

## Eigenvalue Problem Notes on Linear Algebra

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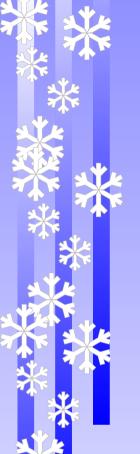


#### Introduction

V Following Ax = b, we turn to the *eigenvalue* problem: Given a square matrix A, we look for  $\lambda$ 's and x's such that

$$Ax = \lambda x$$
.

- $\star$   $\lambda$  is called an *eigenvalue*.
- $\star$  x is called an eigenvector.
- $\checkmark$  Both left and right sides involve unknown x.
- $\checkmark$  Only for some  $\lambda's$  can we find non-zero x for the above equation.



#### Ax = b and $Ax = \lambda x$

- √ Both problems can be simplified solved by transforming to a diagonal or triangular form.
- Now operations do not change the solution of Ax = b. Such cannot be said for  $Ax = \lambda x$ .
- ✓ New operations that leaves the eigenvalues or eigenvectors unchanged will be introduced.



#### **Differential Equations**

Suppose

$$\begin{cases} \frac{dv}{dt} = 4v - 5w, & v = 8 \text{ at } t = 0, \\ \frac{dw}{dt} = 2v - 3w, & w = 5 \text{ at } t = 0. \end{cases}$$

√ We can re-write the above in a matrix form

$$\frac{du}{dt} = Au$$
, with  $u = u(0)$  at  $t = 0$ ,

where 
$$u = \begin{bmatrix} v \\ w \end{bmatrix}$$
 and  $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ .



### An Eigenvalue Problem

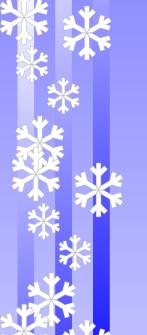
Assume the solution has an exponential form  $u = e^{\lambda t}x$ .

$$\checkmark$$
 Define  $x = \begin{bmatrix} y \\ z \end{bmatrix}$ . Substituting, we have

$$\begin{cases} 4y - 5z = \lambda y \\ 2y - 3z = \lambda z. \end{cases}$$

√ This is an eigenvalue problem

$$Ax = \lambda x!$$



#### **Finding Eigenvalues**

 $\checkmark$  Suppose  $Ax = \lambda x$  has a non-zero solution, then

$$(A - \lambda I)x = 0.$$

 $\sqrt{A-\lambda I}$  must be singular since its nullspace is not empty. Thus,

$$|A - \lambda I| = 0.$$

 $\checkmark$  The above is the *characteristic equation*. It is solved for the eigenvalues of A.



#### **Finding Eigenvectors**

- √ Each eigenvalue has its own set of eigenvectors.
- $\checkmark$  To find the set  $E_{\lambda} = \{x | Ax = \lambda x\}$ , simply solve the system of linear equations.
- $\checkmark$   $E_{\lambda}$  is a vector subspace, called eigenspace.
- $\checkmark$  The dimension of  $E_{\lambda}$  can be more than 1.
- The multiplicity of a root  $\lambda$  of the characteristic equation is called its algebraic multiplicity. The dimension of the eigenspace corresponding to  $\lambda$  is called geometric multiplicity.
- √ They do not always agree.



#### The Example

√ For the example, the characteristic equation is

$$|A - \lambda I| = (4 - \lambda)(3 - \lambda) + 10 = \lambda^2 - \lambda - 2.$$

 $\checkmark$  The roots are  $\lambda_1 = -1$  and  $\lambda_2 = 2$ . For  $\lambda_1 = -1$ ,

$$\begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\checkmark$$
 For  $\lambda_2=2$ ,

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0 \Rightarrow x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

#### **Solution**



$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-t} \begin{vmatrix} 1 \\ 1 \end{vmatrix} + c_2 e^{2t} \begin{vmatrix} 5 \\ 2 \end{vmatrix}.$$

 $\checkmark$  The initial condition determines  $c_1$  and  $c_2$ ,

$$u(0) = c_1 x_1 + c_2 x_2 \Rightarrow \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

 $\checkmark$  The solution is

$$u = \begin{bmatrix} v \\ w \end{bmatrix} = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$



#### The Example: Summary

- $\sqrt{\frac{du}{dt}} = Au$  has an exponential solution  $u = e^{\lambda t}x$ .
- $\checkmark$  The eigenvalue  $\lambda$  gives the rate of growth or decay. The eigenvector x evolves at this rate.
- √ General solutions are mixtures of these pure exponential solutions.
- √ The mixing coefficients are adjusted to fit the initial values.
- √ Each component evolves "independently".



#### **Diagonal Matrix**

√ The eigenvalue problem for an diagonal matrix is easy to solve.

$$\checkmark \text{ Let } A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix}$$
$$|A - \lambda I| = 0 \implies (3 - \lambda)(2 - \lambda) = 0$$
$$\lambda_1 = 3, \ x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \ \lambda_2 = 2, \ x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



#### Triangular Matrix

$$\checkmark \text{ Let } A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 4 & 5 \\ 0 & \frac{3}{4} - \lambda & 6 \\ 0 & 0 & \frac{1}{2} - \lambda \end{bmatrix}$$
$$|A - \lambda I| = 0 \Rightarrow (1 - \lambda)(\frac{3}{4} - \lambda)(\frac{1}{2} - \lambda) = 0$$

√ We often transform a matrix to a triangular form to find the eigenvalues.

