Outline

- What is linear algebra?
- systems of linear equations
- row and column pictures
- Gauss elimination
- matrix notation
- triangular factorization (LU decomposition)
- inverses and transposes

systems of linear equations

- n equations and n unknowns
- linear in the unknowns
- cases of solutions
  - non-singular: unique solution
  - singular case a: no solution
  - singular case b: infinitely many solution

row and column pictures

- row picture
  - each row is a plane (a line in 2-D)
  - solution is the intersection set of these planes
- column picture
  - each column is a vector
  - solution is the right linear combination
- What is the geometrical meaning of the singular cases?

Gauss elimination

- an example (p.12)
- row operations
- back substitution
- when elimination fails
  - no pivots in a column
- cost
  - row operations
  - back substitution

matrix multiplication

- the inner product of two vectors
- multiplication of a matrix A and a vector x
  - inner product between the rows of A and x
  - $Ax =$ a linear combination of the columns of A
  - summation formula $(Ax)_i = \sum_j a_{ij} x_j$
- multiplication of two matrices B and C
  - $(BC)_{ij}$ = (row i of B) $\cdot$ (column j of C)
  - column j of BC = B $\cdot$ (column j of C)
  - row i of BC = (row i of B) $\cdot$ C
  - block multiplication
- $A(B + C) = AB + AC$; $EF \neq FE$ in general

special matrices

- identity matrix, I
- zero matrix
- diagonal matrix, D
- triangular matrices
  - lower-triangular, L
  - upper-triangular, U

triangular factorization

- the elimination matrices E_{ij}
- elimination = applying a sequence of E_{ij} to A
- $Ax = b \longrightarrow Ux = c$ where U is upper-triangular
- inverse of elimination matrix
- Gauss elimination = triangular factorization: $A = E^{-1}U = LU$, assuming no row exchange required
- the entries in L are exactly the multipliers in the elimination process.
- solve $L\underline{c} = b$ (forward substitution), and then $U\underline{x} = c$ (backward)
- $A = LDU$: further factorize U into a diagonal matrix and an upper-triangular matrix with 1's on the diagonal, if A is invertible

row exchanges and permutation matrices

- P = a permutation of rows of identity matrix
- PA: the corresponding permutation of rows of A
- How many distinct $n \times n$ permutation matrices are there?
- What is P^{-1}?

inverses

- definition: A is invertible if there exists B such that $AB = BA = I$. B is called the inverse of A, denoted by A^{-1}.
- what is the inverse of the sum of two invertible matrices?
- what is the inverse of the product of two invertible matrices?
- what is the inverse of A^{-1}?
- how to calculate A^{-1} for given A? (Gauss-Jordan)
- invertible = non-singular (full number of pivots)
  - non-singular $\longrightarrow$ invertible
  - singular $\longrightarrow$ non-invertible

transposes

- definition
- what is the transpose of the sum of two matrices?
- what is the transpose of the product of two matrices?
- what is the transpose of A^T?
- a matrix A is symmetric if $A^T = A$.

further properties

- If $BA = I = AC$, then $B = C$
- If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$.
- If A is symmetric and invertible, then A^{-1} is also symmetric
- If A is lower-triangular and invertible, A^{-1} is also lower-triangular
- If $A = L_1U_1 = L_2U_2$, where the L's are unit-triangular and the U's have non-zero diagonal entries, then $L_1 = L_2$ and $U_1 = U_2$
- If A is symmetric and $A = LDU$, then $U = L^T$