Tree

- *introduction*
- *binary tree*
- *complete binary tree*
  - *max heap, min heap*
  - *Chapter 7 – heap sorting*
  - *Chapter 9 - priority queues*
- *binary search tree(bst)*
- ***AVL tree -*** *Chapter 10 – Efficient BST*

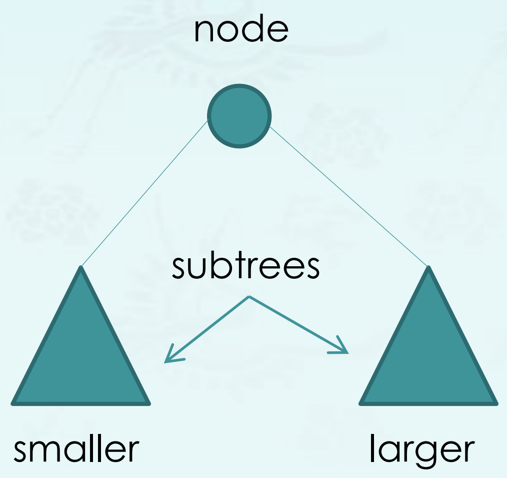
BST

- **Definition: A binary search tree is a binary tree in symmetric order.**

- A binary tree is either
  - empty
  - a key-value pair and two binary trees [neither of which contain that key]

equal keys ruled out

- Symmetric order means that
  - every node has a key
  - every node's key is
    larger than all keys in its left subtree
    smaller than all keys in its right subtree



BST

- **Definition: A binary search tree is a binary tree in symmetric order.**

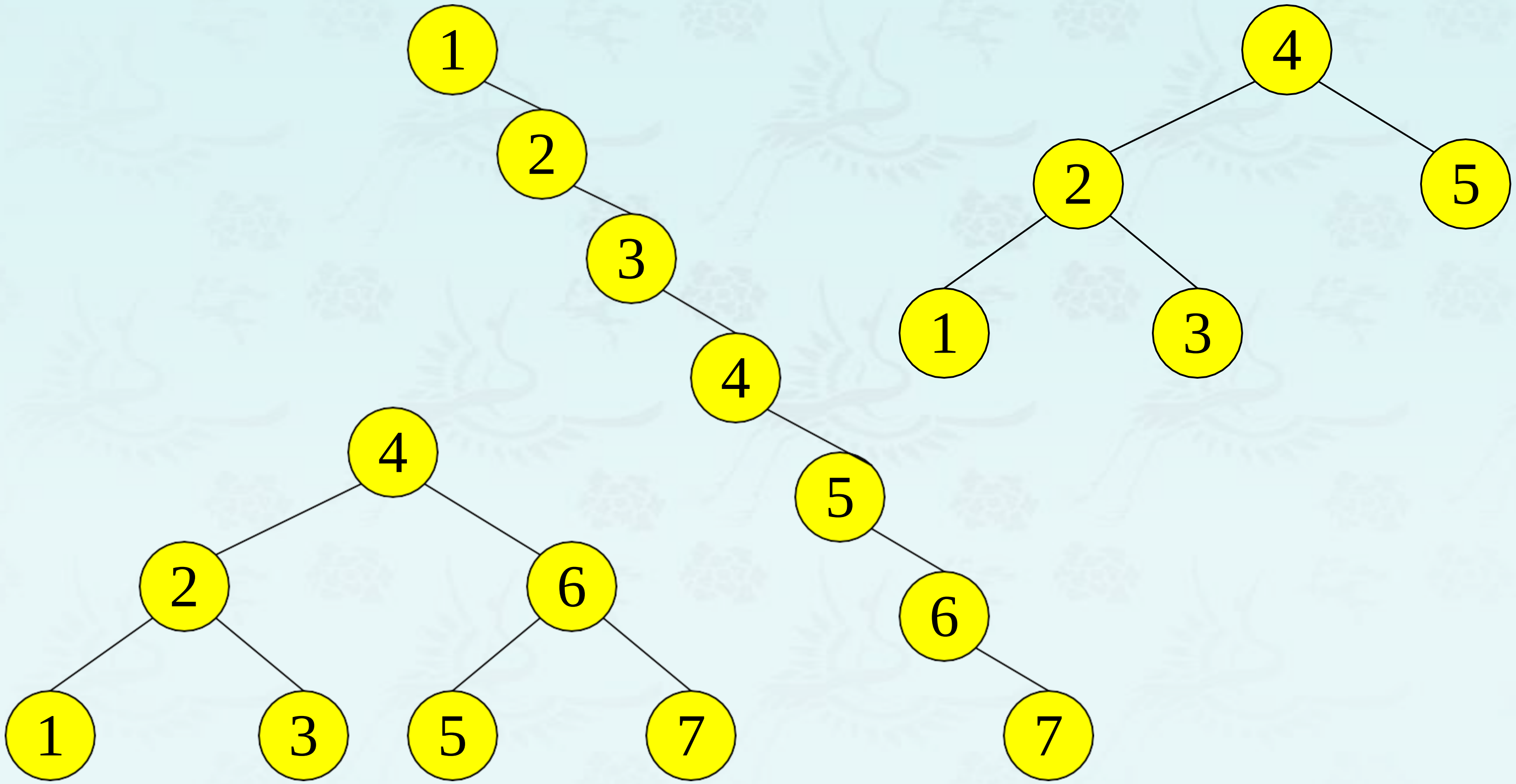
- All BST operations are O(d), where d is tree depth
- Minimum d is $d = \lfloor \log_2 N \rfloor$ for a binary tree with N nodes
  - What is the best case tree?
  - What is the worst case tree?
- So, best case running time of BST operations is O(log N)

BST

Worst case running time is O(N)

- What happens when you Insert elements in ascending order?
  - Insert: 2, 4, 6, 8, 10, 12 into an empty BST
- Problem: Lack of “balance”;
  - compare depths of left and right subtree
- Unbalanced degenerate tree

Balanced and unbalanced BST



Approaches to balancing trees

- Don't balance
  - May end up with some nodes very deep
- Strict balance
  - The tree must always be balanced perfectly
- Pretty good balance
  - Only allow a little out of balance
- Adjust on access
  - Self-adjusting

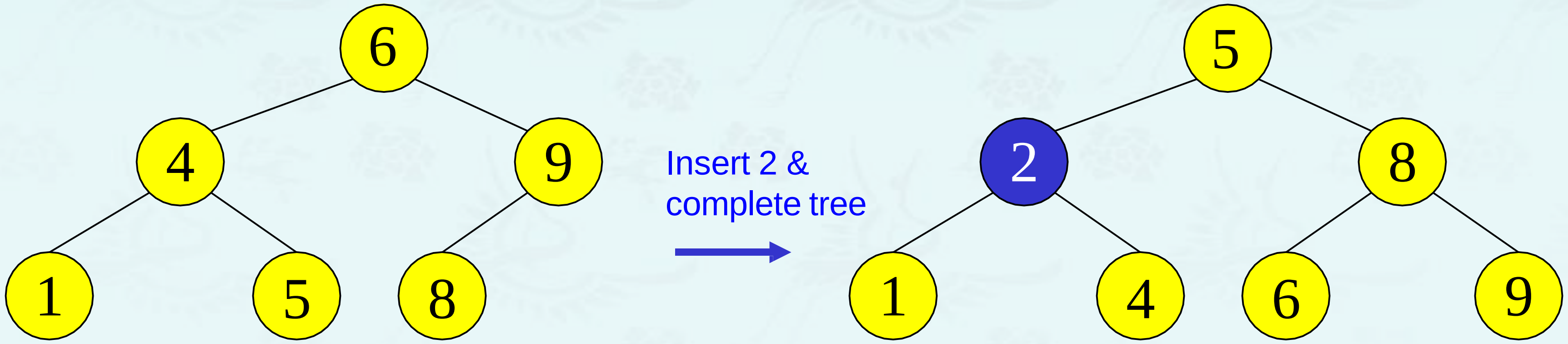
Balancing Binary Search Trees

Many algorithms exist for keeping BST balanced

- Adelson-Velskii and Landis (**AVL**) trees (height-balanced trees)
- Weight-balanced trees
- **Red-black** trees;
- **Splay** trees and other self-adjusting trees
- **B-trees** and other (e.g. 2-4 trees) multiway search trees

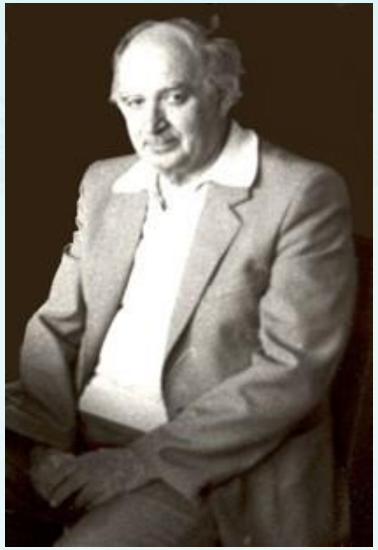
Perfect Balance

- Want a complete tree after every operation
  - tree is full except possibly in the lower right
- This is expensive
  - For example, insert 2 in the tree on the left and then rebuild as a complete tree



AVL Trees (1962)

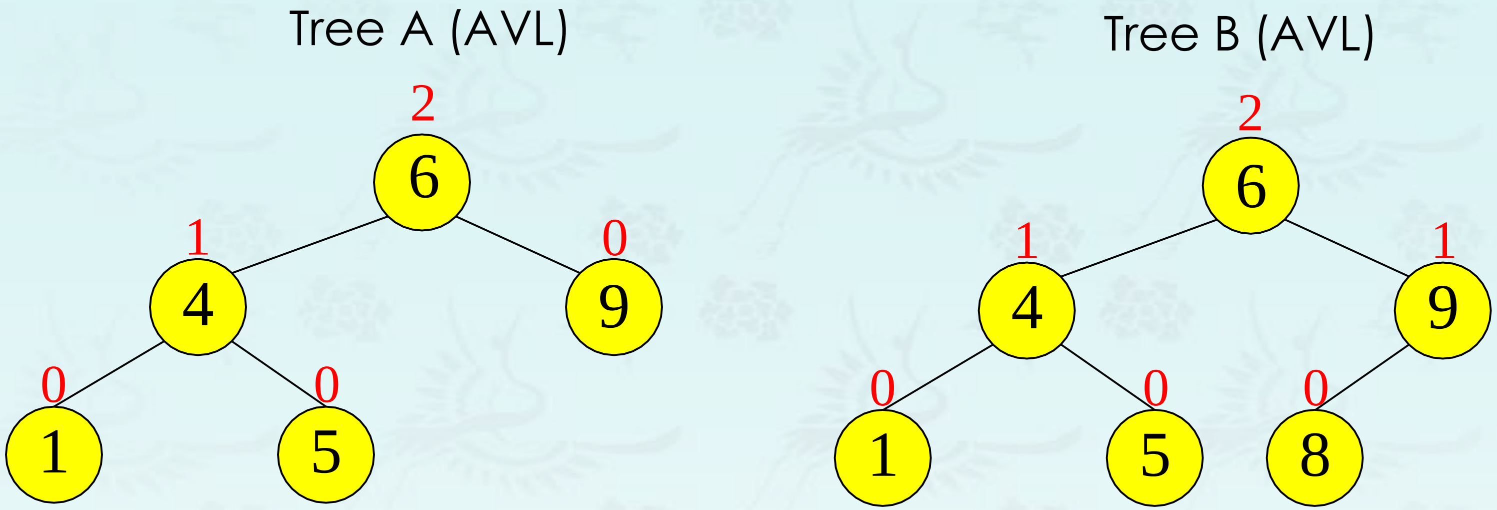
- Named after 2 Russian mathematicians
- Georgii **A**delson-Velsky (1922 - 2014)
- Evgenii Mikhailovich Landis (1921-1997)



AVL - Good but not Perfect Balance

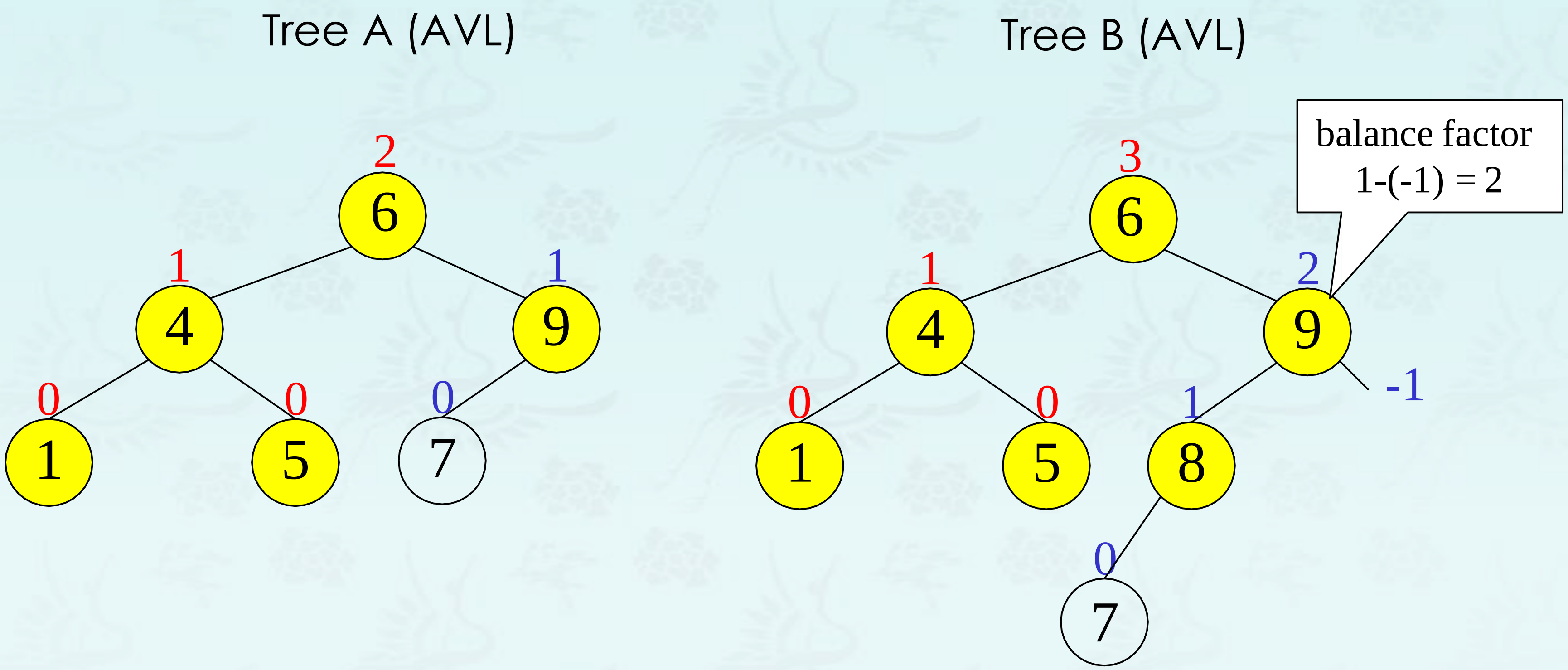
- Height-balanced binary search trees
- Balance factor of a node
  - height(left subtree) - height(right subtree)
- For every node, heights of left and right subtree can differ by no more than 1
  - Store current heights in each node or compute it on the fly

Node Heights



height of node = h
balance factor = h_{left} - h_{right}
empty height = -1

Node Heights after Insert 7

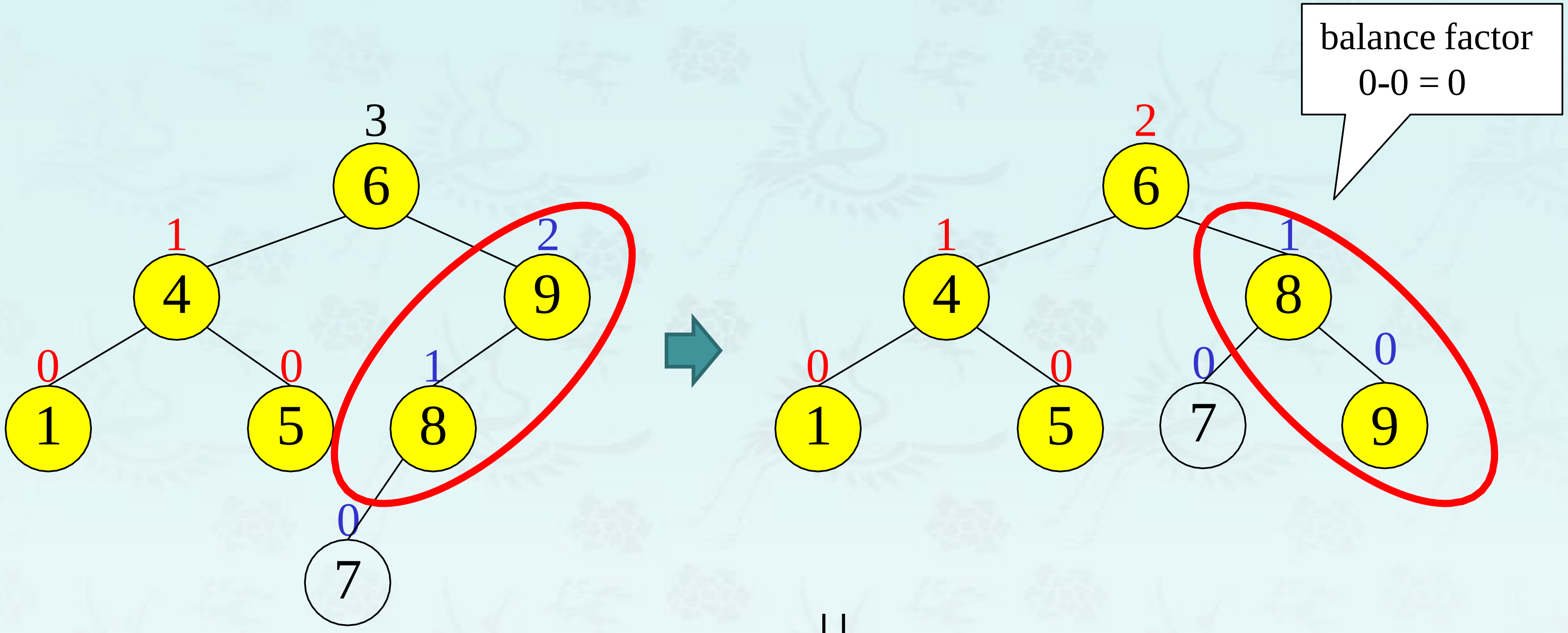


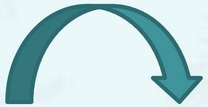
height of node = h
balance factor = h_{left} - h_{right}
empty height = -1

Insert and Rotation in AVL Trees

- Insert operation may cause balance factor to become 2 or –2 for some node
  - Only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, **go back up** to the root node by node.
  - If a new balance factor (the difference $h_{left} - h_{right}$) is 2 or –2, adjust tree by **rotation** around the node

