Signals and Systems (Sem. I / 2014-2015)

Tutorial Solutions

Solutions for Chapter 1

Answer to P 1.1:

Key: $c = x + jy = \rho e^{j\theta}$ with

$$\rho = \sqrt{x^2 + y^2}, \quad tan\theta = \frac{y}{x};$$

$$x = \rho\ cos\theta, \quad y = \rho\ sin\theta$$

- θ is not unique as $e^{j\theta} = e^{j(\theta+2\pi m)}$ for any integer m. Usually, $|\theta| \leq \pi$ is assumed.

- $-5 = 5e^{j\pm\pi} \ \Rightarrow \ \theta = \pm\pi$, $5 = 5e^{j0} \ \Rightarrow \ \theta = 0$; Also,

$$1 + j = \sqrt{2}e^{j\pi/4}, \ 1 - j = \sqrt{2}e^{-j\pi/4}$$

$$1 + j = -(1 - j) = e^{j\pi}\sqrt{2}e^{-j\pi/4} = 1 + j = \sqrt{2}e^{j3\pi/4}$$

End

Answer to P 1.2: Keep the definitions in mind !

$$\textit{Energy} \qquad\qquad \textit{Power}:$$

$$E_x(T) \triangleq \int_{-T}^{T} |x(t)|^2 dt \;\Rightarrow\; E_x = \lim_{T\to\ +\infty} E_x(T); \;\; P_x = \lim_{T\to+\infty} \frac{E_x(T)}{2T}$$

$$E_x[N] \triangleq \sum_{n=-N}^{N} |x[n]|^2 \;\Rightarrow\; E_x = \lim_{N\to\ +\infty} E_x[N]; \;\; P_x = \lim_{N\to+\infty} \frac{E_x[N]}{2N+1}$$

- (a) For $x_1(t) = e^{-2t}$, one has

$$E_{x_1}(T) \triangleq \int_{-T}^{T} |x_1(t)|^2 dt = \int_0^T e^{-4t} dt = \frac{1}{-4} e^{-4t}|_0^T = \frac{1}{4}[1 - e^{-4T}]$$

So, $E_x = \lim_{T\to+\infty} = 1/4$, and hence it is an energy signal. Clearly, $P_x = 0$.

For $x_2[n] = (0.75e^{j\theta})^{|n|}$, one has

$$
\begin{aligned}
E_{x_2}[N] &\triangleq \sum_{n=-N}^{N} |x_2[n]|^2 = \sum_{n=-N}^{N} |(0.75e^{j\theta})^{|n|}|^2 = \sum_{n=-N}^{N} 0.75^{2|n|} \\
&= \sum_{n=-N}^{-1} 0.75^{-2n} + \sum_{n=0}^{N} 0.75^{2n} = \sum_{m=0}^{N} 0.75^{2m} - 1 + \sum_{n=0}^{N} 0.75^{2n} \\
&= 2 \times \frac{1 - 0.75^{2(N+1)}}{1 - 0.75^2} - 1
\end{aligned}
$$

So, $E_x = \lim_{N\to+\infty} = \frac{2}{1-0.75^2} - 1 = \frac{25}{7}$ - an energy signal. Clearly, $P_x = 0$.

- (b) Let $x(t) = x(t+T_0),\ \forall\ t$, then with $T = MT_0$

$$
\begin{aligned}
E_x(T) &\triangleq \int_{-T}^{T} |x_1(t)|^2 dt = \int_{-MT_0}^{-(M-1)T_0} |x(t)|^2 dt + \cdots \\
&+ \int_{-T_0}^{0} |x(t)|^2 dt + \int_{0}^{T_0} |x(t)|^2 dt + \cdots + \int_{(M-1)T_0}^{MT_0} |x(t)|^2 dt
\end{aligned}
$$

It follows from $x(t) = x(t+kT_0)$ and then $\int_{(k-1)T_0}^{kT_0} |x(t)|^2 dt = \int_0^{T_0} |x(t)|^2 dt$

that $E_x = \lim_{T\to+\infty} E_x(T) = \lim_{M\to+\infty} 2M \int_0^{T_0} |x(t)|^2 dt \quad \to \quad +\infty$, and $P_x = \lim_{T\to+\infty} \frac{E_x(T)}{2T} = \lim_{M\to+\infty} \frac{2M \int_0^{T_0} |x(t)|^2 dt}{2MT_0} = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$ - a power signal.

- (c) Let $E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt$. Now, with $y(t) = x(\kappa t)$

$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2 dt = \int_{-\infty}^{+\infty} |x(\kappa t)|^2 dt$$

Define $\tau = \kappa t$, then $dt = \kappa^{-1} d\tau$ and then

$$E_y = \kappa^{-1} \int_{-\infty}^{+\infty} |x(\tau)|^2 d\tau = \kappa^{-1} E_x$$

As required, $E_y = 1$. This means

$$\kappa = E_x$$

End

Answer to P 1.3: As given, the system I/O is $y[n] = nx[n] + \alpha$.

- *Linearity*

  Let $x_k[n] \;\to\; y_k[n] = nx_k[n] + \alpha$, $k = 1, 2$. Clearly, $x[n] = \beta_1 x_1[n] + \beta_2 x_2[n] \;\to\; y[n] = nx[n] + \alpha$, i.e., $y[n] = n\{\beta_1 x_1[n] + \beta_2 x_2[n]\} + \alpha$. So,

$$\begin{aligned} y[n] &= \beta_1\{nx_1[n] + \alpha\} + \beta_2\{nx_2[n] + \alpha\} + \alpha[1 - (\beta_1 + \beta_2)] \\ &= \beta_1 y_1[n] + \beta_2 y_2[n] + \alpha + \alpha[1 - (\beta_1 + \beta_2)] \end{aligned}$$

  Generally, $y[n] \neq \beta_1 y_1[n] + \beta_2 y_2[n]$ as β_1, β_2 are all arbitrary. So, the system is NOT linear unless $\alpha = 0$.

- *Time-invariance*

  Let $x[n] \;\to y[n] = nx[n] + \alpha$. So, $\hat{x}[n] = x[n - n_0] \;\to\; \hat{y}[n] = n\hat{x}[n] + \alpha = nx[n - n_0] + \alpha$ for any n_0 given. Knowing $y[n - n_0] = (n - n_0)x[n - n_0] + \alpha$, we realize that $\hat{y}[n] \neq y[n - n_0]$, which implies that the system is NOT time-invariant.

- *Causality*

  Clearly, the system IS causal as $y[n_0]$ has nothing to do with $x[n]$ for $n > n_0$, where n_0 is any given integer.

- *Stability*

  Let $x[n] = 1$, which is bounded. Note $|y[n]| = |x[n] + \alpha| \geq |n| - |\alpha|$ can be bigger than any given number. This means that $y[n]$ is unbounded and hence the system is UNSTABLE.

End

Answer to P 1.4: Given that $x[n] = cos(\omega_0 n)$, if it is periodic, there should exists N such that

$$x[n] = x[n+N],\ \forall\ n \quad \Leftrightarrow \quad cos(\omega_0 n) = cos(\omega_0 n + \omega_0 N), \quad \forall\ n$$

which implies that there should exist some integer K such that

$$\omega_0 N = 2\pi K \quad \Rightarrow \quad N = \frac{2\pi}{\omega_0} K$$

Keeping that N is integer in mind, the above is impossible for the case when $\frac{\pi}{\omega_0}$ is NOT rational!

Examples: 1) $\omega_0 = 0.3\pi \quad \Rightarrow \quad \frac{\pi}{\omega_0} = 10/3 \quad \Rightarrow \quad N = 20K/3$. So, $K = 3, 6, ..., 3k, ..$ and hence the signal is periodic; 2) when $\omega_0 = 0.3$, there exists no integer K and hence no N, meaning the signal is NOT periodic.

End

Answer to P 1.5 and **Answer tp P 1.8**: To be given on the class on board.

Answer to P 1.6:

- (a) Let $y_1(t) \stackrel{\triangle}{=} x(4 - t/2)$. With $x(t)$ given, one notes $y_1(t) = x(-\frac{1}{2}(t-8))$. Let $\tilde{y}_1(t) = x(-\frac{1}{2}t)$, then $y_1(t) = \tilde{y}_1(t-8)$. Therefore, $\tilde{y}_1(t)$ is obtained by applying a time reversal to $x(t)$ then followed by a time scaling of 1/2. Time-shifting $\tilde{y}_1(t)$ by 8 yields $y_1(t)$.

- $y_2(t) = [x(t) + x(-t)]u(t)$ to be sketch on board.

- Noting

$$
\begin{aligned}
y_3(t) &= x(t)[\delta(t + 3/2) - \delta(t - 3/2)] \\
&= x(-3/2)\delta(t + 3/2) - x(3/2)\delta(t - 3/2)
\end{aligned}
$$

one has $y_2(t) = -0.5\delta(t + 3/2) - 0.5\delta(t - 3/2)$.

Sketch it on board. **End**

Answer to P 1.7:

- Let $y_1[n] \stackrel{\triangle}{=} x[3n+1]$. Observing $x[n]$, one can see $y_1[n] = 0$ for those integer valued n: $3n + 1 < -4 \;\Leftrightarrow\; n < -1 \quad and \quad 3n + 1 > 3 \;\Leftrightarrow\; n \geq 1$

  So, there are two points: $n = -1, 0$, for which $y_1[n]$ may not be zero.

  Knowing $y_1[-1] = x[-2] = 1/2, y_1[0] = x[1] = 1$, we then have

$$y_1[n] = 1/2\delta[n + 1] + \delta[n]$$

- Let $y_2[n] \stackrel{\triangle}{=} x[n]u[3 - n]$. Denote $w[n] = u[1 - n] = u[-(n - 1)]$, obtained by time reversal on $u[n]$, then time-shifting by 1.

  $y_2[n]$ is the signal obtained by the product of $x[n]$ and $w[n]$ point by point.

$$y_2[n] = -\delta[n + 4] - 0.5\delta[n + 3] + 0.5\delta[n + 2] + \delta[n + 1] + \delta[n] + \delta[n - 1]$$

End

Answer to P 1.10:

Noting $w_\tau(t)$ is an even function, i.e., $w_\tau(t) = w_\tau(-t)$, one has $w_\tau(\alpha t) = w_\tau(|\alpha| t)$.

Sketching $w_\tau(|\alpha| t)$, one realizes $w_\tau(|\alpha| t) = w_{\tau/|\alpha|}(t)$.

According to the definition,

$$\delta(\alpha t) = \lim_{\tau \to 0} \frac{w_\tau(\alpha t)}{\tau} = \lim_{\tau \to 0} \frac{w_{\tau/|\alpha|}(t)}{\tau} = \lim_{\tau \to 0} |\alpha|^{-1} \frac{w_{\tau/|\alpha|}(t)}{\tau/|\alpha|}$$

that is

$$\delta(\alpha t) = |\alpha|^{-1} \lim_{\tilde{\tau} \to 0} \frac{w_{\tilde{\tau}}(t)}{\tilde{\tau}} = |\alpha|^{-1} \delta(t)$$

End

Answer to P 1.12:

Key: To memorize and understand the definitions.

- Memoryless: $y(t)$ $(y[n])$ depends only on $x(t)$ $(x[n])$ for all t (n).
- Time-invariant: Let $x(t) \rightarrow y(t)$. Compute the output $\tilde{y}(t)$ when the input is $\tilde{x}(t) = x(t - t_0)$. Calculate $y(t - t_0)$ and check if $\tilde{y}(t)$ is equal to $y(t - t_0)$.
- Linear: Let $x_k \rightarrow y_k, \; k = 1, 2$. Compute the output y when the input is $x = \alpha_1 x_1 + \alpha_2 x_2$. Check if y is equal to $\alpha_1 y_1 + \alpha_2 y_2$.
- Causal: Check if $y[n_0]$ depends *ONLY* on the values of $x[n]$ for $n \leq n_0$ for all n_0 and $x[n]$. If yes, it is causal.
- Stable: For any $|x| < M_x$, check if y is bounded.

See **Problem 1.3**.

Examples

- $x(t) \;\rightarrow\; y_1(t) = x(t-2) + x(2-t)$.

  It is easy to see that it is NOT memoryless, Non-causal, stable, and linear, but is it TI?

