Fundamentals of Signals and Systems

Chapter 6 - Transform-domain Approaches

Outline of Topics

Motivation

It is regarding an extension of Fourier transform and Discrete-time Fourier transform.

Fourier transform works well. Why do we need another transform ?

Questions:

- What is the FT of $x(t) = e^t u(t)$?
- What is the frequency response of the system with $h(t) = e^t u(t)$?

Based on what we have learnt so far, we know that the Fourier transform can not be applied to this type of signals.

However, consider $\tilde{x}(t) \triangleq x(t)e^{-\sigma t} = e^{-(\sigma-1)t}u(t)$.

$\tilde{x}(t)$ has an FT equal to $\frac{1}{(\sigma-1)+j\omega}$ as long as $\sigma > 1$.

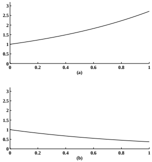

Fig. 6.1: Waveforms of $e^{-(\sigma-1)t}u(t)$. (a) $\sigma = 0$; (b) $\sigma = 2$.

So, for any $x(t)$ consider the FT of $\tilde{x}(t) = x(t)e^{-\sigma t}$

$$\begin{aligned}\tilde{X}(j\omega) &= \int_{-\infty}^{+\infty} \tilde{x}(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}dt \\ &\triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt, \qquad s = \sigma + j\omega\end{aligned}$$

$\boxed{\textit{Definition}}$: The (bilateral) *Laplace transform* (LT) of a signal $x(t)$ is defined as

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt \tag{1}$$

where $s = \sigma + j\omega$ is a complex variable such that the above is finite.

- the FT of a signal, if it exists, is just a special case[1] of the Laplace transform for which $\sigma = 0$, that is $s = j\omega$.
- More importantly, even the FT of $x(t)$ does not exist,

$$X(s) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

may be finite for some values of s. So, the LT is an *extension* of the FT and allows us to handle a *larger* class of signals and systems.

[1] Note though $x(t) = e^{j\omega_0 t} \leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$, we do not consider that $x(t) = e^{j\omega_0 t}$ has an LT this is because that we do not allow impulses in Laplace transforms (in order to keep Laplace transforms analytical).

Region of convergence (ROC) - The key !

The set of vales on the s-plane for which the integral in (1) is finite, i.e.,

$$ROC_x = \{s : |\int_{-\infty}^{+\infty} x(t)e^{-st}dt| < +\infty\} \quad (2)$$

is called *region of convergence* (ROC).

It is crucial to keep in mind that *a Laplace transform should be associated with an ROC*. This is demonstrated by the following example.

$\boxed{\textbf{Example 6.1}}$: Compute the LT of $x_1(t) = e^{\alpha t}u(t)$ and $x_2(t) = -e^{\alpha t}u(-t)$, where $\alpha \in \mathcal{C}$ is *any* constant.

Solution: First of all, for $x_1(t) = e^{\alpha t}u(t)$ we have

$$\begin{aligned} X_1(s) &= \int_{-\infty}^{+\infty} e^{\alpha t}u(t)e^{-st}dt = \int_{0}^{+\infty} e^{-(s-\alpha)t}dt \\ &= \lim_{T\to+\infty}\int_{0}^{T} e^{-(s-\alpha)t}dt = \lim_{T\to+\infty}\frac{1}{s-\alpha}[1-e^{-(s-\alpha)T}] = \frac{1}{s-\alpha} \end{aligned}$$

if $\mathcal{R}_e(s) > \mathcal{R}_e(\alpha)$. Therefore,

$$e^{\alpha t}u(t) \leftrightarrow \frac{1}{s-\alpha}, \ \forall\, s \in ROC_{x_1} = \{s:\ \mathcal{R}_e(\alpha) < \mathcal{R}_e(s)\} \tag{3}$$

For $x_2(t) = -e^{\alpha t}u(-t)$, using a similar procedure one can show

$$-e^{\alpha t}u(-t) \leftrightarrow \frac{1}{s-\alpha}, \ \forall\, s \in ROC_{x_2} = \{s:\ \mathcal{R}_e(s) < \mathcal{R}_e(\alpha)\} \tag{4}$$

As seen, both $X_1(s)$ and $X_2(s)$ converge to the *same* $\frac{1}{s-\alpha}$ but with *different* ROC.

- *Right-sided* signals: if there exists a some finite constant T_r such that

$$x(t) = 0, \ \ \forall t \leq T_r$$

- *Left-sided* signals: if there exists a some finite constant T_l such that

$$x(t) = 0, \ \ \forall t \geq T_l$$

A signal is said *two-sided* if it does not belong to any of the two classes. $x_r(t)$ and $x_l(t)$ are said to be *causal* and *anti-causal* if $T_r = 0$ and $T_l = 0$, respectively.

In the sequel, it is assumed that for $s_0 \in ROC_x$, there exists a small constant $\epsilon > 0$ such that $|s - s_0| < \epsilon \in ROC_x$ and that

$$\int_{-\infty}^{+\infty} |x(t)e^{-st}|dt < +\infty, \ \ \forall\, s \ \in |s - s_0| < \epsilon$$

that is the wide-sense integral absolutely converges around $s = s_0$.

Properties of ROC

Property 1: Let $x_r(t)$ be a right-sided signal and S_r be the set of all s such that $\int_{T_r}^{+\infty} |x_r(t)e^{-st}|dt < +\infty$. Furthermore, denote $\sigma_r \triangleq \inf_{s\in S_r} \mathcal{R}_e(s)$. Then[2]

$$ROC_{x_r} \supseteq \begin{cases} \{s: \sigma_r < \mathcal{R}_e(s)\}, & if\ T_r \geq 0 \\ \{s: \sigma_r < \mathcal{R}_e(s) < +\infty\}, & if\ T_r < 0 \end{cases} \quad (5)$$

which is shown in Fig. 6.2(a).

See the typical LT pair specified by (3).

$$e^{\alpha t}u(t) \leftrightarrow \frac{1}{s-\alpha}, \quad ROC = \{s: \mathcal{R}_e(\alpha) < \mathcal{R}_e(s)\}$$

[2]Note: for right-sided but non-causal signal, the ROC does NOT contain $\mathcal{R}_e(s) = +\infty$ as the first terms is infinite for s with $\mathcal{R}_e(s) = +\infty$:

$$\int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{T_r}^{0} x(t)e^{-st}dt + \int_{0}^{+\infty} x(t)e^{-st}dt$$

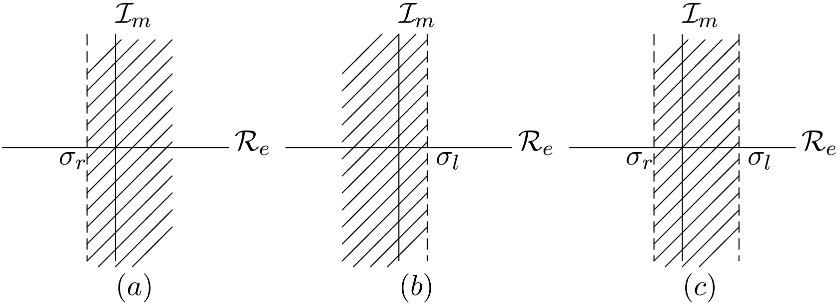

Fig. 6.2: Three types of region of convergence. (a) $\{s : \sigma_r < \mathcal{R}_e(s)\}$ for right-sided signals; (b) $\{s : \mathcal{R}_e(s) < \sigma_l\}$ for left-sided signals; (c) $\{s : \sigma_r < \mathcal{R}_e(s) < \sigma_l\}$ for two-sided signals.

