3.2.

$$x[n] = \begin{cases} n, & 0 \le n \le N-1 \\ N, & N \le n \end{cases} = n\,u[n] - (n-N)u[n-N]$$

$$\begin{aligned}
n\,x[n] &\Leftrightarrow -z\frac{d}{dz}X(z) \Rightarrow n\,u[n] \Leftrightarrow -z\frac{d}{dz}\frac{1}{1-z^{-1}} \quad |z|>1 \\
n\,u[n] &\Leftrightarrow \frac{z^{-1}}{(1-z^{-1})^2} \quad |z|>1 \\
x[n-n_0] &\Leftrightarrow X(z)\cdot z^{-n_0} \Rightarrow (n-N)u[n-N] \Leftrightarrow \frac{z^{-N-1}}{(1-z^{-1})^2} \quad |z|>1
\end{aligned}$$

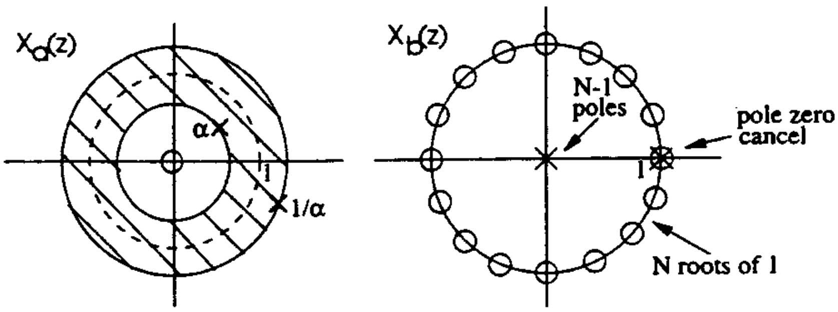
therefore

$$X(z) = \frac{z^{-1} - z^{-N-1}}{(1-z^{-1})^2} = \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}$$

3.3. (a)

$$x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$$

$$\begin{aligned}
X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \qquad |\alpha| < |z| < \frac{1}{|\alpha|}
\end{aligned}$$



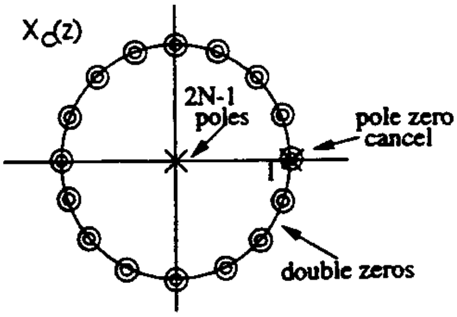
(b)

$$x_b = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & N \le n \\ 0, & n < 0 \end{cases} \Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} \quad z \neq 0$$

(c)

$$x_c[n] = x_b[n-1] * x_b[n] \Leftrightarrow X_c(z) = z^{-1} X_b(z) \cdot X_b(z)$$

$$X_c(z) = z^{-1}\left(\frac{z^N - 1}{z^{N-1}(z-1)}\right)^2 = \frac{1}{z^{2N-1}}\left(\frac{z^N-1}{z-1}\right)^2 \qquad z \neq 0, 1$$



3.6. (a)

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

Partial fractions: one pole $\rightarrow$ inspection, $x[n] = (-\frac{1}{2})^n u[n]$

Long division:

$$\begin{array}{r|llll}
 & 1 & -\frac{1}{2}z^{-1} & +\frac{1}{4}z^{-2} & + \dots \\ \hline
1+\frac{1}{2}z^{-1} & 1 & & & \\
 & 1 & +\frac{1}{2}z^{-1} & & \\ \hline
 & & -\frac{1}{2}z^{-1} & & \\
 & & -\frac{1}{2}z^{-1} & -\frac{1}{4}z^{-2} & \\ \hline
 & & & +\frac{1}{4}z^{-2} & \\
 & & & +\frac{1}{4}z^{-2} & +\frac{1}{8}z^{-3} \\ \hline
\end{array}$$

$$\Longrightarrow x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

(b)

$$X(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| < \frac{1}{2}$$

Partial Fractions: one pole $\rightarrow$ inspection, $x[n] = -(-\frac{1}{2})^n u[-n-1]$

