习题1（仅供参考）

1. 解：(1) $\dfrac{(3+4i)(2-5i)}{2i}=\dfrac{6+20+8i-15i}{2i}=\dfrac{26-7i}{2i}=-\dfrac{7}{2}-13i$.

(2) $\left(\dfrac{3-4i}{1+2i}\right)^2=\left[\dfrac{(3-4i)(1-2i)}{(1+2i)(1-2i)}\right]^2=\left[\dfrac{3-8-4i-6i}{1+4}\right]^2$

$=[-1-2i]^2=(1+2i)^2=-3+4i$.

(3) $i^8-4i^{21}+i=1-4i+i=1-3i$.

2. 解：设 $z=x+iy$．$x, y\in\mathbb{R}$．$z\neq -1$.

则 $\dfrac{z-1}{z+1}=\dfrac{(x-1)+iy}{(x+1)+iy}=\dfrac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$

$=\dfrac{x^2-1+y^2+2iy}{(x+1)^2+y^2}$

$=\dfrac{x^2-1+y^2}{(x+1)^2+y^2}+\dfrac{2y}{(x+1)^2+y^2}\cdot i$.

3. 解：不成立.

当 $y=0$ 即 z 为实数时，$|z|^2=x^2=z^2$.

证明如下：设 $z=x+iy$.

$\therefore |z|^2=x^2+y^2$.

$z^2=(x+iy)^2=x^2-y^2+2xy\cdot i$.

$\therefore |z|^2=z^2 \Leftrightarrow \begin{cases} x^2+y^2=x^2-y^2 \\ 2xy=0 \end{cases} \Leftrightarrow y=0$.

4. 证：(1) $\because |a|<1$．$|b|<1$．$\therefore |\bar{a}b|=|a||b|<1 \Rightarrow 1-\bar{a}b\neq 0$

而证 $\left|\dfrac{a-b}{1-\bar{a}b}\right|<1 \Leftrightarrow |a-b|<|1-\bar{a}b|$

$\Leftrightarrow |a-b|^2<|1-\bar{a}b|^2$

$\because |a-b|^2=(a-b)(\bar{a}-\bar{b})=a\bar{a}+b\bar{b}-\bar{a}b-a\bar{b}$.

$|1-\bar{a}b|^2=(1-\bar{a}b)(1-a\bar{b})=1+a\bar{a}b\bar{b}-\bar{a}b-a\bar{b}$

$\therefore |1-\bar{a}b|^2-|a-b|^2=(1-a\bar{a})(1-b\bar{b})>0$

$\therefore$ 结论成立.

(2) 若 $|a|=1$, $|b|<1$, 则 $|\bar{a}b|=|a||b|<1$. $\Rightarrow 1-\bar{a}b\neq 0$

$$\left|\frac{a-b}{1-\bar{a}b}\right| = |\bar{a}|\left|\frac{a-b}{1-\bar{a}b}\right| = \left|\frac{\bar{a}(a-b)}{1-\bar{a}b}\right| = \left|\frac{a\bar{a}-\bar{a}b}{1-\bar{a}b}\right| = \left|\frac{1-\bar{a}b}{1-\bar{a}b}\right| = 1.$$

若 $|b|=1$. $|a|<1$. 则 $|\bar{a}b|=|a||b|<1$. $\Rightarrow 1-\bar{a}b\neq 0$

$$\left|\frac{a-b}{1-\bar{a}b}\right| = |\bar{b}|\left|\frac{a-b}{1-\bar{a}b}\right| = \left|\frac{\bar{b}(a-b)}{1-\bar{a}b}\right| = \left|\frac{a\bar{b}-b\bar{b}}{1-\bar{a}b}\right|$$

$$= \left|\frac{a\bar{b}-1}{1-\bar{a}b}\right| = \left|\frac{1-a\bar{b}}{1-\bar{a}b}\right| = \frac{|\overline{(1-\bar{a}b)}|}{|1-\bar{a}b|} = 1.$$

5. 解: (1) √

(2) X. 反例: $z=-1$. $\arg\bar{z}=\arg(-1)=\pi$.

而 $-\arg z = -\arg(-1) = -\pi$.

(3) X. $\frac{1}{z}=-z \Rightarrow z^2=-1. \Rightarrow z=\pm i$.

(4) X. 反例: $z_1=1$. $z_2=i$. $|z_1+z_2|=|1+i|=\sqrt{2}$

而 $|z_1|+|z_2|=1+1=2$.

