Tropical Representations of Chinese Monoids with and without Involution

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January 21, 2025

Abstract

Recently, Izhakian and Merlet gave a faithful representation $\widehat{\rho}$ of the Chinese monoid Ch_n of every finite rank n as a submonoid of the monoid $UT_{2,3^n-2}(\mathbb{T})$ of upper triangular matrices over the tropical semiring $\mathbb{T}$. We exhibit another faithful representation ϕ_n of Ch_n as a submonoid of the monoid $UT_{n(n-1)}(\mathbb{T})$ of upper triangular matrices over $\mathbb{T}$. The dimension of ϕ_n is smaller than that of $\widehat{\rho}$ when $n \geq 4$. Further, we give a faithful representation of the Chinese monoid $(Ch_n, \dagger)$ under Schutzenberger's involution $\ddagger$.

Keywords Chinese monoid · Schutzenberger's involution · Representation

1 Introduction

The Chinese monoid appeared in the classification of classes of monoids based on growth properties, whose growth function coincides with that of the plactic monoid [5]. For a positive integer n, the Chinese monoid of rank n, denoted by Ch_n, is generated by a totally ordered alphabet $\mathcal{A}_n = \{a_1 < a_2 < \cdots < a_n\}$ with the relations

$$a_r a_q a_p = a_r a_p a_q = a_q a_r a_p \quad \text{for all } p \leq q \leq r.$$

Let $\mathcal{A}_n^+$ and $\mathcal{A}_n^* = \mathcal{A}_n^+ \cup \{1\}$ be the free semigroup and free monoid generated by $\mathcal{A}_n$ respectively. Clearly, Ch_n is the quotient monoid of the free monoid $\mathcal{A}_n^*$ modulo the congruence $\equiv_{Ch}$ determined by (1.1). It can be seen in [2] that each element of Ch_n has a unique presentation, called the canonical form, written as

$$b_1 b_2 \cdots b_n$$

with

$$b_1 = a_1^{k_1},$$
$$b_2 = (a_2 a_1)^{k_{21}} a_2^{k_{22}},$$
$$b_3 = (a_3 a_1)^{k_{31}} (a_3 a_2)^{k_{32}} a_3^{k_{33}},$$

$$\vdots$$
$$b_n = (a_n a_1)^{k_{n1}} (a_n a_2)^{k_{n2}} \cdots (a_n a_{n-1})^{k_{n(n-1)}} a_n^{k_{nn}},$$

where all exponents k_{ji} are non-negative integers.

Recently, Chinese monoids have attracted much attention and many important results have been obtained. Cassaigne et al. [2] made the first fundamental study of the Chinese monoid. They established an algorithm similar to Schensted's algorithm for the plactic monoids, which yields a characterization of the equivalence classes of Ch_n, and a cross-section theorem was also provided. Kubat and Okniński [9] constructed all irreducible representations of the Chinese monoid Ch_n of any rank n, which are homomorphisms from Ch_n to $End_K(V)$, the monoid of all linear transformations on the linear space V over the field K. A complete rewriting systems was constructed by Cain et al. [1] and they proved the biautomaticity for finite-rank Chinese monoids. Other notable studies have centered around the monoid algebra $K[Ch_n]$ over a field K, which is the unital algebra defined by the algebra presentation determined by relations (1.1): Cedó and Okniński [3] described the structure of the monoid algebra $K[Ch_2]$; Jaszuńska and Okniński [7] studied the structure of the monoid algebra $K[Ch_3]$; Cedó and Okniński [4] and Jaszuńska and Okniński [8] respectively described the minimal prime ideals of the monoid algebra $K[Ch_n]$ for every n.

Here, we focus on tropical representations of Chinese monoids with and without involution. Recall that the tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is the set $\mathbb{R} \cup \{-\infty\}$ under the operations $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ for all $a, b \in \mathbb{T}$, where $\max\{a, -\infty\} = a = \max\{-\infty, a\}$ and $-\infty + a = a + -\infty = -\infty$ for all $a \in \mathbb{R}$. Denote by $M_n(\mathbb{T})$ the monoid of all $n \times n$ matrices with entries from $\mathbb{T}$ together with the matrix multiplication induced from the operations of $\mathbb{T}$ in the obvious way, and by $UT_n(\mathbb{T})$ the submonoid of all $n \times n$ upper triangular matrices over $\mathbb{T}$. A tropical representation of a semigroup S is a semigroup homomorphism $\phi : S \to M_n(\mathbb{T})$. The homomorphism ϕ is said to be faithful if it is injective. The tropical semiring is of interest as a natural carrier for representations of semigroups. For example, the bicyclic monoid $\mathcal{B} = \langle p, q | pq = 1 \rangle$, which is ubiquitous in semigroup theory, admits no faithful finite dimensional representations over any field; however it has a number of faithful tropical representations.

