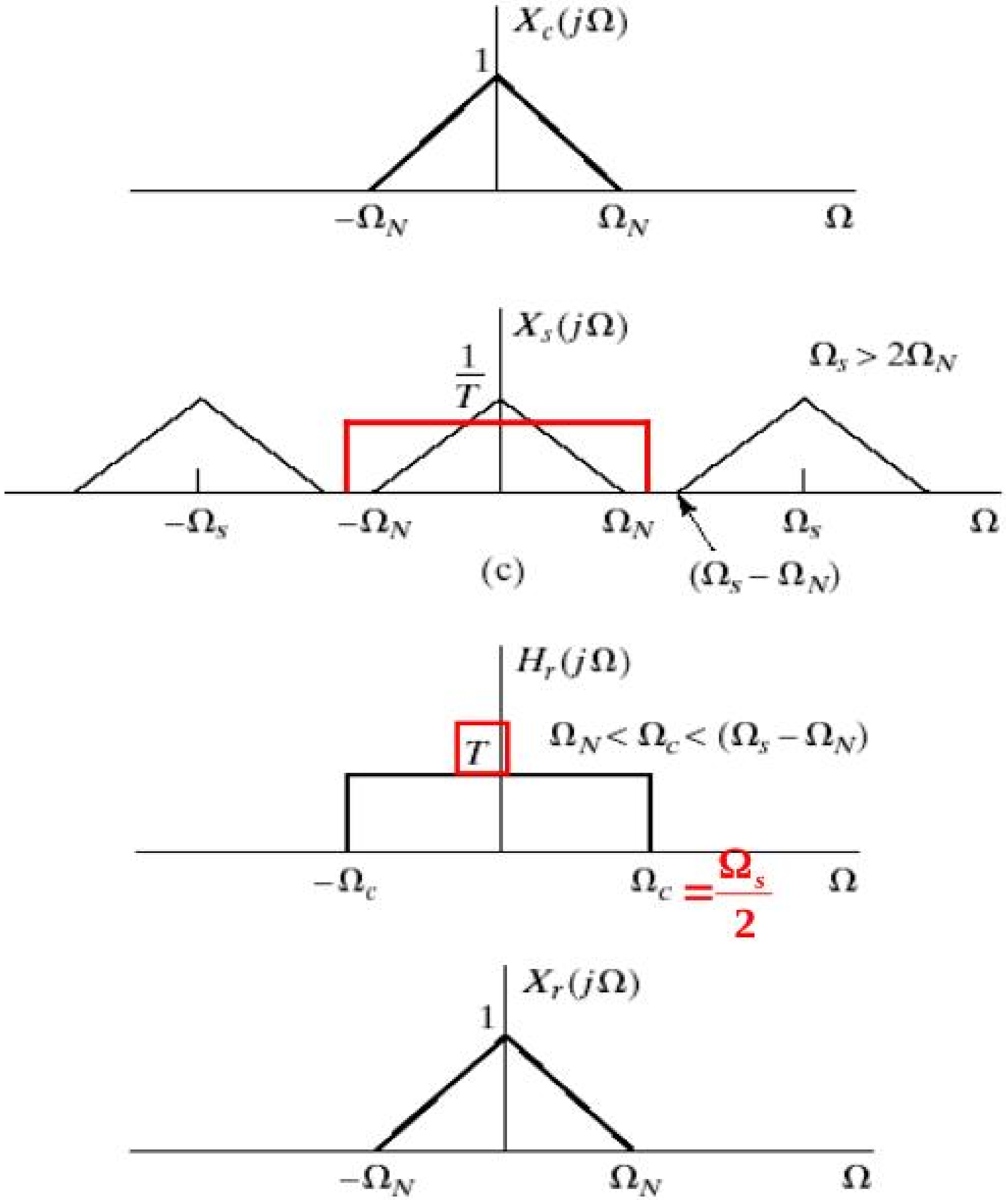


$$\cos(1.1\pi n) = \cos(0.9\pi n)$$

2. Ideal reconstruction

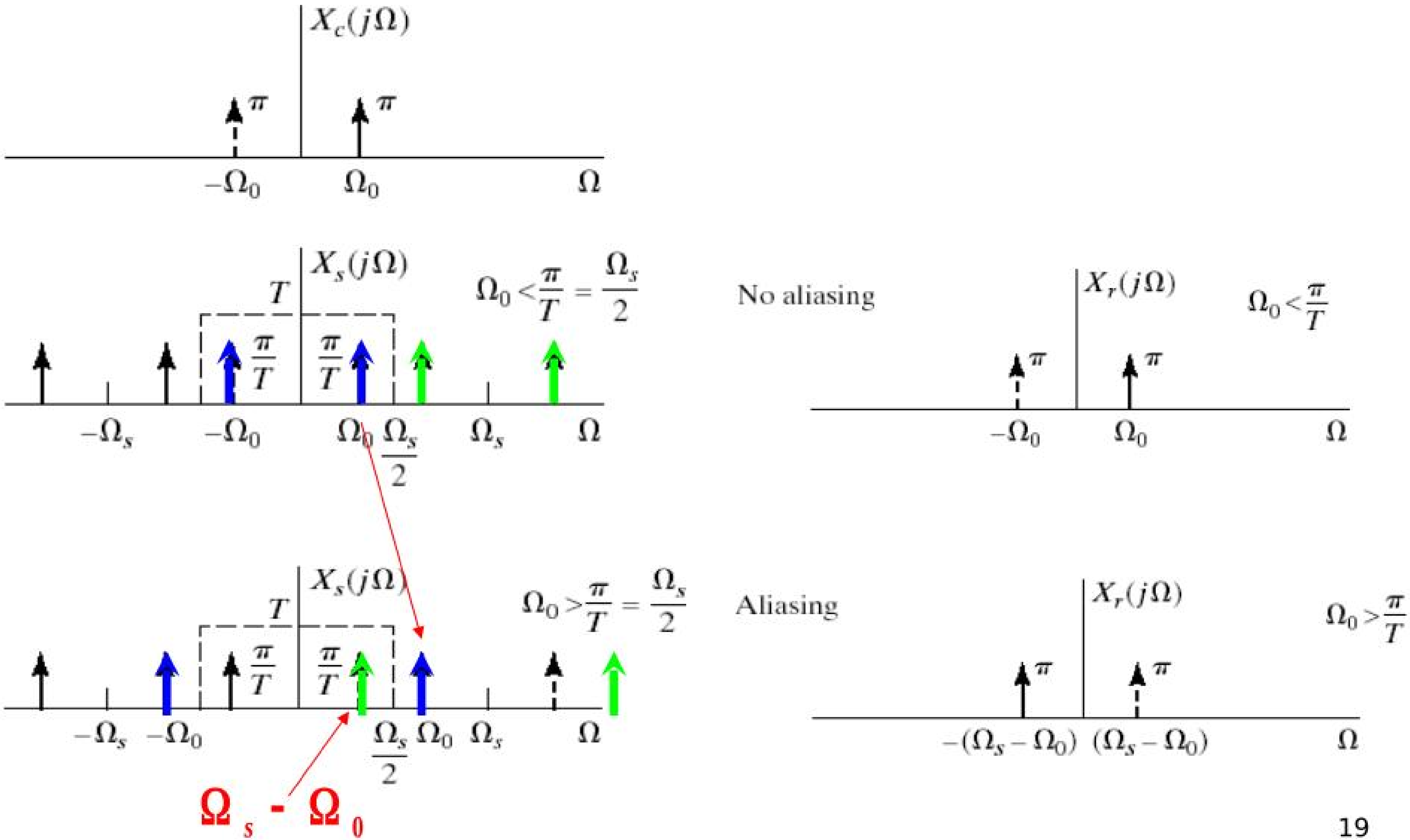


Ideal reconstruction in frequency domain

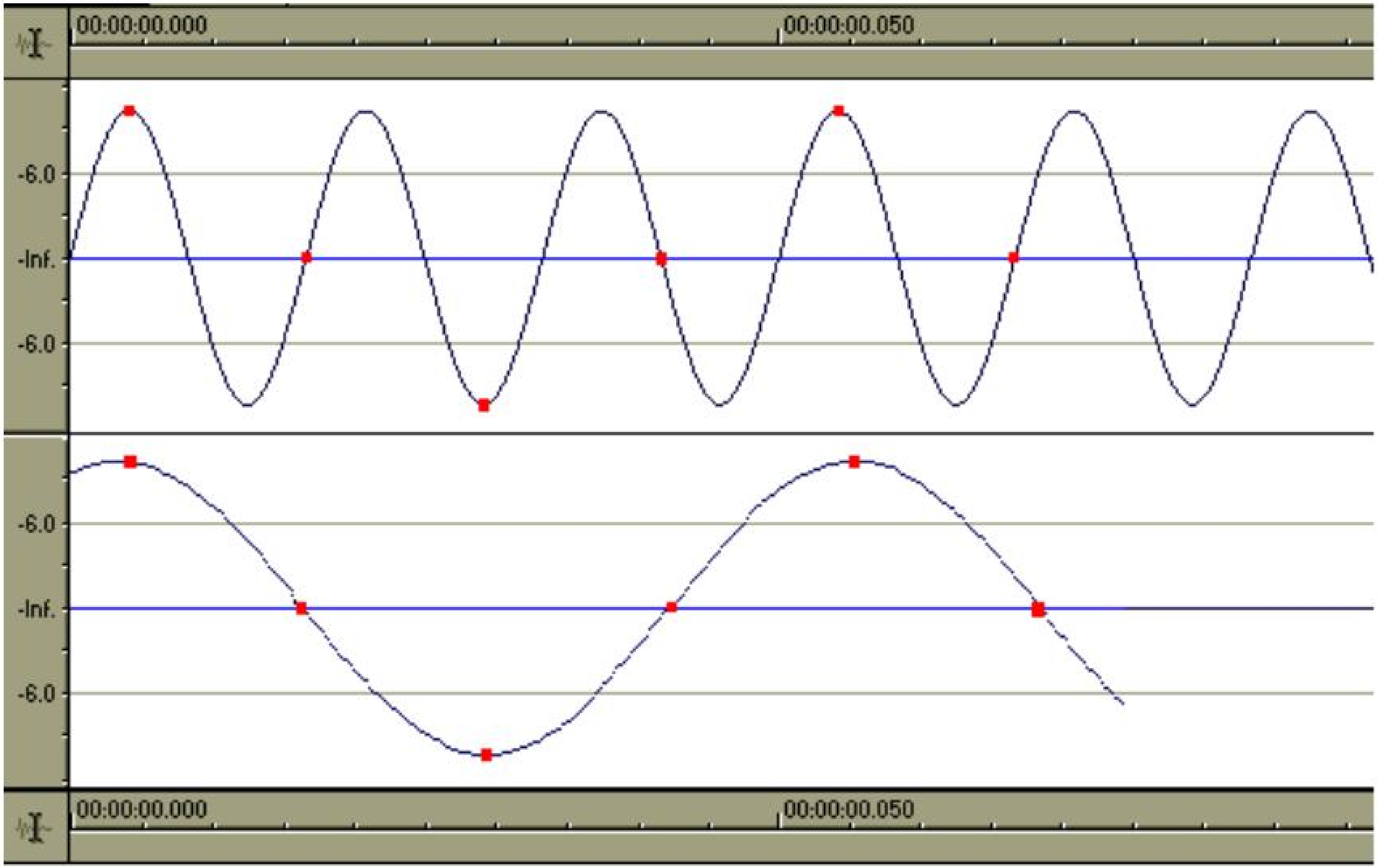


3. Aliasing

e.g. aliasing from frequency domain



e.g. Aliasing from time-domain interpolation

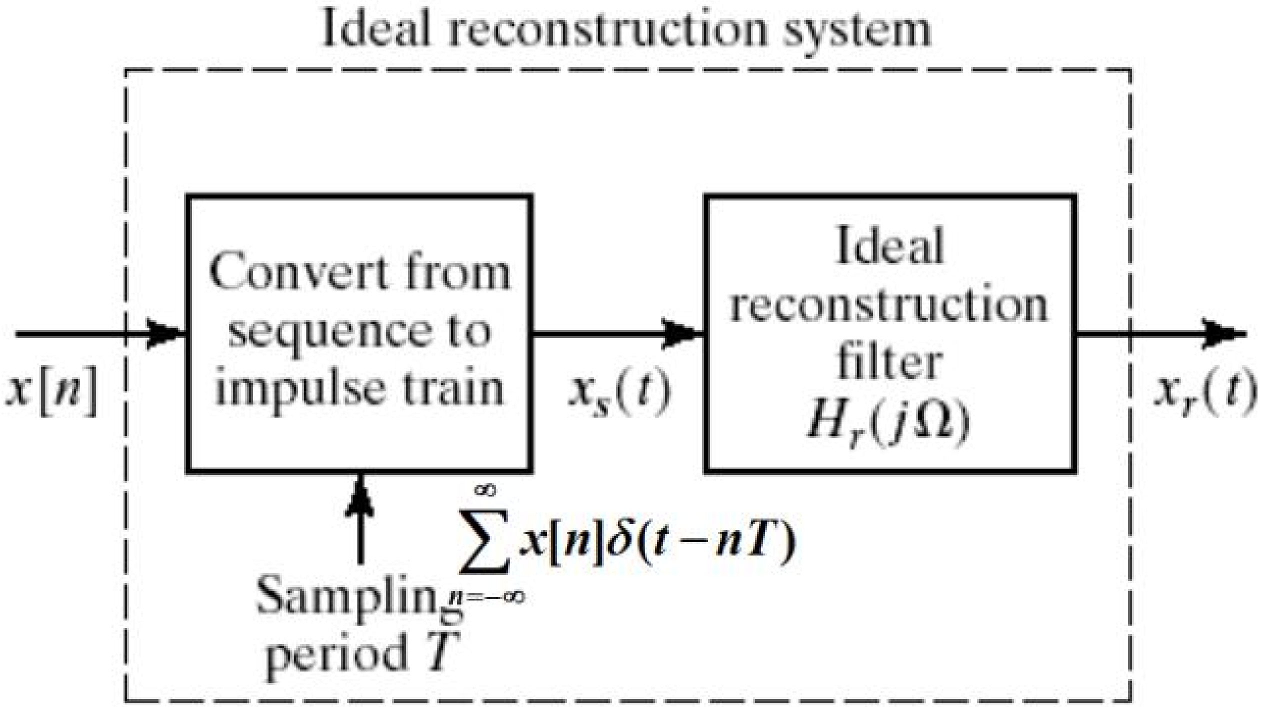


e.g. $x_a(t) = \cos(2\pi * 5t), 0 \le t \le 1,\ f = 5Hz$

Sampling frequency:8Hz

Reconstruct frequency: $f' = 8 - 5 = 3Hz$

4.3 RECONSTRUCTION OF A BANDLIMITED SIGANL FROM ITS SAMPLES



mathematic model for ideal D/C

in f-domain

$$X_r(j\Omega) = X_s(j\Omega)H_r(j\Omega)$$

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \le \Omega_c \\ 0 & |\Omega| > \Omega_c \end{cases}$$

