Math 104B Final Project Report

Xi Sun
Perm Number: 496474-8

June 9, 2018

Introduction

Numerical Analysis is one of the subjects in mathematics that is close related with the real world application. Everything we have seen in the real world are not ideally presented with no errors. However, numerical analysis provides a sufficient way to find out the errors in the real world application and get an estimation about the true outcome. In this project, we will discuss one particular problem in the real world that is related with the content in numerical analysis.

Background Motivation

There are more than 70% of the water on earth. Most of them are saltwater and rest of them are freshwater. Almost in all the countries in the world, we can see the rivers everywhere whether it is natural or artificial, which means the river build for leading the water from one place to another. However, there are some rivers or bands that are built for flood protection which we call that flood band. Numerical analysis provides a sufficient way to simulate either the position of the the flood band, the height of the flood band, or the depth of the river should be within an accuracy of some small number. And the specific area in this project that I will discuss is how do we use numerical analysis to simulate the water flow or the water wave in order to apply the method and the result in the real world.

The question that will being mentioned and being solved in this project will be something like how do we use numerical analysis in the application of flood band, which method in numerical analysis could be used for the application, what kind of result could be derived, and what we can comment on about the application based on the result that we get. And may be some other quick questions being make in the project.

Methods Motivation and Problem Statement

Since we are talking about the flood band or the water wave, the original problem or structure that we had considered is the wave propagation in partial differential equations. For example, consider about the following wave equation in general:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad t > 0, x \in \mathbb{R} \\ u|_{t=0} = \phi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$$

Where $\phi(x)$ and $\psi(x)$ are some functions related with the initial condition and c is some constant.

According to the problem, we know that the general solution for the wave equation is

$$u(t,x) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy.$$

By observing this formula, we can easily understand that a wave equation is usually very hard to solve explicitly in the real world, and even most of the time, we don't have those informations about the initial conditions. Thus we want to use a direct method for differential equations to approximate the actual value by plugging in some data points.

Method Analysis

The method that I am going to use in this project is the Runge-Kutta method with order four. The reason for using this is the following: First, compare with the Taylor's method, it eliminates the need for solving and evaluating the derivatives of the functions which is usually unknown to us. Second, it provides a more accurate result by the convergence theory since the Runge-Kutta method has the rate of convergence $O(h^4)$. By doing the analysis for those two aspects, we figured out that it is true that the higher order Taylor's method might convergents faster than the Runge-Kutta method, however, the more faster it convergents, the more terms and orders of the derivatives we will need which is really inconvenient to us. Moreover, another method we learned is the Adam Bashforth/Moulton method, but I think that is not the best method that we are going to apply in our case. Since the partial differential equation is much more complicated than the ordinary differential equations, and the Adam's methods are involving a huge amount of calculations in the algorithm(assuming we need to meet the same rate of convergence, so we might need the forth order or fifth order), so it might takes a long time in the long run. Therefore, we are going to apply the Runge-Kutta method in our projects.

Next, the question becomes how to apply this method in our case, the conditions we need is a differential equation with it's initial condition. Therefore, in order to achieve this process, our group decided to construct a sample function which is kind of related with the behavior of the ocean waves and pick some point of the interval that we are interested in to compute the approximated value. Next, we are going to verify if this is a good method by comparing the approximated value with the actual value. In addition, since we know that Runge-Kutta method is designed for approximating differential equations, so here comes the question: Are there any other methods that is not designed for differential equations by actually performs better in some special situations? The method we are going to compare is the Cubic Spline Interpolation in 104A since we were using this method in our previous project.

Problem Solving and Results

In order to apply this method, as mentioned above, we will construct a sample function with its initial conditions that is corresponding to the general form of the wave equation mentioned above. So the function that we are going to use is the following:

$$f(t,y) = 5(\cos(10+t) + \cos(10-t))$$

and out original function will be

$$g(t) = 5(\sin(10+t) - \sin(10-t))$$

and we are going to approximate the solution in the interval $[0, 20]$ with $h = 1$ and initial condition $\alpha = 0$.

As for the result, since the data contains 20 outputs, so we are only going to select some of them, and the data that we are gong to use to compare is as the following:

$$\{t = 3, 8, 15, 17\}.$$

Since we have everything ready, so we can apply the Runge-Kutta algorithm(**rk** function in python, see code) to get the approximated solution and then, we are going to compare that with the actual solution and see how large or small the absolute error is. So the solution we get is as the following:

$$\begin{cases} rk(3) = -1.1845215206226891 \\ rk(8) = -8.304393904468641 \\ rk(15) = -5.458332605384895 \\ rk(17) = 8.069699220261548 \end{cases} \begin{cases} act(3) = -1.1840978094607406 \\ act(8) = -8.30142336798679 \\ act(15) = -5.456380123804557 \\ act(17) = 8.066812635616461 \end{cases} \begin{cases} err(3) = 0.00042371116194850345 \\ err(8) = 0.002970536481852193 \\ err(15) = 0.0019524815803375617 \\ err(17) = 0.0028865846450862875 \end{cases}$$

According to the result, we can get the conclusion: The approximated value is close to the actual value and the absolute error is small like 10^{-3} or 10^{-4}. And actually by the observation based on our full out put, the maximum value happens when $t = 5$ and $t = 17$. Therefore, if we use meters for the unit, we can conclude that the maximum height the wave is approximately 8 meters.