Izhakian and Merlet [6] inductively build a faithful tropical representation for Ch_n out of a faithful representation for Ch_{n-1}, that is, constructing a representation by induction on the number of generators n. Based on these representations, it is shown that Ch_n admits the same semigroup identities with the bicyclic monoid $\mathcal{B}$ [6, Corollary II]. The present paper gives another faithful representation of Ch_n for each finite $n \geq 2$ as a submonoid of the monoid $UT_{n(n-1)}(\mathbb{T})$ of upper triangular matrices over $\mathbb{T}$, for which the dimension for each Ch_n is smaller than the dimension of the representation constructed by Izhakian and Merlet when $n \geq 4$; see Sect. 2.

Further, we consider a tropical representation of the Chinese monoid with involution. Recall that a unary operation * on a semigroup S is an involution if S satisfies the identities

$$(x^*)^* \approx x, \quad (xy)^* \approx y^*x^*.$$

An involution semigroup is a pair $(S, *)$ where S is a semigroup with involution $*$, and S is called the semigroup reduct of $(S, *)$. Common examples of involution semigroups include groups with inversion and multiplicative matrix semigroups over any field with transposition. For any matrix $A \in M_n(\mathbb{T})$, denote by A^D the matrix obtained by reflecting A with respect to the secondary diagonal (from the top right to the bottom left corner), that is, $(A^D)_{ij} = A_{(n+1-j)(n+1-i)}$. It is easy to verify that this unary operation D (called the skew transposition) is an involution operation of $M_n(\mathbb{T})$. In Sect. 3, we give a faithful tropical representation of the Chinese monoid $(Ch_n, \dagger)$ under Schützenberger's involution for each finite $n \geq 2$ as a submonoid of the monoid $(UT_{2n(n-1)}(\mathbb{T}), ^D)$ of upper triangular matrices over $\mathbb{T}$ under the skew transposition.

2 Tropical representations of Ch_n

In this section, we exhibit a faithful tropical representation of the Chinese monoid Ch_n for each finite n.

By (1.1), each element $u \in Ch_n$ can be converted into a unique element in canonical form. We say $u \equiv ch$ if the elements $u, v \in Ch_n$ have the same canonical form. Clearly if the elements $u, v \in Ch_n$ are in canonical form and $u \equiv ch$ then $u = v$.

Denote by

$$P = \begin{pmatrix} 1 & 0 \\ -\infty & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ -\infty & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -\infty \\ -\infty & 0 \end{pmatrix}.$$

Clearly I is the identity matrix in $M_2(\mathbb{T})$. It is routine to show that

$$QP^2 = PQP = \begin{pmatrix} 2 & 1 \\ -\infty & 1 \end{pmatrix}, \quad Q^2P = QPQ = \begin{pmatrix} 1 & 1 \\ -\infty & 2 \end{pmatrix}$$

and for any non-negative integer $k \geq 1$,

$$P^k = \begin{pmatrix} k & k-1 \\ -\infty & 0 \end{pmatrix}, \quad Q^k = \begin{pmatrix} 0 & k-1 \\ -\infty & k \end{pmatrix}, \quad (QP)^k = \begin{pmatrix} k & k-1 \\ -\infty & k \end{pmatrix}.$$

First we introduce a tropical representation of the Chinese monoid Ch_1 of rank 1. Define a map $\phi_1 : \mathcal{A}_1 \cup \{1\} \to UT_2(\mathbb{T})$ given by $1 \mapsto I$ and $a_1 \mapsto QP$. Clearly, the map ϕ_1 induces a faithful tropical representation of the Chinese monoid Ch_1 of rank 1.

