

# Time Complexity

**Data Structures**  
**C++ for C Coders**

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## Performance Analysis

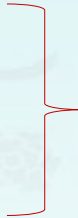
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The program we write should

1. meet the specification.
2. work correctly.
3. be documented properly.
4. run effectively
5. be readable.

**6. use the storage effectively – space**

**7. run timely – time**



space & time complexity

The **space complexity** of a program is the amount of **memory** that it needs to run to completion.

The **time complexity** of a program is the amount of computer **time** that it needs to run to completion.

## Performance Analysis

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### Space complexity:

1. Fixed space requirements :  $c$ 
  - that do not depend on input size, simple or fixed-size variables
2. Variable space requirements :  $S_p(I)$ 
  - that depend on the instance  $I$ , stack, variable

The total space requirement for the program  $P$ :

$$S(P) = c + S_p(I)$$

where  $c$  is a constant for fixed space and variable space for the instance  $I$ .

We are concerned about only  $S_p(I)$ , but not  $c$ . **Why?**

Because we usually **compare** the algorithms of the programs.

## Performance Analysis

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**Space complexity:**  $S(P) = c + S_p(I)$

**Example:**  $S_{\text{sum}}(\mathbf{n}) = ?$

Program1.11

```
float sum(float list[], int n) {  
    float total = 0;  
    for (int i=0; i<n; i++)  
        total += list[i];  
    return total;  
}
```

$S_{\text{sum}}(\mathbf{n}) = 0$  since the C passes list[] by its address.

## Performance Analysis

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**Space complexity:**  $S(P) = c + S_p(I)$

**Example:**  $S_{\text{rsum}}(n=\text{MAX\_SIZE}) = ?$

Program1.12

```
float rsum(float list[], int n) {  
    if (n)  
        return rsum(list, n-1) + list[n-1];  
    return 0;  
}
```

The variable space requirement are for **two** parameters and **one** return address are saved in the system stack **per recursive call**:

$$\text{sizeof}(n) + \text{list[] address} + \text{return address} = 12$$

$$S_{\text{sum}}(n) = 12 * n$$

# Performance Analysis

**Time complexity:** The time taken by the program P:

$$T(P) = \text{compile time } c + \text{execution time } T_p(n)$$

Similarly, we are concerned about only  $T_p(n)$ , but not  $c$ .

**Example:**  $T_p(n) = c_a ADD(n) + c_s SUB(n) + c_l LDA(n) + c_{st} STA(n)$

where  $n$  – number of execution,  $c$  for constant time for operation

°°°  
We are not concerned  
about this, but ...

**Program step:** a meaningful program segment whose execution time is independent of the instance characteristics.

**Example:**

$a = 2;$

⇒ 1 step!!

$a = 2 * b + 3 * c/d - e + f/g/a/b/c ;$

⇒ 1 step!!

## Performance Analysis

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**Example:** How many **program steps** required?

Program sum	$2n+3$
<pre>float sum(float list[], int n) {     float total = 0;     for (int i=0; i&lt;n; i++)         total += list[i];     return total; }</pre>	<pre>1 n+1 n 1</pre>

## Performance Analysis

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**Exercise:** How many **program steps** required?

Program rsum	$2n + 2$
<pre>float rsum(float list[], int n) {     if (n)         return rsum(list, n-1) + list[n-1];     return 0; }</pre>	$n + 1$ $n$ $1$



## Performance Analysis

### Comparison:

Program    sum

```
float sum(float list[], int n) {  
    float total = 0;  
    for (int i=0; i<n; i++)  
        total += list[i];  
    return total;  
}
```

Program    rsum

```
float rsum(float list[], int n) {  
    if (n)  
        return rsum(list, n-1) + list[n-1];  
    return 0;  
}
```

$$2n + 3 > 2n + 2$$

$$\text{sum} > \text{rsum}$$

$$(\text{iterative}) > (\text{recursive})$$

$$\Rightarrow T_{\text{iterative}} > T_{\text{recursive}}$$

## Performance Analysis

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**Example:** How many **program steps** required?

Program	sum of matrix
<pre>void add(int a[][MAX_SIZE], int b[][MAX_SIZE],         int c[][MAX_SIZE], int rows, int cols) {     for(int i=0; i&lt;rows; i++)         for(int j=0; j&lt;cols; j++)             c[i][j] = a[i][j] + b[i][j]; }</pre>	<pre>rows + 1 rows * (cols+1) rows * cols</pre>

step count = **2 rows\*cols + 2 rows + 1**

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

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Why step count?

