

Recursion

Data Structures
C++ for C Coders

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Data Structures

Chapter 1

- **algorithm specification**
recursive algorithm
- data abstraction
- performance analysis - time complexity

Algorithm Specification

- Input
- Output
- Definiteness – clear and unambiguous
- Finiteness – it terminates after a finite number of steps
- Effectiveness – it is carried out and feasible
- Ex. **program = algorithms + data structures**
flowchart is not an algorithm.

Recursion

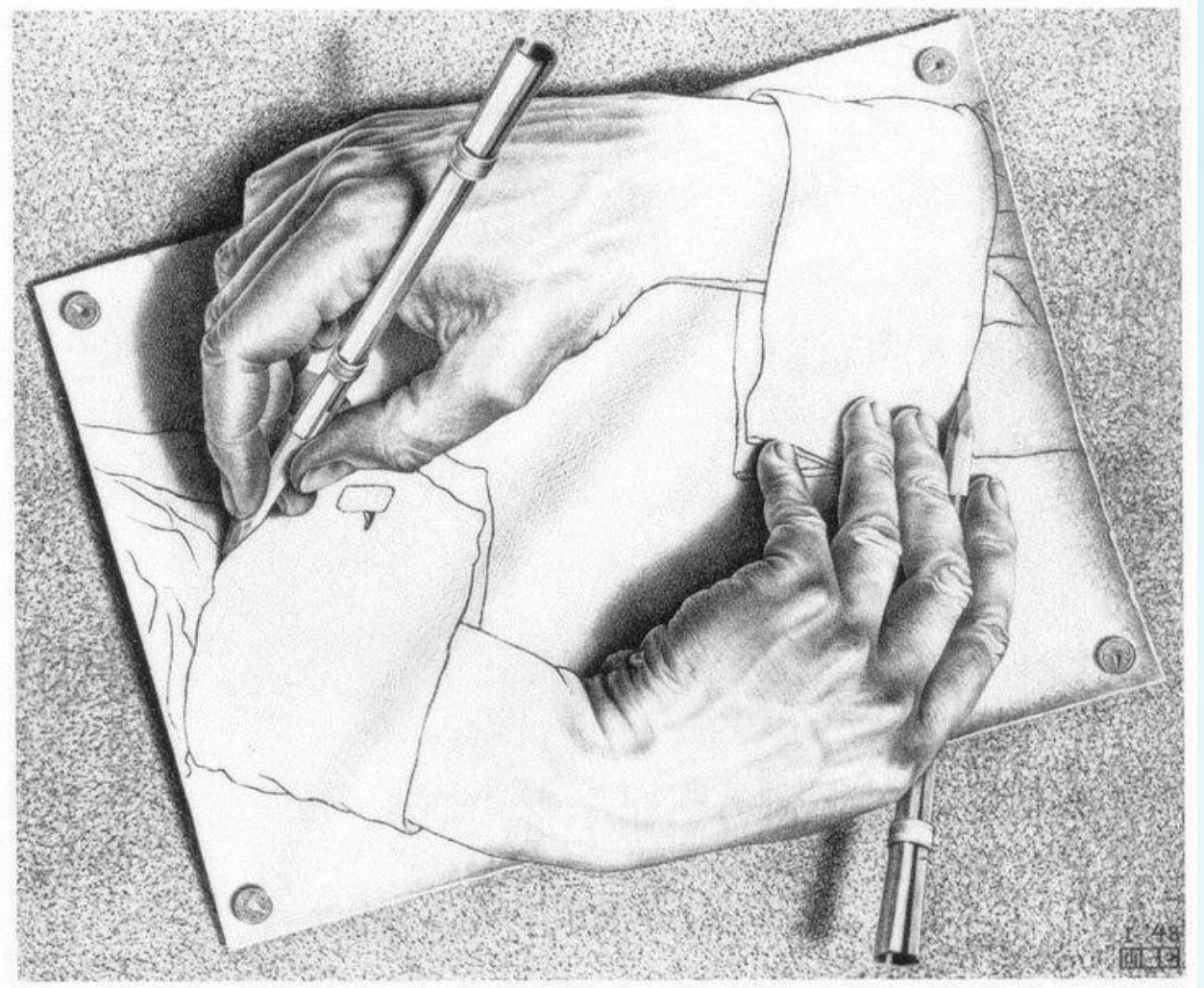
- See recursion
- A process in which the result of each repetition is dependent upon the result of the next repetition
- Simplifies program structure at a cost of function calls

