Discrete-Time Signal Processing Midterm 2011.4.20

Solution

1.

We desire the step response to a system whose impulse response is

$$h[n] = a^{-n}u[-n], \text{ for } 0 < a < 1.$$

The convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The step response results when the input is the unit step:

$$x[n] = u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Substitution into the convolution sum yields

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$

For $n \leq 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} a^{-k} \\ &= \sum_{k=-n}^{\infty} a^{k} \\ &= \frac{a^{-n}}{1-a} \end{aligned}$$

For $n > 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{0} a^{-k} \\ &= \sum_{k=0}^{\infty} a^{k} \\ &= \frac{1}{1-a} \end{aligned}$$

2.

The difference equation:

$$y[n]-\frac{1}{2}y[n-1]=x[n]+2x[n-1]+x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1-\frac{1}{2}e^{-j\omega}]=X(e^{j\omega})[1+2e^{-j\omega}+e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1+2e^{-j\omega}+e^{-j2\omega}}{1-\frac{1}{2}e^{-j\omega}}. \end{aligned}$$

3.

We have

$$w[n]=\begin{cases} \frac{1}{2}(1+\cos(\frac{2\pi n}{M}), & \text{for } 0\le n\le M \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} W(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\frac{1}{2}(1+\cos(\frac{2\pi n}{M}))e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{2}(1+\frac{1}{2}e^{j\frac{2\pi n}{M}}+\frac{1}{2}e^{-j\frac{2\pi n}{M}})e^{-j\omega} \\ &= (\frac{1}{2}\delta(\omega)+\frac{1}{4}\delta(\omega+\frac{2\pi}{M})+\frac{1}{4}\delta(\omega-\frac{2\pi}{M})) \end{aligned}$$

4.

For an LTI system, we use the convolution equation to obtain the output:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Let $n = m + N$:

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m+N-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} x[(m-k)+N]h[k] \end{aligned}$$

Since $x[n]$ is periodic, $x[n] = x[n+rN]$ for any integer r. Hence,

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m-k]h[k] \\ &= y[m] \end{aligned}$$

So, the output must also be periodic with period N.

5.

Let the input be $x[n] = \delta[n-1]$, if the system is causal then the output, $y[n]$, should be zero for $n < 1$. Let's evaluate $y[0]$:

$$\begin{aligned} y[0] &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} Y(e^{j\omega})\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-j\omega}e^{-j\omega/2}\,d\omega \\ &= -\frac{2}{3\pi} \\ &\neq 0. \end{aligned}$$

This proves that the system is not causal.

6.

$$x[n] = \begin{cases} n, & 0 \le n \le N-1 \\ N, & N \le n \end{cases} = n\,u[n] - (n-N)u[n-N]$$

$$\begin{aligned}
n\,x[n] &\Leftrightarrow -z\frac{d}{dz}X(z) \Rightarrow n\,u[n] \Leftrightarrow -z\frac{d}{dz}\frac{1}{1-z^{-1}} \quad |z|>1 \\
n\,u[n] &\Leftrightarrow \frac{z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \\
x[n-n_0] &\Leftrightarrow X(z)\cdot z^{-n_0} \Rightarrow (n-N)u[n-N] \Leftrightarrow \frac{z^{-N-1}}{(1-z^{-1})^2} \quad |z|>1
\end{aligned}$$

therefore

$$X(z) = \frac{z^{-1} - z^{-N-1}}{(1-z^{-1})^2} = \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}$$

7.

$$\begin{aligned}
G(z) &= \sin(z^{-1})(1+3z^{-2}+2z^{-4}) \\
&= (z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!})(1+3z^{-2}+2z^{-4}) \\
&= \sum_n g[n]z^{-n}
\end{aligned}$$

$g[11]$ is simply the coefficient in front of z^{-11} in this power series expansion of $G(z)$:

$$g[11] = -\frac{1}{11!} + \frac{3}{9!} - \frac{2}{7!}.$$

8.

$$X(z) = e^z + e^{1/z} \quad z \neq 0$$

$$X(z) = \sum_{n=0}^{\infty}\frac{1}{n!}z^n + \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0}\frac{1}{(-n)!}z^{-n} + \sum_{n=0}^{\infty}\frac{1}{n!}z^{-n} \Longrightarrow x[n] = \frac{1}{|n|!} + \delta[n]$$

9.

$$
\begin{aligned}
x[n] &= x_c(nT) \\
&= \sin\left(2\pi(100)n\frac{1}{400}\right) \\
&= \sin\left(\frac{\pi}{2}n\right)
\end{aligned}
$$

10.

Notice first that $H(e^{j\omega}) = 10j\omega, -\pi \le \omega < \pi$.

(i) After sampling,

$$
\begin{aligned}
x[n] &= \cos(\frac{3\pi}{5}n), \\
y[n] &= |H(e^{j\frac{3\pi}{5}})|\cos(\frac{3\pi}{5}n + \angle H(e^{j\frac{3\pi}{5}})) \\
&= 6\pi\cos(\frac{3\pi}{5}n + \frac{\pi}{2}) \\
&= -6\pi\sin(\frac{3\pi}{5}n) \\
y_c(t) &= -6\pi\sin(6\pi t).
\end{aligned}
$$