Property 2: Let $x_l(t)$ be a left-sided signal and S_l be the set of all s such that $\int_{-\infty}^{T_l} |x_l(t)e^{-st}|dt < +\infty$. Denote $\sigma_l \triangleq \sup_{s\in S_l} \mathcal{R}_e(s)$. Then

$$ROC_{x_l} \supseteq \begin{cases} \{s: \quad \mathcal{R}_e(s) < \sigma_l\}, & if \quad T_l \leq 0 \\ \{s: -\infty < \mathcal{R}_e(s) < \sigma_l\}, & if \quad T_l > 0 \end{cases} \tag{6}$$

as shown in Fig. 6.2(b). See the typical LT pair specified by (4).

Property 3: Let $x(t)$ be a two-sided signal and S be the set of all s such that $\int_{-\infty}^{+\infty} |x(t)e^{-st}|dt < +\infty$. Denote $\sigma_r \triangleq \inf_{s\in S} \mathcal{R}_e(s)$, $\sigma_l \triangleq \sup_{s\in S} \mathcal{R}_e(s)$. Then

$$ROC_x \supseteq \{s: \sigma_r < \mathcal{R}_e(s) < \sigma_l\} \tag{7}$$

Note: $x(t)e^{-st}$ is absolutely integrable on its ROC_x, then ROC_x is equal to the right side of (5), (6) and (7), respectively.

Example 6.2 : Consider $x(t) = e^{-\beta|t|}$ with β real. Find out its Laplace transform and the region of convergence.

Solution: Let $x_r(t) = e^{-\beta t}u(t)$, $x_l(t) = e^{\beta t}u(-t)$. Clearly,

$$X(s) \;=\; \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{-\infty}^{+\infty} x_l(t)e^{-st}dt \;+\; \int_{-\infty}^{+\infty} x_r(t)e^{-st}dt$$

Note that the 1st term is the LT of $x_l(t)$, which, according to (4), converges to

$$x_l(t) = e^{\beta t}u(-t) \;\Leftrightarrow\; X_l(s) = \frac{-1}{s-\beta}, \quad ROC_l = \{s : \mathcal{R}_e(s) < \beta\}$$

and the 2nd term is the LT of $x_r(t)$ and, according to (3), converges to

$$x_r(t) = e^{-\beta t}u(t) \;\Leftrightarrow\; X_r(s) = \frac{1}{s+\beta}, \; ROC_r = \{s : -\beta < \mathcal{R}_e(s)\}$$

One then concludes that $x(t)$ has LT if and only if $\beta > 0$:

$$x(t) = e^{-\beta|t|} \;\Leftrightarrow\; X(s) = X_l(s) + X_r(t) = \frac{-2\beta}{s^2-\beta^2}$$

Property 4: For a finite duration and absolutely integrable signal, the region of convergence is the entire s-plane.

Example 6.3 : Consider $x(t) = e^{-\alpha t} w_T(t - T/2)$, where $T > 0$ is finite. Find the Laplace transform of this signal and the corresponding region of convergence.

Solution: By definition, we have

$$X(s) = \int_0^T e^{-\alpha t} e^{-st} dt = \frac{1}{s+\alpha}[1 - e^{-(s+\alpha)T}]$$

Note that at $s = -\alpha$, it follows from L'hôpital's rule that

$$X(-\alpha) = \lim_{s \to -\alpha} X(s) = \lim_{s \to -\alpha} \frac{\frac{d}{ds}(1 - e^{-(s+\alpha)T})}{\frac{d}{ds}(s+\alpha)} = T$$

So, $X(s)$ is finite for any finite s.

Poles & zeros

The complex number $s = p_k$ is said to be a *pole* of $X(s)$ if $X(p_k) = \infty$, and $s = z_k$ is a *zero* of $X(s)$, if $X(z_k) = 0$.

For example, as $x(t) = e^{-2t}u(t) \;\; \leftrightarrow \;\; X(s) = \frac{1}{s+2}$ for $-2 < \mathcal{R}_e(s)$, $s = -2$ is a (finite) pole of $X(s)$ and $s = \infty$ is a (infinite) zero of $X(s)$.

When $X(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are two co-prime polynomials in s, the *finite poles* of $X(s)$ are the roots of $D(s)$:

$$(finite \;\; poles): \;\; D(s) = 0 \;\; \Rightarrow \;\; \{p_k\}$$

and the *finite zeros* of $X(s)$ are given by

$$(finite \;\; zeros): \;\; N(s) = 0 \;\; \Rightarrow \;\; \{z_k\}$$

Example 6.4 : For $x(t) = [3e^{2t} - 2e^{-t}]u(t) + e^{3t}u(-t)$. Compute the Laplace transform $X(s)$ and determine the poles and zeros of $X(s)$.

Solution: Note $x(t) = 3x_1(t) - 2x_2(t) - x_3(t)$ with $x_k(t)$ defined below. According to (3) and (4), we have

$$\begin{aligned} x_1(t) &\triangleq e^{2t}u(t) \leftrightarrow X_1(s) = \frac{1}{s-2}, \ \{s: \ 2 < \mathcal{R}_e(s)\} \\ x_2(t) &\triangleq e^{-t}u(t) \leftrightarrow X_2(s) = \frac{1}{s+1}, \ \{s: \ -1 < \mathcal{R}_e(s)\} \\ x_3(t) &\triangleq -e^{3t}u(-t) \leftrightarrow X_3(s) = \frac{1}{s-3}, \ \{s: \ \mathcal{R}_e(s) < 3\} \end{aligned}$$

It then follows from $x(t) = 3x_1(t) - 2x_2(t) - x_3(t)$ that

$$\begin{aligned} X(s) &= 3\int_{-\infty}^{+\infty} x_1(t)e^{-st}dt - 2\int_{-\infty}^{+\infty} x_2(t)e^{-st}dt \\ &\quad - \int_{-\infty}^{+\infty} x_3(t)e^{-st}dt \\ &= \frac{3}{s-2} - \frac{2}{s+1} - \frac{1}{s-3} \\ &= \frac{5s-19}{(s+1)(s-2)(s-3)} \end{aligned}$$

with an ROC containing the intersection of the three above: $2 < \mathcal{R}_e(s) < 3$ and poles at $s = -1,\ 2,\ 3$, zeros: one at $s = 19/5$ and two at $s = \infty$.

Let $x(t) \leftrightarrow X(s)$. Generally speaking,
$ROC_x = \{s : \sigma_r < Re(s) < \sigma_l\}$.

Depending on σ_r, σ_l, there are three types of possible ROCs

- containing no (finite) poles: $\sigma_r = -\infty, \sigma_l = +\infty$, that is the s-plane;
- bounded by (finite) poles: with finite σ_r, σ_l;
- extending infinity: one of σ_r, σ_l is infinite.

Now, let us consider the following example.