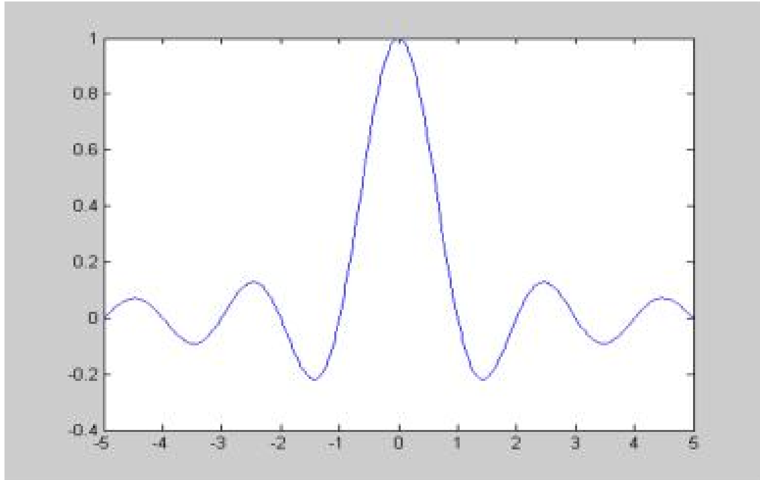
$$X_r(j\Omega) = X_s(j\Omega)H_r(j\Omega)$$

in t-domain

$$h_r(t) = IFT\{H_r(j\Omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\Omega_C}^{\Omega_C} Te^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\pi t/T} = \frac{\sin(\pi t/T)}{\pi t/T}$$

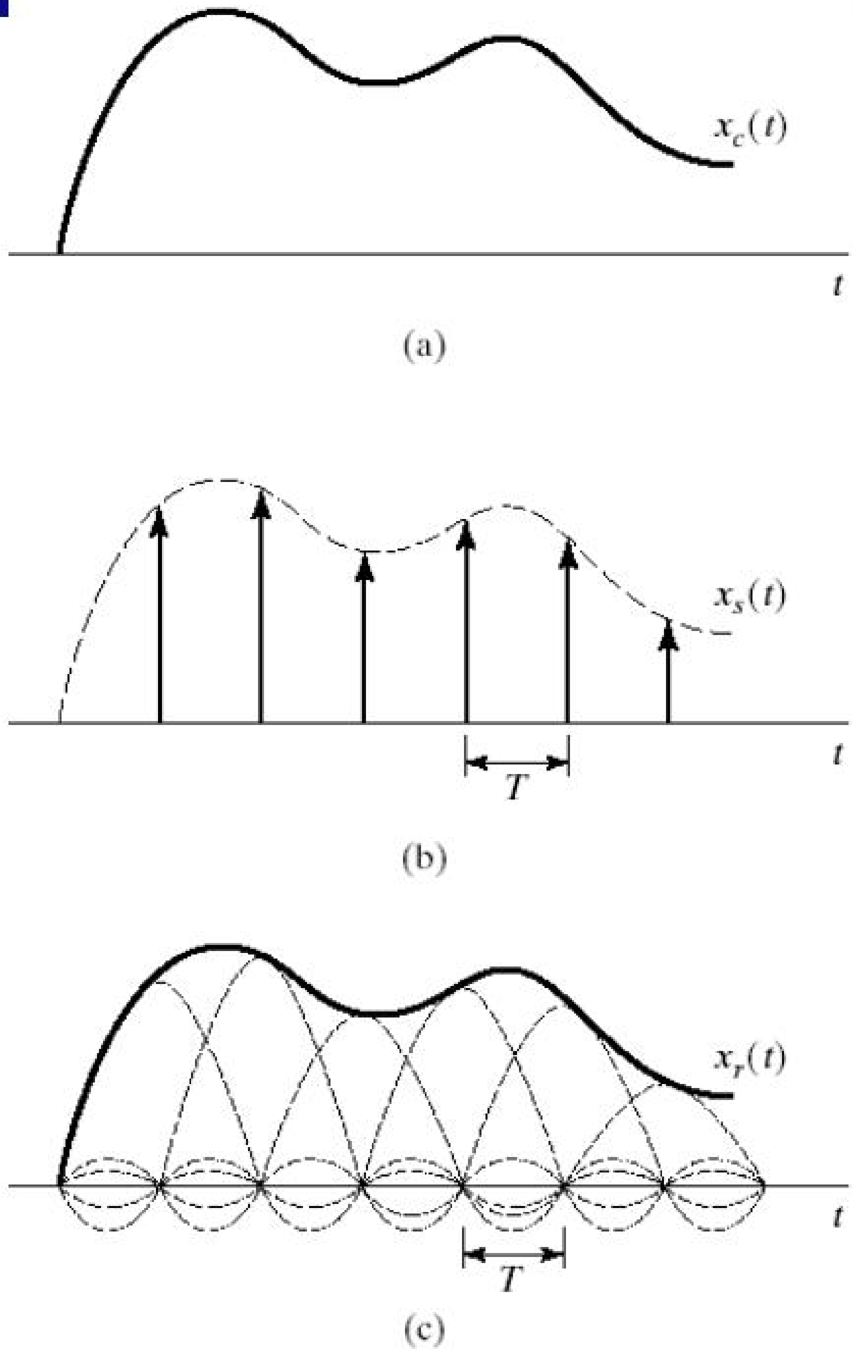


$$x_r(t) = x_s(t) * h_r(t)$$

$$= \left[\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\right] * \frac{\sin(\pi t/T)}{\pi t/T}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left[\delta(t - nT) * \frac{\sin(\pi t/T)}{\pi t/T}\right]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$



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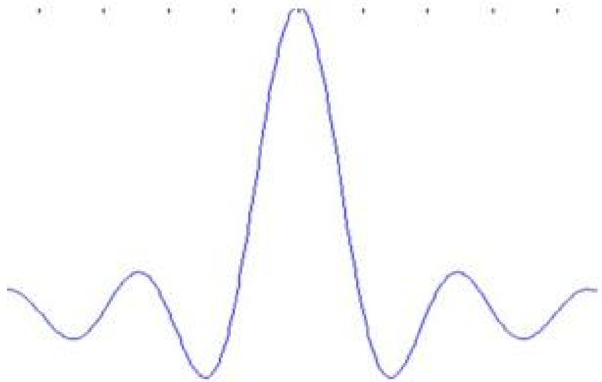
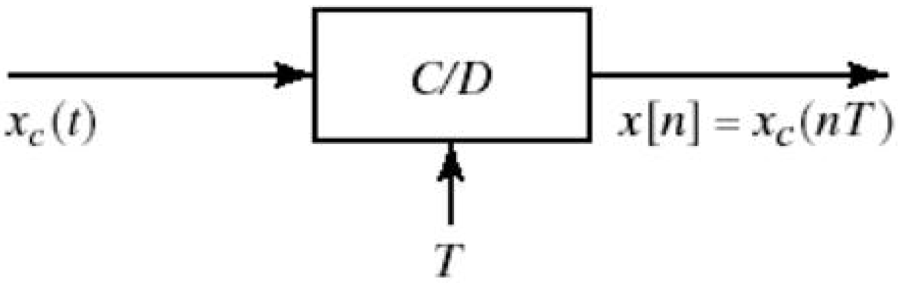
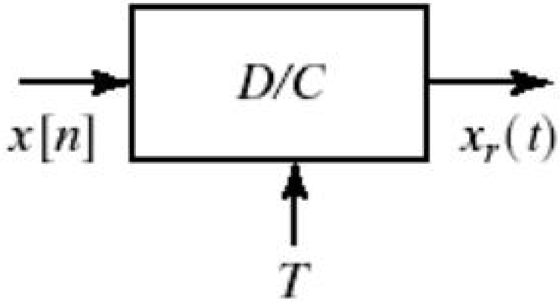


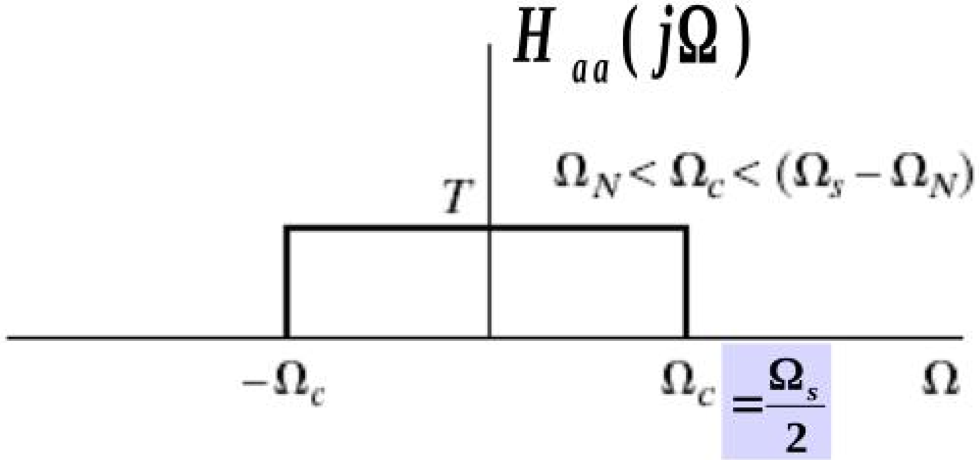
Interpolation in t-domain

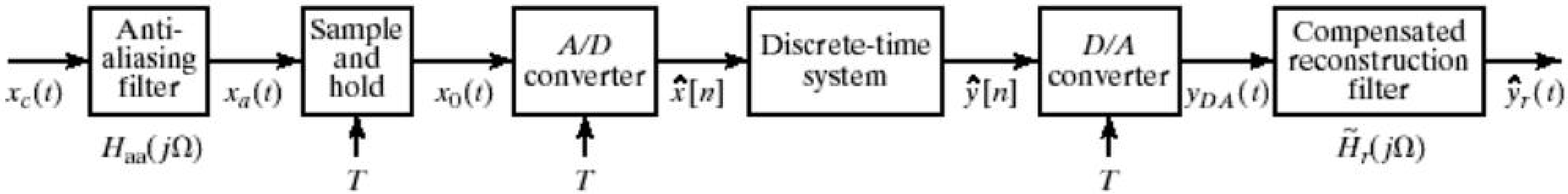




$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - k2\pi)/T)$$

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega T n}$$
$$= H_r(j\Omega) X(e^{j\Omega T})$$





Digital processing of analog signals

e.g.