Long division:

$$\begin{array}{r|llll}
 & 2z & -4z^2 & +8z^3 & + \dots \\
\hline
\frac{1}{2}z^{-1} + 1 & 1 & & & \\
 & 1 \quad +2z & & & \\
\cline{2-2}
 & \quad -2z & & & \\
 & \quad -2z & -4z^2 & & \\
\cline{2-3}
 & & +4z^2 & & \\
 & & +4z^2 & +8z^3 & \\
\cline{3-4}
\end{array}$$

$$\Longrightarrow x[n] = -\left(-\frac{1}{2}\right)^n u[-n-1]$$

(c)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned}
X(z) &= \frac{-3}{1 + \frac{1}{4}z^{-1}} + \frac{4}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\
x[n] &= \left[-3\left(-\frac{1}{4}\right)^n + 4\left(-\frac{1}{2}\right)^n\right] u[n]
\end{aligned}$$

Long division:

$$\begin{array}{r|llll}
 & 1 \quad + (-\frac{3}{4} - \frac{1}{2})z^{-1} & + (-\frac{3}{16} + 1)z^{-2} & & + \dots \\
\hline
1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} & 1 \quad - \frac{1}{2}z^{-1} & & & \\
 & 1 \quad + \frac{3}{4}z^{-1} & + \frac{1}{8}z^{-2} & & \\
\cline{2-3}
 & (-\frac{3}{4} - \frac{1}{2})z^{-1} & - \frac{1}{8}z^{-2} & & \\
 & (-\frac{3}{4} - \frac{1}{2})z^{-1} & + \frac{3}{4}(-\frac{3}{4} - \frac{1}{2})z^{-2} & + \frac{1}{8}(-\frac{3}{4} - \frac{1}{2})z^{-3} & \\
\cline{2-5}
 & & [-\frac{1}{8} + \frac{3}{4}(\frac{3}{4} + \frac{1}{2})]z^{-2} & + \frac{1}{8}(\frac{3}{4} + \frac{1}{2})z^{-3} & \\
\end{array}$$

$$\Longrightarrow x[n] = \left[-3(\left(-\frac{1}{4}\right)^n + \left(-\frac{1}{2}\right)^{n-2}\right] u[n]$$

(d)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned}
X(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\
x[n] &= \left(-\frac{1}{2}\right)^n u[n]
\end{aligned}$$

Long division: see part (i) above.

(e)

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a} \quad |z| > |a^{-1}|$$

Partial Fractions:

$$X(z) = -a - \frac{a^{-1}(1 - a^2)}{1 - a^{-1}z^{-1}} \qquad |z| > |a^{-1}|$$

$$x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

Long division:

$$\begin{array}{r|l} & -\frac{1}{a} \quad - (\frac{a^{-1}-a}{a})z^{-1} \quad - (\frac{a^{-1}-a}{a^2})z^{-2} \quad + \ldots \\ \cline{2-2} -a + z^{-1} & 1 \qquad - az^{-1} \\ & 1 \qquad - az^{-1} \\ \cline{2-2} & \qquad (a^{-1} - a)z^{-1} \qquad\qquad \ldots \end{array}$$

$$\Longrightarrow \qquad x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

3.8. The causal system has system function

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}$$

and the input is $x[n] = (\frac{1}{3})^n u[n] + u[-n-1]$. Therefore the z-transform of the input is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - z^{-1})} \qquad \frac{1}{3} < |z| < 1$$

(a) $h[n]$ causal $\Rightarrow$

$$h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]$$

(b)

$$\begin{aligned} Y(z) &= X(z)H(z) = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{3}{4}z^{-1})} \qquad \frac{3}{4} < |z| \\ &= \frac{-\frac{8}{13}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{8}{13}}{1 + \frac{3}{4}z^{-1}} \end{aligned}$$

Therefore the output is

$$y[n] = -\frac{8}{13}\left(\frac{1}{3}\right)^n u[n] + \frac{8}{13}\left(-\frac{3}{4}\right)^n u[n]$$

(c) For $h[n]$ to be causal the ROC of $H(z)$ must be $\frac{3}{4} < |z|$ which includes the unit circle. Therefore, $h[n]$ absolutely summable.