$\boxed{\textbf{Example 6.5}}$: Given $X(s) = \frac{s+3}{(s+1)(s^2-2s+2)(s-2)}$, what are the possible ROC$_x$?

Solution: First of all, there are one zero: $z_1 = -3$ and four poles: $p_1 = -1$, $p_2 = 1 + j$, $p_3 = 1 - j$, $p_4 = 2$, as shown in the *pole-zero plot* by Fig. 6.3.

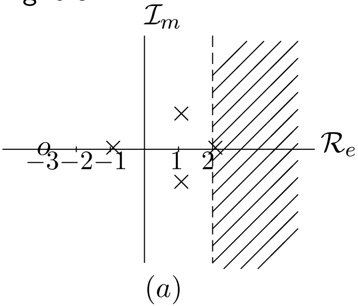

Fig. 6.3: Pole-zero plot ("×" for poles and "o" for zeros) and 4 possible ROCs (shadowed areas) for **Example 6.5**. (a) $x(t)$ is causal; (b) $x(t)$ is anti-causal.

Two possible two-sided signals:

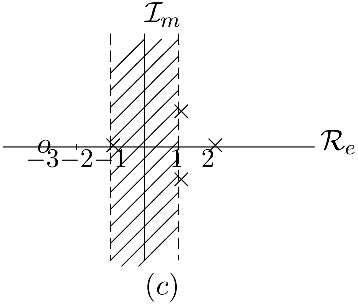

Fig. 6.3: Pole-zero plot ("×" for poles and "o" for zeros) and 4 possible ROCs (shadowed areas) for **Example 6.5**. (a) $x(t)$ is causal; (b) $x(t)$ is anti-causal.

Does there exist any other possible signal for it ?

Inverse Laplace transform (ILT)

It can be shown that

$$x(t) = \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \tag{8}$$

where σ is any constant such that $\sigma + j\omega \in ROC_x$. This is called the inverse Laplace transform (ILT) of $X(s)$.

When $X(s)$ is rational in s, i.e., $X(s) = \frac{N(s)}{D(s)}$ with the numerator and denominator $N(s), D(s)$ two polynomials in s, the ILT can be obtained easily without evaluating the integral (8). This is done with the help of *partial fraction expansion* (see **Appendix E**) of $X(s)$ and some LT pairs associating with their ROC.

Let us demonstrate it using the following example.

Example 6.6 : Let $X(s) = \frac{2s^2 - s + 3}{(s+2)(s+1)^2(s-2)}$ be the Laplace transform of a signal $x(t)$ with $ROC_x = \{s : -1 < Re(s) < 2\}$. Determine $x(t)$.

Solution: Note that

$$X(s) = \frac{2s^2 - s + 3}{(s+2)(s+1)^2(s-2)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{s-2} \quad (9)$$

Each of the terms that

- have poles in the region defined by $\mathcal{R}_e(s) \leq -1$ should have an ROC of the form $\{s : \sigma_r < \mathcal{R}_e(s)\}$, leading to a right-sided signal;
- have poles in the region defined by $\mathcal{R}_e(s) \geq 2$ should have an ROC of the form $\{s : \mathcal{R}_e(s) < \sigma_l\}$ and hence yields a left-sided signal.

Therefore, the inverse Laplace transform of $X(s)$ is

$$x(t) = [Ae^{-2t} + Be^{-t} + Cte^{-t}]u(t) - De^{2t}u(-t)$$

Multiplying both sides of (9) with $s+2$ and then letting $s \to -2$ yields

$$A = \lim_{s \to -2} (s+2)X(s) = \frac{2s^2 - s + 3}{(s+1)^2(s-2)}|_{s=-2} = -13/4$$

Similarly, we have

$$\begin{aligned} C &= \lim_{s \to -1} (s+1)^2 X(s) = \frac{2s^2 - s + 3}{(s+2)(s-2)}|_{s=-1} = -2 \\ D &= \lim_{s \to 2} (s-2)X(s) = \frac{2s^2 - s + 3}{(s+1)^2(s+2)}|_{s=2} = 1/4 \end{aligned}$$

For the coefficient B, multiplying both sides of (9) with $(s+1)^2$, differentiating them *w.r.t.* s, and letting s go to -1, we have

$$B = \lim_{s \to -1} \frac{d}{ds}[(s+1)^2 X(s)] = 3$$

What is the $x(t)$ if $ROC_x = \{s : -2 < \mathcal{R}_e(s) < -1\}$ is given ?

Properties of Laplace transform

Property	LT pair $x(t) \leftrightarrow X(s),\ x_k(t) \leftrightarrow X_k(s)$	ROC
Linearity	$\sum_k \alpha_k x_k(t) \ \leftrightarrow\ \sum_k \alpha_k X_k(s)$	$\supseteq \cap_k ROC_{x_k}$
Time shift	$x(t-\tau) \ \leftrightarrow X(s)e^{-s\tau}$	ROC$_x$
Time scaling	$x(\alpha t) \ \leftrightarrow\ \frac{1}{\lvert\alpha\rvert} X(\frac{s}{\alpha}),\ \alpha \neq 0$	$\sigma_r < \mathcal{R}_e(s/\alpha) < \sigma_l$
Multiplication by e^{ct}	$x(t)e^{ct} \ \leftrightarrow\ X(s-c)$	ROC$_x$ shifted with c
$\frac{d}{dt}$	$\frac{dx(t)}{dt} \ \leftrightarrow\ sX(s)$	$\supseteq ROC_x$
$\frac{d}{ds}$	$tx(t) \ \leftrightarrow\ -\frac{dX(s)}{ds}$	ROC$_x$
$x(t) = x_1(t) * x_2(t)$	$X(s) = X_1(s)X_2(s)$	$\supseteq ROC_{x_1} \cap ROC_{x_2}$

Table: Properties of Laplace transform

The procedure of proof for each property is therefore very much like that in Fourier transform.

For example, the following property

$$x(t) \leftrightarrow X(s), \quad ROC_x \quad \Rightarrow \quad tx(t) \leftrightarrow -\frac{dX(s)}{ds}, \; ROC_x$$

can be obtained by differentiating both sides of (1) with respect to s directly. Since $X(s)$ is analytical, $\frac{dX(s)}{ds}$ has the same ROC as $X(s)$ does.

Applying to this property to the Laplace transform pairs (3) and (4), we have

$$\begin{cases} t^m e^{-\alpha t} u(t) & \leftrightarrow & \frac{m!}{(s+\alpha)^{m+1}}, \; \{s : -\mathcal{R}_e(\alpha) < \mathcal{R}_e(s)\} \\ -t^m e^{-\alpha t} u(-t) & \leftrightarrow & \frac{m!}{(s+\alpha)^{m+1}}, \; \{s : \mathcal{R}_e(s) < -\mathcal{R}_e(\alpha)\} \end{cases} \qquad (10)$$

where m is any positive integer and $m! = 1 \times 2 \times \cdots \times m$.

The following two properties regard the behavior of $x(t)$ satisfying $x(t) = 0, \ \forall \, t < 0$.

