Singular Value Decomposition

Ground Truth

Any $m \times n$ real matrix A can be factored into

$$A = Q_1 \Sigma Q_2^T,$$

where Q_1 is an $m \times m$ orthogonal matrix, Q_2 is an $n \times n$ orthogonal matrix, and Σ is an $m \times n$ matrix whose diagonal entries are non-negative and its off diagonal entries are 0.

Moreover, the non-zero entries in Σ are the square roots of the eigenvalues of both AA^T and $A^T A$.

Implications of SVD

- $AQ_2 = Q_1\Sigma$
- $AA^T = Q_1\Sigma\Sigma^T Q_1^T$, and $A^TA = Q_2\Sigma^T\Sigma Q_2^T$. Therefore Q_1 is the eigenvector matrix of AA^T. Similarly, Q_2 is the eigenvector matrix of A^TA.

  The eigenvalues for AA^T is the diagonal entries of $\Sigma\Sigma^T$, which are σ_i^2 and 0.

(Proof by) Construction of SVD

$A^T A$ is a symmetric matrix so it has a complete set of orthonormal eigenvectors $\{x_1, \ldots, x_n\}$ (going into Q_2) and eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

$$\begin{aligned} A^T A x_j = \lambda_j x_j &\Rightarrow x_j^T A^T A x_j = \lambda_j x_j^T x_j \\ &\Rightarrow ||Ax_j||^2 = \lambda_j \geq 0 \end{aligned}$$

Suppose $\lambda_1, \ldots, \lambda_r$ are positive and the rest are 0. Let $\sigma_j = \sqrt{\lambda_j}$ and $q_j = \frac{Ax_j}{\sigma_j}$. These $q_j's$ are orthonormal. Furthermore, they can be expanded to a basis for R^m and go into Q_1. The ij-entry of $Q_1 A Q_2$ is

$$q_i^T A x_j = \begin{cases} 0, & j > r \\ 0, & j \leq r \text{ and } i \neq j \\ q_i^T \sigma_j q_j, & j \leq r \text{ and } i = j \end{cases} \Rightarrow \Sigma = Q_1^T A Q_2 \Rightarrow A = Q_1 \Sigma Q_2^T$$