$$x_a(t) = \cos(10\pi t), 0 \le t < 1, f = 5Hz$$

$$f_s = 10Hz(T = 0.1s)$$

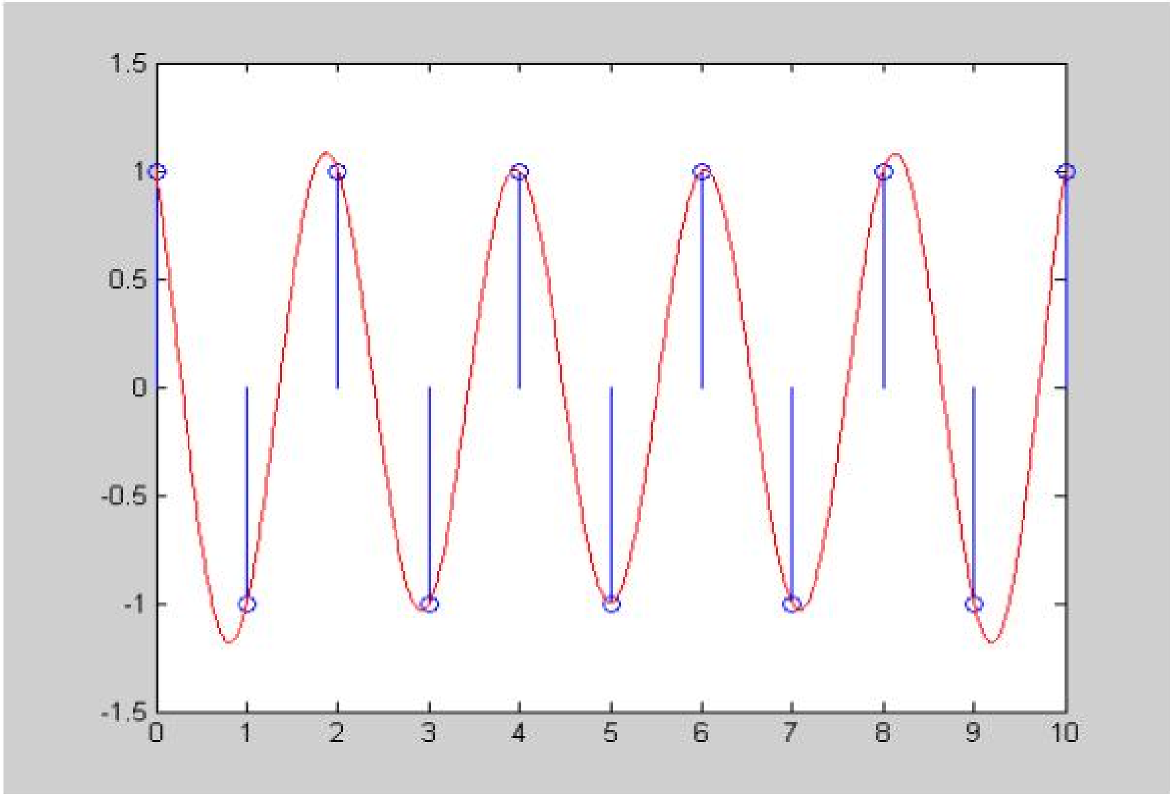
$$draw \quad x[n] = x_a(nT) = \cos(10\pi nT) = \cos(\pi n)$$

$$draw \quad reconstruction \quad signal:$$

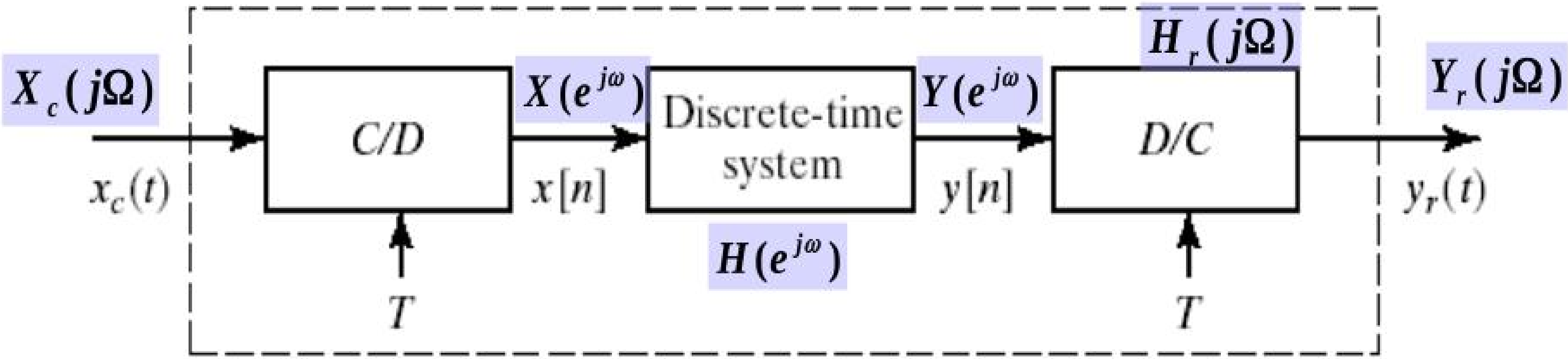
$$y(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

Matlab codes to realize interpolation

```
T=0.1;
n=0:10;
x=cos(10*pi*n*T);
stem(n,x);
dt=0.001;
t=ones(11,1)* [0:dt:1];
n=n'*ones(1,1/dt+1);
y=x*sinc((t-n*T)/T);
hold on;
plot(t/T,y,'r')
```



4.4 Discrete-Time Processing of Continuous-Time Signals



$$
\begin{aligned}
Y_r(j\Omega) &= H_r(j\Omega)Y\left(e^{j\Omega T}\right) \\
&= H_r(j\Omega)H\left(e^{j\Omega T}\right)X\left(e^{j\Omega T}\right) \\
&= H_r(j\Omega)H\left(e^{j\Omega T}\right)\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\Omega - j\frac{2\pi k}{T}\right) \\
&\overset{k=0}{=} H\left(e^{j\Omega T}\right)X_c(j\Omega) \qquad |\Omega| < \frac{\pi}{T}
\end{aligned}
$$

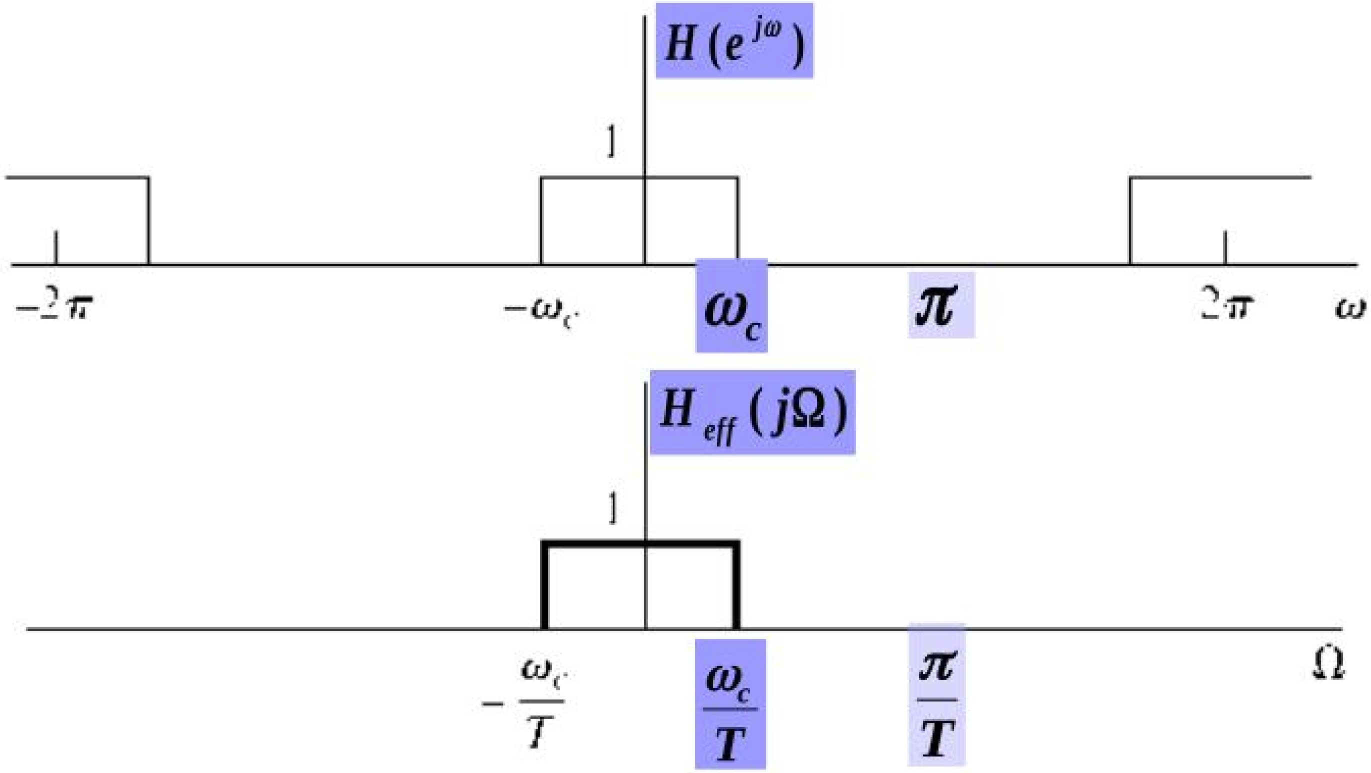
$$
H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}
$$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & |\Omega| \geq \pi / T \end{cases}$$

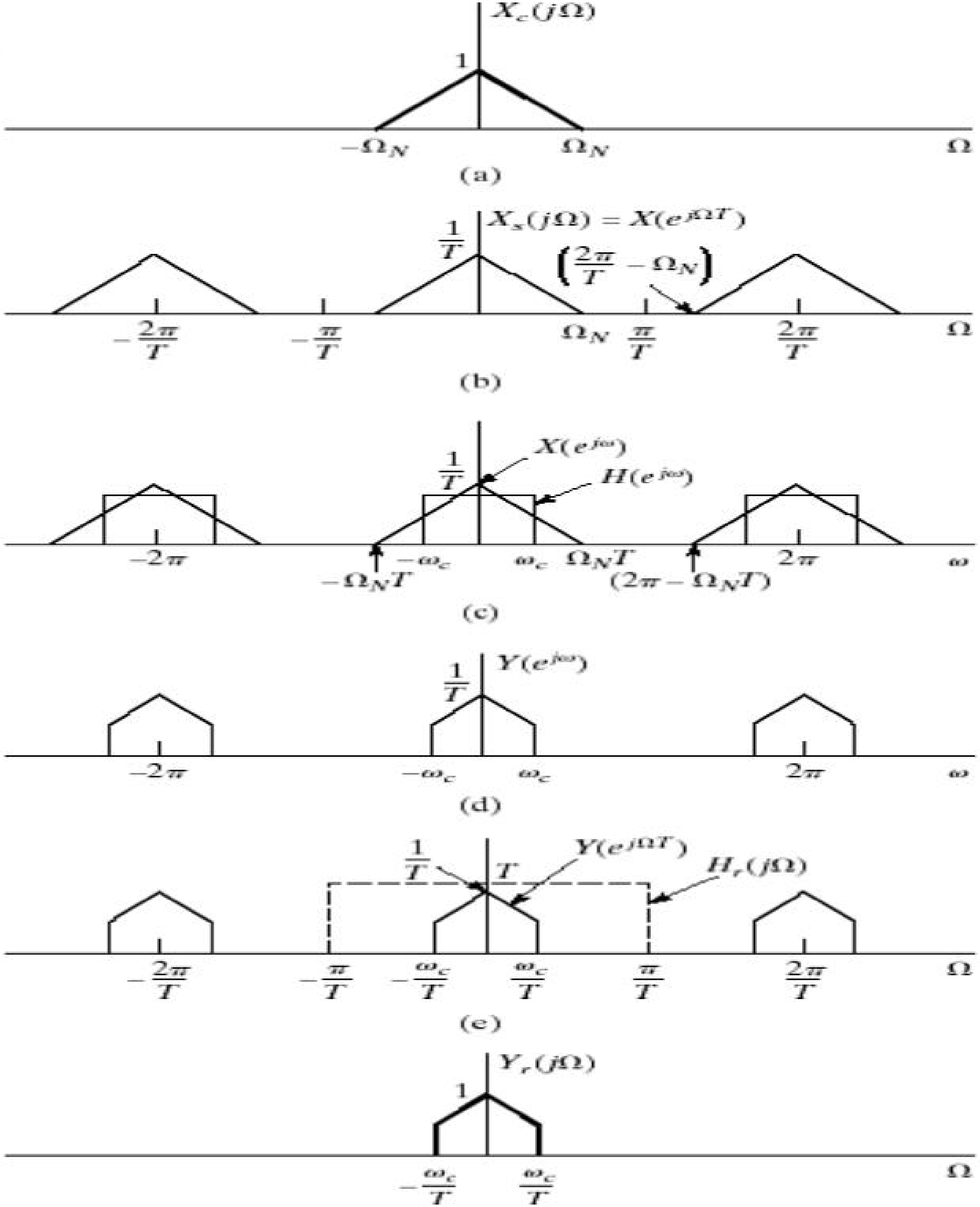
conditions : LTI ;

no aliasing or aliasing occurred outside the pass band of filters

e.g.

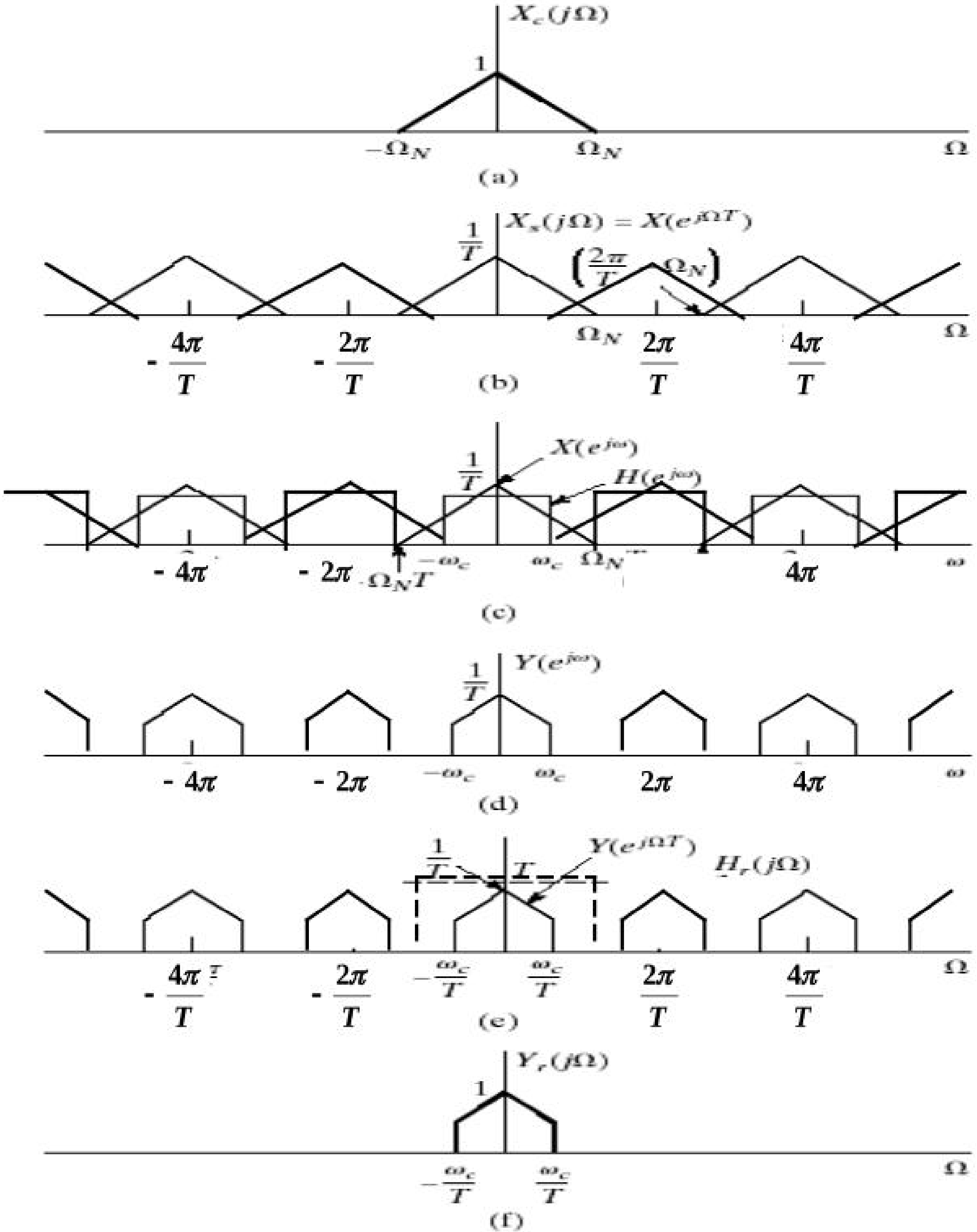


e.g.