  Let $\tilde{x}(t) = x(t-t_0)$, then $\tilde{y}(t) = \tilde{x}(t-2) + \tilde{x}(2-t)$. Noting that

  $$\tilde{x}(t) = x(t-t_0) \;\Rightarrow\; \tilde{x}(t-2) = x((t-2)-t_0), \;\; \tilde{x}(2-t) = x((2-t)-t_0)$$

  we have

  $$\tilde{y}(t) = x((t-2)-t_0) + x((2-t)-t_0) = x((t-t_0)-2) + x(2-(t+t_0))$$

  Since $y_1(t) = x(t-2) + x(2-t)$,

  $$y_1(t-t_0) = x((t-t_0)-2) + x(2-(t-t_0))$$

  Clearly, $\tilde{y}_1(t) \neq y_1(t-t_0)$. Therefore, it is NOT time-invariant!

- $x(t) \ \rightarrow \ y_3(t) = \frac{dx(t)}{dt}$.

It is easy to see that it is time-invariant, unstable $(u(t) \ \rightarrow \ \delta(t))$ and linear, but is it causal and memoryless?

$$y_3(t) = \lim_{\Delta t \ \rightarrow \ 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

So, it is Not memoryless because it depends on $x(t + \Delta t)$ when $|\Delta t| \neq 0$ very small, and it is NOT causal since it depends on $x(t + \Delta t)$, where Δt can be positive.

Both cases are *Marginal*!

End

Answer to P 1.13: As seen, $y_1(t)$ is the output of the LTI system excited by $x_1(t) = u(t) - u(t-2)$:

$$x_1(t) \quad \rightarrow \quad y_1(t)$$

- Note that $x_2(t)$ can be expressed in terms of $x_1(t)$:

$$x_2(t) = x_1(t) - x_1(t-2) \quad \rightarrow \quad y_2(t) = \quad ??$$

If $q(t) \stackrel{\triangle}{=} x_1(t-2) \quad \rightarrow \quad p(t)$, *linearity* implies

$$y_2(t) = y_1(t) - p(t)$$

$y_1(t)$ has been given, what about $p(t)$?

Time invariance suggests $p(t) = y_1(t-2)$ and consequently,

$$y_2(t) = y_1(t) - y_1(t-2)$$

- Observing carefully, one can see that

$$x_3(t) = x_1(t+1) + x_1(t)$$

and immediately

$$y_3(t) = y_1(t+1) + y_1(t)$$

With $y_1(t)$ given, one can sketch $y_2(t)$ and $y_3(t)$ obtained above with little difficulty.

End

Answer to P 1.14:

- For the system: $x(t) \;\rightarrow\; y(t) = x(\alpha t)$, we have

$$\tilde{x}(t) \triangleq x(t - t_0) \;\rightarrow\; \tilde{y}(t) = \tilde{x}(\alpha t)$$

  On the one hand, $\tilde{x}(t) \triangleq x(t-t_0) \;\Rightarrow\; \tilde{x}(\alpha t) = x(\alpha t - t_0)$, $\tilde{y}(t) = x(\alpha t - t_0)$.

  On the other hand, $y(t-t_0) = x(\alpha t - \alpha t_0)$. Clearly, $\tilde{y}(t) \neq y(t-t_0)$, which implies that the system is NOT time-invariant, unless $\alpha = 1$.

  Let $x_k(t) \;\rightarrow\; y_k(t) = x_k(\alpha t),\; k = 1, 2.$ Noting that

$$\begin{aligned} x(t) = \beta_1 x_1(t) + \beta_2 x_2(t) \;\;\rightarrow\;\; y(t) = x(\alpha t) &= \beta_1 x_1(\alpha t) + \beta_2 x_2(\alpha t) \\ &= \beta_1 y_1(t) + \beta_2 y_2(t) \end{aligned}$$

  we conclude that the system is LINEAR.

- As given, the system $x(t) \;\to\; y(t)$ is TI, that is $x(t+t_0) \;\to\; y(t+t_0)$ for ANY t_0.

  Particularly, for the period T_0 we have

$$x(t+T_0) \;\to\; y(t+T_0)$$

  Knowing $x(t) = x(t+T_0) \;\to\; y(t)$, one concludes

$$y(t) = y(t+T_0) \quad \Rightarrow \quad y(t) \;\; is \;\; periodic$$

  and the period, denoted as $\tilde{T}_0$, according to definition, should not be larger than T_0.

End

Answer to P 1.16: Given that

$$x_p(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_p), \quad T_p > 0$$

Since

$$x_p(t + T_p) = \sum_{k=-\infty}^{+\infty} x_0((t + T_p) - kT_p) = \sum_{k=-\infty}^{+\infty} x_0(t - (k-1)T_p)$$

With $m = k - 1$, one has

$$x_p(t + T_p) = \sum_{m=-\infty}^{+\infty} x_0(t - mT_p) = x_p(t), \quad \forall \ t$$

This tells us that $x_p(t)$ is periodic and the period is not bigger than T_p.

With $x_0(t)$ given, sketch $x_p(t)$ on board for different T_p.

Answer to P 1.17: Based on the diagram given, $e[n] = x[n] - y[n]$. As $y[n] = 0.75e[n-1]$, one finally has

$$y[n] = 0.75x[n-1] - 0.75y[n-1], \quad y[n] = 0, \ \ \forall\, n < 0$$

- When $x[n] = \delta[n]$, $y[0] = 0.75\delta[-1] - 0.75y[-1] = 0, y[1] = 0.75\delta[0] - 0.75y[0] = 0.75$, and $y[n] = -0.75y[n-1], \quad \forall\, n \geq 2$. Clearly,

$$y[n] = 0.75(-0.75)^{n-1}u[n-1]$$

- When $x[n] = u[n]$, $y[0] = 0.75u[-1] - 0.75y[-1] = 0$, and

$$y[n] = 0.75 - 0.75y[n-1]), \quad \forall\, n \geq 1$$

Clearly, $y[1] = 0.75, \ y[2] = 0.75 - 0.75^2, \cdots,$. A closed-form expression

$$y[n] = \frac{0.75}{1.75}\,[0.75(-0.75)^{n-1} + 1]u[n-1]$$

can be obtained easily using the technique in Chapter 2. **End**

Solutions for Chapter 2

Answer to P 2.1: By a graphical approach, one can see that

$$\begin{cases} 0, & n < 0 \\ n+1, & 0 \geq n, n-N \geq 0 \;\Rightarrow\; 0 \leq n \leq N \\ N+1, & n-N > 0, n \leq 9 \;\Rightarrow\; N < n \leq 9 \\ 9-(n-N)+1, & 9 < n, n-N \leq 9 \;\Rightarrow\; 9 < n \leq 9+N \\ 0, & n-N > 9 \;\Rightarrow\; n > 9+N \end{cases}$$

Now,

$$y[4] = 5 \quad \Rightarrow \quad n = 4 \; belongs \; to \quad 0 \leq n \leq N \quad \Rightarrow \quad 4 \leq N$$

$$y[14] = 0 \quad \Rightarrow \quad n = 14 \; belongs \; to \quad n > 9+N \quad \Rightarrow \quad 14 > 9+N$$

Therefore,

$$N = 4$$

Answer to P 2.2: Given

$$x(t) = (t+1)w_1(t-1/2) + (2-t)w_1(t-3/2), \quad h(t) = \delta(t+2) + 2\delta(t+1)$$

Consider $x(t)$ as the unit impulse response of an LTI system: $\delta(t) \;\to\; x(t)$ and $h(t)$ is the input. So, one has

$$y(t) = x(t) * h(t) = h(t) * x(t) = x(t+2) + 2x(t+1)$$

Sketch $x(t)$, then $2x(t+1), x(t+2)$. $y(t)$ can be obtained by adding the two one-by-point.

End

Answer to P 2.3: Given $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$.

- Direct approach

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{3}^{5} 1\ e^{-3(t-\tau)}u(t-\tau)d\tau \\
&= \begin{cases} 0, & t < 3 \\ \int_3^t 1\ e^{-3(t-\tau)}d\tau, & 3 < t < 5 \\ \quad \int_3^5 1\ e^{-3(t-\tau)}d\tau, & 5 < t \end{cases}
\end{aligned}
$$

Therefore,

$$
y(t) = \frac{1}{3}[1 - e^{-3(t-3)}]\ u(t-3) - \frac{1}{3}[1 - e^{-3(t-5)}]\ u(t-5)
$$

- As $\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$,

$$
\begin{aligned}
g(t) &\stackrel{\triangle}{=} (\frac{dx(t)}{dt}) * h(t) = h(t-3) - h(t-5) \\
&= e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5)
\end{aligned}
$$

- Clearly, $\frac{dy(t)}{dt} = g(t)$. So,

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{t} g(\tau)d\tau + y(-\infty) \\
&= \int_{-\infty}^{t} e^{-3(\tau-3)}u(\tau-3)d\tau - \int_{-\infty}^{t} e^{-3(\tau-5)}u(\tau-5)d\tau \\
&= \int_{3}^{t} e^{-3(\tau-3)}d\tau u(t-3) - \int_{5}^{t} e^{-3(\tau-5)}d\tau u(t-5) = ...
\end{aligned}
$$

End

Answer to P 2.6 Given three sub LTI systems with UIRs

$$h_1(t) = tu(t), \quad h_2(t) = \delta(t-1), \quad h_3(t) = -\delta(t)$$

- How to design a system with the three s.t the system has an UIR below?

$$h(t) = -(t-1)u(t-1)$$

By observation, $h(t) = -h_1(t-1) = h_1(t) * h_2(t) * h_3(t)$, which is a cascade of the three sub-systems.

- How to design a system with the three s.t the system has an UIR below?

$$h(t) = tu(t) - \delta(t-1) - (t-1)u(t-1)$$

By observation, $h(t) = h_1(t) - h_2(t) + h_1(t) * h_2(t) * h_3(t)$.

Draw the block-diagrams on board.

Answer to P 2.7: Note

$$x(t) = u(t-1) - u(t-2), \quad y(t) = (t-2)[u(t-2) - u(t-3)] + u(t-3)$$

and hence

$$\frac{dx(t)}{dt} = \delta(t-1) - \delta(t-2), \quad \frac{dy(t)}{dt} = u(t-2) - u(t-3)$$

Since

$$y(t) = x(t) * h(t) \quad \Rightarrow \quad \frac{dy(t)}{dt} = (\frac{dx(t)}{dt}) * h(t)$$

one has

$$u(t-2) - u(t-3) = h(t-1) - h(t-2)$$

By observation,

$$h(t) = u(t-1)$$

Answer to P 2.10: Given that $y[n] = \sum_{k=-\infty}^{n} x[k]$,

- Obviously,

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n]$$

- Note

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{+\infty} x[k]u[n-k] = x[n] * u[n]$$

According to the theorem, the system is LTI with $h[n] = u[n]$.

Answer to P 2.11: Recall the relationship between causality and stability of an LTI system with its unit impulse response h.

For the LTI with $h[n] = 5^n u[3-n]$, it is

- not causal because of $h[-3] = 5^{-3} \neq 0$ - against the condition of initial rest;
- stable because of $\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} 5^n u[3-n]$, equal to

$$\sum_{n=-\infty}^{3} 5^{(n-3)} 5^3 = 5^3 \sum_{k=0}^{+\infty} 5^{-k} = \frac{5^3}{1-1/5} < +\infty$$

For the LTI with $h(t) = e^{-6|t|}$, it is

- not causal because of $h(t) \neq 0$ for all $t < 0$.
- stable because of $\int_{-\infty}^{+\infty} |h(t)| dt = 2 \int_0^{+\infty} e^{-6t} dt = 1/3 < +\infty$.

End

Answer to P 2.12: Consider the signal $x[n] = \alpha^n \; u[n]$.

- Clearly, one has

$$g[n] \stackrel{\triangle}{=} x[n] - \alpha x[n-1] = \alpha^n \; u[n] - \alpha\alpha^{(n-1)} \; u[n-1]$$
$$= \alpha^n \; u[n] - \alpha^n \; u[n-1] = \alpha^n\{u[n] - u[n-1]\} = \delta[n]$$

- As given, $x[n] * h[n] = (\frac{1}{2})^n\{u[n+2] - u[n-2]\} \stackrel{\triangle}{=} p[n]$. On the one hand,

$$g[n] * h[n] = (x[n] - \alpha x[n-1]) * h[n]$$
$$= x[n] * h[n] - \alpha x[n-1] * h[n] = p[n] - \alpha p[n-1]$$

On the other hand, the fact that $g[n] = \delta[n]$ implies $g[n] * h[n] = h[n]$. So,

$$h[n] = p[n] - \alpha p[n-1]$$

Note: The solution depends on the value of α.