It is to compare the **time complexities** of two programs that compute the same function and also to predict the **growth rate** in run time.

**Example:** Let's compute the step count for three programs and compare their time complexities.

1.  $T_{\text{add}}(n)$  – adding two numbers
2.  $T_{\text{sum}}(n)$  – adding list of numbers
3.  $T_{\text{mtx}}(n)$  – adding two matrix

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점근표기법

Program <b>add</b>	step count
<pre>float add(int a, int b) {     return    a + b; }</pre>	1

Program <b>sum of list</b>	step count
<pre>float sum(float list[], int n) {     float total = 0;     int i;     for (i=0; i&lt;n; i++)         total += list[i];     return total; }</pre>	1  n + 1 n 1

Program <b>sum of matrix</b>	step count
<pre>void add(int a[][MAX_SIZE], int b[][MAX_SIZE],          int c[][MAX_SIZE], int rows, int cols) {     for(int i=0; i&lt;rows; i++)         for(int j=0; j&lt;cols; j++)             c[i][j] = a[i][j] + b[i][j]; }</pre>	rows + 1 rows * (cols+1) rows * cols

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

What is **the exact number of times** `sum++` executed?

	Step count	
<pre>int sum = 0; for (int i = 1; i &lt;= n*n; i++)     for (int j = 1; j &lt;= i; j++)         sum++;</pre>	<pre>1 n * n + 1 2 + 3 + ... + n*n+1 ?</pre>	

**Useful formulas:**

$$1 + 2 + 3 + \dots + N = N(N+1)/2$$

$$1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1$$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

What is **the exact number of times** `sum++` executed?

	Step count
<pre>int sum = 0; for (int i = 1; i &lt;= n; i++)     for (int j = n; j &gt;= i; j--)         sum++;</pre>	<pre>1 n + 1 (n + 1) + n + (n-1) + ... + 2 ?</pre>

**Useful formulas:**

$$1 + 2 + 3 + \dots + N = N(N+1)/2$$

$$1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1$$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

What is **the exact number of times** `sum++` executed?

	Step count
<pre>int sum = 0; while (n &gt; 1) {     sum++;     n /= 2; }</pre>	$n / 2^k = 1$

We have to find the smallest  $k$  such that  $n / 2^k = 1$

### Useful formulas:

$$1 + 2 + 3 + \dots + N = N(N+1)/2$$

$$1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1$$

$$n / 2^k = 1$$

$$n = 2^k$$

$$\log(n) = \log(2^k)$$

$$\log(n) = k$$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

Compute the following series:

a)  $1 + 2 + 3 + \dots + 9 + 10 =$

b)  $1 + 2 + 3 + \dots + (N - 1) + N =$

c)  $1 + 2 + 4 + \dots + 16 =$

Compute the following series and express the result in term of  $N$  but without log expression. (Hint:  $N = 2^{\log N}$ )

Then use the result and to compute the series shown above in c):