Algorithm Specification

- Input
- Output
- Definiteness – clear and unambiguous
- Finiteness – it terminates after a finite number of steps
- Effectiveness – it is carried out by a human or machine
- Ex. **program = algorithms + data**
flowchart is not an algorithm

Recursion

- See recursion
- A process in which the result of the current step depends on the result of the next repetition
- Simplifies program structure at times



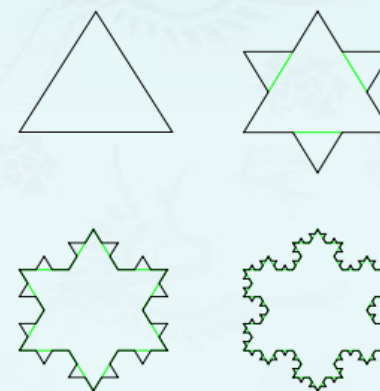
recursion is when a function calls itself

Recursive algorithms

Recursion is a method where the solution to a problem depends on solutions to **smaller** instances of the same problem (as opposed to iteration).

Recursive algorithm is expressed in terms of

1. **base case(s)** for which the solution can be stated **non-recursively**,
2. **recursive case(s)** for which the solution can be expressed in terms of a **smaller version of itself**.



Four stages in the construction of a **Koch snowflake**. The stages are obtained via a recursive definition.

Recursive algorithms

Example: Factorial

$$\text{fact}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot \text{fact}(n - 1) & \text{if } n > 0 \end{cases}$$

factorial(n)

function factorial

input: integer n such that $n \geq 0$

output: $[n \times (n-1) \times (n-2) \times \dots \times 1]$

1. if n is 0, **return** 1

2. otherwise, **return** $[n \times \text{factorial}(n-1)]$

end factorial

factorial ($n = 4$)

$$\begin{aligned} f_4 &= 4 * f_3 \\ &= 4 * (3 * f_2) \\ &= 4 * (3 * (2 * f_1)) \\ &= 4 * (3 * (2 * (1 * f_0))) \\ &= 4 * (3 * (2 * (1 * 1))) \\ &= 4 * (3 * (2 * 1)) \\ &= 4 * (3 * 2) \\ &= 4 * 6 \\ &= 24 \end{aligned}$$

Exercise: With four students, compute $4!$ using recursion.

