Chapter 5: Eigenvalues and Eigenvectors

- Eigenvalue Problems
- Diagonal Form
- Difference Equations and A^k
- Differential Equations and e^{At}
- Complex Matrices
- Similarity Transformations

Eigenvalue Problems

The Old and The New

- Old problem: to find the solutions for a system of linear equations
  - Given A and b, find x such that $Ax = b$.
  - Operations on A without affecting the solutions OK.
- New problem: to find the eigenvectors and eigenvalues of a matrix
  - Given A, find λ and non-zero x such that $Ax = \lambda x$
  - The row operations are no longer useful since such alter the eigenvalues and eigenvectors. The new useful operation is called diagonalization.

How to Solve an Eigenvalue Problem?

- Given A, find λ and non-zero x such that

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$$

We are not interested in 0 as an eigenvector.

- In order to have a non-zero x for the above equation, the matrix $A - \lambda I$ must be singular. Therefore

$$\det(A - \lambda I) = 0.$$

From this equation we solve for λ's and then solve for the corresponding x's.

Example: Diagonal Matrix

$$Ax = \lambda x, \text{ where } A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix} = (3 - \lambda)(2 - \lambda) = 0$$

$$\Rightarrow \lambda_1 = 3, x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \;\; \lambda_2 = 2, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example: Projection Matrix

$$Px = \lambda x, \text{ where } P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(P - \lambda I) = \det \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{bmatrix} = \lambda^2 - \lambda = 0$$

$$\Rightarrow \lambda_1 = 1, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 0, x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A projection matrix always has eigenvalues of 0 or 1.

Properties of the Eigenvalues

- $\sum_i \lambda_i = \sum_i a_{ii} \triangleq tr(A)$, called the trace of A. This can be seen by comparing the coefficients of $(-\lambda)^{n-1}$ on both sides of

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn}-\lambda \end{bmatrix} = \prod_i (\lambda_i - \lambda).$$

- $\prod_i \lambda_i = \det\ A$. This can be seen by setting λ to 0 in

$$\det(A - \lambda I) = \prod_i (\lambda_i - \lambda).$$

Diagonal Form of a Matrix

The Eigenvector Matrix

Suppose A has n linearly independent eigenvectors. Let

$$S \triangleq [u_1| \dots |u_n],$$

then

$$AS = A[u_1| \dots |u_n] = [\lambda_1 u_1| \dots |\lambda_n u_n] = S\Lambda,$$

where Λ is the diagonal matrix formed by the eigenvalues. We say that S is the eigenvector matrix and that A is diagonalizable (by S). The above equation is equivalent to

$$A = S\Lambda S^{-1}, \;\; \Lambda = S^{-1}AS.$$

Note that A and Λ have the same eigenvalues.

Notes

- A matrix with distinct eigenvalues is diagonalizable, since the corresponding eigenvectors must be linearly independent.

$$\begin{aligned} & c_1x_1 + c_2x_2 = 0 \\ & \Rightarrow A(c_1x_1 + c_2x_2) = 0,\ \lambda_1(c_1x_1 + c_2x_2) = 0 \\ & \Rightarrow c_2(\lambda_1 - \lambda_2)x_2 = 0 \Rightarrow c_2 = 0. \end{aligned}$$

- S is not unique
- S must be an eigenvector matrix
- There are "defective matrices" which do not have n linearly independent eigenvectors.

Diagonalization Examples

1. $$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

2. $$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, (\lambda = \pm i, x = \begin{bmatrix} 1 \\ \mp i \end{bmatrix})$$

$$\Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

Diagonalization of Power of A

- If x is a eigenvector of A, then it is an eigenvector of A^2.

$$A^2 x = A(Ax) = A\lambda x = \lambda A x = \lambda^2 x.$$

  Therefore, A^2 can be diagonalized by the eigenvector matrix of A.

- This fact is also indicated by the following identity,

$$A^k = (S\Lambda S^{-1})^k = S\Lambda^k S^{-1} = S \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} S^{-1}.$$

Diagonalization of Product

Suppose A and B are diagonalizable. A and B share the same eigenvector matrix iff A, B commutes, i.e.,

$$AB = BA.$$

("if")

$$Ax = \lambda x \Rightarrow ABx = BAx = \lambda Bx \Rightarrow Bx = \mu x.$$

("only if")

$$\begin{aligned} AB &= S\Lambda_A S^{-1} S\Lambda_B S^{-1} = S\Lambda_A \Lambda_B S^{-1} \\ &= S\Lambda_B \Lambda_A S^{-1} = S\Lambda_B S^{-1} S\Lambda_A S^{-1} = BA. \end{aligned}$$

Difference Equations

Fibonacci Sequence I

$$F_{k+2} = F_{k+1} + F_k, \quad 0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

Let

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \Rightarrow u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k = Au_k.$$

If the matrix A is diagonalizable with S, then

$$u_k = A^k u_0 = S\Lambda^k S^{-1} u_0 = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2,$$

where $c = S^{-1}u_0$. To find S, we need to solve the eigenvalue problem for A.

