Linear Algebra Quiz 1

1. Given the system of linear equations

$$\begin{array}{rrrrl} 2w+ & x- & y+ & z & =1 \\ -2w & + & 2y & & =1 \\ & 2x+ & 5y+ & 3z & =1 \\ 2w+ & x+ & 5y- & z & =1 \end{array}$$

   (a) What are the pivots?

   (b) What are the multipliers?

   (c) What is the LU decomposition of the coefficient matrix, say C?

   (d) What is the LDU decomposition of C?

   (e) What is the solution?

   (a) Define R$_{ij}^{k}$ $means\ that\ the\ i_{th}\ row\ *k\ add\ to\ the\ j_{th}\ row$

$$\begin{bmatrix} 2 & 1 & -1 & 1 & \mid & 1 \\ -2 & 0 & 2 & 0 & \mid & 1 \\ 0 & 2 & 5 & 3 & \mid & 1 \\ 2 & 1 & 5 & -1 & \mid & 1 \end{bmatrix} \to R_{12}^{1}, R_{14}^{-1} \to \begin{bmatrix} 2 & 1 & -1 & 1 & \mid & 1 \\ 0 & 1 & 1 & 1 & \mid & 2 \\ 0 & 2 & 5 & 3 & \mid & 1 \\ 0 & 0 & 6 & -2 & \mid & 0 \end{bmatrix} \to$$

$$R_{23}^{-2} \to \begin{bmatrix} 2 & 1 & -1 & 1 & \mid & 1 \\ 0 & 1 & 1 & 1 & \mid & 2 \\ 0 & 0 & 3 & 1 & \mid & -3 \\ 0 & 0 & 6 & -2 & \mid & 0 \end{bmatrix} \to R_{34}^{-2} \to \begin{bmatrix} 2 & 1 & -1 & 1 & \mid & 1 \\ 0 & 1 & 1 & 1 & \mid & 2 \\ 0 & 0 & 3 & 1 & \mid & -3 \\ 0 & 0 & 0 & -4 & \mid & 6 \end{bmatrix}$$

$\therefore pivots = 2, 1, 3, -4$

(b) by(b)

l(2,1)=-1, l(3,1)=0, l(4,1)=1, l(3,2)=2, l(4,2)=0, l(4,3)=2

(c) C = $\begin{bmatrix} 2 & 1 & -1 & 1 \\ -2 & 0 & 2 & 0 \\ 0 & 2 & 5 & 3 \\ 2 & 1 & 5 & -1 \end{bmatrix} \to R_{12}^{1}, R_{14}^{-1} \to \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 6 & -2 \end{bmatrix} \to R_{23}^{-2} \to$

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & -2 \end{bmatrix} \to R_{34}^{-2} \to \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$

$\Rightarrow R_{34}^{-2}R_{23}^{-2}R_{14}^{-1}R_{12}^{1} * C = U$

$\Rightarrow C = (R_{34}^{-2}R_{23}^{-2}R_{14}^{-1}R_{12}^{1})^{-1} * U$

$$\therefore C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} = LU.$$

(d) by(c)

C = LU= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ =LDU.

(e) by(a)

$$\begin{aligned} 2w + x - y + z &= 1 \\ x + y + z &= 2 \\ 3y + z &= -3 \\ -4z &= 6 \end{aligned}$$

$\therefore w = -1, x = 4, y = -1/2, z = -3/2$

2. Find the inverse of the following matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & | & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & | & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow R_{13}^{-1} \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & | & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow R_{21}^{1}, R_{23}^{-1} \rightarrow \begin{bmatrix} -1 & 0 & 1 & 0 & | & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & | & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & | & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow R_{41}^{1}, R_{42}^{1} \rightarrow$$

$$\begin{bmatrix} -1 & 0 & 0 & 2 & | & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 2 & | & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & | & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & | & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow R_{3_1}^{-2}, R_{3_2}^{-2}, R_{3_4}^{-2} \rightarrow \begin{bmatrix} -1 & 0 & 0 & 0 & | & 3 & 3 & -2 & 1 \\ 0 & -1 & 0 & 0 & | & 2 & 3 & -2 & 1 \\ 0 & 0 & 0 & 1 & | & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & | & 2 & 2 & -2 & 1 \end{bmatrix}$$

$$\rightarrow R_1^{-1}, R_2^{-1}, R_{34} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & -3 & -3 & 2 & -1 \\ 0 & 1 & 0 & 0 & | & -2 & -3 & 2 & -1 \\ 0 & 0 & 1 & 0 & | & -2 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & | & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & -3 & 2 & -1 \\ -2 & -3 & 2 & -1 \\ -2 & -2 & 2 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

3. Compute the matrix multiplication AB

   (a) using the linear combination of columns and

   (b) using the linear combination of rows

   where

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

   (a) by columns

$$\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \text{*(-1)+} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{*(-1)+} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{*(-1)+} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{*(0)=} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \text{*(1)+} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{*(1)+} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{*(0)+} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{*(0)=} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \text{*(0)+} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{*(0)+} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{*(1)+} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{*(-1)=} \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

(b) by Rows

$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$*(-1)+$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$*(1)+$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$*(0)+$\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$*(0)=$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$*(-1)+$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$*(0)+$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$*(1)+$\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$*(1)=$\begin{bmatrix} 0 & -1 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$*(0)+$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$*(0)+$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$*(-1)+$\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$*(2)=$\begin{bmatrix} 1 & 0 & -3 \end{bmatrix}$

$$\therefore AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$