#### **Projection Matrix**

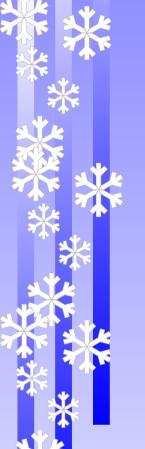
- ✓ From  $P^2 = P$ , a projection matrix always has eigenvalues of 0 or 1!
- $\checkmark$  The projection point of an eigenvector of  $\lambda = 1$  is itself. It must be in the column space of P.
- $\checkmark$  An eigenvector of  $\lambda = 0$  is projected to the zero vector, so it must be in the nullspace of P.
- $\checkmark$  The column space is the eigenspace of  $\lambda = 1$ , and the nullspace is the eigenspace for  $\lambda = 0$ .
- $\checkmark$  There is no other eigenvalues as the sum of eigenspaces of  $\lambda = 0, 1$  is  $R^n$ .



#### Example

✓ Let 
$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
.

$$P - \lambda I = \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix}$$
$$|P - \lambda I| = 0 \implies \lambda(\lambda - 1) = 0$$
$$\lambda_1 = 1, \ x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ \lambda_2 = 0, \ x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$



#### 0 as An Eigenvalue

- Recall that we are only interested in non-zero eigenvectors.
- √ However, 0, like any other numbers, can be an eigenvalue.
- √ A zero eigenvalue does not imply that the eigenvector is zero.
- $\checkmark$  If A is singular, then 0 must be an eigenvalue.
- $\checkmark$  If A is non-singular (invertible), then 0 cannot be an eigenvalue.



#### Trace

√ The trace of a matrix is the sum of diagonal entries

$$tr(A) = a_{11} + \dots + a_{nn}.$$

√ The sum of eigenvalues equals trace,

$$\sum_{i=1}^{n} \lambda_i = tr(A).$$

√ The product of eigenvalues equals determinant,

$$\prod_{i=1}^{m} \lambda_i = |A|.$$



#### **Characteristic Equation**

 $\checkmark$  Let  $\{\lambda_i\}_{i=1}^n$  be the eigenvalues, then

$$f(\lambda) = |A - \lambda I| = (-1)^n (\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0)$$
$$= \prod_{i=1}^n (\lambda_i - \lambda).$$

$$\checkmark$$
 So  $|A| = f(\lambda = 0) = \prod_{i=1}^n \lambda_i$ .

 $\checkmark$  Comparing the coefficients of  $\lambda^{n-1}$ , we have

$$\sum_{i=1}^{n} \lambda_i = tr(A).$$



#### **Eigenvector Matrix**

V Suppose A has independent eigenvectors  $u_1, \ldots, u_n$ . Using the eigenvectors as columns of S, then

$$AS = S\Lambda$$
, where  $\Lambda = \operatorname{diag} \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \end{bmatrix}$ 

- $\checkmark$  S is the eigenvector matrix and  $\Lambda$  is the eigenvalue matrix.
- $\checkmark$  Since S is invertible,

$$AS = S\Lambda \Leftrightarrow A = S\Lambda S^{-1} \Leftrightarrow \Lambda = S^{-1}AS.$$



#### Diagonalization

- $\checkmark$  If  $AS = S\Lambda$  for diagonal λ, then A is said to be diagonalized by S.
- $\checkmark$  S is not unique, but it must be an eigenvector matrix. The columns of S must be eigenvectors.
- √ Not all matrices can be diagonalized. Some matrices do not have enough independent eigenvectors.
- √ A matrix with no repeated eigenvalues can be diagonalized, as we show next.



#### Distinct Eigenvalues

V Eigenvectors of different eigenvalues are linearly independent. This can be proved by induction on the number of eigenvalues. For n = 2,

$$A(c_1x_1 + c_2x_2) = 0, \lambda_2(c_1x_1 + c_2x_2) = 0$$
  

$$\Rightarrow c_1(\lambda_1 - \lambda_2)x_1 = 0 \Rightarrow c_1 = 0 = c_2.$$

Suppose it is true for k. For k + 1, since

$$A(\sum_{i=1}^{k+1} c_i x_i) = 0, \text{ and } \lambda_{k+1}(\sum_{i=1}^{k+1} c_i x_i) = 0,$$

$$\sum_{i=1}^{k} c_i (\lambda_i - \lambda_{k+1}) x_i = 0 \Rightarrow c_1 = \dots = c_k = 0 = c_{k+1}.$$



#### **Defective Matrices**

- A defective matrix does not have enough linearly independent eigenvectors.
- Consider

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

- $\sqrt{\lambda} = 3$  is a double root to  $|A \lambda I| = 0$ , but there is only one independent vector for  $\lambda = 3$ .
- $\checkmark$  The existence S leads to contradiction

$$S^{-1}AS = \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow SS^{-1}ASS^{-1} = A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$



#### **Diagonalization Examples**

A projection matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

 $\checkmark$  A 90° rotation matrix

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \ S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

✓ Note that a real matrix may have imaginary (complex) eigenvalues and eigenvectors.