End

Answer to P 2.13: Given an LTI system: $x(t) \ \rightarrow \ y(t)$ and

$$e(t) = e^{\alpha t}u(t) \ \rightarrow \ r(t), \quad \frac{de(t)}{dt} \ \rightarrow \ \beta r(t) + e^{-2t}u(t)$$

It follow from $\frac{de(t)}{dt} = \alpha e^{\alpha t}u(t) + e^{\alpha t}\delta(t) = \alpha e^{\alpha t}u(t) + \delta(t)$ that the output in response of such an input should be

$$\alpha r(t) + h(t)$$

and as given,

$$\alpha r(t) + h(t) = \beta r(t) + e^{-2t}u(t) \ \Rightarrow \quad h(t) = (\beta - \alpha)r(t) + e^{-2t}u(t)$$

End

Answer to P 2.15: This exercise is used to strengthen your understanding of the concepts.

- *If $h(t)$ is the impulse response of of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable* - **true**!

  Justification: As the sufficient and necessary condition for LTI systems to be stable if

$$\int_{-\infty}^{+\infty} |h(t)|dt < +\infty$$

  If $h(t+T) = h(t)$ with T the period, then

$$\int_{-\infty}^{+\infty} |h(t)|dt = \lim_{N \to +\infty} \sum_{k=-N}^{N} \int_{(k-\frac{1}{2})T}^{(k+\frac{1}{2})T} |h(t)|dt = +\infty$$

  because of

$$\int_{(k-\frac{1}{2})T}^{(k+\frac{1}{2})T} |h(t)|dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} |h(t)|dt \neq 0$$

- *The inverse of a causal LTI system is always causal* - **false**!

  Inverse: If $x(t) \rightarrow y(t) \rightarrow x(t)$, then the system: $y(t) \rightarrow x(t)$ is said the inverse of the system: $x(t) \rightarrow y(t)$.

  Justification: As $y(t) = x(t - t_0)$ with $t_0 > 0$ is causal and LTI, and $w(t) = y(t + t_0)$ is LTI, the second system is the inverse of the first one (because of $w(t) = x(t)$) but it is non-causal.

- *If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with $h[n]$ as unit impulse response is stable* - *false*!

  Justification:

$$|u[n]| \leq 1 \;\Rightarrow\; \sum_n |h[n]| = +\infty$$

  which implies that the system is unstable.

- *If an LTI system has an impulse response $h[n]$ of finite duration, the system is stable* - **true** - as long as $|h[n]| < +\infty$!

  Justification: Such a $h[n]$ is of form

$$h[n] = \sum_{k=N_1}^{N_2} h[k]\delta[n-k]$$

  where both N_1, N_2 are finite with $N_2 \geq N_1$. Clearly, $\sum_n |h[n]| < +\infty$ is always met.

- *If an LTI system is causal, then it is stable* - **false**!

  Justification: $\sum_n |h[n]| < +\infty$ may not be met. Say the causal system $h[n] = 2^n u[n]$, which is unstable.

- The cascade of a non-causal LTI system with a causal one is necessarily non-causal - **false**!

Justification: Assume $h_1[n] = \delta[n-3]$ - causal and $h_2[n] = u[n+1]$ - non-causal. Clearly, the cascade of the two has a unit impulse response $h[n] = h_1[n] * h_2[n] = u[n-2]$ - causal.

- *An LTI system with $u(t) \rightarrow s(t)$ – unit step response stable if and only if $\int_{-\infty}^{+\infty} |s(t)|\, dt < +\infty$* - **false**!

  Justification: The system $h(t) = e^{-t}u(t)$ is stable but $s(t) = h(t) * u(t) = [1 - e^{-t}]u(t)$ is not absolutely integrable.

- *An LTI system with $u[n] \rightarrow s[n]$ is causal if and only if* $s[n] = 0,\ \forall\ n < 0$ - **true**!

  Justification: *Sufficient condition:* Assume $s[n] = 0,\ \forall\ n < 0$. As $[n]$ is the output of the system when the input is $u[n]$ and $h[n]$ is the output when the input is $\delta[n] = u[n] - u[n-1]$, the system being LTI means

$h[n] = s[n] - s[n-1]$. Clearly,

$$h[n] = 0, \ \forall \ n < 0 \quad \Leftrightarrow \quad \textit{system causal}$$

Necessary condition: Assume the system is causal, then $h[n] = 0, \ \forall \ n < 0$, that is $h[n] = h_0[n] \ u[n]$. Note

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{+\infty} h_0[k]u[k]u[n-k] = \sum_{k=0}^{n} h_0[k] \ u[n]$$

Therefore,

$$s[n] = 0, \ \forall \ n < 0$$

End

Answer to P 2.16:

With the information given, one has

$$y[n]/y[n-1] = 1/2 \;\Rightarrow\; y[n]-1/2\,y[n-1] = 0, \quad y[0] = h \;\Rightarrow\; y[n] = (1/2)^n\,h$$

With $h = 3$, $y[n] < 0.1$ implies

$$(1/2)^n\,3 < 0.1 \qquad \Rightarrow \quad 30 \geq 2^n \quad \Rightarrow \quad n \geq 5$$

Answer to P 2.17: By analysis,

$$y[n] = x[n] + (1+r)y[n-1]$$

with $y[0] = 100, \quad r = 0.25\%, \; x[n] = 1000 + 100n.$

Solutions for Chapter 3

Answer to P 3.1: Basically, all the problems can be solved with

$$\frac{d(te^{\alpha t})}{dt} = e^{\alpha t} + \alpha te^{\alpha t} \;\Rightarrow\; \int te^{\alpha t}dt = \frac{1}{\alpha}[te^{\alpha t} - \frac{1}{\alpha}e^{\alpha t}]$$

(a): The period is $T_0 = 2$ and $\omega_0 = 2\pi/T_0 = \pi$. The FS coefficients are

$$c[k] = \frac{1}{2}\int_{-1}^{1} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{-1}^{1} te^{-jk\omega_0 t}dt$$

$$= \cdots = \begin{cases} 0 & , k = 0 \\ \frac{j}{k\omega_0}[cos(k\omega_0) - \frac{sin(k\omega_0)}{k\omega_0}] & , k \neq 0 \end{cases}$$

Observations: 1) $x(t)$ is odd and $c[k] = j\beta[k], \;\; \forall\, k$. 2) This problem can also be solved using $c[k] = G(jk\omega_0)/T_0$, where $G(j\omega) = j\frac{d}{d\omega}(\frac{sin\omega}{\omega})$ is the FT of $g(t) = tw_2(t)$.

(b): Note that $x(t)$ has a fundamental period $T_0 = 2$. We then have

$$c[k] = \frac{1}{2}\int_{-0.5}^{1.5} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2} - e^{-jk\omega_0}$$

(c): The period is $T_0 = 3$ and $\omega_0 = 2\pi/T_0 = 2\pi/3$. The FS coefficients are

$$\begin{aligned} c[k] &= \frac{1}{3}\int_{0}^{3} x(t)e^{-jk\omega_0 t}dt = \frac{1}{3}\int_{0}^{1} 2e^{-jk\omega_0 t}dt + \frac{1}{3}\int_{1}^{2} e^{-jk\omega_0 t}dt \\ &= \begin{cases} 1 & , k = 0 \\ \frac{2-e^{-j\omega_0 k}-e^{-j2\omega_0 k}}{j3\omega_0 k} & , k \neq 0 \end{cases} \end{aligned}$$

Remark: $x(t)$ is neither even nor odd, so $c[k] = \alpha[k]+j\beta[k], \ \forall\, k$.

Answer to P 3.3:

(a) For $x_1[n]$, the period is $N_0 = 7$ and $\Omega_0 = 2\pi/N_0 = 2\pi/7$. The DTFS coefficients $X_1[k],\ k = 0, \cdots, 6$ are[1]

$$X_1[k] = \frac{1}{7}\sum_{0}^{6} x_1[n]e^{-jk\Omega_0 n} = \frac{1}{7}\sum_{0}^{4} 1\ e^{-jk\Omega_0 n}$$

$$= \frac{1}{7} \times \begin{cases} 5 & , k = 0 \\ \frac{1-e^{-j5k\Omega_0}}{1-e^{-jk\Omega_0}} & , k \neq 0 \end{cases} = \frac{1}{7} \times \begin{cases} 5 & , k = 0 \\ \frac{sin(5\Omega_0 k/2)}{sin(\Omega_0 k/2)}\ e^{-j2k\Omega_0} & , k \neq 0 \end{cases}$$

(b) For $x_2[n]$, the period is $N_0 = 6$ and $\Omega_0 = 2\pi/N_0 = \pi/3$. The FS coef.s are

$$X_2[k] = \frac{1}{6}\sum_{0}^{5} x_2[n]e^{-jk\Omega_0 n} = \frac{1}{6}\sum_{0}^{3} 1\ e^{-jk\Omega_0 n}$$

$$= \frac{1}{6} \times \begin{cases} 4 & , k = 0 \\ \frac{sin(2\Omega_0 k)}{sin(\Omega_0 k/2)}\ e^{-j3k\Omega_0/2} & , k = 1, 2, 3, 4, 5 \end{cases}$$

[1] Note: $1 - e^{j\theta} = e^{j\theta/2}(e^{-j\theta/2} - e^{j\theta/2}) = -jsin(\theta/2)e^{j\theta/2}$.

(c) For $x_3[n]$, the period is $N_0 = 6$ and $\Omega_0 = 2\pi/N_0 = \pi/3$. The FS coeffi.s are

$$
\begin{aligned}
X_3[k] &= \frac{1}{6}\sum_{0}^{5} x_3[n]e^{-jk\Omega_0 n} = \frac{1}{6}[e^{-jk\Omega_0 0} + 2\ e^{-jk\Omega_0 1} - e^{-jk\Omega_0 2} \\
&\quad +0\ e^{-jk\Omega_0 3} - e^{-jk\Omega_0 4} + 2\ e^{-jk\Omega_0 5}] \\
&= \frac{1}{6}[1 + 4cos(\Omega_0 k) - 2cos(2\Omega_0 k)]
\end{aligned}
$$

(d): As given, $N_0 = 4 \ \Rightarrow \ \ \Omega_0 = \frac{\pi}{2}$. Note $x_4[n] = 1 - \frac{1}{j2}[e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}],\ n = 0, 1, 2, 3$. The DTFS coefficients are given by

$$
\begin{aligned}
X_4[k] &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\Omega_0 n} \\
&= \frac{1}{4}\{\frac{1 - e^{-jk\Omega_0 4}}{1 - e^{-jk\Omega_0}} - \frac{1}{j2}[\frac{1 - e^{-j(k\Omega_0 - \frac{\pi}{4})4}}{1 - e^{-j(k\Omega_0 - \frac{\pi}{4})}} - \frac{1 - e^{-j(k\Omega_0 + \frac{\pi}{4})4}}{1 - e^{-j(k\Omega_0 + \frac{\pi}{4})}}]\}
\end{aligned}
$$

(Hint: $\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$ if $r \neq 1$.) **End**

Answer to P 3.6: As given, $x[n]$ is periodic with $N_0 = 8$ and its FS coefficients $X[k]$ satisfy $X[k] = -X[k-4]$, that is

$$X[0] = -X[-4] = -X[4],\ X[1] = -X[-3] = -X[5]$$

$$X[2] = -X[-2] = -X[6],\ X[3] = -X[-1] = -X[7]$$

Then

$$x[n] = \sum_{k=0}^{7} X[k]e^{jk\Omega_0 n} = X[0][1 - e^{j4\Omega_0 n}] + X[1][e^{j\Omega_0 n} - e^{j5\Omega_0 n}]$$
$$+X[2][e^{j2\Omega_0 n} - e^{j6\Omega_0 n}] + X[3][e^{j3\Omega_0 n} - e^{j7\Omega_0 n}]$$

and hence

$$x[2n] = X[0][1 - e^{j8\Omega_0 n}] + X[1][e^{j2\Omega_0 n} - e^{j10\Omega_0 n}]$$
$$+X[2][e^{j4\Omega_0 n} - e^{j12\Omega_0 n}] + X[3][e^{j6\Omega_0 n} - e^{j14\Omega_0 n}]$$

Note $e^{j(m+8p)\Omega_0 n} = e^{jm\Omega_0 n}$ for any integers p, m, then $x[2n] = 0$.