Theorem

Let $X(s)$ *be the LT of* $x(t)$ *satisfying* $x(t) = 0, \ \forall \, t < 0$. *Then*

- *If* $\frac{d^k x(t)}{dt^k}|_{t=0_+}$ *exist for all* $k = 0, 1, \cdots$, *then*

$$x(0_+) = \lim_{s \to \infty} sX(s) \tag{11}$$

- *If* $\mathcal{R}_e(s) = 0 \in ROC_x$, *then*

$$x(+\infty) = \lim_{s \to 0} sX(s) \tag{12}$$

The first claim is called *initial-value theorem* and the 2nd one is usually referred to as *final-value theorem*.

Example 6.7 : Determine the right-sided signals of the following LTs, then verify the theorems: i) $X_1(s) = \frac{1}{s+2}$; ii) $X_2(s) = \frac{s+1}{s^2+5s+6}$; iii) $X_3(s) = \frac{1}{s-2}$.

Solution: As all the signals have an ROC of form $\{s : \sigma_r < \mathcal{R}_e(s)\}$,

- $X_1(s) = \frac{1}{s+2}, \;\; \sigma_r = -2 \leftrightarrow \;x_1(t) = e^{-2t}u(t)$ and hence

$$\lim_{s\to+\infty} sX_1(s) = 1 = x_1(0_+), \;\; \lim_{s\to 0} sX_1(s) = 0 = x_1(+\infty)$$

- $X_2(s) = \frac{-1}{s+2} + \frac{2}{s+3}, \;\; \sigma_r = -2 \;\leftrightarrow\; x_2(t) = [2e^{-3t} - e^{-2t}]u(t)$ and

$$\lim_{s\to+\infty} sX_2(s) = 1 = x_2(0_+), \;\; \lim_{s\to 0} sX_2(s) = 0 = x_2(+\infty)$$

- $X_3(s) = \frac{1}{s-2}, \;\; \sigma_r = 2 \;\leftrightarrow\; x_3(t) = e^{2t}u(t)$ and hence

$$\lim_{s\to+\infty} sX_3(s) = 1 = x_3(0_+), \;\; \lim_{s\to 0} sX_1(s) = 0 \neq x_3(+\infty) = +\infty$$

Why does the inequality above occur?

$\boxed{\textit{Definition}}$: The (bilateral) z-transform of $x[n]$ is defined as

$$x[n] \ \leftrightarrow \ X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \tag{13}$$

where $z = re^{j\Omega}$ is a complex variable with both $r \geq 0, \Omega \in \mathcal{R}$.

Like the Laplace transform, the concept of region of convergence, which is defined as

$$ROC_x = \{z : |\sum_{n=-\infty}^{+\infty} x[n]z^{-n}| < +\infty\} \tag{14}$$

plays a crucial role in z-transform. This is demonstrated by the following example.

Example 6.8: Find the z-Ts of $x_1[n] = \alpha^n u[n], x_2[n] = -\alpha^n u[-n-1]$.

Solution: Clearly,

$$
\begin{aligned}
X_1(z) &= \sum_{n=-\infty}^{+\infty} x_1[n] z^{-n} = \sum_{n=0}^{+\infty} \alpha^n z^{-n} \\
&= \lim_{N\to\infty} \sum_{n=0}^{N-1} (\alpha z^{-1})^n = \lim_{N\to\infty} \frac{1-(\alpha z^{-1})^N}{1-\alpha z^{-1}} = \frac{1}{1-\alpha z^{-1}} \\
& \quad ROC_{x_1} = \{z : \ |\alpha z^{-1}| < 1\} = \{z : \ |\alpha| < |z|\}
\end{aligned}
$$

$$
\begin{aligned}
X_2(z) &= \sum_{n=-\infty}^{+\infty} x_2[n] z^{-n} = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{m=1}^{+\infty} (\alpha^{-1} z)^m \\
&= -[\sum_{m=0}^{+\infty} (\alpha^{-1} z)^m - 1] = -[\frac{1}{1-\alpha^{-1} z} - 1] = \frac{1}{1-\alpha z^{-1}} \\
& \quad ROC_{x_2} = \{z : \ |\alpha^{-1} z| < 1\} = \{z : \ |z| < |\alpha|\}
\end{aligned}
$$

Region of convergence (summarized)

Signals: i) *Right sided*: $x[n] \equiv 0, \forall\, n < N_r < 0$; ii) *Left sided*: $x[n] \equiv 0, \forall\, n > N_l$; iii) the rest.

Consequently, there are three types of ROC.s, See Fig. 6.4,

- right-sided signals: $ROC_x \supseteq \{z : \ \rho_r < |z|\}$ (causal) or $ROC_x \supseteq \{z : \ \rho_r < |z| < +\infty\}$ (non-causal).
- left-sided signals: $ROC_x \supseteq \{z : \ |z| < \rho_l\}$ (anti-causal) or $ROC_x \supseteq \{z : \ 0 < |z| < \rho_l\}$ (non-anti-causal).
- two-sided signals: $ROC_x \supseteq \{z : \ \rho_r < |z| < \rho_l\}$

$\boxed{\textbf{Example 6.9}}$: Compute the z-transform of $x[n] = \beta^{|n|} + 0.5^n u[n]$ and specify its region of convergence.

Solution: Denote $x_l[n] = -[-(\beta^{-1})^n u[-n-1]$, $x_r[n] = [\beta^n + 0.5^n]u[n]$. Note

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=-\infty}^{-1} x[n]z^{-n} + \sum_{n=}^{+\infty} x[n]z^{-n} = X_l(z) + X_r(z)$$

As shown before,

$$\begin{aligned} x_l[n] \quad &\leftrightarrow \quad X_l(z) = -\frac{1}{1-\beta^{-1}z^{-1}}, \quad ROC_{x_l} \supseteq \{z : \ |z| < |\beta^{-1}|\} \\ x_r[n] \quad &\leftrightarrow \quad X_r(z) = \frac{1}{1-\beta z^{-1}} + \frac{1}{1-0.5z^{-1}} \\ &\qquad\qquad ROC_{x_r} \supseteq \{z : \max\{0.5, |\beta|\} < |z|\} \end{aligned}$$

The ROC_{x_r} should contain the intersection of $\{z : |\beta| < |z|\}$ and $\{z : 0.5 < |z|\}$. Note that

- when $|\beta| < 1$, ROC_{x_l} and ROC_{x_r} have a common area (intersection) which is not empty. $X(z)$ then exists with this area as a part of its ROC and

$$X(z) = -\frac{1}{1 - \beta^{-1}z^{-1}} + \frac{1}{1 - \beta z^{-1}} + \frac{1}{1 - 0.5z^{-1}}$$

  with $ROC_x \supseteq \{z : \max\{0.5, |\beta|\} < |z| < |\beta^{-1}|\}$, which is not empty if $0.5 < |\beta|^{-1}$.
- When $|\beta| \geq 1$, the z-transform of such a signal does not exist.

Let $X(z)$ be the z-transform of a signal $x[n]$. The definition for *poles/zeros* of $X(z)$ is the same as that in the Laplace transform.

- *Poles*: if $X(p) = \infty$, then $z = p$ is said a pole of $X(z)$.
- *Zeros:* if $X(q) = 0$, then $z = q$ is said a zero of $X(z)$.

