NSYSU CSE   Linear Algebra Quiz 1 Solution

1. Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix},\ \text{and } C = AB.$$

   Show *explicitly* that the following ways to compute C yield the same results.

   (a) $C_{ij} =$ (row i of A) $\cdot$ (column j of B)

   (b) column j of C is the linear combination of columns of A using column j of B as combination coefficients

   (c) row i of C is the linear combination of rows of B using row i of A as combination coefficients

   (d) $C = \sum_{k=1}^{3} C_k$, where $C_k =$ (column k of A) (row k of B)

   <u>solution</u>

   (a) For example, $C_{11} =$ (row 1 of A) $\cdot$ (column 1 of B) $= 18$. There are nine such calculations and the result is
$$\begin{bmatrix} 18 & 18 & 22 \\ 42 & 37 & 42 \\ 22 & 18 & 18 \end{bmatrix}$$

   (b) For column 1 of C,
$$\begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} = 5 * \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + 2 * \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} + 3 * \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 42 \\ 22 \end{bmatrix}.$$

   Similarly for the other columns.

   (c) For row 1 of C,
$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \end{bmatrix} = 1*\begin{bmatrix} 5 & 4 & 3 \end{bmatrix} + 2*\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} + 3*\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 18 & 22 \end{bmatrix}.$$

   Similarly for the other rows.

   (d)
$$\begin{aligned} C &= \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 4 & 3 \\ 20 & 16 & 12 \\ 15 & 12 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 4 \\ 10 & 5 & 10 \\ 4 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 9 & 12 & 15 \\ 12 & 16 & 20 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 18 & 22 \\ 42 & 37 & 42 \\ 22 & 18 & 18 \end{bmatrix}. \end{aligned}$$

2. Find A^{-1} with Gauss-Jordan method or otherwise.

   solution: the last three columns of the last matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 4 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -8 & -4 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -8 & -4 & 1 & 0 \\ 0 & 0 & \frac{8}{3} & \frac{7}{3} & \frac{-4}{3} & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -8 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{8} & \frac{-1}{2} & \frac{3}{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{8}{3} & \frac{4}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & \frac{7}{8} & \frac{-1}{2} & \frac{3}{8} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{3}{8} & \frac{-1}{2} & \frac{7}{8} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & \frac{7}{8} & \frac{-1}{2} & \frac{3}{8} \end{bmatrix}$$

3. Find the LU and LDU factorizations of A.

   solution: From the first line in the solution of the previous problem, the LU factorization of A is

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -8 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}.$$

   From this, the LDU factorization is

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{8}{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Solve the following system of equations

$$\begin{cases} u - 2v + 2w - x + 3y = 1, \\ 2u + v + w + 3x - y = 2, \\ 3u - v + 2w + x + 2y = 3, \end{cases}$$

   by

   (a) finding a particular solution,

   (b) finding the homogeneous solution and then the general solution.

   solution:

   (a) $x_p = [1\ 0\ 0\ 0\ 0]^T$

   (b) $x_h = c_1 \begin{bmatrix} \frac{-1}{5} & \frac{-8}{5} & -1 & 1 & 0 \end{bmatrix}^T + c_2 \begin{bmatrix} \frac{-1}{5} & \frac{7}{5} & 0 & 0 & 1 \end{bmatrix}^T$,
   and $x_g = x_h + x_p$

5. Let

$$D = \begin{bmatrix} 0 & 1 & 1 & 2 \\ -1 & 2 & -2 & 3 \\ 3 & -3 & 1 & 1 \end{bmatrix}.$$

(a) Find a vector that is orthogonal to the rows of D.

(b) Find the nullspace of D.

solution:

(a) A vector orthogonal to the rows of D is in the nullspace of D. So we can use any vector found in the next subproblem.

(b) To find the nullspace of D, we use the Gauss elimination method and solve $Ux = 0$.

$$\begin{bmatrix} -1 & 2 & -2 & 3 \\ 3 & -3 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & -2 & 3 \\ 0 & 3 & -5 & 10 \\ 0 & 0 & \frac{8}{3} & \frac{-4}{3} \end{bmatrix}$$

$$\Rightarrow \mathcal{N}(D) = \{x | x = c[-6 \ -5 \ 1 \ 2]^T, \ c \in R\}$$