Next we consider a tropical representation of the Chinese monoid Ch_n of rank $n \geq 2$. Let $i, j \geq 1$ and $i + j \leq n$. Define

$$\phi_{ij} : \mathcal{A}_n \cup \{1\} \to UT_2(\mathbb{T})$$

given by $1 \mapsto I$ and

$$a_k \mapsto \begin{cases} P, & \text{if } 1 \leq k \leq i, \\ I, & \text{if } i+1 \leq k \leq n-j, \\ Q, & \text{if } n-j+1 \leq k \leq n. \end{cases}$$

Clearly the map $\phi_{ij} : \mathcal{A}_n \cup \{1\} \to UT_2(\mathbb{T})$ can be extended to a homomorphism from $\mathcal{A}_n^*$ to $UT_2(\mathbb{T})$. In the following, we prove that the map ϕ_{ij} induces a tropical representation of Ch_n for each $n \geq 2$.

Lemma 1. *The map $\phi_{ij} : Ch_n \to UT_2(\mathbb{T})$ is a tropical representation of Ch_n.*

Proof. Note that ϕ_{ij} is a homomorphism from $\mathcal{A}_n^*$ to $UT_2(\mathbb{T})$. Then to show that ϕ_{ij} induces a tropical representation of Ch_n, we only need to show that for any $u, v \in \mathcal{A}_n^*$, if $u \equiv_{Ch} v$, then $\phi_{ij}(u) = \phi_{ij}(v)$.

Let $1 \leq p \leq q \leq r \leq n$. Note that $i, j \geq 1$ and $i + j \leq n$. If $i + j < n$, then by the definition of ϕ_{ij} and (2.1), it is routine to show that

$$\begin{aligned} & \phi_{ij}(a_r a_q a_p) = \phi_{ij}(a_r a_p a_q) = \phi_{ij}(a_q a_r a_p) \\ = & \begin{cases} P^3, & \text{if } 1 \leq p \leq q \leq r \leq i, \\ P^2, & \text{if } 1 \leq p \leq q \leq i, i+1 \leq r \leq n-j, \\ QP^2, & \text{if } 1 \leq p \leq q \leq i, n-j+1 \leq r \leq n, \\ P, & \text{if } 1 \leq p \leq i, i+1 \leq q \leq r \leq n-j, \\ QP, & \text{if } 1 \leq p \leq i, i+1 \leq q \leq n-j, n-j+1 \leq r \leq n, \\ Q^2P, & \text{if } 1 \leq p \leq i, n-j+1 \leq q \leq r \leq n, \\ I, & \text{if } i+1 \leq p \leq q \leq r \leq n-j, \\ Q, & \text{if } i+1 \leq p \leq q \leq n-j, n-j+1 \leq r \leq n, \\ Q^2, & \text{if } i+1 \leq p \leq n-j, n-j+1 \leq q \leq r \leq n, \\ Q^3, & \text{if } n-j+1 \leq p \leq q \leq r \leq n. \end{cases} \end{aligned}$$

If $i + j = n$, then by the definition of ϕ_{ij} and (2.1), it is routine to show that

$$\begin{aligned} & \phi_{ij}(a_r a_q a_p) = \phi_{ij}(a_r a_p a_q) = \phi_{ij}(a_q a_r a_p) \\ = & \begin{cases} P^3, & \text{if } 1 \leq p \leq q \leq r \leq i, \\ Q^3, & \text{if } i+1 \leq p \leq q \leq r \leq n, \\ QP^2, & \text{if } 1 \leq p \leq q \leq i, i+1 \leq r \leq n, \\ Q^2P, & \text{if } 1 \leq p \leq i, i+1 \leq q \leq r \leq n. \end{cases} \end{aligned}$$

Hence for any $u, v \in \mathcal{A}_n^*$, if $u \equiv_{Ch} v$, then $\phi_{ij}(u) = \phi_{ij}(v)$. Therefore, the map ϕ_{ij} is a tropical representation of Ch_n. $\square$

