SEMIRING FRAMEWORKS AND ALGORITHMS FOR SHORTEST-DISTANCE PROBLEMS

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Introduction

- The single-source shortest-path problem in a directed graph consists of determining the shortest path from a fixed source vertex s to all other vertices.
- The classical single-source shortest paths problem is denoted by Bellman-Ford equations with real-valued weights and specific operations:
  - the weights are added along the paths using addition of real numbers (+ operation)
  - the solution of the equation gives the shortest distance to each vertex q belong to Q (min operation).

Introduction(cont.)

- Classical shortest-paths problems can be generalized to other weight sets, and to other operations.
- The weights, elements of a set $\mathbb{K}$ may be
  - real numbers
  - strings, regular expressions
  - subsets of another set
  - any other quantity that can be multiplied along a path using an $\otimes$ operation , and that can be summed using an $\oplus$operation .
- The weight of a path is obtained by multiplying edge weights along that path using $\otimes$
- The shortest distance from a vertex p to a vertex q is the sum of the weights of all paths from p to q using $\oplus$.

Definition of semiring

Definition 1 *A* semiring *is a system* $(\mathbb{K}, \oplus, \otimes, \overline{0}, \overline{1})$ *such that*

1. $(\mathbb{K}, \oplus, \overline{0})$ *is a commutative monoid with* $\overline{0}$ *as the identity element for* $\oplus$,
2. $(\mathbb{K}, \otimes, \overline{1})$ *is a monoid with* $\overline{1}$ *as the identity element for* $\otimes$,
3. $\otimes$ *distributes over* $\oplus$*: for all* a, b, c *in* $\mathbb{K}$,

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c),$$
$$c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b),$$

4. $\overline{0}$ *is an annihilator for* $\otimes$*:* $\forall a \in \mathbb{K},\ a \otimes \overline{0} = \overline{0} \otimes a = \overline{0}$.

Property of semiring

Definition 2 *Let* $(\mathbb{K}, \oplus, \otimes, \overline{0}, \overline{1})$ *be a semiring. An element* $a \in K$ *is* idempotent *if* $a + a = a$. $\mathbb{K}$ *is said to be* idempotent *when all elements of* $\mathbb{K}$ *are idempotent.*

Definition 5 *Let* $(\mathbb{K}, \oplus, \otimes, \overline{0}, \overline{1})$ *be a semiring.* $\mathbb{K}$ *is* bounded *if* $\overline{1}$ *is an annihilator for* $\oplus$*:* $\forall a \in \mathbb{K},\ \overline{1} \oplus a = \overline{1}$.

Property of semiring

Definition 6 *Let $k \geq 0$ be an integer. A semiring $(\mathbb{K}, \oplus, \otimes, \overline{0}, \overline{1})$ is* k-closed *if*

$$\forall a \in \mathbb{K}, \quad \bigoplus_{n=0}^{k+1} a^n = \bigoplus_{n=0}^{k} a^n.$$

Lemma 4 *Let $(\mathbb{K}, \oplus, \otimes, \overline{0}, \overline{1})$ be a k-closed semiring, then for any integer $l > k$ and $a \in \mathbb{K}$*

$$\forall a \in \mathbb{K}, \quad \bigoplus_{n=0}^{l} a^n = \bigoplus_{n=0}^{k} a^n.$$

- *G=(Q, E, w)* : weighted directed graph
- *Q:* set of vertices of G
- E: set of edges of *G*
- *w: E* $\rightarrow \mathbb{K}$ weight function
- *P(q)*: set of paths from *s* (source vertex) to q
- π: path $\pi = e_1 e_2 \ldots e_k, e_i \in E,\ i = 1 \ldots k$
- $w[\pi]$: weight of path π

$$w[\pi] = \bigotimes_{i=1}^{k} w[e_i]$$

- *d[p]*: shortest distance from *s* to *q*

Shortest distance definition

- We denote a general algebraic framework for single-source shortest-distance problems based on the structure of semirings.

- For any vertex *q* belong to *Q*, we denote by *δ(s,q)* the shortest distance from *s* to *q* associated to the weighted directed graph *G* and define it by

$$\begin{cases} \delta(s,s) = \overline{1} \\ \forall q \in Q - \{s\}, \delta(s,q) = \bigoplus_{\pi \in P(q)} w[\pi] \end{cases}$$

Extend the definition of k-closed semirings for graph

Definition 8 *Let* $(\mathbb{K}, \oplus, \otimes, \overline{0}, \overline{1})$ *be a commutative semiring,* $G = (Q, E, w)$ *a weighted directed graph over* $\mathbb{K}$ *and* $k \geq 0$ *an integer.* $\mathbb{K}$ *is* k-closed for G *if for any cycle* π *in* G*:*

$$\bigoplus_{n=0}^{k+1} w[\pi]^n = \bigoplus_{n=0}^{k} w[\pi]^n.$$

- For each vertex q belong to Q, we denote by $P_k(q)$ the set of paths from s to q with at most k occurrences of a cycle. It is not hard to show that the set of paths in $P_k(q)$ is finite.

