Math 104A Final Project Report

Xi Sun
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Introduction

Numerical Analysis is one of the subjects in mathematics that is close related with the real world application. Everything we have seen in the real world are not ideally presented with no errors. However, numerical analysis provides a sufficient way to find out the errors in the real world application and get an estimation about the true outcome. In this project, we will discuss one particular problem in the real world that is related with the content in numerical analysis.

Background Motivation

There are more than 70% of the water on earth. Most of them are saltwater and rest of them are freshwater. Almost in all the countries in the world, we can see the rivers everywhere whether it is natural or artificial, which means the river build for leading the water from one place to another. However, there are some rivers that is built for flood protection which we call that flood band. Numerical analysis provides a sufficient way to simulate either the position of the the flood band, the height of the flood band, or the depth of the flood band should be within an accuracy of some small number. And the specific area in this project that I will discuss is how to we use numerical analysis to simulate the water flow or the water wave in order to apply the method and the result in a the real world.

The question that will being mentioned and being solved in this project will be something like how do we use numerical analysis in the application of flood band, which method in numerical analysis could be used for the application, what kind of result could be derived, and what we and comment on about the application based on the result that we get. And may be some other quick questions being make in the project.

Methods Motivation and Problem Statement

Since we are talking about the flood band or the water wave, the original problem or structure that we had considered is the wave propagation in partial differential equations. For example, consider about the following wave equation in general:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad t > 0, x \in \mathbb{R} \\ u|_{t=0} = \phi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

Where $\phi(x)$ and $\psi(x)$ are some functions related with the initial condition and c is some constant.

According to the problem, we know that the general solution for the wave equation is

$$u(t,x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy.$$

By observing this formula, we can easily understand that a wave equation is usually very hard to solve explicitly in the real world, and even most of the time, we don't have those informations about the initial conditions. Thus we want to use a polynomial to approximate $u(t,x)$, which will gives us a simplified model of wave propagation which is easier to solve.

Methods Analysis

The method we are going to use in numerical analysis is the Cubic Spline Interpolation method. And the reason I use this method is because the cubic spline interpolation gives us not only continuously differentiable, but also has a continuous second order derivative. And I think that somehow supports our idea, since the wave equation involves a second order differentiation on both t and x. Another thing for using this method is because the clamped cubic spline requires the information about the boundary conditions, since most of the time in the real world, and also for the problem in this project, we want our flood band be within some fixed interval so that we only need to approximate the equation and estimate the result on part of the function instead of the whole real line.

The cubic spline interpolation method will be used in the following way. First, we create some sample points, and use those points to construct several cubic spline interpolant with the undetermined coefficients for the corresponding interval. Then, use the function mentioned in the initial condition being the function that we are interested in to solve and plug those into the function in python together with the sample points. Finally, we can get several sequences of coefficients from python so that we can construct the concrete interpolant by using the coefficient that are being calculated. In addition, in order to verify how accuracy our result is, I will use the same interpolant but with different points and see how the result being changed and how the error being changed.

In order to use cubic spline interpolation to approximate the wave equation, we need to know some points on that equation or close to that equation, since the cubic spline interpolation will be constructed as a curve which passes through some points. Moreover, since the range of the flood band is fixed most of the time, so we are only interested in some finite intervals. However, that is not the boundary conditions, or actually is, but in our problem, we are interested in how to find those boundary conditions.In addition, we need a function, not necessarily being the same as the general solution, but we can use the form of the general solution and make several assumptions and see which one is the best for simulation. Next, the thing becomes how do we actually achieve our goal.

Problem Solving and Results

In order to give a better visualization and understanding, we will construct a sample question as above, and for simplicity, let $c = 1$. And let $\phi(x)$ and $\psi(x)$ be some concrete functions which will lead the problem being the following:

$$\begin{cases} u_{tt} - u_{xx} = 0 \qquad t > 0, x \in \mathbb{R} \\ u|_{t=0} = 0 \\ u_t|_{t=0} = \cos(x) \end{cases}$$

Next, we take some sample points shown below in order to estimate the equation that was being provided:

$$\{(0, f(0)), (1, f(1)), (2, f(2))\}.$$

Then, we get our points and the problem, so we can do an estimation and finally verify the result at the end.

