Vector Space

Notes on Linear Algebra

Chia-Ping Chen

Department of Computer Science and Engineering

National Sun Yat-Sen University

Kaohsiung, Taiwan ROC

Vector Spaces

- Formally, a vector space is
  - a set of vectors
  - closed under addition and scalar multiplication
- The following are vector spaces.
  - the set of all 3×2 real matrices (vector = matrix)
  - the set of all functions defined on an interval (vector = function)
  - Everyone knows the spaces R^1, R^2, R^3. They are vector spaces.

Subspaces

- A (vector) subspace is
  - a subset of a vector space and
  - a vector space
- For example, R^3 is a vector space. Consider a plane S through the origin in R^3. S is a subspace, since it satisfies the conditions, which we repeat as follows
  - if $x, y \in S$, then $x + y \in S$.
  - if $x \in S$, then $cx \in S$.

Examples of Subspaces

Which of the following is a subspace?

1. the subset of all lower-triangular matrices in the space of all $n \times n$ matrices? the subset of all symmetric matrices?
2. the set of all functions of the form $a \sin x + b \cos x$ in the set of all functions defined on $(0, \pi)$?
3. the first quadrant in R^2?
4. the union of the first and third quadrants?

Existence and Uniqueness

- existence: Is there a solution?
- uniqueness: Are there other solutions?
- Questions regarding existence and uniqueness are common in mathematical theories.
- In this chapter we seek an understanding of linear equations through the concept of vector space.

Column Space

- Suppose the size of a matrix A is $m \times n$.
  - There are n column vectors, each of which is m-dimensional.
- The column space of A, denoted by $\mathcal{C}(A)$, is the set of vectors which are linear combinations of the column vectors of A.
- $\mathcal{C}(A)$ is a subspace of R^m.

Existence of Solution

- (theorem) $Ax = b$ has a solution if and only if b is in the column space of A.
- (proof) Recall that Ax is a linear combination of the columns of A using x_i's as coefficients.
  - (if part) If b is a linear combination of the column vectors of A, then the combination coefficients constitute a solution for x.
  - (only if part) If there is a solution, say x_0, then $b = Ax_0$ is a linear combination of the column vectors of A.

Size of Column Space

- The size of column space varies from matrix to matrix.
  - the smallest case: $\{\mathbf{0}\}$
  - the largest case: R^m
- If $m = n$, we have a square matrix. From Chapter 1, we know that
  - a non-singular matrix A has $\mathcal{C}(A) = R^m$, since $Ax = b$ can be solved for any b.
  - for a singular matrix A, $\mathcal{C}(A)$ is a proper subset of R^m: some vector in R^m is not in $\mathcal{C}(A)$.

Nullspace

- The nullspace of A, denoted by $\mathcal{N}(A)$, is the set

$$\{x \mid Ax = 0\}.$$

- $\mathcal{N}(A)$ is a subspace of R^n, since

$$x_1, x_2 \in \mathcal{N}(A) \Rightarrow A(x_1 + x_2) = 0 \Rightarrow x_1 + x_2 \in \mathcal{N}(A),$$
$$x \in \mathcal{N}(A) \Rightarrow A(cx) = 0 \Rightarrow cx \in \mathcal{N}(A).$$

- $Ax = 0$ is also known as the homogeneous equation. Accordingly, a solution x_0 to the homogeneous equation is in $\mathcal{N}(A)$.

Examples

$$A_1 = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{cases} \mathcal{C}(A_1) = \{x | x = c_1 \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\}; \\ \mathcal{N}(A_1) = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}. \end{cases}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow \begin{cases} \mathcal{C}(A_2) = \mathcal{C}(A_1); \\ \mathcal{N}(A_2) = \{x | x = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\}. \end{cases}$$

General System of Linear Equations

- When $m = n$, i.e. A is square, we use the Gaussian elimination to solve a system of equations $Ax = b$.
- What if $m \neq n$? We will generalize the treatment of a system of linear equations to this case.
- Let's look at an example. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

Row Echelon Form

- We apply the Gaussian elimination to make entries below pivots zero as we did for the square case.

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

- We say that U is in an echelon (staircase) form. All entries under the staircase are zeros.
- The pivots are the leading non-zero elements of the rows of U. There are two pivots in this example.

Basic and Free Variables

- First, we need to solve the corresponding homogeneous equation $Ax = 0$. For this example,

$$Ax = 0 \Leftrightarrow Ux = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = 0$$

- u and w correspond to columns with pivots. They are called the *basic variables*.
- v and y correspond to columns without pivots. They are called the *free variables*.

