Fundamentals of Signals and Systems

Chapter 5 - Discrete Processing of Analog Signals

Outline of Topics

Analog-to-Discrete Conversion

To convert a continuous-time signal, i.e., analog signal, into a discrete one.

Mathematically, sampling is described with

$$x(t) \;\rightarrow\; x[n] = x(nT_s) \tag{1}$$

Physically, a sampling can be implemented with an electronic switch.

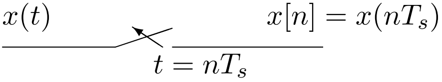

Fig. 5.1: Implementation of sampling operation in time domain

Ideally, sampling should satisfy: $x[n] \Leftrightarrow x(t)$.

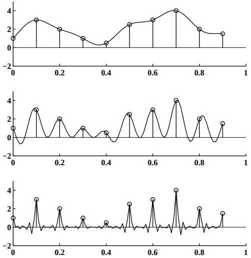

Fig. 5.2: Three continuous-time signals yielding the same $x[n]$ when sampled with the same period T_s.

Fig. 5.2 implies that all

$$x_1(t),\ x_2(t),\ x_3(t)\ \textit{sampling with}\ T_s\ \Rightarrow\ x[n]$$

Under what conditions $x(t)$ can be uniquely determined by its samples $x[n]$? And how serious the situation is if these conditions are not satisfied?

Let $x(t) \leftrightarrow \tilde{X}(j\omega)$, $x[n] \leftrightarrow X(e^{j\Omega})$. Note

$$x[n] \Leftrightarrow x(t) \Leftrightarrow X(e^{j\Omega}) \Leftrightarrow \tilde{X}(j\omega)$$

It follows from the IFT of $\tilde{X}(j\omega)$ and (1) that

$$x[n] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{X}(j\omega)e^{j\omega n T_s} d\omega$$

Divide $(-\infty, +\infty)$ for ω into a set of intervals with the points

$$\omega_k \triangleq \frac{2\pi k + \pi}{T_s}, \quad k = \cdots, -2, -1, 0, 1, 2, \cdots$$

Then $x[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int_{(2\pi k-\pi)/T_s}^{(2\pi k+\pi)/T_s} \tilde{X}(j\omega)e^{j\omega n T_s} d\omega$ and hence with $\xi = T_s\omega$

$$x[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\frac{1}{T_s}\int_{(2\pi k-\pi)}^{(2\pi k+\pi)} \tilde{X}(j\xi/T_s)e^{j\xi n} d\xi$$

With some manipulations, it can be shown that

$$\begin{cases} X(e^{j\Omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty}\tilde{X}(j\frac{\Omega+2k\pi}{T_s}) \\ (\omega = \frac{\Omega}{T_s}, \ \ \omega_s \triangleq \frac{2\pi}{T_s}) \\ X(e^{jT_s\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty}\tilde{X}(j(\omega + k\omega_s)) \end{cases} \quad (2)$$

As known, $x(t) = \frac{\omega_M}{2\pi}(\frac{sin(\omega_M t/2)}{\omega_M t/2})^2 \leftrightarrow \tilde{X}(j\omega) = (1 - \frac{|\omega|}{\omega_M})w_{2\omega_M}(\omega)$.

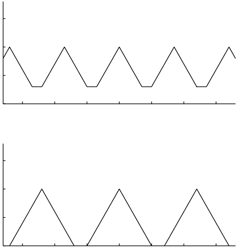

Fig. 5.3: (a) $|X(e^{j\omega T_s})|$ with $T_s = \frac{4\pi}{3\omega_M}$; (b) $|X(e^{j\omega T_s})|$ with $T_s = \frac{4\pi}{5\omega_M}$.

Theorem

Suppose $x(t)$ *is bandlimited:*

$$\tilde{X}(j\omega) = 0, \ \forall \ |\omega| \ \geq \ \omega_M$$

Then $x(t)$ *can be recovered from its samples* $x[n] = x(nT_s)$ *if*

$$\omega_s \triangleq \frac{2\pi}{T_s} > 2\omega_M \triangleq \ \omega_N \tag{3}$$

where ω_N *is usually referred to as the* Nyquist rate.

This is also called *Nyquist Theorem* or Sampling theorem.

Example 5.1 : Let $x(t)$ be a signal with a band limited to ω_M and $p_\tau(t)$ is a periodic signal:

$$p_\tau(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{\tau} w_\tau(t - kT)$$

Analyze the spectrum of $\tilde{x}(t) = x(t)p_\tau(t)$.