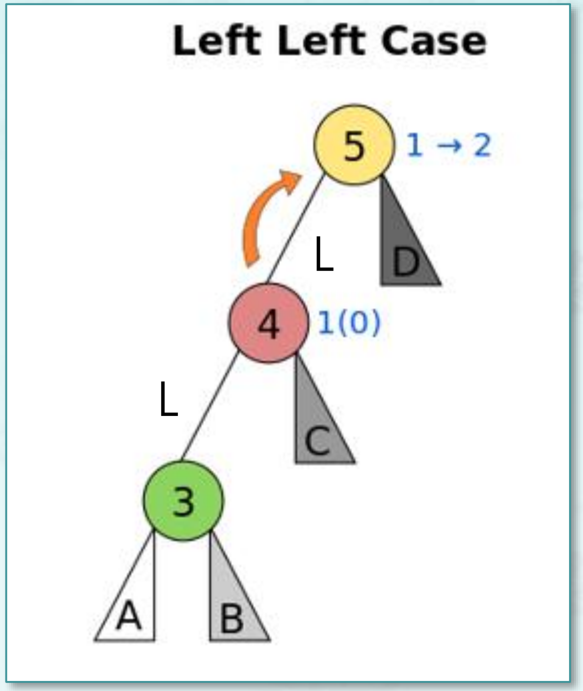
Single Rotation in an AVL Tree

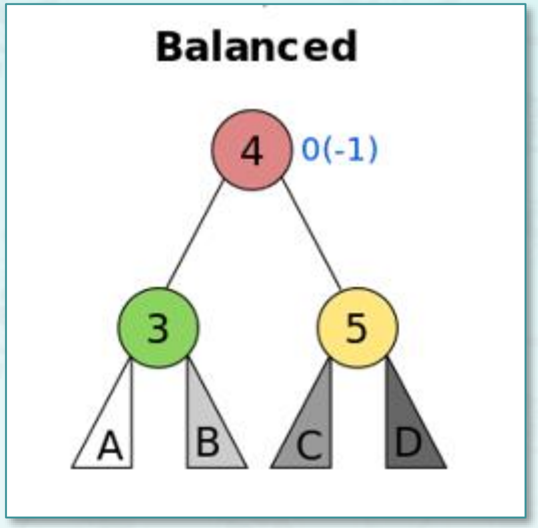




LL
Single Rotation
Right Rotation

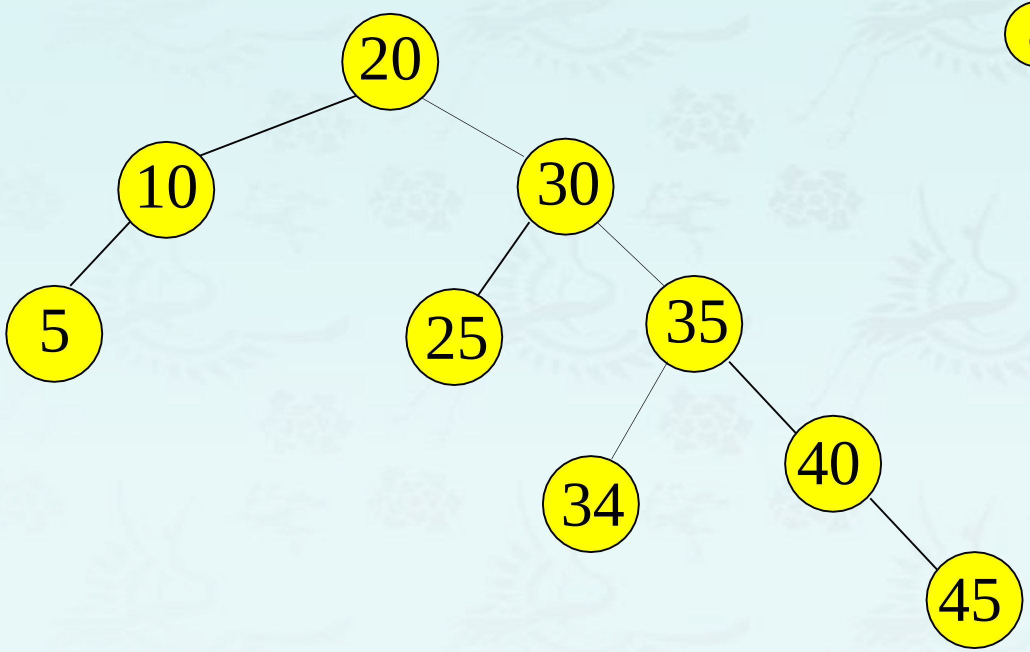
Single Rotation in an AVL Tree

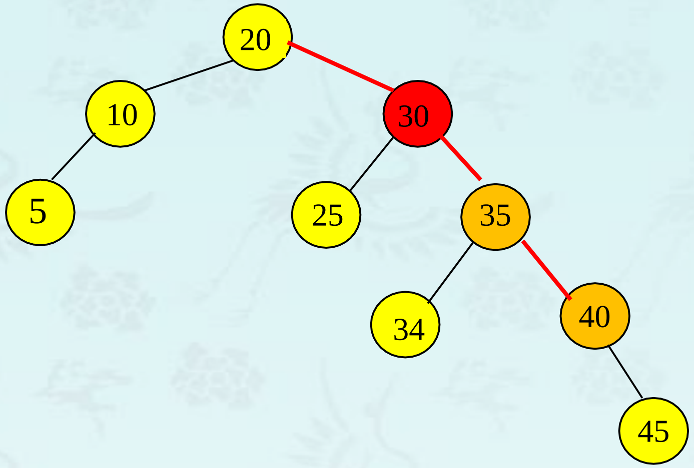




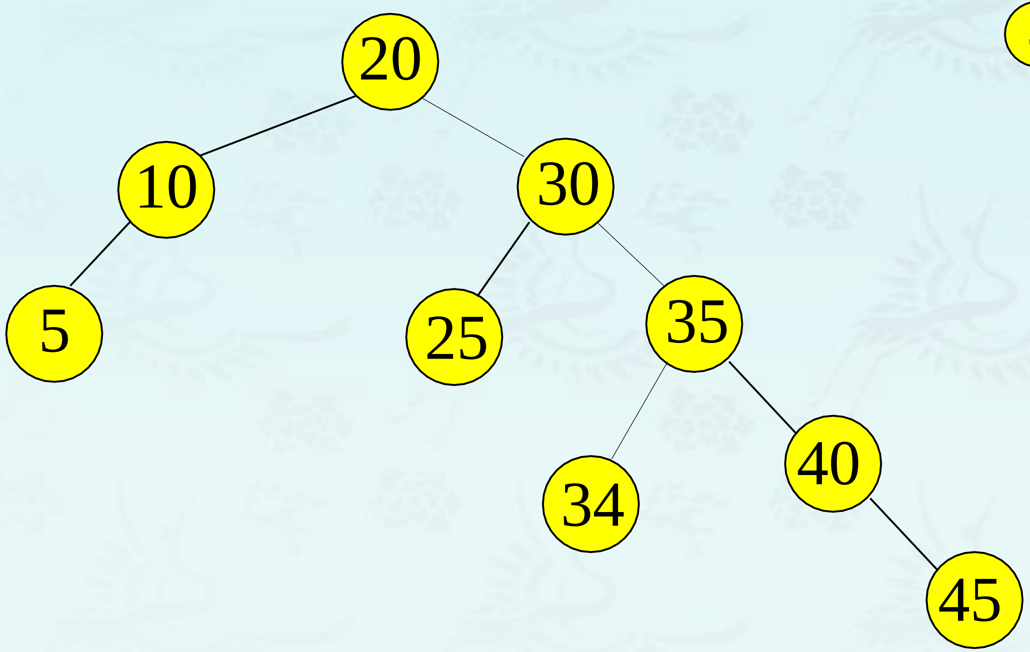
LL Case
Single Right Rotation

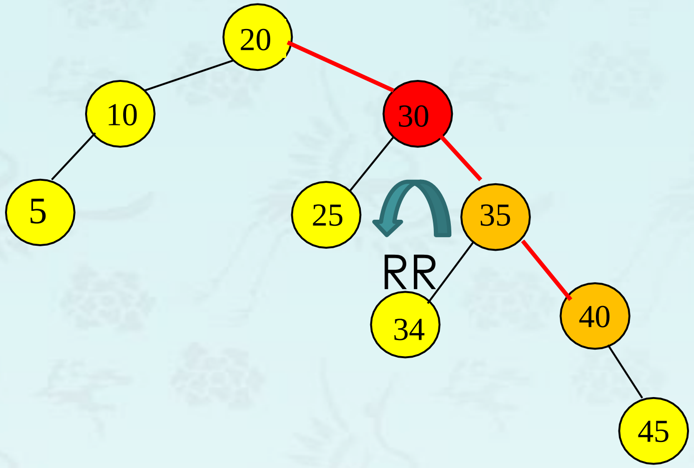
AVL Tree Balanced?



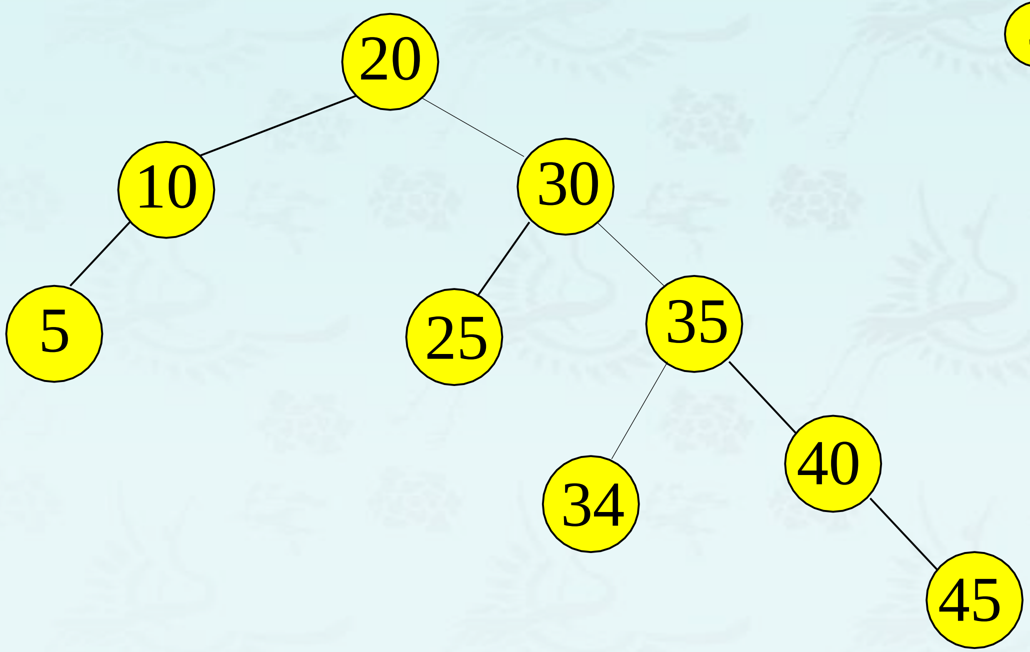


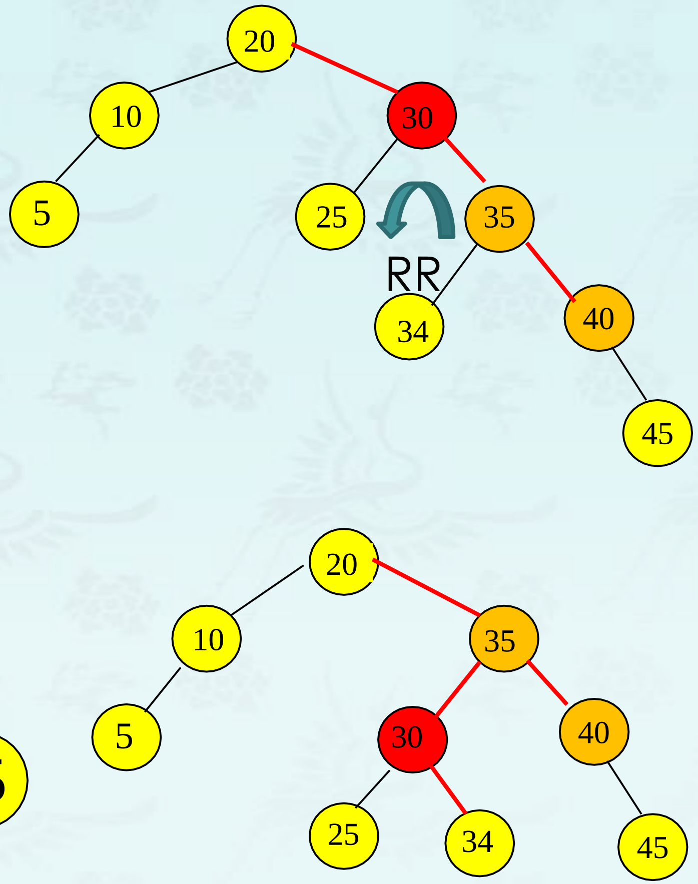
AVL Tree Balanced?



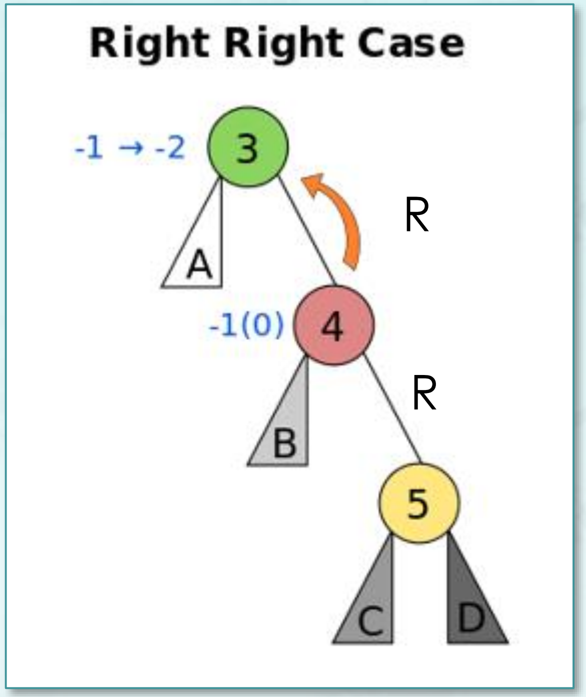


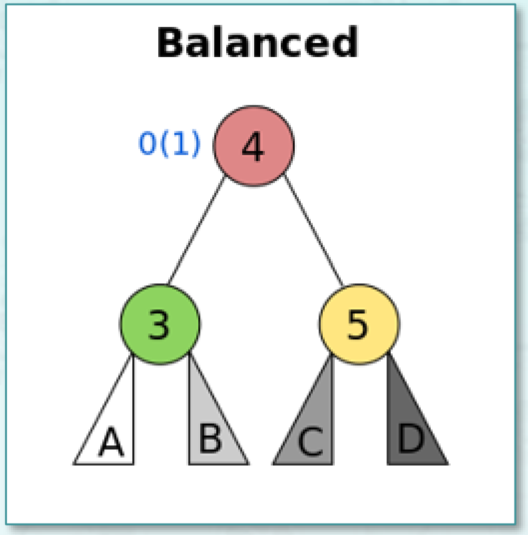
AVL Tree Balanced?





Single Rotation in an AVL Tree

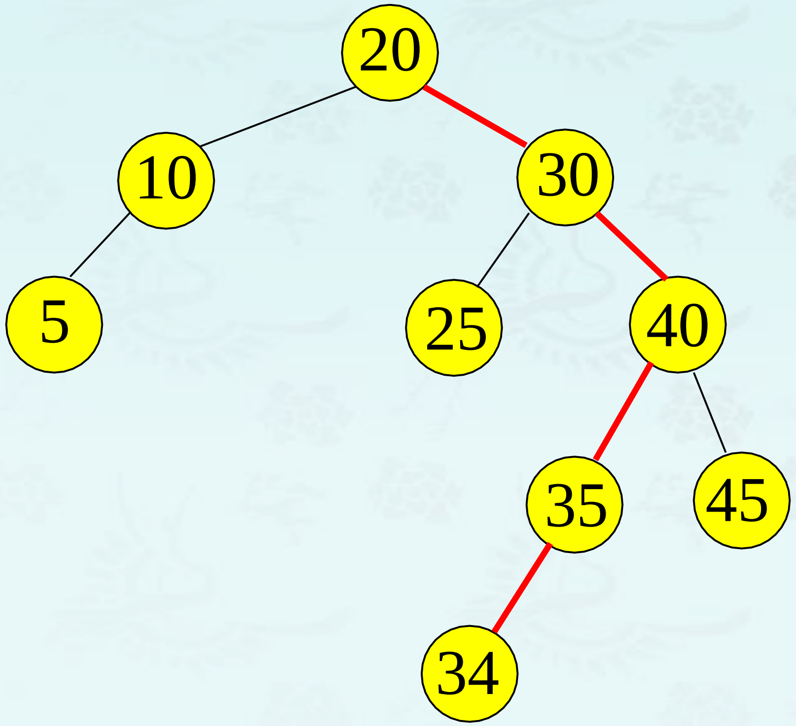




RR Case
Single Left Rotation

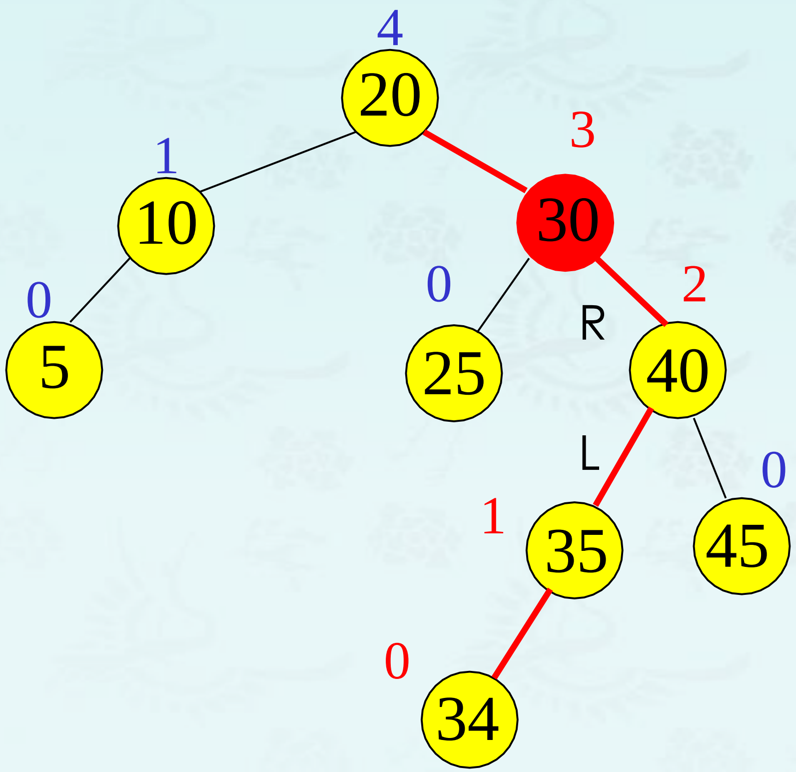
AVL Tree Balanced?

Insertion of 34
Imbalance at 30
Balance factor at 30 = -2



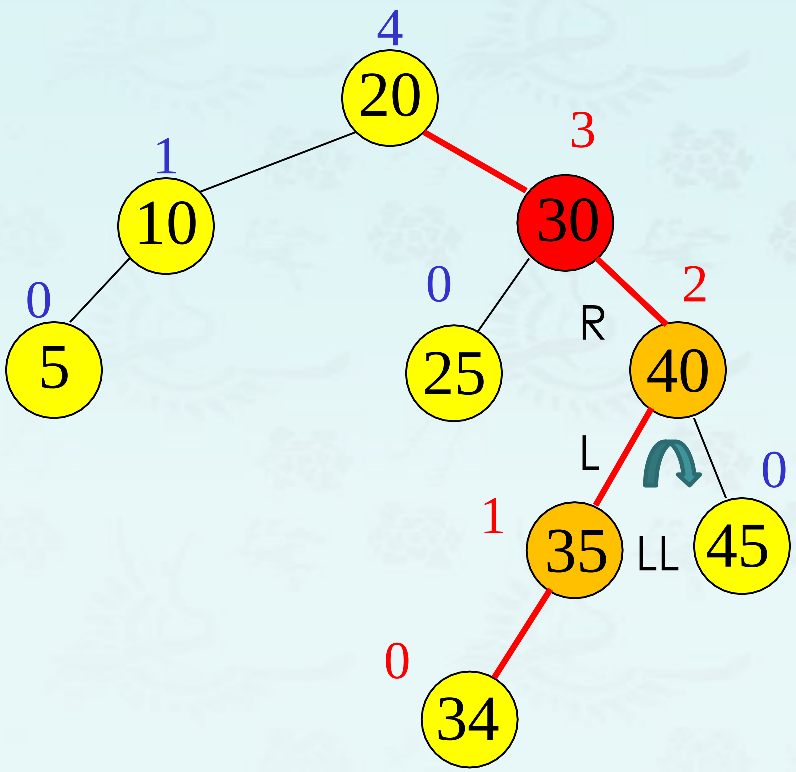
AVL Tree Balanced?