Solutions to $Ax = 0$

- Solving the basic variables in terms of free variables, one gets the general solution

$$w = -\frac{1}{3}y,\ u = -3v - y$$

$$\Rightarrow x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v - y \\ v \\ -\frac{1}{3}y \\ y \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}.$$

- The vectors are found by setting the free variables to $(v = 1, y = 0)$ and $(v = 0, y = 1)$.

Step by Step

- Steps to find the solutions to $Ax = 0$
  1. use elimination to reach $Ux = 0$ and identify the basic and free variables.
  2. set one free variable to 1 and others to 0, and solve $Ux = 0$ for the basic variables.
  3. Every free variable produces a solution in step 2. The linear combinations of these solutions form the solution set for $Ax = 0$.
- If $n > m$, then there must be free variables, since each column has at most one pivot. It follows that $Ax = 0$ must have non-zero solutions (we will use this result soon).

Solving $Ax = b \neq 0$

- Not every b is solvable. This can be seen from the triangular system $Ux = L^{-1}b$,

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}.$$

  If $b_3 - 2b_2 + 5b_1 \neq 0$ then there is no solution.

- Can you characterize all such $b's$?

Example

Let $b = \begin{bmatrix} 1 & 5 & 5 \end{bmatrix}'$. Then

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} w = 1 - \frac{1}{3}y \\ u = -2 - 3v - y \end{cases}$$

$$\Rightarrow \quad x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}.$$

General Solution

- Every solution to $Ax = b$ is the sum of a particular solution and a homogeneous solution.

$$x_g = x_p + x_h.$$

This is true since

$$Ax_g = Ax_p + Ax_h = b + 0 = b.$$

- We already know how to find x_h.
- To find an x_p, one can set all free variables to 0 and solve for the basic variables.

Rank

- After the elimination process, the number of pivots in U, say r, is called the *rank* of A.
- r is the number of basic variables. $n - r$ is the number of free variables.
- If $r = m$, the column space is the entire R^m and there is *at least* one solution since every b is in the column space.
- If $r = n$, there are no free variables. The nullspace contains only 0 and there is *at most* one solution.

Independent Equations

- The numbers m is not precise in giving the size of the system of equations. Some equations may be redundant in the sense that they can be derived from other equations.
- In our example, the last row is 0 after row operations, therefore it is only a linear combination of the other rows. This indicates that the last equation is either redundant or contradicting.
- We want to find the independent equations in a system. We introduce the concept of “linearly independent”.

Linearly Independent

- The vector set $\{v_i\}_{i=1}^n$ is said to be *linearly independent* if

$$\sum_i c_i v_i = 0 \Leftrightarrow c_1 = \cdots = c_n = 0.$$

  Otherwise it is said to be linearly dependent.

- If a set of vectors is linearly dependent, then at least one of them is a linear combination of the others.

Examples

- Examples of linearly dependent vectors: two (three) vectors are linearly dependent if they lie on the same line (plane).
- Examples of linearly independent vectors
  - the opposite of the above cases
  - non-zero rows in a matrix in row-echelon form
  - columns with pivots in a matrix in echelon form

Checking Independent

- We can determine whether a set of vectors is linearly dependent by putting the vectors as columns in a matrix A and solve $Ax = 0$.
  - If $x = 0$ is the only solution, then it is linearly independent.
  - If there are non-zero solution, then it is not.
- That is, the nullspace of A tells whether the columns of A are linearly independent!

Span

- A set of vectors $\{w_1, \ldots, w_l\}$ spans a vector space V if every vector $v \in V$ is a linear combination of the w's,

$$v = c_1 w_1 + \cdots + c_l w_l.$$

- Such a set is called a spanning set (of V).
- For example,
  - $\{(1,0,0),(0,1,0),(-2,0,0)\}$ spans a plane. (what plane?)
  - The set of columns of A spans $\mathcal{C}(A)$.

Spanning Set and Vector Space

- Given a spanning set, the vector space it spans is unique.
- Given a vector space (except for the trivial case), a spanning set is not unique: there are infinitely many spanning sets for a particular space.
- Adding linearly dependent vectors into a spanning set spans the same vector space.

Basis and Dimension

- Given a vector space, a *basis*
  1. is a spanning set
  2. is linearly independent
- A basis consists of the minimum number of vectors to span the given vector space.
- The number of vectors in a basis is called the *dimension* of the vector space.
  - What are the dimensions of the vector spaces we have seen?