As given $x[2m+1] = (-1)^m$ for $m = \cdots, -2, -1, 0, 1, 2, \cdots$, one finally has

$$x[n] = \begin{cases} 0 & , n = 2m \\ (-1)^m & , n = 2m+1 \end{cases}$$

$x[n]$ can then be sketched easily.

Note: As observed, the fundamental period of $x[n]$ is 4. Also, since $X[k]$ is the coefficients of the FS expanded with $N_0 = 8$, it satisfies $X[k] = X[k-8]$ but not necessarily $X[k] = X[k-4]$ even $x[n]$ is of a fundamental period 4.

Let $x[n]$ have a fundamental period of N_0. Denote $\Omega_0 \triangleq 2\pi/N_0$ and $\tilde{\Omega}_0 \triangleq 2\pi/(2N_0) = \Omega_0/2$. Let

$$x[n] = \sum_{k=0}^{N_0-1} X[k]e^{j\Omega_0 kn} = \sum_{m=0}^{2N_0-1} \tilde{X}[m]e^{j\tilde{\Omega}_0 mn}$$

Find the expression of $\tilde{X}[k]$ in terms of $X[k]$. **End**

Answer to P 3.8:

This exercise is intended to help students get familiar with the properties of FTs. Given that $x(t) \quad \leftrightarrow \quad X(j\omega)$.

- As given, $x_1(t) = x(1-t) + x(-1-t)$. Noting

$$g(t) \stackrel{\triangle}{=} x(-t) \quad \leftrightarrow \quad G(j\omega) = \frac{1}{|-1|} X(\frac{j\omega}{-1})$$

  and

$$g(t-\tau) \quad \leftrightarrow \quad G(j\omega)e^{-j\omega\tau}$$

  we then have $x_1(t) = g(t-1) + g(t+1)$ and hence

$$\begin{aligned} X_1(j\omega) &= G(j\omega)e^{-j\omega\ 1} + G(j\omega)e^{-j\omega(-1)} \\ &= G(j\omega)\ 2cos\omega = 2cos\omega\ X(-j\omega) \end{aligned}$$

- Note $x_2(t) = x(3t-6)$. It follows from

$$g(t) \stackrel{\triangle}{=} x(3t) \quad \leftrightarrow \quad \frac{1}{3}X(\frac{j\omega}{3}), \quad g(t-\tau) \quad \leftrightarrow \quad G(j\omega)e^{-j\omega\tau}$$

that

$$x_2(t) = g(t-2) \quad \leftrightarrow \quad X_2(j\omega) = \frac{1}{3}X(\frac{j\omega}{3})e^{-j2\omega}$$

- Given that $x_3(t) = \frac{d^2x(t-1)}{dt^2}$. Denote $g(t) \stackrel{\triangle}{=} x(t-1)$, then $G(j\omega) = X(j\omega)e^{-j\omega}$. It follows from $\frac{d^m g(t)}{dt^m} \quad \leftrightarrow \quad (j\omega)^m G(j\omega)$ that

$$x_3(t) = \frac{d^2 g(t)}{dt^2} \quad \leftrightarrow \quad X_3(j\omega) = (j\omega)^2 G(j\omega) = -\omega^2 X(j\omega)e^{-j\omega}$$

End

Answer to P 3.9:

Given $x(t) = e^{-|t|} \quad \leftrightarrow \quad X(j\omega) = \frac{2}{1+\omega^2}$, then

(a): $x_1(t) = t\,x(t) \;\leftrightarrow\; X_1(j\omega) =$?. Noting $-jt\,x(t) \quad \leftrightarrow \quad \frac{dX(j\omega)}{d\omega}$, one has

$$x_1(t) \quad \leftrightarrow \quad X_1(j\omega) = j\frac{dX(j\omega)}{d\omega} = \frac{-j4\omega}{(1+\omega^2)^2}$$

(b): The *duality* suggests that based on the above,

$$X_1(jt) = \frac{-j4t}{(1+t^2)^2} \quad \leftrightarrow \quad 2\pi x_1(-\omega)$$

So,

$$\frac{4t}{(1+t^2)^2} \quad \leftrightarrow \quad j2\pi\; x_1(-\omega) = -j2\pi\omega e^{-|\omega|}$$

End

Answer to P 3.10: By definition, one can show that

$$y(t) = x(t)e^{-\alpha t} \leftrightarrow Y(j\omega) = X(\alpha + j\omega)$$

where α is any complex number such that $Y(j\omega)$ exists.

- Note that $x_1(t) = e^{-\alpha t}cos(\omega_0 t)u(t) \stackrel{\triangle}{=} \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]x(t)$, where

  $$x(t) = e^{-\alpha t}u(t) \;\leftrightarrow\; X(j\omega) = \frac{1}{j\omega + \alpha} \;\Rightarrow\; X_1(j\omega) = \frac{1}{2}[X(j(\omega{-}\omega_0)){+}X(j(\omega{+}\omega_0))$$

- The FT of $x_2(t) = e^{-3|t|}sin2t$ can be obtained using the same approach.

- Note that $x_3(t) = \sum_{n=-\infty}^{+\infty} g(t-2n)$ is periodic with a period of 2, where $g(t) = e^{-|t|}$. By FS, we have

  $$x_3(t) = \sum_{k=-\infty}^{+\infty} c[k]e^{jk\omega_0 t},\ \omega_0 = 2\pi/2 = \pi \;\leftrightarrow\; X_3(j\omega) = \sum_{k=-\infty}^{+\infty} c[k]2\pi\delta(\omega{-}k\omega_0)$$

  where $c[k] = G(jk\omega_0)/2$ with $G(j\omega) = 1/(1+\omega^2)$.

- $x_4(t) = \sum_{k=0}^{+\infty} \alpha^k \delta(t - kT), \quad |\alpha| < 1$ yields

$$X_4(j\omega) = \sum_{k=0}^{+\infty} \alpha^k e^{-j\omega kT} = \sum_{k=0}^{+\infty} (\alpha \; e^{-j\omega T})^k = \frac{1}{1 - \alpha e^{-j\omega T}}$$

- $x_5(t) = te^{-2t} sin(4t) \; u(t) \stackrel{\triangle}{=} t \; g_1(t)$, where $g_1(t) = e^{-2t} sin(4t) \; u(t) \stackrel{\triangle}{=} g_2(t) sin(4t)$. Clearly,

$$G_2(j\omega) = \frac{1}{2 + j\omega}, \quad G_1(j\omega) = \frac{1}{j2} [G_2(j(\omega - 4)) - G_2(j(\omega + 4))]$$

and

$$x_5(t) = t \; g_1(t) \;\; \leftrightarrow \;\; X_5(j\omega) = j \frac{dG_1(j\omega)}{d\omega} = ...$$

- $x_6(t) = \frac{sin(\pi t)}{\pi t} \; \frac{sin[2\pi(t-1)]}{\pi(t-1)} \stackrel{\triangle}{=} sinc(t) \; 2 sinc[2(t-1)]$. As known,

$$w_\tau(t) \quad \leftrightarrow \quad W_\tau(j\omega) = \tau \; \frac{sin(\omega\tau/2)}{\omega\tau/2}$$

and hence $w_{2\pi}(t) \quad \leftrightarrow \quad W_{2\pi}(j\omega) = 2\pi \; \frac{sin(\omega\pi)}{\omega\pi} = 2\pi sinc(\omega)$.

The *duality* implies

$$sinc(t) \leftrightarrow w_{2\pi}(\omega) \Rightarrow sinc(2t) \leftrightarrow \frac{1}{2}w_{2\pi}(\omega/2) = \frac{1}{2}w_{4\pi}(\omega)$$

Note that

$$x_6(t) = sinc(t)sinc(t-1) \leftrightarrow X_6(j\omega) = \frac{1}{2\pi}\ 2\ w_{2\pi}(\omega) * [\frac{1}{2}w_{4\pi}(\omega)e^{-j\omega}]$$

we have by taking advantages of the convolution property

$$X_6(j\omega) = \frac{1}{2\pi}\ \frac{dw_{2\pi}(\omega)}{d\omega} * \int_{-\infty}^{\omega}[w_{4\pi}(\xi)e^{-j\xi}]d\xi$$

It follows from

$$G(j\omega) \stackrel{\triangle}{=} \int_{-\infty}^{\omega}[w_{4\pi}(\xi)e^{-j\xi}]d\xi = \int_{-2\pi}^{\omega} e^{-j\xi}d\xi\ w_{4\pi}(\omega) + \int_{-2\pi}^{2\pi} e^{-j\xi}d\xi\ u(\omega-2\pi) =?$$

one has

$$X_6(j\omega) = \frac{1}{2\pi}[G(j(\omega+\pi)) - G(j(\omega-\pi))]$$

- As to $x_7(t) = u(t+\pi) - u(t-\pi)$, noting $u(t) \ \leftrightarrow \ \ U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$ one has

$$X_7(j\omega) = U(j\omega)[e^{j\pi\omega} - e^{-j\pi\omega}] = \cdots$$

- Noting $x_8(t) = (1+sint)w_{2\pi}(t) = [1+\frac{1}{j2}(e^{jt}-e^{-jt})]w_{2\pi}(t)$ and $w_\tau(t) \ \leftrightarrow \ \tau\frac{sin(\omega\tau/2)}{\omega\tau/2} \triangleq \tau s_\tau(\omega)$, one realizes that

$$X_8(j\omega) = 2\pi[s_{2\pi}(\omega) + \frac{1}{j2}(s_{2\pi}(\omega+1) - s_{2\pi}(\omega-1))$$

End

Answer to P 3.11:

(a): As $w_\tau(t) \;\leftrightarrow\; W_\tau(j\omega) = \tau\; s_\tau(\omega)$, it follows from $z(t)e^{j\omega_0 t} \;\leftrightarrow\; Z(j(\omega-\omega_0))$ that

$$X(j\omega) = 2\,\frac{sin(3(\omega-2\pi))}{\omega-2\pi} = 6s_6(\omega-2\pi) \quad\leftrightarrow\quad x(t) = w_6(t)e^{j2\pi t}$$

(b): As $\delta(t-t_0) \;\leftrightarrow\; e^{-j\omega t_0}$ and $X(j\omega) = cos(4\omega+\pi/3) = \frac{e^{j(4\omega+\pi/3)}+e^{-j(4\omega+\pi/3)}}{2}$,

$$x(t) = \frac{e^{j\pi/3}}{2}\delta(t+4) + \frac{e^{-j\pi/3}}{2}\delta(t-4)$$

(c): Direct approach: This is essential as well as important!

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|e^{j\phi_x(\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}[\int_{-1}^{0} -\omega e^{-j3\omega}e^{j\omega t}d\omega + \int_{0}^{1} \omega e^{-j3\omega}e^{j\omega t}d\omega] = \cdots =?
\end{aligned}$$

Indirect approach: As given $X(j\omega) = X_0(j\omega)e^{-j3\omega}$, where $X_0(j\omega) \stackrel{\triangle}{=} -\omega w_1(\omega + 1/2) + \omega w_1(\omega - 1/2)$. As $w_1(t) \quad \leftrightarrow \; s_1(\omega) = \frac{sin(\omega/2)}{\omega/2}$, the duality means

$$z(t) \stackrel{\triangle}{=} \frac{1}{2\pi}s_1(\omega) \;\leftrightarrow\; \quad w_1(\omega) \Rightarrow \; z(t)e^{j\omega_0 t} \;\leftrightarrow\; \quad w_1(\omega - \omega_0)$$

It follows from $-j\frac{dv(t)}{dt} \;\leftrightarrow\; \omega V(j\omega)$ that

$$-j\frac{d}{dt}[z(t)e^{-jt/2}] \quad \leftrightarrow \; \omega w_1(\omega + 1/2), \quad -j\frac{d}{dt}[z(t)e^{jt/2}] \quad \leftrightarrow \; \omega w_1(\omega - 1/2)$$

Therefore, $x_0(t) \; = \; j\{\frac{d}{dt}[z(t)e^{-jt/2}] - \frac{d}{dt}[z(t)e^{jt/2}]\} \; = \; 2\frac{d}{dt}[z(t)sin(t/2)] \; = \; \cdots$

and hence

$$x(t) = x_0(t-3)$$

(d): Using the direct approach to IFT

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} ce^{-t_0\omega}e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} ce^{j\omega(t-t_0)}d\omega = \cdots
\end{aligned}
$$

(e): Easy as $e^{j\omega_0 t} \quad \leftrightarrow \quad \delta(\omega-\omega_0)$

.