Insertion of 34
Imbalance at 30
Balance factor at 30 = -2



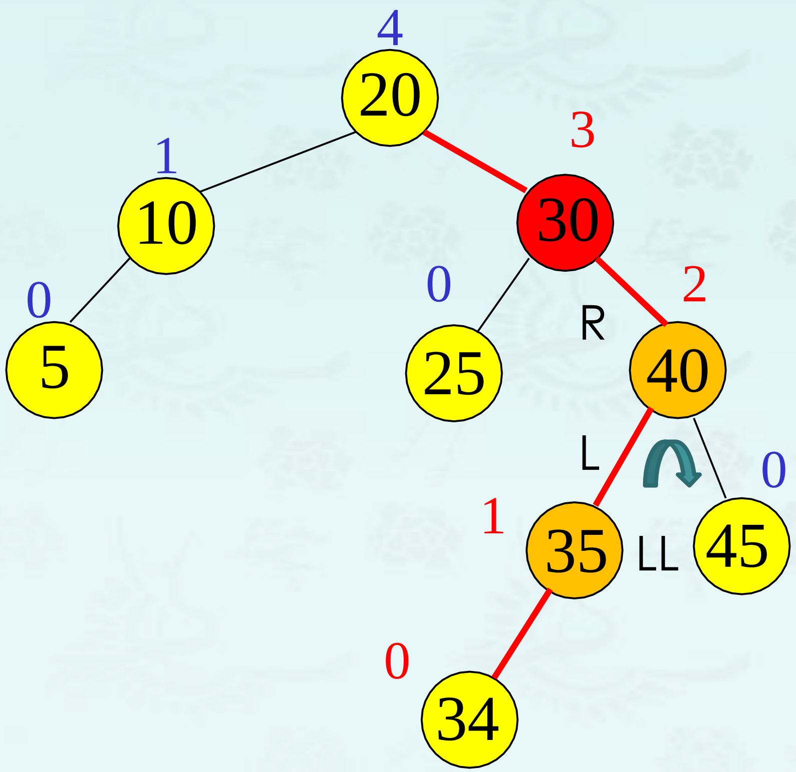
Double rotation RL

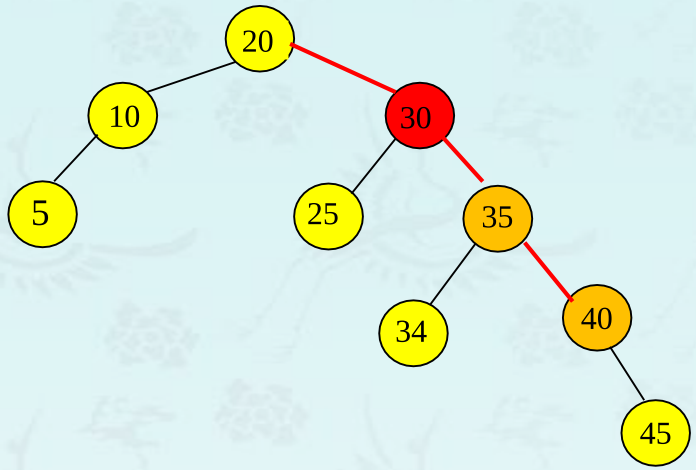
Insertion of 34
Imbalance at 30
Balance factor at 30 = -2



Double rotation RL

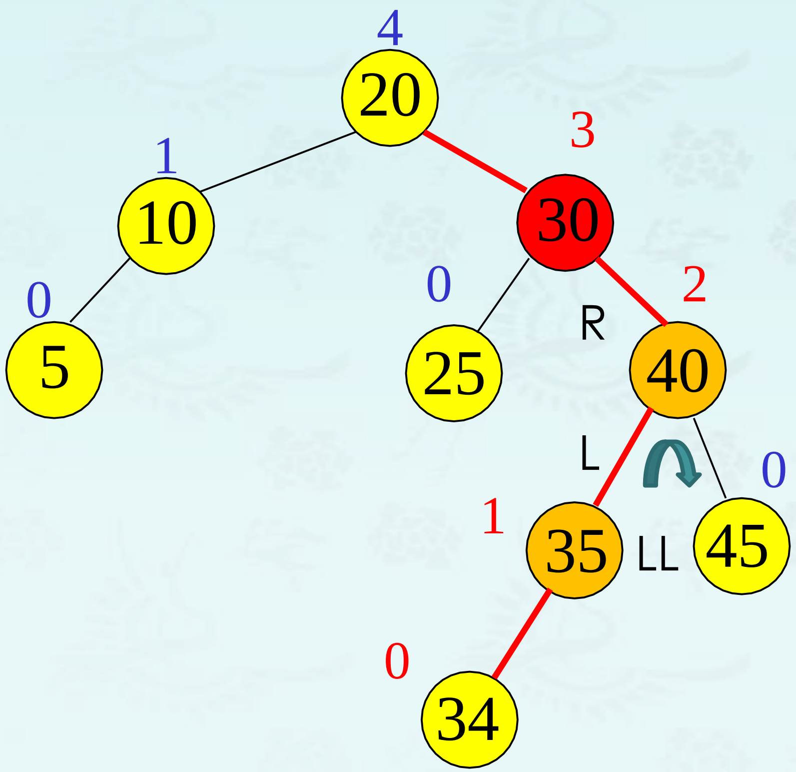
Insertion of 34
Imbalance at 30
Balance factor at 30 = -2

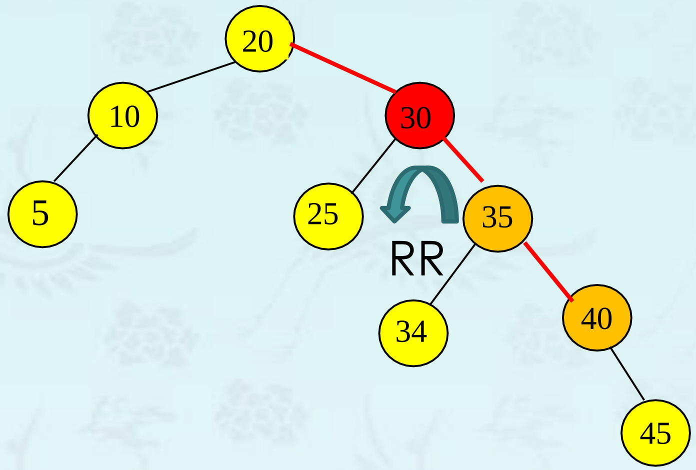




Double rotation RL

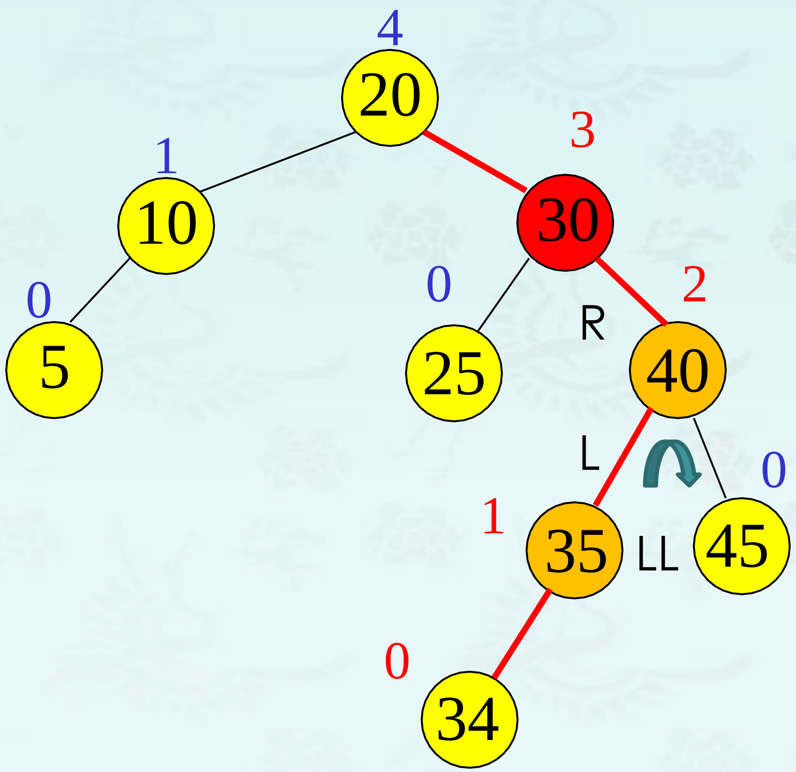
Insertion of 34
Imbalance at 30
Balance factor at 30 = -2

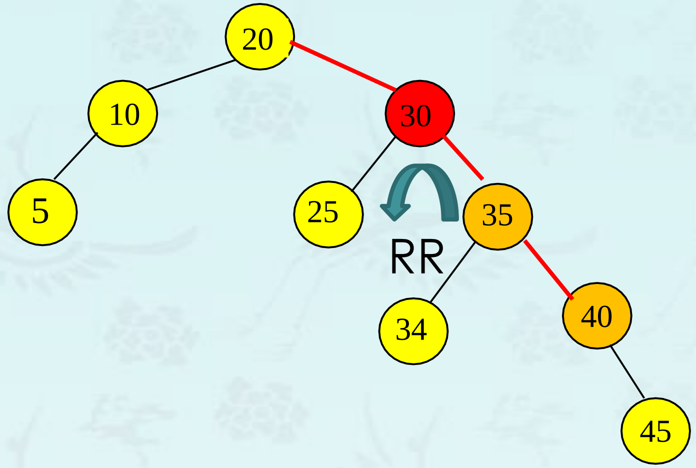


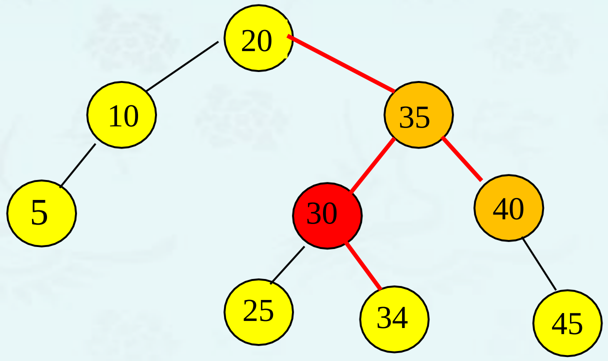


Double rotation RL

Insertion of 34
Imbalance at 30
Balance factor at 30 = -2

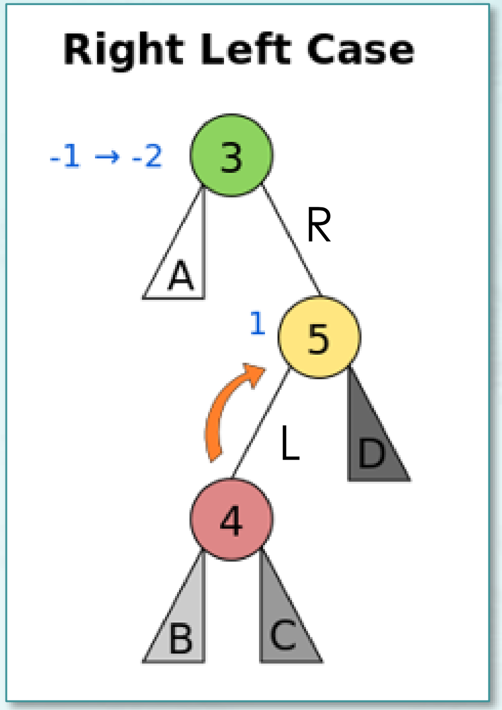




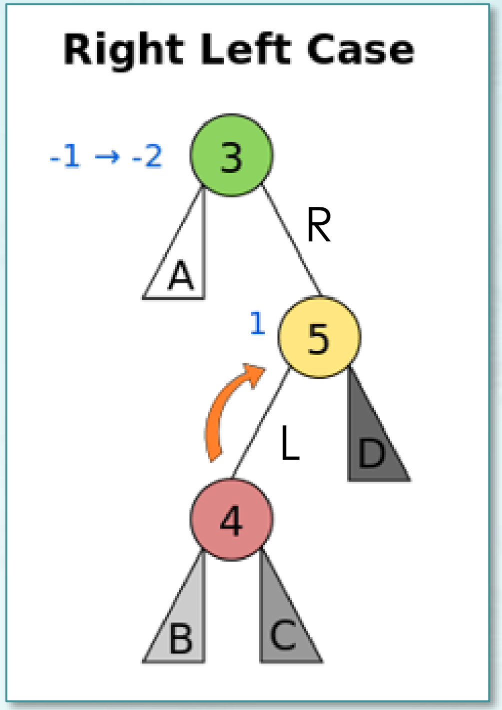


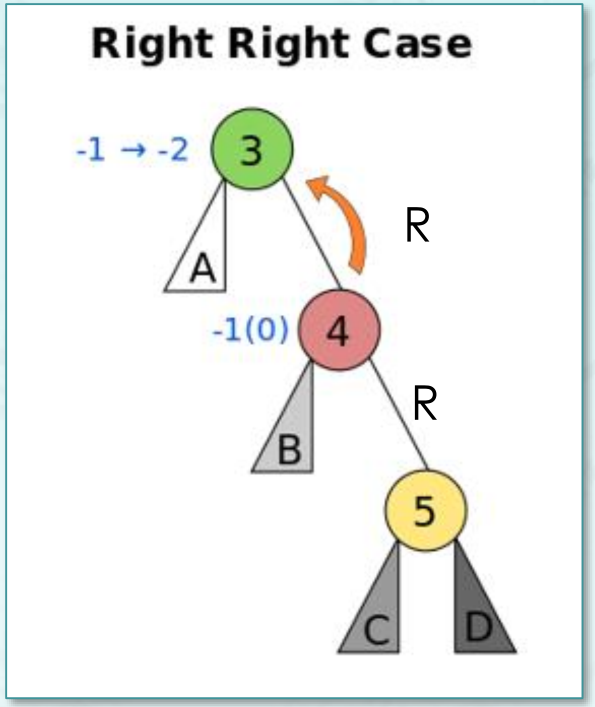
RL
double rotation
LL rotation + RR rotation