In **Example 6.9** above, $X(z)$ has three poles located at $p_1 = \beta^{-1}$, $p_2 = \beta$, and $p_3 = 0.5$. As known, the ROC of $X(z)$ is in general of the form $ROC_x = \{z : \ \rho_r < |z| < \rho_l\}$, where ρ_r, ρ_l are closely related to the poles of $X(z)$. In fact, in **Example 6.9**

$$\rho_l = |p_1|, \ \ \rho_r = \max\{|p_2|, |p_3|\}$$

The relationship between the poles and ROC is further demonstrated with the following example.

$\boxed{\textbf{Example 6.10}}$: Determine the number of signals that have the same z-transform $X(z) = \frac{3z^3 - \frac{5}{6}z^2}{(z-1/4)(z+1/3)(z-1)}$ and specify the possible ROCs.

Solution: First of all,

Zeros: $3z^3 - \frac{5}{6}z^2 = 0 \;\Rightarrow\; q_1 = 0, q_2 = 0,\; q_2 = 5/18.$

Poles:

$$(z - 1/4)(z + 1/3)(z - 1) = 0 \;\Rightarrow\; p_1 = 1/4,\;\; p_2 = -1/3,\;\;\; p_3 = 1.$$

As known, there are three types of signals/ROCs:

- Right-sided signal: ρ_r is $\rho_r = \max_k |p_k| = 1.$
- Left-sided signal: clearly, the only choice for ρ_l is $\rho_l = \min_k |p_k| = 1/4.$
- Two-sided signal: we have the following choices for (ρ_r, ρ_l):

$$(1/4, 1/3), \quad (1/3,\; 1)$$

Properties

Property	$ROC_x = \{\rho_r < \|z\| < \rho_l\}$	ROC
Linearity	$\sum_k \alpha_k x_k[n]) \leftrightarrow \sum_k \alpha_k X_k(z)$	$\supseteq \cap_k ROC_{x_k}$
Time shift	$x[n-n_0] \leftrightarrow X(z)z^{-n_0}$	ROC_x except for possible $z=0$
Time reversal	$x[-n] \leftrightarrow X(z^{-1})$	$\{\rho_l^{-1} < \|z\| < \rho_r^{-1}\}$
Multiplication by κ^n	$x[n]\kappa^n \leftrightarrow X(\kappa^{-1}z)$	$\{\|\kappa\| \times \rho_r < \|z\| < \|\kappa\| \times \rho_l\}$
Convolution (time)	$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z)$	$\supseteq ROC_{x_1} \cap ROC_{x_2}$
Summation	$\sum_{m=-\infty}^{n} x[m] \leftrightarrow \frac{1}{1-z^{-1}}X(z)$	$\supseteq ROC_x \cap \{1 < \|z\|\}$
$\frac{d}{dz}$	$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$	ROC$_x$

Table: Some properties of z-Transform

Inverse of z-transform:

the same techniques used for the inverse of Laplace transform - partial fraction and properties. See Example 6.11

It can be shown that for an LTI system

$$\left\{ \begin{array}{lcl} y[n] & = & \sum_{m=-\infty}^{+\infty} h[m]x[n-m] \\ y(t) & = & \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \end{array} \right. \Leftrightarrow \begin{array}{lcl} Y(z) & = & X(z)H(z) \\ Y(s) & = & X(s)H(s) \end{array} \quad (15)$$

where $Y(.)$ has an $ROC_y \supseteq ROC_h \cap ROC_x$.

Transfer/system function: $H(.) = Y(.)/X(.)$.

As understand, it is a generalization of frequency response of an LTI system.

$\boxed{\textbf{Example 6.12}}$: An LTI system has an impulse response $h(t) = e^{3t}u(t)$. Compute the output $y(t)$ of the system when excited by $x(t) = Ke^{-2t}u(t)$, where K is a constant.

Solution: First of all, one has

$$h(t) = e^{3t}u(t) \leftrightarrow H(s) = \frac{1}{s-3}, \;\; ROC_h = \{s : 3 < \mathcal{R}_e(s)\}$$

Noting that

$$x(t) = Ke^{-2t}u(t) \leftrightarrow X(s) = \frac{K}{s+2}, \;\; ROC_x = \{s : -2 < \mathcal{R}_e(s)\}$$

and $ROC_y \supseteq ROC_h \cap ROC_x = ROC_h$ is not empty, we have

$$Y(s) = H(s)X(s) = \frac{-K/5}{s+2} + \frac{K/5}{s-3}, \;\; ROC_y \supseteq \{s : 3 < \mathcal{R}_e(s)\}$$

and hence $y(t) = \frac{K}{5}[e^{3t} - e^{-2t}]u(t)$.

Example 6.13 : Compute $y[n] = \sum_{k=-\infty}^{n} x[k]$ for (i) $x[n] = \alpha^n u[n]$; (ii) $x[n] = -\alpha^n u[-n-1]$, where α is a constant.

Solution: As an LTI with $h[n] = u[n]$, the transfer function is

$$H(z) = \frac{1}{1-z^{-1}}, \quad ROC_h = \{z : 1 < |z|\}$$

Therefore, the z-transform of $y[n]$ is

$$Y(z) = H(z)X(z) = \frac{1}{1-z^{-1}}X(z) \quad ROC_y \supseteq \{1 < |z|\} \cap ROC_x$$

When $x[n] = \alpha^n u[n]$, $X(z) = \frac{1}{1-\alpha z^{-1}}$, $\quad ROC_x = \{z : |\alpha| < |z|\}$. So,

$$Y(z) = \frac{1}{1-z^{-1}} \frac{1}{1-\alpha z^{-1}}, \quad ROC_y \supseteq \{z : \max\{1, |\alpha|\} < |z|\}$$

Clearly, $y[n]$ is a right-sided signal and

- If $\alpha = 1$, $y[n] = (n+1)u[n]$.
- if $\alpha \neq 1$, $Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-\alpha z^{-1}} \Rightarrow A = \frac{1}{1-\alpha},\ \ B = \frac{-\alpha}{1-\alpha}$, leading to $\quad y[n] = [A + B\alpha^n]u[n]$.

When $x[n] = -\alpha^n u[-n-1]$, $X(z) = \frac{1}{1-\alpha z^{-1}} \quad ROC_x = \{z : |z| < |\alpha|\}$.

- For $|\alpha| > 1$, $Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-\alpha z^{-1}} \ \ ROC_y \supseteq \{z :\ 1 < |z| < |\alpha|\}$ with A, B obtained above and therefore,

$$y[n] = Au[n] - B\alpha^n u[-n-1]$$

- If $|\alpha| \leq 1$, then ROC_x has no intersection with ROC_h and hence the transfer domain approach can not be used for computing $y[n]$ in this situation.

Inverse system of an LTI and de-convolution

Let $x[n] = \alpha^n u[n] - \beta\alpha^{n-1}u[n-1]$ with both α and β constant and $\alpha \neq \beta$. Consider

- Denote $X(z)$ as the z-transform of $x[n]$ and $G(z) \triangleq X^{-1}(z)$ be the z-transform of the signal $g[n]$ such that

$$x[n] * g[n] = \delta[n]$$

  Find out all possible such $g[n]$.

- Assume that $y[n] = \gamma^n u[n]$ with γ constant is the output of an LTI system $h[n]$ in response to the $x[n]$ given above. Determine all possible such $h[n]$.