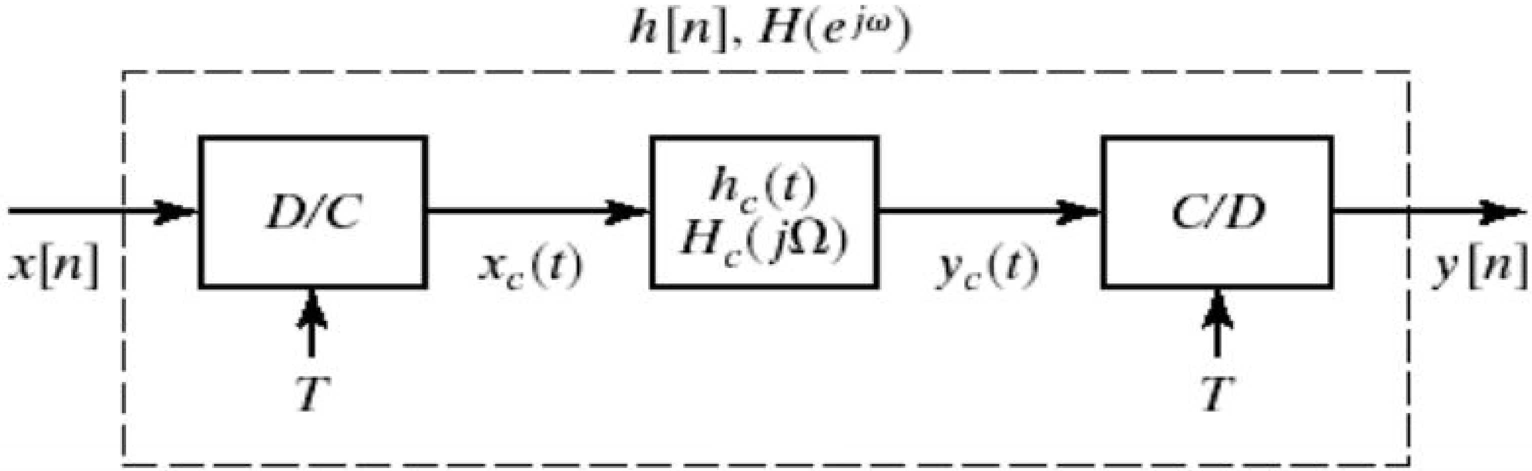
$X_c(j\Omega)$
1
$-\Omega_N$
Ω_N
Ω
(a)
$X_s(j\Omega) = X(e^{j\Omega T})$
$\frac{1}{T}$
$\left(\frac{2\pi}{T} - \Omega_N\right)$
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
Ω_N
$\frac{\pi}{T}$
$\frac{2\pi}{T}$
Ω
(b)
$X(e^{j\omega})$
$H(e^{j\omega})$
$\frac{1}{T}$
-2π
$-\omega_c$
ω_c
$\Omega_N T$
2π
ω
$-\Omega_N T$
$(2\pi - \Omega_N T)$
(c)
$Y(e^{j\omega})$
$\frac{1}{T}$
-2π
$-\omega_c$
ω_c
2π
ω
(d)
$\frac{1}{T}$
T
$Y(e^{j\Omega T})$
$H_r(j\Omega)$
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
$-\frac{\omega_c}{T}$
$\frac{\omega_c}{T}$
$\frac{\pi}{T}$
$\frac{2\pi}{T}$
Ω
(e)
$Y_r(j\Omega)$
1
$-\frac{\omega_c}{T}$
$\frac{\omega_c}{T}$
Ω

e.g.



Aliasing occurred outside the pass band of digital filters satisfies the equivalent relation of frequency response mentioned before.

4.5 Continuous-Time Processing of Discrete-Time Signal



$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

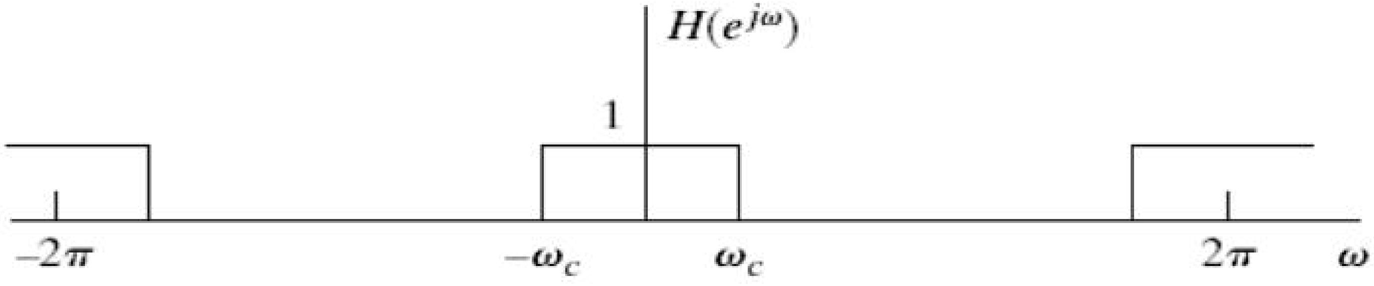
$$X_c(j\Omega) = TX(e^{j\Omega T}) \qquad |\Omega| < \frac{\pi}{T}$$

$$Y_c(j\Omega) = X_c(j\Omega)H_c(j\Omega) \qquad |\Omega| < \frac{\pi}{T}$$

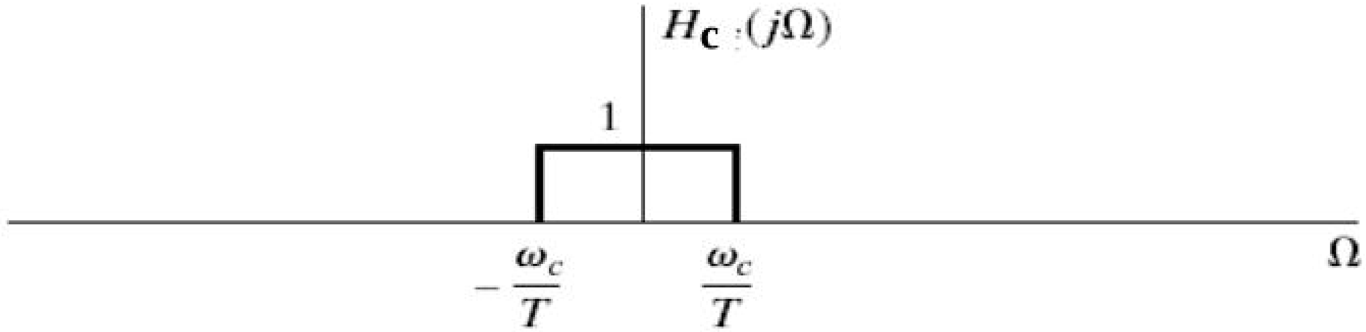
$$Y(e^{j\omega}) = \frac{1}{T} Y_c(\frac{j\omega}{T}) \qquad |\omega| < \pi$$

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad for\ |\omega| < \pi$$

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad for\ |\omega| < \pi$$



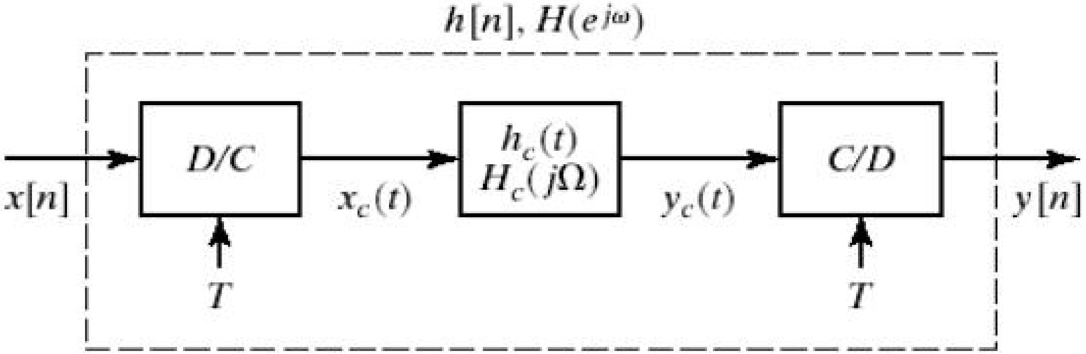
(a)



(b)

e.g. noninteger delay

$$H(e^{j\omega}) = e^{-j\omega\Delta}, \quad |\omega| < \pi$$

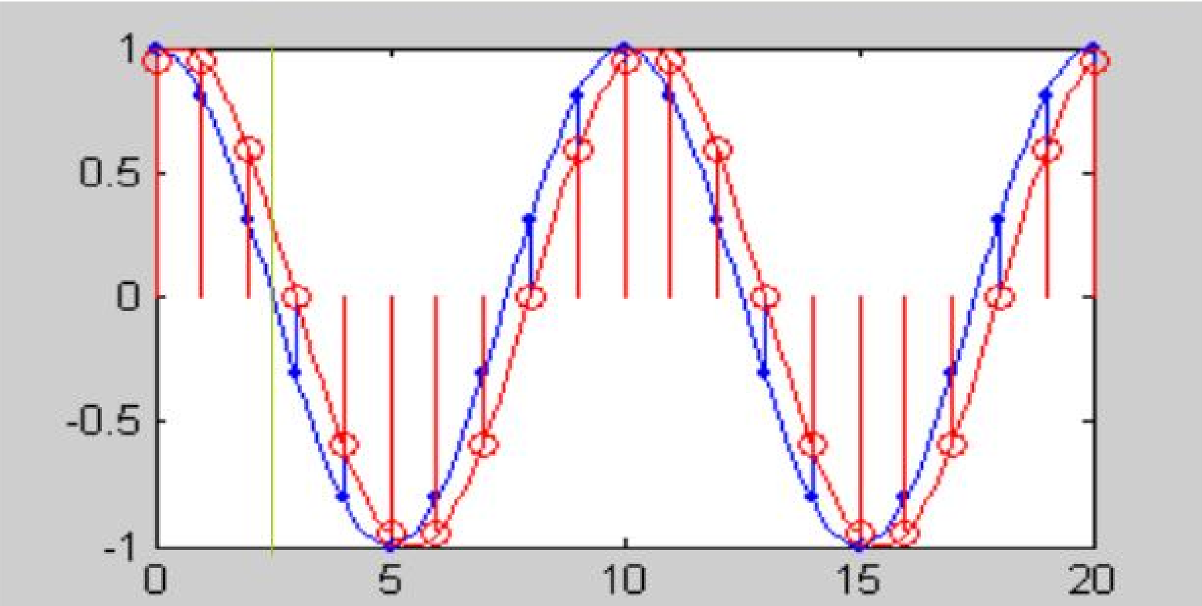


$$H_c(j\Omega) = H(e^{j\Omega T}) = e^{-j\Omega\Delta T}$$

$$y_c(t) = x_c(t - \Delta T)$$

$$y[n] = y_c(nT) = x_c(nT - \Delta T)$$

$$h[n] = \frac{\sin\pi(n-\Delta)}{\pi(n-\Delta)}, \quad -\infty < n < \infty$$



4.6 Changing The Sampling Rate Using Discrete-time

1. Sampling rate reduction by an integer factor
(downsampling, decimation)

2. Increasing the sampling rate by an integer factor
(upsampling, interpolation)

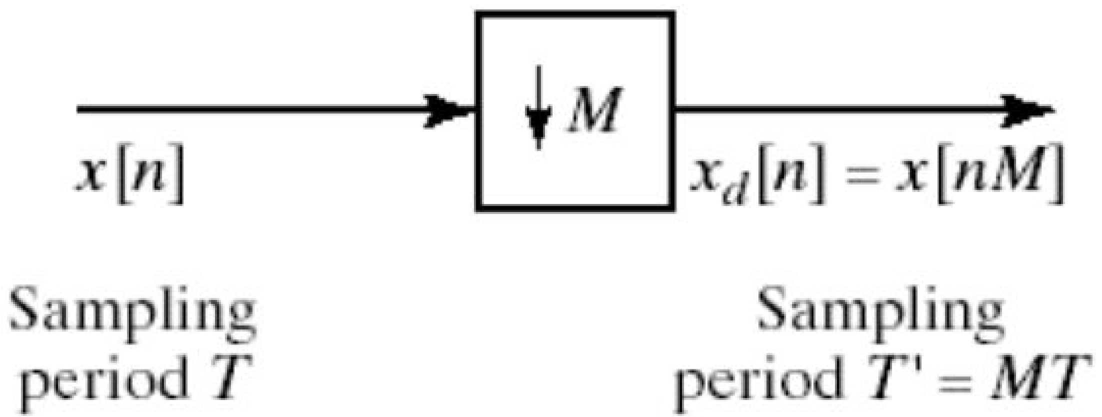
3. Changing the sampling rate by a noninteger factor