According to the points mentioned above, we can see that the points involves in the interval $[0, 2]$ on x, so we construct two polynomials as the following:

$$\begin{cases} S_0(x) = a_0 + b_0x + c_0x^2 + d_0x^3 & x \in [0, 1] \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 & x \in [1, 2] \end{cases}$$

Now, we use our sample points mentioned above, and we use the **cubicspline** function in python(see codes part 1). Since one of our initial condition gives us $u_t|_{t=0} = \cos(x)$, therefore, according to the general solution, we get

$$u(t, x) = \frac{1}{2}(\sin(x+t) - \sin(x-t)),$$

and the derivative of that respect to t will be

$$u_t(t, x) = \frac{1}{2}(\cos(x+t) + \cos(x-t)).$$

Thus, we can derive two sequences of coefficients such we can construct the polynomial like the following:

$$\begin{cases} S_0 = -0.83907x - 0.0097306x^2 - 0.12329x^3 & x \in [0, 1] \\ S_1 = -0.70605 - 0.44975(x-1) + 0.37959(x-1)^2 - 0.013249(x-1)^3 & x \in [1, 2] \end{cases}$$

In order to verify our result, we need to take some other points that are different from the points above. And actually, in the real world, as I mentioned above, the distance or the position of the band are usually fixed, so in this case, I will fix $x = 10$ as an example. Then, I need to construct a sequence of t since we have two variables. so let t being a list of $\{0.3, 0.8, 1.5, 1.7\}$ and then apply in to both actual functions and the approximated functions(see codes part 2), and also by computing the absolute error, we get the following:

$$\begin{cases} S_0(0.3) = -0.25593 \\ S_0(0.8) = -0.74061 \\ S_1(1.5) = -0.83768 \\ S_1(1.7) = -0.83942 \end{cases} \quad \begin{cases} f(0.3) = -0.24796 \\ f(0.8) = -0.60191 \\ f(1.5) = -0.83697 \\ f(1.7) = -0.83208 \end{cases} \quad \begin{cases} e_{abs}(0.3) = 0.00796 \\ e_{abs}(0.8) = 0.139 \\ e_{abs}(1.5) = 0.000714 \\ e_{abs}(0.3) = 0.00734 \end{cases}$$

According to the data result shown above, we can see that the absolute error is small enough which is less than 10^{-2} except when $t = 0.8$. That is accepted, since we usually get some outliers in out experiment, when we doing modeling approximations. Overall, we get our expected approximation which is most of the absolute errors are small enough. Therefore, we can conclude that cubic spline interpolation is a sufficient way to approximate the wave propagation.

Related with the real world problem. According to our result, since we get all negative numbers for all the points we tried, I think that could means that the depth of the river or the flood band should be, for example, 0.25593 meters with and accuracy within 10^{-2} below ground level at

$t = 0.3$ and $x = 10$, which means our designed depth or the height of the flood band should be $0.25593 + 0.01 = 0.25693$ meters to make sure that the flood is successfully protected in the flood band area at a certain position.

Summary

According to everything we found above, we can conclude that there is a sufficient method in numerical analysis that could solve our problem. Moreover, there are a lot of different method in numerical analysis that could solve variety of applications in the real world. Each method has its own goodness since there is no one method that is the best for all the applications. Back to our problem, I think there is more than one method could being used in numerical analysis to approximate the decision of the flood band. Because of the length of this report, I will not mention other methods anymore. However, I believe that there is a method in numerical analysis called Numerical Simulation with Partial Differential Equation, I hope I can get more idea about the question being answered in this project when I finally finish learning that method.