(5) √

6. 解: (1) $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{i\frac{\pi}{2}}$

(2) $-1=\cos\pi+i\sin\pi=e^{i\pi}$.

(3) $$1+i\sqrt{3}=2\left[\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]=2\cdot\left[\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right]$$
$$=2\cdot e^{i\frac{\pi}{3}}=e^{\ln 2+i\frac{\pi}{3}}.$$

(4) $$\frac{(\cos\varphi-i\sin\varphi)^3}{(\cos2\varphi+i\sin2\varphi)^2}=\frac{(\cos(-\varphi)+i\sin(-\varphi))^3}{(\cos2\varphi+i\sin2\varphi)^2}=\frac{[e^{i(-\varphi)}]^3}{(e^{i2\varphi})^2}=e^{i(-7\varphi)}$$
$$=\cos7\varphi-i\sin7\varphi.$$

(5) $$1-\cos\varphi+i\sin\varphi=2\sin^2\frac{\varphi}{2}+i\cdot2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}$$
$$=2\sin\frac{\varphi}{2}\left[\cos\left(\frac{\varphi}{2}-\frac{\pi}{2}\right)+i\sin\left(\frac{\varphi}{2}-\frac{\pi}{2}\right)\right]$$
$$=2\sin\frac{\varphi}{2}\cdot e^{i(\frac{\varphi}{2}-\frac{\pi}{2})}$$
$$=e^{\ln2+\ln\sin\frac{\varphi}{2}+i(\frac{\varphi}{2}-\frac{\pi}{2})}$$

7. 解：(1) $(\sqrt{3}-i)^5 = \left[2\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)\right]^5$

$= 2^5 \cdot \left(\cos(-\frac{5\pi}{6}) + i\sin(-\frac{5\pi}{6})\right)$

$= 32 \cdot (-\frac{\sqrt{3}}{2} - \frac{1}{2}i) = -16\sqrt{3} - 16i$

(2) $(1+i)^8 = [\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^8$

$= (\sqrt{2})^8 \cdot (\cos 2\pi + i\sin 2\pi)$

$= 16$

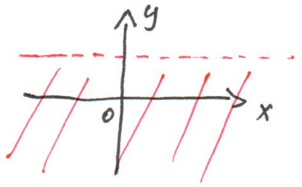
(3) $\sqrt[6]{-1} = [-1]^{\frac{1}{6}} = [e^{i(\pi+2k\pi)}]^{\frac{1}{6}} = e^{i(\frac{\pi}{6}+\frac{k}{3}\pi)}$

$= e^{i\frac{\pi}{6}}, e^{i\frac{\pi}{2}}, e^{i\frac{5}{6}\pi}, e^{i\frac{7}{6}\pi}, e^{i\frac{3}{2}\pi}, e^{i\frac{11}{6}\pi}$.

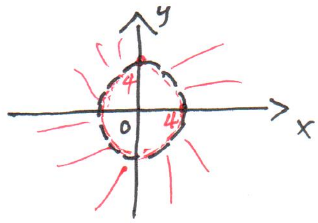
(4) $(-1-i)^{\frac{1}{3}} = [\sqrt{2} \cdot e^{i(-\frac{3}{4}\pi+2k\pi)}]^{\frac{1}{3}} = 2^{\frac{1}{6}} \cdot e^{i(-\frac{\pi}{4}+\frac{2}{3}k\pi)}$

$= 2^{\frac{1}{6}} \cdot e^{-\frac{\pi}{4}i}, 2^{\frac{1}{6}} \cdot e^{\frac{5}{12}\pi i}, 2^{\frac{1}{6}} e^{\frac{13}{12}\pi i}$.

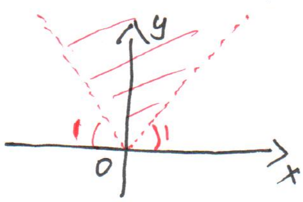
8. 解：(1) $\mathrm{Im}(z) < 1 \iff y < 1$