d)  $1 + 2 + 4 + \dots + N =$

**Useful formulas:**

$$1 + 2 + 3 + \dots + (N-1) + N = N(N+1)/2$$

$$1 + 2 + 4 + 8 + \dots + 2^{n-1} + 2^n = 2^{n+1} - 1$$

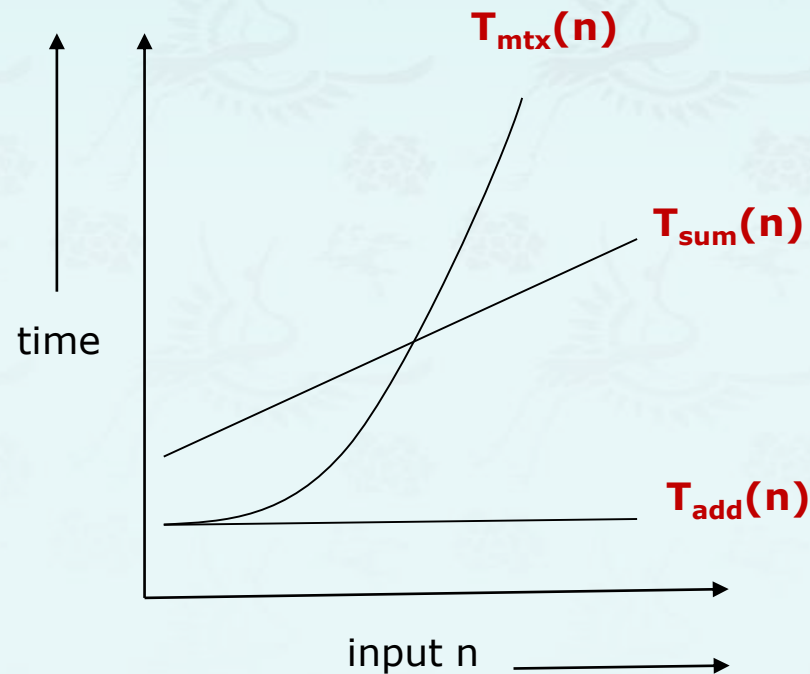


## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

$$T_{add}(n) = 2 \quad \rightarrow O(1)$$

$$\begin{aligned} T_{sum}(n) &= 1 + 2(n + 1) + 2n + 1 = 4n + 4 \quad \rightarrow O(n) \\ &= c * n + c' \end{aligned}$$

$$\begin{aligned} T_{mtx}(n) &= 2 \text{ rows} * \text{cols} + 2 \text{ rows} + 1 \quad \rightarrow O(n^2) \\ &= a * n^2 + b * n + c \end{aligned}$$

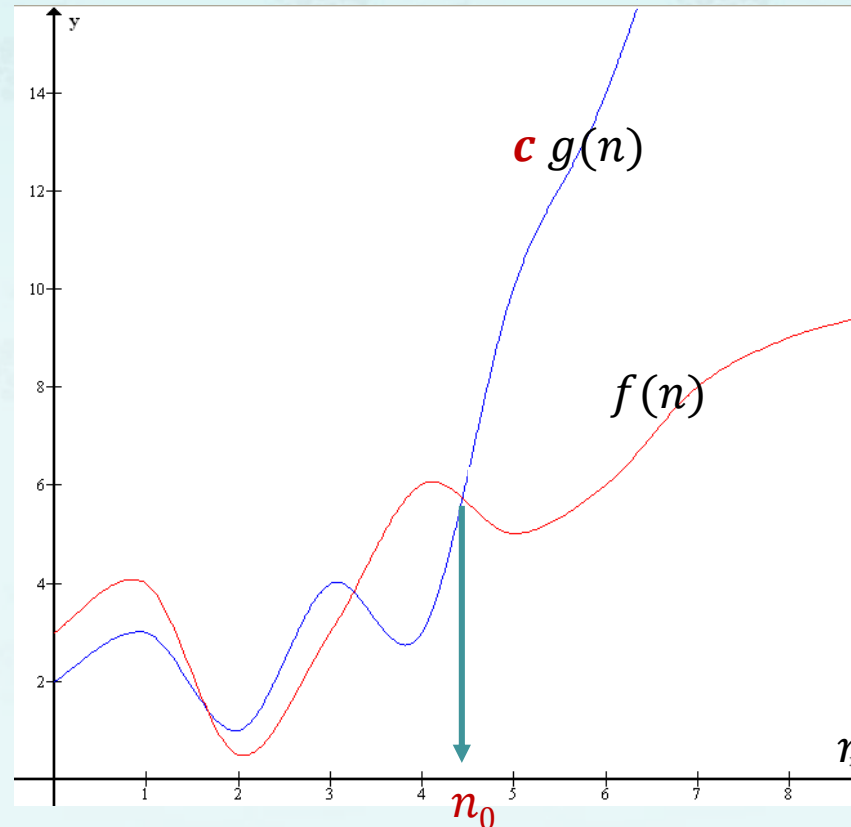


## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

### The "Big-Oh" Notation:

Let  $f(n)$  and  $g(n)$  be functions mapping nonnegative integers to real numbers. We say that  **$f(n)$  is  $O(g(n))$**  iff there are positive constants  **$c$**  and  **$n_0$**  such that

