5.1

$\lambda = 0, 1$ $trace(A) = 0 + 1 = 1$ and $det(A) = 0 * 1 = 0$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$$

5.5

$\lambda_1 = 4$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 2$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$u(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

if $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2} \Rightarrow u(t) = \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

if $u(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{-1}{2}$ and $c_2 = \frac{1}{2} \Rightarrow u(t) = \frac{-1}{2} e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $e^{At} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

5.10

(1)

Let $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$det(A + cI) = det \begin{bmatrix} x + c & y \\ z & w + c \end{bmatrix} = (x + c)(w + c) - yz$$

$\because$ the front term is complex. Let yz is real then $A + cI$ is nonsingular

(2)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

5.13

$\because P^2 = P$ and $ImP = S$

$\Rightarrow \forall$v in s

$\Rightarrow v = Pw, w \in R^n$

$\Rightarrow v = Pw = PPw = Pv \Rightarrow$ v is eigenvector of P and eigenvalue is 1

5.16
$k^T = -k$

$[(I-k)(I+k)^{-1}]^T(I_k)(I+k)^{-1}$

$(I+k)^{-1^T}(I-k)^T(I-k)(I+k)^{-1} = (I-k)^{-1}(I+k)(I-k)(I+k)^{-1} = I$

$Q = (I-k)(I+k)^{-1} = \frac{1}{5}\begin{bmatrix}-3 & -4\\ 4 & -3\end{bmatrix}$

5.18

(1)

$x(t) = e^{At}x(0)e^{Bt}$
$\frac{d}{dt}x(t) = Ae^{At}x(0)e^{Bt} + e^{At}x(0)e^{Bt}B = Ax(t) + x(t)B$

(2)

The solution is $x(t) = e^{At}x(0)e^{-At}$

$\Rightarrow x(t)$ keep the same eignvalue of all time

5.23
5.24
(1) $A^2 = s\Lambda s^{-1} = -I \Rightarrow \Lambda^2 = -I$
$\Rightarrow \lambda = i$
(2)
If A is real
$\because det(A^2) = {\lambda_1}^2{\lambda_2}^2 \vdots {\lambda_n}^2 = det(-I) = (-1)^n$
$\forall {\lambda_i}^2 \geq 0$
$\therefore$ n is even

5.26

$$S^TS = \frac{1}{2}\begin{bmatrix} \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \\ 1 & 0 & -1 \\ \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \\ 1 & 0 & -1 \\ \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \end{bmatrix} = I$$

$S^T = S^{-1}$

5.30

$Ax = \lambda x \Rightarrow (Ax)^T = (\lambda_1 x)^T \Rightarrow x^T A^T = \lambda_1 x^T$
$\lambda_1 x^T y = x^T A^T y = x^T \lambda_2 y = \lambda_2 x^T y$

$\because \lambda_1 \neq \lambda_2 \Rightarrow (\lambda_1 - \lambda_2) x^T y = 0$

$x^T y = 0$