Key points: How to express a given $x(t)$ (or $X(j\omega)$) in terms of some spacial functions and then use the FT properties - *it is not easy but you have plenty of time to learn*!

Answer to P 3.12:

(a): Clearly,

$$x_1(t) = w_2(t) \;\Rightarrow\; X_1(j\omega) = 2s_2(\omega); \;\; X_2(j\omega) = \frac{1}{2\pi}2\pi s_{2\pi}(\omega) \;\Rightarrow\; x_2(t) = \frac{1}{2\pi}w_{2\pi}(t)$$

(b): The convolution property simply tells that

$$X_3(j\omega) \stackrel{\triangle}{=} X_1(j\omega)X_2(j\omega) \quad \Rightarrow \quad x_3(t) = x_1(t) * x_2(t) = \cdots$$

(c): By observation, we have

$$\int_{-\infty}^{+\infty} \pi X_1(j\omega)X_2(j\omega)e^{-j\omega}d\omega = \pi^2 \, \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_3(j\omega)e^{j\omega(-1)}d\omega = \pi^2 x_3(-1)$$

End

Answer to P 3.14: Given $x[n] = \alpha^{|n-1|}, \quad \alpha = 1/2$, we then have

$$X(e^{j\Omega}) = \sum_n x[n]e^{-j\Omega n} = \sum_n \alpha^{|n-1|}e^{-j\Omega n} = \sum_{n=-\infty}^{0} \alpha^{-(n-1)}e^{-j\Omega n} + \sum_{n=1}^{+\infty} \alpha^{n-1}e^{-j\Omega n}$$

$$= \alpha \sum_{n=0}^{+\infty} (\alpha e^{j\Omega})^n + e^{-j\Omega} \sum_{n=0}^{+\infty} (\alpha e^{-j\Omega})^n = \frac{\alpha}{1-\alpha e^{j\Omega}} + \frac{e^{-j\Omega}}{1-\alpha e^{-j\Omega}}$$

$$= \frac{(1-\alpha^2)e^{-j\Omega}}{(1-\alpha e^{j\Omega})(1-\alpha e^{-j\Omega})}$$

Clearly,

$$|X(e^{j\Omega})| = \frac{1-\alpha^2}{1-2\alpha cos\Omega + \alpha^2}$$

With $\alpha = 1/2$, one can sketch it.

Answer to P 3.16: (a): By definition,

$$X(e^{j\Omega}) = \sum_{-\infty}^{+\infty} x[n]e^{-jn\Omega} \quad \Rightarrow \quad X(1) = \sum_{-\infty}^{+\infty} x[n]$$

(b): How to compute $A = \sum_{n=2}^{+\infty} n0.5^n$?

$$\sum_{n=2}^{+\infty} n0.5^n = \sum_{n=-\infty}^{+\infty} n0.5^n u[n-2] = X(1)$$

Denote $g[n] \stackrel{\triangle}{=} 0.5^n u[n]$ and then $G(e^{j\Omega}) = \frac{1}{1-0.5e^{-j\Omega}}$. Clearly,

$$x[n] = n0.5^n u[n-2] = 0.5^2 nw[n], \quad w[n] = g[n-2] \;\leftrightarrow\; X(e^{j\Omega}) = 0.5^2\, j\frac{dW(e^{j\Omega})}{d\Omega}$$

where

$$W(e^{j\Omega}) = G(e^{j\Omega})e^{-j2\Omega} \quad \Rightarrow \quad X(e^{j\Omega}) = \cdots$$

End

Answer to P 3.22: By definition,

$$x_p(t) = \sum_{k=-\infty}^{+\infty} g(t - kT_0), \qquad g(t) \stackrel{\triangle}{=} x_0(t) + x_)(-t)$$

Clearly,

$$g(t) \quad \leftrightarrow \quad G(j\omega) = X_0(j\omega) + X_0(-j\omega)$$

and hence

$$x_p(t) = \sum_{m=-\infty}^{\infty} c[m] = e^{jm\omega_0 t}, \qquad \omega_0 = 2\pi/T_0$$

with

$$c[m] = \frac{G(jm\omega_0)}{T_0} = \frac{X_0(jm\omega_0) + X_0(-jm\omega_0)}{T_0}$$

End

Solutions for Chapter 4

The objective is to enhance the understanding the concept:

$$y(t) = h(t) * x(t) \;\Rightarrow\; Y(j\omega) = H(j\omega)X(j\omega) \;\Rightarrow\; y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)e^{j\omega t}d\omega$$

and how to use it for studying LTI systems.

Answer to P 4.1: Important FT pairs:

$$e^{-\alpha t}u(t) \;\rightarrow\; \frac{1}{j\omega + \alpha},\; \mathcal{R}e(\alpha) > 0, \quad -e^{-\beta t}u(-t) \;\rightarrow\; \frac{1}{j\omega + \beta},\; \mathcal{R}e(\beta) < 0$$

and hence with $tx(t) \quad \leftrightarrow \quad j\frac{dX(j\omega)}{d\omega}$

$$te^{-\alpha t}u(t) \;\rightarrow\; \frac{1}{(j\omega + \alpha)^2},\; \mathcal{R}e(\alpha) > 0,\; -te^{-\beta t}u(-t) \;\rightarrow\; \frac{1}{(j\omega + \beta)^2},\; \mathcal{R}e(\beta) < 0$$

Just take (a) as example. $x(t) = te^{-2t}u(t) \ \leftrightarrow \ X(j\omega) = \frac{1}{(j\omega+2)^2}$ and $h(t) = e^{-4t}u(t) \ \leftrightarrow \ H(\mathrm{j}\omega) = \frac{1}{j\omega+4}$

Note $Y(j\omega) = X(j\omega)H(j\omega)$, that is

$$Y(j\omega) = \frac{1}{(j\omega+2)^2}\frac{1}{j\omega+4} = \frac{A}{j\omega+2} + \frac{B}{(j\omega+2)^2} + \frac{C}{j\omega+4}$$

then

$$y(t) = Ae^{-2t}u(t) + Bte^{-2t}u(t) + Ce^{-4t}u(t)$$

where (with $\rho = j\omega$)

$$B = [Y(\rho)(\rho{+}2)^2]_{\rho=-2} = \frac{1}{-2+4} = \frac{1}{2}, \ \ C = [Y(\rho)(\rho{+}4)]_{\rho=-4} = \frac{1}{(-4+2)^2} = \frac{1}{4}$$

and

$$A = \frac{d[Y(\rho)(\rho+2)^2]}{d\rho}|_{\rho=-2} = -\frac{1}{(\rho+4)^2}|_{\rho=-2} = -\frac{1}{4}$$

End

Answer to P 4.2: As known, $w_\tau(t) \;\leftrightarrow\; \tau\frac{sin(\omega\tau/2)}{\omega\tau/2} \triangleq \tau s_\tau(\omega)$. So, the duality suggests

$$\tau s_\tau(-t) = s_\tau(t) \;\leftrightarrow\; 2\pi w_\tau(\omega) \Leftrightarrow \frac{\tau}{2\pi} s_\tau(t) \;\leftrightarrow\; w_\tau(\omega)$$

Now, given that $h(t) = \frac{sin(4(t-1)}{\pi(t-1)}$, i.e., $h(t) = \frac{4}{\pi}\; s_8(t-1)$

$$h(t) \leftrightarrow H(j\omega) = w_8(\omega)e^{-j\omega}$$

Clearly, $|H(j\omega)| = w_8(\omega), \;\; \phi(\omega) = -\omega$.

Sketch the frequency response on the board.

Now, let us consider each sub-question one by one.

First of all, note that if we input

$$x(t) = cos(\omega_c t+\phi) = \frac{e^{j\phi}}{2}e^{j\omega_c t}+\frac{e^{-j\phi}}{2}e^{-j\omega_c t} \Leftrightarrow X(j\omega) = \frac{e^{j\phi}}{2}\delta(\omega-\omega_c)+\frac{e^{-j\phi}}{2}\delta(\omega+\omega_c)$$

the corresponding output $y(t)$ has an FT

$$Y(j\omega) = H(j\omega)X_1(j\omega) = \frac{e^{j\phi}}{2}H(j\omega_c)\delta(\omega - \omega_c) + \frac{e^{-j\phi}}{2}H(-j\omega_c)\delta(\omega + \omega_c)$$

and hence by IFT

$$\begin{aligned} y(t) &= \frac{e^{j\phi}}{2}H(j\omega_c)e^{j\omega_c t} + \frac{e^{-j\phi}}{2}H(-j\omega_c)e^{-j\omega_c t} \\ &= w_8(\omega_c)cos(\omega_c t + \phi - \omega_c) \end{aligned}$$

- For $x_1(t) = cos(6t + \pi/2)$, $\omega_c = 6$. As $w_8(\omega_c) = w_8(6) = 0$, $y_1(t) = 0$.

- $x_2(t) = \sum_{k=0}^{\infty}(1/2)^k sin(3kt) = \sum_{k=0}^{\infty}(1/2)^k cos(3kt - \pi/2)$ - a linear combination of $\{cos(3kt - \pi/2)\}$.

  Note that $cos(3kt - \pi/2) \quad \to \quad w_8(3k)cos(3kt - \pi/2 - 3k)$, then

$$y_2(t) = \sum_{k=0}^{\infty}(1/2)^k w_8(3k)cos(3kt - \pi/2 - 3k)$$

  It follows from $w_8(3k) = 1, \; k = -1, 0, 1$ and $w_8(3k) = 0, \; \forall \; |k| \geq 2$ that

$$y_2(t) = cos(-\pi/2) + 1/2cos(3t - \pi/2 - 3) = 1/2sin(3t - 3)$$

- As to $x_3(t) = \frac{sin(4(t+1)}{\pi(t+1)} = \frac{4}{\pi} \, s_8(t+1)$, we have $X_3(j\omega) = w_8(\omega)e^{j\omega}$. So,

$$Y_3(j\omega) = H(j\omega)X_3(j\omega) = w_8(\omega) \quad \leftrightarrow \quad y_3(t) = \frac{4}{\pi} \, s_8(t) = \frac{4}{\pi} \, \frac{sin(4t)}{4t}$$

- Knowing that $x_4(t) = (\frac{sin(2t)}{\pi t})^2 = (\frac{2}{\pi}\ s_4(t))^2$ and

$$\frac{2}{\pi}\ s_4(t) \ \leftrightarrow\ w_4(\omega)$$

we have

$$X_4(j\omega) = \frac{1}{2\pi} w_4(\omega) * w_4(\omega)$$

As $w_4(\omega) = 0,\ \forall\ |\omega| > 2$, we have $X_4(j\omega) = 0,\ \forall\ |\omega| > 4$ and therefore,

$$Y_4(j\omega) = H(j\omega)X_4(j\omega) = e^{-j\omega}X_4(j\omega) \ \leftrightarrow\ y_4(t) = x_4(t-1)$$

End

Answer to P 4.3 Knowing that causal and stable LTI system

$$H(j\omega) = \frac{4 + j\omega}{6 - \omega^2 + j5\omega}$$

- Determine the corresponding differential equation:

  Since $H(\jmath\omega) = \frac{Y(\jmath\omega)}{X(j\omega)} = \frac{4+j\omega}{6-\omega^2+j5\omega}$, we have

$$(6 - \omega^2 + j5\omega)Y(j\omega) = (4 + j\omega)X(j\omega)$$

  that is

$$[6 + (j\omega)^2 + j5\omega]Y(j\omega) = (4 + j\omega)X(j\omega)$$

  It follows from

$$\frac{d^k s(t)}{dt^k} \quad \leftrightarrow \quad (j\omega)^k S(j\omega)$$

  that the IFT of the above is

$$6y(t) + \frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} = 4x(t) + \frac{dx(t)}{dt}$$

- By partial fraction, $H(\jmath\omega) = \frac{4+j\omega}{6-\omega^2+j5\omega} = \frac{A}{2+j\omega} + \frac{B}{3+j\omega}$, where by comparing the numerators, $A = 2,\ B = -1$. Therefore, $h(t) = [Ae^{-\alpha_1 t} + Be^{-\alpha_2 t}]u(t)$.