Let

$$w = a_1^{k_{11}} \cdot (a_2 a_1)^{k_{21}} a_2^{k_{22}} \cdots (a_n a_1)^{k_{n1}} (a_n a_2)^{k_{n2}} \cdots (a_n a_{n-1})^{k_{n(n-1)}} a_n^{k_{nn}}$$

be any element of Ch_n in canonical form by (1.2). Then by computation,

$$\phi_{ij}(w) = P^m \cdot (QP)^{l_{11}} Q^{l_{12}} \cdot (QP)^{l_{21}} Q^{l_{22}} \cdots (QP)^{l_{j1}} Q^{l_{j2}} = \begin{pmatrix} A_{ij} & B_{ij} \\ -\infty & C_{ij} \end{pmatrix}$$

where

$$A_{ij} = m + \sum_{r=1}^{j} l_{r1},$$

$$B_{ij} = m + \sum_{r=1}^{j} (l_{r1} + l_{r2}) - 1,$$

$$C_{ij} = \sum_{r=1}^{j} (l_{r1} + l_{r2}),$$

with

$$m = k_{11} + \sum_{r=2}^{i} \left(\sum_{t=1}^{r-1} 2k_{rt} + k_{rr} \right) + \sum_{r=i+1}^{n-j} \sum_{t=1}^{i} k_{rt},$$

$$l_{r1} = \sum_{t=1}^{i} k_{(n-j+r)t},$$

$$l_{r2} = \sum_{t=i+1}^{n-j} k_{(n-j+r)t} + \sum_{t=n-j+1}^{n-j+r-1} 2k_{(n-j+r)t} + k_{(n-j+r)(n-j+r)}.$$

Let

$$\phi_n = (\phi_{11}, \ldots, \phi_{n-1,1}, \phi_{22}, \ldots, \phi_{n-2,2}, \ldots, \phi_{1,n-2}, \phi_{2,n-2}, \phi_{1,n-1}).$$

Then ϕ_n is a map from $\mathcal{A}_n \cup \{1\}$ to $(UT_2(\mathbb{T}))^{\frac{n(n-1)}{2}}$. By Lemma 2.1, ϕ_n induces a semigroup homomorphism from Ch_n to $(UT_2(\mathbb{T}))^{\frac{n(n-1)}{2}}$.

Theorem 1. *The representation $\phi_n : Ch_n \to (UT_2(\mathbb{T}))^{\frac{n(n-1)}{2}}$ is faithful.*

Proof. Let

$$\mathbf{u} = a_{11}^{k_{11}} \cdot (a_{21}a_{12})^{k_{21}} a_{22}^{k_{22}} \cdots (a_n a_1)^{k_{n1}} (a_n a_2)^{k_{n2}} \cdots (a_n a_{n-1})^{k_{n(n-1)}} a_n^{k_{nn}}$$

$$\mathbf{v} = a_{11}^{k'_{11}} \cdot (a_{21}a_{12})^{k'_{21}} a_{22}^{k'_{22}} \cdots (a_n a_1)^{k'_{n1}} (a_n a_2)^{k'_{n2}} \cdots (a_n a_{n-1})^{k'_{n(n-1)}} a_n^{k'_{nn}}$$

be any elements of Ch_n in canonical form by (1.2). Then the elements $\mathbf{u}$ and $\mathbf{v}$ are uniquely determined by $\frac{n(n+1)}{2}$-tuples

$$(k_{11}, k_{21}, k_{22}, \ldots, k_{n1}, k_{n2}, \ldots, k_{nn})$$

and

$$(k'_{11}, k'_{21}, k'_{22}, \ldots, k'_{n1}, k'_{n2}, \ldots, k'_{nn})$$

respectively. Hence to show that ϕ_n is faithful, it suffices to show that for any $\mathbf{u}, \mathbf{v} \in \mathcal{A}_n^*$, if $\phi_n(\mathbf{u}) = \phi_n(\mathbf{v})$, then $k_{ng} = k'_{ng}$ for each $1 \leq g \leq h \leq n$.

It follows from $\phi_n(\mathbf{u}) = \phi_n(\mathbf{v})$ that $\phi_{ij}(\mathbf{u}) = \phi_{ij}(\mathbf{v})$ for each $i, j \geq 1$ and $i + j \leq n$. By (2.3), denote

$$\phi_{ij}(\mathbf{u}) = \begin{pmatrix} A_{ij} & B_{ij} \\ -\infty & C_{ij} \end{pmatrix}$$

and $\phi_{ij}(\mathbf{v}) = \begin{pmatrix} A'_{ij} & B'_{ij} \\ -\infty & C'_{ij} \end{pmatrix}$.