# Pov

#### Power of a Matrix

 $\checkmark$  An eigenvector of A is an eigenvector of  $A^2$ , with the eigenvalue squared.

$$A^2x = A(Ax) = A\lambda x = \lambda Ax = \lambda^2 x.$$

- $\checkmark$  This result can be generalized to  $A^k$ .
- $\checkmark$  A matrix S diagonalizing A also diagonalizes  $A^k$ ,

$$A^k = (S\Lambda S^{-1})^k = S\Lambda^k S^{-1} \Rightarrow \Lambda^k = S^{-1}A^k S.$$

 $\checkmark$  If A is invertible, then

$$Ax = \lambda x \Rightarrow x = \lambda A^{-1}x \Rightarrow A^{-1}x = \frac{1}{\lambda}x.$$

#### **Product** AB

- ✓ Suppose A has eigenvalue  $\mu$  and B has eigenvalue  $\lambda$ . Does AB has eigenvalue  $\mu\lambda$ ?
  - ★ No in general.
  - \* Yes if they have the same eigenvector.
- ✓ Suppose A and B are diagonalizable. A and B have the same eigenvector matrix iff AB = BA.

$$(\Rightarrow) AB = S\Lambda_A S^{-1} S\Lambda_B S^{-1} = S\Lambda_A \Lambda_B S^{-1}$$
$$= S\Lambda_B \Lambda_A S^{-1} = S\Lambda_B S^{-1} S\Lambda_A S^{-1} = BA$$

 $(\Leftarrow)$  For simplicity, suppose all eigenvalues of A are distinct.

$$Ax = \lambda_a x \Rightarrow ABx = BAx = B\lambda_a x = \lambda_a Bx.$$

x and Bx are eigenvectors of A (for  $\lambda_a$ ), so  $Bx = \lambda_b x$ .



### **Uncertainty Principle**

- ✓ In quantum mechanics operators are represented by matrices.
- $\checkmark$  Position P (symmetric) and momentum Q (skew-symmetric) do not commute,

$$QP - PQ = I.$$

✓ The uncertainty principle is a result of the Schwartz inequality  $(Qx)^T(Px) \le |Qx||Px|$ . Let x be a wave function,

$$|x|^2 = x^T x = x^T (QP - PQ)x \le 2|Qx||Px| \Rightarrow \frac{|Qx|}{|x|} \frac{|Px|}{|x|} \ge \frac{1}{2}.$$

#### **Discrete and Continuous**

 $\checkmark$  Suppose you invest 1000 at 6% rate for five years.

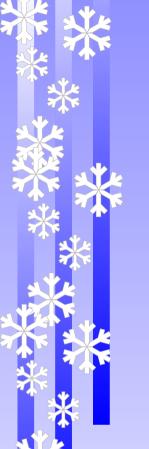
compounded once a year 
$$1000(1.06)^5 = 1338$$

once a month 
$$1000(1.005)^{5*12} = 1349$$

once a day 
$$1000(1 + \frac{0.06}{365})^{5*365} = 1349.83$$

every instant 
$$\lim_{N \to \infty} 1000(1 + \frac{0.06}{N})^{5*N} = 1349.87$$

√ They are not that different. Discrete picture is more concrete.



#### **Difference Equations**

- ✓ Difference equation evolves in finite steps.
- ✓ Difference equations can be used to approximate differential equations.
- ✓ Difference equations can also arise when the underlying problem is discrete in nature, as in a sequence.



#### Fibonacci Sequence

- √ The Fibonacci sequence is defined recursively by,
  - $\star$  basis:  $F_0 = 0, F_1 = 1.$
  - $\star$  recursion:  $F_{k+2} = F_k + F_{k+1}$ .
- √ First few numbers in the sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, \dots$$

- √ This sequence is defined by a difference equation with an initial condition.
- $\checkmark$  Let's find  $F_k$  as a function of k.



#### **Matrix Representation**

√ The recursion can be represented by a matrix. Let

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow u_{k+1} = Au_k.$$

 $\checkmark$  Suppose A is diagonalized by S, then

$$u_k = A^k u_0 = S\Lambda^k S^{-1} u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2,$$

where  $\lambda_i$  is the *i*th eigenvalue and  $x_i$  is the corresponding eigenvector of A.

$$c = S^{-1}u_0.$$



#### Solution

 $\checkmark$  For the eigenvalue  $\lambda$ 's, we solve  $|A - \lambda I| = 0$ .

$$|A - \lambda I| = \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}.$$

- $\checkmark$  The eigenvector for  $\lambda$  is  $(\lambda, 1)$  since the second row of  $A \lambda I$  is  $(1, -\lambda)$ .
- ✓ In addition,

$$c = S^{-1}u_0 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \\ \frac{-1}{\lambda_1 - \lambda_2} \end{bmatrix}.$$

#### Formula



$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = c_1 \lambda_1^k \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \lambda_2^k \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\Rightarrow F_k = \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^k - \lambda_2^k) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right].$$

- ✓ Since the second term is always less then  $\frac{1}{2}$ , we can say  $F_k$  is the nearest integer to  $\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^k$ .
- $\checkmark$  The ratio of  $\frac{F_{k+1}}{F_k}$  approaches  $\lambda_1$ .



#### **Comments**

We can write  $u_0$  as a linear combination of the eigenvectors of A. It follows that  $u_k$  is the same combination with  $\lambda_i^k x_i$ 's. That is,

If 
$$u_0 = c_1 x_1 + \dots + c_n x_n$$
,  
then  $u_k = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$ .

- $\checkmark$  The role of c is to match the initial condition.
- $\checkmark$  As a special case, if the initial  $u_0$  happens to be an eigenvector x with eigenvalue  $\lambda$ , then  $u_k = \lambda^k x$ .



#### Markov Process

- √ We introduce another class of problems whose solutions depend on using matrices wisely.
- $\checkmark$  Suppose for city X every year
  - $\star \frac{1}{10}$  of the people outside move in;
  - $\star$   $\frac{2}{10}$  of the people inside move out.
- This is a *Markov process*. The population at the end of year i, given the population at the end of year i-1, does not depend on any populations before that.



#### **Markov Matrix**

 $\checkmark$  Let  $y_i$  be the people outside and  $z_i$  be the people inside at the end of year i. Then

$$\begin{cases} y_{i+1} = 0.9y_i + 0.2z_i \\ z_{i+1} = 0.1y_i + 0.8z_i \end{cases} \Rightarrow x_{i+1} = Ax_i,$$

where 
$$A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$
.