Double rotation – RL Case

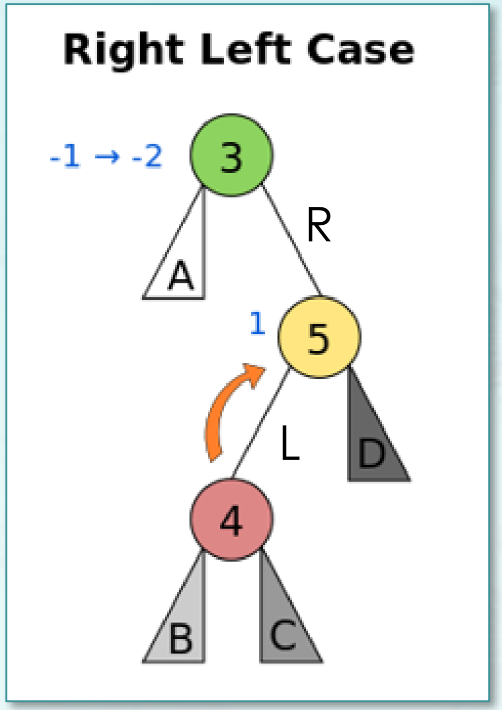


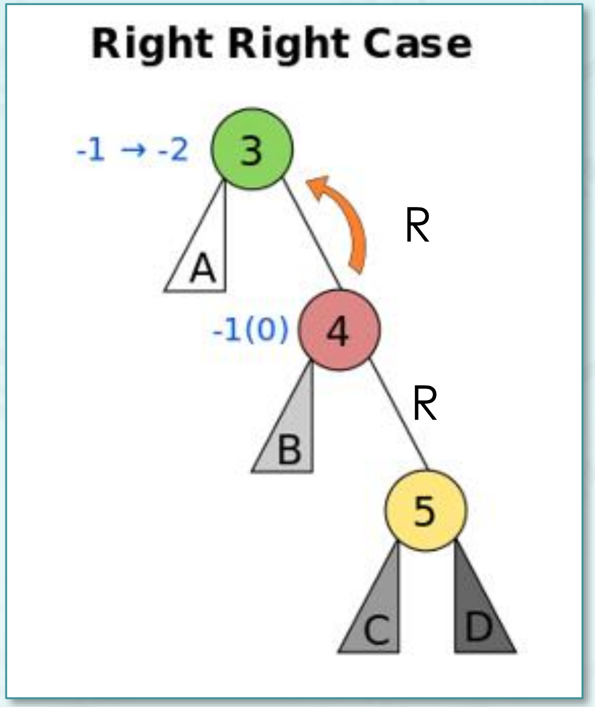
Double rotation – RL Case

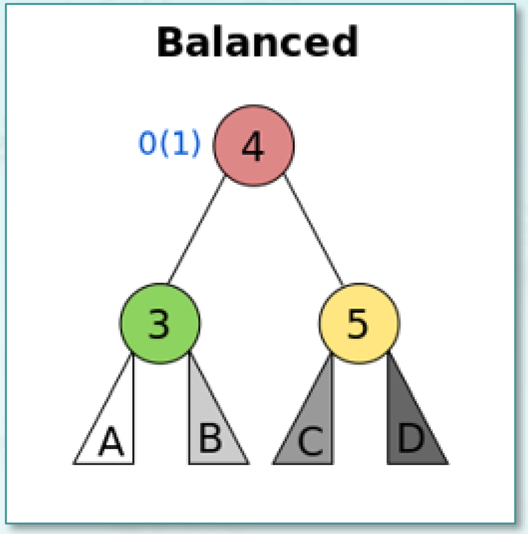




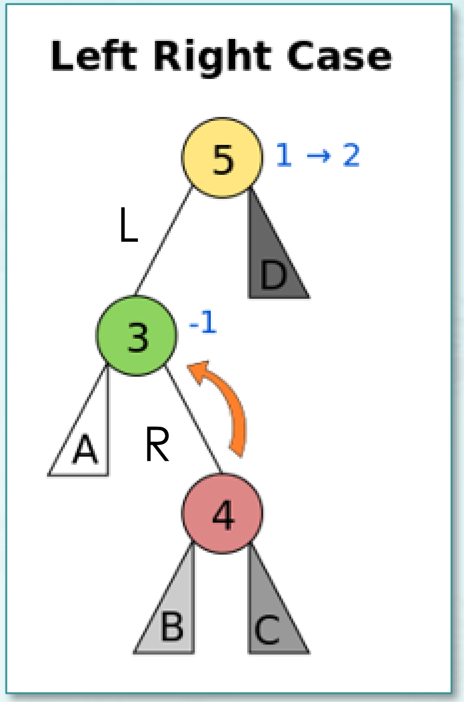
Double rotation – RL Case



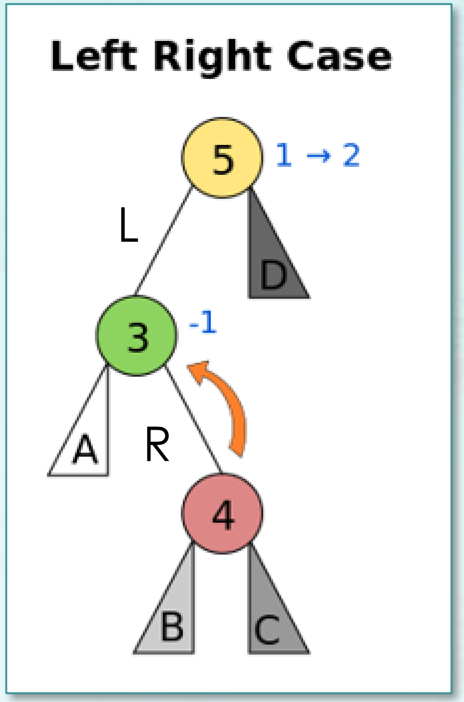


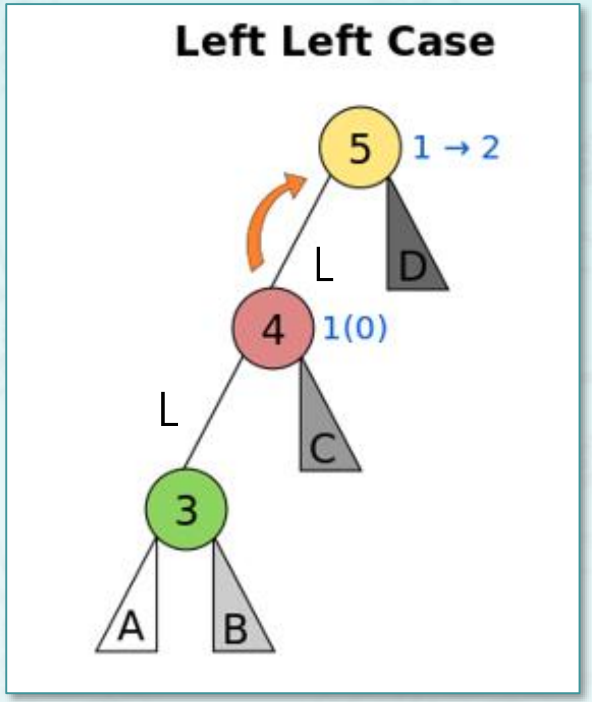


Double rotation – LR Case

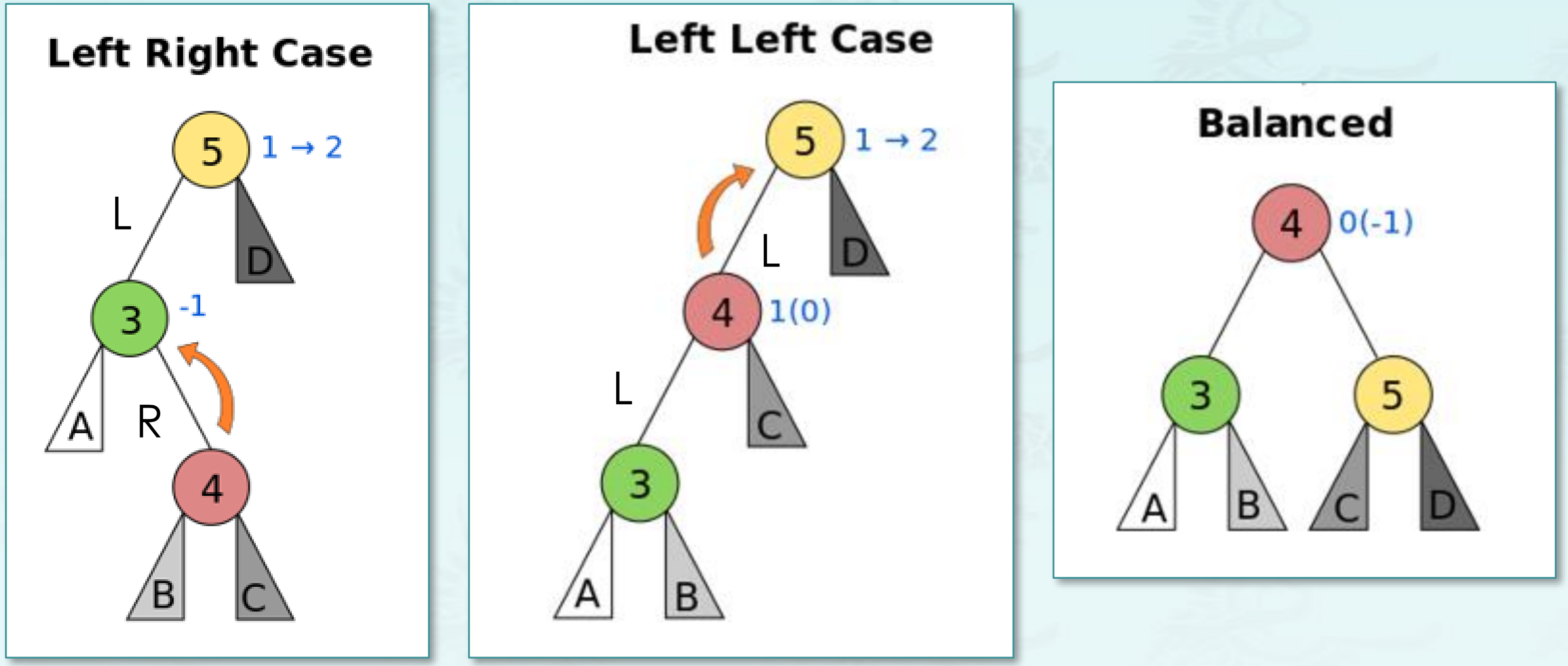


Double rotation – LR Case





Double rotation – LR Case



Insertions in AVL Trees

Let the node that needs rebalancing be a.

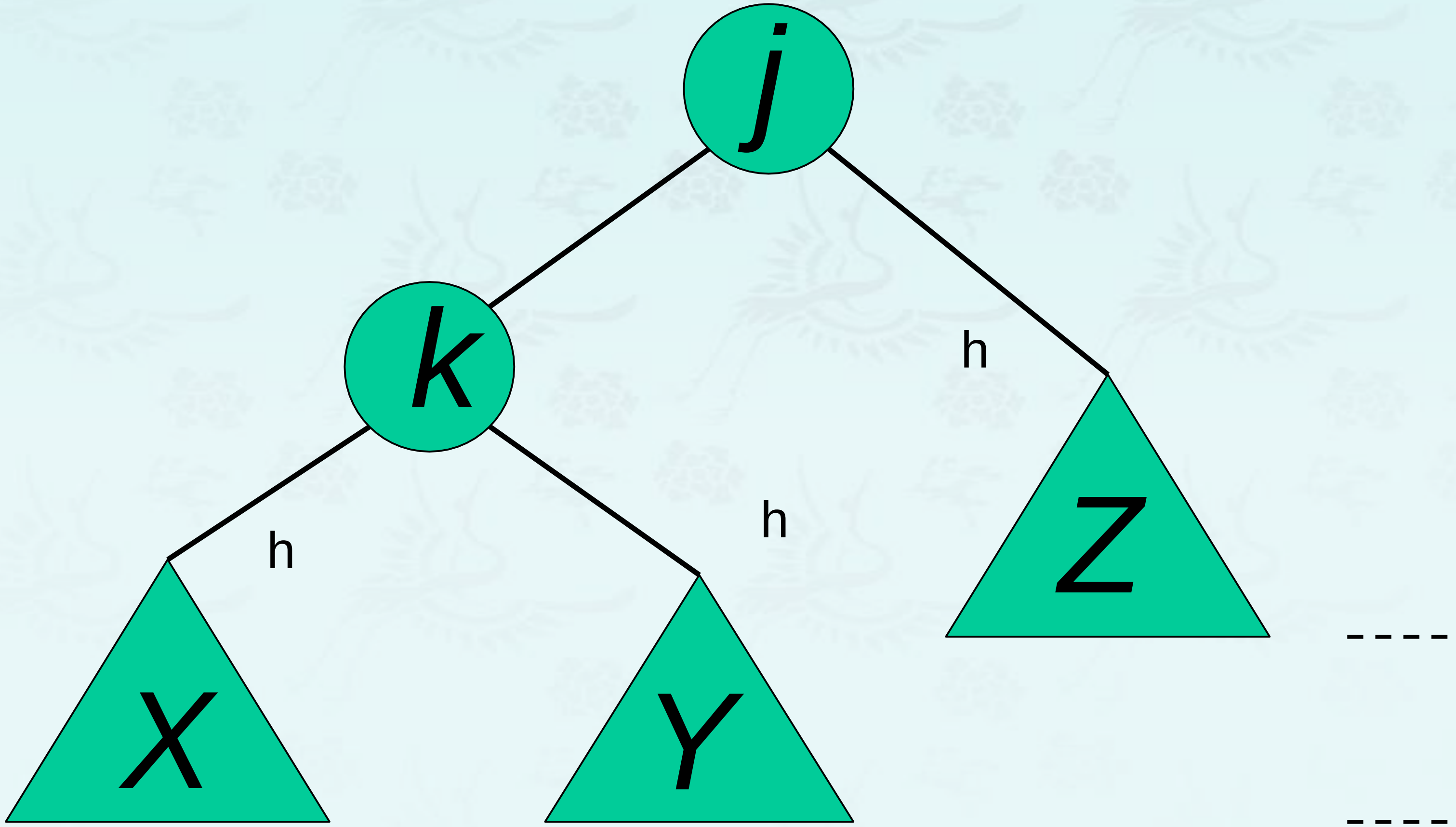
There are 4 cases:

- Outside Cases (require single rotation) :
  1. Insertion into left subtree of left child of a.
  2. Insertion into right subtree of right child of a.
- Inside Cases (require double rotation) :
  1. Insertion into right subtree of left child of a.
  2. Insertion into left subtree of right child of a.

The rebalancing is performed through four separate rotation algorithms.

AVL Insertion: Outside Case

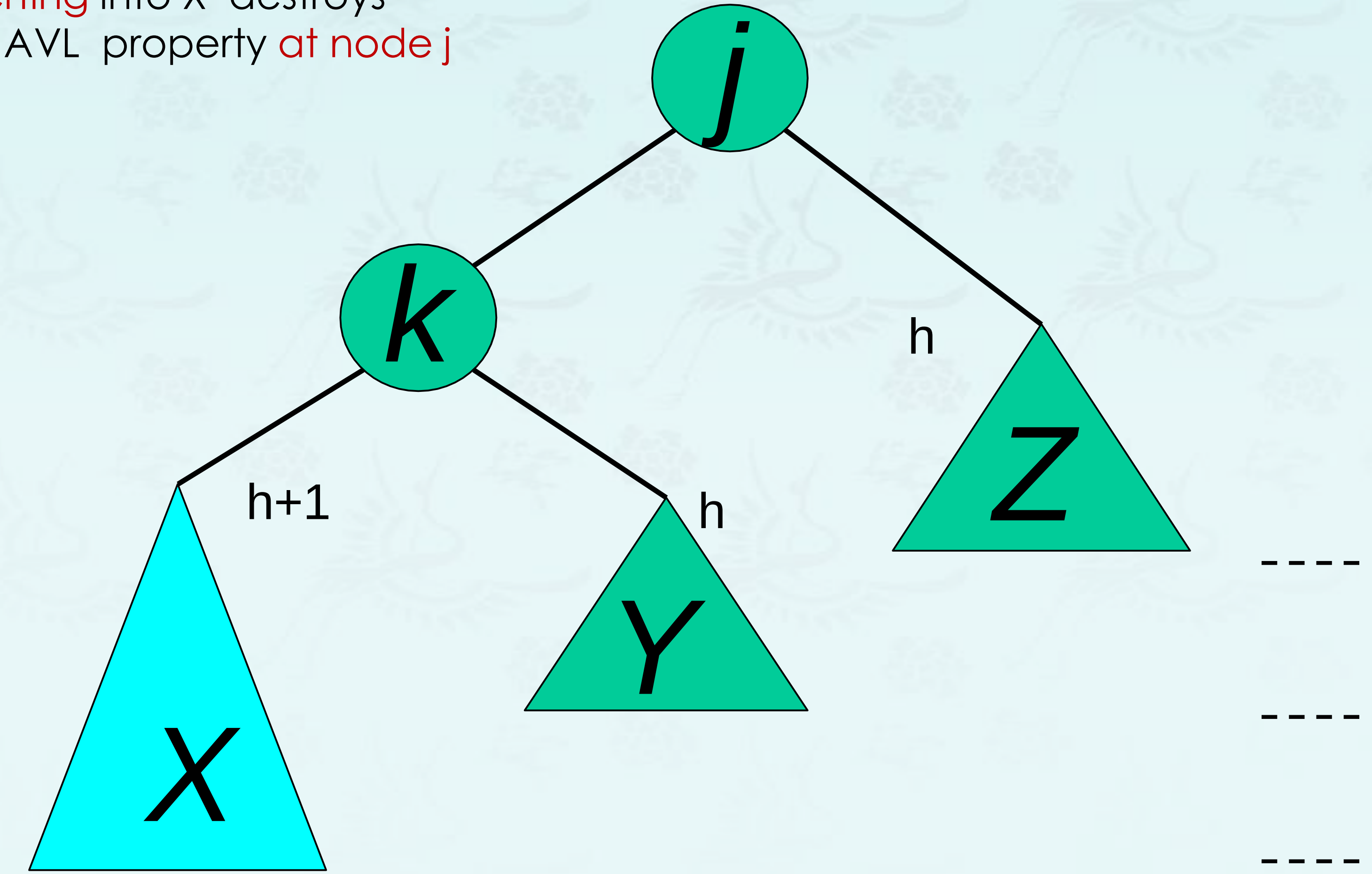
Consider a valid AVL subtree



AVL Insertion: Outside Case

Consider a valid AVL subtree

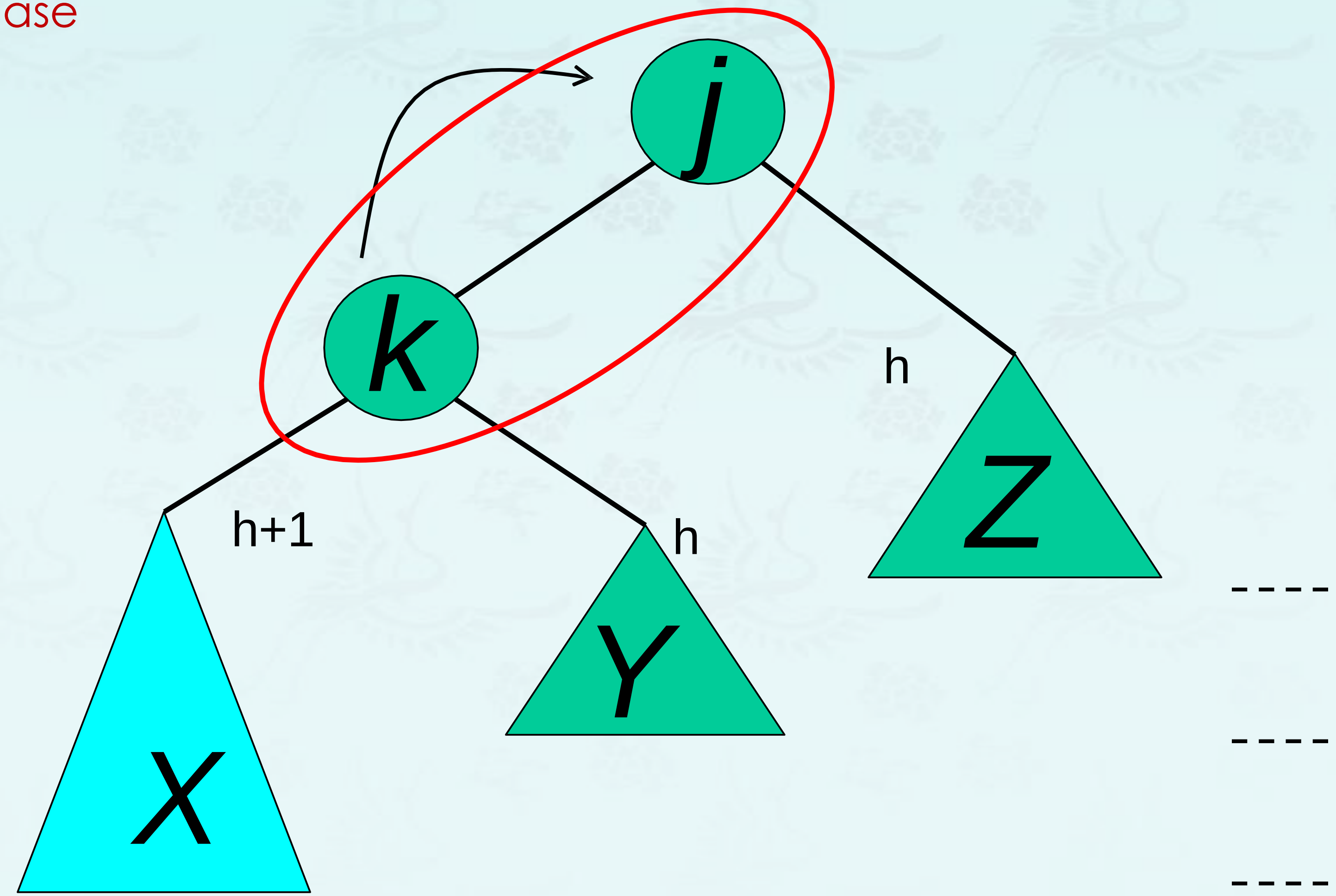
Inserting into X destroys
the AVL property at node j



AVL Insertion: Outside Case

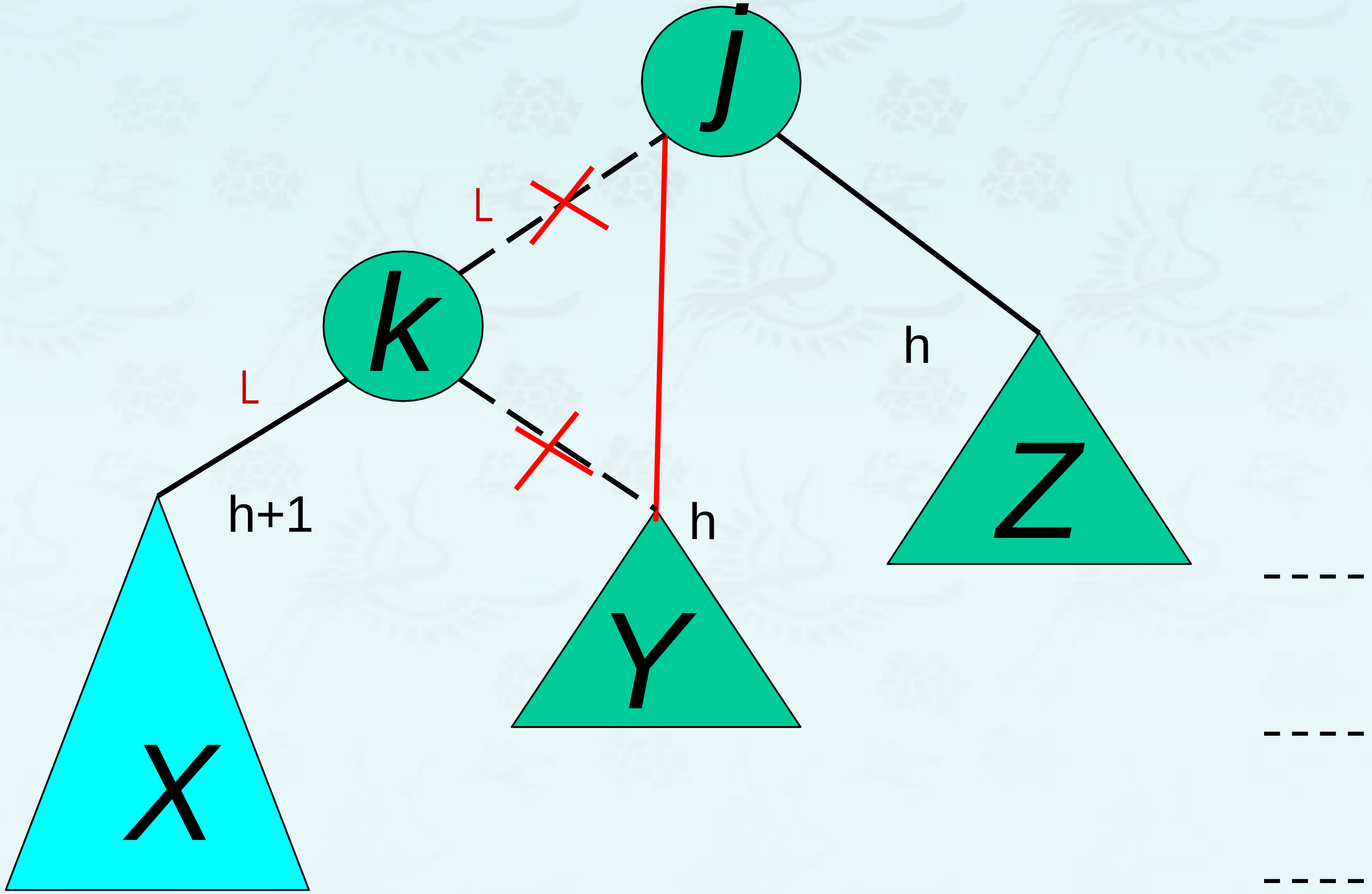
Do a "right rotation"