- It follows from $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$ that

$$X(j\omega) = \frac{1}{j\omega+4} - j\frac{d(j\omega+4)^{-1}}{d\omega} = \frac{j\omega+3}{(j\omega+4)^2}$$

and similarly by partial fraction,

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{4+j\omega}{6-\omega^2+j5\omega}\frac{j\omega+3}{(j\omega+4)^2} = \frac{C_1}{2+j\omega} + \frac{C_2}{4+j\omega}$$

Once again, comparing the numerators lead to we can specify the four constants (please do it yourself) and hence

$$y(t) = [c_1e^{-2t} + C_2e^{-4t}]u(t)$$

End

Answer to P 4.5 As given, the frequency response of an ideal BPF is

$$H(j\omega) = w_{2\omega_c}(\omega + 2\omega_c) + w_{2\omega_c}(\omega - 2\omega_c)$$

- Denote $h_0(t) \leftrightarrow H_0(j\omega) = w_{2\omega_c}(\omega)$. Clearly,

$$h(t) = h_0(t)e^{-j2\omega_c} + h_0(t)e^{j2\omega_c} = 2h_0(t)cos(2\omega_c t)$$

As known before that the IFT of $H_0(j\omega) = w_{2\omega_c}(\omega)$ is

$$h_0(t) = \frac{1}{2\pi}2\omega_c\frac{sin(t2\omega_c/2)}{t2\omega_c/2} \Rightarrow h(t) = \frac{sin(\omega_c t)}{\pi t}2cos(2\omega_c t) \Rightarrow g(t) = 2cos(2\omega_c t)$$

$g(t)$ is usually referred to as *carrier*, where $\frac{sin(\omega_c t)}{\pi t}$ is the *envelope* of $h(t)$.

- The first *zero* of the envelope is found from: $\omega_c t_0 = \pi \quad \Rightarrow \quad t_0 = \frac{\pi}{\omega_c}$.

When ω_c gets bigger, t_0 becomes smaller, close to the origin $t = 0$, and hence $h(t)$ is more concentrated about $t = 0$.

End

Answer to P 4.6 First of all, we note that $H_1(j\omega)$ has the following critical frequencies:

$$\omega_1 = 1, \omega_2 = 8, \omega_3 = 40$$

which implies that

$$H_1(j\omega) = \kappa_1(1 + j\frac{\omega}{\omega_1})\frac{1}{1 + j\frac{\omega}{\omega_2}}\frac{1}{1 + j\frac{\omega}{\omega_3}} = \kappa_1\frac{1 + j\frac{\omega}{\omega_1}}{(1 + j\frac{\omega}{\omega_2})(1 + j\frac{\omega}{\omega_3})}$$

where $20log_{10}|\kappa_1| = 6$ dB, i.e., $\kappa_1 = \pm 10^{0.3}$.

Similarly, the fact that $H(j\omega)$ has only one critical frequency at $\omega_p = 8$ suggests that $H(j\omega) = \frac{\kappa}{(1+j\frac{\omega}{\omega_p})^2}$, where $20log_{10}|\kappa| = -20$ dB, i.e., $\kappa = \pm 10^{-1}$.

Since $H(j\omega) = H_1(j\omega)H_2(j\omega)$, we finally have

$$H_2(j\omega) = H(j\omega)/H_1(j\omega) = \frac{\kappa/\kappa_1(1 + j\frac{\omega}{\omega_2})(1 + j\frac{\omega}{\omega_3})}{(1 + j\frac{\omega}{\omega_1})(1 + j\frac{\omega}{\omega_p})^2}$$

End

Answer to P 4.8 Note the following holds for any LTI systems

$$y = x * h \quad \leftrightarrow \quad Y = XH \quad \Rightarrow \quad H = \frac{Y}{X}$$

- For the 1st LIT, as given

$$x[n] = (1/2)^n u[n] + 2^n u[-n-1] \;\rightarrow\; y[n] = x[n] * h_1[n] = 6(1/2)^n u[n] - 6(3/4)^n u[n]$$

we have

$$X(e^{j\Omega}) = \frac{1}{1-(1/2)e^{-j\Omega}} - \frac{1}{1-2e^{-j\Omega}}, \;\; Y(e^{j\Omega}) = \frac{6}{1-(1/2)e^{-j\Omega}} - \frac{6}{1-(3/4)e^{-j\Omega}}$$

and hence

$$H_1(e^{j\Omega}) = \frac{\frac{6}{1-(1/2)e^{-j\Omega}} - \frac{6}{1-(3/4)e^{-j\Omega}}}{\frac{1}{1-(1/2)e^{-j\Omega}} - \frac{1}{1-2e^{-j\Omega}}} = \frac{6(1/2-3/4)}{1/2-2} \frac{1-2e^{-j\Omega}}{1-(3/4)e^{-j\Omega}}$$

Therefore,

$$H_1(e^{j\Omega}) = \frac{1-2e^{-j\Omega}}{1-(3/4)e^{-j\Omega}} \;\leftrightarrow\; h_1[n] = (3/4)^n u[n] - 2(3/4)^{(n-1)} u[n-1]$$

- For the 2nd LTI, it follows from $y[n] = 0.9y[n-1] + x[n] + 0.9x[n-1]$ that

$$Y(e^{j\Omega}) = 0.9Y(e^{j\Omega})e^{-j\Omega} + X(e^{j\Omega}) + 0.9X(e^{j\Omega})e^{-j\Omega}$$

and hence

$$H_2(e^{j\Omega}) = Y(e^{j\Omega})/X(e^{j\Omega}) = \frac{1+0.9e^{-j\Omega}}{1-0.9e^{-j\Omega}} \leftrightarrow h_2[n] = 0.9^n u[n] + 0.9^n u[n-1]$$

- When the two LTI are cascaded, the overall system has its impulse response $h[n] = h_1[n] * h_2[n]$, therefore,

$$H(e^{j\Omega}) = H_1(e^{j\Omega})H_2(e^{j\Omega}) = \frac{1-2e^{-j\Omega}}{1-(3/4)e^{-j\Omega}} \frac{1+0.9e^{-j\Omega}}{1-0.9e^{-j\Omega}}$$

When the input is $x[n] = e^{j\Omega_0 n}$, the output of the overall system $H(e^{j\omega})$ is

$$y[n] = H(e^{j\Omega_0})e^{j\Omega_0 n}$$

End

Answer to P 4.10 It follows from

$$
\begin{aligned}
H(e^{j\Omega}) &= b_0 + b_1 e^{-j\Omega} + b_1 e^{-j2\Omega} + b_0 e^{-j3\Omega} \\
&= b_0[1 + e^{-j3\Omega}] + b_1[e^{-j\Omega} + e^{-j2\Omega}] \\
&= b_0 e^{-j3\Omega/2}[e^{j3\Omega/2} + e^{-j3\Omega/2}] + b_1 e^{-j3\Omega/2}[e^{j\Omega/2} + e^{-j\Omega/2}] \\
&= 2[b_0 cos(3\Omega/2) + b_1 cos(\Omega/2)]e^{-j3\Omega/2} \triangleq H_r(\Omega)e^{-j3\Omega/2}
\end{aligned}
$$

where $H_r(\Omega)$ is real and hence $H_r(\Omega) = |H_r(\Omega)|e^{j\theta}$ with $\theta = 0$ if $H_(\Omega) \geq 0$, and $\theta = \pi$ if $H_(\Omega) < 0$. Therefore,

$$
H(e^{j\Omega}) = |H_r(\Omega)|e^{-j(3\Omega/2-\theta)}
$$

which implies that $H(e^{j\Omega})$ is of (piece-wise) linear phase response.

End

Solutions for Chapter 5

Recall: Let $x(t) \leftrightarrow X_a(j\omega)$. Then $x[n] = x(nT_s) \leftrightarrow X(e^{j\Omega}) = \sum_n x[n]e^{j\Omega n}$ satisfies

$$X(e^{jT_s\omega}) = (1/T_s) \sum_{k=-\infty}^{+\infty} X_a(j(\omega - k\omega_s)), \quad \omega_s \triangleq \frac{2\pi}{T_s}$$

Sampling Theorem: For a signa $x(t)$ of frequency limited to ω_M, it can be recovered from $x[n] = x(nT_s)$ if

$$\omega_s \triangleq \frac{2\pi}{T_s} \geq 2\omega_M \triangleq \omega_N \quad \Leftrightarrow \quad T_s \leq \frac{\pi}{\omega_M}$$

where ω_s is called **sampling frequency**; ω_N, the **Nyquist rate** of the signal $x(t)$, determined uniquely by the CT signal $x(t)$.

- Speech: $\omega_M = 2\pi \times 4 \times 10^3 \Rightarrow \omega_s = 2\pi \times 8 \times 10^3$, or even higher.
- Music: $\omega_M = 2\pi \times 20 \times 10^3 \Rightarrow \omega_s = 2\pi \times 44 \times 10^3$, or even higher.

Answer to P 5.1 Do it on board !

$$X(e^{jT_s\omega}) = (1/T_s)\sum_{k=-\infty}^{+\infty} X_a(j(\omega - k\omega_s))$$

Answer to P 5.2 As $x(t)$ is the output of the ideal low-pass filter of cutoff frequency $\omega_c = 1,000\ \pi$, its spectrum must be limited to $\omega_M = \omega_c$. Therefore, the Nyquiste frequency is $\omega_N = 2\omega_M = 2,000\pi \quad \Rightarrow \quad T_s \leq \frac{\pi}{\omega_M} = 10^{-3}\ (sec.s)$.

- $T = 0.5\ \times 10^{-3} \quad \Rightarrow \qquad OK$
- $T = 2\ \times 10^{-3} \quad \Rightarrow \qquad KO$
- $T = 1 \times 10^{-4} \quad \Rightarrow \qquad OK$

End

Answer to P 5.3: In what follows, denote $\omega_N^{y_k}$ as the Nyquist rate of signal $y_k(t)$ that is band limited to ω_{y_k}. Clearly,

$$\omega_N^{y_k} = 2\omega_{y_k}$$

For $x(t)$, we have $\omega_N^x = \omega_0$ given. So, $\omega_x = \omega_0/2$.

Now, let us determine the Nyquist rate for each of the following signals generated based on $x(t)$.

The key: to find the freq. to which the signal is band limited.

- Note $y_1(t) = x(t) + x(t-1)$ has a spectrum (i.e., FT)

$$Y_1(j\omega) = X(j\omega) + X(j\omega)e^{-j\omega} = X(j\omega)(1 + e^{-j\omega})$$

As $x(t)$ is band limited to ω_x, the above means that

$$\omega_{y_1} = \omega_x = \omega_0/2 \quad \Rightarrow \quad \omega_N^{y_1} = 2\omega_{y_1} = \omega_0$$

- As to the 2nd signal, we have

$$y_2(t) = \frac{dx(t)}{dt} \quad \leftrightarrow \quad Y_2(j\omega) = j\omega X(j\omega)$$

With the same argument used for $y_1(t)$, we can see

$$\omega_{y_2} = \omega_x \;\Rightarrow\; \omega_N^{y_2} = \omega_x = \omega_0$$

- For the 3rd signal, since

$$y_3(t) = x^2(t) \quad \leftrightarrow \quad Y_3(j\omega) = \frac{1}{2\pi}X(j\omega) * X(j\omega)$$

As $Y_3(j\omega)$ is band limited to $2\omega_x$ (**it can be shown graphically**), the Nyquist rate of this signal is $\omega_N^{y_3} = 2 \times 2\omega_x = 2\omega_0$.

- What about $y_4(t) = x(t)cos(\omega_0 t)$?

End

Answer to P 5.6: Given $x(t) \leftrightarrow X_a(j\omega)$ and $p(t) = \sum_{k=-\infty}^{\infty}(-1)^k\delta(t - kT)$.