Uniqueness of Linear Combination

- Suppose V is a basis for a vector space S.
- Then any $s \in S$ has a unique expansion

$$s = \sum_i \lambda_i v_i.$$

- This can be proved by the property of V being linearly independent.
- Note that a basis is not unique for a given vector space. However, once a basis is fixed, the expansion is unique.

Uniqueness of Dimension

- (theorem) For a given vector space, all bases have the same number of vectors.
- (proof) Suppose there are two bases U and W of a vector space with $|W| = n > m = |U|$. Let the matrices M_U and M_W use the vectors in U, W as column vectors. Since U is a basis,

$$M_W = M_U A$$

for some $m \times n$ matrix A. Since $n > m$, there exists non-zero c such that $Ac = 0$, or $M_W c = 0$. This contradicts the assumption that W is a basis.

Dual

- Given V, a linearly independent set of vectors in V can be extended to be a basis of V, by adding vectors as necessary.
- A spanning set of V can be reduced to be a basis of V, by removing dependent vectors as necessary.
- A basis is a maximal independent set and a minimal spanning set.
  - Adding (removing) any vector to (from) a basis will lose the property of being linearly independent (spanning).

Fundamental Subspaces

- We define four fundamental subspaces associated with a matrix A

  column space: $\mathcal{C}(A) = \{y \mid \exists\, c \text{ s.t. } y = Ac\}$

  nullspace: $\mathcal{N}(A) = \{x \mid Ax = 0\}$

  row space: $\mathcal{C}(A^T) = \{x \mid \exists\, c \text{ s.t. } x = A^T c\}$

  left nullspace: $\mathcal{N}(A^T) = \{y \mid A^T y = 0\}$

- Note that

$$\begin{cases} \mathcal{C}(A), \mathcal{N}(A^T) \subset R^m \\ \mathcal{N}(A), \mathcal{C}(A^T) \subset R^n \end{cases}$$

Basis for Row Space

- Each row operation leaves the row space unchanged.
  - It cannot be bigger since the new rows are linear combination of old rows.
  - It cannot be smaller as the old rows are linear combination of the new rows.
- Row space of A = Row space of U
- The non-zero rows of U constitute a basis for the row space of A.
- It follows that the dimension of the row space is equal to the rank r of a matrix.

Basis for Nullspace

- $\mathcal{N}(A) = \mathcal{N}(U)$, since

$$Ax = 0 \Leftrightarrow Ux = 0.$$

- construction of basis for $\mathcal{N}(A)$
  1. Gauss elimination $Ax = 0 \longrightarrow Ux = 0$
  2. set a free variable to 1 and others to 0; solve for the basic variables to obtain a vector in $\mathcal{N}(U)$
  3. repeat for each free variable
  4. the $n - r$ vectors thus obtained constitute a basis for the nullspace of A
- Obviously the dimension of $\mathcal{N}(A)$ is $n - r$.

Basis for Column Space

- $Ux = 0 \Leftrightarrow Ax = 0$: a linear combination of columns of U giving the zero vector also produces 0 when using the columns of A.
- If a column of U is some linear combination of the other columns, the same relation holds with the columns of A.
- If a set of column vectors of U is independent, so is the corresponding set of columns of A.
- The columns with pivots in U constitute a basis for $\mathcal{C}(U)$. It follows that the corresponding columns of A also constitute a basis for $\mathcal{C}(A)$.
- Note $\mathcal{C}(A) \neq \mathcal{C}(U)$.

Rank and Dimension

- From the above discussion, we see, for a matrix A

$$\dim \mathcal{C}(A) = \dim \mathcal{R}(A) = \text{rank of } A$$

- They are all equal to the number of pivots in U.
- Rank has been defined computationally, to be the number of basic variables.
- Here we see another meaning of rank: the maximum number of linearly independent rows (or columns) of a matrix.

Basis for Left Nullspace

- The left nullspace of A equals the nullspace of A^T,

$$y^T A = 0 \Leftrightarrow A^T y = 0.$$

- The column space of A^T has the same dimension as the row space of A, which is r.
- It follows that the dimension of nullspace of A^T is $m - r$, the same as that of the left nullspace of A.
- The last $m - r$ rows of L^{-1} constitute a basis since they are independent and they are left null-vectors of A as

$$L^{-1} A = U.$$

Fundamental Theorem, Part I

- The dimension of the column space of a matrix A equals the rank of A, say r.
- The dimension of the nullspace of A is $n - r$.
- The dimension of the row space of A is r.
- The dimension of the left nullspace of A is $m - r$.