LL Case



Single right rotation

Do a "right rotation"

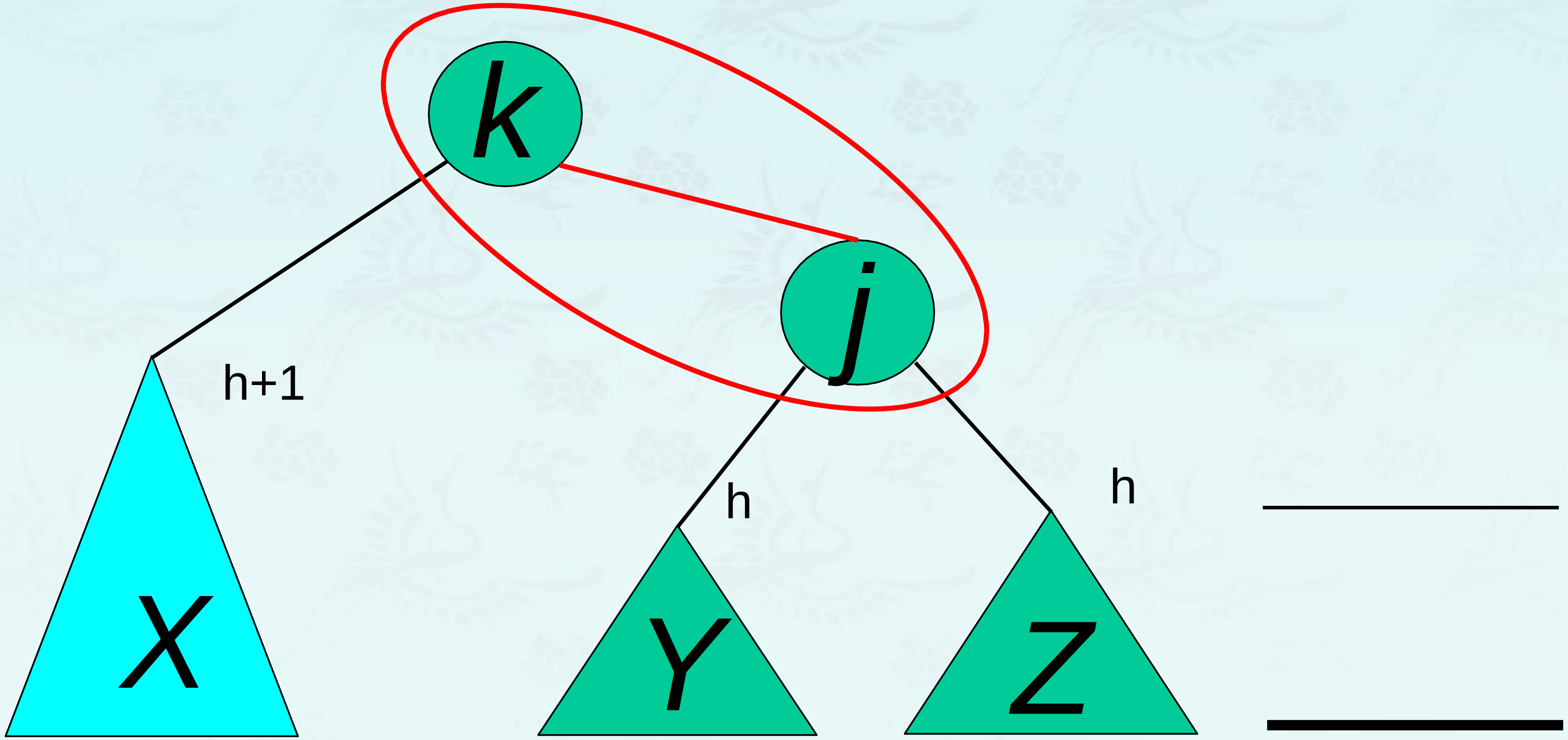


Outside Case Completed

LL Case – Single Rotation

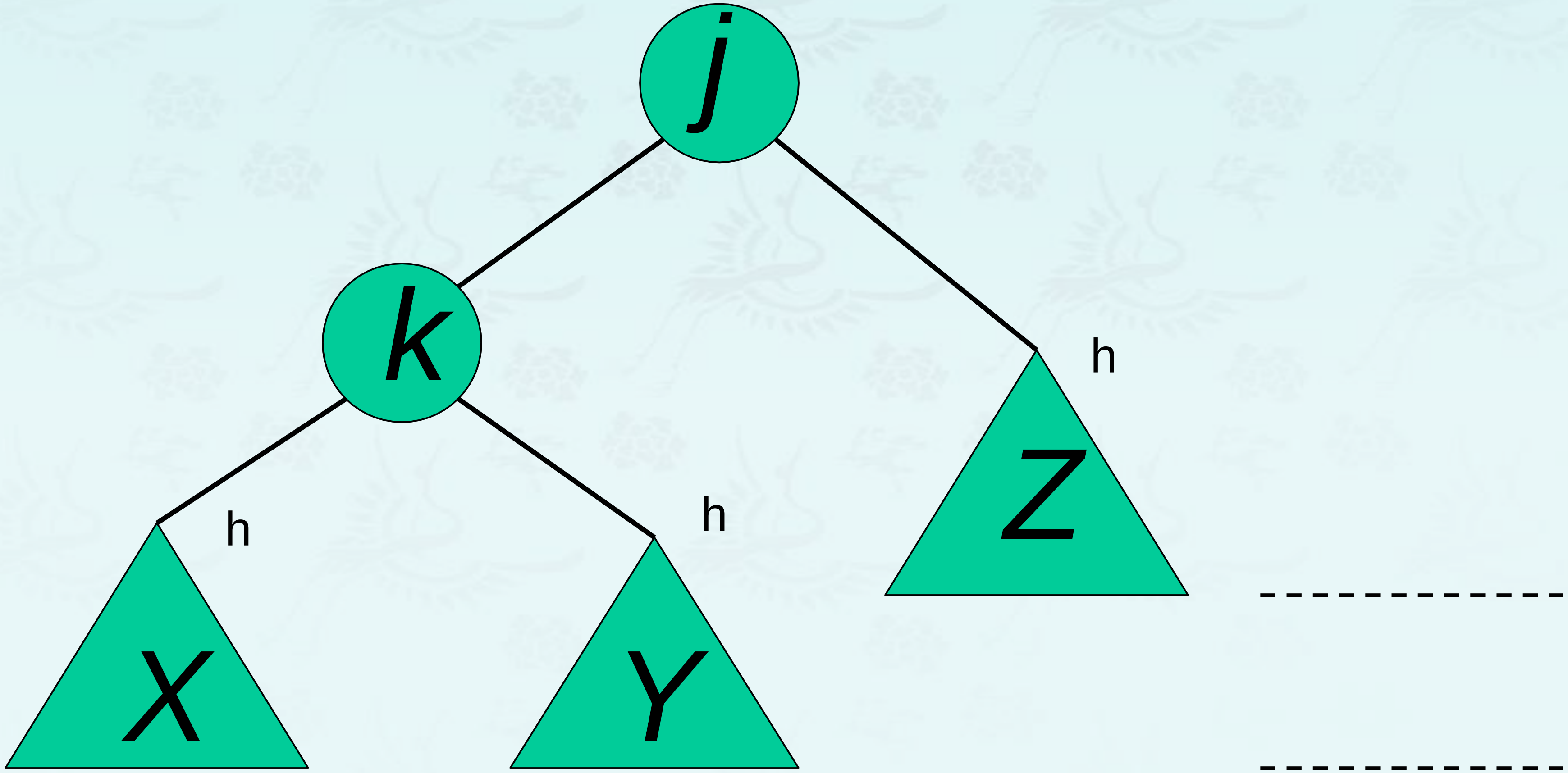
AVL property has been restored!

"Right rotation" done!
("Left rotation" is mirror symmetric)



AVL Insertion: Inside Case

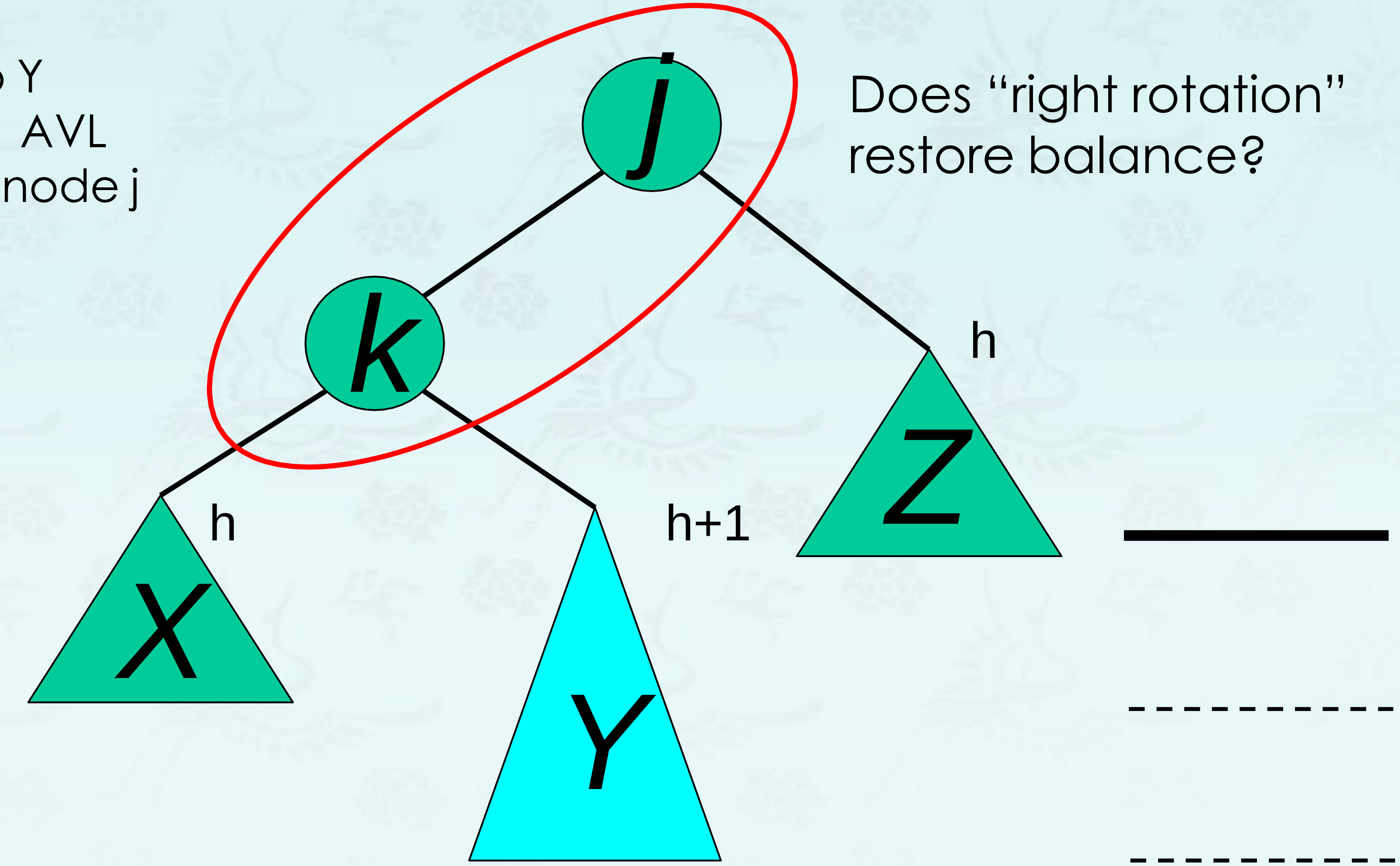
Consider a valid AVL subtree



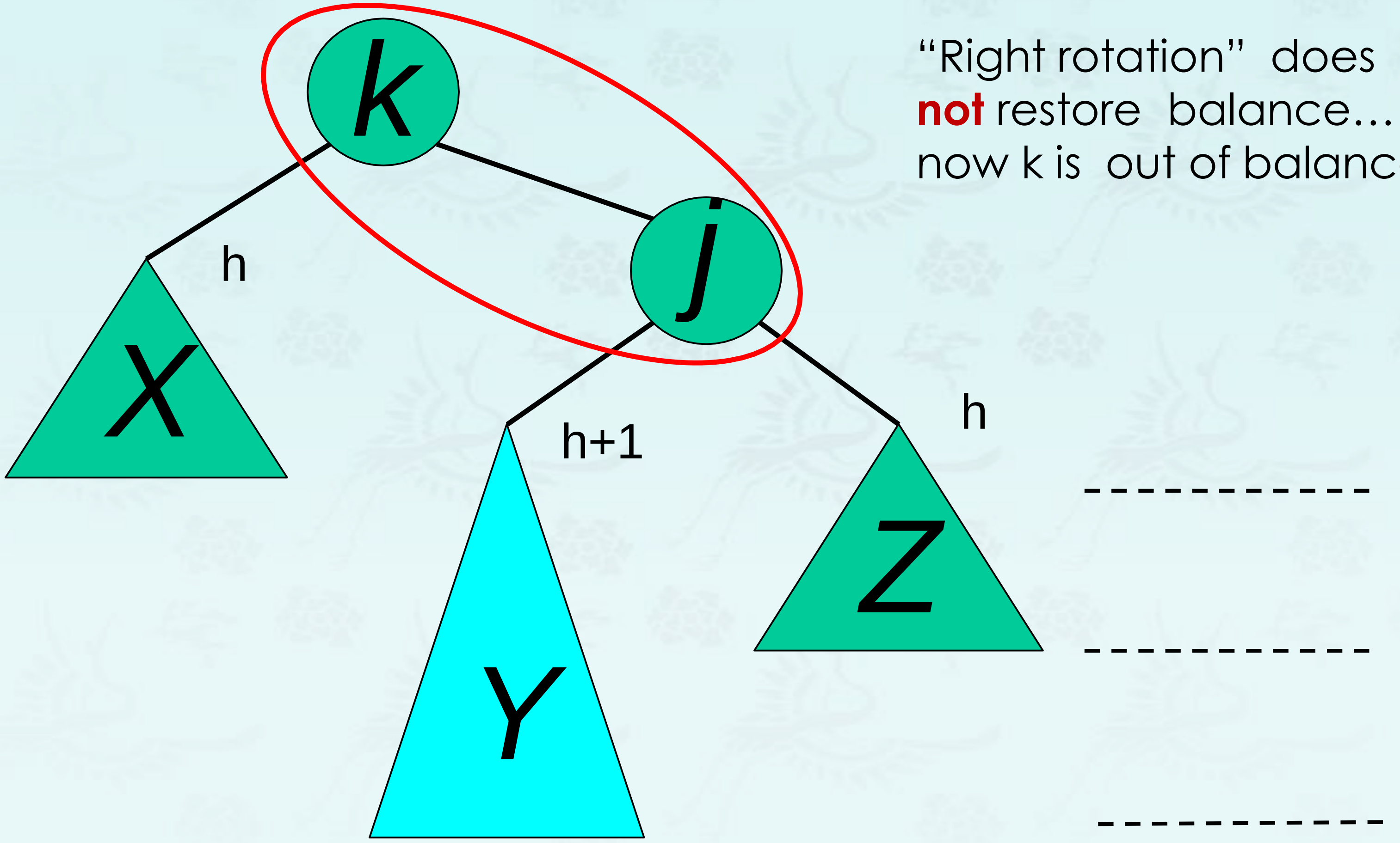
AVL Insertion: Inside Case

Inserting into Y destroys the AVL property at node j

Does "right rotation" restore balance?