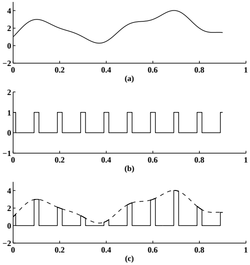

Fig. 5.4: A mixer equivalent to a sampler. (a) $x(t)$; (b) $\tau p_\tau(t)$; (c) $\tau\tilde{x}(t)$.

Solution: First of all, $p_\tau(t) = \sum_k c[k]e^{j\omega_s kt}$, where

$$c[k] = \tau^{-1}\frac{1}{T_s}\tau\frac{sin(k\omega_s\tau/2)}{k\omega_s\tau/2} = \frac{1}{T_s}\frac{sin(k\omega_s\tau/2)}{k\omega_s\tau/2}, \ \forall\ k$$

Therefore, $\tilde{x}(t) = x(t)p_\tau(t) \ \leftrightarrow \ \tilde{X}(j\omega) = \sum_k c[k]X(j(\omega - k\omega_s))$. One observes

$$\lim_{\tau/T_s\to 0} c[k] = 1/T_s \ \Rightarrow \ x[n] = x(nT_s) \ \leftrightarrow \ X(e^{jT_s\omega}) \approx \tilde{X}(j\omega)$$

This implies that the ideal sampling can be realized using a mixer.

Discrete-to-Analog Conversion

This is done with

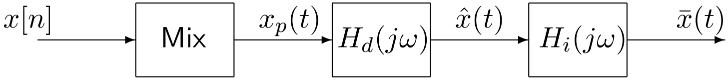

Fig. 5.5: Structure of the discrete-analog-convertor (DAC).

where Mix is a device to execute $x_p(t) = \sum_{m=-\infty}^{+\infty} x[m]\delta(t - mT_s)$,
$h_d(t) \leftrightarrow H_d(j\omega)$ and

$$x[n] \rightarrow \hat{x}(t) = \sum_{m=-\infty}^{+\infty} x[m]h_d(t - mT_s) \tag{4}$$

How is $\hat{x}(t)$ close to the original $x(t)$?

Applying the FT to both sides of the above equation yields

$$\hat{X}(j\omega) \; = \; \sum_{m=-\infty}^{+\infty} x[m]H_d(j\omega)e^{-jm\omega T_s} = H_d(j\omega)X(e^{j\omega T_s}) \qquad (5)$$

Take

$$H_d(j\omega) = T_s w_{2\omega_l}(\omega) \triangleq H_0(j\omega) \; \Leftrightarrow \; h_0(t) = T_s \frac{sin(\omega_l t)}{\pi t} \qquad (6)$$

It turns out from (4) that the corresponding reconstructed signal is

$$\hat{x}_0(t) = T_s \sum_{m=-\infty}^{+\infty} x[m] \frac{sin(\omega_l(t - mT_s))}{\pi(t - mT_s)} \qquad (7)$$

Under the recovery condition (3), it follows from (2), (5), and (6) that $\hat{X}(j\omega) = \tilde{X}(j\omega)$ and hence $\hat{x}_0(t) \equiv x(t)$ as long as

$$w_M < \omega_l \leq \frac{\omega_s}{2} \tag{8}$$

Note that $\hat{x}_0(nT_s) = x[n]$ is always true if $\omega_l = \frac{k\omega_s}{2}$ for any integer $k > 1$ which violates (8), even though $\hat{x}_0(t) \neq x(t)$ for all $t \neq nT_s$.

As seen, (4) can be rewritten into

$$\hat{x}(t) = \sum_{m=-\infty}^{n} x[m]h_d(t - mT_s) + \sum_{m=n+1}^{+\infty} x[m]h_d(t - mT_s)$$

In an *on-line* (*real-time*) signal reconstruction system, for $t < (n+1)T_s$ the samples of discrete-time sequence $x[m]$ are available just for $m \leq n$.

To make $\hat{x}(t)$ implementable, it suffices to ensure

$$h_d(t) \equiv 0, \; \forall \, t < 0$$

This means that *the reconstruction system* $H_d(j\omega)$ *should be causal*!

$h_0(t) = T_s \frac{sin(\omega_l t)}{\pi t}$ is non-causal. Though of theoretical importance, in practice it has to be replaced with an implementable reconstruction system.

Zero order hold (ZOH)

$$h_1(t) \triangleq w_{T_s}(t - T_s/2) \Rightarrow \hat{x}_1(t) = \sum_{m=-\infty}^{+\infty} x[m] h_1(t - mT_s)$$

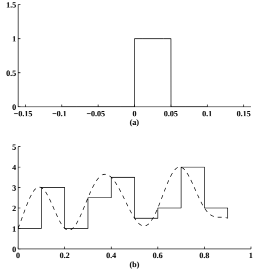

Fig. 5.6: Reconstruction of $x(t)$ (dotted-line) using ZOH, where $T_s = 0.1$.