- $\checkmark$  A is a Markov matrix:
  - \* A has only non-negative entries.
  - $\star$  The entries in any column of A sum to 1.



#### **Eigenvalues of Markov Matrix**

 $\checkmark$  The eigenvalues of A are

$$|A - \lambda I| = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 0.7.$$

√ The eigenvectors are

$$x_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \ x_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{3} \end{bmatrix}.$$

✓ Note that  $A \ge 0 \Rightarrow A^k \ge 0$  (having no negative entries), and we are guaranteed that if the initial  $x_0$  are non-negative, all  $x_i$  will be non-negative.

#### **Solution**

√ The solution is a linear combination of two terms.

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = S\Lambda^k S^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$
$$= (y_0 + z_0) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + (y_0 - 2z_0)(0.7)^k \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{3} \end{bmatrix}$$

- √ The first term in invariant with time, while the second term is exponentially small.
- √ The first term dominates the second term as time goes by.



## Steady State

- ✓ Since the column sum is 1, the column sum of A I is 0. So  $\lambda = 1$  is an eigenvalue of A.
- $\checkmark$  The eigenvector x for  $\lambda = 1$  is a steady state. Once the system is a steady state, it remains in that state,

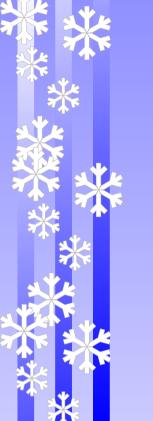
$$Ax = x$$
.

## Stability of a Difference Equation

- ✓ Suppose we are given  $u_{k+1} = Au_k$ , and we want to study the behavior of  $u_k$  as  $k \to \infty$ .
- $\checkmark$  Assuming A can be diagonalized, then

$$u_k = A^k u_0 = S\Lambda^k S^{-1} u_0 = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n.$$

- $\checkmark$  The growth of  $u_k$  depends on the  $\lambda_i$ 's.
  - \* stable case:  $|\lambda_i| < 1$  for all  $i. u^k \to 0$ .
  - \* neutrally stable case:  $|\lambda_i| = 1$  for some i and  $|\lambda_i| < 1$  for others.
  - \* unstable case:  $\exists i \ |\lambda_i| > 1$ .  $u_k$  is unbounded.



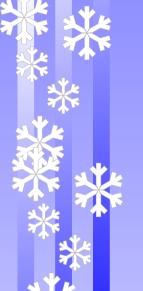
## Leontief's Input-Output Matrix

- ✓ Consumption matrix C:  $c_{ij}$  is the amount of product i needed to create one unit of product j.
- ✓ Consider the case of steel, service, and labor.
- $\checkmark$  Can we meet *external* demand of  $y_1$  of steel,  $y_2$  of service and  $y_3$  of labor?
- $\checkmark$  Let p be total amount. Then

$$p = Cp + y,$$

 $\checkmark$  p has to be non-negative and

$$p - Cp = y \text{ or } p = (I - C)^{-1}y.$$



## **Condition on Eigenvalue**

✓ Note that  $(I - C)^{-1}$  has the same eigenvector as C, with eigenvalue  $\frac{1}{1-\lambda}$ .

$$(I - C)x = (1 - \lambda)x \Rightarrow \frac{1}{1 - \lambda}x = (I - C)^{-1}x.$$

 $\checkmark$  A sufficient condition is  $\lambda_1 < 1$ ,

$$I + C + C^2 + \dots$$
 converges to  $(I - C)^{-1}$ ,

and it is nonnegative.



#### **Prices**

- $\checkmark$  We may be more interested in price than in production. We can use p to represent prices instead of production levels.
- $\checkmark$  Let  $p_0$  be a vector of prices. Then  $p_1 = Cp_0$  is the vector of costs.  $p_1$  is a reasonable vector of prices.
- ✓ In equilibrium, the price reflects the cost, so

$$p = Cp$$
.

### Perron-Frobenius Theorem

✓ Let A be a positive matrix  $(a_{ij} > 0, \forall i, j)$ . The largest eigenvalue  $\lambda_1$  is positive and  $x_1$  is positive,

$$A > 0 \Rightarrow \lambda_1 > 0, x_1 > 0.$$

- V Define  $T = \{t \mid \exists x \ge 0 \text{ s.t. } Ax \ge tx\}$  and let  $t_M = \sup_{t \in T} t$ . We show  $Ax = t_M x$  for some  $x \ge 0$ .
- ✓ Suppose  $Ax > t_M x$ ,  $\forall x \ge 0$ . Multiplying both sides by A we have  $A^2x > t_M Ax$ , or  $Ay > t_M y$  where y = Ax. This contradicts that  $t_M$  is the least upper bound.
- $\checkmark$  x is positive since Ax is sure to be positive.



## A Differential Equation

√ Suppose

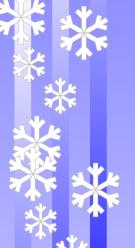
$$\frac{du}{dt} = Au = \begin{bmatrix} -2 & 1\\ 1 & -2 \end{bmatrix} u.$$

 $\checkmark$  The eigenvalues and eigenvectors of A are,

$$\lambda_1 = -1, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \lambda_2 = -3, x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

√ So a general solution is

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$



#### Solution

 $\checkmark$  We apply the initial condition to find  $c_1, c_2$ ,

$$u(t=0) = c_1x_1 + c_2x_2$$
, or  $Sc = u(0)$ .