Recursive algorithms

Example: Factorial

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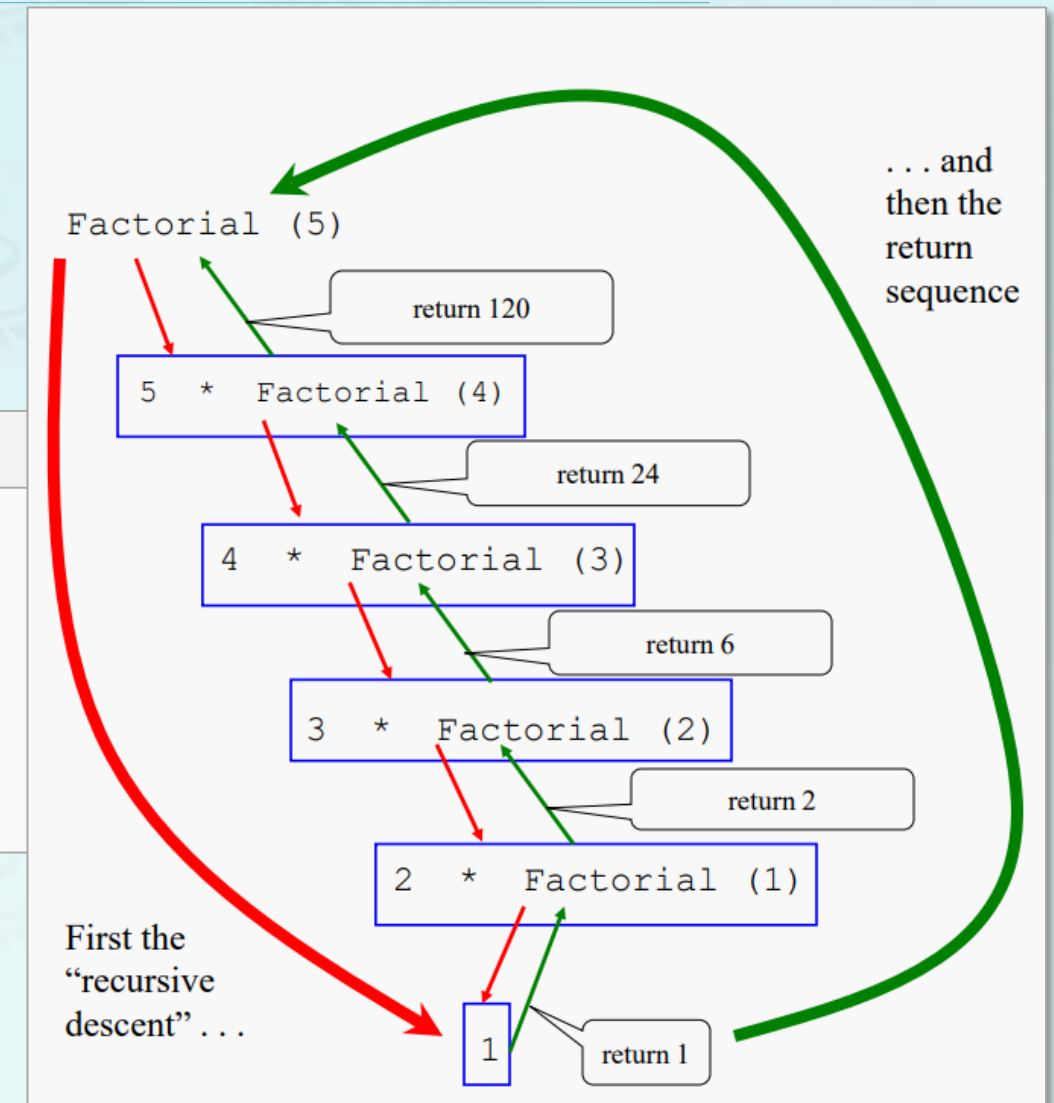
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Recursive algorithms

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end factorial

Exercise: GCD recursively with $\text{gcd}(x=259, y=111) = ?$

Recursive algorithms

Example: GCD (Great common divisor)

$$\text{gcd}(x, y) = \begin{cases} x & \text{if } y = 0 \\ \text{gcd}(y, \text{remainder}(x, y)) & \text{if } y > 0 \end{cases}$$

gcd(x, y)
function gcd input: integer x, y such that $x \geq y$, $y > 0$ output: gcd of x and y 1. if y is 0, return x 2. otherwise, return [gcd (y, $x \% y$)] end gcd

gcd (x=259, y=111)
gcd(259, 111) = gcd(111, $259 \% 111$) = gcd(111, 37) = gcd(37, $111 \% 37$) = gcd(37, 0) = 37

Exercises: gcd(91, 52)

Exercises: Fibonacci, Binomial coefficients, Akerman's function

Execution sequence of recursive functions:

Exercise: What is the output of the function (num=0)?

execution sequence

```
void recursiveFunction(int num) {  
    cout << num << endl;  
    if (num < 4)  
        recursiveFunction(num + 1);  
}
```

Execution sequence of recursive functions:

Exercise: What is the output of the function (num=0)?

execution sequence	
void	recursiveFunction(int num) {
	cout << num << endl;
	if (num < 4)
	recursiveFunction(num + 1);
	}

1	recursive_fun(0)			
2	cout << 0			
3	recursive_fun(0+1)			
4	cout << 1			
5	recursive_fun(1+1)			
6	cout << 2			
7	recursive_fun(2+1)			
8	cout << 3			
9	recursive_fun(3+1)			
10	cout << 4			



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execution sequence

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Execution sequence of recursive functions:

Exercise: What is the output of the function (num=0)?