$$f(n) \leq c g(n), \text{ for } n \geq n_0.$$



**$f(n)$  is  $O(g(n))$**

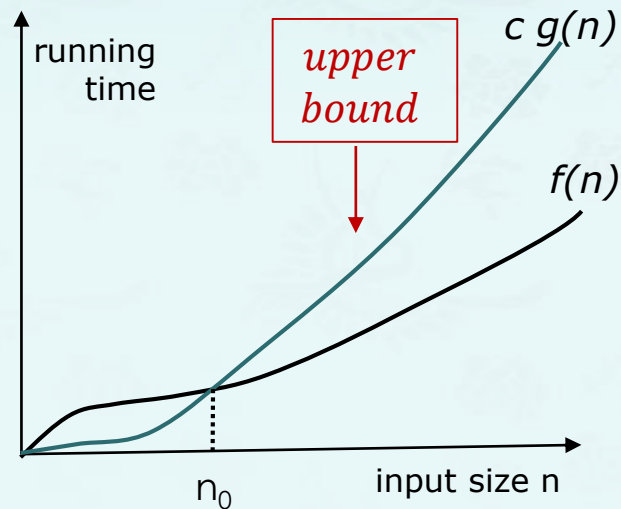
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$$f(n) \leq c g(n), \text{ for } n \geq n_0.$$

Then it is pronounced as " $f(n)$  **is big Oh** of  $g(n)$  or  $f(n) = O(g(n))$ "



**Example:** Justify that the function  **$8n - 2$  is  $O(n)$** .

Given  $f(n) = 8n - 2$ ,  $g(n) = n$ , we need to find  **$c$**  and  **$n_0$**  such that  **$8n - 2 \leq c n$**  for every integer  $n \geq n_0$ .

An easy choice among many is  $c = 8$  and  $n_0 = 1$ . Therefore,  $f(n) = 8n - 2$  is  **$O(n)$** .

$g(n) = n$

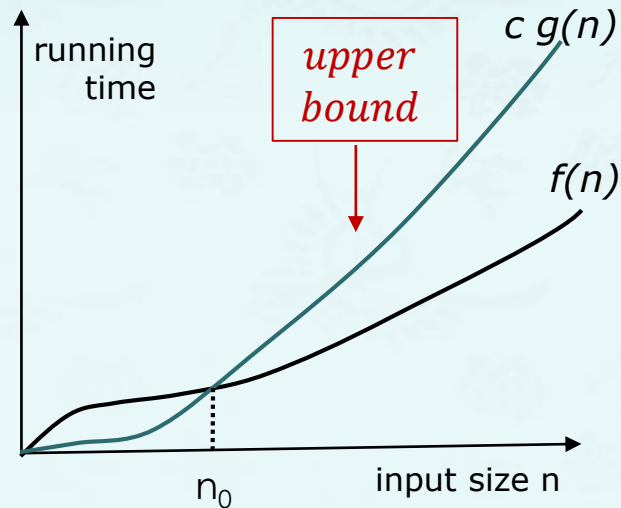
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$$f(n) \leq c g(n), \text{ for } n \geq n_0.$$

Then it is pronounced as " $f(n)$  **is big Oh** of  $g(n)$  or  $f(n) = O(g(n))$ "



Find  **$c$**  and  **$n_0$**  to justify that the function  $7n + 5$  **is  $O(n)$** .

**$7n + 5$  is  $O(n)$** , we have to find  **$c$**  and  **$n_0$**  such that

$$7n + 5 \leq c n \text{ for } n \geq n_0$$

$$7n + 5 \leq 7n + n$$

$$7n + 5 \leq 8n, \text{ for } n \geq n_0 = 6$$

Therefore,  $7n + 5 \leq c n$  for  $c = 8$  and  **$n_0 = 6$**

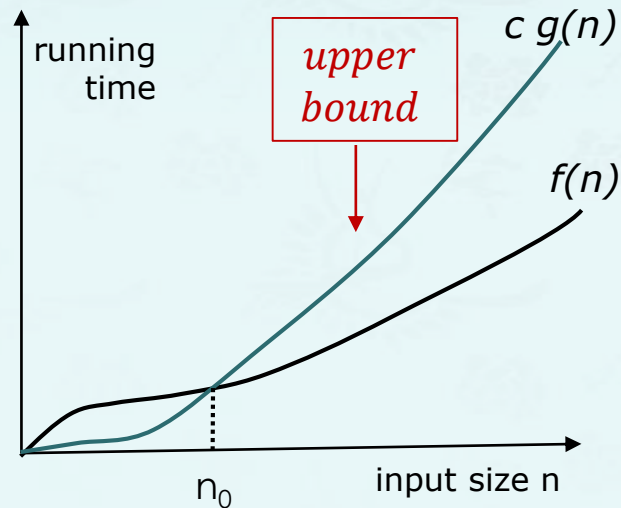
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### The "Big-Oh" Notation:

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$$f(n) \leq c g(n), \text{ for } n \geq n_0.$$

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$$7n + 5 \leq c n \text{ for } n \geq n_0$$

$$7n + 5 \leq 7n + n$$

$$7n + 5 \leq 8n, \text{ for } n \geq n_0 = 6$$

Therefore,  $7n + 5 \leq c n$  for  $c = 8$  and  **$n_0 = 6$**

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

### Examples:

1)  $3n + 2 =$

2)  $3n + 3 =$

3)  $100n + 6 =$

4)  $10n^2 + 4n + 2 =$

5)  $6 * 2^n + n^2 =$

✗ 6)  $3n + 3 =$

✗ 7)  $10n^2 + 4n + 2 =$

8)  $3n + 2 \neq O(1)$

9)  $10n^2 + 4n + 2 \neq O(n)$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

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### Preferred Big-Oh usage:

- **Pick the tightest bound.** If  $f(N) = 5N$ , then:

$$f(N) = O(N^5)$$

$$f(N) = O(N^3)$$

$$f(N) = O(N \log N)$$

$$\mathbf{f(N) = O(N)} \quad \leftarrow \text{preferred or right!}$$

- **Ignore constant factors and low order terms:**

$$f(N) = \mathbf{O(N)}, \quad \text{not } f(N) = O(5N)$$

$$f(N) = \mathbf{O(N^3)}, \quad \text{not } f(N) = O(N^3 + N^2 + 15)$$

- Wrong:  $f(N) \leq O(g(N))$
- Wrong:  $f(N) \geq O(g(N))$
- **Right:**  $\mathbf{f(N) = O(g(N))}$

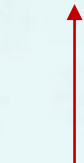


## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

Suppose two algorithms, A and B, solving the same problem have the running time of  $O(n)$  and  $O(n^2)$ , respectively.  
Then algorithm A is asymptotically better than algorithm B.

$$\times O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$$

constant



정수함수

logarithmic



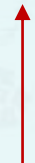
대수

linear



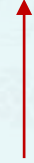
선형함수

linearithmic



선형대수  
loglinear

quadratic



2/3승함수

cubic



exponential



지수함수



## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

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**[Omega]**  $f(n) = \Omega(g(n))$  iff there exist positive constants  $c$  and  $n_0$  such that

$$f(n) \geq c g(n), \text{ for } n \geq n_0.$$

**Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$

For all  $n \geq 0$ , this  $(2n + 1)$  will be  $\geq$  to 1, **if** we have  $c = 5$  and  $n_0 = 0$ .

Then,  $5n^2 \leq f(n)$ , for all  $n \geq 0$

**Therefore**, we can say that the time complexity of  $f(n)$  is  $\Omega(n^2)$ ;

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

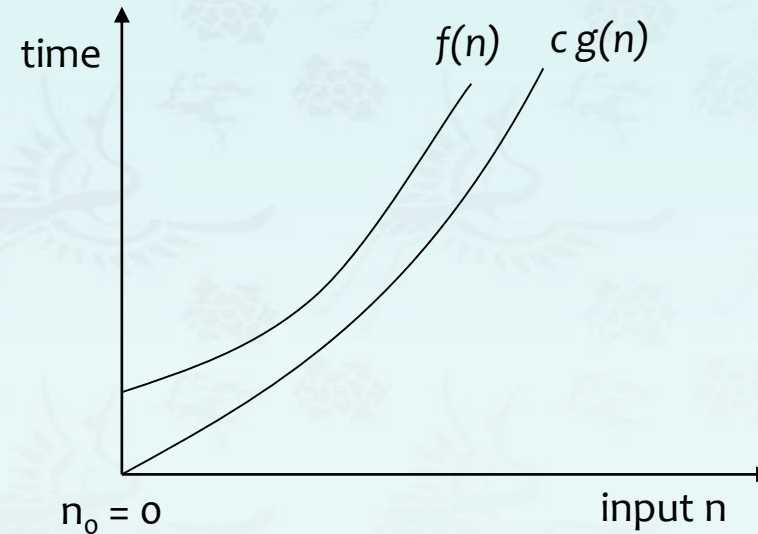
**[Omega]**  $f(n) = \Omega(g(n))$  iff there exist positive constants  $c$  and  $n_0$  such that

$$f(n) \geq c g(n), \text{ for } n \geq n_0.$$

**Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$


$$g(n) = n^2$$



❖ **Omega** notation gives us the **lower bound** of the growth rate of a function.