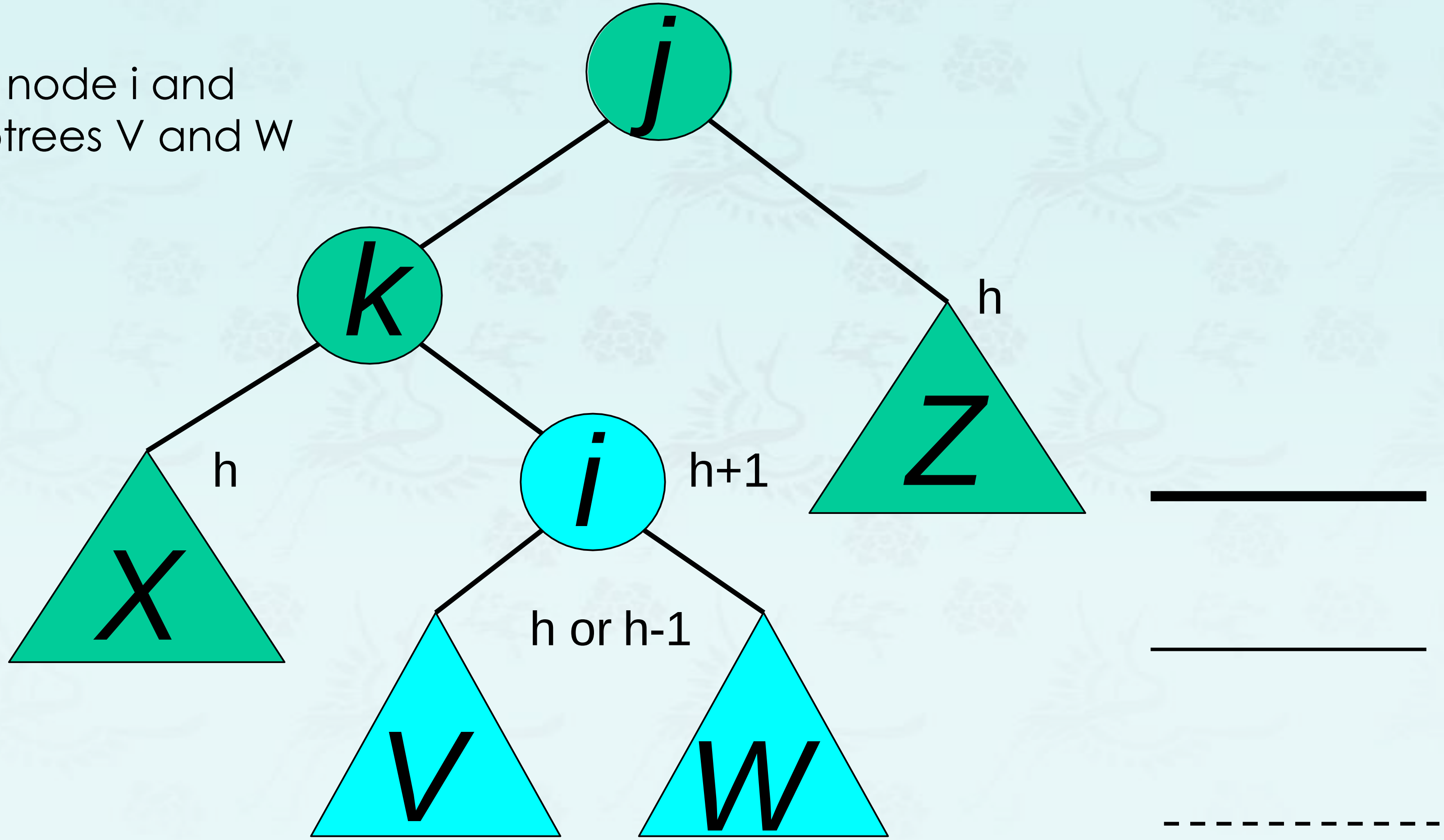
AVL Insertion: Inside Case



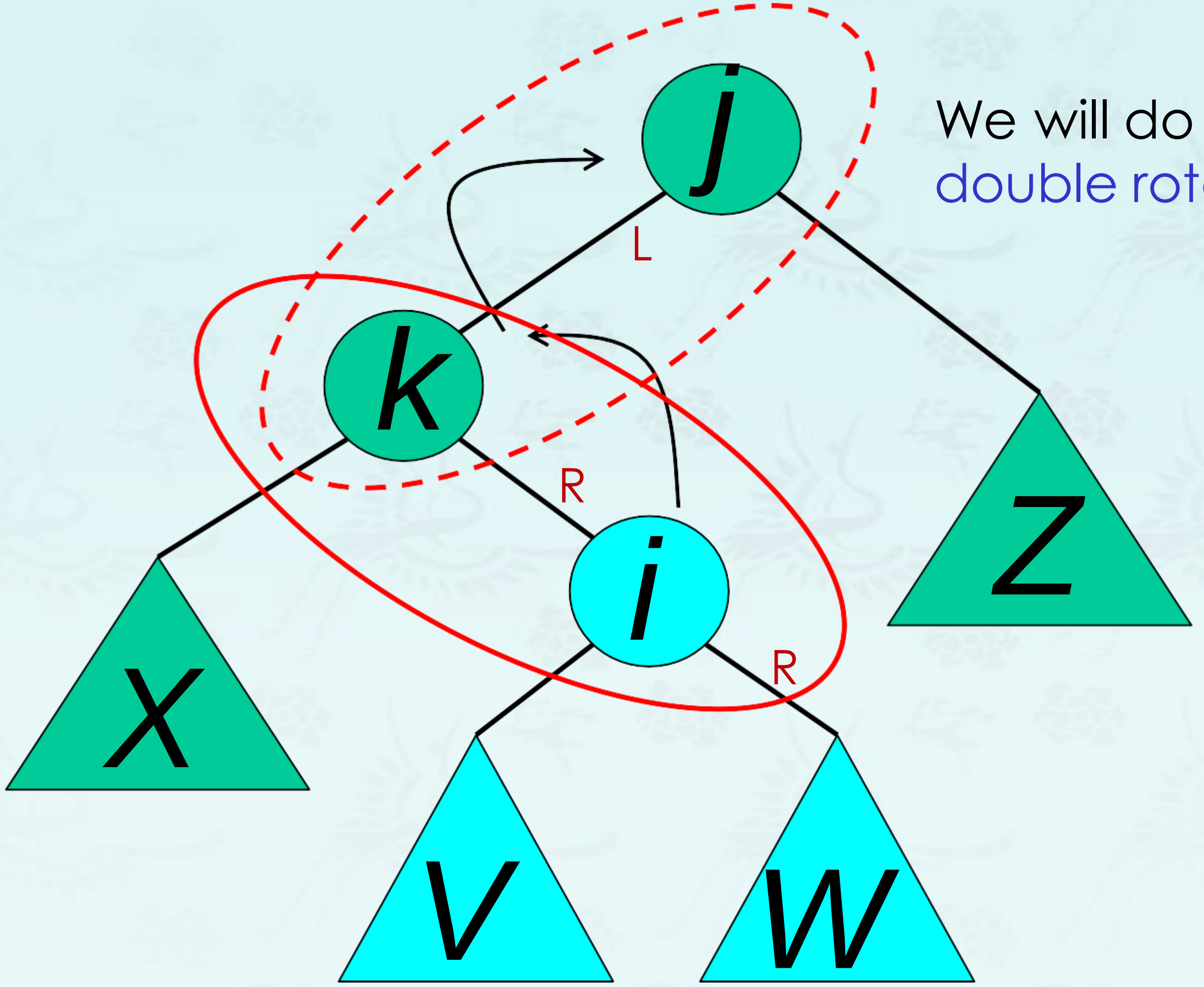
"Right rotation" does **not** restore balance… now k is out of balance

AVL Insertion: Inside Case

Y = node i and
subtrees V and W



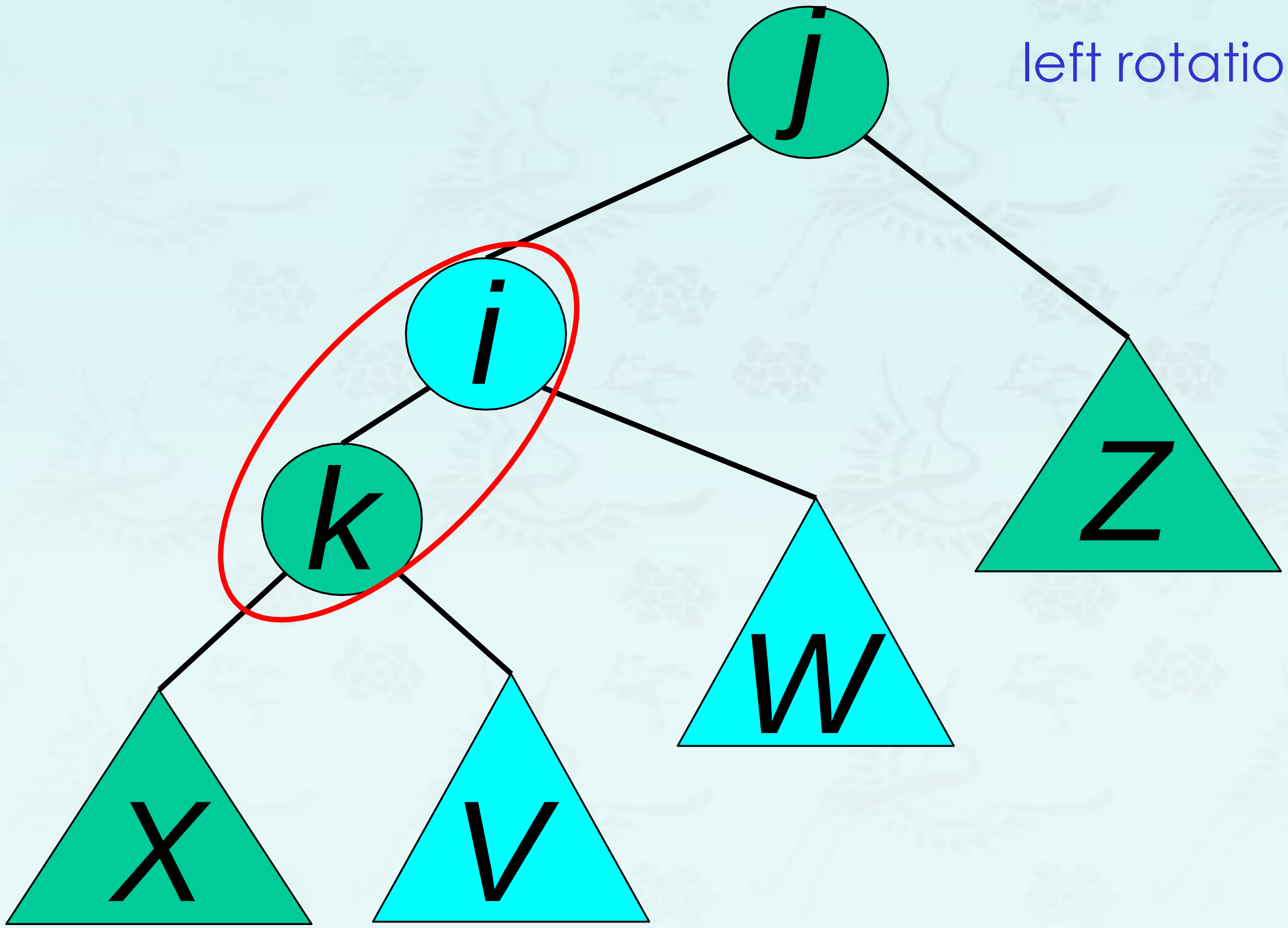
AVL Insertion: Inside Case



We will do a left-right "double rotation" . . .

LR Case

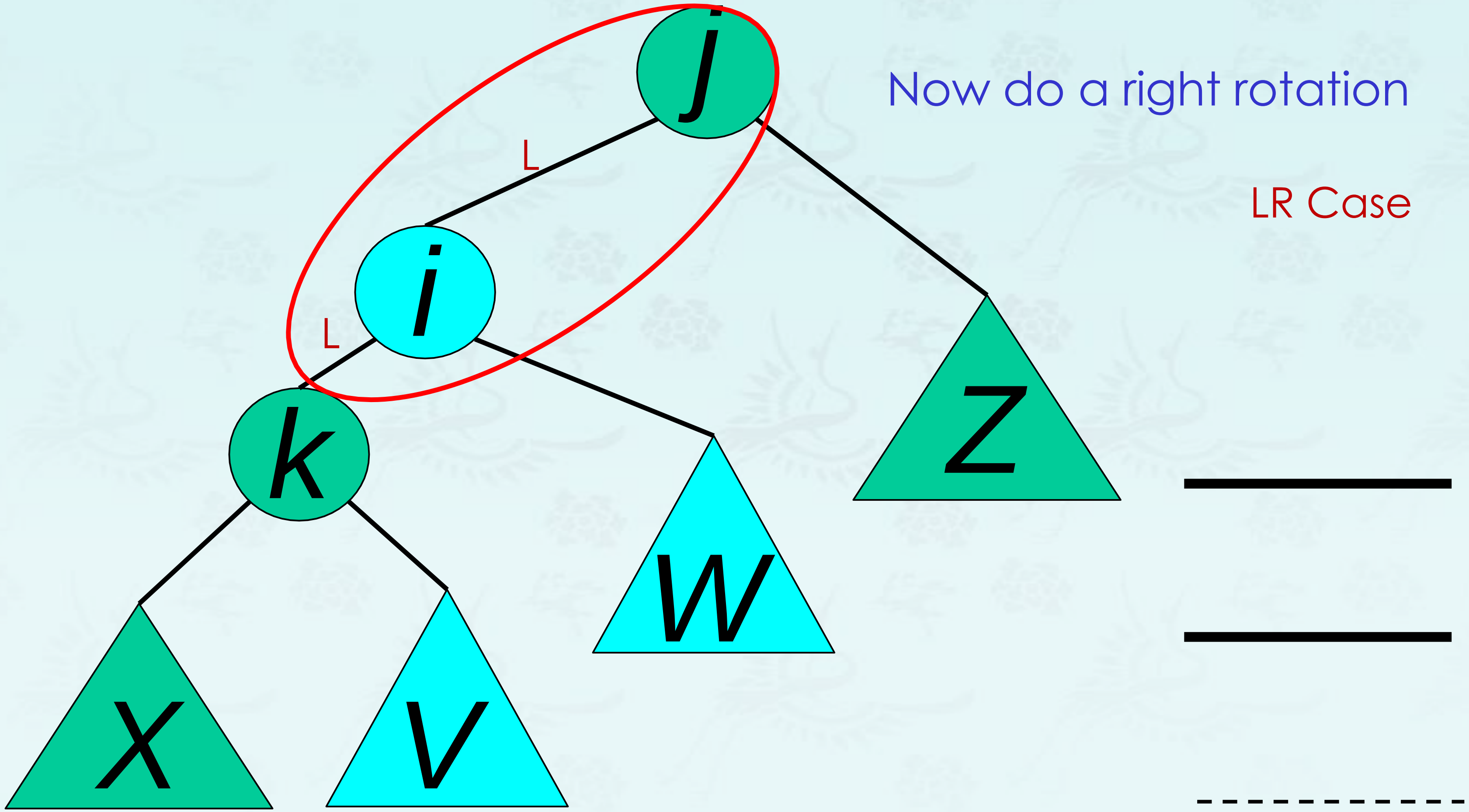
Double rotation : first rotation



left rotation complete

LR Case

Double rotation : second rotation



Now do a right rotation

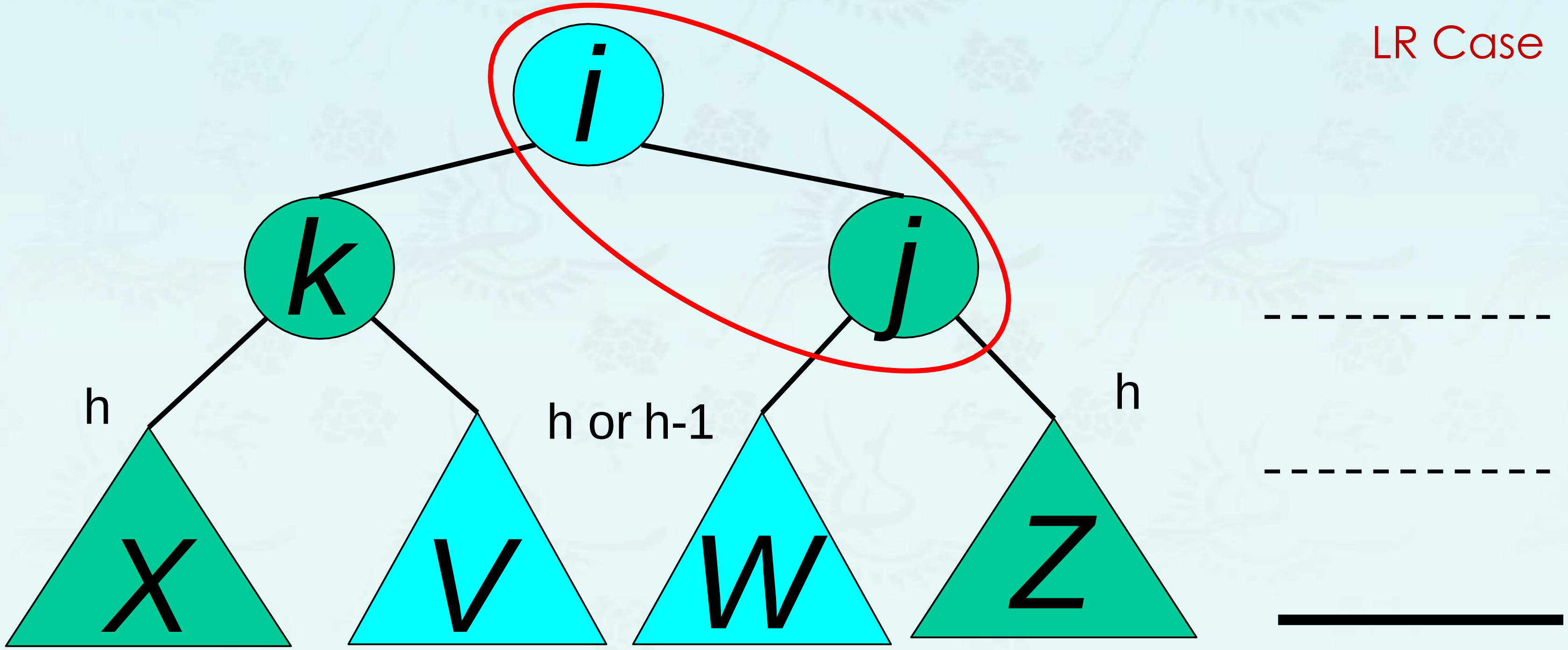
LR Case

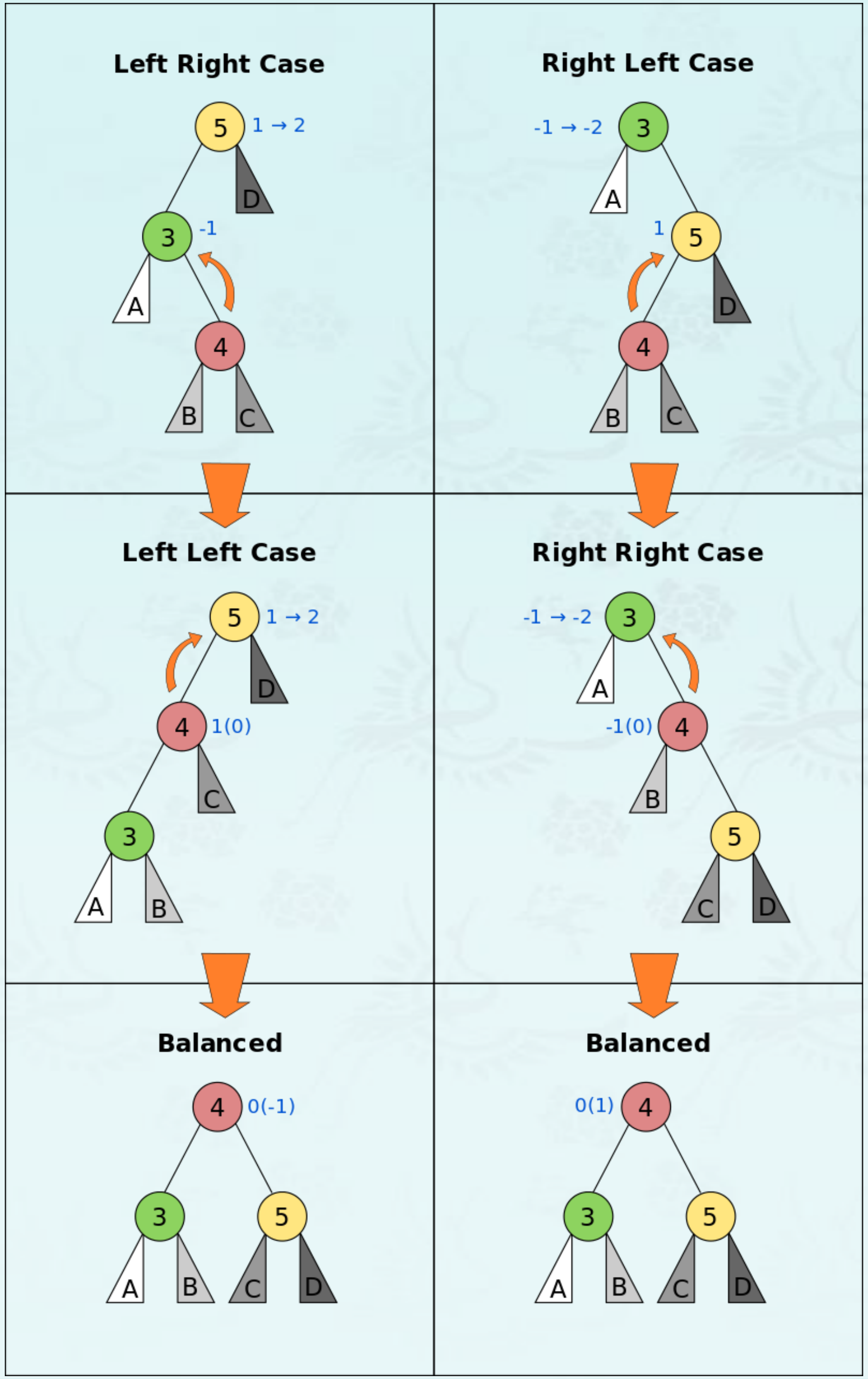
Double rotation : second rotation

right rotation complete

Balance has been restored

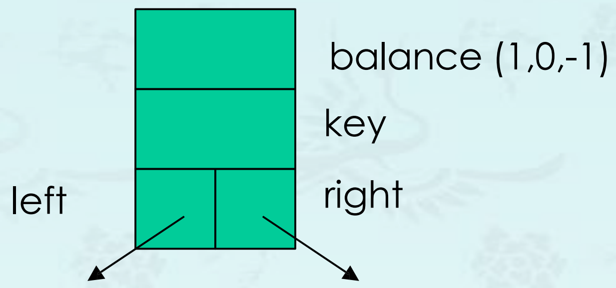
LR Case





- The numbered circles represent the nodes being rebalanced.
- The lettered triangles represent subtrees which are themselves balanced AVL trees.
- A blue number next to a node denotes possible balance factors
- (those in parentheses occurring only in case of deletion).
- Source: www.wikipedia.com

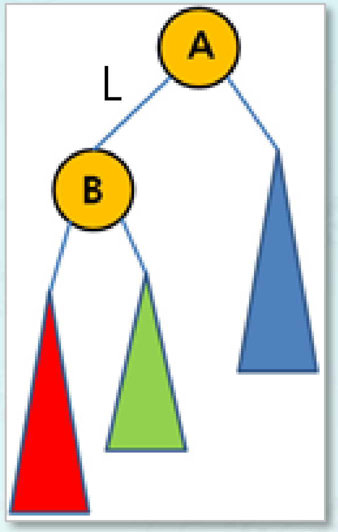
Implementation



- You can either keep the height or just the difference in height,
  - i.e. the balance factor; this has to be modified on the path of insertion even if you don't perform rotations
  - Once you have performed a rotation (single or double) you won't need to go back up the tree
- You may compute the balance factor on the fly after the insert is done during the recursion.