4.6.1 Sampling rate reduction by an integer factor

(downsampling, decimation)

$$x_d[n] = x[nM]$$

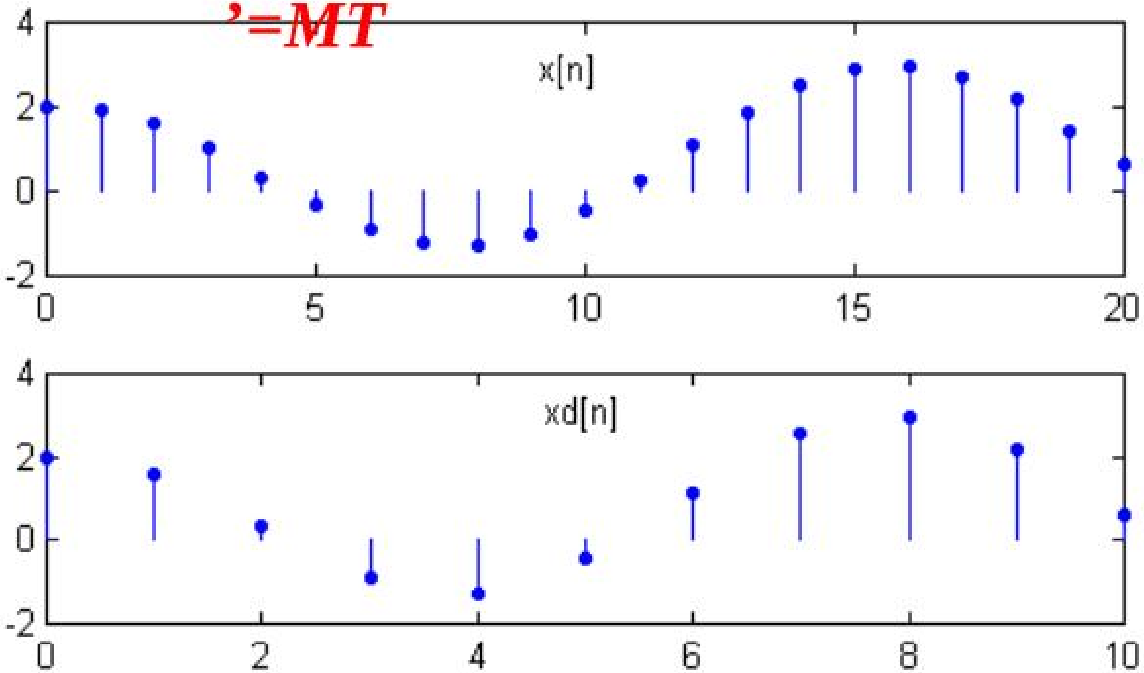
a sampling rate compressor :



time-domain of downsampling ：

decrease the data ， reduce the sampling rate

M=2 ， f_s'=f_s /M, T'=MT

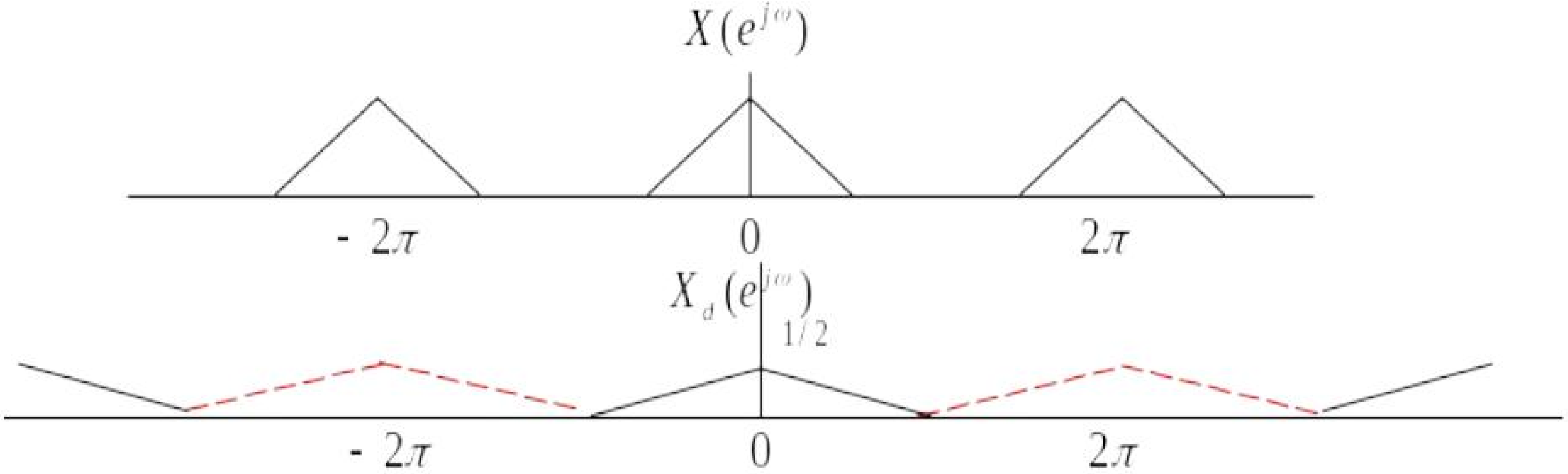


frequency-domain of downsampling ：

take aliasing into consideration

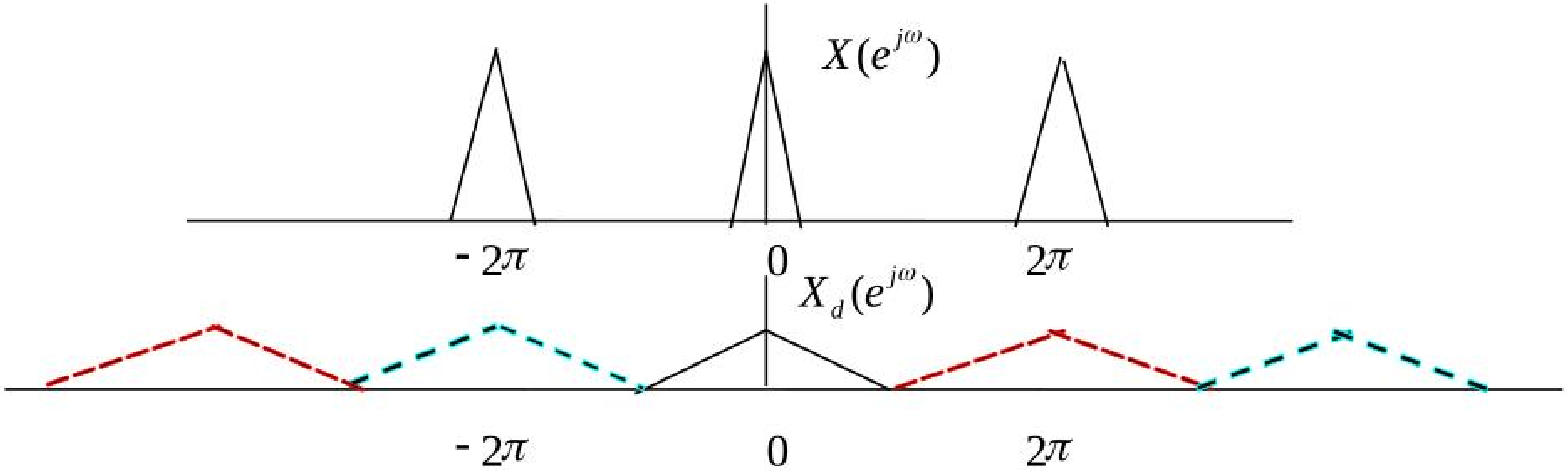
$$X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega-2\pi i)/M}\right)$$

e.g. $M=2$ $\quad X_d(e^{j\omega}) = \frac{1}{2}\left[X\left(e^{j\omega/2}\right) + X\left(e^{j(\omega-2\pi)/2)}\right)\right]$

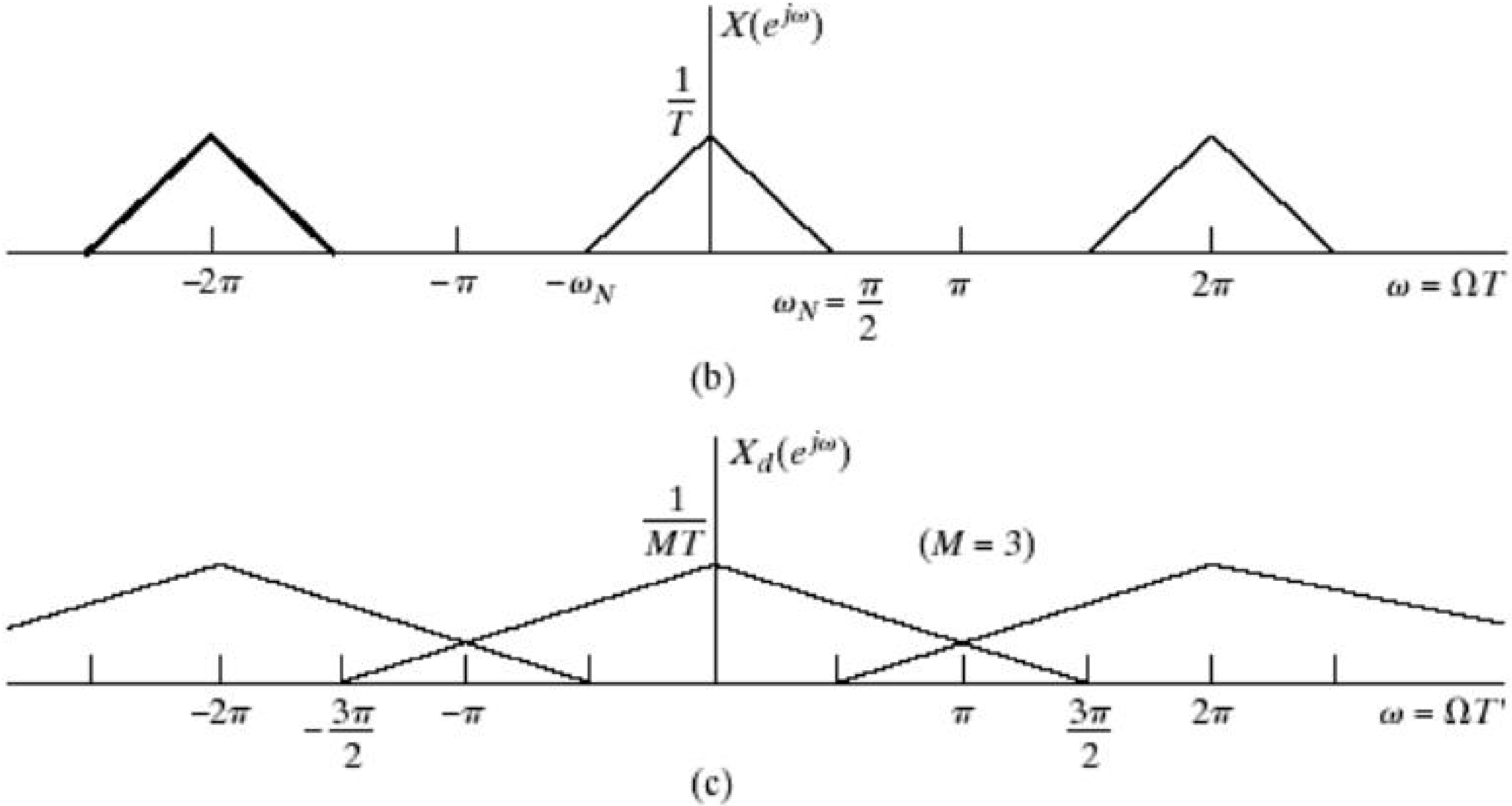


e.g. $M=3$

$$X_d(e^{j\omega}) = \frac{1}{3}\left[X\left(e^{j\omega/3}\right) + X\left(e^{j(\omega-2\pi)/3)}\right) + X\left(e^{j(\omega-4\pi)/3)}\right)\right]$$



e.g. $M=3$, aliasing



frequency spectrum after decimation :

period=2π , *M* times wider , 1/*M* times higher

4.6.2 Increasing the sampling rate by an integer factor

(upsampling, interpolation)

$$x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L \cdots \\ 0 & other \end{cases}$$