- By Lemma 4, for a semiring $\mathbb{K}$ k-closed for G, we have

$$\forall l \geq k, \bigoplus_{\pi \in P_l(q)} w[\pi] = \bigoplus_{\pi \in P_k(q)} w[\pi]$$

- Since $P_\infty(q) = P(q)$,

$$\bigoplus_{\pi \in P(q)} w[\pi] = \bigoplus_{\pi \in P_k(q)} w[\pi]$$

- and thereby define the shortest distances $\delta(s,q)$ when $\mathbb{K}$ is k-closed for G

Classical shortest distance algorithm over semiring framework

- Before presenting our generic algorithm, let us first mention that a straightforward extension of the classical algorithms based on a relaxation technique would not produce the correct result for non-idempotent semirings.

for $i \leftarrow 1$ to $|Q|$
    do $d[i] \leftarrow \overline{0}$
$d[s] \leftarrow \overline{1}$
$S \leftarrow \{s\}$
while $S \neq \phi$
    do $q \leftarrow head(S)$
        DEQUEUE(S)
        for each $e \in E[q]$
            do if $d[n[e]] \neq d[n[e]] \oplus (d[p[e]] \otimes w[e])$
                $d[n[e]] \leftarrow d[n[e]] \oplus (d[p[e]] \otimes w[e])$
                if $n[e] \notin S$
                    then ENQUEUE($S, n[e]$)

Example

- The successive values of a tentative shortest distance from the source 0 to the vertex 1 will be:

$$\text{After first while loop: } d[1] = a$$

$$\text{After second while loop: } d[1] = a \oplus (a \otimes b) = a \otimes (\bar{1} \oplus b)$$

$$\text{After third while loop: } d[1] = \underline{a \otimes (\bar{1} \oplus b)} \oplus \underline{a \otimes (\bar{1} \oplus b) \otimes b} = a \otimes (\bar{1} \oplus b)^2$$

$$= a \oplus \underline{a \otimes b} \oplus \underline{a \otimes b} \oplus a \otimes b^2 \neq a \oplus a \otimes b \oplus a \otimes b^2$$

$$\vdots$$

$$a \otimes (\bar{1} \oplus b)^n$$

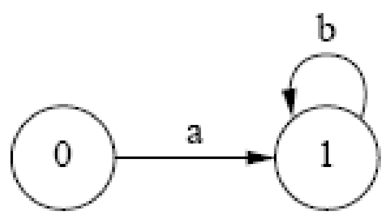

Figure 1: Single-source shortest distances for non-idempotent semirings

Generic Single-Source Shortest-Distance Algorithm

- To deal properly with multiplicities in the case of non-idempotent semirings, we keep track of the changes to the tentative shortest distance from s to q after the last extraction of q from the queue

Generic Single-Source Shortest-Distance Algorithm

```
GENERIC-SINGLE-SOURCE-SHORTEST-DISTANCE (G, s)
  for i ← 1 to |Q|
      do d[i] ← r[i] ← 0̄
  d[s] ← r[s] ← 1̄
  S ← {s}
  while S ≠ ∅
        do q ← head(S)
           DEQUEUE(S)
           r′ ← r[q]
           r[q] ← 0̄
           for each e ∈ E[q]
               do if d[n[e]] ≠ d[n[e]] ⊕ (r′ ⊗ w[e])
                     then d[n[e]] ← d[n[e]] ⊕ (r′ ⊗ w[e])
                          r[n[e]] ← r[n[e]] ⊕ (r′ ⊗ w[e])
                          if n[e] ∉ S
                             then ENQUEUE(S, n[e])
  d[s] ← 1̄
```

Figure 2: Pseudocode of a generic algorithm for solving single-source shortest-distance problems

- We use a queue S to maintain the set of vertices whose leaving edges are to be relaxed.
- For each vertex q belong to Q, we maintain two attributes:
  - $d[q]$ belong to $\mathbb{K}$, an estimate of the shortest distance from s to q, and
  - $r[q]$ belong to $\mathbb{K}$, the total weight added to $d[q]$ since the last time q was extracted from S