So

$$c = S^{-1}u(0).$$

The solution is

$$u(t) = S \begin{bmatrix} e^{-t} \\ e^{-3t} \end{bmatrix} S^{-1}u(0) = Se^{\Lambda t}S^{-1}u(0).$$



## The Exponential of a Matrix

- In the previous slide we introduce a notation using a matrix as exponent.
- √ Formally, the exponential of a matrix is defined by

$$e^{M} = I + M + \frac{M^{2}}{2!} + \frac{M^{3}}{3!} + \dots$$

 $\checkmark$  Specifically, if A can be diagonalized, then

$$e^{At} = I + At + \frac{(At)^2}{2!} + \dots = I + S\Lambda S^{-1}t + \frac{(S\Lambda S^{-1}t)^2}{2!} + \dots$$
$$= S(I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \dots)S^{-1} = Se^{\Lambda t}S^{-1}$$



## Example



$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

 $\checkmark$  What is  $e^A$ ? Is it

$$\begin{bmatrix} e^{-2} & e^1 \\ e^1 & e^{-2} \end{bmatrix}$$
?



## Example

$$e^{A} = Se^{\Lambda}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} e^{\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$
$$= \frac{1}{2} \begin{bmatrix} e^{-1} + e^{-3} & e^{-1} - e^{-3} \\ e^{-1} - e^{-3} & e^{-1} + e^{-3} \end{bmatrix}.$$





✓ If A can be diagonalized, then the equation  $\frac{du}{dt} = Au$ ,  $u(0) = u_0$  has the solution

$$u(t) = Se^{\Lambda t}S^{-1}u_0 = e^{At}u_0.$$

 $\checkmark$  The columns of S are eigenvectors of A, so

$$u(t) = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n.$$

✓ It is a combination of exponentials, with coefficients chosen to match the initial condition  $c = S^{-1}u_0$ .



## $e^{At}$

 $\checkmark$   $e^{At}$  is never singular.

$$|e^{At}| = |e^{\Lambda t}| = \prod e^{\lambda_i t} = e^{tr(A)t} \neq 0.$$

 $\checkmark$   $e^{At}$  sometimes acts like a scalar,

$$(e^{At})(e^{As}) = e^{A(t+s)}$$
$$(e^{At})^{-1} = e^{-At}$$
$$\frac{d}{dt}(e^{At}) = Ae^{At}$$



## **Linearly Independent Solutions**

- $\checkmark$  If n solutions are linearly independent at t = 0, then they remain linearly independent at all times.
- $\checkmark$  This is a result of the non-singularity of  $e^{At}$ . If

$$| [v_1(0) \ v_2(0) \ \dots \ v_n(0)] | \neq 0,$$

then

$$| \begin{bmatrix} v_1(t) & \dots & v_n(t) \end{bmatrix} | = |e^{At}| | \begin{bmatrix} v_1(0) & \dots & v_n(0) \end{bmatrix} |$$

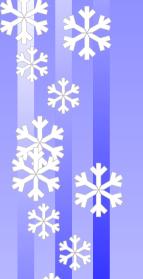
$$\neq 0.$$

## Stability of a Differential Equation

- ✓ What is the asymptotic behavior of  $\frac{du}{dt} = Au$  as  $t \to \infty$ ?
- $\checkmark$  Assuming A can be diagonalized,

$$u(t) = c_1 e^{\lambda_1 t} x_1 + \dots + c_n e^{\lambda_n t} x_n.$$

- ✓ Stability is controlled by  $e^{\lambda_i t}$ 's. The equation is said to be
  - \* stable if Re  $\lambda_i < 0$  for all  $i. e^{At} \rightarrow 0$ .
  - \* neutrally stable if Re  $\lambda_i = 0$  for some i and Re  $\lambda_i \leq 0$  for all i.
  - \* unstable if Re  $\lambda_i > 0$  for some i. u(t) is unbounded.



### $2 \times 2$ Matrix

 $\checkmark$  Stability is easy to decide for a real  $2 \times 2$  matrix.

$$\frac{du(t)}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} u = Au.$$

✓ If tr(A) is negative and |A| is positive, then the above equation is stable. Otherwise it cannot be stable.



#### **Rotation Matrix**

✓ An interesting example is

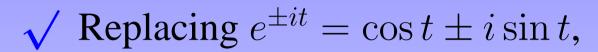
$$\frac{du(t)}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u = Au.$$

 $\checkmark$  The eigenvalues are i and -i, with eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

$$u(t) = c_1 e^{it} x_1 + c_2 e^{-it} x_2 = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

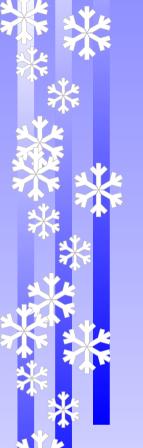




$$u(t) = \begin{bmatrix} (c_1 + c_2)\cos t + i(c_1 - c_2)\sin t \\ -i(c_1 - c_2)\cos t + (c_1 + c_2)\sin t \end{bmatrix}.$$

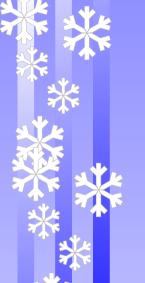
$$\checkmark$$
 Let  $u(0) = \begin{bmatrix} a \\ b \end{bmatrix}$ .

$$\begin{cases} a = c_1 + c_2 \\ b = -i(c_1 - c_2) \end{cases} \Rightarrow u(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$



## Real vs. Complex

- ✓ In eigenvalue problems, it is no longer possible to work only with real numbers.
- √ The set of complex numbers includes the set of real numbers. Definitions and properties for complex matrices are valid for real matrices as well.
- √ Theorems valid for the real set can be extended to the complex set.



