Make-Up Exam of Linear Algebra

1. Given
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix},$$
find the matrices for

   (a) projection to the column space of A.
$$P = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}.$$

   (b) projection to the left nullspace of A. ($I - P$).

2. Find the 4 fundamental subspaces of matrix B,
$$B = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$
$$\begin{aligned} \text{row space } &= \{y \mid y = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4\} \\ \text{nullspace } &= \{y \mid y = c(1\ 1\ 1\ 1\ 1)\} \\ \text{column space } &= \{x \mid x_1 + x_2 + x_6 = 0, x_3 - x_5 - x_6 = 0\} \\ \text{left nullspace } &= \{x \mid x = c_1(1\ 1\ 0\ 0\ 0\ 1) + c_2(0\ 0\ 1\ 0\ -1\ -1)\} \end{aligned}$$

3. (20%) Suppose a linear transformation T transforms a to c and b to d, where
$$a = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ b = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ c = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix},\ d = \begin{bmatrix} -5 \\ 10 \\ 0 \end{bmatrix}.$$
Find the matrix M such that
$$T(x) = Mx \text{ for any } x \in R^2.$$

$$M = \begin{bmatrix} 3 & -1 \\ 0 & 5 \\ 2 & 1 \end{bmatrix}$$

4. Find the best approximation (closest in the function space) to the function $x^4 + 2x^3$ by a linear combination of $1, x$ and x^2 in the interval $[0, 1]$.

$$A^T Ax = A^T b \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} x = \begin{bmatrix} \frac{7}{10} \\ \frac{17}{30} \\ \frac{10}{21} \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix} \begin{bmatrix} \frac{7}{10} \\ \frac{17}{30} \\ \frac{10}{21} \end{bmatrix} \doteq \begin{bmatrix} 0.19 \\ -2.11 \\ 4.71 \end{bmatrix}.$$

5. Find the QR factorization of the following matrix,

$$C = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 4 & 5 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{-4}{5} & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 5 & \frac{26}{5} & \frac{13}{5} \\ 0 & 0 & \frac{7}{5} & \frac{-9}{5} \\ 0 & 0 & 0 & 6 \end{bmatrix} = QR$$