Outline

- vector spaces
- systems of m equations in n unknowns
- linear independence, basis and dimension
- fundamental subspaces related to a matrix
- graphs and incidence matrices
- linear transformations

vector spaces

- space containing vectors
- addition and multiplication by a scalar
- rules (a.k.a. axioms, see ex 2.1.5)
- familiar examples R^1, R^2, R^3
- not-so-familiar examples
  - infinite-dimensional space R^∞
  - the set of all matrices of a given size (vector = matrix!)
  - the set of all functions defined on an interval (vector = function)

subspaces

- a subset S of vector space satisfying
  - If $x, y \in S$, then $x + y \in S$.
  - If $x \in S$, then $cx \in S$.
- examples
  - plane (containing origin) in R^3
  - lower triangular matrices, symmetric matrices
  - all functions of the form $a \sin x + b \cos x$ defined on $(0, \pi)$
  - Is the first quadrant (Q1) a subspace of R^2? How about all points in either Q1 or Q3?

column space and nullspace of an $m \times n$ matrix

- the column space of A, denoted by $\mathcal{R}(A)$
  - the set of all linear combinations of the columns of A
  - a subspace of R^m
  - if $b_1, b_2 \in \mathcal{R}(A)$, then $c_1 b_1 + c_2 b_2 \in \mathcal{R}(A)$
- the nullspace of A, denoted by $\mathcal{N}(A)$
  - the set of all x's such that $Ax = 0$
  - a subspace of R^n
  - if $x_1, x_2 \in \mathcal{N}(A)$, then $c_1 x_1 + c_2 x_2 \in \mathcal{N}(A)$
- the system $Ax = b$ is solvable iff $b \in \mathcal{R}(A)$

column space and nullspace: examples

- $A_1 = \begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix}$
  - $\mathcal{R}(A) = \{x | x = c_1(1\ 5\ 2) + c_2(0\ 4\ 4)\}$
  - $\mathcal{N}(A) = \{(0\ 0)\}$
- $A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{pmatrix}$
  - $\mathcal{R}(A_2) = \mathcal{R}(A_1)$
  - $\mathcal{N}(A) = \{x | x = c(1\ 1\ \ -1)\}$

solving $Ax = b$, A is $m \times n$

- m equations, n unknowns
- same elimination process
- $PA = LU$, where L is $m \times m$ and U is $m \times n$
- row-echelon form
  - non-zero rows come first
  - below each pivot is a column of zeros
  - each pivot lies to the right of the pivots above
- modifications in the back substitution
- illustrative example

example

- $Ax = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & -3 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = 0$

- $LUx = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = 0$

homogeneous case $Ax = 0$

- $Ax = 0 \to Ux = 0$
- *basic variables* correspond to columns with pivots
- *free variables* correspond to columns without pivots
- systematic method to find the solution set
  - identify the basic and free variables
  - set one free variable to 1 and others to 0, and solve $Ux = 0$; repeat for each free variable
  - $\mathcal{N}(A)$ = all linear combinations of these solutions
- If $n > m$, then $Ax = 0$ has non-trivial solutions

example continued

- $x_1 = \begin{pmatrix} u \\ 1 \\ w \\ 0 \end{pmatrix}$ solving $u, w \quad \to x_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
- $x_2 = \begin{pmatrix} u \\ 0 \\ w \\ 1 \end{pmatrix}$ solving $u, w \quad \to x_2 = \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$
- $x = c_1 x_1 + c_2 x_2$

non-homogeneous case $Ax = b \neq 0$

- two equivalent conditions for b for $Ax = b$ to be solvable
  - $b \in \mathcal{R}(A)$ (old)
  - b must satisfy some constraints (new)
- solution $x_g = x_p + x_h$
  - x_h is the solution to the homogeneous equation
  - to find x_p, set all free variables to the value 0 (or others) and solve for the basic variables
  - $Ax_g = Ax_p + Ax_h = b + 0 = b$

illustrative example

- $$Ax = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & -3 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$
- $$Ux = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix}$$
- the last entry on the right side must be 0
- have x_h; set the free variables to 0 to have $x_p = (-2\ 0\ 1\ 0)^T$ when $b = (1\ 5\ 5)^T$

Rank

After the elimination process, the number of pivots in U, say r, is called the rank of A.

- r = the number of basic variables
- r = the number of non-zero rows
- $n - r$ = the number of free variables
- if $r = m$, there is at least a solution (why?)
- if $r = n$, there is at most a solution (why?)

linearly dependent / independent

- a set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent if

$$\sum_i c_i v_i = 0 \text{ iff } c_1 = \cdots = c_n = 0,$$

  otherwise the set is linearly dependent.

