Final of Linear Algebra

1. Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} = LU,$$

   find the elementary matrices $E_{1,2,3}$ such that $E_3E_2E_1A = U$. (Hint: an elementary matrix corresponds to a row operation.)

   solution:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

2. Given a set of vectors $V = \{x \in R^6 \mid x_1 + x_3 = 0, x_2 - x_5 = 0\}$.

   (a) Show that V a vector subspace in R^6.

   (b) Give a basis for $V^\perp$, the orthogonal complement of V.

   solution:

   (a) Need to show that for any $x, y \in V, x + y \in V$ and $cx \in V$.

   (b) $\{[1, 0, 1, 0, 0, 0], [0, 1, 0, 0, -1, 0]\}$.

3. Find the best approximation to the function $f(x) = 1$ by a linear combination of $\sin x$ and $\cos x$ in the interval $(0, \frac{\pi}{4})$.

   solution: $f(x) \sim a \sin x + b \cos x$, where a, b is determined by

$$\begin{bmatrix} \frac{\pi-2}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{\pi+2}{8} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix}.$$

4. Find the QR factorization for matrix

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 2 & 0 & -2 \end{bmatrix}.$$

solution:

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} \end{bmatrix}, R = \begin{bmatrix} 3 & 2 & \frac{1}{3} \\ 0 & 2 & \frac{8}{3} \\ 0 & 0 & \frac{4}{3} \end{bmatrix},$$

5. Suppose we have a 10×10 matrix W, where

$$w_{ij} = 2\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}, \text{ where } \delta_{m,n} = \begin{cases} 1, & m = n \\ 0, & m \neq n. \end{cases}$$

Compute the determinant of W.

solution: $|W| = 11$. See p.60 of text.

6. (20%) Find the singular value decomposition $F = U\Sigma V^T$, where

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Make sure that a larger singular value appears higher in Σ.

solution: U is the eigenvector matrix of FF^T, Σ contains the singular values and V is the eigenvector matrix of F^TF. Note that the eigenvectors are related by $u_j = \frac{Fv_j}{\sigma_j}$, so one needs not solve both eigenvalue problems.

$$U = \begin{bmatrix} -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}.$$

7. Find the projection of b to the column space of A, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

solution:

$$A^TA\bar{x} = A^Tb \Rightarrow \bar{x} = \begin{bmatrix} \frac{5}{9} \\ \frac{1}{3} \end{bmatrix} \Rightarrow p = A\bar{x} = \begin{bmatrix} \frac{11}{9} \\ \frac{16}{9} \\ \frac{10}{9} \end{bmatrix}.$$

8. Express matrix Z as the sum of two matrices of rank 1.

$$Z = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

solution: The eigenvalues are $0, 3, 3$. Orthonormal eigenvectors for 3 are

$$\begin{bmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{6}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}, \text{ so } Z = 3 \begin{bmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \\ 0 \end{bmatrix}^T + 3 \begin{bmatrix} \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{6}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{6}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

9. Find the stationary distribution of the Markov process with the following transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.6 & 0 \\ 0.2 & 0 & 0.8 \end{bmatrix}.$$

solution: Solving $Px = x$,

$$x = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$