Fibonacci Sequence II

The eigenvalues and eigenvectors of A are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}; \quad x_{1,2} = \begin{bmatrix} \lambda_{1,2} \\ 1 \end{bmatrix}$$

It follows that

$$\begin{aligned} u_k &= S\Lambda^k S^{-1} u_0 \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \\ & \lambda_2^k \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \\ \frac{1}{\lambda_2 - \lambda_1} \end{bmatrix} \\ \Rightarrow F_k &= \frac{1}{\lambda_1 - \lambda_2}(\lambda_1^k - \lambda_2^k). \end{aligned}$$

Markov Process

Every year, 0.1 of the population outside move in and 0.2 of the population inside move out. Let $y_i(z_i)$ be the fraction of total population outside (inside) at year i, then

$$\begin{bmatrix} y_{i+1} \\ z_{i+1} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} y_i \\ z_i \end{bmatrix} = A \begin{bmatrix} y_i \\ z_i \end{bmatrix}$$

The matrix A has non-negative entries and the entries in any column sum to 1. It is called the transition probability matrix (t.p.m.). Note that $y_i + z_i = 1$ for all i.

Properties of Transition Probability Matrices

Suppose that A is a t.p.m., then

- If A is a t.p.m., so is A^k.
- 1 is an eigenvalue of A, since $\det(A - I) = 0$;
- The eigenvector $Ax = 1\,x$ is stationary: it stays the same when repeatedly multiplied by A from the left.
- $|\lambda_i| \leq 1$ for other eigenvalues;
- If $A^k > 0$ for some $k > 0$, then for any y_0, z_0

$$\lim_{n\to\infty} A^n \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = cx_1.$$

Differential Equations

An Example with Differential Equation

$$\frac{du}{dt} = Au = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$$

Solving the eigenvalue problem for A,

$$\lambda_1 = -1, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \lambda_2 = -3, x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The solution for $u_0 = c_1 x_1 + c_2 x_2$ is

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = S \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution of Differential Equations

Applying the initial condition $u(0) = u_0$, we have

$$\begin{bmatrix} x_1 & \mid & x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = u_0, \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} u_0,$$

$$\Rightarrow u = S \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} c = S e^{\Lambda t} S^{-1} u_0 = e^{At} u_0.$$

Some mathematical details are given in the next slide.

Exponential of a Matrix

The exponential of a matrix is defined by $e^M = \sum_{i=0}^{\infty} \frac{M^i}{i!}$. Therefore,

$$e^{\Lambda t} = \sum_{i=0}^{\infty} \frac{(\Lambda t)^i}{i!} = \sum_{i=0}^{\infty} \frac{\begin{bmatrix} (\lambda_1 t)^i & \\ & (\lambda_2 t)^i \end{bmatrix}}{i!} = \sum_{i=0}^{\infty} \begin{bmatrix} \frac{(\lambda_1 t)^i}{i!} & \\ & \frac{(\lambda_2 t)^i}{i!} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_i \frac{(\lambda_1 t)^i}{i!} & \\ & \sum_i \frac{(\lambda_2 t)^i}{i!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix}.$$

$$S e^{\Lambda t} S^{-1} = \sum_{i=0}^{\infty} \frac{S(\Lambda t)^i S^{-1}}{i!} = \sum_{i=0}^{\infty} \frac{S(\Lambda t)S^{-1}S(\Lambda t)S^{-1} \ldots S(\Lambda t)S^{-1}}{i!}$$

$$= \sum_{i=0}^{\infty} \frac{(At)^i}{i!} = e^{At}.$$

Complex Matrices

Complex Numbers

A complex number z is defined by

$$z = a + ib$$

where a, b are real numbers and i is the square root of -1.

- addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$
- multiplication: $(a + ib)(c + id) = (ac - bd) + i(bc + ad)$
- complex conjugate: $\overline{a + ib} = a - ib$
- absolute value: $|(a + ib)|^2 = a^2 + b^2$

Complex Vectors and Matrices

- A complex vector has entries of complex numbers.
  - inner product: $(x, y) = \bar{x}^T y$
  - length: $||x||^2 = (x, x)$
  - orthogonal: x, y are orthogonal if $(x, y) = 0$
- A complex matrix has entries of complex numbers. The complex conjugate of a matrix is defined by complex-conjugating the entries.

Inner Products and Hermitians

- The Hermitian (as a *noun*) of a matrix A is defined by the transpose of the complex conjugate of A.

$$A^H = \bar{A}^T.$$

- The following properties for the inner products are true.
  - $(x, y) = \overline{(y, x)}$
  - $(Ax, y) = (x, A^H y)$

Hermitian Matrices

A matrix A is Hermitian (as an *adjective*) if $A = A^H$. If A is Hermitian, the following properties are true.

- $x^H Ax$ is real for any x.
  $(x, Ax) = (x, A^H x) = (Ax, x) = \overline{(x, Ax)}$.
- The eigenvalues of A are real. $(x, Ax) = \lambda(x, x)$.
- Two eigenvectors of A are orthogonal if they correspond to distinct eigenvalues.
  $(Ax_1, x_2) = (x_1, Ax_2) \Rightarrow (\overline{\lambda_1} - \lambda_2)(x_1, x_2) = 0$.

Spectral Theorem

- A real symmetric matrix is Hermitian, so the eigenvalues are real. Furthermore, the eigenvectors are real since they are solutions to $Ax = \lambda x$.
- We can further normalize the eigenvectors to unit length, therefore, $A = Q\Lambda Q^T$, where Q is an orthogonal matrix.
- (Spectral theorem) A real symmetric matrix can be written as
$$A = \sum_i \lambda_i x_i x_i^T,$$
where λ_i, x_i are the ith eigenvalue and eigenvector.

Unitary Matrices

- A unitary matrix is a complex matrix with orthonormal column vectors. I.e., $U^H U = I$.
- If U is unitary, then
  - $||Ux|| = ||x||$. $(Ux, Ux) = x^H U^H U x = x^H x$.
  - the eigenvalues satisfy $|\lambda| = 1$.
    $||x|| = ||Ux|| = ||\lambda x|| = |\lambda| ||x||$.
  - Eigenvectors of U corresponding to different eigenvalues are orthogonal.
    $(Ux_1, Ux_2) = (x_1, U^H U x_2) \Rightarrow (1 - \overline{\lambda_1}\lambda_2)(x_1, x_2) = 0$

Skew-Hermitian

- A matrix is skew-symmetric if

$$K^H = -K.$$

- If A is Hermitian, then iA is skew-symmetric.

$$(iA)^H = -iA^H = -(iA).$$

  A and iA share the same set of eigenvectors.

- The eigenvalues of a skew-Hermitian matrix are pure imaginary.

Similarity Transforms

Definition of Similarity Transform

- Let A be a square matrix and M be invertible, then

  A and $M^{-1}AM$ are said to be similar and

  $A \to B = M^{-1}AM$ is called similarity transform

- We have seen a few examples:
  - $\Lambda = S^{-1}AS$
  - $\Lambda = Q^TAQ$
  - $\Lambda = U^HAU$

Questions about Similarity Transform

Let A and B be similar via M, i.e.

$$B = M^{-1}AM.$$

- What do A and B have in common?
  (the same eigenvalues)

- How can M be chosen to make B have specific forms?
  (Jordan form)

Eigenvalues of Similar Matrices

- The eigenvalues of two similar matrices A, B are the same.
  - If x is an eigenvector of A, then $M^{-1}x$ is an eigenvector of B.
    $Ax = \lambda x \Rightarrow MBM^{-1}x = \lambda x \Rightarrow BM^{-1}x = \lambda M^{-1}x$
  - The characteristic equations for A and B have the same roots.

$$\det(A - \lambda I) = 0 \Leftrightarrow \det(MBM^{-1} - \lambda I) = 0$$
$$\Leftrightarrow \det(B - \lambda I) = 0$$

Examples of Similar Matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$M_1 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \ M_1^{-1} = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix};$$

$$M_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \ M_2^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Change of Basis = Similarity Transform