- Note that $p(t)$ is periodic with a period of $2T$, then $p(t) = \sum_{m=-\infty}^{+\infty} c[m]e^{jm\omega_0 t}$ where $c[m] = \frac{1}{2T}[1 + (-1)e^{-jm\pi}]$, $\omega_0 = \frac{2\pi}{2T} = \pi/T$. Therefore,

$$y(t) = x(t)p(t) \quad \leftrightarrow \quad Y(j\omega) = \sum_{m=-\infty}^{\infty} c[m]X_a(j(\omega - m\omega_0))$$

- Similarly, $q[n] \stackrel{\triangle}{=} \sum_{k=-\infty}^{+\infty}(-1)^k\delta[n-k]$ is periodic with a period of 2. Thus, $q[n] = \sum_{m=0}^{2-1} d[m]e^{jm\pi n} = e^{jn\pi}$ as $d[m] = \frac{1}{2}[1 + (-1)e^{-j\omega m}]$.

$$w[n] = x[n]q[n] \quad \leftrightarrow \quad W(e^{j\Omega}) = X(e^{j(\Omega-\pi)})$$

Since for $x[n] = x(nT)$, one has

$$X(e^{j\omega T}) = T^{-1}\sum_{k=-\infty}^{+\infty} X_a(j(\omega - k\omega_s))$$

So,

$$W(e^{j\omega T}) = X(e^{j(T\omega-\pi)}) = X(e^{jT(\omega-\pi/T)}), \quad \pi/T = \omega_s/2$$

that is

$$W(e^{j\omega T}) = T^{-1}\sum_{k=-\infty}^{+\infty} X_a(j(\omega-(k+1/2)\omega_s))$$

Clearly, if $\omega_s > 2\omega_M$, there is no spectral overlapping and hence it is possible to recover $x(t)$ from $w[n]$.

End

Answer to P 5.7:

- Define $y(t) \stackrel{\triangle}{=} \frac{1}{T_s}\int_{t-T_s}^{t} x(\tau)d\tau$, then it can be seen that $y(t) = x(t)*\{\frac{1}{T_s}[u(t)-u(t-T_s)]\} \stackrel{\triangle}{=} x(t) * h(t)$, that is

$$h(t) = \frac{1}{T_s}[u(t) - u(t-T_s)] \quad \leftrightarrow \quad H(j\omega) = ??$$

- Since $\tilde{x}[n] = y(nT_s)$, one should have

$$\tilde{X}(e^{jT_s\omega}) = (1/T_s)\sum_{k=-\infty}^{+\infty} Y(j(\omega - k\omega_s)), \quad Y(j\omega) = H(j\omega)X(j\omega)$$

- Let $w[n]$ be the output of $Q(e^{j\Omega})$ to be designed, then $W(e^{j\Omega}) = Q(e^{j\Omega})\tilde{X}(e^{j\Omega})$ and hence $\tilde{W}(e^{jT_s\omega}) = (1/T_s)\sum_{k=-\infty}^{+\infty} Q(e^{jT_s\omega})Y(j(\omega - k\omega_s))$. So, $w[n] = x[n]$ when $Q(e^{j\Omega})$ is chosen such that

$$Q(e^{jT_s\omega})H(j\omega)X(j\omega) = X(j\omega) \;\Rightarrow\; Q(e^{jT_s\omega}) = H^{-1}(j\omega), \;\; |\omega| \le \omega_s/2$$

End

Solutions for Chapter 6

Answer to P 6.2: Remember the following important pairs

$$
\begin{aligned}
e^{\gamma t}u(t) \quad &\leftrightarrow \quad \frac{1}{s-\gamma}, \quad \mathcal{R}_e(\gamma) < \mathcal{R}_e(s) \\
e^{\gamma t}u(-t) \quad &\leftrightarrow \quad -\frac{1}{s-\gamma}, \quad \mathcal{R}_e(s) < \mathcal{R}_e(\gamma)
\end{aligned}
$$

It follows from

$$
x(t) \;\Leftrightarrow\; X(s) = \frac{1}{s+5} + \frac{1}{s+\beta}
$$

with an ROC_x containing

$$
\{-5 < \mathcal{R}_e(s)\} \bigcap \{\mathcal{R}_e(\beta) < \mathcal{R}_e(s)\} = \{\max\{\mathcal{R}_e(\beta),\; -5\} < \mathcal{R}_e(s)\}
$$

As given, $ROC_x = \{-3 < \mathcal{R}_e(s)\}$. Therefore, there should be $\mathcal{R}_e(\beta) = -3$ and no constraint on $I_m(\beta)$.

End

Answer to P 6.3: Given that $X(s) = \frac{2(s+2)}{s^2+7s+12}$, then

$$X(s) = \frac{2(s+2)}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

where $A = \lim_{s\to -3}(s+3)X(s) = -2, \quad B = \lim_{s\to -4}(s+4)X(s) = 4.$

Zeros: $2(s+2) = 0 \quad \Rightarrow \quad Z_1 = -2$ and $Z_2 = \infty$

Poles: $s^2 + 7s + 12 = 0 \;\Rightarrow\; P_1 = -4, P_2 = -3.$

- *Right-sided*: $ROC_x \supseteq \{\sigma_r < \mathcal{R}e(s) < +\infty\}$

$$\sigma_r = -3 \quad \Rightarrow \quad x(t) = [Ae^{-4t} + Be^{-3t}]u(t)$$

- *Left-sided*: $ROC_x \supseteq \{-\infty < \mathcal{R}e(s) < \sigma_l\}$

$$\sigma_l = -4 \quad \Rightarrow \quad x(t) = -[Ae^{-4t} + Be^{-3t}]u(-t)$$

- *Two-sided*: $ROC_x \supseteq \{\sigma_r < \mathcal{R}e(s) < \sigma_l\}$

$$\sigma_r = -4,\; \sigma_l = -3 \;\Rightarrow\; x(t) = Ae^{-4t}u(t) - Be^{-3t}u(-t)$$ **End**

Answer to P 6.4: Try to learn how to derive LT properties based on the definitions

$$X(s) = \int x(t)e^{-st}dt, \qquad x(t) = \frac{1}{j2\pi}\int X(s)e^{st}ds$$

- $x_1(t) = t\frac{d}{dt}\{e^{-t}u(t)\} \triangleq t\ y_1(t)$. Note

  $$y_2(t) = e^{-t}u(t) \quad \leftrightarrow \quad Y_2(s) = \frac{1}{s+1},\ y_1(t) = \frac{dy_2}{dt} \ \leftrightarrow\ Y_1(s) = sY_2(s)$$

  then $x_1(t) = t\ y_1(t) \ \leftrightarrow\ X_1(s) = -\frac{dY_1(s)}{ds}$ can be obtained easily.

- Note $x_4(t) = \int_0^t e^{-2\xi}cos(3\xi)d\xi = [e^{-2t}cos(3t)u(t)] * u(t) \triangleq y_1(t) * u(t)$.

  Then $X_4(t) = Y_1(s)\frac{1}{s}$, where $y_1(t) = e^{-2t}cos(3t)u(t)$, i.e.,

  $$y_1(t) = \frac{e^{-2t}}{2}[e^{j3t} + e^{-j3t}]u(t) \triangleq \frac{1}{2}[e^{-(2-j3)t} + e^{-(2+j3)t}]u(t)$$

  which yields

  $$Y_1(s) = \frac{1}{s+(2-j3)} + \frac{1}{s+(2+j3)}$$

  End

Answer to P 6.5:

- Since $x(\alpha t) \;\leftrightarrow\; \frac{1}{|\alpha|}X(s/\alpha)$, $X_1(s) = \frac{1/4}{1/4}X(\frac{s}{1/4}) \quad\leftrightarrow\quad x_1(t) = \frac{1}{4}x(t/4)$.

- As $\frac{dx}{dt} \quad\leftrightarrow\quad sX(s)$, one has $(s^2+2s+1)X(s) \quad\leftrightarrow\quad x_2(t) = \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x(t)$.

- Noting $e^{-\alpha t}x(t) \quad\leftrightarrow\quad X(s+\alpha)$, one has $X_3(s) = X(s+2) \;\leftrightarrow\; x_3(t) = e^{-2t}x(t)$.

- Since $u(t) \;\leftrightarrow\; s^{-1}$, $s^{-2} \;\leftrightarrow\; y_1(t) = u(t)*u(t) = \delta(t)*\int_{-\infty}^{t} u(\tau)d\tau = tu(t)$, we have

$$X_4(s) = s^{-2}X(s) \quad\leftrightarrow\quad x_4(t) = y_1(t)*x(t)$$

- Let $Y_1(s) = e^{-3s}X(s)$, then $y_1(t) = x(t-3)$. Then

$$X_5(s) = \frac{dY_1(s)}{ds} \quad\leftrightarrow\quad x_5(t) = -ty_1(t) = -tx(t-3)$$

End

Answer to P 6.8: Remember the following important pairs

$$x_1[n] = \alpha^n u[n] \quad \leftrightarrow \quad X_1(z) = \frac{1}{1-\alpha z^{-1}}, \quad |\alpha z^{-1}| < 1 \;\Leftrightarrow\; |\alpha| < |z|$$

$$x_2[n] = -\beta^n u[-n-1] \quad \leftrightarrow \quad X_2(z) = \frac{1}{1-\beta z^{-1}}, \quad |\beta z^{-1}| > 1 \;\Leftrightarrow\; |z| < |\beta|$$

For $x[n] = (1/2)^n u[n] + \beta^n u[-n] \stackrel{\triangle}{=}$, we have

$$x_1[n] = (1/2)^n u[n] \quad \leftrightarrow \quad X_1(z) = \frac{1}{1-1/2\; z^{-1}}, \quad ROC_{x_1} = \{z : 1/2 < |z|\}$$

and denoting $\tilde{x}_2[n] = -\beta^n u[-n-1]$, then $\tilde{X}_2 = \frac{1}{1-\beta z^{-1}}, \quad |\alpha z^{-1}| > 1 \;\Leftrightarrow\; |z| < |\alpha|$ and hence

$$x_2[n] = \beta^n u[-n] = -\beta^{-1}\tilde{x}_2[n-1] \quad \leftrightarrow \quad X_2(z) = -\beta^{-1}\tilde{X}_2(z)z^{-1}$$

with $ROC_{x_2} \supseteq \{|z| < |\beta|\} \cap \{|z| \neq 0\} = \{0 < |z| < |\beta|\}$. So, $X(z) = X_1(z) + X_2(z)$ has a non-empty ROC if and only if $|\beta| > 1/2$.

End

Answer to P 6.9: Given that

$$X(z) = \frac{2 - 1.5z^{-1}}{(1 - 2z^{-1})(1 + 0.5z^{-1})} = \frac{z(2z - 1.5)}{(z - 2)(z + 0.5)} = \frac{A}{1 - 2z^{-1}} + \frac{B}{1 + 0.5z^{-1}}$$

where $A =$?? and $B =$??

- Poles: $X(z_p) = \infty \;\Rightarrow\; z_p = 0.5, -2$ and Zeros: $X(z_q) = 0 \;\Rightarrow\; z_q = 0.75, z = 0.$

- right-sided: $\rho_r < |z| \;\Rightarrow\; \rho_r = \max\{|z_p|\} = 2$ and hence

$$x[n] = [A2^n + B(-0.5)^n]u[n]$$

- left-sided: $|z| < \rho_l \;\Rightarrow\; \rho_l = \min\{|z_p| = 0.5\} =$ and hence

$$x[n] = -[A2^n + (-0.5)^n]u[-n-1]$$

- two-sided: $\rho_r < |z| < \rho_l \;\Rightarrow\; \rho_r = 0.5, \rho_l = 2$ and hence

$$x[n] = -A2^n u[-n-1] + +B(-0.5)^n u[n]$$

End

Answer to P 6.10: Key: to use the properties of z-transform.

- $x_1[n] = w[n] * v[n] \quad \leftrightarrow \quad X_1(z) = W(z)V(z)$
- $x_2[n] = n\tilde{x}[n] \;\leftrightarrow\quad X_2(z) = -z\frac{d\tilde{X}(z)}{dz}$
- Note $X_3(z) = \sum \frac{x[n]}{n+1} z^{-n} \quad \Leftrightarrow \quad \tilde{X}_3(z) \triangleq z^{-1}X_3(z) = \sum \frac{x[n]}{n+1} z^{-(n+1)}$. Since $x_3[n]$ is causal, $X_3(z)|_{z=\infty} = x_3[0]$ and $X_3(z)|_{z=\infty} = 0$. Note $\frac{d\tilde{X}_3(z)}{dz} = \sum x[n](-z^{-2})z^{-n} = z^{-2}X(z)$, then with $X(z) = \frac{1}{1-\alpha z^{-1}}$ one has

$$\tilde{X}_3(z) = \int_{+\infty}^{z} \xi^{-2}X(\xi)d\xi = \int_{+\infty}^{z} \xi^{-2}(1-\alpha\xi^{-1})d\xi \;\Rightarrow\; X_3(z) = z\tilde{X}_3(z)$$

- $x_4[n] = (n-1)^2u[n-1] \triangleq w[n-1]$ with $w[n] = n^2u[n] \triangleq nv[n]$, where $v[n] = nu[n]$. So, ...
- $x_5[n] = \sum_{m=0}^{n}(-1)^m u[n] = \frac{1}{2}[(-1)^n + 1]u[n] \quad X_5(z) = ...$

End

Answer to P 6.12:

- As given, $x[n]$ being causal implies $x[n] = 0,\ \forall\ n < 0$ and hence

$$X(z) = x[0] + x[1]z^{-1} + \cdots + x[k]z^{-k} + \cdots$$

If $X(z) = \frac{z(z-3/2)}{(z-1/3)(z-1/2)}$, $x[0] = X(\infty) = 1$ and $x[1] = \lim_{z\to\infty}[X(z) - 1]z$.

