2.2. For an LTI system, the output is obtained from the convolution of the input with the impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(a) Since $h[k] \neq 0$, for $(N_0 \leq n \leq N_1)$,

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input, $x[n] \neq 0$, for $(N_2 \leq n \leq N_3)$, so

$$x[n-k] \neq 0, \text{ for } N_2 \leq (n-k) \leq N_3$$

Note that the minimum value of $(n-k)$ is N_2. Thus, the lower bound on n, which occurs for $k = N_0$ is

$$N_4 = N_0 + N_2.$$

Using a similar argument,

$$N_5 = N_1 + N_3.$$

Therefore, the output is nonzero for

$$(N_0 + N_2) \leq n \leq (N_1 + N_3).$$

(b) If $x[n] \neq 0$, for some $n_o \leq n \leq (n_o + N - 1)$, and $h[n] \neq 0$, for some $n_1 \leq n \leq (n_1 + M - 1)$, the results of part (a) imply that the output is nonzero for:

$$(n_o + n_1) \leq n \leq (n_o + n_1 + M + N - 2)$$

So the output sequence is $M + N - 1$ samples long. This is an important quality of the convolution for finite length sequences as we shall see in Chapter 8.

2.5. (a) The homogeneous difference equation:

$$y[n] - 5y[n-1] + 6y[n-2] = 0$$

Taking the Z-transform,

$$1 - 5z^{-1} + 6z^{-2} = 0$$
$$(1 - 2z^{-1})(1 - 3z^{-1}) = 0.$$

The homogeneous solution is of the form

$$y_h[n] = A_1(2)^n + A_2(3)^n.$$

(b) We take the z-transform of both sides:

$$Y(z)[1 - 5z^{-1} + 6z^{-2}] = 2z^{-1}X(z)$$

Thus, the system function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \\ &= \frac{-2}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}}, \end{aligned}$$

where the region of convergence is outside the outermost pole, because the system is causal. Hence the ROC is $|z| > 3$. Taking the inverse z-transform, the impulse response is

$$h[n] = -2(2)^n u[n] + 2(3)^n u[n].$$

(c) Let $x[n] = u[n]$ (unit step), then

$$X(z) = \frac{1}{1 - z^{-1}}$$

and

$$\begin{aligned} Y(z) &= X(z) \cdot H(z) \\ &= \frac{2z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Partial fraction expansion yields

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{4}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}.$$

The inverse transform yields:

$$y[n] = u[n] - 4(2)^n u[n] + 3(3)^n u[n].$$

2.7. $x[n]$ is periodic with period N if $x[n] = x[n + N]$ for some integer N.

(a) $x[n]$ is periodic with period 12:

$$e^{j(\frac{\pi}{6}n)} = e^{j(\frac{\pi}{6})(n+N)} = e^{j(\frac{\pi}{6}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{6}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 12$ shows that $x[n]$ has period 12.

(b) $x[n]$ is periodic with period 8:

$$e^{j(\frac{3\pi}{4}n)} = e^{j(\frac{3\pi}{4})(n+N)} = e^{j(\frac{3\pi}{4}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{3\pi}{4}N, \text{for integers } k, N$$

$$\Longrightarrow N = \frac{8}{3}k, \text{for integers } k, N$$

The smallest k for which both k and N are integers are is 3, resulting in the period N being 8.

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$ is not periodic because the denominator term is linear in n.

(d) We will show that $x[n]$ is not periodic. Suppose that $x[n]$ is periodic for some period N:

$$e^{j(\frac{\pi}{\sqrt{2}}n)} = e^{j(\frac{\pi}{\sqrt{2}})(n+N)} = e^{j(\frac{\pi}{\sqrt{2}}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{\sqrt{2}}N, \text{for integers } k, N$$

$$\Longrightarrow N = 2\sqrt{2}k, \text{for some integers } k, N$$

There is no integer k for which N is an integer. Hence $x[n]$ is not periodic.

2.8. We take the Fourier transform of both $h[n]$ and $x[n]$, and then use the fact that convolution in the time domain is the same as multiplication in the frequency domain.

$$\begin{aligned} H(e^{j\omega}) &= \frac{5}{1 + \frac{1}{2}e^{-j\omega}} \\ Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{5}{1 + \frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{3}e^{-j\omega}} \\ &= \frac{3}{1 + \frac{1}{2}e^{-j\omega}} + \frac{2}{1 - \frac{1}{3}e^{-j\omega}} \\ y[n] &= 2(\frac{1}{3})^n u[n] + 3(-\frac{1}{2})^n u[n] \end{aligned}$$

2.10. (a)

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k] \\
&= \begin{cases} \sum_{k=-\infty}^{n} a^k, & n \leq -1 \\ \sum_{k=-\infty}^{-1} a^k, & n > -1 \end{cases} \\
&= \begin{cases} \dfrac{a^n}{1-1/a}, & n \leq -1 \\ \dfrac{1/a}{1-1/a}, & n > -1 \end{cases}
\end{aligned}
$$

(b) First, let us define $v[n] = 2^n u[-n-1]$. Then, from part (a), we know that

$$
w[n] = u[n] * v[n] = \begin{cases} 2^{n+1}, & n \leq -1 \\ 1, & n > -1 \end{cases}
$$

Now,

$$
\begin{aligned}
y[n] &= u[n-4] * v[n] \\
&= w[n-4] \\
&= \begin{cases} 2^{n-3}, & n \leq 3 \\ 1, & n > 3 \end{cases}
\end{aligned}
$$

(c) Given the same definitions for $v[n]$ and $w[n]$ from part(b), we use the fact that $h[n] = 2^{n-1}u[-(n-1)-1] = v[n-1]$ to reduce our work:

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= x[n] * v[n-1] \\
&= w[n-1] \\
&= \begin{cases} 2^n, & n \leq 0 \\ 1, & n > 0 \end{cases}
\end{aligned}
$$

(d) Again, we use $v[n]$ and $w[n]$ to help us.