Clearly, the reconstructed signal $\hat{x}_1(t)$ is very different from $x(t)$.

Why is that ?

Note: $H_1(j\omega) = 2\frac{sin(\omega T_s/2)}{\omega} e^{-j\omega T_2/2}$, $\hat{x}_1(t)$ different from $x(t)$ due to: i) there is a time delay of $T_s/2$; ii) $|H_1(j\omega)|$ is not flat within $[-\omega_M, \omega_M]$, and iii) $|H_1(j\omega)|$ is not nil constantly outside $[-\omega_M, \omega_M]$. This is demonstrated with Fig. 5.6.

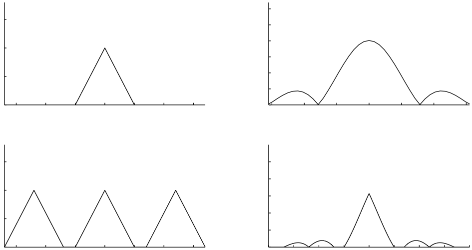

Fig. 5.7: Demonstration of the imperfection of the ZOH reconstruction.

$\boxed{\textit{First order hold (FOH)}}$: $h_2(t) \triangleq \frac{t}{T_s}[w_{T_s}(t-\frac{T_s}{2}) - w_{T_s}(t-\frac{3T_s}{2})]$

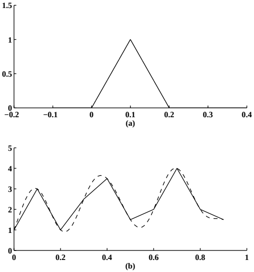

Fig. 5.8: Reconstruction of $x(t)$ (dotted-line) using $\hat{x}_2(t) = \sum_n x[n]h_2(t-nT_s)$.

It can be shown that

$$\hat{x}_2(t) = \sum_n [x[n] + \frac{x[n+1]-x[n]}{T_s}(t-nT_s)]h_1(t-nT_s)$$

Clearly, the first order hold yields a better result. Why is that?

To remove these undesired high frequency components, the continuous-time signal $\hat{x}(t)$ from the hold is usually smoothed by a smoothing analog low-pass filter - *anti-imaging* filter $H_i(j\omega)$.

Based on (5), we have

$$\bar{X}(j\omega) = H_i(j\omega)H_d(j\omega)X(e^{j\omega T_s}) \tag{9}$$

What is the best $H_i(j\omega)$?

Note: Sampling Theorem yields a sufficient condition only.

Aliasing effect

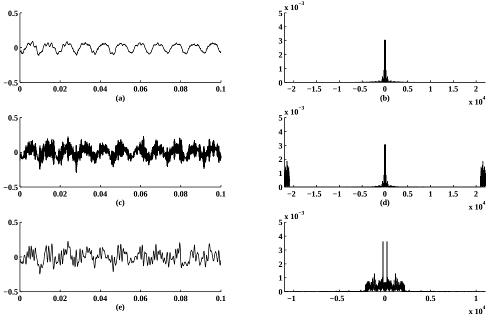

Fig. 5.9: (a)-(b) — for $x_0(t)$; (c)-(d) — for $x_1(t)$; (e)-(f) — for $\hat{x}_0(t)$.

If we discretize this signal with a sampling frequency $8.00\ kHz$, the high frequency spectrum than $20\ kHz$ will be folded around the frequency ω_s. This aliasing effect can damage badly the reconstructed signal.

In many applications, the analog signals have a very wide bandwidth of spectrum. In order to reduce the effect of the aliasing phenomenon, the signals are pre-filtered before sampled. Such a filter, denoted as $H_a(j\omega)$, is usually called *aliasing filter* and is intended to limit the spectrum of the output $x_a(t)$ to $\omega_s/2$ without damaging too much the information of $x(t)$.

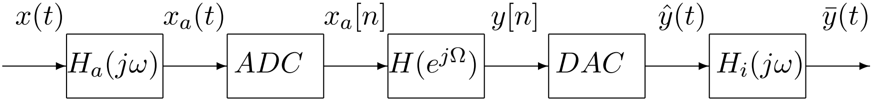

Fig. 5.10: Block-diagram of discrete processing of continuous-time signals.

The DTFT $X(e^{j\Omega})$ of $x[n]$ is a function of the continuous variable Ω.