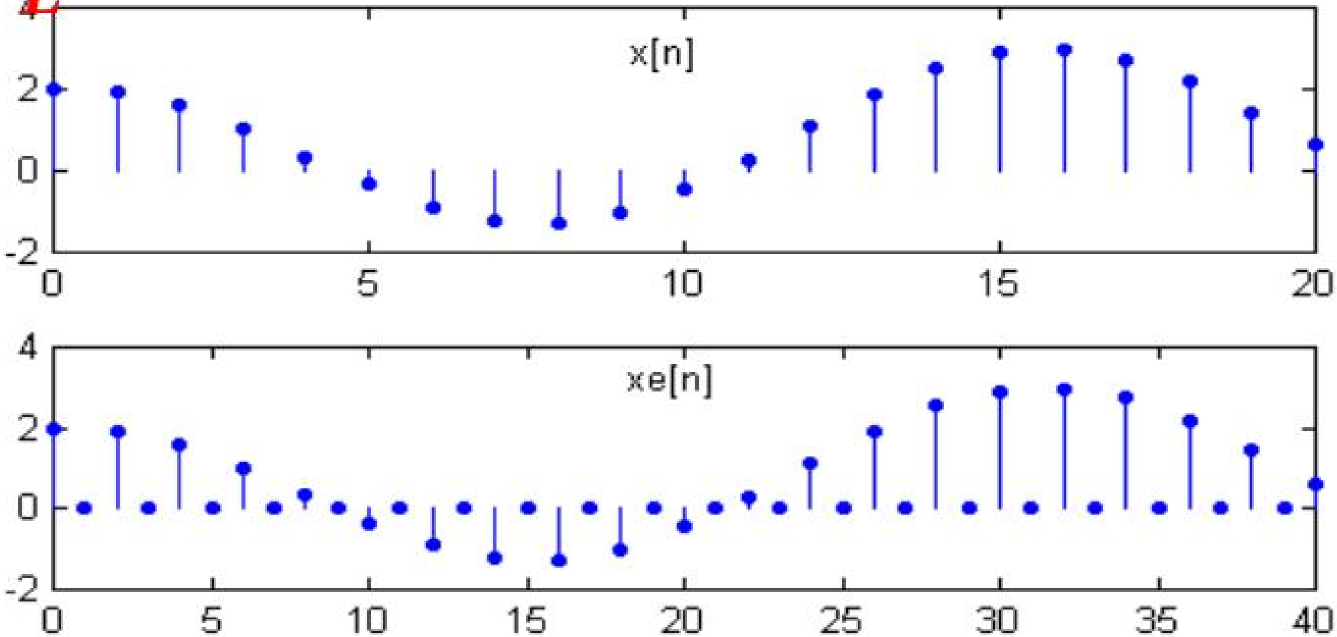
$$or,\ x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]$$

a sampling rate **exp*ander*** **:**

$$x[n] \longrightarrow \boxed{\uparrow\ L} \longrightarrow x_e[n]$$

Time-domain of upsampling ：
increase the data ， raise the sampling rate

$L=2$ ， $f_s' = Lf_s$, $T' = T/L$

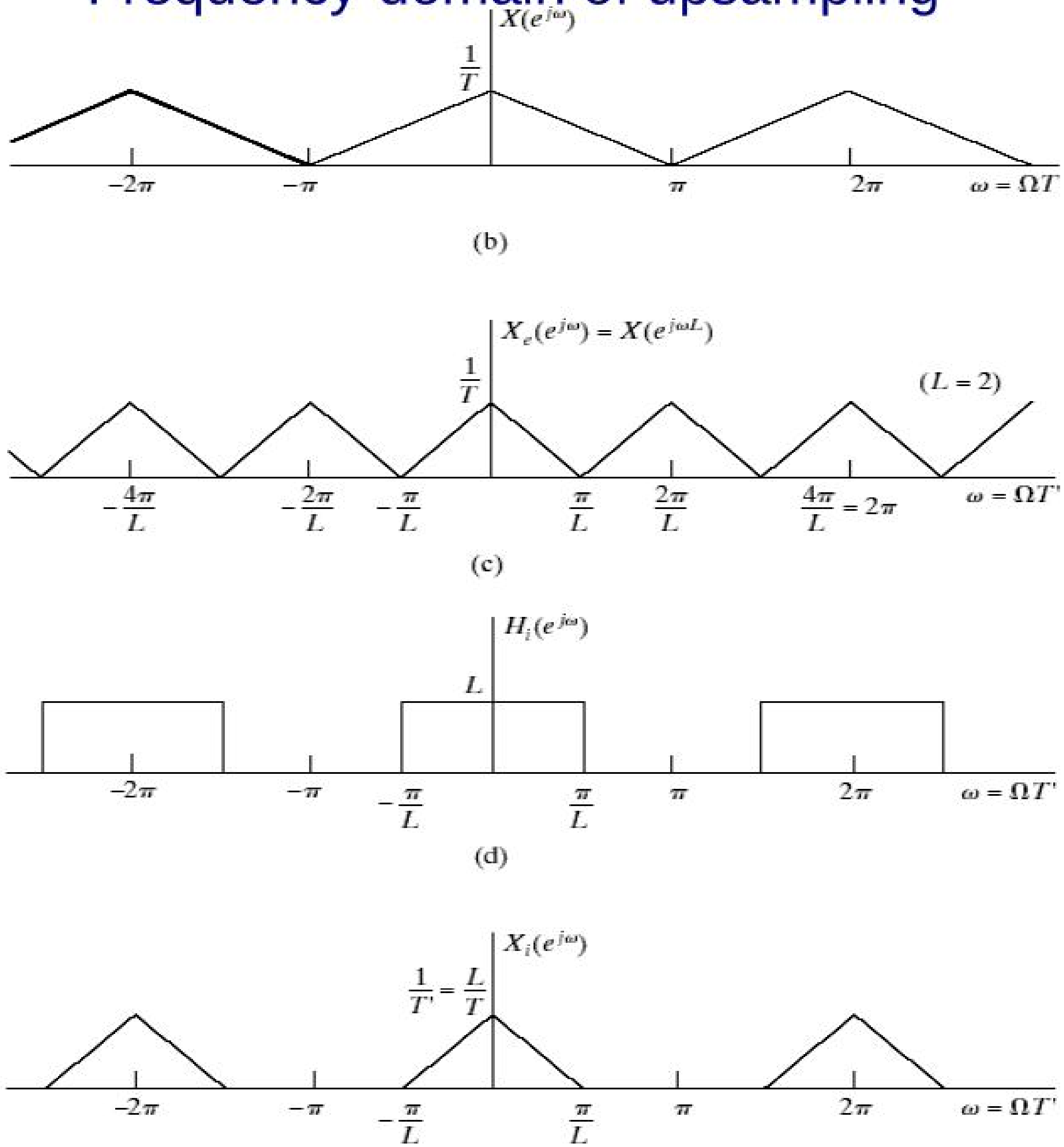


Frequency-domain of upsampling： need not take aliasing into consideration

$$X_e(e^{j\omega}) = X(e^{jL\omega})$$

Frequency-domain of upsampling

e.g. $L=2$

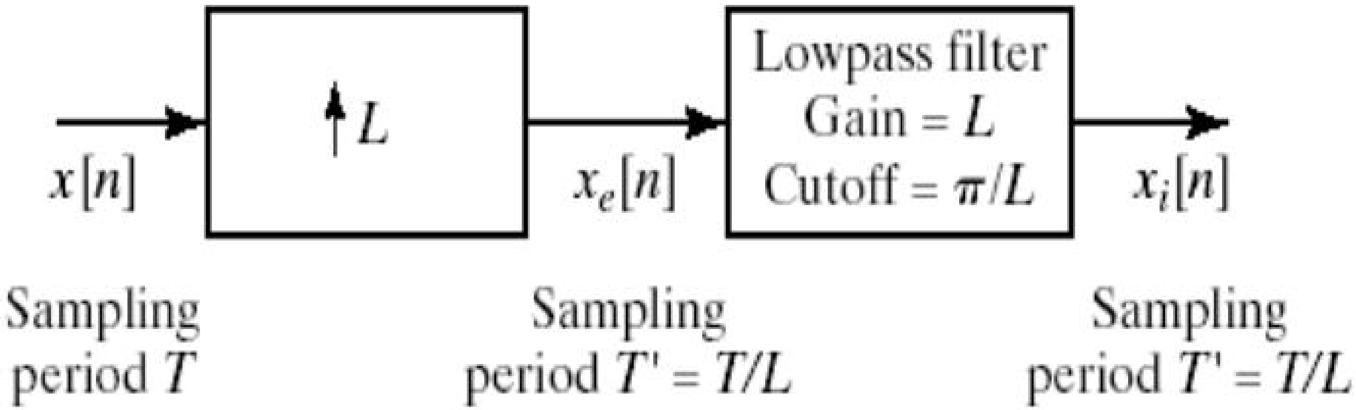


Transverse axis is 1/*L* timer shorter.

Magnitude has no change.

Period=2π , also period =2 π /*L*

total upsampling system : total system



time-domain explanation of reverse mirror-image filter :

slowly-changed signal by interpolation

$$h_i[n] = IFT[H(e^{j\omega})] = \frac{\sin(\pi n / L)}{\pi n / L}$$

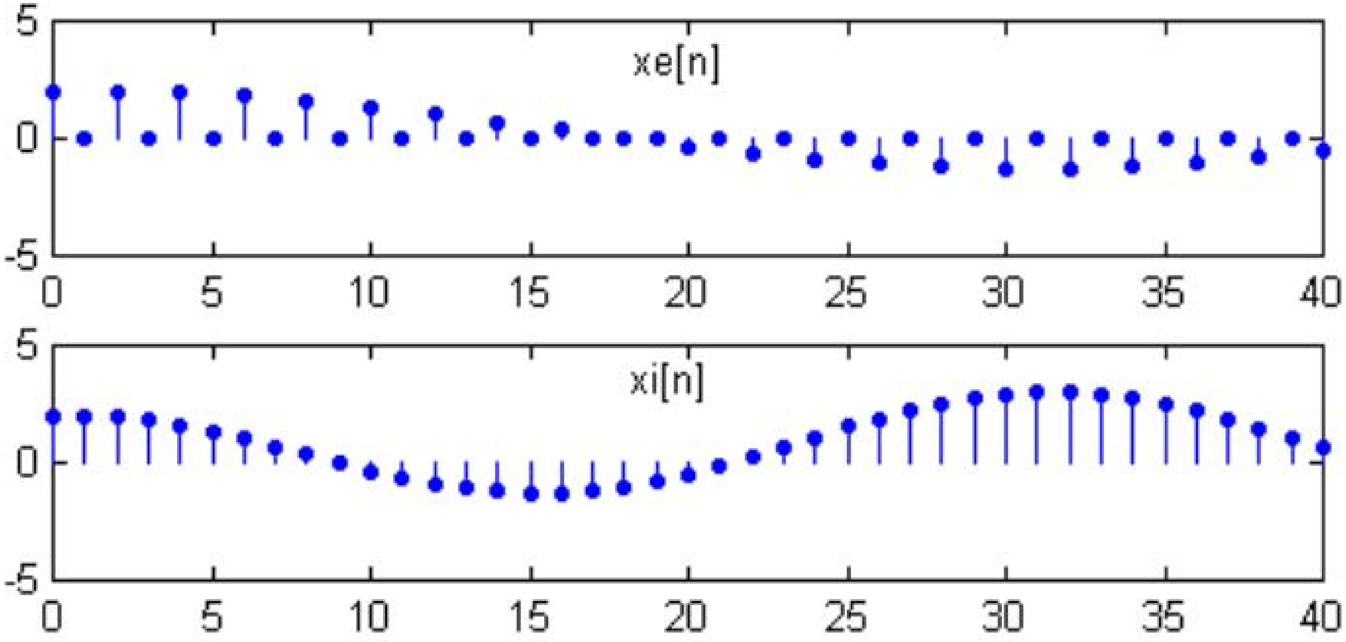
$$x_i[n] = x_e[n] * h_i[n] = \left(\sum_{k=-\infty}^{\infty} x[k]\delta[n - kL]\right) * h_i[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k](\delta[n - kL] * h_i[n])$$

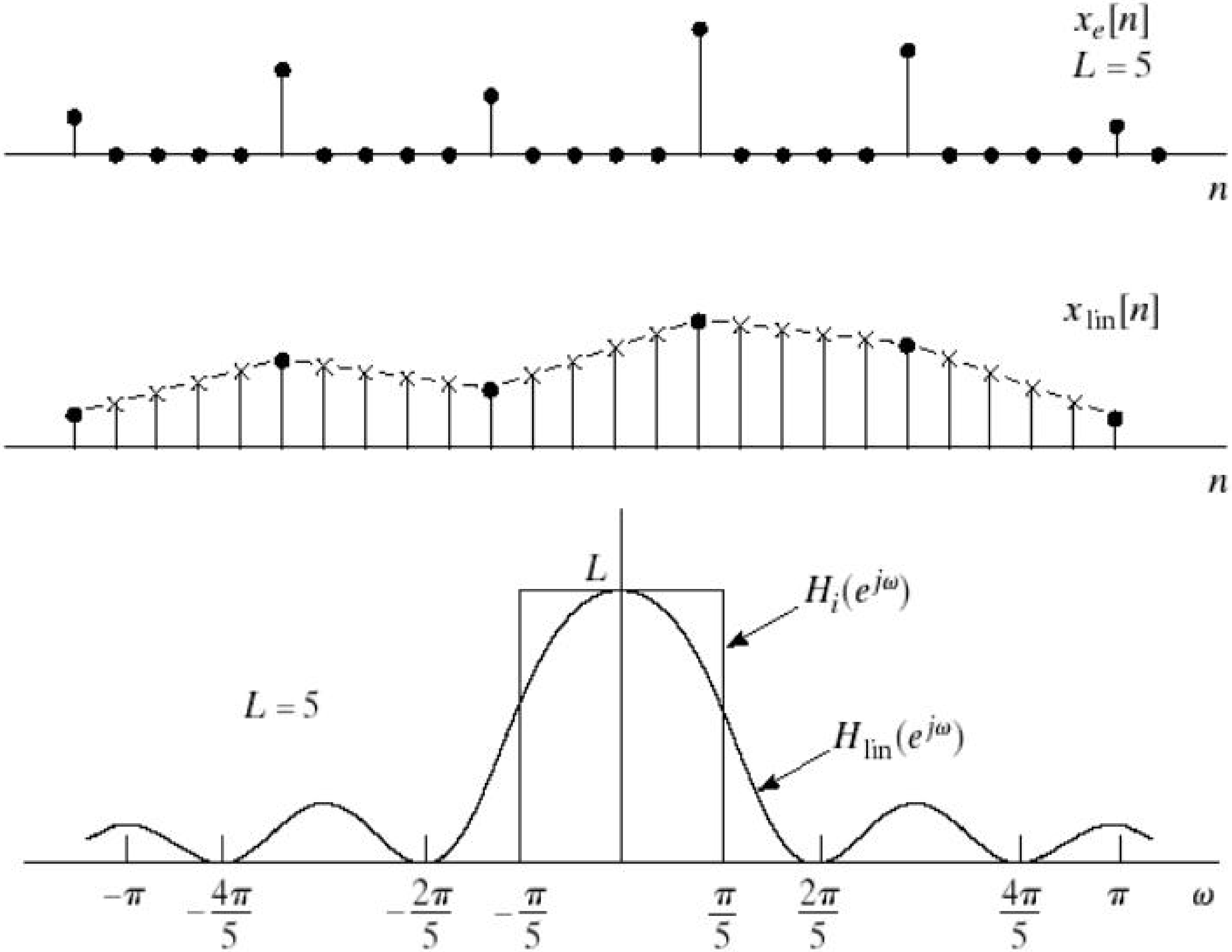
$$= \sum_{k=-\infty}^{\infty} x[k]h_i[n - kL] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin\left(\frac{\pi(n - kL)}{L}\right)}{\frac{\pi(n - kL)}{L}}$$

e.g.