## **Complex Vectors**

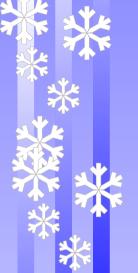
 $\checkmark$  A complex vector in  $\mathbb{C}^n$  has n complex components,

$$x = [x_1 \ x_2 \ \dots \ x_n]^T, \ x_k = a_k + ib_k.$$

√ The length of a complex vector is defined by

$$|x|^2 = |x_1|^2 + \dots + |x_n|^2$$
, where  $|x_k|^2 = a_k^2 + b_k^2$ .

 $\checkmark$  It agrees with the original definition when x is real.



#### **Inner Product**

√ The inner product of two vectors is defined by

$$\overline{x}(x,y) = \overline{x}^T y = \overline{x}_1 y_1 + \dots + \overline{x}_n y_n.$$

✓ Note that

$$|x|^2 = (x, x)$$
  
 $(y, x) = (x, y)^*.$ 

√ Two complex vectors are said to be orthogonal if

$$(x,y) = (y,x)^* = 0.$$



#### Hermitian of a Matrix

✓ The Hermitian of a matrix A, denoted by  $A^H$ , is the conjugate transpose of A. That is,

$$A^H = \overline{A}^T, \ (A_{ij}^H = \overline{A}_{ji})$$

✓ Inner product of two vectors can be written as

$$(x,y) = x^H y.$$

 $\checkmark$  For any x, A, y,

$$(x, Ay) = x^{H}Ay = (A^{H}x)^{H}y = (A^{H}x, y).$$



### Hermitian Matrix

 $\checkmark$  A matrix A is said to be Hermitian if

$$A = A^H$$
.

√ The following matrix is Hermitian,

$$\begin{bmatrix} 3 & 2-2i \\ 2+2i & 3 \end{bmatrix}.$$

√ Hermitian matrix is the complex counterpart of symmetric matrix.

## **Property**

- $\checkmark$  Let A be Hermitian.
- $\checkmark$   $x^H Ax$  is real for any complex vector x.

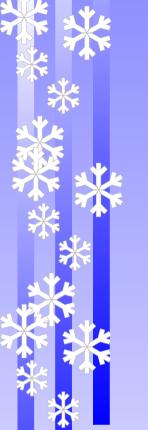
$$x^{H}Ax = (x, Ax) = (x, A^{H}x) = (Ax, x) = \overline{(x, Ax)}.$$

√ The eigenvalues are real.

$$(x, Ax) = \lambda(x, x) \Rightarrow \lambda = \frac{(x, Ax)}{(x, x)}.$$

√ The eigenvectors are orthogonal.

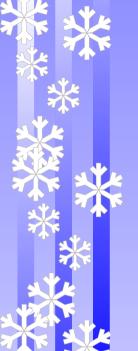
$$(Ax_1, x_2) = (x_1, Ax_2) \Rightarrow (\lambda_1 - \lambda_2)(x_1, x_2) = 0.$$



## Real Symmetric Matrices

- ✓ Since a real symmetric matrix is Hermitian by definition, the eigenvalues are real and the eigenvectors are orthogonal.
- The eigenvectors can be chosen orthonormal. They go to the columns of an orthogonal matrix Q which diagonalizes A,

$$A = Q\Lambda Q^T.$$

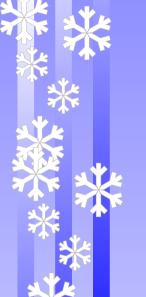


## **Spectral Theorem**

✓ If we multiply columns by rows (block matrix multiplication) of  $A = Q\Lambda Q^T$ , then

$$A = \lambda_1 x_1 x_1^T + \dots + \lambda_n x_n x_n^T.$$

- √ This is also known as the spectral theorem.
- √ A real symmetric matrix is a linear combination of one-dimensional projection matrices (to the eigenvectors). These projection matrices are symmetric with rank 1.



## **Unitary Matrices**

 $\checkmark$  By definition, a matrix U is unitary if

$$U^H U = U U^H = I.$$

 $\checkmark$  This is a generalization of orthogonal matrices. If U is real, then U is orthogonal.



## **Property**

✓ Inner product (so is length) is preserved by a unitary transformation,

$$(Ux, Uy) = x^H U^H Uy = x^H y.$$

 $\checkmark$  Eigenvalues of U have unit modulus  $|\lambda|=1$ ,

$$|x| = |Ux| = |\lambda x| = |\lambda||x|.$$

 $\checkmark$  Eigenvectors of U corresponding to different eigenvalues are orthogonal, since

$$(Ux_1, Ux_2) = (x_1, U^H Ux_2) = (x_1, x_2) \Rightarrow (1 - \overline{\lambda_1}\lambda_2)(x_1, x_2) = 0.$$

# Similarity Transformations

- ✓ Suppose A is diagonalizable so  $\Lambda = S^{-1}AS$ .
  - $\star$  If A is symmetric, then we write Q instead of S since it is orthogonal.
  - $\star$  If A is Hermitian, then we write U instead of Q since it is unitary.
- ✓ The transforming matrix needs not be an eigenvector matrix: We look at matrix  $B = M^{-1}AM$  where M is invertible. A and B are said to be *similar*. Going from A to B is a *similarity transformation*.

