NSYSU CSE Linear Algebra Quiz 2

1. Let
$$D = \begin{bmatrix} 2 & 1 & 0 & -1 & -2 \\ -1 & 2 & 0 & 2 & -1 \end{bmatrix}.$$

   (a) Find an orthonormal basis for the row space of D.
   Sol: Since the rows are orthogonal, one can simply normalize each row vector. An orthonormal basis is
   $$\{\frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 & 0 & -1 & -2 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} -1 & 2 & 0 & 2 & -1 \end{bmatrix}\}$$

   (b) Find an orthonormal basis for the orthogonal complement of the row space of D.
   Sol: The orthogonal complement of the row space of a matrix is the null space. One can find the basis for the nullspace and then apply the Gram-SchmidtProcedure. It turn out to be
   $$\{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{4}{5} & \frac{-3}{5} & 0 & 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 & 0 & 1 \end{bmatrix}\}$$

2. Find the least squares approximation for the function $\sin x$ by $a + bx + cx^2$ in the interval $[0, 1]$. In other words, find a, b, c such that
$$\int_0^1 (\sin x - a - bx - cx^2)^2 dx$$
is minimized.
Sol: The least squares approximation is given by
$$A^T A\bar{x} = A^T b \Rightarrow \bar{x} = (A^T A)^{-1} A^T b,$$
where $A^T A$ is the inner product matrix of functions $1, x$, and x^2, and $A^T b$ is the inner product vector of the above functions and $\sin x$. Using that $\int x \sin x dx = -x \cos x + \sin x$ and $\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x$, one can obtain the solution.

3. Find the projection point of $x = [1, 1, 1, 1]$ on the space spanned by $v_1 = [1, 2, 3, 2]$ and $v_2 = [1, -1, 1, -1]$.
Sol: Since v_1 is orthogonal to v_2, the projection to the plane is the sum of projections to the vectors.
$$p = P_{v_1} x + P_{v_2} x = \frac{4}{9}[1, 2, 3, 2].$$

4. (permutation and sorting)

   (a) Is the permutation $(3, 8, 9, 1, 4, 6, 7, 5, 2)$ even or odd?
   Sol: Even. 6 exchanges of objects are required (see below).

   (b) How do you swap objects in this permutation to make it $(1, 2, 3, 4, 5, 6, 7, 8, 9)$?
   Sol: In the ith epoch, one swaps i with the object in the ith position,
   or does nothing when i is already in that position.

$$\begin{aligned}(3,8,9,1,4,6,7,5,2) &\to (1,8,9,3,4,6,7,5,2) \to (1,2,9,3,4,6,7,5,8)\\ &\to (1,2,3,9,4,6,7,5,8) \to (1,2,3,4,9,6,7,5,8)\\ &\to (1,2,3,4,5,6,7,9,8) \to (1,2,3,4,5,6,7,8,9)\end{aligned}$$

5. Compute the determinant of A, where

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & -1 & -3 \end{bmatrix}.$$

$$\det(A) = 73,$$

$$\det \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} = 55$$