Study this part by yourself.

Revisit of LTI system's stability and causality

Theorem

Let $H(s)$ be the transfer function of an LTI system with ROC_h. We have

- *The system is stable* if and only if

$$ROC_h \supseteq \mathcal{R}_e(s) = 0 \tag{16}$$

- *Assume that there exists a non-negative integer p such that $H(s)/s^p$ has no infinite poles, then the system is* causal *if and only if*

$$ROC_h \supseteq \{\sigma_r < \mathcal{R}_e(s) < +\infty\} \tag{17}$$

- *Furthermore, a causal rational $H(s)$ is stable* if and only if *its (finite) poles are all within the region $\mathcal{R}_e(s) < 0$.*

It should be noted that (17) is not a sufficient condition for *any* LTI systems to be causal.

Consider $H_1(s) = \frac{e^{3s}}{s+1}$ with $ROC_{h_1} = \{s: \ -1 < \mathcal{R}_e(s) < +\infty\}$ that satisfies (17), but it is not causal as $h_1(t) = e^{-(t+3)}u(t+3)$. While $H_2(s) = s + \frac{1}{s+1}$ with the same $ROC_{h_1} = \{s: \ -1 < \mathcal{R}_e(s) < +\infty\}$ is causal since $H_2(s)/s$ has no pole at $s = \infty$. In fact, $h_2(t) = \frac{d\delta(t)}{dt} + e^{-t}u(t)$.

(17) is a sufficient-necessary condition for *any* LTI systems of rational $H(s)$ to be causal.

Theorem

Let $H(z)$ be the transfer function of an LTI system with an ROC_h. Then the system is

- *stable if and only if*

$$ROC_h \supseteq \{|z| = 1\} \tag{18}$$

- *causal* if and only if

$$ROC_h = \{z : \rho_r < |z|\} \tag{19}$$

- *Furthermore, as a consequence of the two claims above, a causal LTI system is stable if and only if the poles of $H(z)$ are all inside the unit circle $|z| = 1$.*

It is noted that if $H(z) = N(z)/D(z)$ is causal, then $|H(\infty)| < +\infty$, which implies that the order of $N(z)$ (in z) should not be greater than that of $D(z)$.

For a causal $H(z)$, its stability can be determined by computing the poles of $H(z)$ or roots of $D(z) = 0$ and then checking if all of them are inside $|z| = 1$.

$\boxed{\textit{Stability triangle}}$:

Consider the class of 2nd order *causal* LTI systems:

$$H(z) = \frac{\beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha z^{-1} + \alpha_2 z^{-2}}$$

As known, it is stable *iff* its poles $\lambda_k,\ k = 1, 2$ satisfy $|\lambda_k| < 1,\ k = 1, 2.$

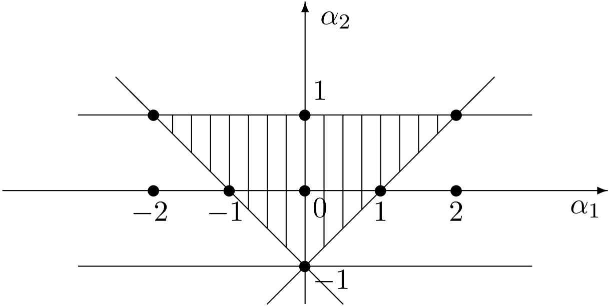

Fig. 6.7: Stability triangle.

The stability region, called *stability triangle*, is defined with

$$|\alpha_2| < 1, \quad |\alpha_1| < 1 + \alpha_2 \tag{20}$$

which are the *iff* for a 2nd order causal LTI discrete-time system to be stable.

Transfer function of LTI systems by LCCDEs

As mentioned in Chapter 2, the following LCCDE can characterize a class of LTI systems

$$\frac{d^N y(t)}{dt^N} + \sum_{k=1}^{N} \alpha_k \frac{d^{(N-k)} y(t)}{dt^{(N-k)}} = \sum_{k=0}^{M} \beta_k \frac{d^{(M-k)} x(t)}{dt^{(M-k)}} \tag{21}$$

Since $y(t) = h(t) * x(t)$ and $\frac{d^k y(t)}{dt^k} = h(t) * \frac{d^k x(t)}{dt^k}$, (21) can be rewritten as

$$h(t) * [\frac{d^N x(t)}{dt^N} + \sum_{k=1}^{N} \alpha_k \frac{d^{(N-k)} x(t)}{dt^{(N-k)}}] = \sum_{k=0}^{M} \beta_k \frac{d^{(M-k)} x(t)}{dt^{(M-k)}}$$

It then follows from $\frac{d^k x(t)}{dt^k} \leftrightarrow s^k X(s)$ that the transfer function of any LTI system described by (21), if it exists, is characterized by

$$H(s) = \frac{\sum_{k=0}^{M} \beta_k s^{M-k}}{s^N + \sum_{k=1}^{N} \alpha_k s^{N-k}} \tag{22}$$

$\boxed{\textit{How to characterize } h(t)}$:

Without loss of generality, it is assumed that the numerator and the denominator of $H(s)$ are co-prime with $N \geq 1$.

Based on the distribution of the poles, we can find all such $h(t)$ that $h(t) \rightarrow H(s)$, among which a *unique* causal one and a *unique* anti-causal one, denoted as $h_c(t)$ and $h_a(t)$, respectively.

All these $h(t)$ represent just a sub-class of the LTI systems described with (21). In general, any $h(t)$ is characterized by

$$h(t) = h_c(t) + h_h(t)$$

where $h_h(t)$ is any homogenous solution of (21). See Chapter 2.

$\boxed{\textbf{Example 6.14}}$: Characterize the unit impulse response set of all the LTI systems that obey the following LCCDE

$$\frac{dy(t)}{dt} - 3y(t) = x(t)$$

Solution: According to (22), the transform function of any such an LTI system has its Laplace transform given by

$$H(s) = \frac{1}{s-3}$$

As $H(s)$ has one pole, there are two possible ROCs, which are

- $ROC_h = \{s : \ 3 < \mathcal{R}_e(s)\}$, leading to a *causal* LTI system:

$$h(t) = e^{3t}u(t) \triangleq h_c(t)$$

- $ROC_h = \{s : \ \mathcal{R}_e(s) < 3\}$, yielding an *anti-causal* LTI system:

$$h(t) = -e^{3t}u(-t) \triangleq h_a(t)$$

From the view-point of LCCDE, $h_c(t)$ and $h_a(t)$ are just two particular solutions of this equation for $x(t) = \delta(t)$. All the solutions of this equation to the unit impulse input are given by

$$h(t) = h_c(t) + \mu e^{3t}$$

where μe^{3t} is the *homogeneous* solution of the LCCDE with μ being *any* constant.

It is noted that when $\mu = -1$, $h(t) = h_a(t)$, and when $\mu \neq 0, -1$, the Laplace transform of the corresponding $h(t)$ does not exist!