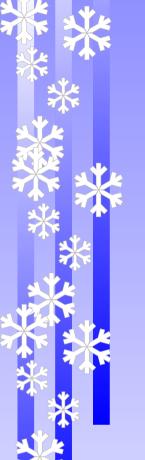


## Example

$$\checkmark \text{ Let } A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

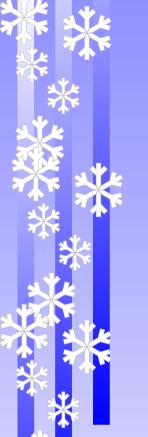
$$\begin{cases}
M_1 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, M_1^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, & M_1^{-1}AM_1 = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \\
M_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, M_2^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, & M_2^{-1}AM_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.
\end{cases}$$

$$\checkmark$$
  $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$  are similar to  $A$ .



## **Key Questions**

- $\checkmark$  What do A and  $M^{-1}AM$  have in common?
- ✓ The eigenvalues of A and  $M^{-1}AM$  are the same. In the example, one can see that  $A, M_1^{-1}AM_1$  and  $M_2^{-1}AM_2$  have the same eigenvalues 0 and 1.
- ✓ With a special choice of M, what special form can be achieved by  $M^{-1}AM$ ?
- $\checkmark$  The Jordan form can be achieved with special choice of M.



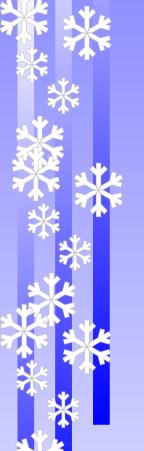
## **Eigenvalues of Similar Matrices**

- ✓ A and  $B = M^{-1}AM$  have the same eigenvalues. Furthermore, if x is an eigenvector of A, then  $M^{-1}x$  is an eigenvector of B.
- $\checkmark$  A and B have the same characteristic equations, so they have the same roots.

$$|B - \lambda I| = |M^{-1}(A - \lambda I)M| = |A - \lambda I|.$$

 $\checkmark$  The eigenvectors are related via M, as

$$Ax = \lambda x \Rightarrow M^{-1}Ax = \lambda M^{-1}x \Rightarrow M^{-1}AMM^{-1}x = \lambda M^{-1}x$$
$$\Rightarrow BM^{-1}x = \lambda M^{-1}x.$$



## **Change of Basis**

- Recall that a linear transformation can be represented by a matrix.
- √ The matrix representation is dependent on the basis. When the basis is changed, the matrix representation needs to be changed accordingly.
- ✓ We will see that similar matrices represent the same linear transformations in different bases.



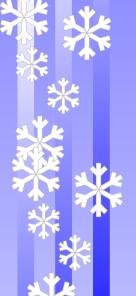
#### Two Bases

✓ Suppose we have a basis  $v = \{v_1, \dots, v_n\}$  and a linear transform defined by

$$Tv_j = a_{1j}v_1 + \dots + a_{nj}v_n, \ j = 1 \dots n.$$

 $\checkmark$  Consider another basis  $V = \{V_1, \dots, V_n\}$ . In this basis, T is defined by

$$TV_{i} = b_{1i}V_{1} + \cdots + b_{ni}V_{n}, \ j = 1 \dots n.$$



#### **Matrices**

 $\checkmark$  Since v is a basis, one can write

$$V_j = m_{1j}v_1 + \cdots + m_{nj}v_n, \ j = 1 \dots n.$$

✓ Identical vectors are transformed identically, so

$$\begin{cases} TV_j = b_{1j}V_1 + \dots + b_{nj}V_n = \sum_{p,l} b_{pj} m_{lp} v_l \\ TV_j = T(\sum_k m_{kj} v_k) = \sum_k m_{kj} T(v_k) = \sum_{k,l} m_{kj} a_{lk} v_l \end{cases}$$

✓ It follows that

$$AM = MB \Rightarrow B = M^{-1}AM.$$

## Example



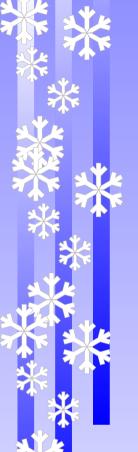
$$v = \{v_1, v_2\} = \{ \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix} \}$$

$$V = \{V_1, V_2\}$$

$$= \{\cos \theta \ v_1 + \sin \theta \ v_2, -\sin \theta \ v_1 + \cos \theta \ v_2 \}.$$

Then

$$M = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, M^{-1} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow A = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}.$$



# **Computation of Eigenvalues**

- √ If we apply a sequence of similar transformation, the sequence of similar matrices obtained have the same eigenvalues.
- ✓ If we can make the matrix form go toward a triangular form, the eigenvalues will be obvious.
- ✓ This is much better than solving  $|A \lambda I| = 0$  directly.



#### Schur's Lemma

- ✓ Given A, we want to find a M such that  $M^{-1}AM$  is triangular.
- √ The Schur's lemma guarantees the existence of such a matrix.
- $\checkmark$  For any square matrix A there exists a unitary matrix U such that

$$U^{-1}AU = T$$

is upper-triangular.

✓ Note that unitary matrices are used.



- $\checkmark$  A matrix A, say of size  $4 \times 4$ , has at least one eigenvalue  $\lambda_1$  with a unit-length eigenvector  $x_1$ .
- V Place  $x_1$  in the first column of  $U_1$ , and fill in the other columns with entries so that  $U_1$  is unitary. Then

$$AU_{1} = U_{1} \begin{bmatrix} \lambda_{1} & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \Rightarrow U_{1}^{-1}AU_{1} = \begin{bmatrix} \lambda_{1} & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$



- ✓ Let  $B = U_1^{-1}AU_1$  and  $B_{lr}$  be the lower-right  $3 \times 3$  sub-matrix of B.
- $\checkmark$  Let  $\lambda_2, x_2$  be an eigenvalue and eigenvector of  $B_{lr}$ .
- $\checkmark$  Let  $U_2$  be a unitary matrix whose lower-right submatrix  $M_2$  is unitary using  $x_2$  as the first column, then

$$BU_2 = \begin{bmatrix} \lambda_1 & v^T \\ 0 & B_{lr} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & M_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & v^T M_2 \\ 0 & B_{lr} M_2 \end{bmatrix}.$$



