Outline

- Properties of Determinants (axiomatic approach)
- Formula (Calculations) of Determinants
- Applications of Determinants

Properties of Determinants

Basic Properties

1. (a1) $\det(A)$ depends linearly on the first row of A.

$$\begin{vmatrix} ta+t'a' & tb+t'b' \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix} + t'\begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

2. (a2) $\det(A)$ changes its sign if two rows of A are exchanged.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

3. (a3) $\det(I) = 1$

Derived Properties

4. (p4) If there are two equal rows, say row i and row j, then $\det(A) = 0$. Exchanging row i and j we get the same matrix, and by (a1) $\det(A) = -\det(A) = 0$.

5. (p5) Row operations (as in Gauss elimination) leaves $\det(A)$ unchanged. This follows from property (a1) and (p4).

6. (p6) If there is a zero row in A then $\det(A) = 0$. This follows from (p4) and (p5).

7. (p7) If A is triangular, then $\det(A) = \prod_i a_{ii}$. This follows from (p5), (a1) and (a3). Singular case is trivially true.

More Derived Properties

8. (p8) If A is singular, then $\det(A) = 0$. If A is not singular, then $\det(A) \neq 0$.
9. (p9) Since $d(A) = \det(AB)/\det(B)$ satisfies (a1) - (a3), we have
$$\det(AB) = \det(A)\det(B).$$
10. (p10) $\det\ A^T = \det\ A$. Trivially true for A singular. For A non-sigular, let $PA = LDU$ and note all matrices P, L, D, U have the same determinants as their transposes.

Formula for Determinants

Gauss Elimination: Reducing to Pivots

- If A is non-singular, $PA = LU$, so

$$\begin{aligned} \det\ A &= \det\ P^{-1}\ \det\ L\ \det\ U \\ &= \pm \det\ U = \pm \text{ product of pivots} \end{aligned}$$

Example: (Toeplitz matrix)

$$\det \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} = n+1$$

Summing over Permutations

- Expand $\det(A)$ into a sum of n^n terms, of which at most $n!$ terms can be non-zero. These are the terms where there is no more than 1 non-zero entry in every column and row. Let the non-zero entry in row i have column index j_i, then the j_i' must be distinct. $(j_1, j_2, \ldots, j_n)$ is called a permutation of $(1, 2 \ldots, n)$. The determinant of A is thus equal to

$$\det(A) = \sum_{\sigma} (a_{1\alpha} a_{2\beta} \ldots a_{n\nu}) \det(P_\sigma),$$

where $\sigma = (\alpha, \beta, \ldots, \nu)$ is a permutation of $(1, 2, \ldots, n)$, and P_σ is the permutation matrix.

Cofactor Expansions

- From the formula summing over permutation,

$$\det\ A = a_{11}A_{11} + \cdots + a_{1n}A_{1n},$$

where $A_{11} = \sum_{\sigma'} (a_{2\beta} \ldots a_{n\nu}) \det P_{\sigma'}$

- A_{1j} does not depend on row 1 and column j
- Let M_{1j} be the submatrix formed by knocking out row 1 and column j of A. Then

$$\det(A) = \sum_{j=1}^{n} a_{1j}A_{1j},$$

where $A_{1j} = (-1)^{1+j} \det(M_{1j})$, the cofactor of a_{1j}.

Applications for Determinants

Finding A^{-1}

$$
\begin{aligned}
A\,A_{cof} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det(A) \end{bmatrix} = \det(A)\,I \\
&\Rightarrow A^{-1} = \frac{1}{\det(A)} A_{cof}
\end{aligned}
$$

- A_{cof} is called the cofactor matrix (or adjunct matrix)
- Note the arrangement of cofactors in the cofactor matrix!

Solving $Ax = b$ with the Cramer's Rule

- Crammer's Rule

$$x_j = \frac{\det(B_j)}{\det(A)}, \text{ where } B_j = \begin{bmatrix} a_1 & \dots & a_{j-1} & b_j & a_{j+1} & \dots \end{bmatrix}.$$

- Proof

$$x = A^{-1}b = \frac{1}{\det(A)} A_{cof}\, b,$$

and

$$(A_{cof}\, b)_j = A_{1j}b_1 + \dots A_{nj}b_n = \det\ B_j.$$

A Formula for the Pivots

- The first k pivots are completely determined by the submatrix A_k in the upper-left corner of A. In fact,

$$A_k = L_k D_k U_k$$

if there are no row exchanges required. Note by block multiplication that

$$\begin{bmatrix} L_k & 0 \\ B & C \end{bmatrix} \begin{bmatrix} D_k & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} U_k & F \\ 0 & G \end{bmatrix} = \begin{bmatrix} L_k D_k U_k & X \\ Y & Z \end{bmatrix}$$

- Gauss elimination can be carried out without row exchanges if $A_1, A_2, \ldots, A_n$ are non-singular.