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

**[Omega]**  $f(n) = \Omega(g(n))$  iff there exist positive constants  $c$  and  $n_0$  such that

$$f(n) \geq c g(n), \text{ for } n \geq n_0.$$


**Example:**

1)  $3n + 2 = \Omega(n)$  since  $3n + 2 \geq 3n$  for  $n \geq 1$

2)  $3n + 3 = \Omega(n)$  since  $3n + 3 \geq 3n$  for  $n \geq 1$

3)  $100n + 6 = \Omega(n)$  since  $100n + 6 \geq 100n$  for  $n \geq 1$

4)  $100n^2 + 4n + 2 = \Omega(n^2)$  since  $100n^2 + 4n + 2 \geq n^2$  for  $n \geq 1$

5)  $6 * 2^n + n^2 = \Omega(2^n)$  since  $6 * 2^n + n^2 \geq 2^n$  for  $n \geq 1$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

**[Theta]**  $f(n) = \Theta(g(n))$  iff there exist positive constants  $c$  and  $n_0$  such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ for } n \geq n_0.$$

**Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$

Then, we can choose  $c_1 = 5$ ,  $c_2 = 8$ , and  $n_0 = 1$ ; and our inequality will hold. Therefore we can say that the time complexity of

$$f(n) = 5n^2 + 2n + 1 = \Theta(n^2)$$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

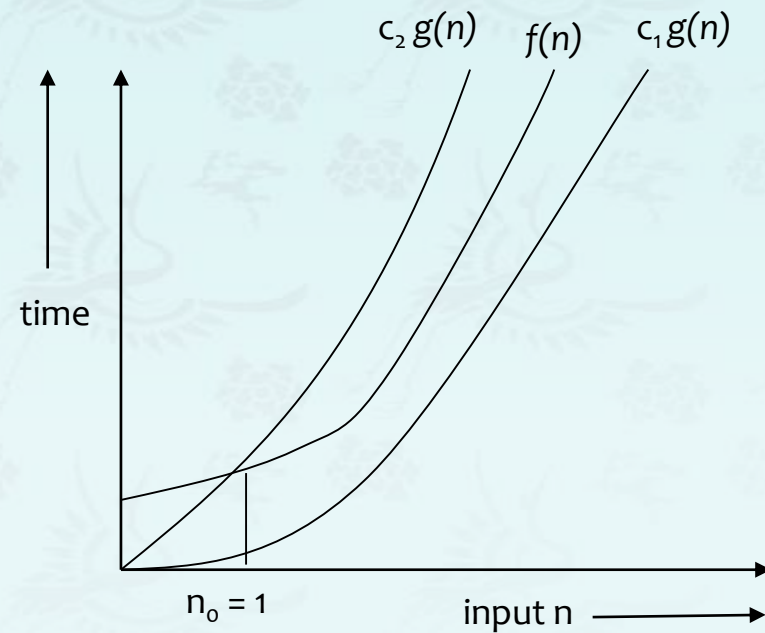
**[Theta]**  $f(n) = \Theta(g(n))$  iff there exist positive constants  $c$  and  $n_0$  such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ for } n \geq n_0.$$

**Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$



❖  **$\Theta$  notation** best describes or give the best idea about the growth rate of the function because it gives us a **tight bound** unlike  **$O$  and  $\Omega$**  which give us **upper bound** and **lower bound**, respectively.

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

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**[Theta]**  $f(n) = \Theta(g(n))$  iff there exist positive constants  $c$  and  $n_0$  such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ for } n \geq n_0.$$

**Example:**

1)  $3n + 2 = \Theta(n)$

since  $3n \leq 3n + 2 \leq 4n$  for all  $n \geq 2$ ,  $c_1 = 3$ ,  $c_2 = 4$ , and  $n_0 = 2$

2)  $3n + 3 = \Theta(n)$

3)  $10n^2 + 4n + 2 = \Theta(n^2)$

4)  $6 * 2^n + n^2 = \Theta(2^n)$

5)  $10 * \log n + 4 = \Theta(\log n)$

## Performance Analysis – Linear search

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The time complexity of the linear search:

- **Best Case:** Find at first place - one comparison
- **Worst Case:** Find at  $n$ th place or not at all -  $n$  comparisons
- **Average Case:** It is shown below that this case takes -  $(n+1)/2$  comparisons
- In considering the average case there are  $n$  cases that can occur, i.e. find at the first place, the second place, the third place and so on up to the  $n$ th place. If found at the  $i$ th place then  $i$  comparisons are required. Hence the average number of comparisons over these  $n$  cases is:

$$\text{average} = (1 + 2 + 3 \dots + n) / n$$

$$= n(n + 1)/2 / n,$$

since  $(1 + 2 + 3 + \dots + n)$  is equal to  $n(n + 1)/2$ .