 $\checkmark$  The first column vector of  $M_2$  is an eigenvector of  $B_{lr}$ , so

$$B_{lr}M_2 = B_{lr} [x_2 * *] = [\lambda_2 x_2 * *]$$

$$= [x_2 * *] \begin{bmatrix} \lambda_2 * * * \\ 0 * * * \\ 0 * * * \end{bmatrix}$$

$$= M_2 \begin{bmatrix} \lambda_2 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$



It follows that

$$BU_2 = \begin{bmatrix} \lambda_1 & v^T M_2 \\ \mathbf{0} & B_{lr} M_2 \end{bmatrix}$$

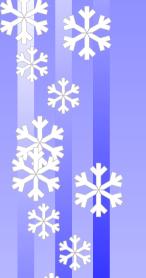
$$= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & M_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & v^T M_2 \\ \mathbf{0} & \begin{bmatrix} \lambda_2 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} = U_2 C.$$

✓ One more step for  $CU_3 = U_3T$ .  $U = U_1U_2U_3$  is unitary and  $U^{-1}AU = T$  is triangular.



# **Complete Set of Eigenvectors**

- √ Any Hermitian matrix (including real symmetric matrices) has a complete set of orthonormal eigenvectors.
- ✓ It does not matter whether the eigenvalues are distinct or not.
- This follows from the Schur's lemma.

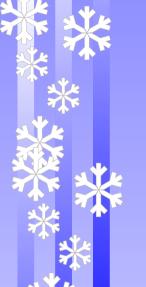


#### Diagonalizing Hermitian Matrices

- $\checkmark$  Let A be Hermitian and it is triangularized by U.
- $\sqrt{U^{-1}AU}$  is Hermitian, since

$$(U^{-1}AU)^H = U^H A^H (U^{-1})^H = U^{-1}AU.$$

- ✓ A triangular Hermitian matrix must be diagonal.
- ✓ U is indeed an eigenvector matrix of A, since  $U^{-1}AU$  diagonalizes A.
- $\checkmark$  The columns of U consist a complete set of orthonormal eigenvectors of A.



### **Spectral Decomposition**

 $\checkmark$  For a Hermitian matrix A,

$$A = U\Lambda U^H$$
, where U is unitary.

Very Hermitian matrix with k different eigenvalues can be expressed as a sum of k projection matrices,

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_k P_k,$$

where  $P_i$  projects to the eigenspace of  $\lambda_i$ . Since the eigenspaces are orthogonal,  $P_iP_j=0,\ i\neq j$ .



#### **Normal Matrices**

 $\checkmark$  A matrix N is normal if N commutes with  $N^H$ ,

$$NN^H = N^H N.$$

- √ As a special case, a Hermitian matrix is a normal matrix.
- √ A normal matrix has a complete set of orthonormal eigenvectors. We prove this by showing the triangular form is diagonal.

# Normal Matrices

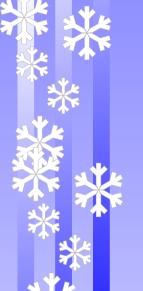
 $\checkmark$  If N is normal, then  $U^{-1}NU$  is normal.

$$(U^{H}NU)(U^{H}NU)^{H} = U^{H}NUU^{H}N^{H}U = U^{H}NN^{H}U$$
  
 $(U^{H}NU)^{H}(U^{H}NU) = U^{H}N^{H}UU^{H}N^{H}U = U^{H}N^{H}NU$ 

- $\checkmark$  Let T be normal and (upper-)triangular.
  - ★ Comparing  $(TT^H)_{11}$  and  $(T^HT)_{11}$ , one can see that  $T_{1j} = 0, j \neq 1$ .
  - ★ Comparing  $(TT^H)_{22}$  and  $(T^HT)_{22}$ , one can also see that  $T_{2j} = 0, j \neq 2$ .
  - $\star$  Continuing, T is shown to be diagonal.

#### **Defective Matrices**

- ✓ If A has a complete set of eigenvectors, we take M = S and  $S^{-1}AS = \Lambda$  is diagonal.
- √ For a defective matrix, this is impossible, since there are not enough eigenvectors.
- ✓ We want to make  $M^{-1}AM = J$  as diagonal as possible.
- √ Result: in addition to the eigenvalues on the main diagonal (we do have enough eigenvalues), there will be a 1 just above the main diagonal for every missing eigenvector.

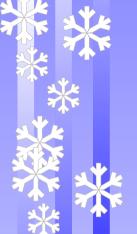


#### Jordan Form

 $\checkmark$  If A has s independent eigenvectors, it is similar to a matrix with s blocks,

$$J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}.$$

- $\checkmark$  J is said to be in Jordan form.
- $\checkmark$   $J_i$  is a triangular matrix with eigenvalue  $\lambda_i$  for the diagonal entries and 1's just above the diagonal.



# Examples



Matrices

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

share the same Jordan form

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

 $\checkmark$  T, A, B are similar.



#### **Comments on Jordan Form**

- √ Every missing eigenvector has a "1" just above the main diagonal.
- $\checkmark$  J is diagonal iff each block is of size  $1 \times 1$ .
- $\checkmark$  Only a repeated eigenvalue may require off-diagonal 1's in J.
- $\checkmark$  Multiple blocks may have the same  $\lambda$ .
- √ Two matrices may have the same set of eigenvalues but different Jordan forms.