- examples
  - if $v_i = 0$ for some i
  - the rows in the former example
  - columns in a diagonal matrix with non-zero diagonal entries
  - non-zero rows in a matrix in row-echelon form
  - columns with pivots in a matrix in row-echelon form
- how to check linear independence? solve $Ax = 0$!

spanning sets

- given a subspace V, a set of vectors $W = \{w_1, \ldots, w_n\}$ is a spanning set of V if every vector $v \in V$ is a linear combination of vectors in W
- examples
  - $\{(1,0,0),(0,1,0),(-2,0,0)\}$ spans a plane in R^3
  - $\mathcal{R}(A)$ is spanned by the columns of A; every vector in $\mathcal{R}(A)$ can be written as Ax
- a spanning set of a vector space is apparently not unique

basis and dimension of a space

- by definition, a basis $B = \{b_1, \ldots, b_n\}$ for a space V, satisfies two properties
  1. B is a linearly independent set
  2. B is a spanning set of V
- all bases for a given subspace contain the same number of vectors
- proof: suppose $|W| = n > m = |U|$. $W = UA$ for some $m \times n$ matrix A. Then $Wc = UAC = 0$ for some non-zero c, contradiction to W being a basis
- this number is called the dimension of V

further properties

- every linearly independent set in V can be extended to a basis of V
- every spanning set of V can be reduced to a basis of V
- dimension = degree of freedom
- the rank of a matrix A = the dimension of $\mathcal{R}(A)$

fundamental subspaces of a matrix A

1. column space $\mathcal{R}(A)$
   the space spanned by the column vectors of A
2. nullspace $\mathcal{N}(A)$
   the set of vectors $\{x|Ax = 0\}$
3. row space $\mathcal{R}(A^T)$
   the space spanned by the row vectors of A
4. left nullspace $\mathcal{N}(A^T)$
   the set of vectors $\{y|A^Ty = 0\}$

Note that $\mathcal{R}(A)$ and $\mathcal{N}(A^T) \subset R^m$, $\mathcal{N}(A)$ and $\mathcal{R}(A^T) \subset R^n$.

the row space of A

- Gauss elimination = triangular factorization
- $A = LU$ and $U = L^{-1}A$
- row space of A = row space of U
- non-zero rows in U are linearly independent
- dimension (of row space) = rank

the nullspace of A

- nullspace of A = nullspace of U
- construction of basis for the nullspace of A
  1. Gauss elimination $Ax = 0 \rightarrow Ux = 0$
  2. set one free variable to 1 and others to 0, to obtain a non-zero vector in the nullspace
  3. repeat for each free variable
  4. the set of the $n - r$ vectors thus obtained is a basis
- example
- kernel of A and nullity of A

the column space of A

- $Ux = 0 \Leftrightarrow Ax = 0$, so a linear combination of columns of U giving the zero vector also produces 0 when using the columns of A, and vice versa
- if a set of column vectors of U is independent, so is the corresponding set of columns of A
- the set of columns of U with pivots is a basis for $\mathcal{R}(U)$; the corresponding set of columns of A is a basis for $\mathcal{R}(A)$
- *FOR ANY matrix,*
  dimension of column space = dimension of row space !

the left nullspace of A

- $A^T y = 0 \Leftrightarrow y^T A = 0$, so the left nullspace of A is equivalent to the nullspace of A^T
- the dimension of the left nullspace is the dimension of the nullspace of A^T
- dimension of column space + dimension of nullspace = number of columns
- the dimension of the left nullspace is $m - r$
- the last $m - r$ rows of L^{-1} are left nullvectors of A; furthermore, they are independent

existence of inverses

- left inverse and right inverse of A
- $r = m \Leftrightarrow$ columns span R^m
- $r = n \Leftrightarrow$ columns are linearly independent
- when $r = m$, there exists a right inverse $AC = I_m$
- when $r = m$, $Ax = b$ has *at least* one solution
- when $r = n$, there exists a left inverse $BA = I_n$
- when $r = n$, $Ax = b$ has *at most* one solution
- $(A^T A)^{-1} A^T$ and $A^T (AA^T)^{-1}$

conditions for invertibility

- columns span R^n, $Ax = b$ has a solution for every b
- nullspace is $\{0\}$
- the rows span R^n
- the rows are linearly independent
- all diagonal entries in U are non-zero
- non-zero determinant
- non-zero eigenvalues
- $A^T A$ is positive definite