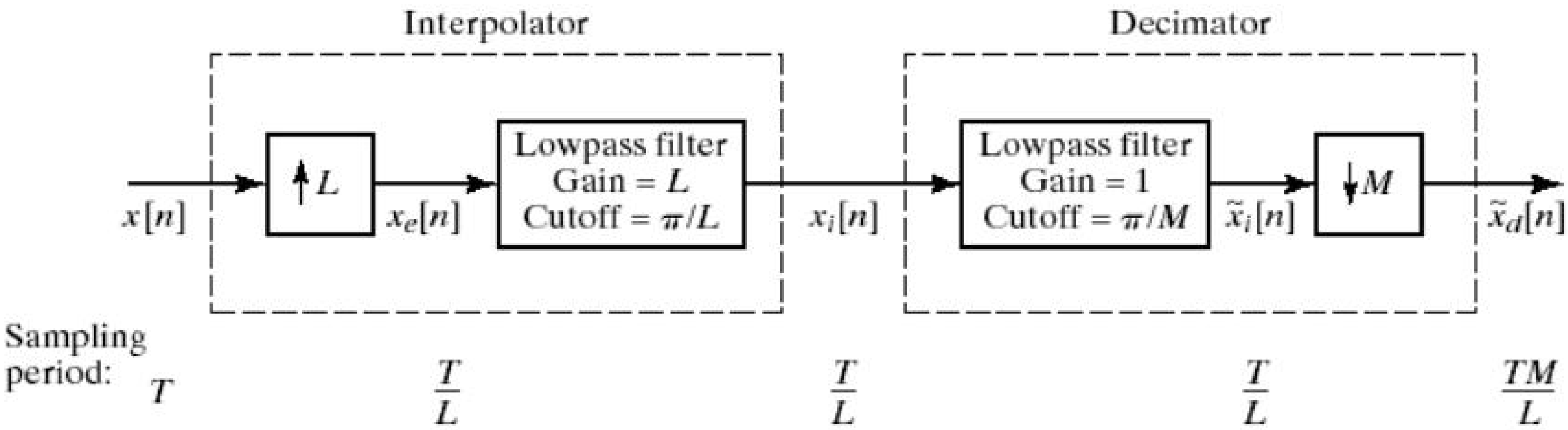
Time-domain process of mirror-image filter