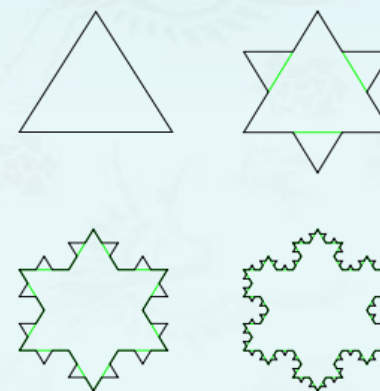
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void recursiveFunction(int num) {  
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1	recursive_fun(0)
2	recursive_fun(0+1)
3	recursive_fun(1+1)
4	recursive_fun(2+1)
5	recursive_fun(3+1)
6	cout << 4
7	cout << 3
8	cout << 2
9	cout << 1
10	cout << 0

Recursive algorithms

Recursion is a method where the solution to a problem depends on solutions to **smaller** instances of the same problem (as opposed to iteration).



Four stages in the construction of a **Koch snowflake**. The stages are obtained via a recursive definition.

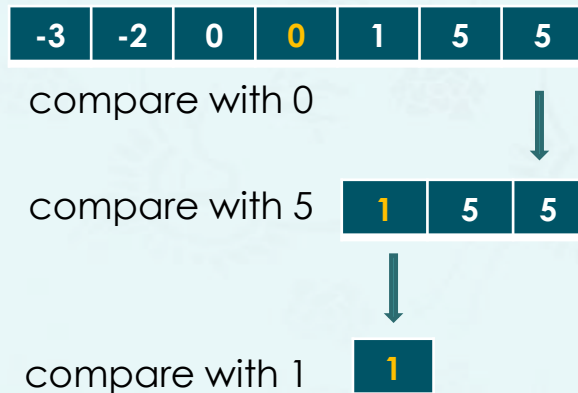
Recursive algorithms



Example: Recursive binary search

It searches a *sorted* array of **ints** for a particular **int**. Let **i** be an array of **ints** sorted from least to greatest. For instance, $\{-3, -2, 0, 0, 1, 5, 5\}$. We want to search **the array for the value** "wally". If we find "wally", we return its array *index*; otherwise, we return FAILURE(-1).

Let's suppose "wally" is 1.



Exercise: Base case(s) & recursive case(s):?

```
int binarySearch(int list[], int wally, int lo, int hi)
```

Recursive algorithms



Example: Recursive binary search

Exercise: Base case(s) & recursive case(s):?

```
int binarySearch(int list[], int wally, int lo, int hi)

if (lo > hi) return -1;                // base case

mid = (lo + hi)/2;
if (wally == list[mid]) return mid;    // base case
if (wally < list[mid])
    return binarySearch(list, wally, lo, mid-1 ); // recursive
else
    return binarySearch(list, wally, mid+1, hi ); // recursive
```

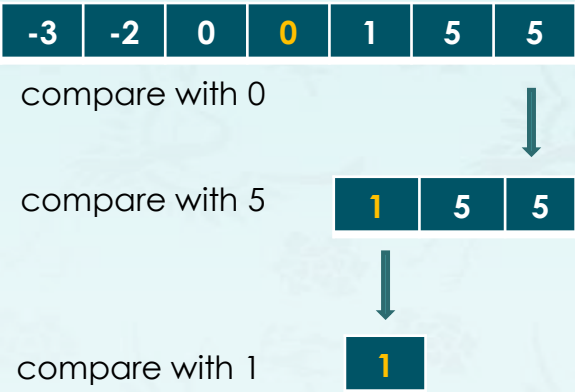
How long does the binarySearch() take?

In one call to binarySearch(), we eliminate at least half the elements from consideration. Hence, it takes $\log_2 n$ (the base 2 logarithm of n) binarySearch() calls to pare down the possibilities to one. Therefore binarySearch takes time proportional to $\log_2 n$.