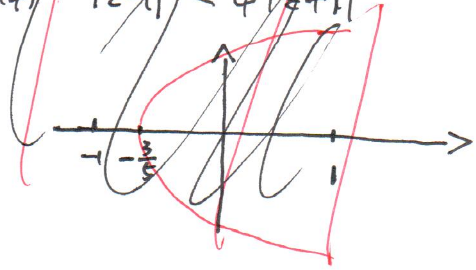
(2) $|z-1| > 4$



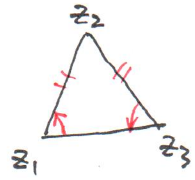
(3) $1 < \arg z < \pi - 1$



(4) $|z-1| < 4|z+1|$



9. 证明：$\because \dfrac{z_2-z_1}{z_3-z_1}=\dfrac{z_1-z_3}{z_2-z_3}$



$\therefore \arg\dfrac{z_2-z_1}{z_3-z_1}=\arg\dfrac{z_1-z_3}{z_2-z_3}$

即 $\angle z_2z_1z_3=\angle z_2z_3z_1$

$\therefore |z_1-z_2|=|z_2-z_3|$

又有 $\left|\dfrac{z_2-z_1}{z_3-z_1}\right|=\left|\dfrac{z_1-z_3}{z_2-z_3}\right|$

$\therefore |z_1-z_3|^2=|z_2-z_1|\,|z_2-z_3|$

$\therefore |z_1-z_3|=|z_1-z_2|=|z_2-z_3|.$

10. 证明：因为 $\mathrm{Im}\left(\dfrac{z_1-z_4}{z_1-z_2}\cdot\dfrac{z_3-z_2}{z_3-z_4}\right)=0$

$\therefore \arg\dfrac{z_1-z_4}{z_1-z_2}+\arg\dfrac{z_3-z_2}{z_3-z_4}=k\pi,\quad k\in\mathbb{Z}.$

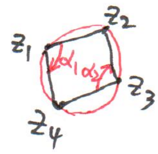
$\because$ arg 的取值在 $(-\pi,\pi]$.

$\therefore$ k 只能取 $-1,0,1,2$.

若 $k=-1$，则 $\arg\dfrac{z_1-z_4}{z_1-z_2}\in(-\pi,0)$. $\arg\dfrac{z_3-z_2}{z_3-z_4}\in(-\pi,0)$

此时 $\angle\alpha_1+\angle\alpha_2=\pi$.

$\therefore$ z_1,z_2,z_3,z_4 在圆周上.



若 $k=0$，则有如下三种情形.

① $\arg\dfrac{z_1-z_4}{z_1-z_2}=0$. $\arg\dfrac{z_3-z_2}{z_3-z_4}=0$

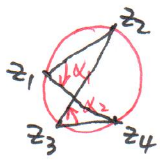
$\therefore$ z_1,z_2,z_4 共线 并且 z_3,z_2,z_4 共线

$\therefore$ $z_1\ z_2\ z_3.\ z_4$ 在同一直线上.

② $\arg\dfrac{z_1-z_4}{z_1-z_2}\in(-\pi,0)$. $\arg\dfrac{z_3-z_2}{z_3-z_4}\in(0,\pi)$.

则 $\angle\alpha_1=\angle\alpha_2$

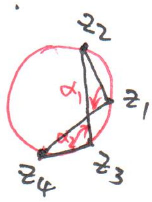
$\therefore$ 四点共圆.



③ $\arg\dfrac{z_1-z_4}{z_1-z_2}\in(0,\pi)$. $\arg\dfrac{z_3-z_2}{z_3-z_4}\in(-\pi,0)$.

则 $\angle\alpha_1=\angle\alpha_2$

$\therefore$ 四点共圆.



若 $k=1$，则有如下三种情况

① $\arg\dfrac{z_1-z_4}{z_1-z_2}=0$，$\arg\dfrac{z_3-z_2}{z_3-z_4}=\pi$。

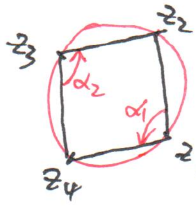
则四点共线。

② $\arg\dfrac{z_1-z_4}{z_1-z_2}=\pi$，$\arg\dfrac{z_3-z_2}{z_3-z_4}=0$

则四点共线。

③ $\arg\dfrac{z_1-z_4}{z_1-z_2}\in(0,\pi)$，$\arg\dfrac{z_3-z_2}{z_3-z_4}\in(0,\pi)$。

则 $\angle\alpha_1+\angle\alpha_2=\pi$。



∴ 四点共圆。

若 $k=2$，则 $\arg\dfrac{z_1-z_4}{z_1-z_2}=\arg\dfrac{z_3-z_2}{z_3-z_4}=\pi$。

此时四点共线。

11. 证：z 平面上过 z_0，z_1 的直线方程为

$$z-z_0=k(z_1-z_0)。\quad k \text{ 为一实数}。$$

$$\therefore \frac{z-z_0}{z_1-z_0}=k=\overline{\left(\frac{z-z_0}{z_1-z_0}\right)}$$

$$\therefore \frac{1}{z_1-z_0}z-\frac{z_0}{z_1-z_0}=\frac{1}{\overline{z_1}-\overline{z_0}}\overline{z}-\frac{\overline{z_0}}{\overline{z_1}-\overline{z_0}}。$$

$$\therefore \frac{i}{z_1-z_0}z+\frac{-i}{\overline{z_1}-\overline{z_0}}\cdot\overline{z}+\frac{-iz_0}{z_1-z_0}+\frac{i\overline{z_0}}{\overline{z_1}-\overline{z_0}}=0$$

