Determinants

Notes on Linear Algebra

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Introduction

- The determinant of a matrix is a scalar
  - very limited information
  - for square matrices only
- If 0, it does say quite a bit about the matrix.
- It provides an explicit formula for A^{-1}, giving a closed-form solution for $Ax = b$.
- It used to be an interesting and important subject in linear algebra.

What Determinants Can Do

- The determinant gives a test for invertibility for matrices

$$|A| = 0 \text{ if and only if } A \text{ is singular.}$$

  - This property is used in the treatment of eigenvalue problems.
- It gives a formula for pivots.
- It gives a formula for each component of $A^{-1}b$.

Formulas and Properties

- One can give an explicit formula for determinants.
- However, the real simple things about determinants are the properties they have.
- Usually, a treatment of determinants starts with a formula as definition and then derives the properties.
- Here we start with the properties and then derive some formulas.

Axiom I

- (A1) *The determinant of an identity matrix is* 1.

$$|I_n| = 1, \;\; n = 1, 2, \ldots$$

- For examples,

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

- Note that we use vertical bars for to represent a determinant.

Axiom II

- (A2) *The determinant of a matrix changes sign if two rows are exchanged.*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}.$$

- For a permutation matrix P, $|P| = \pm 1$, since we can turn P into an identity matrix by row exchanges.

Axiom III

- (A3) *The determinant of a matrix depends linearly on the first row.*
- That is, suppose we have two matrices that differ only in the first rows, then

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Derived Properties

With the above axioms, the following properties are true.

- If two rows of A are equal, then $|A| = 0$.
- Subtracting a multiple of one row from another (as in elimination) leaves the determinant unchanged.
- If A has a zero row, then $|A| = 0$.
- If A is triangular, then $|A|$ is the product of entries on the main diagonal.
- A is non-singular (full set of pivots) iff $|A| \neq 0$.
- $|AB| = |A|\,|B|$.
- $|A^T| = |A|$.

Equal Rows

- (D4) *If two rows of A are identical, then* $|A| = 0$.
- (proof) Exchange the two identical rows of A to get B. Then from (A2),

$$|B| = -|A|.$$

But $B = A$ (as the rows are identical) so

$$|B| = |A|.$$

It follows that

$$|A| = -|A| = 0.$$

Row Operation

- (D5) *Subtracting a multiple of one row from another row leaves the determinant unchanged.*
- (proof) With (A3),

$$\begin{vmatrix} a - lc & b - ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} lc & ld \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} c & d \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Zero Row

- (D6) *If* A *has a zero row, then* $|A| = 0$.
- (proof) Add another row of A to the zero row to get B. Then from (D5),

$$|B| = |A|.$$

But B has two identical rows, so from (D4)

$$|B| = 0.$$

It follows that

$$|A| = 0.$$

Triangular Matrices

- (D7) *If A is triangular, then* $|A| = \prod_i a_{ii}$.
- (proof) If all diagonal entries are non-zero, then elimination (consisting of row operations) can make all off-diagonal entries 0 without changing the determinant. Eventually we get a diagonal matrix D of the a_{ii}'s. The determinant of D is

$$|D| = a_{11} \ldots a_{nn} \, |I_n| = \prod_i a_{ii}.$$

If any diagonal entry is zero, then we get a matrix with a zero row, so the determinant is 0, which still equals $\prod_i a_{ii}$.

Singular Matrices

- (D8) *If A is singular, then $|A| = 0$. If A is invertible, then $|A| \neq 0$.*
- (proof) If A is singular, elimination process leads to a zero row in U, so

$$|A| = \pm|U| = 0.$$

If A is non-singular, elimination process leads to a full set of pivots, and

$$|A| = \pm|U| = \pm\prod_i d_i \neq 0.$$

Product Rule

- (D9) $|AB| = |A||B|$.
- (proof) For $n = 2$, it can be patiently checked that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix}.$$

For a general n, we can prove that

$$\frac{|AB|}{|B|}$$

satisfies the properties A1-A3 for any A, so it must be $|A|$.

Proof of the Product Rule

- The axioms are satisfied as follows

  (A1): $\dfrac{|IB|}{|B|} = \dfrac{|B|}{|B|} = 1$

  (A2): $\dfrac{|AB|}{|B|} = -\dfrac{|A'B|}{|B|}$, since $|A'B| = -|AB|$

  (A3): $\dfrac{|AB|}{|B|} = t_1\dfrac{|A_1B|}{|B|} + t_2\dfrac{|A_2B|}{|B|}$, since $|AB| = t_1|A_1B| + t_2|A_2B$

  where
  - A' is obtained from A by a row exchange
  - the first row of A is a linear combination of the first rows of A_1 and A_2 by t_1, t_2 while the other rows are identical