Left and Right Inverses

- A matrix A has a left inverse if there exists a matrix B such that $BA = I_n$.
- A matrix A has a right inverse if there exists a matrix C such that $AC = I_m$.
- The existence of left and right inverses is related to the rank of a matrix: we will see that an inverse exists if and only if the rank is as large as possible (m or n).

Theorems

- (throrem) A has a right inverse if and only if $r = m$.
- (proof) If $r = m$, then the columns of A spans R^m, so the columns of C for $AC = I_m$ can be solved. Conversely, if $AC = I_m$, then columns of A spans R^m, so $r = m$.
- (theorem) A has a left inverse if and only if $r = n$.
- (proof) A has a left inverse if and only if A^T has a right inverse. So the rank of A^T is n. The theorem follows since the rank of A^T is the same as the rank of A.

Existence and Rank

- The existence and uniqueness of the solution of a system of linear equations $Ax = b$ is related to the rank and the inverses of the coefficient matrix A.
- (existence theorem) $Ax = b$ has at least one solution for every b if and only if A has a right inverse.
- (proof) $r = m$. Columns of A span R^m.

Uniqueness and Rank

- (uniqueness theorem) $Ax = b$ has at most one solution for every b if and only if A has a left inverse.
- (proof) $r = n$. Columns of A are linearly independent.
- It can be verified that $(A^T A)^{-1} A^T$ and $A^T (A A^T)^{-1}$ are left and right inverses, when they exist.

Non-singularity

- An $n \times n$ square matrix A is non-singular if and only if any of the following equivalent conditions is true
  - the columns (rows) span R^n
  - the columns (rows) are linearly independent
  - a full set of pivots exists after elimination
  - A is invertible, i.e., there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$.
- This list of necessary and sufficient conditions for non-singularity will grow as we continue on linear algebra.

Matrix of Rank One

- The smallest rank for a non-zero matrix is 1.
- If a matrix A is of rank 1, every row is a multiple of the first non-zero row.
- Similarly, every column must be a multiple of the first non-zero column.
- It follows that a rank-1 matrix A has the simple form

$$A = uv^T, \text{ where } u \text{ is } m \times 1, v \text{ is } n \times 1.$$

- The columns are multiples of u and the rows are multiples of v^T.

Edge-Node Incidence Matrix

- A graph with m directed edges and n nodes can be represented by an $m \times n$ matrix A, where

$$A_{ij} = \begin{cases} 1, & \text{if edge } i \text{ ends at node } j \\ -1, & \text{if edge } i \text{ starts at node } j \\ 0, & \text{otherwise} \end{cases}$$

- Each row corresponds to an edge and each column corresponds to a node.
- In each row, one entry is 1 and one entry is -1. The rest are 0.

Example

- The following matrix defines a graph.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

- Can you draw it?

Nullspace of A

- Is there a non-zero solution to $Ax = 0$?
- That is, are the columns linearly dependent?
- Since the row sum is zero for each row, the sum of columns is the zero vector, and a solution is

$$x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}'.$$

- Given $b_1, \ldots, b_5$ as the potential differences across edges, we want to find the potentials at the nodes. This is the same as solving $Ax = b$, where x_j is the potential at node j.
- Adding a constant to a solution remains a solution.

Column Space of A

- For what $b_1, \ldots, b_5$ can we solve $Ax = b$?
- For each column, the sum of the first and second entries equals the third entry: $b_3 = b_1 + b_2$ for b to be in the column space of A.
- We are given potential differences $b_1, \ldots, b_5$ and we want to find x to satisfy these differences.
- The potential difference b_3 must be equal to the sum of b_1 and b_2 for $Ax = b$ to have a solution x.
- Similarly $b_5 = b_3 + b_4$.

Left Nullspace of A

- $y^T A = 0$: What combination (y) of the rows of A gives the zero vector?
- From the previous discussion, two solutions are

$$y_1^T = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \end{bmatrix}, \; y_2^T = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

- y_1, y_2 correspond to loops in the graph. That is, each loop produces a vector y in the left nullspace.
- A component of y indicates whether the edge is in the same direction as the loop.
- Here we see a relation between the column space and the left nullspace: for $b \in \mathcal{C}(A), y^T b = 0$.

Row Space of A

- Since the row sum is zero for each row, a vector in the row space must satisfy

$$f_1 + f_2 + f_3 + f_4 = 0.$$

- From above and previous discussion, $f^T x = 0$ for x in the nullspace and f in the row space.