Single Rotation - LL case

outside case



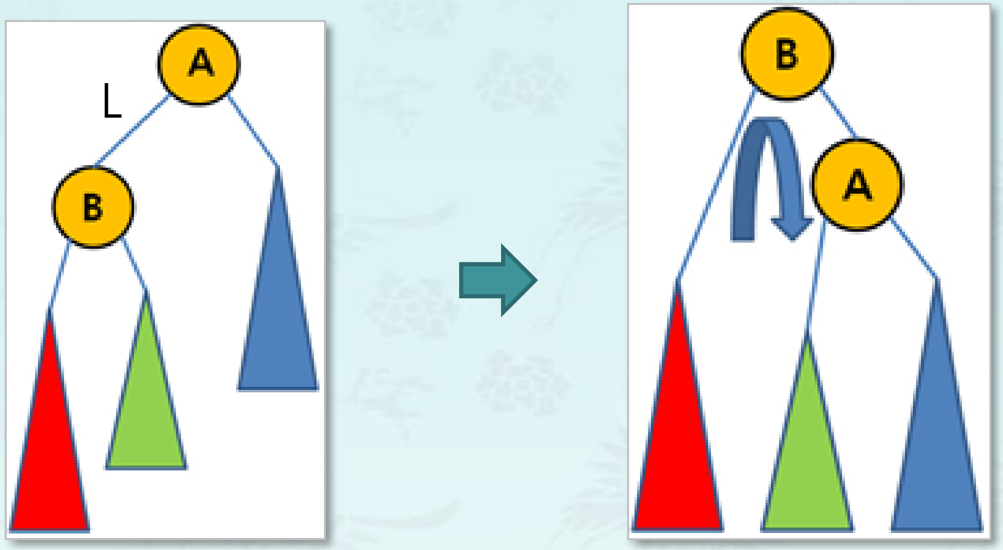

```
node rotateLL(node A)
{



    return
}
```

Single Rotation - LL case

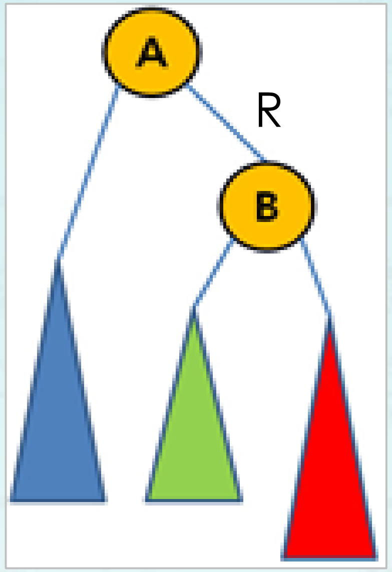
outside case




```
node rotateLL(node A)
{
  node B    = A->left;
  A->left   = B->right;
  B->right = A;
  return B;
}
```

Single Rotation – RR case

outside case

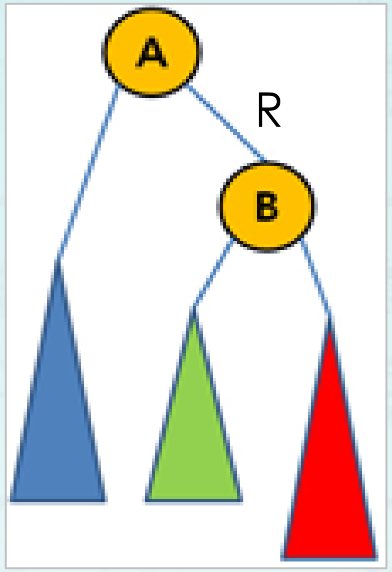


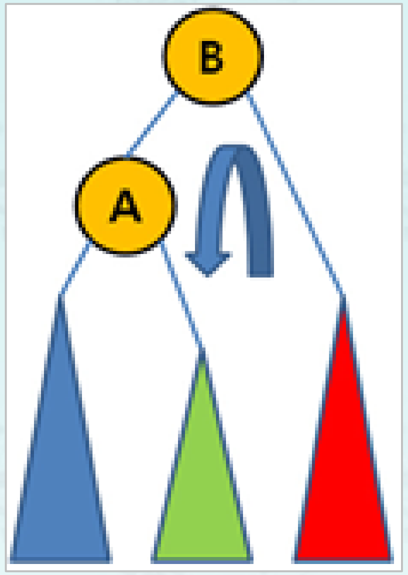

```
node rotateRR(node A)
{
    ?

    return ?
}
```

Single Rotation – RR case

outside case

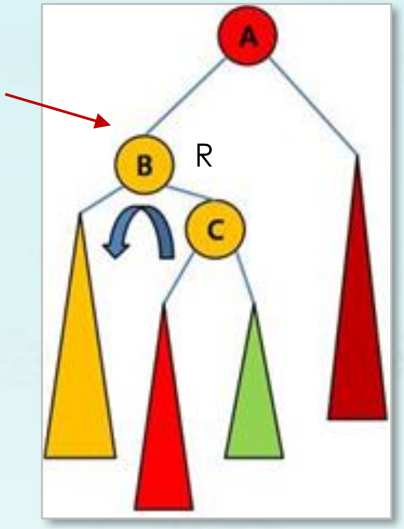





```
node rotateRR(node A)
{
  node B    = A->right;
  A->right = B->left;
  B->left  = A;
  return B;
}
```

Double Rotation - LR

inside case

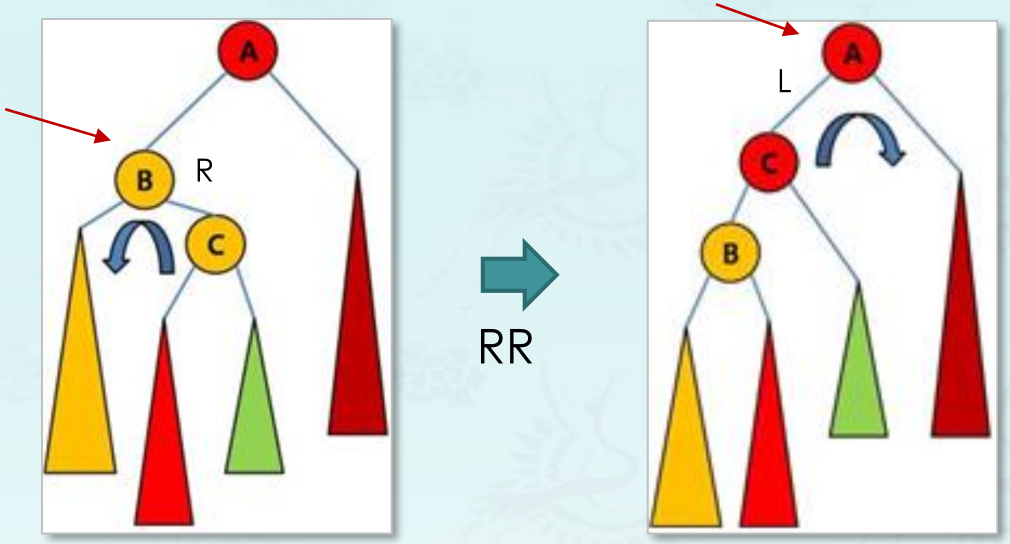



```
node rotateLR(node A) // RR and LL
{
}
```

Two rotations?

Double Rotation - LR

inside case



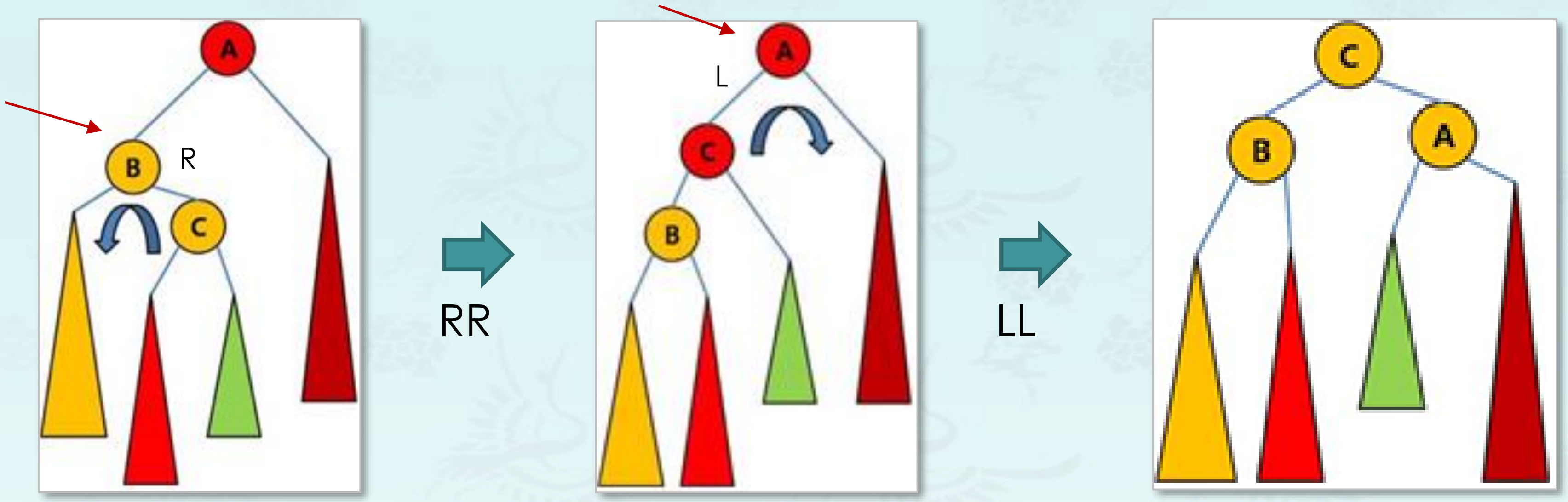

```
node rotateLR(node A) // RR and LL
{

}
```

Two rotations?

Double Rotation - LR

inside case



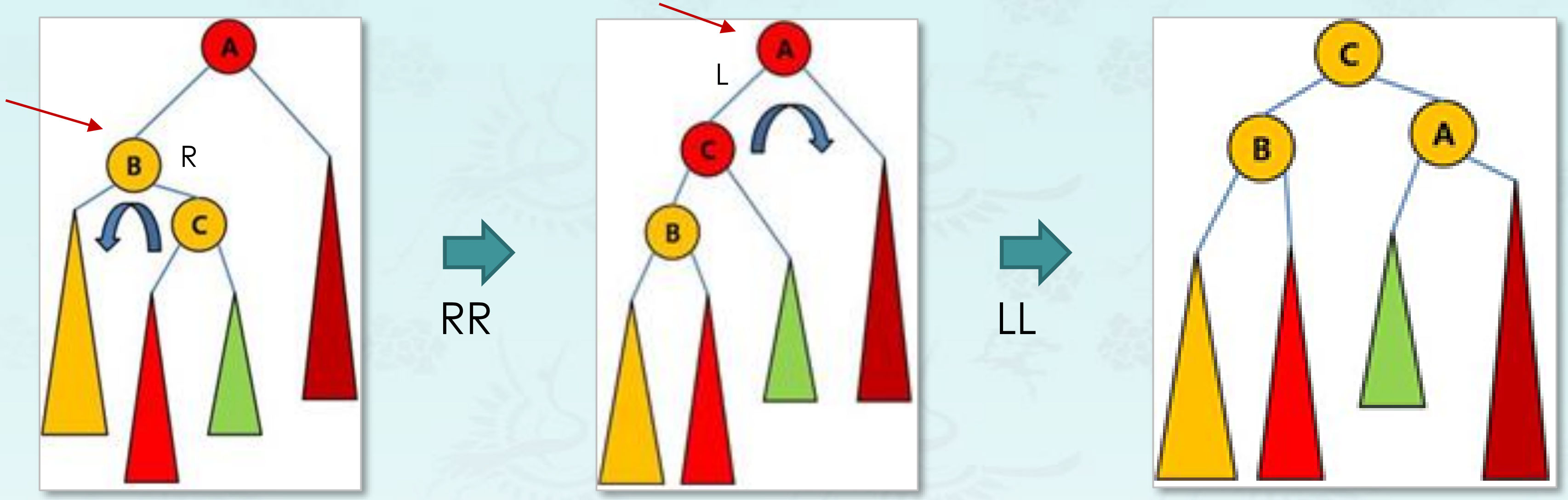

```
node rotateLR(node A) // RR and LL
{

}
```

Two rotations?

Double Rotation - LR

inside case

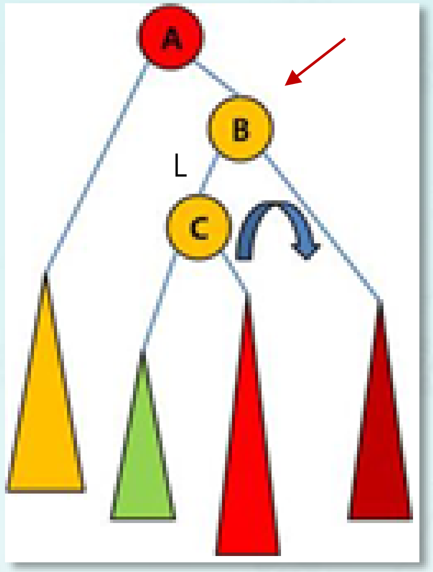



```
node rotateLR(node A) // RR and LL
{
  node B  = A->left;
  A->left = rotateRR(B);
  return rotateLL(A);
}
```

What will return eventually?

Double Rotation - RL

inside case



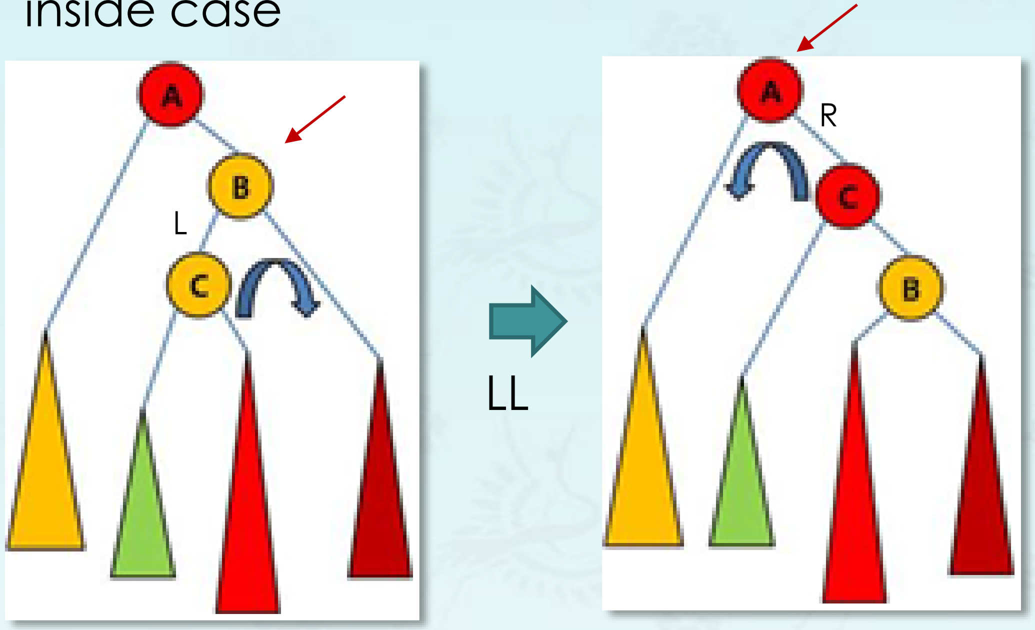

```
node rotateRL(node A) { // LL and RR
{

}
```

Two rotations?

Double Rotation - RL

inside case



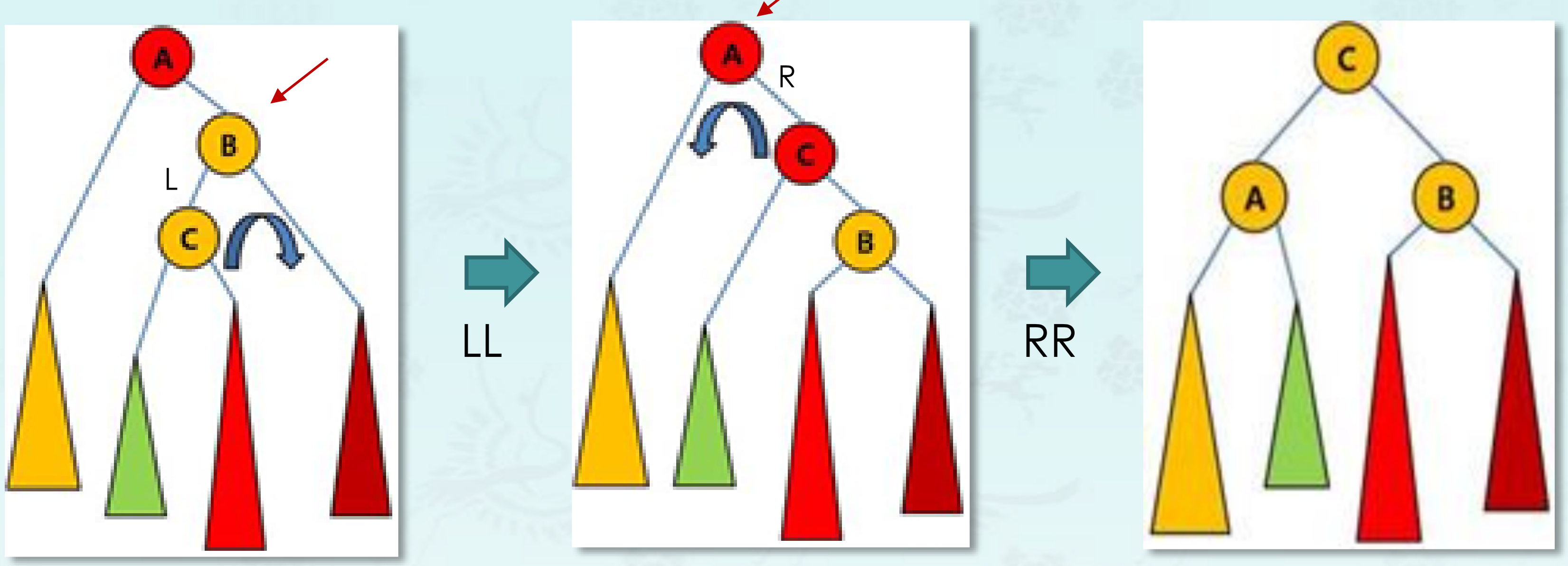

```
node rotateRL(node A) { // LL and RR
{

}
```

Two rotations?