- linear transform = matrix multiplication
- the matrix representation depends on the basis
- change of basis = similarity transform
- Let $T(v_j) = \sum A_{ij} v_i$, $V_j = \sum M_{ij} v_i$, i.e. $(V = vM, v = VM^{-1})$, then

$$T(V_j) = T(\sum_k M_{kj} v_k) = \sum_k M_{kj} T(v_k) = \sum_{k,l} M_{kj} A_{lk} v_l$$

$$= \sum_{k,l,i} M_{kj} A_{lk} M^{-1}_{il} V_i \triangleq \sum_i B_{ij} V_i,$$

$$\Rightarrow B = M^{-1} A M.$$

Examples of Change of Basis

$$
\begin{aligned}
v_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
V_1 &= \cos\theta\, v_1 + \sin\theta\, v_2, \\
V_2 &= -\sin\theta\, v_1 + \cos\theta\, v_2, \\
M &= \begin{bmatrix} c & -s \\ s & c \end{bmatrix},\ M^{-1} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \\
B(\text{in } V) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A(\text{in } v) = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}
\end{aligned}
$$

Triangular Similar Matrix via Unitary M

- Given A, we want to find M such that $M^{-1}AM$ is triangular. (The eigenvalues are trivial to find in this case.)
- (Schur's Lemma) For any square matrix A there exists a unitary matrix U such that $T = U^{-1}AU$ is upper-triangular. T and A have the same eigenvalues which appear along the main diagonal of T.

Proof of Schur's Lemma

A matrix A has at least one eigenvalue with non-zero eigenvector, say λ_1 and x_1. We build a unitary matrix U_1 by using x_1 as the first column, and apply Gram-Schmidt process for the other columns. At step i we choose U_i to be

$$U_i = \begin{bmatrix} I_{i-1} & 0 \\ 0 & M_i \end{bmatrix},$$

where the first column vector of M_i is the eigenvector of the lower-right submatrix (Z_i) in

$$U_{i-1}^{-1} \dots U_1^{-1} A U_1 \dots U_{i-1} = \begin{bmatrix} T_{i-1} & * \\ 0 & Z_i \end{bmatrix}.$$

Constructing a Triangular Similar Matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 1, x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow U_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \Rightarrow U_1^{-1} A U_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Diagonalizing Hermitian Matrices

- Any Hermitian (including real symmetric) matrix has a complete set of orthonormal eigenvectors.
- This follows from:
  - If A is Hermitian, so is $U^{-1}AU$.
  - If a Hermitian matrix is triangular, it must be diagonal.
  - So the unitary matrix resulted from the construction of triangular similar matrix is actually an eigenvector matrix of A, since it diagonalizes A.

Spectral Decomposition

- Any symmetric matrix can be written as

$$A = \sum_i \lambda_i P_i$$

  where

  - $\sum P_i = I$ (complete)
  - $P_i P_j = 0 \quad \forall i \neq j$ (orthogonality)

- P_i is the projection matrix to the eigenspace of A corresponding to λ_i.

- Example of p.310

Normal Matrices

- (definition) A matrix is normal if it commutes with its Hermitian, i.e.

$$NN^H = N^H N.$$

  - unitary matrices ($U^H U = I = UU^H$)
  - Hermitian matrices ($A^H A = A^2 = AA^H$)

- (theorem) Normal matrices have complete set of orthonormal eigenvectors.
  - If N is normal, so is $T = U^{-1}NU$
  - A triangular matrix that is normal must be diagonal. ($||N^H x|| = ||Nx||$, setting $x = e_i, i = 1, \ldots, n$ to show the off-diagonal entries are 0)

The Jordan Form

- If a matrix A has s independent eigenvectors, it is similar to a matrix with s blocks:

$$J = M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix},$$

where each Jordan block J_i is a triangular matrix with one eigenvalue λ_i and an independent eigenvector:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}.$$

When a Jordan block has order $m > 1$, the eigenvalues λ_i is repeated m times and there are $m - 1$ $1's$ above the diagonal. The same λ_i may appear in different blocks if it corresponds to several independent eigenvectors. Two matrices are similar if they have the same Jordan form.

Examples

1. $\lambda = 1$ is a double root with one eigenvector

$$T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

2. $\lambda = 0$ is a triple root

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are three possible Jordan forms. A has one eigenvector while B has 2.

Powers/Exponentials with Jordan Form

Let J_i be a Jordan block of a triple eigenvalue λ,

$$J_i^n = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & n(n-1)\lambda^{n-2}/2 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

$$e^{J_i t} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & t^2 e^{\lambda t}/2 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

These can be used in difference and differential equations.