- Denote $y[n] \stackrel{\triangle}{=} x[n] - x[n-1]$. Then $Y(z) = (1 - z^{-1})X(z)$ and hence

$$\begin{aligned}(1 - z^{-1})X(z) &= \sum^{+\infty} y[n]z^{-n} = \cdots + y[-1]z + y[0] + y[1]z^{-1} + \cdots + y[n]z^{-n} + \cdots \\ &= \cdots + (x[-1] - x[-2])z + (x[0] - x[-1])y[0] + (x[1] - x[0])z^{-1} + \cdots \\ &\quad +(x[n] - x[n-1])z^{-n} + \cdots\end{aligned}$$

Clearly, with $x[-\infty] = 0$

$$\begin{aligned}\lim_{z\to 1}(1 - z^{-1})X(z) &= x[-\infty] + \cdots + (x[0] - x[-1])y[0] + (x[1] - x[0]) \\ &\quad + \cdots + (x[n] - x[n-1]) + \cdots = x[+\infty] \qquad \textbf{End}\end{aligned}$$

Answer to P 6.16: Note

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5y(t) = \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x(t) \quad \Rightarrow \quad H(s) = \frac{s^2 - 2s + 1}{s^2 + s + 5}$$

- Find the poles and zeros:

$$s^2 + s + 5 = 0 \;\Rightarrow\; poles : p_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \times 5}}{2} = -0.5 \pm j\sqrt{19}$$

$$s^2 - 2s + 1 = 0 \;\Rightarrow\; zeros : q_1 = q_2 = 1$$

If $G(s)$ is the inverse, then $H(s)G(s) = 1$, that is

$$G(s) = \frac{1}{H(s)} = \frac{s^2 + s + 5}{s^2 - 2s + 1}$$

which has two poles at $s = 1$. If it is causal, $ROC_g = \{s : 1 < \mathcal{R}e(s)\}$ does not contain $s = j\omega$ and hence it is unstable. Therefore, the system has NO stable and causal inverss (system).

- Nothing needs to be done as such an inverse does not exist. **End**

Answer to P 6.17: Note

$$y = h * x \quad \Leftrightarrow \quad Y = HX \quad \Rightarrow \quad H = \frac{Y}{X}, \quad ROC_y \supseteq ROC_h \cap ROC_x$$

- $x[n] = \delta[n] + \frac{1}{4}\delta[n-1] \;\; \leftrightarrow \;\; X(z) = 1 + \frac{1}{4}z^{-1}, \;\; ROC_x = \{z : z \neq 0\}$ and $y[n] = \delta[n] - \frac{3}{4}\delta[n-1] \;\; \leftrightarrow \;\; Y(z) = 1 - \frac{3}{4}z^{-1}, \;\; ROC_y = \{z : z \neq 0\}$. So,

$$H(z) = \frac{1 - \frac{3}{4}z^{-1}}{1 + \frac{1}{4}z^{-1}} = D + \frac{A}{1 + \frac{1}{4}z^{-1}}$$

with two possible ROC_h, yielding

$$h_c[n] = D\delta[n] + A(\frac{1}{4})^n u[n], \quad h_a[n] = D\delta[n] - A(\frac{1}{4})^n u[-n-1]$$

- For the 2nd situation, do it yourself using the same approach.

When $x[n] = \alpha^n u[-n] = \delta[n] + \alpha^n u[-n-1]$, $X(z) = 1 + \frac{1}{1-\alpha z^{-1}}, \quad |z| < |\alpha|$.

- If $|\alpha| < 1/4$, ROC_{h_c} and $|z| < |\alpha|$ have no intersection. So, the only possible output is

$$Y(z) = H(z)X(z), \quad |z| < |\alpha|, \quad y[n] = x[n] + [A(1/4)^n + B\alpha^n]u[-n-1]$$

- If $|\alpha| > 1/4$, both ROC_{h_c} and ROC_a have a non-empty intersection with $|z| < |\alpha|$ and hence there are two possible outputs:

$$y_1[n] = h_c[n] * x[n], \quad y_2[n] = h_a[n] * x[n]$$

Please find them using transfer-domain approach as in the first part.

End

Answer to P 6.20: The rational $H(z)$ has a general form

$$H(z) = \kappa \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

- As the inverse system is $G(z) = 1/H(z)$, Conditions (iii) and (iv) suggest $M = 1, N = 2$, that is $H(z) = \kappa \frac{z - z_1}{(z + j/2)(z - p_2)}$.

- Condition (ii) implies $p_2 = p_1^* = j/2$, leading to

$$H(z) = \kappa \frac{z^{-1}(1 - z_1 z^{-1})}{(1 + j/2 z^{-1})(1 - j/2 z^{-1})} = \kappa \frac{z^{-1}(1 - z_1 z^{-1})}{1 + 1/4 \ z^{-2}}$$

- Condition (v) tells:

$$\sum_{n=0}^{+\infty} h[n] 2^{-n} = H(2) = 0 \quad \Leftrightarrow \quad z_1 = 2 \quad \Rightarrow H(z) = \kappa \frac{z^{-1}(1 - 2z^{-1})}{1 + 1/4 \ z^{-2}}$$

and[2] $h[1] = 1$ means $\tilde{H}(z) = z(H(z) - h[0])$ satisfies $\tilde{H}(\infty) = 1$. Since

[2] There is a typo in the question: replace $h[0] = 0$ with $h[1] = 1$.

$h[0] = H(\infty) = 0$, we have $\kappa = 1$ and finally

$$H(z) = \frac{z^{-1}(1 - 2z^{-1})}{1 + 1/4\ z^{-2}}, \quad 1/2 < |z|$$

- As the system is causal and has its (finite) poles all are inside $|z| = 1$, it is stable.

- The inverse system is $G(z) = 1/H(z)$, i.e.,

$$G(z) = \frac{z^2 + 1/4}{z(z-2)} = D + \frac{B}{z} + \frac{A}{z-2} \quad \leftrightarrow \quad g[n] = \ D\delta[n] + B\delta[n-1] + A\tilde{g}[n]$$

where $\tilde{g}[n] \quad \leftrightarrow \quad \tilde{G}(z) = \frac{1}{z-2}$. Clearly, there are two possible $\tilde{g}[n]$:

$$\tilde{g}_1[n] = 2^{n-1}u[n-1], \quad \tilde{g}_2[n] = -2^{n-1}u[-n]$$

As seen, no matter $g[n] = D\delta[n] + B\delta[n-1] + A\tilde{g}_1[n]$ or $g[n] = D\delta[n] + B\delta[n-1] + A\tilde{g}_2[n]$, the system is non-causal, while the system is stable for $g[n] = D\delta[n] + B\delta[n-1] + A\tilde{g}_2[n]$. **End**

Answer to P 6.22: Given that

$$\frac{d^3y(t)}{dt^3} + 6\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 6y(t) = x(t)$$

and $x(t) = e^{-4t}u(t)$ with $y(0_-) = 1$, $\frac{dy(t)}{dt}|_{t=0_-} = -1$, $\frac{d^2y(t)}{dt^2}|_{t=0_-} = 1$, the output for $t \geq 0_-$ is $y(t) = y_{zs}(t) + y_{zi}(t)$, where

- Zero-state response $y_{zs}(t)$ can be obtained using (bilateral) LT:

  Applying LT both sides gives

$$Y_{zs}(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}\, X(s) \stackrel{\triangle}{=} H(s)\, \frac{1}{s+4}$$

  with $ROC_x = \{s: \ -4 < Re(s)\}$ and

$$H(s) = \frac{1}{(s+1)(s^2+5s+6)} = \frac{1}{(s+1)(s+2)(s+3)}$$

  with $ROC_h = \{s: \ -1 < Re(s)\}$ since the system is causal.

As $ROC_{y_{zs}} \supseteq ROC_x \cap ROC_h = \{s : -1 < Re(s)\}$.

$$Y_{zs}(s) = H(s)X(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

and therefore, the zero-state response is

$$y_{zs}(t) = [Ae^{-t} + Be^{-2t} + Ce^{-3t} + De^{-4t}]u(t)$$

As to the constants A, B, C, D, you should have no problem to find them, my dear friends!

- Zeros-input response $y_{zi}(t)$ can be obtained using unilateral LT:

  With $\alpha = y(0_-) = 1$, $\beta = \frac{dy(t)}{dt}|_{t=0_-} = -1$, $\gamma = \frac{d^2y(t)}{dt^2}|_{t=0_-} = 1$, the zero-input response is the solution of the following

$$\frac{d^3y_{zi}(t)}{dt^3} + 6\frac{d^2y_{zi}(t)}{dt^2} + 11\frac{dy_{zi}(t)}{dt} + 6y_{zi}(t) = 0$$

which can be solved using the unilateral LT:

$$s[s[sY_{zi}(s)-\alpha)]-\beta]]-\gamma+6[s[sY_{zi}(s)-\alpha]-\beta]$$
$$+11[sY_{zi}(s)-\alpha]+6Y_{zi}(s)=0$$

Equivalently,

$$Y_{zi}(s)=\frac{as^2+bs+c}{s^3+6s^2+11s+6}=\frac{d_1}{s+1}+\frac{d_2}{s+2}+\frac{d_3}{s+3}$$

Please find a, b, c and hence d_1, d_2, d_3 yourself.

The zeros-input response $y_{zi}(t)$, which is a right-sided signal, is then

$$y_{zi}(t)=[d_1e^{-t}+d_2e^{-2t}+d_3e^{-3t}]u(t)$$

- The total output $y(t)$ is then given by

$$y(t)=y_{zi}(t)+y_{zs}(t)=[(A+d_1)e^{-t}+(B+d_2)e^{-2t}+(C+d_3)e^{-3t}+De^{-4t}]u(t)$$

End

Answer to P 6.23: Given that

$$y[n-1] + 2y[n] = x[n]$$

- Zeros-input response $y_{zi}[n]$ with $y[-1] = 2$:

  With $\alpha = y[-1] = 2$, the zero-input response is the solution of the following

  $$y_{zi}[n-1] + 2y_{zi}[n] = 0$$

  which can be solved using the unilateral ZT:

  $$z^{-1}Y_{zi}(z) + \alpha + 2Y_{zi}(z) = 0$$

  Equivalently,

  $$Y_{zi}(z) = \frac{\alpha}{z^{-1}+2} = \frac{1}{1+1/2z^{-1}} \quad \rightarrow \quad y_{zi}[n] = (-1/2)^n u[n]$$

- Zero-state response $y_{zs}[n]$ can be obtained using (bilateral) ZT:

  Applying LT both sides gives

$$z^{-1}Y_{zs}(z) + 2Y_{zs}(z) = X(z) \Rightarrow H(z)\ \frac{1/2}{1+(1/2)z^{-1}}$$

  with $ROC_h = \{z:\ 1/2 < |z|\}$ (why ?) and for $x[n] = (1/4)^n u[n]$,

$$X(z) = \frac{1}{1-(1/4)z^{-1}}, \quad ROC_x = \{z:\ 1/4 < |z|\}$$

  As $ROC_{y_{zs}} \supseteq ROC_x \cap ROC_h = \{z : 1/2 < |z|\}$.

$$Y_{zs}(z) = H(z)X(z) = \frac{A}{1+1/2z^{-1}} + \frac{B}{1-1/4z^{-1}}$$

  and therefore, the zero-state response is

$$y_{zs}[n] = [A(-1/2)^n + B(1/4)^n]u[n]$$

- The total output $y[n]$ is then given by

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

End

Solutions for Chapter 7

To be given.