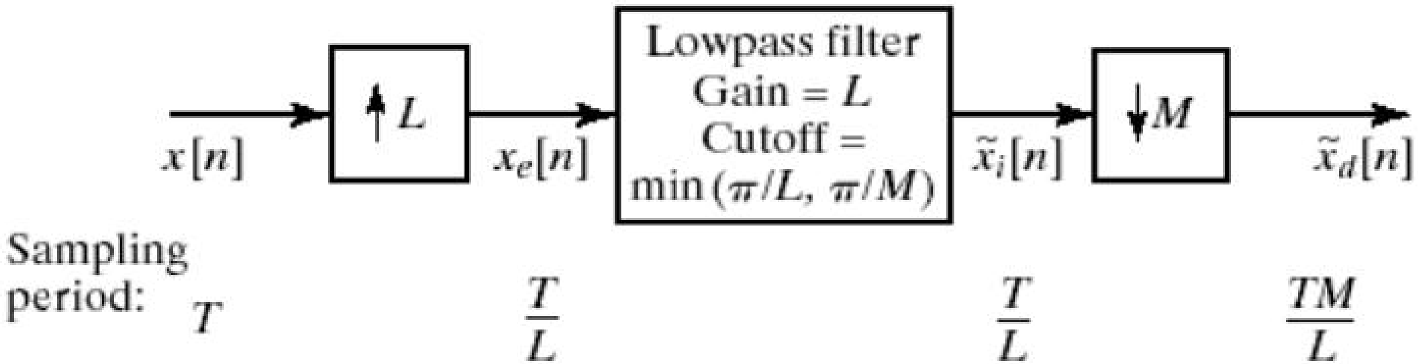
Use linear interpolation actually



4.6.3 Changing the sampling rate by a noninteger factor

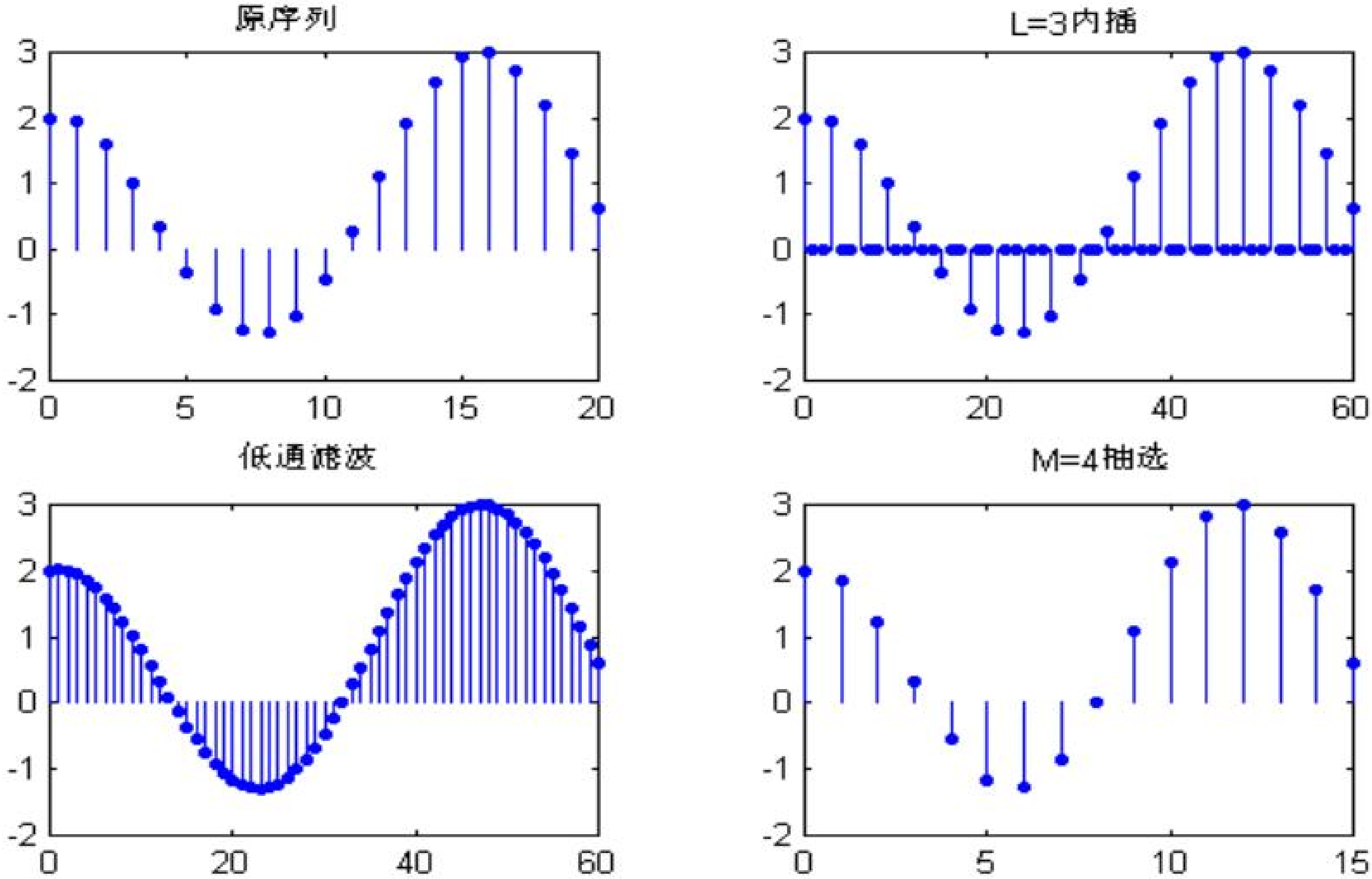


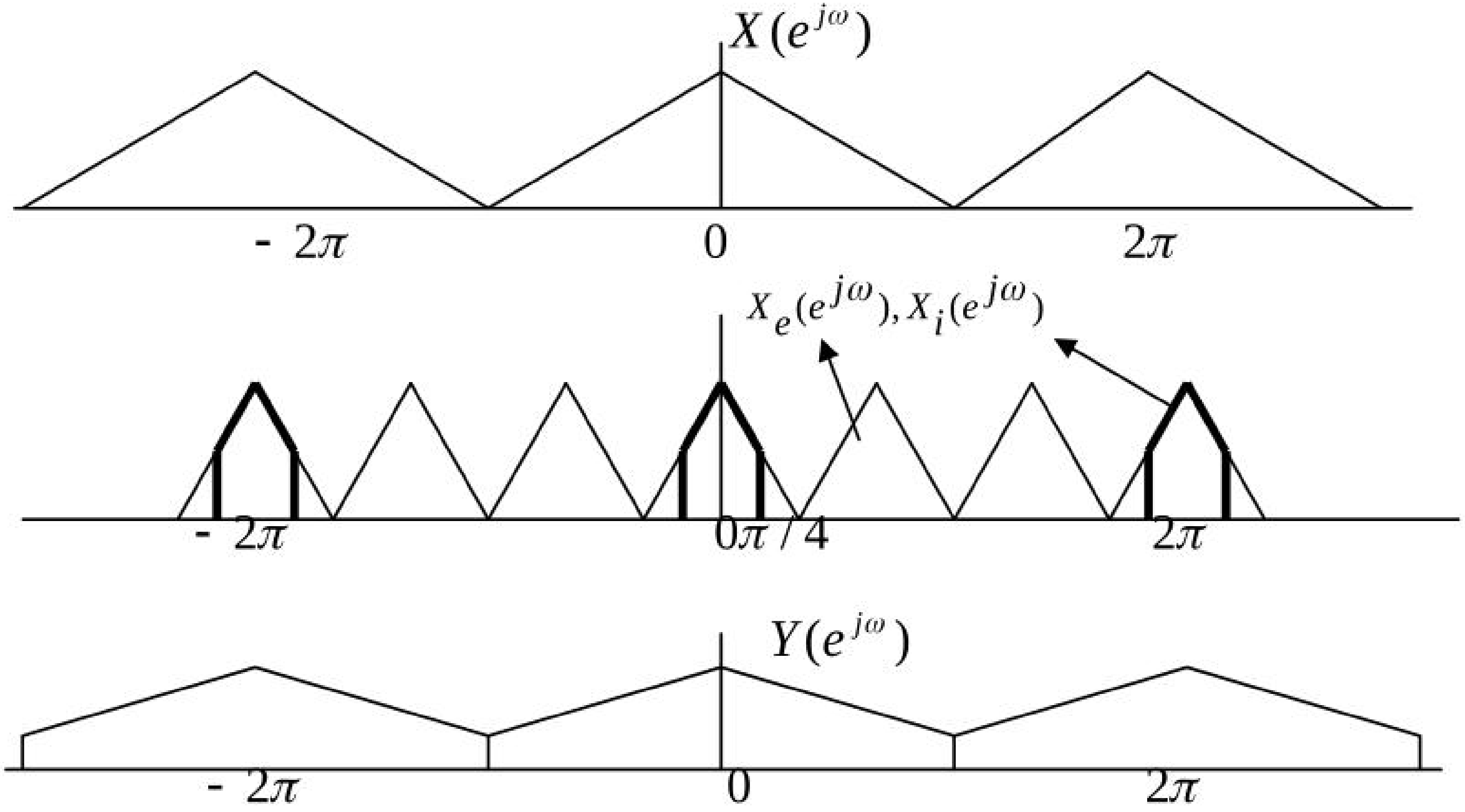
$$f_s' = f_s \frac{L}{M}$$



e.g. $change \quad 400Hz's \quad signal \quad to \quad 300Hz$

$L=3, M=4$



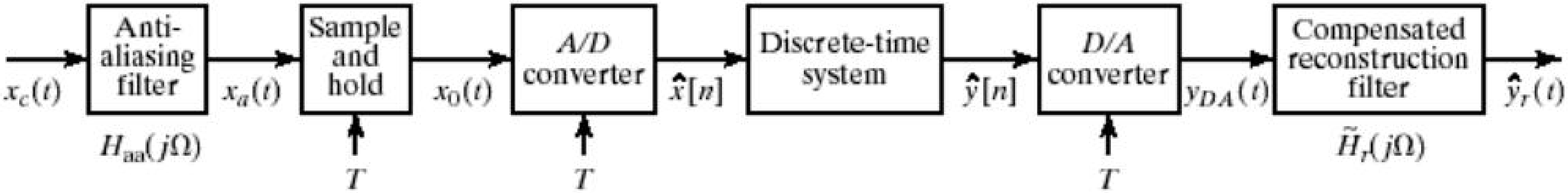


Advantages of decimation after interpolation：

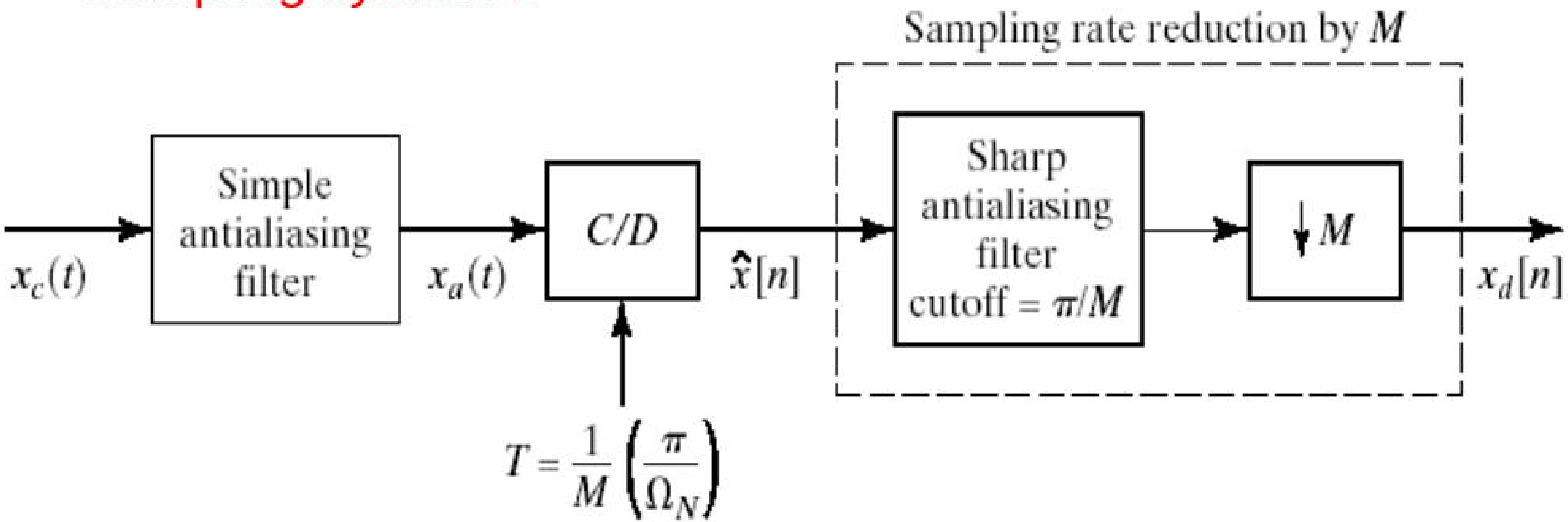
1.Combine antialiasing and reverse mirror-image filter

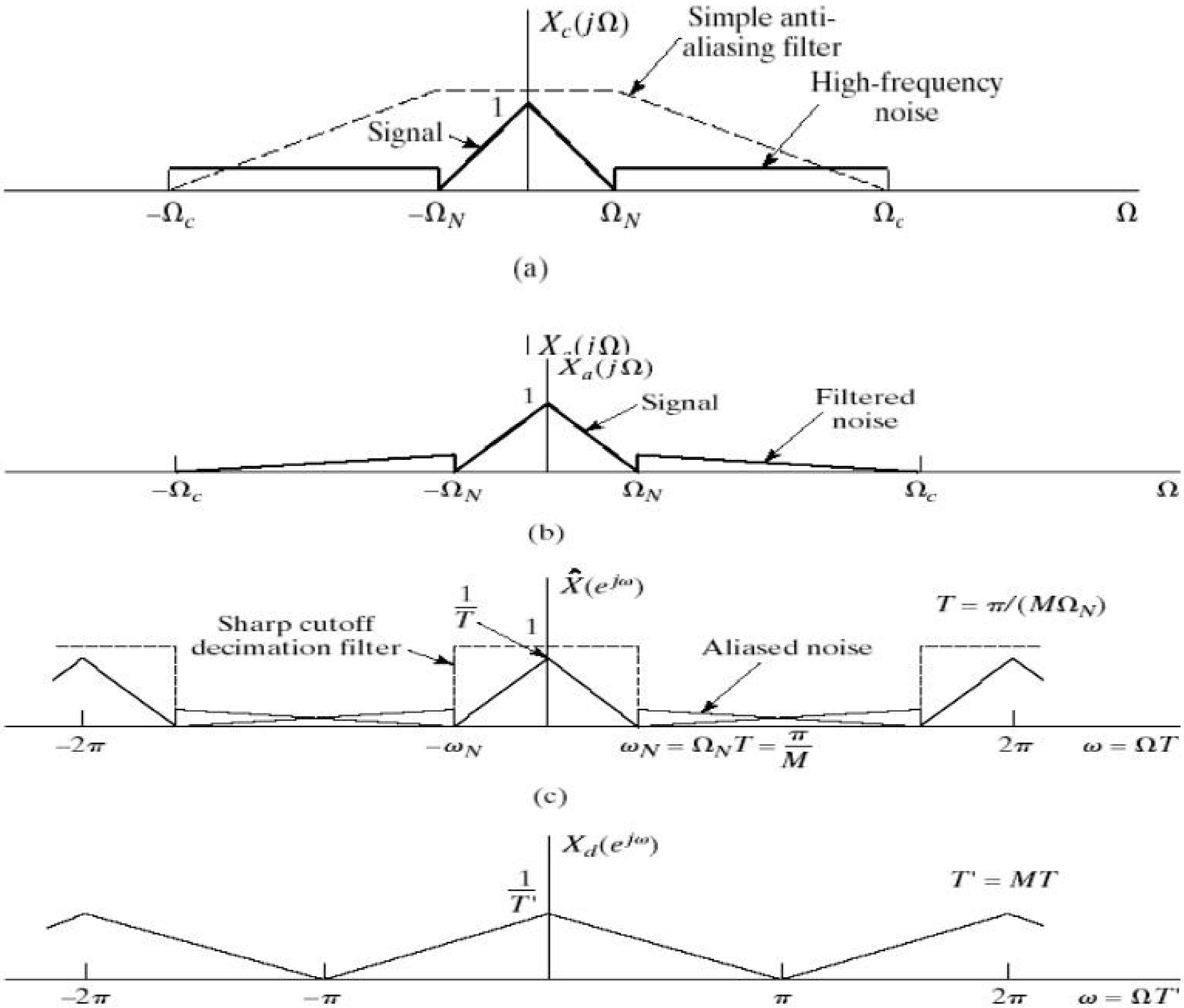
2.Lossless information for upsampling

Application of multi-rate signal processing



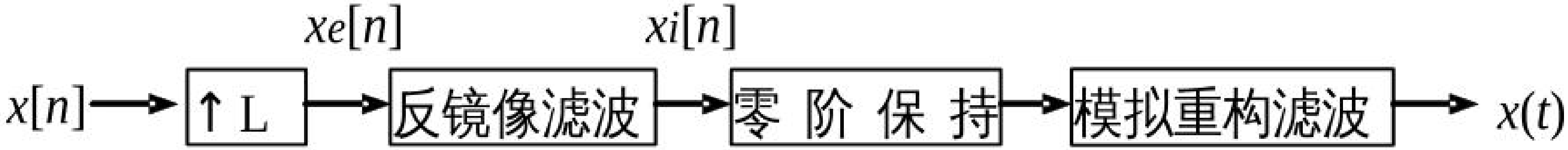
Sampling system：

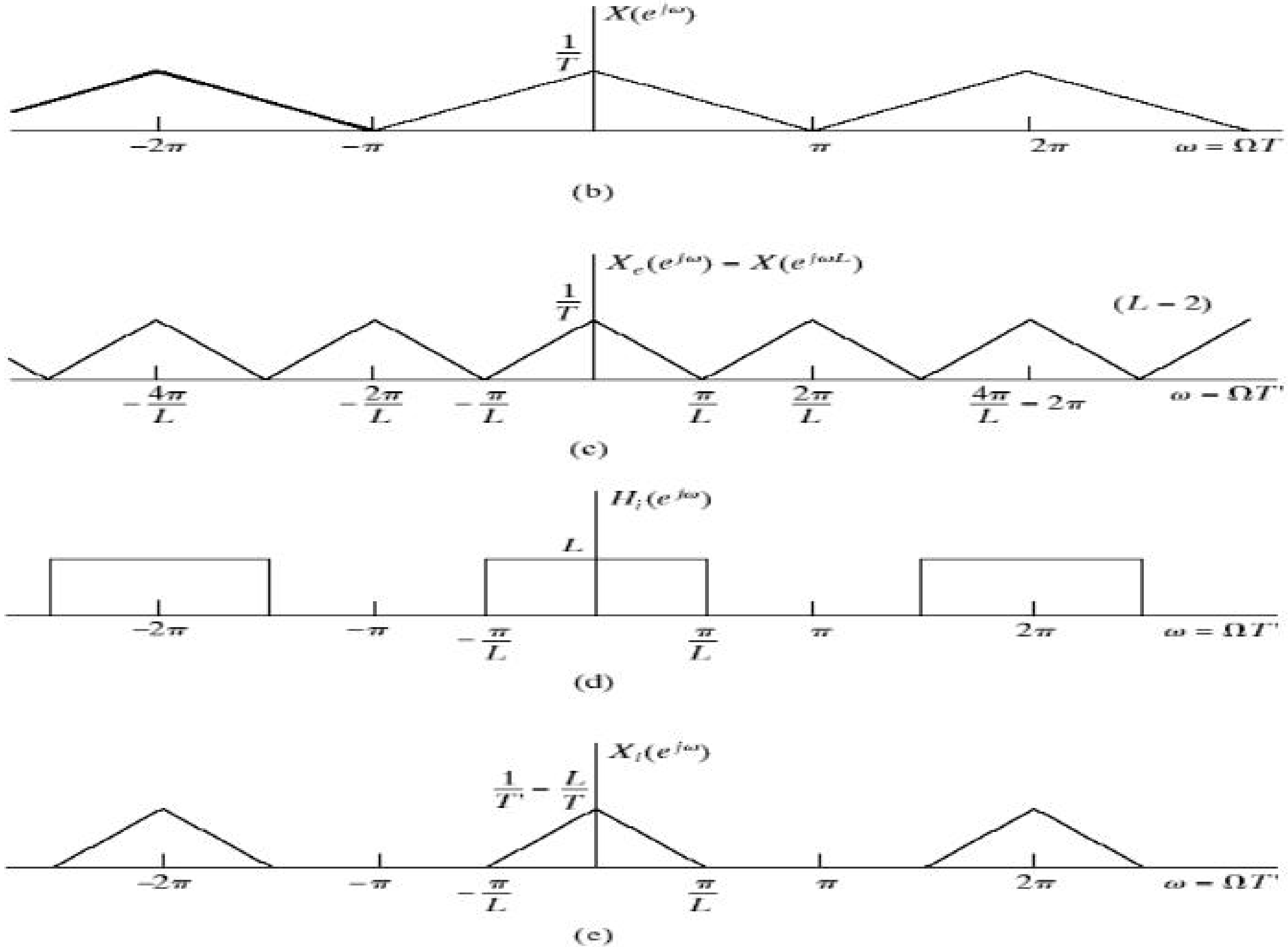




$X_c(j\Omega)$
Simple anti-aliasing filter
High-frequency noise
1
Signal
$-\Omega_c$
$-\Omega_N$
Ω_N
Ω_c
Ω
(a)
$X_a(j\Omega)$
1
Signal
Filtered noise
$-\Omega_c$
$-\Omega_N$
Ω_N
Ω_c
Ω
(b)
$\hat{X}(e^{j\omega})$
$T = \pi/(M\Omega_N)$
$\frac{1}{T}$
1
Sharp cutoff decimation filter
Aliased noise
-2π
$-\omega_N$
$\omega_N = \Omega_N T = \frac{\pi}{M}$
2π
$\omega = \Omega T$
(c)
$X_d(e^{j\omega})$
$\frac{1}{T'}$
$T' = MT$
-2π
$-\pi$
π
2π
$\omega = \Omega T'$

Reconstruction system：





Requirements and difficulties：

- sampling processing in time and frequency domain ， frequency spectrum chart;
- comprehension and application of sampling theorem;
- frequency response in discrete-time processing system of continuous-time signals;

summary

1. Representation in time domain and changes in frequency domain of sampling and reconstruction.
2. Sampling theorem educed from aliasing in frequency domain.
3. Analog signal processing in digital system or digital signal in analog system , to explain some digital systems , their frequency responses are linear in dominant period

Exercises

第二版	第三版
4.2	4.2
4.15	4.15
4.17	4.17
4.24	4.30