Rows of Incidence Matrix

- With an elimination step on an edge-node incidence matrix, the row sum is 0 and there are two non-zero elements: it is still an edge-node incidence matrix.
- The graph do change with the incidence matrix. For example,

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

- Draw the graphs to get the idea.

Spanning Tree

- A tree is a graph with no loops.
- A spanning tree of a connected graph is a tree that contains all nodes in the graph.
- A spanning tree of a graph of n nodes has $n-1$ edges.
- $n-1$ independent rows for the edge-node incidence matrix for a connected graph: the rank is $n-1$.
- It follows that the dimension of the nullspace is 1 and the dimension of the left nullspace is $m-n+1$.

Block Matrices

- We can partition a matrix into blocks, each block being a matrix.
- In an $I \times J$ block matrix, there are I blocks in each column of blocks (block column) and J blocks in each row.
- The numbers of rows of matrices in a block row must agree. Similarly for the numbers of columns of matrices in a block column.
- The number of columns and rows are not specified, so there are many ways to partition a given matrix into a block matrix.

Block Matrice Multiplication

- Let A be partitioned into an $I \times J$ block matrix, with blocks A_{ij}. Let B be partitioned into a $J \times K$ block matrix, with blocks B_{jk}.
- The multiplication AB is an $I \times K$ block matrix C, following the same formula as treating the blocks as scalar entries, i.e.,

$$AB = \begin{bmatrix} C_{11} & \ldots & C_{1K} \\ \vdots & \ddots & \vdots \\ C_{I1} & \ldots & C_{IK} \end{bmatrix}, \text{ where } C_{ik} = \sum_{j=1}^{J} A_{ij} B_{jk}.$$

Linear Transformations

- Multiplication by a matrix A from left transforms a vector x of dimension n into another vector of dimension m in the column space of A.
- That is,

$$x \longrightarrow Ax.$$

- This transformation is linear in the sense that

$$A(cx + dy) = c(Ax) + d(Ay).$$

Examples of L. T.

- 90° rotation counterclockwise

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- reflection against $x = y$

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- projection to the x axis

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

L. T. on Basis

- Ax represents an L. T. of x.
- How about the opposite direction: does every L. T. lead to a matrix? The answer is a resounding yes.
- If we know the result of T on basis vectors, then we know the result of T on any vectors

$$T(x_i) = y_i,\ x = \sum_i c_i x_i \Rightarrow T(x) = \sum_i c_i y_i.$$

- All we need to know about an L. T. T is its effect on a basis.

Matrix Representation for L. T.

- Suppose T maps vectors in V with basis $\{x_1, \ldots, x_n\}$ to W, with basis $\{z_1, \ldots, z_m\}$.

$$Tx_j = y_j = a_{1j}z_1 + a_{2j}z_2 + \cdots + a_{mj}z_m = \sum_{i=1}^{m} a_{ij}z_i.$$

- Consider $x = \sum_{j=1}^{n} d_j x_j$.

$$y = T(x) = \sum_{j=1}^{n} d_j y_j = \sum_{j=1}^{n} d_j \sum_{i=1}^{m} a_{ij} z_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} d_j \right) z_i$$

$$= \sum_{i=1}^{m} c_i z_i \quad \Rightarrow c_i = \sum_{j=1}^{n} a_{ij} d_j, \text{ or } c = Ad.$$

Examples

- Rotation by an angle θ.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}\Rightarrow Q_\theta=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

- Projection on the θ-line.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}\cos^2\theta\\ \cos\theta\sin\theta\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}\sin\theta\cos\theta\\ \sin^\theta\end{bmatrix}\Rightarrow P_\theta=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$$

- Reflection with the θ-line.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\cos^2\theta-1\\ 2\cos\theta\sin\theta\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2\sin\theta\cos\theta\\ 2\sin^2\theta-1\end{bmatrix}\Rightarrow H_\theta=?$$

Composite L. T.

- If an L. T. can be decomposed into a sequence of L. T., then the matrix for the L. T. is the product of the matrices for the component L. T. That is, if T is defined by T_1 followed by T_2, then

$$M(T) = M(T_2)M(T_1).$$

- This is quite obvious from the perspective of matrix representation.

Examples

- rotation of a vector by angle θ followed by ϕ

$$Q_{\theta+\phi} = Q_\phi Q_\theta$$

- repeated projections

$$P_\theta P_\theta = P_\theta$$

- repeated reflections

$$H_\theta H_\theta = I$$

- They are correct algebraically and geometrically.