3.12. (a)

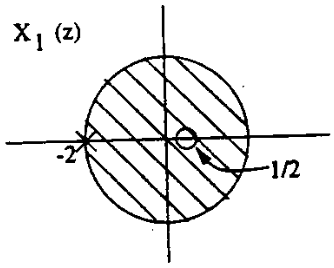
$$X_1(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$$

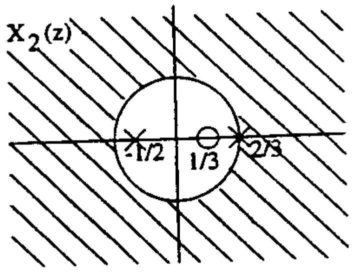
The pole is at -2, and the zero is at 1/2.

(b)

$$X_2(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})}$$

The poles are at -1/2 and 2/3, and the zero is at 1/3. Since $x_2[n]$ is causal, the ROC is extends from the outermost pole: $|z| > 2/3$.

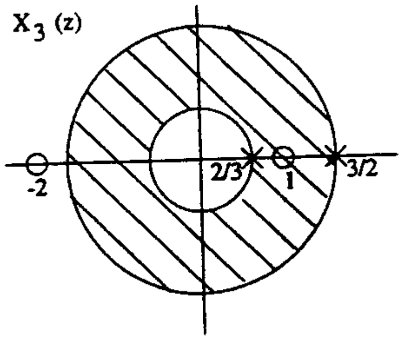




(c)

$$X_3(z) = \frac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$$

The poles are at 3/2 and 2/3, and the zeros are at 1 and -2. Since $x_3[n]$ is absolutely summable, the ROC must include the unit circle: $2/3 < |z| < 3/2$.



3.13.

$$\begin{aligned} G(z) &= \sin(z^{-1})(1 + 3z^{-2} + 2z^{-4}) \\ &= (z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!})(1 + 3z^{-2} + 2z^{-4}) \\ &= \sum_n g[n]z^{-n} \end{aligned}$$

$g[11]$ is simply the coefficient in front of z^{-11} in this power series expansion of $G(z)$:

$$g[11] = -\frac{1}{11!} + \frac{3}{9!} - \frac{2}{7!}.$$

3.18. (a)

$$
\begin{aligned}
H(z) &= \frac{1 + 2^z - 1 + z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1})} \\
&= -2 + \frac{\frac{1}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{\frac{8}{3}}{1 - z^{-1}}
\end{aligned}
$$

Taking the inverse z-transform:

$$h[n] = -2\delta[n] + \frac{1}{3}(-\frac{1}{2})^n u[n] + \frac{8}{3}u[n].$$

(b) We use the eigenfunction property of the input:

$$y[n] = H(e^{j\pi/2})x[n],$$

where

$$
\begin{aligned}
H(e^{j\pi/2}) &= -2 + \frac{\frac{1}{3}}{1 + \frac{1}{2}e^{-j\pi/2}} + \frac{\frac{8}{3}}{1 - e^{-j\pi/2}} \\
&= -2 + \frac{\frac{1}{3}}{1 - \frac{1}{2}j} + \frac{\frac{8}{3}}{1 + j} \\
&= \frac{-2j}{\frac{3}{2} + \frac{j}{2}}.
\end{aligned}
$$

Putting it together,

$$y[n] = \frac{-2j}{\frac{3}{2} + \frac{j}{2}} e^{j(\pi/2)n}.$$

3.19. The ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$Y(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

The intersection of ROCs of $H(z)$ and $X(z)$ is $|z| > \frac{1}{2}$. So the ROC of $Y(z)$ is $|z| > \frac{1}{2}$.

(b) The ROC of $Y(z)$ is exactly the intersection of ROCs of $H(z)$ and $X(z)$: $\frac{1}{3} < |z| < 2$.

