Linear Algebra: solution

- 4.3 Via Gauss elimination or otherwise,

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \end{bmatrix} = -6.$$

  Since the column vectors in the second matrix sum to 0, it has linearly dependent columns so the determinant is 0.

- 4.9 If all entries in a matrix are integers, its determinant, as a sum of integers, must be an integer. So both A and A^{-1} are integers. In addition,

$$\det\ A^{-1} = \frac{\det\ AA^{-1}}{\det\ A} = \frac{1}{\det\ A},$$

  so $\det A$ must be 1 or -1 to guarantee that $\frac{1}{\det A}$ is an integer. In addition, here $\det A = \det A^{-1}$.

- 4.14 One can establish a one-to-one correspondence between the set of even permutations, say E, and the set of odd permutations, say O, of $(1, \ldots, 9)$ by mapping an element e in E to the element o in O. Here o has the same ordering as e except that the last two numbers in e are swapped. Thus the total number of permutations is even and half of them are odd permutations.

- 4.15

$$\begin{aligned} \det(P_1 + P_2) &= \det P_1(P_1^T + P_2^T)P_2 \\ &= \det P_1 \det(P_1^T + P_2^T) \det P_2 \\ &= -\det(P_1^T + P_2^T) = -\det(P_1^T + P_2^T)^T \\ &= -\det(P_1 + P_2), \end{aligned}$$

  So $\det(P_1 + P_2) = 0$.

- 4.16 If A is $1 \times 1, \det A = 2$. If A is $2 \times 2, \det A = -1$. Otherwise, $\det A = 0$, since the rows are linearly dependent: the difference between adjacent rows is a constant vector.