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= (u[n] - u[n-10]) * v[n] \\
&= w[n] - w[n-10] \\
&= (2^{n+1}u[-(n+1)] + u[n]) - (2^{n-9}u[-(n-9)] + u[n-10]) \\
&= \begin{cases} 2^{(n+1)} - 2^{(n-9)}, & n \leq -2 \\ 1 - 2^{(n-9)}, & -1 \leq n \leq 8 \\ 0, & n \geq 9 \end{cases}
\end{aligned}
$$

2.11. First we re-write $x[n]$ as a sum of complex exponentials:

$$x[n] = \sin(\frac{\pi n}{4}) = \frac{e^{j\pi n/4} - e^{-j\pi n/4}}{2j}.$$

Since complex exponentials are eigenfunctions of LTI systems,

$$y[n] = \frac{H(e^{j\pi/4})e^{j\pi n/4} - H(e^{-j\pi/4})e^{-j\pi n/4}}{2j}$$

Evaluating the frequency response at $\omega = \pm\pi/4$:

$$\begin{aligned} H(e^{j\frac{\pi}{4}}) &= \frac{1 - e^{-j\pi/2}}{1 + 1/2e^{-j\pi}} = 2(1-j) = 2\sqrt{2}e^{-j\pi/4} \\ H(e^{-j\frac{\pi}{4}}) &= \frac{1 - e^{j\pi/2}}{1 + 1/2e^{j\pi}} = 2(1+j) = 2\sqrt{2}e^{j\pi/4} \end{aligned}$$

We get:

$$\begin{aligned} y[n] &= \frac{2\sqrt{2}e^{-j\pi/4}e^{j\pi n/4} - 2\sqrt{2}e^{j\pi/4}e^{-j\pi n/4}}{2j} \\ &= 2\sqrt{2}\sin(\pi n/4 - \pi/4). \end{aligned}$$

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(c) To determine if the system is time-invariant, consider the input:

$$x[n] = \delta[n-1]$$

the recursion yields

$$y[n] = 0, \text{ for } n < 0$$
$$y[0] = 0$$
$$y[1] = 1$$
$$y[2] = 2$$
$$y[3] = 6$$
$$y[4] = 24$$

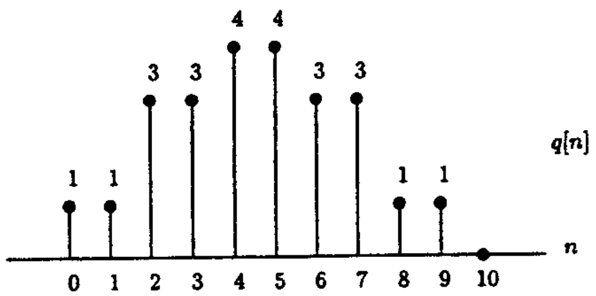
Using $h[n]$ from part (a),

$$h[n-1] = (n-1)!u[n-1] \neq y[n]|_{x[n]=\delta[n-1]}$$

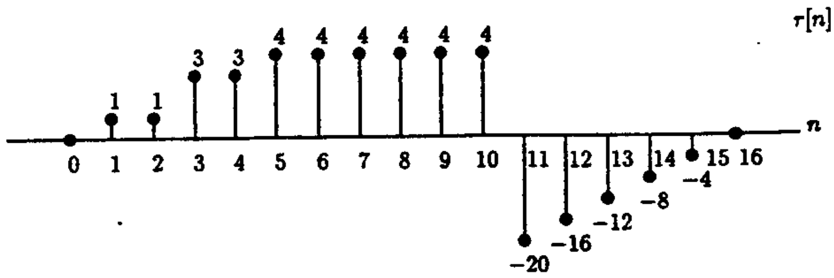
Conclude: NOT TIME INVARIANT.

2.18. $h[n]$ is causal if $h[n] = 0$ for $n < 0$. Hence, (a) and (b) are causal, while (c), (d), and (e) are not.

2.50. (a) Carrying out the convolution sum, we get the following sequence $q[n]$:



(b) Again carrying out the convolution sum, we get the following sequence $r[n]$:



(c) Let $a[n] = v[-n]$ and $b[n] = w[-n]$, then:

$$\begin{aligned} a[n] * b[n] &= \sum_{k=-\infty}^{+\infty} a[k]b[n-k] \\ &= \sum_{k=-\infty}^{+\infty} v[-k]w[k-n] \\ &= \sum_{r=-\infty}^{+\infty} v[r]w[-n-r] \text{ where } r = -k \\ &= q[-n]. \end{aligned}$$

We thus conclude that $q[-n] = v[-n] * w[-n]$.

2.72. The analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

(a) The Fourier transform of $x^*[n]$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* \\ &= X^*(e^{-j\omega n}).\end{aligned}$$

(b) The Fourier transform of $x^*[-n]$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} &= \sum_{l=-\infty}^{\infty} x^*[l]e^{j\omega l} \\ &= \left(\sum_{l=-\infty}^{\infty} x[l]e^{-j\omega l}\right)^* \\ &= X^*(e^{j\omega}).\end{aligned}$$