edge-node incidence matrices

- $G = (V, E)$, graph = nodes + edges
- A graph with m directed edges and n nodes can be represented by an $m \times n$ matrix A, where

$$A_{ij} = \begin{cases} 1, & \text{if edge } i \text{ ends at node } j \\ -1, & \text{if edge } i \text{ starts at node } j \\ 0, & \text{otherwise} \end{cases}$$

$$\text{e.g.,} \quad A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

nullspace of an incidence matrix

- let n_j denotes the column j of the incidence matrix, corresponding to node j of the graph
- $\sum_j n_j = 0 \Rightarrow (1\ 1\ 1\ 1)^T \in$ nullspace of A
- physical meaning
  - let x_j be the potential at node j, then Ax is the potential difference across the edges
  - $Ax = 0$ means 0 potential differences across all edges
  - constant potentials $c(1\ 1\ 1\ 1)^T$ satisfy this condition
  - any solution to $Ax = b$ can be added $c(1\ 1\ 1\ 1)^T$ and remains a solution

column space of an incidence matrix

- for which $b = (b_1\ b_2\ b_3\ b_4\ b_5)^T$ can we solve $Ax = b$?
- let e_i be row i of A, corresponding to edge i of the graph
  - $e_1 + e_2 = e_3 \Rightarrow b_1 + b_2 = b_3$
  - $e_3 + e_4 = e_5 \Rightarrow b_3 + b_4 = b_5$
  - $e_1 + e_2 + e_4 = e_5$ is not an independent condition
- we are given potential differences $b_1, \ldots, b_5$ and we want to find x to satisfy these differences
- to have a solution, potential differences around a loop must add to 0 (*Kirchhoff voltage law*)

left nullspace of an incidence matrix

- $y^T A = 0$: what combinations of the rows of A gives the zero vector?
- $y_1^T = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \end{pmatrix}, y_2^T = \begin{pmatrix} 0 & 0 & 1 & 1 & -1 \end{pmatrix}$
- each loop in G corresponds to a vector in the left nullspace of A. For example,

$$y_1^T + y_2^T = \begin{pmatrix} 1 & 1 & 0 & 1 & -1 \end{pmatrix}$$

- $y^T b = 0$ for y in the left nullspace and b in the column space of A

row space of an incidence matrix

- the row sum is zero for each row of an incidence matrix
- a vector f in the row space is a linear combination of rows of A so it must satisfy

$$f_1 + f_2 + f_3 + f_4 = 0$$

- $f^T x = 0$ for x in the nullspace and f in the row space
- $A^T y = f$, where y_i is the current on edge i and f_j is the current source at node j, is solvable iff f is in the row space of A (the column space of A^T)
- the net current into every node is 0 *(Kirchoff current law)*

linear transformations

- linear transforms are defined by the rule of linearity

$$T(cx + dy) = cT(x) + dT(y).$$

- examples in 2-D vectors
  - stretch
  - rotation
  - reflection
  - projection
- matrix multiplication $\Leftrightarrow$ linear transformation

construction of matrix for a linear transformation

- suppose the vectors $x_1, \ldots, x_n$ are a basis for V and $y_1, \ldots, y_m$ are a basis for W. A linear transformation T from V to W can be represented by a matrix A, where $Tx_j = \sum_{i=1}^{m} a_{ij} y_i$. That is, column j of A is the vector by applying T on the jth basis vector x_j.
  - an example from R^2 to R^3

composite linear transformations

- composite linear transformation = matrix multiplication
- examples
  - rotation of a vector by angle θ and ϕ

$$Q_{\theta+\phi} = Q_\theta Q_\phi$$

  - repeated projections

$$P = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \Rightarrow P^2 = P$$

  - repeated reflections: $H = 2P - I$, $H^2 = I$

multiplication of block matrices

- A block matrix is a matrix defined by (smaller) matrices. That is,

$$M = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}, \quad \text{where } W, X, Y, Z \text{ are matrices.}$$

- Let A be $m \times n$ and B be $n \times r$. If A is partitioned into a $I \times J$ block matrix (with entries A_{ij}), and B is partitioned into a $J \times K$ block matrix, then

$$AB = \begin{bmatrix} C_{11} & \dots & C_{1K} \\ \vdots & \vdots & \vdots \\ C_{I1} & \dots & C_{IK} \end{bmatrix},$$

where $C_{ik} = \sum_{j=1}^{J} A_{ij} B_{jk}$.