Then

$$A_{ij} = A'_{ij}, \quad B_{ij} = B'_{ij} \quad \text{and} \quad C_{ij} = C'_{ij}$$

for each $i, j \geq 1$ and $i + j \leq n$.

First we show that $k_{h1} = k'_{h1}$ for each $h = n, n-1, \ldots, 1$. Put $i = 1$ in (2.4). Then by (2.4) that

$$A_{1j} + C_{1j} - B_{1j} = A'_{1j} + C'_{1j} - B'_{1j}$$

for each $j = 1, 2, \ldots, n-1$, and so by (2.3) and a simple calculation we have

$$\begin{aligned} & k_{n1} + k_{(n-1)1} + \cdots + k_{(n-j+1)1} \\ = {} & k'_{n1} + k'_{(n-1)1} + \cdots + k'_{(n-j+1)1}. \end{aligned}$$

Thus, if $j = 1$, then (2.5) implies that $k_{n1} = k'_{n1}$. If $j = 2$, then (2.5) implies that $k_{n1} + k_{(n-1)1} = k'_{n1} + k'_{(n-1)1}$, and so $k_{(n-1)1} = k'_{(n-1)1}$ by $k_{n1} = k'_{n1}$. Continue this process until $j = n-1$, we can gradually prove that $k_{h1} = k'_{h1}$ for each $h = n, n-1, \ldots, 2$. To show that $k_{11} = k'_{11}$. Let $i = 1$ and $j = n-1$ in (2.4). Then by (2.4) that

$$B_{1,n-1} - C_{1,n-1} = B'_{1,n-1} - C'_{1,n-1}.$$

Hence by (2.3) and a simple calculation we can deduce that $k_{11} = k'_{11}$. Therefore

$$k_{h1} = k'_{h1}$$

for each $h = n, n-1, \ldots, 1$.

Next show that $k_{h2} = k'_{h2}$ for each $h = n, n-1, \ldots, 2$. Put $i = 2$ in (2.4). Then by (2.4) that

$$A_{2j} + C_{2j} - B_{2j} = A'_{2j} + C'_{2j} - B'_{2j}$$

for each $j = 1, 2, \ldots, n-2$, and so by (2.3) and a simple calculation we have

$$\begin{aligned} & k_{n1} + k_{(n-1)1} + \cdots + k_{(n-j+1)1} + k_{n2} + k_{(n-1)2} + \cdots + k_{(n-j+1)2} \\ = {} & k'_{n1} + k'_{(n-1)1} + \cdots + k'_{(n-j+1)1} + k'_{n2} + k'_{(n-1)2} + \cdots + k'_{(n-j+1)2}. \end{aligned}$$

Thus, if $j = 1$, then (2.7) implies that

$$k_{n1} + k_{n2} = k'_{n1} + k'_{n2}.$$

Since $k_{n1} = k'_{n1}$ by (2.6), it follows that $k_{n2} = k'_{n2}$. If $j = 2$, then (2.7) implies that

$$k_{n1} + k_{n2} + k_{(n-1)1} + k_{(n-1)2} = k'_{n1} + k'_{n2} + k'_{(n-1)1} + k'_{(n-1)2}.$$

Since $k_{n1} = k'_{n1}$ and $k_{(n-1)1} = k'_{(n-1)1}$ by (2.6) and $k_{n2} = k'_{n2}$, it follows that $k_{(n-1)2} = k'_{(n-1)2}$. Continue this process until $j = n-2$, we can gradually prove that $k_{h2} = k'_{h2}$ for $h = n, n-1, \ldots, 3$. To show that $k_{22} = k'_{22}$. Let $i = 2$ and $j = n-2$ in (2.4). Then by (2.4) that

$$B_{2,n-2} - C_{2,n-2} = B'_{2,n-2} - C'_{2,n-2}.$$

Then by (2.3) and a simple calculation we can deduce that

$$k_{11} + 2k_{21} + k_{22} = k'_{11} + 2k'_{21} + k'_{22}.$$

Since $k_{11} = k'_{11}$ and $k_{21} = k'_{21}$ by (2.6), it follows that $k_{22} = k'_{22}$. Therefore

$$k_{h2} = k'_{h2}$$

for each $h = n, n-1, \ldots, 2$.