Recursive algorithms



Example: Recursive binary search – revisited



	Stack	Stack	Heap
binsearch()	lo[4] hi[4] mid[4]	wally[1] list[.]	
binsearch()	lo[4] hi[6] mid[5]	wally[1] list[.]	
binsearch()	lo[0] hi[6] mid[3]	wally[1] list[.]	
binsearch()	wally[1]	list[.]	[-3 -2 0 0 1 5 5]
main()		args[.]	args[]

Most operating systems give a program enough stack space for a few thousand stack frames. If you use a recursive procedure to walk through a million-node list, the program will try to create a million stack frames, and **the stack will run out of space**. The result is a run-time error.

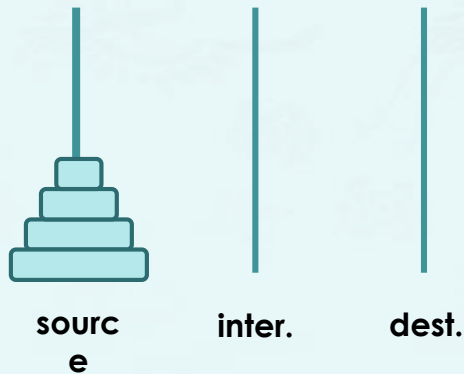
Recursive algorithms

Example: Tower of Hanoi (Refer to p.17, Ex11)

Given three pegs, one with a set of N disks of increasing size, determine the minimum (optimal) number of steps it takes to move all the disks from their initial position to a single **stack** on another peg *without placing a larger disk on top of a smaller one*. Only one disk can be moved at any time.

Recursive algorithm:

- (1) Move the top **$n-1$** disks from **source** to **intermediate**.
- (2) Move the remaining (**largest**) disk from **source** to **destination**.
- (3) Move the **$n-1$** disks from **intermediate** to **destination**.