Transpose Rule

- (D10) $|A^T| = |A|$.
- (proof) If A is singular, then A^T is singular, $|A| = |A^T| = 0$. Otherwise, let $PA = LDU$, where P is a permutation matrix, L is unit-lower-triangular, U is unit-upper-triangular, and D is diagonal. Then

$$|P|\,|A| = |L|\,|D|\,|U| = |D|.$$

$$|A^T|\,|P^T| = |U^T|\,|D^T|\,|L^T| = |D^T| = |D|.$$

The equality follows since

$$PP^T = I \;\Rightarrow\; |P| = |P^T| = \pm 1.$$

Formulas for Determinants

- product of pivots

$$|A| = \pm \prod_i d_i.$$

- sum over of permutations

$$|A| = \sum_{\text{all } P\text{'s}} (a_{1\alpha} a_{2\beta} \ldots a_{n\nu}) |P|.$$

- cofactor expansion

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Determinants by Pivots

- If A is non-singular, then $A = P^T LDU$, and

$$|A| = |P^T|\,|L|\,|D|\,|U| = \pm|D|$$
$$= \pm(\text{product of pivots})$$

  The sign $\pm$ is the determinant of the P^T, or P.

- One can check that for $n = 2$,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & (ad-bc)/a \end{vmatrix} = ad - bc$$

Expansion

- Consider the case of $n = 2$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

- For general n, we can expand $|A|$ into a sum of n^n determinants.
- In each matrix, there are at most n non-zero entries.
- At most $n!$ determinants are non-zero. The other determinants are zero as there are zero columns in the matrices.

Permutation

- For the determinant of a matrix in the above expansion to be non-zero, the non-zero entries $a_{1\alpha}, a_{2\beta}, \ldots, a_{n\nu}$ have to satisfy

$$\alpha \neq \beta \neq \cdots \neq \nu.$$

- $(\alpha, \beta, \ldots, \nu)$ is called a permutation of $(1, \ldots, n)$.
- Now the determinant can be written by

$$|A| = \sum_{\text{all } P\text{'s}} (a_{1\alpha} a_{2\beta} \ldots a_{n\nu}) |P|,$$

where P is the permutation matrix that moves row 1 to row α, row 2 to row β, and so on.

Determinants by Cofactors

- Consider the case of $n = 3$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

- More generally, for $n \times n$ matrix A

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

- This is the cofactor expansion. C_{ij} is the *cofactor* of a_{ij}.

Cofactors and Submatrices

- A submatrix of a matrix is formed by knocking out rows and columns from the matrix.
- In particular, we call M_{1j} the submatrix formed by removing row 1 and column j.
- C_{1j} and M_{1j} are related by

$$C_{1j} = (-1)^{j-1}|M_{1j}|.$$

- $j - 1$ column exchanges are required to bring $a_{1j}C_{1j}$ to a form similar to $a_{11}C_{11}$.

General Cofactor Expansion

- Not limited to row 1, $|A|$ can be expanded along any row.
- The cofactor expansion along row i is

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},$$

  where

$$C_{ij} = (-1)^{i+j}|M_{ij}|.$$

- Since $|A^T| = |A|$, $|A|$ can be expanded along any column as well.

Inverse by Determinants

- Define the cofactor matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}.$$

- We have

$$AC^T = |A|\, I.$$

- It follows that

$$A^{-1} = \frac{1}{|A|} C^T.$$

Cramer's Rule

- The solution of $Ax = b$ can be written by

$$x = A^{-1}b = \frac{1}{|A|}C^T b \;\Rightarrow\; x_j = \frac{|B_j|}{|A|},$$

where

$$B_j = \begin{bmatrix} a_{11} & a_{12} & \ldots b_1 \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots b_n \ldots & a_{nn} \end{bmatrix}.$$

- This can be proved by the cofactor expansion of $|B_j|$ along column j and identifying with the jth component of $C^T b$.

A Formula for Pivots

- If row exchanges are not required, the first k pivots are determined by the upper-left $k \times k$ submatrix A_k of A.
- That is,

$$A = LDU \;\Rightarrow\; A_k = L_k D_k U_k.$$

- This is shown by

$$LDU = \begin{bmatrix} L_k & 0 \\ B & C \end{bmatrix} \begin{bmatrix} D_k & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} U_k & F \\ 0 & G \end{bmatrix} = \begin{bmatrix} L_k D_k U_k & H \\ J & K \end{bmatrix}$$

Pivots by Determinants

- Recall that

$$|A_k| = d_1 d_2 \dots d_k$$

- It follows that the pivot d_k is a ratio of determinants,

$$\frac{|A_k|}{|A_{k-1}|} = \frac{d_1 d_2 \dots d_{k-1} d_k}{d_1 d_2 \dots d_{k-1}} = d_k .$$

- If $|A_k| \neq 0$ for all k, then Gauss elimination can be completed without row exchange.

The Parity of A Permutation

- There are different sequences of (pair-wise) swaps to transform $\sigma = (\alpha, \beta, \ldots, \nu)$ to $\sigma_0 = (1, 2, \ldots, n)$.
- Do all such sequences have the same "parity"?
- Let $N(\sigma)$ be the number of pairs in σ in which the larger number comes first in σ. We show that σ has the same parity as $N(\sigma)$.
- Every swap changes N by an odd number (since swapping neighbors changes N by 1 and swapping non-neighbors is achieved by an odd number of neighbor swapping). So if σ has an odd $N(\sigma)$, then only an odd number of swaps can bring it to σ_0 with $N(\sigma_0) = 0$.