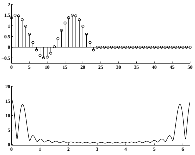

Fig. 5.11: (a) a finite duration $x[n]$ of 24 samples; (b) the magnitude spectrum of $X(e^{j\Omega})$.

The *discrete Fourier transform* (DFT) is the sampled DTFT of the same signal at the frequency points

$$\Omega_k = \frac{2\pi k}{N_s} \triangleq k\Omega_s, \quad k \in \mathcal{Z} \tag{10}$$

where $\Omega_s = 2\pi/N_s$ is the sampling period with N_s a positive integer.

The N_s-point DFT of $x[n]$ is then defined as

$$X[k] \quad \triangleq \quad \sum_{n=-\infty}^{+\infty} x[n]e^{-j\frac{2\pi kn}{N_s}} = X(e^{j\Omega_s k}) \tag{11}$$

Note $X[k]$ is periodic with period N_s. So, the DTFS suggests

$$X[k] = \sum_{m=0}^{N_s-1} \tilde{x}[m] e^{-j\Omega_s mk} \tag{12}$$

where[1] the $\tilde{x}[m]$ (i.e., the DTFS coefficients of $X[k]$) are given by

$$\tilde{x}[m] = \frac{1}{N_s} \sum_{k=0}^{N_s-1} X[k] e^{j\Omega_s km}, \; m = 0, 1, \cdots, N_s - 1 \tag{13}$$

called the N_s-point *inverse discrete Fourier transform* (IDTF) of $X[k]$.

It can be shown that

$$\tilde{x}[m] = \sum_{l=-\infty}^{+\infty} x[m - lN_s] \tag{14}$$

[1] Here, the periodic signal $X[k]$ is decomposed using bases $\{e^{-j\Omega_s m}, \; m = 0, 1, \cdots, N_s - 1\}$ rather than $\{e^{j\Omega_s m}, \; m = 0, 1, \cdots, N_s - 1\}$ as they are identical.

It is easy understand that (14) is in general not invertible, this is to say that $x[n]$ can not be uniquely determined from $\tilde{x}[n]$ for any N_s.

However, if $x[n]$ is of finite duration L, say $x[n] = 0,\ \forall n < N_1,\ n > N_1 + L - 1$. Then

$$\tilde{x}[n] = x[n], \quad n = N_1, N_1 + 1, \cdots, N_1 + L - 1$$

as long as

$$N_s \geq L \tag{15}$$

The above is actually consistent with the sampling theorem specified with (3), while (14) is the discrete counterpart of (2). It should be noted that (15) is just a sufficient condition for the sampling number N_s to ensure the reconstruction of $x[n]$ from its DFT.

Example 5.2: Consider the signal $x[n]$ shown in Fig. 5.11, which is of a duration of $L = 24$ samples. Computed its N_s-DFT and then corresponding IDFT $\tilde{x}[n]$ for different $N_s = 12, 24, 28$.

Case I: $N_s = 12 < L$, corresponding a under-sampling situation.

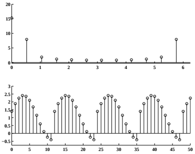

Fig. 5.12: (a) 12-point DFT of $x[n]$ shown in Fig. 5.11 **The x-axis seems wrong**; (b) the IDFT $\tilde{x}[n]$.

$\tilde{x}[n]$ is totally different from $x[n]$ for $n = 13, 14, \cdots, 23$ due to aliasing effect.

Case II: $N_s = 24 = L$, corresponding to the critical sampling. The results are represented in Fig. 5.13.

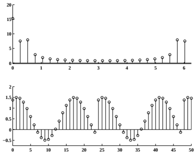

Fig. 5.13: (a) 24-point DFT of $x[n]$ shown in Fig. 5.11; (b) the IDFT $\tilde{x}[n]$.

For this case, as observed, $\tilde{x}[n] = x[n],\ n = 0, 1, 2, \cdots, 23$, which means that $x[n]$ can be recovered from $X[k]$ as there is no overlapping between $x[n]$ and $x[n + 24m]$ for any non-zero integer.

Case III: $N_s = 28 > L$, corresponding to an over-sampling. The results are represented in Fig. 5.14.

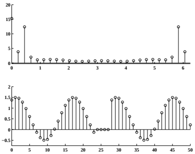

Fig. 5.14: (a) 28-point DFT of $x[n]$ shown in Fig. 5.11; (b) the IDFT $\tilde{x}[n]$.

As expected, $\tilde{x}[n] = x[n],\ n = 0, 1, 2, \cdots, 23$, which is confirmed in Fig. 5.14(a). Though over-sampling provides a better resolution than the critical-sampling, both have exactly the same information on $x[n]$.