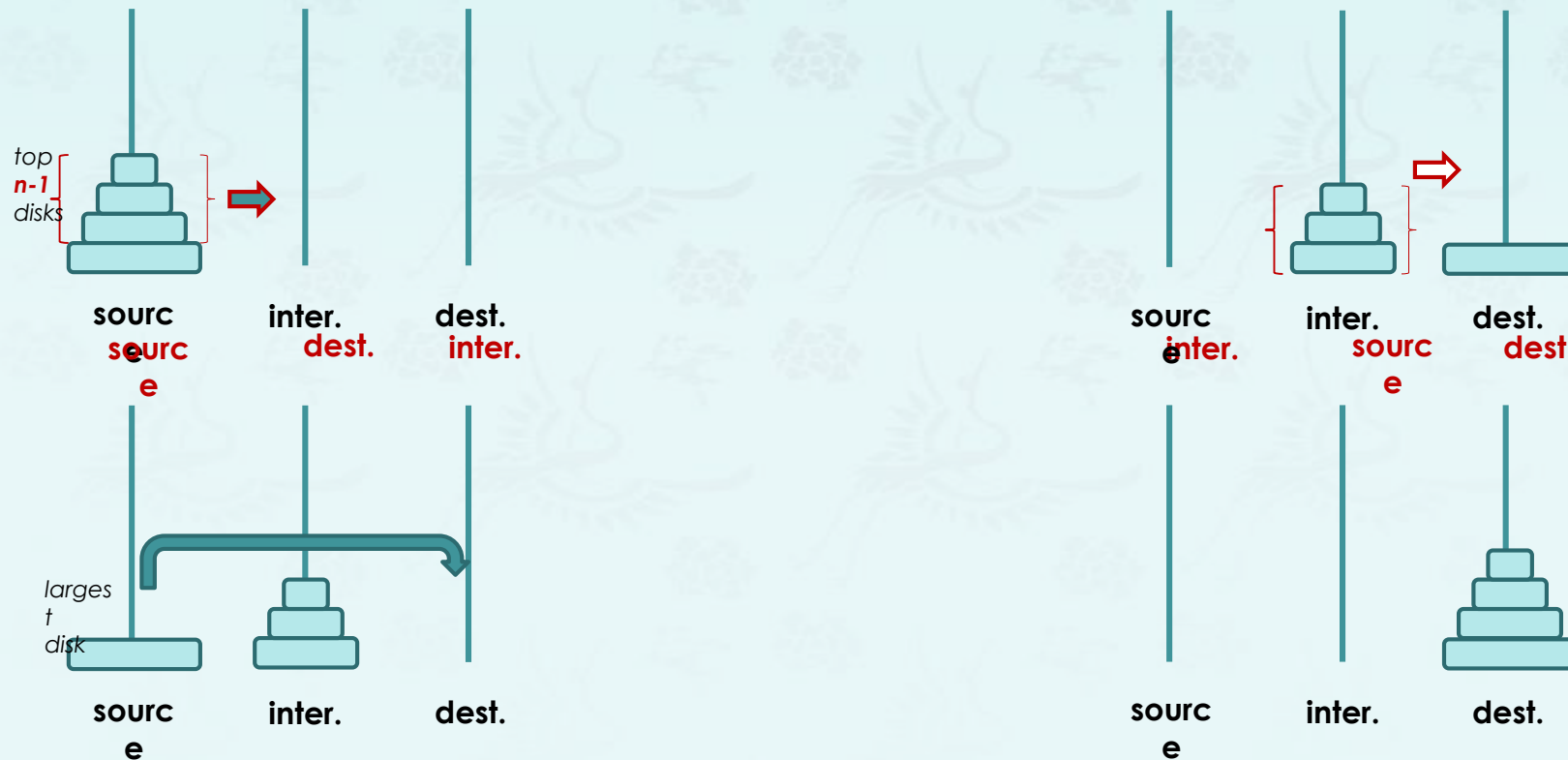


Recursive algorithms

Example: Tower of Hanoi

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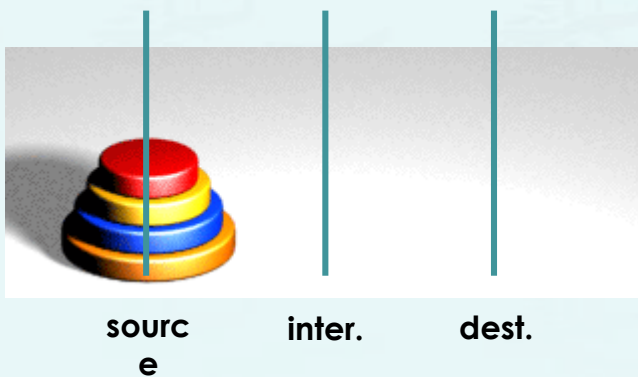


Recursive algorithms

Example: Tower of Hanoi

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Recursive algorithms

Exercise: Tower of Hanoi – revisited

Recursive algorithm:

- (1) Move the top ***n-1*** disks from **source** to **intermediate**.
- (2) Move the remaining (**largest**) disk from **source** to **destination**.
- (3) Move the ***n-1*** disks from **intermediate** to **destination**.

How do you program this to have the output as shown below?

Disk 1 from A to C
Disk 2 from A to B
Disk 1 from C to B
Disk 3 from A to C
Disk 1 from B to A
Disk 2 from B to C
Disk 1 from A to C

```
hanoi()

void hanoi(int n, char from, char inter, char to) {
    if (n == 1)
        printf("Disk 1 from %c to %c\n", from, to);
    else {
        hanoi(n - 1, from, to, inter);
        printf("Disk %d from %c to %c\n", n, from, to);
        hanoi(n - 1, inter, from, to);
    }
}
```



Recursive algorithms

Exercise: How many moves for n disks in Tower of Hanoi, $\text{hanoi}(n)$?

Recursive algorithm:

- (1) Move the top $n-1$ disks from **source** to **intermediate**.
- (2) Move the remaining (**largest**) disk from **source** to **destination**.
- (3) Move the $n-1$ disks from **intermediate** to **destination**.

$\text{hanoi}(n-1)$ move

$\text{hanoi}(1)$ move

$\text{hanoi}(n-1)$ move

$$\text{hanoi}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 \cdot \text{hanoi}(n-1) + 1 & \text{if } n > 1 \end{cases}$$

Exercise:

$\text{hanoi}(2) = 3$

$\text{hanoi}(4) = 15$

$\text{hanoi}(32) = 4,294,967,295$

$\text{hanoi}(64) = 18,446,744,073,709,600,000$

$\text{hanoi}(n = 4)$

$\text{hanoi}(4)$

$= 2 \cdot \text{hanoi}(3) + 1$

$= 2 \cdot (2 \cdot \text{hanoi}(2) + 1) + 1$

$= 2 \cdot (2 \cdot (2 \cdot \text{hanoi}(1) + 1) + 1) + 1$

$= 2 \cdot (2 \cdot (2 \cdot 1 + 1) + 1) + 1$

$= 2 \cdot (2 \cdot (3) + 1) + 1$

$= 2 \cdot (7) + 1 = 15$

How many years will take to move 64 disks?