(c)

$$Y(z) = \frac{1}{(1 - \frac{1}{5}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

The ROC is $|z| > \frac{1}{3}$.

3.26. (a) $x[n]$ is right-sided and

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

Long division:

$$\begin{array}{r|l}
 & 1 \quad - \frac{2}{3}z^{-1} \quad + \frac{2}{9}z^{-2} \quad + \ldots \\ \hline
1 + \frac{1}{3}z^{-1} & 1 \quad - \frac{1}{3}z^{-1} \\
 & 1 \quad - \frac{1}{3}z^{-1} \\ \cline{2-2}
 & \qquad - \frac{2}{3}z^{-1} \\
 & \qquad - \frac{2}{3}z^{-1} \quad - \frac{2}{9}z^{-2} \\ \cline{2-2}
 & \qquad\qquad\qquad + \frac{2}{9}z^{-2}
\end{array}$$

Therefore, $x[n] = 2(-\frac{1}{3})^n u[n] - \delta[n]$

(b)

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}} = \frac{3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 - \frac{1}{4}z^{-1}}$$

Poles at $\frac{1}{2}$, and $-\frac{1}{4}$. $x[n]$ stable, $\Rightarrow |z| > \frac{1}{2}$ $\Rightarrow$ causal.
Therefore,

$$x[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{4}\right)^n u[n]$$

(c)

$$\begin{aligned}
X(z) &= \ln(1 - 4z) \qquad |z| < \frac{1}{4} \\
&= -\sum_{i=1}^{\infty} \frac{(4z)^i}{i} = -\sum_{\ell=-\infty}^{-1} \frac{1}{\ell}(4z)^{-\ell}
\end{aligned}$$

Therefore,

$$x[n] = \frac{1}{n}(4)^{-n} u[-n-1]$$

(d)

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-3}} \qquad |z| > (3)^{-\frac{1}{3}} \Rightarrow \text{causal}$$

By long division:

$$\begin{array}{r|l}
 & 1 \quad + \frac{1}{3}z^{-3} \quad + \frac{1}{9}z^{-6} \quad + \ldots \\ \hline
1 - \frac{1}{3}z^{-3} & 1 \\
 & 1 \quad - \frac{1}{3}z^{-3} \\ \cline{2-2}
 & \qquad + \frac{1}{3}z^{-3} \\
 & \qquad + \frac{1}{3}z^{-3} \quad - \frac{1}{9}z^{-6} \\ \cline{2-2}
 & \qquad\qquad\qquad + \frac{1}{9}z^{-6}
\end{array}$$

$$\Longrightarrow \qquad x[n] = \begin{cases} (\frac{1}{3})^{\frac{n}{3}}, & n = 0, 3, 6, \ldots \\ 0, & \text{otherwise} \end{cases}$$

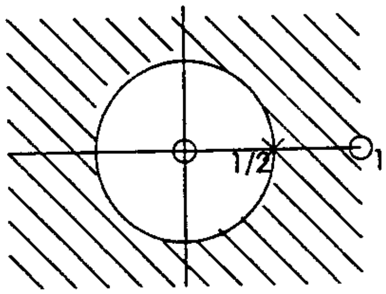
3.36.

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1] = 4\left(\frac{1}{2}\right)^{n+1} u[n+1] \Leftrightarrow Y(z) = \frac{4z}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

(a)

$$H(z) = \frac{Y(z)}{X(Z)} = \frac{4z(1 - z^{-1})}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$



(b)

$$\begin{aligned} H(z) &= \frac{4z}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2} \\ h[n] &= 4\left(\frac{1}{2}\right)^{n+1} u[n+1] - 4\left(\frac{1}{2}\right)^{n} u[n] \\ &= 4\delta[n+1] - 2\left(\frac{1}{2}\right)^{n} u[n] \end{aligned}$$

(c) The ROC of $H(z)$ includes $|z| = 1 \Rightarrow$ stable.

(d) From part (b) we see that $h[n]$ starts at $n = -1 \Rightarrow$ not causal