取 $\overline{a}=\dfrac{i}{z_1-z_0}$，$a=\dfrac{-i}{\overline{z_1}-\overline{z_0}}$。

$c=\dfrac{iz_0}{z_1-z_0}-\dfrac{i\overline{z_0}}{\overline{z_1}-\overline{z_0}}$ 为实数。

则 $\overline{a}z+a\overline{z}=c$。

12. 证：z 平面上以 z_0 为中心，R 为半径的圆为 $|z-z_0|=R$。

$$\therefore (z-z_0)\overline{(z-z_0)}=R^2$$

即 $z\overline{z}-\overline{z_0}z-z_0\overline{z}+z_0\overline{z_0}=R^2$。

取 $a=-\overline{z_0}$，$\overline{a}=-z_0$，$c=z_0\overline{z_0}-R^2$。

则 $z\overline{z}+a\overline{z}+\overline{a}z+c=0$。

13. 解：(1) $\lim\limits_{z\to 2+i}\dfrac{\bar{z}}{z}=\dfrac{\lim\limits_{z\to 2+i}\bar{z}}{\lim\limits_{z\to 2+i}z}=\dfrac{2-i}{2+i}=\dfrac{3}{5}-\dfrac{4}{5}i$.

(2) $\lim\limits_{z\to i}\dfrac{z\bar{z}+z^3-\bar{z}+2}{z^2-1}=\dfrac{\lim\limits_{z\to i}(z\bar{z}+z^3-\bar{z}+2)}{\lim\limits_{z\to i}(z^2-1)}=\dfrac{1+i^3+i+2}{-1-1}=-\dfrac{3}{2}$.

14. 证：取 $z=iy$. 则 $\dfrac{\operatorname{Re}z}{z}=\dfrac{0}{iy}=0$.

$\therefore \lim\limits_{\substack{z\to 0\\ z=iy}}\dfrac{\operatorname{Re}z}{z}=0$

取 $z=x$. 则 $\dfrac{\operatorname{Re}z}{z}=\dfrac{x}{x}=1$

$\therefore \lim\limits_{\substack{z\to 0\\ z=x}}\dfrac{\operatorname{Re}z}{z}=1$.

$\therefore \lim\limits_{z\to 0}\dfrac{\operatorname{Re}z}{z}$ 不存在.

15. 证：$\because \lim\limits_{\substack{x=y^3\\ y\to 0}}f(z)=\lim\limits_{y\to 0}\dfrac{y^6}{y^6+y^6}=\dfrac{1}{2}\neq 0=f(0)$.

$\therefore f(z)$ 在原点不连续.

16. 证：当 $z\neq 0$ 时. $|f(z)-f(0)|=\left|\dfrac{(\operatorname{Re}(z^2))^2}{|z|^2}\right|$

$=\dfrac{|\operatorname{Re}z^2|^2}{|z|^2}\leq\dfrac{|z^2|^2}{|z|^2}=|z|^2$

$\therefore \forall\varepsilon>0$. 取 $\delta=\delta(\varepsilon)=\sqrt{\varepsilon}>0$. 当 $|z-0|<\delta=\sqrt{\varepsilon}$ 时.

$|f(z)-f(0)|\leq|z|^2<\delta^2=\varepsilon$.

$\therefore f(z)$ 在 $z=0$ 连续.

17. 证：$\because f(z)$ 在 z_0 连续.

$\therefore \forall\varepsilon>0$, $\exists\delta>0$, s.t 当 $|z-z_0|<\delta$ 时

$|f(z)-f(z_0)|<\varepsilon$.

$\therefore \big||f(z)|-|f(z_0)|\big|\leq|f(z)-f(z_0)|<\varepsilon$.

$\Rightarrow |f(z)|$ 在 z_0 连续.

$|\overline{f(z)}-\overline{f(z_0)}|=|f(z)-f(z_0)|<\varepsilon$.

$\Rightarrow \overline{f(z)}$ 在 z_0 连续.