Now, by continue this process, we can gradually show that $k_{h3} = k'_{h3}$ for $h = n, n-1, \ldots, 3$; $k_{h4} = k'_{h4}$ for $h = n, n-1, \ldots, 4$; $\ldots$; and $k_{hn} = k'_{hn}$ for $h = n$. Therefore, the representation $\phi_n : Ch_n \to (UT_2(\mathbb{T}))^{\frac{n(n-1)}{2}}$ is faithful. $\square$

Let $M_i \in UT_2(\mathbb{T}), i = 1, 2, \ldots, \frac{n(n-1)}{2}$. Define the map

$$t : (UT_2(\mathbb{T}))^{\frac{n(n-1)}{2}} \to UT_{n(n-1)}(\mathbb{T})$$

given by

$$(M_1, M_2, \ldots, M_{\frac{n(n-1)}{2}}) \mapsto \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_{\frac{n(n-1)}{2}} \end{pmatrix}.$$

It is easy to show that the map t is injective. Consider the map $\tilde{\phi}_n = t \circ \phi_n$ from Ch_n to $UT_{n(n-1)}(\mathbb{T})$, we have the following theorem.

Theorem 2. *The map $\tilde{\phi}_n : Ch_n \to UT_{n(n-1)}(\mathbb{T})$ is a faithful tropical representation of Ch_n.*

Proof. Note that the homomorphisms ϕ_n and t are injective. Then $\tilde{\phi}_n = t \circ \phi_n$ is a faithful tropical representation of Ch_n. $\square$

Remark 1. *In [6], Izhakian and Merlet gave a faithful representation $\widehat{\rho}$ of the Chinese monoid Ch_n as a submonoid of the monoid $UT_{2,3^n-2}(\mathbb{T})$ of upper triangular matrices over $\mathbb{T}$. It is obvious that the dimension of $\tilde{\phi}_n$ is smaller than that of $\widehat{\rho}$ when $n \geq 4$, and equal to that of $\widehat{\rho}$ for $n = 2, 3$.*

3 Tropical representations of $(Ch_n, \dagger)$

In this section, we exhibit a faithful tropical representation of the Chinese monoid $(Ch_n, \dagger)$ under Schützenberger's involution $\dagger$ for each finite n.

Consider the anti-automorphism $\dagger$ of $\mathcal{A}_n^*$ given by reversing the linear order on $\mathcal{A}_n$, that is $a_1^\dagger = a_n, a_2^\dagger = a_{n-1}, \ldots, a_n^\dagger = a_1$. By [2, Subsection 1.2], the relation $\equiv_{Ch}$ is compatible with the unary operation $\dagger$, that is, for any $u, v \in \mathcal{A}_n^*, u \equiv_{Ch} v$ if and only if $u^\dagger \equiv_{Ch} v^\dagger$. Hence $(Ch_n, \dagger)$ is an involution monoid, and the involution $\dagger$ is called Schützenberger's involution.

Proposition 1. *Schützenberger's involution is the unique involution on the Chinese monoid Ch_n for each $n \geq 1$.*

Proof. Suppose that * is an involution operation on Ch_n. Note that the relations in Ch_n are length-preserving and each element in Ch_n with length 1 is a generator of Ch_n. Then the involution of a generator in Ch_n is still a generator in Ch_n. Since Ch_1 has only one generator a_1, we have $a_1^* = a_1$. Thus the involution on Ch_1 is trivial. For Ch_n with $n \geq 2$, let $a < b \leq n$. Then $(aba)^* = a^*b^*a^* = _{Ch} a^*a^*b^* = (baa)^*$ by $aba = _{Ch} baa$. This implies $a^*b^*a^* = _{Ch} a^*a^*b^*$, and so $b^* < a^*$. Hence for any $a < b$, there must be $b^* < a^*$ under the involution *. Therefore * is just Schützenberger's involution. $\square$