$\sim 584,942,417,355$ years
https://hanoi.aimary.com/index_en.php

Recursive algorithms

Q: Is the recursive version usually faster?

A: No -- it's usually slower (due to the overhead of maintaining the stack)

Q: Does the recursive version usually use less memory?

A: No -- it usually uses **more** memory (for the stack).

Q: Then **why** use recursion?

A: Sometimes it is much simpler to write the recursive version.

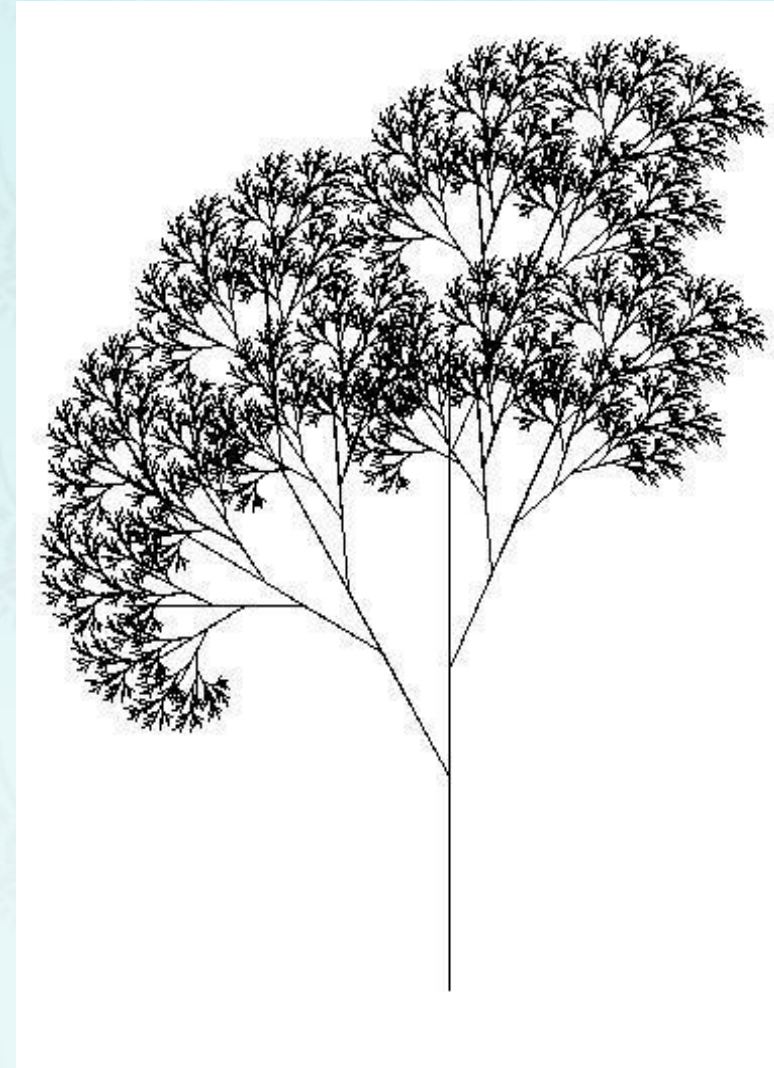
How the function call work? See[System Stack] in p.108.

*Because the recursive version causes an **activation record** to be pushed onto the system stack for every call, it is also more limited than the iterative version (it will fail, with a "stack overflow" error), for large values of N.*



Sierpinski Triangle:
a confined recursion of
triangles to form a geometric
lattice

Recursive algorithms



Recursion
GNU

see *Recursion*
GNU's not Unix.

Data Structures

Chapter 1

- algorithm specification
recursive algorithm
- **data abstraction**
- performance analysis - time complexity