Consider the LTI systems characterized by the following LCCDE

$$y[n] + \sum_{k=1}^{N} \alpha_k y[n-k] = \sum_{k=0}^{M} \beta_k x[n-k] \tag{23}$$

If $y[n] = h[n] * x[n]$ and the z-transform of $h[n]$ exists, then

$$H(z) = \frac{\sum_{k=0}^{M} \beta_k z^{-k}}{1 + \sum_{k=1}^{N} \alpha_k z^{-k}} \tag{24}$$

Once again, it is claimed that

- among the infinite number of LTI systems described by (23), there are a unique *causal* one and a unique *anti-causal* one;
- any of such LTI systems can be characterized by the unique causal impulse response plus a homogeneous solution of (23).

$\boxed{\textbf{Example 6.15}}$: Identify all possible LTI systems governed with the LCCDE $y[n] - 0.75\, y[n-1] = x[n]$ by specifying their unit impulse response set.

Solution: First of all, one has $H(z) = \frac{1}{1-0.75\ z^{-1}}$. With one pole at $z = 0.75$, there are two possible ROCs.

- If $ROC_h = \{z :\ 0.75 < |z|\}$, then

$$h[n] = 0.75^n u[n] \triangleq h_c[n]$$

  This is the unique *causal* LTI system.

- If $ROC_h = \{z :\ |z| < 0.75\}$, we have

$$h[n] = -0.75^n u[-n-1] \triangleq h_a[n]$$

  which corresponds to an anti-causal LTI.

As understood, all possible unit impulse responses are characterized with

$$h[n] = h_c[n] + \mu 0.75^n$$

One can see that there is an infinite set of LTI systems whose input/output relationship is constrained by the same LCCDE and the one mentioned above (corresponding to $\mu = -1$) is just one of them. With $\mu \neq 0, -1$, the z-transform of any $h[n]$ does not exist at all.

Transform domain approach to LCCDEs

As pointed out in Chapter 2, any complete solution of an LCCDE is of form

$$y = h * x + y_h$$

where $y_p = h * x$ and y_h are also referred to as the *forced response* and the *natural response*.

- y_h can be obtained by solving the characteristic equation of the LCCDE.
- As to h, note

$$LCCDE \;\Rightarrow\; H \;\Rightarrow\; h_c$$

where h_c is the causal inverse transform of H.

See the two examples in the textbook by yourself.

Decomposition of LTI responses

Let 0_- be the number smaller than 0 but infinitely close to the latter. Define

$$u_{0_-}(t) \triangleq \begin{cases} 1, & \forall\, t \geq 0_- \\ 0, & \forall\, t < 0_- \end{cases} \tag{25}$$

For any signal $x(t)$, $\hat{x}(t) \triangleq x(t)u_{0_-}(t)$, $\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$ are the future and past of the signal $x(t)$. Clearly,

$$x(t) = \tilde{x}(t) + \hat{x}(t) \;\; \rightarrow \;\; y(t) = h(t) * \tilde{x}(t) + h(t) * \hat{x}(t) \triangleq \tilde{y}(t) + \hat{y}(t)$$

Define $\bar{y}(t) \triangleq y(t)u_{0_-}(t)$ as the *complete response* to $x(t)$. Then,

$$\bar{y}(t) = \tilde{y}(t)u_{0_-}(t) + \hat{y}(t)u_{0_-}(t) \triangleq y_{zi}(t) + y_{zs}(t) \tag{26}$$

where $y_{zi}(t)$ and $y_{zs}(t)$ are the *zero-input* and *zero-state* responses to $x(t)$.

When the LTI is *causal*,

$$\hat{y}(t) = h(t) * \hat{x}(t) = \int_0^{t-0_-} h(\tau)\hat{x}(t-\tau)d\tau \, u_{0_-}(t)$$

where the factor $u_{0_-}(t)$ is due to $\hat{y}(t) = 0$ for $t - 0_- < 0$, i.e., $t < 0_-$, for which $\hat{x}(t) = 0$. So, one has[3]

- $\hat{y}(0_-) = 0$. This implies $\bar{y}(0_-) = y(0_-) = \tilde{y}(0_-)$ is independent of the future input $\hat{x}(t)$;
- $y_{zs}(t) = \hat{y}(t)u_{0_-}(t) = \hat{y}(t)$, leading to

$$\bar{y}(t) = h(t) * \hat{x}(t) + y_{zi}(t)$$

Can we compute $\bar{y}(t)$ using Laplace transform ?

[3]Similar claims apply to discrete-time causal LTI systems.

Consider the *causal* LTI system described using

$$\frac{dy(t)}{dt} + 3y(t) = 3x(t)$$

with $y(0_-) = \alpha$. Evaluate its *complete response* $\bar{y}(t) = y(t)u_{0_-}(t)$.

Multiplying both sides of the above with $u_{0_-}(t)$ leads to

$$\frac{dy(t)}{dt}u_{0_-}(t) + 3y(t)u_{0_-}(t) = 3x(t)u_{0_-}(t)$$

Applying the LT to both sides of the above, we then have

$$\int_{-\infty}^{+\infty} [\frac{dy(t)}{dt}u_{0_-}(t) + 3y(t)u_{0_-}(t)]e^{-st}dt = \int_{-\infty}^{+\infty} 3x(t)u_{0_-}(t)e^{-st}dt$$

that is

$$\int_{0_-}^{+\infty} \frac{dy(t)}{dt}e^{-st}dt + 3\int_{0_-}^{+\infty} y(t)e^{-st}dt = 3\int_{0_-}^{+\infty} x(t)e^{-st}dt$$

involving 3 integrations of the same form - *unilateral Laplace transform*.

Unilateral Laplace transform

$\boxed{\textit{Definition}}$: Let $x(t)$ a signal on $\mathcal{R}$. The unilateral Laplace transform is defined as

$$\begin{aligned} x(t) \quad &\rightarrow \quad \bar{x}(t) = x(t)u(t-0_-) \\ &\leftrightarrow \quad \bar{X}(s) = \int_{-\infty}^{+\infty} \bar{x}(t)e^{-st}dt \\ &\qquad\qquad = \int_{0_-}^{+\infty} x(t)e^{-st}dt \triangleq \mathcal{X}(s) \end{aligned} \tag{27}$$

It is important to note that $\mathcal{X}(s)$ is the (bilateral) LT of $x(t)u(t-0_-)$. Therefore, if $x(t) = 0,\ \forall\ t < 0$, then $\mathcal{X}(s) = X(s)$.

Example 6.19 : Compute the unilateral LT for each of the following signals: (i) $x_1(t) = e^{-\alpha t}u(t)$; (ii) $x_2(t) = e^{-\alpha(t+1)}u(t+1)$; (iii) $x_3(t) = e^{-\alpha(t-1)}u(t-1)$.

Solution: First of all,

- as $x_1(t) = 0,\ t < 0$, we have

$$\mathcal{X}_1(s) = X_1(s) = \frac{1}{s+\alpha},\ \ ROC_{x_1} = \{s :,\ -\mathcal{R}_e(\alpha) < \mathcal{R}_e(s)\}$$

- Though $x_2(t) = e^{-\alpha(t+1)}u(t+1)$ is not nil within $(-1, 0)$, the unilateral LT is the Laplace of $x_2(t)u(t-0_-)$. So,

$$\begin{aligned}\mathcal{X}_2(s) &= \int_{0_-}^{+\infty} e^{-\alpha(t+1)}u(t+1)e^{-st}dt = e^{-\alpha}\int_{0_-}^{+\infty} e^{-\alpha t}e^{-st}dt \\ &= \frac{e^{-\alpha}}{s+\alpha},\ \ ROC_{x_2} = \{s :,\ -\mathcal{R}_e(\alpha) < \mathcal{R}_e(s)\}\end{aligned}$$

not equal to the LT of $x_2(t) = x_1(t+1)$, i.e., $X_2(s) = \frac{e^s}{s+\alpha}$.