It is easy to check that neither the homomorphism $\tilde{\phi}_n$ defined in Sect. 2 nor the homomorphism $\widehat{\rho}$ defined in [2] are compatible with Schützenberger's involution. Now we use ϕ_n to construct a homomorphism from $(Ch_n, \dagger)$ to $((UT_2(\mathbb{T}))^{n(n-1)}, *)$ that is compatible with Schützenberger's involution. Clearly the map ϕ_1 induces a faithful tropical representation of $(Ch_1, \dagger)$. Next we consider the tropical representation of $(Ch_n, \dagger)$ for each $n \geq 2$. Let

$$\psi_n = (\phi_{n-1,1}, \phi_{n-2,2}, \phi_{n-2,1}, \ldots, \phi_{2,n-2}, \ldots, \phi_{21}, \phi_{1,n-1}, \ldots, \phi_{11})$$

and $\xi_n = (\phi_n, \psi_n)$. By Theorem 2.2, the map ξ_n is an injective homomorphism from Ch_n to $(UT_2(\mathbb{T}))^{n(n-1)}$. Let $M_i \in UT_2(\mathbb{T}), i = 1, 2, \ldots, n(n-1)$. Define the operation * on $(UT_2(\mathbb{T}))^{n(n-1)}$ by

$$(M_1, M_2, \ldots, M_{n(n-1)})^* = (M_{n(n-1)}^D, \ldots, M_2^D, M_1^D).$$

Now by the definition of ϕ_{ij}, it is routine to verify that $\phi_{ij}(a_k) = \phi_{ji}(a_{n-k+1}) = \phi_{ji}(a_k^\dagger)$ for each $a_k \in Ch_n$. Hence $\phi_n(a_k) = \psi_n(a_k^\dagger)^*, \psi_n(a_k) = \phi_n(a_k^\dagger)^*$, whence

$$\xi_n(a_k^\dagger) = (\phi_n(a_k^\dagger), \psi_n(a_k^\dagger)) = (\psi_n(a_k)^*, \phi_n(a_k)^*) = \xi_n(a_k)^*$$

for each $a_k \in Ch_n$, that is, ξ_n is compatible with Schützenberger's involution. Thus, the following result holds.

Lemma 2. *The map $\xi_n : \mathcal{A}_n \cup \{1\} \to (UT_2(\mathbb{T}))^{n(n-1)}$ extends to an injective homomorphism from $(Ch_n, \dagger)$ to $((UT_2(\mathbb{T}))^{n(n-1)}, *)$.*

Let $M_i \in UT_2(\mathbb{T}), i = 1, 2, \dots, n(n-1)$. Define the map

$$t' : (UT_2(\mathbb{T}))^{n(n-1)} \to UT_{2n(n-1)}(\mathbb{T})$$

given by

$$(M_1, M_2, \dots, M_{n(n-1)}) \mapsto \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_{n(n-1)} \end{pmatrix}.$$

Clearly the map t' is an injective semigroup homomorphism. It is routine to verify that

$$t'((M_1, M_2, \dots, M_{n(n-1)})^*) = t(M^D_{n(n-1)}, \dots, M^D_2, M^D_1) = t(M_1, M_2, \dots, M_{n(n-1)})^D$$

whence the map t' can be extended to an injective homomorphism from $(UT_2(\mathbb{T}))^{n(n-1)}, *)$ to $(UT_{2n(n-1)}(\mathbb{T}), D)$. Consider the map $\rho_n = t' \circ \xi_n : (Ch_n, \dagger) \to (UT_{2n(n-1)}(\mathbb{T}), D)$. We have the following result.

Lemma 3. *The map $\rho_n : (Ch_n, \dagger) \to (UT_{2n(n-1)}(\mathbb{T}), D)$ is a faithful tropical representation of $(Ch_n, \dagger)$.*

Proof. Note that the map t' is an injective homomorphism from $(UT_2(\mathbb{T}))^{n(n-1)}, *)$ to $(UT_{2n(n-1)}(\mathbb{T}), D)$ and the map ξ_n is an injective homomorphism from $(Ch_n, \dagger)$ to $(UT_2(\mathbb{T}))^{n(n-1)}, *)$ by Lemma 3.2. Now the result holds. □

Theorem 3. *The Chinese monoid $(Ch_n, \dagger)$ under Schützenberger's involution has a faithful representation by tropical triangular matrices in $UT_{2n(n-1)}(\mathbb{T})$ under the skew transposition.*

Acknowledgements

The authors are very grateful to the anonymous referee for his/her careful reading and valuable suggestions.

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