Double Rotation - RL

inside case



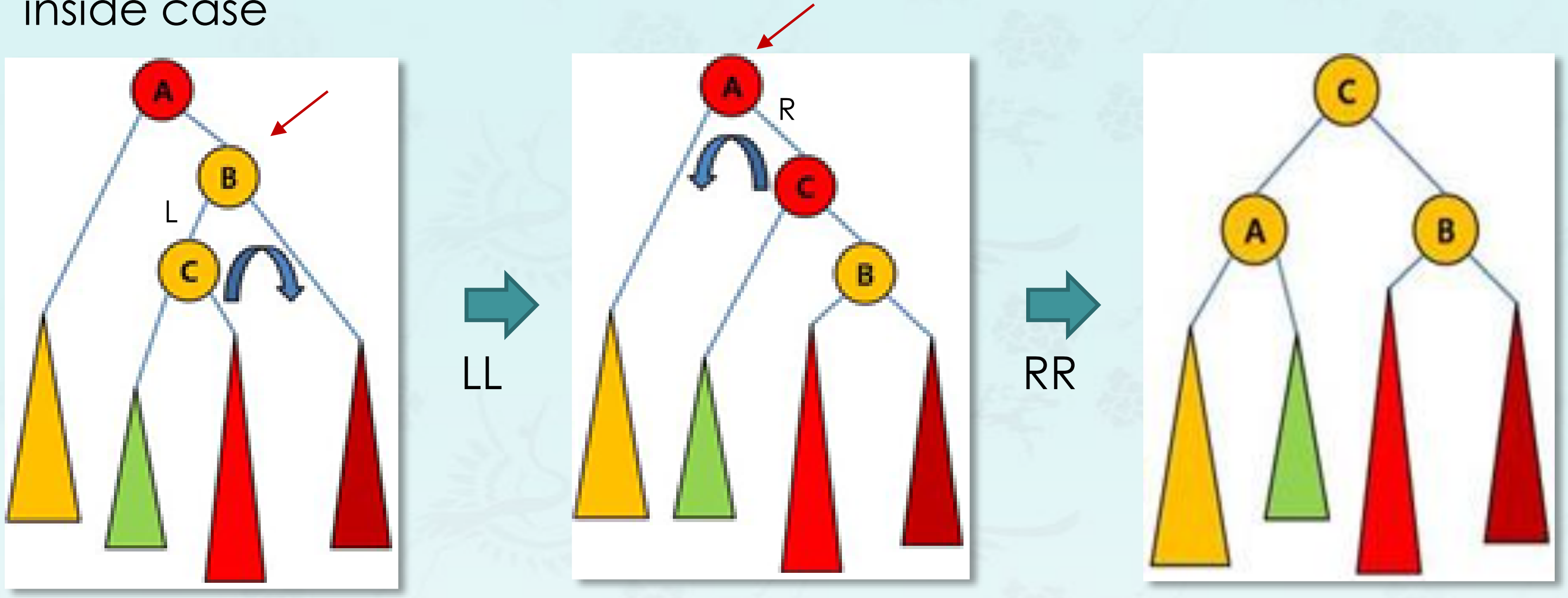

```
node rotateRL(node A) { // LL and RR
{


}
```

Two rotations?

Double Rotation - RL

inside case




```
node rotateRL(node A) { // LL and RR
{
  node B  = A->right;
  A->right = rotateLL(B);
  return rotateRR(A);
}
```

Two rotations?

rotateLL(B)

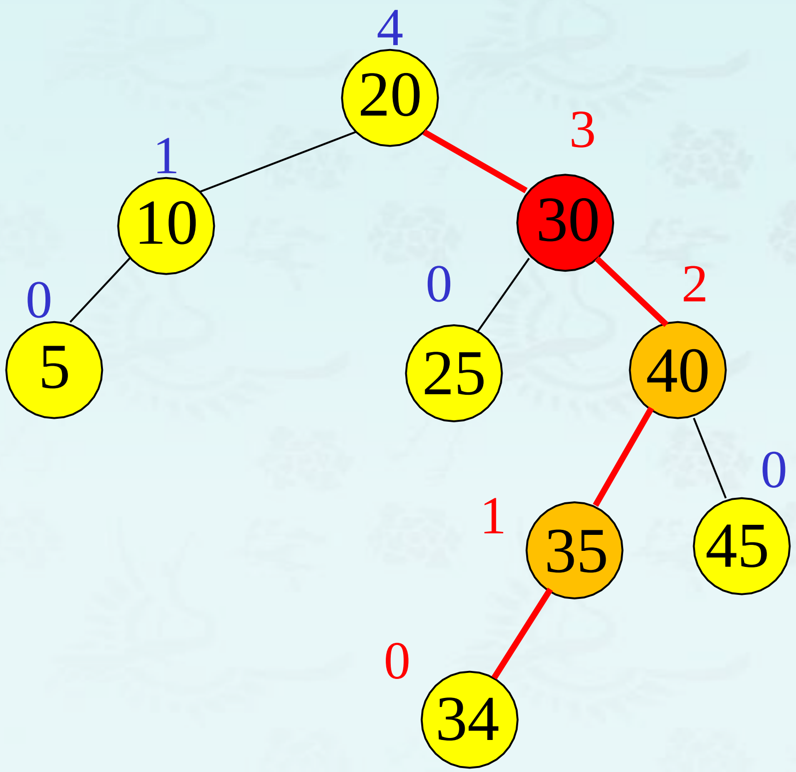
rotateRR(A)

Insertion of 34

Imbalance at 30

Double rotation RL

Balance factor at 30 = -2



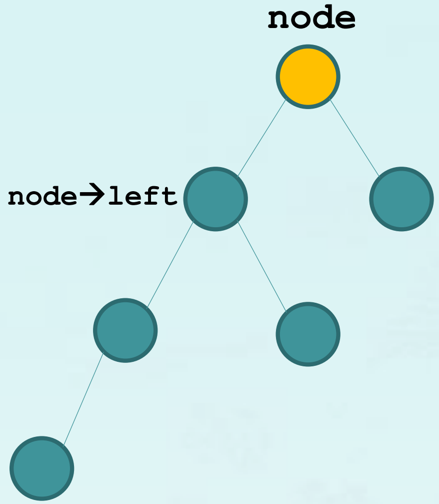
Balance Factor and Height

```
int getHeight(tree node) {
  if (node == NULL) return 0;
  int left  = getHeight (node->left);
  int right = getHeight(node->right);
  return (left > right) ? left + 1 : right + 1;
}
```

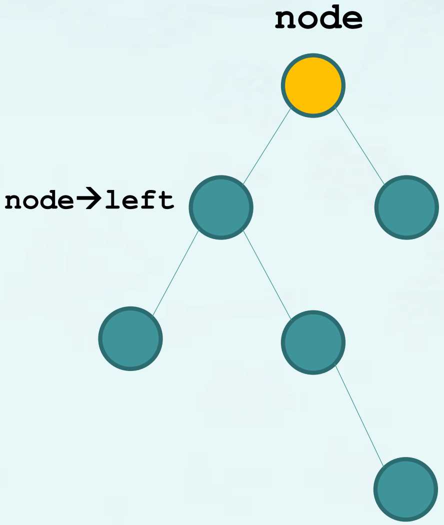
```
int balanceFactor(tree node) {
  if (node == NULL) return 0;
  int left  = getHeight(node->left);
  int right = getHeight(node->right);
  return left - right;
}
```

Rebalance

outside case



inside case




```
node rebalance(tree node) {
  bf = balanceFactor(node);
  if (bf >= 2) {
    if (balanceFactor(node->left) >= 1)
      node = rotateLL(node);      // LL
    else
      node = rotateLR(node);      // LR
  }
  else if (bf <= -2) {
    if (balanceFactor(node->right) <= -1)
      node = rotateRR(node);
    else
      node = rotateRL(node);
  }
  return node;
}
```

checking single or double rotation

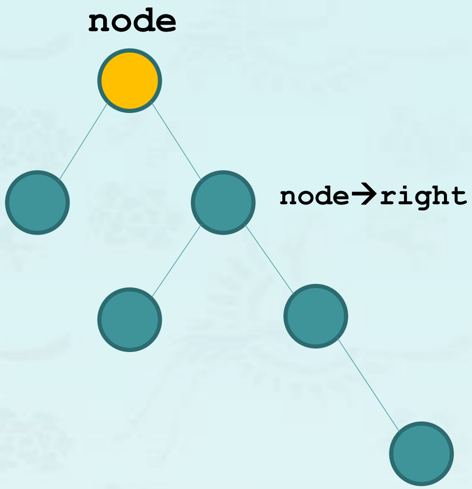
outside case

inside case

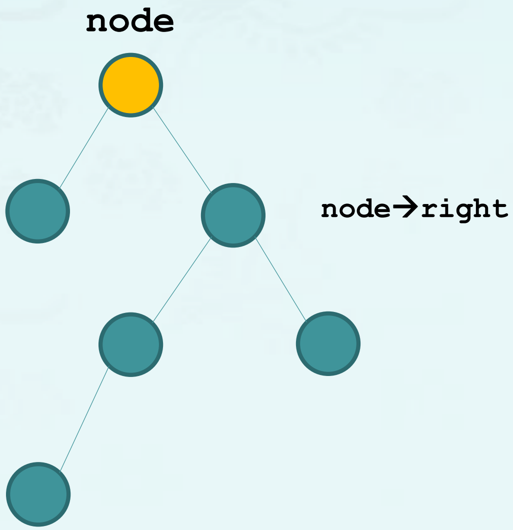
outside case

inside case

outside case



inside case

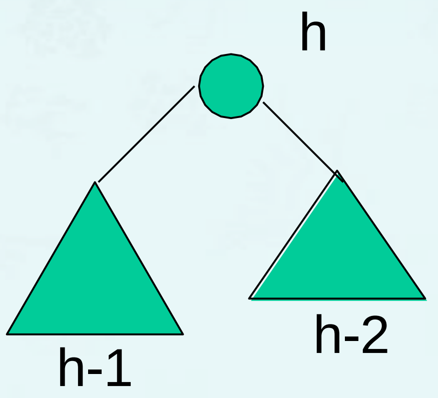


Height of an AVL Tree

AVL trees are binary search trees that balances itself every time an element is **inserted or deleted**. **Each node** of an AVL tree has the property that the heights of the sub-tree rooted at its children differ **by at most one**.

N(**h**) = minimum number of nodes in an AVL tree of height h.

- Base cases
  - N(1) = 1, N(2) = 2
- Induction
  - N(h) = **N(h-1)** + N(h-2) + 1
- Solution (compare it with Fibonacci analysis)
  - N(h) $\geq \phi^h$ ($\phi \approx 1.62$)



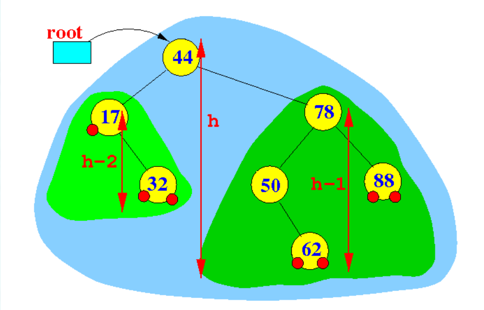
Suppose we have **n nodes** in an AVL tree of height h.

- $n \geq N(h)$
- $n \geq \phi^h$ hence $log_\phi n \geq h$
  (relatively well balanced tree!!)
- $\boldsymbol{h} \leq 1.44\ log_2 n$ (i.e., 'Find' operation takes O(log n))

Height of an AVL Tree

What is the maximum height of an AVL tree having exactly n nodes?

- To answer this question, we must ask this question first:
  What is the minimum number of nodes (sparsest possible AVL tree) an AVL?
- Consider **the minimum number of nodes** in an AVL tree of height h:



- We can get the recurrence relationship:

$$n(1) = 1$$
$$n(2) = 2$$
$$n(h) = n(h-1) + n(h-2) + 1$$

- This approximate solution (or the minimum of nodes in an AVL tree of height h) is known as $\mathrm{n(h)} \cong 1.62^h$
- Solve the equation above for h to get **the max height of an AVL tree** with n nodes?

$\log_2 n \geq \mathrm{h} * \log_2 1.62$
$\mathrm{h} \leq 1/\log_2 1.62 * \log_2 n$
$\mathrm{h} \leq 1.44 * \log_2 n$

Height of an AVL Tree

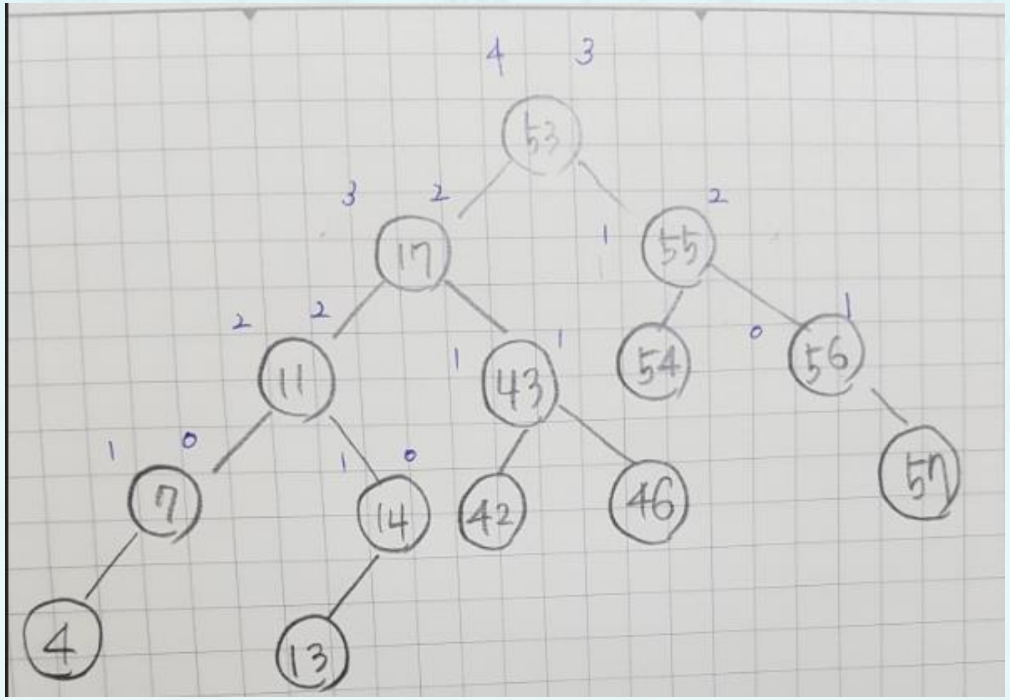
AVL trees are binary search trees that balances itself every time an element is **inserted or deleted**. **Each node** of an AVL tree has the property that the heights of the sub-tree rooted at its children differ **by at most one**.

- If there are n nodes in AVL tree, minimum height of AVL tree is floor($\log_2 n$).
- If there are n nodes in AVL tree, maximum height can't exceed $1.44 * \log_2 n$.
- If height of AVL tree is h, maximum number of nodes can be $2^{h+1} - 1$.
- Minimum number of nodes in a tree with height h can be represented as:
  N(h) = N(h-1) + N(h-2) + 1, where N(1) = 1 and N(2) = 2.
- The complexity of searching, inserting and deletion in AVL tree is O($\log_2 n$).
- The cost of balancing AVL tree is O(1).
  What is the time complexity of adding N elements to an empty AVL tree?
  Time complexity: log(1) + log(2) + …. + log(n) <= log(n) + log(n) + … + log(n) = n log (n)

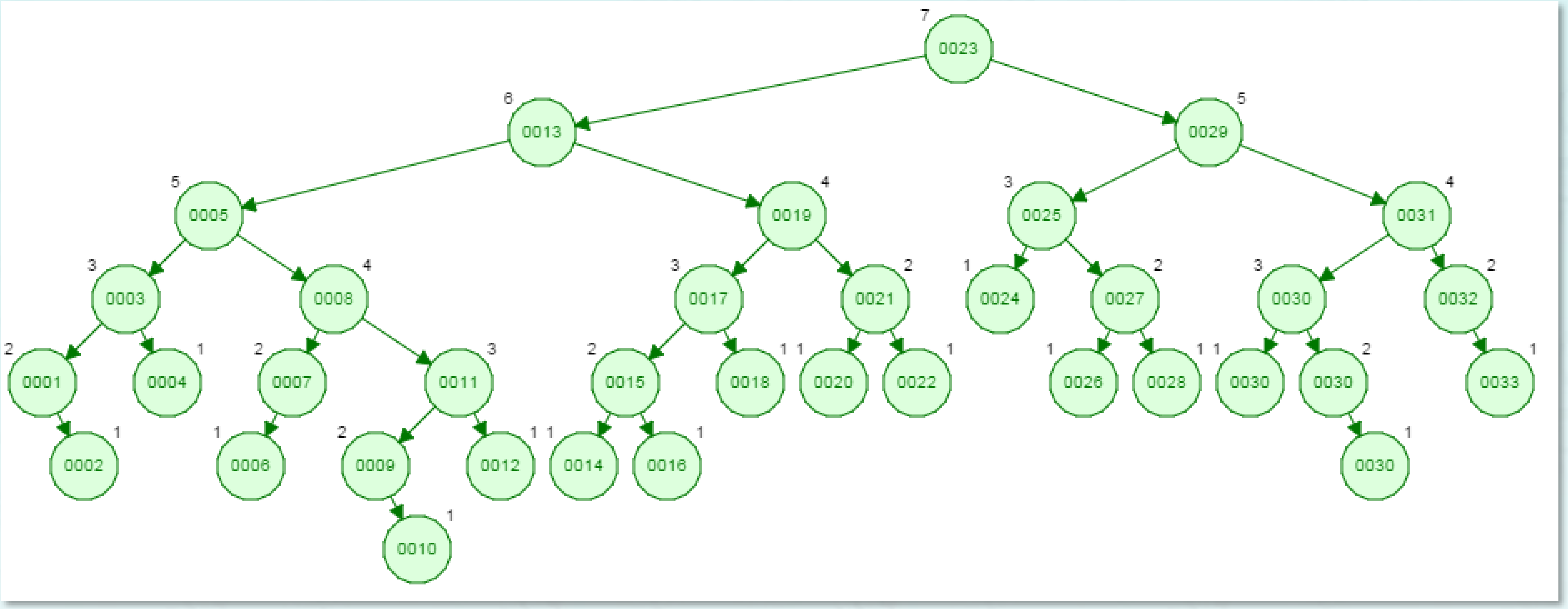
이것도 AVL tree가 될수 있을까요?

저는 AVL tree가 모든 노드의 왼쪽과 오른쪽의 height의 차이가 절대값 1을 넘어서지 않는 것이라고 알고있습니다. 근데 이 트리는 모든 노드에서 왼쪽과 오른쪽의 height의 차이가 절대값 1을 넘지는 않지만 55-54가 연결되어 있는 부분의 높이가 다른 쪽에 비해 2이상 차이나는 것을 보았습니다. 제가 아는 정의상으로는 AVL tree인거 같으면서도 저렇게 height가 2이상 차이가 나니... 결론을 내릴 수가 없어 질문 드립니다.

제가 AVL tree의 정의를 잘못 알고 있는건가요?



Example with leaf 24 on level 3 and leaf 10 on level 6:



AVL maintain the maximum height difference of 1 between two children subtree, not any two leaves.

The difference in levels of any two leaves can be any value!
The definition of AVL describes height difference only on two sub-trees from one node.

https://stackoverflow.com/questions/28964971/height-difference-between-leaves-in-an-avl-tree

Pros and Cons of AVL Trees

Arguments **for** AVL trees:

- Search is O(log n) since AVL trees are always balanced.
- Insertion and deletions are also O(log n)
- The height balancing adds no more than a constant factor to the speed of insertion.

Arguments **against** using AVL trees:

- **Difficult** to program & debug; more space for balance factor.
- Asymptotically faster but rebalancing costs time.
- Most large searches are done in database systems on disk and use other structures (e.g. **B-trees**).
- May be OK to have O(N) for a single operation if total run time for many consecutive operations is fast (e.g. Splay trees).

Homework: Draw AVL trees whenever the tree changes its shape by insertion and deletion. Include trees before and after its rotation and the type of rotation.

(1) [1.0p] Insert the sequence of elements (10, 20, 15, 25, 30, 16, 18, 19) into an AVL tree.
Delete 30 in the AVL tree that you got above and rebalance it.

(2) [0.5p] Delete 32 in the AVL tree shown below and rebalance it.

- Tree가 모양을 바꿀 때마다 AVL tree들을 그려서 다음 시간에 제출합니다.
- 각 단계별로 LL, RR, LR, RL을 분명히 표시하십시오. A4 한 두 장 혹은 앞뒤 면을 사용하면 한 장으로도 가능함