- As $x_3(t) = e^{-\alpha(t-1)}u(t-1) = 0 = x_1(t-1)$, $\forall\, t < 0$, we have

$$\mathcal{X}_3(s) = X_3(s) = X_1(s)e^{-s} = \mathcal{X}_1(s)e^{-s}$$

with $ROC_{x_3} = \{s :, -\mathcal{R}_e(\alpha) < \mathcal{R}_e(s)\}$

This example shows that the time shift property of the Laplace transform holds for the unilateral one *only* for $\tau \geq 0$: $x(t-\tau) \;\leftrightarrow\; \mathcal{X}(s)e^{-\tau s}$.

Let $x(t) \leftrightarrow \mathcal{X}(s)$. Note the *partial differential* formula:

$$u'v = (uv)' - uv'$$

Then (with $u = x(t), v = e^{-st}$)

$$\int_{0_-}^{+\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st}|_{0_-}^{+\infty} + s\mathcal{X}(s) = s\mathcal{X}(s) - x(0_-)$$

as $x(\infty)e^{-s\infty} = 0$ due to the existence of $\mathcal{X}(s)$. Therefore,

$$\frac{dx(t)}{dt} \Leftrightarrow s\mathcal{X}(s) - x(0_-) \tag{28}$$

Consequently,

$$\frac{d^2x(t)}{dt^2} \leftrightarrow s[s\mathcal{X}(s) - x(0_-)] - \frac{dx(t)}{dt}|_{t=0_-}$$

$\boxed{\textbf{Example 6.20}}$: A causal LTI system given by

$$\frac{d^2y(t)}{dt^2} + \alpha_1 \frac{dy(t)}{dt} + \alpha_0 y(t) = \beta_1 \frac{dx(t)}{dt} + \beta_0 x(t)$$

with $x(t) = \kappa u(t)$ and initials conditions $\frac{dy(t)}{dt}|_{t=0_-} = \gamma_1, \; y(0_-) = \gamma_0$. Compute the system output $y(t)$ for $t > 0$, that is $\bar{y}(t) = y(t)u(t - 0_-)$.

Solution: Note $\frac{d^m x(t)}{dt^m}|_{t=0_-} = 0, \; m = 0, 1.$ Applying the unilateral LT

$$s[s\bar{\mathcal{Y}}(s) - \gamma_0] - \gamma_1 + \alpha_1[s\bar{\mathcal{Y}}(s) - \gamma_0] + \alpha_0\bar{\mathcal{Y}}(s) = (\beta_1 s + \beta_0)\mathcal{X}(s)$$

Equivalently,

$$\begin{aligned} \bar{\mathcal{Y}}(s) &= \frac{\gamma_0 s + 3\gamma_0 + \gamma_1}{s^2 + \alpha_1 s + \alpha_0} + \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0} X(s) \\ &= \frac{\gamma_0 s + 3\gamma_0 + \gamma_1}{s^2 + \alpha_1 s + \alpha_0} + H(s)X(s) \triangleq \mathcal{Y}_{zi}(s) + \mathcal{Y}_{zs}(s) \end{aligned}$$

Noting all the signals and system involved are all *causal*, the inverse LT can be done easily.

Take $\alpha_1 = 3, \alpha_0 = 2, \beta_1 = 0, \beta_0 = 1, \kappa = 2$ and $\gamma_0 = 3, \gamma_1 = -5$. Computations show

$$y_{zs}(t) = [e^{-2t} - e^{-t} + 1]u(t), \;\; y_{zi}(t) = 2e^{-2t} + e^{-t}, t > 0_-$$

which are the *zero input* response and *zero state* response, and

$$\bar{y}(t) = y_{zi}(t) + y_{zs}(t)$$

Unilateral z-transform

Defined in the way, it is given by

$$x[n] \;\rightarrow\; \bar{x}[n] = x[n]u[n] \;\leftrightarrow\; \mathcal{X}(z) \triangleq \sum_{n=0}^{+\infty} x[n]z^{-n} \tag{29}$$

- Comparing with (13), we realize that $\mathcal{X}(z)$ is the (bilateral) z-transform of $x[n]u[n]$ and therefore, the ROC for $\mathcal{X}(z)$ is *always* outside a circle and including $z = \infty$.
- For causal LTI systems, i.e., $h[n] = h[n]u[n]$, we then have $\mathcal{H}(z) = H(z)$.

Let $x[n] \;\leftrightarrow\; \mathcal{X}(z)$, it can then be shown that

$$x[n-1] \;\leftrightarrow\; z^{-1}\mathcal{X}(z) + x[-1], \;\; x[n+1] \;\leftrightarrow\; z[\mathcal{X}(z) - x[0]] \tag{30}$$

In a similar manner, we can show that the response $y[n]$ for $n \geq 0$ of a causal LTI system described by (23) with initial conditions

$$y[-k] = \gamma_k, \; k = 1, 2, \cdots, N$$

is given by

$$\bar{y}[n] = y[n]u[n] = y_{zs}[n] + y_{zi}[n]$$

with $y_{zs}[n]$ and $y_{zi}[n]$ called *zero-state* and *zero-input* responses of the system:

$$y_{zs}[n] \; \leftrightarrow \; \mathcal{Y}_{zs} = H(z)X(z)$$

and $y_{zi}[n]$ is the homogenous solution of (23) satisfying the initial conditions, which is the inverse unilateral z-transform of

$$\mathcal{Y}_{zi}(z) = \frac{\mathcal{I}(z)}{1 + \sum_{k=1}^{N} \alpha_k z^{-k}} \qquad (31)$$

$\boxed{\textbf{Example 6.21}}$: Given a *causal* LTI described by $y[n] + 2y[n-1] = x[n]$ with $x[n] = \kappa u[n]$. Evaluate $y[n], n \geq 0$ with the initial condition $y[-1] = \gamma_1$.

Solution: Applying unilateral z-transform to both sides yields

$$\bar{\mathcal{Y}}(z) + 2[z^{-1}\bar{\mathcal{Y}}(z) \quad + \quad \gamma_1] = \frac{\kappa}{1 - z^{-1}}$$

$$\Downarrow$$

$$\bar{\mathcal{Y}}(z) = -\frac{2\gamma_1}{1 + 2z^{-1}} \quad + \quad \frac{\kappa}{(1 + 2z^{-1})(1 - z^{-1})}$$

It follows from $\frac{\kappa}{(1+2z^{-1})(1-z^{-1})} = \frac{\kappa}{3}\left[\frac{2}{1+2z^{-1}} + \frac{1}{1-z^{-1}}\right]$ that

$$\bar{y}[n] = -2\gamma_1(-2)^n u[n] + \frac{\kappa}{3}\left[2(-2)^n + 1\right]u[n]$$

The first term is due to the initial condition $y[-1] = \gamma_1$, while the second term is the response to the (future) input. **End**