Transform-domain Approaches.

Rev

$e^{-t}u(t) \longleftrightarrow \frac{1}{j\omega+1}$

$x(t) = e^{t}u(t) \longleftrightarrow ?$

$\tilde{x}(t) = x(t)\,e^{-\sigma t} \longleftrightarrow$ 收敛域 ROC: region of convergence

$$\tilde{X}(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-\sigma t} e^{-j\omega t} dt = \int_{-\infty}^{+\infty} x(t)\, e^{-(\sigma + j\omega)t} dt$$

$s = j\omega$ 时 $\longrightarrow$ FT Fourier Transform

$\sigma = 0$

$$\overset{s=\sigma+j\omega}{=\!=} \int_{-\infty}^{+\infty} x(t)\, e^{-st} dt$$

Laplace transform

$$\begin{cases} e^{\alpha t}u(t) \longleftrightarrow \frac{1}{s-\alpha}, & \forall s \in ROC_{x_1} = \{s : Re(\alpha) < Re(s)\} \\ -e^{\alpha t}u(-t) \longleftrightarrow \frac{1}{s-\alpha}, & \forall s \in ROC_{x_2} = \{s : Re(s) < Re(\alpha)\} \end{cases}$$

- 虚轴上的Laplace ⟶ 连续时间的FT.
- 单位圆上的ZF ⟶ 离散时间的FT.

$$x(t) = e^{-\beta|t|} \Longrightarrow \begin{cases} x_l(t) = e^{\beta t}u(-t) \Longleftrightarrow \frac{-1}{s-\beta} & ROC_l = \{s : Re(s) < \beta\} \\ x_r(t) = e^{-\beta t}u(t) \Longleftrightarrow \frac{1}{s+\beta} & ROC_r = \{s : -\beta < Re(s)\} \end{cases}$$

$$X(s) = X_l(s) + X_r(t) = \frac{-2\beta}{s^2 - \beta^2} \qquad ROC_x = ROC_r \cap ROC_l = \{s : -\beta < Re(s) < \beta\}$$

Properties:

① right-sided signal

② Left-sided signal

③ two-sided signal

- Poles 极点
- zeros 零点

$X(s) = \frac{N(s)}{D(s)}$ $\quad D(s) = 0 \Longrightarrow$ Poles $\quad N(s) = 0 \Longrightarrow$ zeros

ROC
- 不能包含Poles
- 可包含 zeros

Theorem 10:

- 初值定理 If $\left.\frac{d^k x(t)}{dt^k}\right|_{t=0_+}$ exist for all $k = 0, 1, \cdots$, then (0和+∞在收敛域内才能用)

$$x(0_+) = \lim_{s\to\infty} sX(s)$$

- 终值定理 If $Re(s) = 0 \in ROC_x$, then

$$x(+\infty) = \lim_{s\to 0} sX(s)$$

The z-Transform

$$x[n] \longleftrightarrow X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$\sum_{n=-\infty}^{+\infty} \boxed{x[n]\, r^{-n}} e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x[n] (re^{j\Omega})^{-n} \overset{z=re^{j\Omega}}{=\!=\!=} \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = X(z)$$

$$\begin{cases} \alpha^n u[n] \longleftrightarrow \dfrac{1}{1-\alpha z^{-1}} & |\alpha| < |z| \\ -\alpha^n u[-n-1] \longleftrightarrow \dfrac{1}{1-\alpha z^{-1}} & |\alpha| > |z| \end{cases}$$ 半径比较 $\delta[n] \longleftrightarrow 1$

$$\begin{cases} n\alpha^n u[n] \longleftrightarrow \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} & |\alpha| < |z| \\ -n\alpha^n u[-n-1] \longleftrightarrow \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} & |\alpha| > |z| \end{cases}$$

Transform function of LTI system.

$x(t) = e^{pt} \longrightarrow y(t) = H(p)\, e^{pt}$, $\forall p \in ROC_h$.

$x[n] = c^n \longrightarrow y[n] = H(c)\, c^n$, $\forall c \in ROC_h$.