Next, we are going to validate our result and answer the question of the claim that is mentioned above, we are going to validate this approximation by using Cubic Spline Interpolation since the interpolation method is not originally designed for approximate differential equations, and we are going to check if the method not for differential equations actually performs better. We will divide the interval $[0, 20]$ into 5 subintervals, and by the Cubic Spline Algorithm, we can construct 5 polynomials as the following:

$$\begin{cases} S_0(x) = -8.39072x + 5.17086x^2 - 0.66908x^3 & x \in [0, 4] \\ S_4(x) = 6.35011 + 0.86057(x-4) - 2.85804(x-4)^2 + 0.4318(x-4)^3 & x \in [4, 8] \\ S_8(x) = -8.30142 - 1.27761(x-8) + 2.3235(x-8)^2 - 0.30097(x-8)^3 & x \in [8, 12] \\ S_{12}(x) = 4.50223 + 2.86398(x-12) - 1.2881(x-12)^2 + 0.11042(x-12)^3 & x \in [12, 16] \\ S_{16}(x) = 2.41571 - 2.14044(x-16) + 0.036997(x-16)^2 - 0.032909(x-16)^3 & x \in [16, 20] \end{cases}$$

In order to validate the points that we picked before, we only need S_0, S_4, S_{12}, S_{16}. By plug in the points $\{3, 8, 15, 17\}$, we get the following:

$$\begin{cases} S_0(3) = 3.30042 \\ S_4(8) = -8.3015 \\ S_{12}(15) = 4.48261 \\ S_{16}(17) = 0.279358 \end{cases} \begin{cases} act(3) = -1.1840978094607406 \\ act(8) = -8.30142336798679 \\ act(15) = -5.456380123804557 \\ act(17) = 8.066812635616461 \end{cases}$$

Now the question becomes: Why we get this amazing answer? As we can see that actually there is one point that is very close to the actual value which is at 8, and all other results are far away from the actual result. So the answer is: The point 8 is on the boundary of the interval in which the polynomial $S_4(x)$ falls in. So it is in our expectation that the approximated solution should be very close to the actual value. So what happens to other points? The thing is, if we want to simulate a function by a polynomial, we have to separate the interval into smaller and smaller pieces in order to get a more accurate result. However, in the long run, it take a long time to construct those polynomials. Assuming we are going to separate the interval with the difference between the boundaries equals to 1, then in our case, we need to construct 20 polynomials which we really didn't want to do that. And of course, in the real world application, if we want to simulate the water wave in the nearest 100 meters from us, then we need to construct 100 polynomials which is obviously not a good idea. Therefore, the answer for the claim is in our situation or in any kind of real world application that need to simulate a long-run behavior. The interpolation method is not the best. So from now on, we will go back to our Runge-Kutta method and then make some conclusion.

Interpretation and Conclusion

According to the approximated value by the Runge-Kutta method, the highest value of the water wave in a period amount of time is approximately 8 meters, and the absolute error between the approximated value and the actual value at the highest point is approximately 3×10^{-3} which is close, which means that our approximation succeeded. Next, consider about the actual situation, in order to build the flood band for protecting from the water wave, I will make that should be 9 meters above the sea level(assuming the sea level is 0 meters). And the reason for that is because we don't want the highest water wave equals the flood band which will make part of the water flow over the band. And actually 9 meters is just a pending result in the actual situation, what I want to conclude is the actual result should be greater than our approximated value, since by physics, if a fluid hits a wall, the fluid might goes above a little bit depending on the instantaneous velocity when the fluid hit the wall, and also, the whether or the wind might also affect the water flows, especially for the wind that is strong enough. If we have all the information above, we will able to build a "perfect" flood band.

Summary

According to everything we found above, we can conclude that there is a sufficient method in numerical analysis that could solve our problem. Moreover, there are a lot of different method in numerical analysis that could solve variety of applications in the real world. Each method has its own goodness since there is no one method that is the best for all the applications. Back to our problem, I think there is more than one method could being used in numerical analysis to approximate the decision of the flood band. The more information we can get, the closer the result that approaches the real situation.