Revist of LTI system's stability and causality

① Causality:

<1> $y[n] \to x[m]$, $m \leq n$

<2> $h(t) = 0$, $t < 0$ 小于0时 h 等于 0

$h[n] = 0$, $n < 0$

<3> $H(s)$ $ROC_h \supseteq \{\sigma_r < Re(s) < +\infty\}$

$H(z)$ $ROC_h = \{z : p_r < |z|\}$

> Poles
zeros

② Stability

<1> if $|x(t)| \leq M$, $|y(t)| \leq N$ 有界

<2> $\int_{-\infty}^{+\infty} |h(t)| dt < M$ $\sum_{n=-\infty}^{+\infty} |h[n]| \leq M$

绝对可积 绝对可和

<3> $H(s)$: $ROC_h \supseteq Re(s) = 0$ 包含虚轴

$H(z)$: $ROC_h \supseteq \{|z| = 1\}$ 包含单位圆

Unilateral transforms 单边LT

$-\infty < t < +\infty \implies t \geq 0_-$

complete response $\bar{y}(t) = y(t) u_{0-}(t)$

Defination: $x(t) \longleftrightarrow \mathcal{X}(s) \triangleq \int_{0_-}^{+\infty} x(t) e^{-st} dt$

Compute (i) $x_1(t) = e^{-\alpha t} u(t)$

(ii) $x_2(t) = e^{-\alpha(t+1)} u(t+1)$

(iii) $x_3(t) = e^{-\alpha(t-1)} u(t-1)$

Solution: as $x_1(t) = 0$, $t < 0$

$\mathcal{X}_1(s) = X_1(s) = \frac{1}{s+\alpha}$ $ROC_{x_1} = \{s : -Re(\alpha) < Re(s)\}$

$x_2(t) = e^{-\alpha(t+1)} u(t+1)$

$\mathcal{X}_2(s) = \int_{0_-}^{+\infty} e^{-\alpha(t+1)} u(t+1) e^{-st} dt = e^{-\alpha} \int_{0_-}^{+\infty} e^{-\alpha t} e^{-st} dt$

$= \frac{e^{-\alpha}}{s+\alpha}$, $ROC_{x_2} = \{s : -Re(\alpha) < Re(s)\}$

$\neq X_2(s) = \frac{e^s}{s+\alpha}$

$x_3(t) = e^{-\alpha(t-1)} u(t-1) = 0 = x_1(t-1)\ \forall t < 0$

we have $\mathcal{X}_3(s) = X_3(s) = X_1(s) e^{-s} = \mathcal{X}_1(s) e^{-s}$ $ROC_{x_3} = \{s : -Re(\alpha) < Re(s)\}$

分部积分 $u'v = (uv)' - uv'$

$$\int_{0_-}^{+\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t) e^{-st} \Big|_{0_-}^{+\infty} + s\mathcal{X}(s) = s\mathcal{X}(s) - x(0_-)$$

$(u = x(t),\ v = e^{-st})$ $-(-s)\boxed{e^{-st} \int_{0_-}^{+\infty} x(t) dt}$ $\mathcal{X}(s)$

△ 1. $\frac{dx(t)}{dt} \Longleftrightarrow s\mathcal{X}(s) - x(0_-)$

2. $\frac{d^2 x(t)}{dt^2} \Longleftrightarrow s[s\mathcal{X}(s) - x(0_-)] - \frac{dx(t)}{dt}\Big|_{t=0_-}$

Unilateral z-transform

$x[n] \longleftrightarrow \mathcal{X}(z) \triangleq \sum_{n=0}^{+\infty} x[n] z^{-n}$

for a causal signal $x[n]$, i.e. $x[n] = x[n]u[n]$ then $\mathcal{X}(z) = X(z)$.

当$x[n]$只在$n\geq 0$有非零值与双边相同.

而当$x[n]$在$n<0$时也有非零值时,双边与单边不同.

$$\begin{cases} x[n-1] \longleftrightarrow z^{-1}\mathcal{X}(z) + x[-1] \\ x[n+1] \longleftrightarrow z[\mathcal{X}(z) - x[0]] \end{cases}$$

Ex. LTI $y[n] + 2y[n-1] = x[n]$ $\quad y[-1] = \gamma_1$, $x[n] = K u[n]$ for $n \geq 0$

Solution: $\mathcal{Y}(z) - [z^{-1}\mathcal{Y}(z) + y[-1]] = \frac{K}{1-z^{-1}}$

$$\mathcal{Y}(z) = -\frac{2\gamma_1}{1+2z^{-1}} + \frac{K}{(1+2z^{-1})(1-z^{-1})} \triangleq \mathcal{Y}_{zi}(z) + \mathcal{Y}_{zs}(z)$$

$\frac{K}{3}\left(\frac{2}{1+2z^{-1}} + \frac{1}{1-z^{-1}}\right)$

with $ROC_{\mathcal{Y}} = \{z : 2 < |z|\}$

$$y[n] = -2\gamma_1(-2)^n u[n] + \frac{K}{3}[2(-2)^n + 1]u[n]$$

$$\triangleq y_{zi}[n] + y_{zs}[n]$$