*Hence linear search is an order( $n$ ) process or  $T(n) = O(n)$ .*



## Recurrence Relations

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**Recurrence Relations** is an equation that recursively defines a sequence or multidimensional array of values, once one or more initial terms are given: each further term of the sequence or array is defined as a function of the preceding terms.

**For example:**

$$T(1) = c$$

$$T(n) = T(n - 1) + c$$

**Useful formulas:**

$$1 + 2 + 3 + \dots + N = N(N+1)/2$$

$$1 + 2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 1$$





## Performance Analysis – Linear search

We may describe that the time complexity of the linear search is

$$T(1) = c$$

$$T(n) = T(n - 1) + c \quad \longleftarrow \text{Recurrence equation}$$

- The cost of searching **n** elements is the cost of looking at **1** element, plus the cost of searching **n - 1** elements.
- Let's "telescoping" a few of these:

$$T(n) = T(n - 1) + c$$

$$T(n - 1) = T(n - 2) + c$$

$$T(n - 2) = T(n - 3) + c$$

...

$$T(2) = T(1) + c$$

- Then add each side,

$$T(n) = T(1) + (n - 1)c$$

$$T(n) = c + nc - c$$

$$T(n) = O(n)$$

## Performance Analysis – Selection sort

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$$T(1) = 1$$
$$T(n) = n + T(n - 1)$$

```
public static void selectionSort(int[]a) {           // Java syntax
    int min;
    for (int i = 0; i < a.length-1; i++)
        min = i;
        for (int j = i+1; j < a.length; j++)
            if (a[j] < a[min])
                min = j;
        swap(a[i], a[min]);    // exchange a[i] with a[min] found
    }
}
```

## Performance Analysis – Selection sort

$$T(1) = 1$$

← Recurrence equation

$$T(n) = n + T(n - 1)$$

- **Unfolding** makes repeated substitutions applying the recursive rule until the base case is reached.

Substitute  $n-1$  everywhere we see an  $n$  in the recurrence relation:

$$T(n - 1) = (n - 1) + T(n - 2)$$

$$T(n) = n + (n - 1) + T(n - 2)$$

Making this substitution one more time we get

$$T(n) = n + (n - 1) + (n - 2) + T(n - 3)$$

We repeat this process until we reaches  $T(1)$ , base case

$$T(n) = n + (n - 1) + \dots + (n - (n - 2)) + \boxed{\phantom{000000}}$$

$$T(n) = n + (n - 1) + \dots + 2 + \boxed{\phantom{000000}}$$

$$= n + (n - 1) + \dots + 2 + 1$$

$$= \frac{n(n + 1)}{2}$$

$$= O(n^2)$$

## Performance Analysis – Selection sort

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$$T(1) = 1$$

$$T(n) = n + T(n - 1)$$

- **Telescoping**

$$T(n) = n + T(n - 1)$$

$$T(n - 1) = n - 1 + T(n - 2)$$

$$T(n - 2) = n - 2 + T(n - 3)$$

...

$$T(2) = 2 + T(1)$$

- **Add all terms in each side and cancel the equal terms, then it becomes**

$$T(n) = n + (n - 1) + \dots + 2 + T(1)$$

$$= \frac{n(n + 1)}{2} - 1 + T(1)$$

$$= O(n^2)$$

## Performance Analysis – Binary search

Recurrence equation



**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + \text{teal\_square}$  where  $n$  is  $hi - lo$

- $O(\log n)$  where  $n$  is *array.length*
- Solve *recurrence equation* to know that...

```
// returns whether k is in array, array a is sorted
boolean binarySearch(int *a, int k, size){
    return _binarySearch(a,k,0,size-1);
}

boolean _binarySearch(int *a, int k, int lo, int hi) {
    int mid = (hi+lo)/2;
    if (lo==hi) return false;
    if (a[mid]==k) return true;
    if (a[mid]< k) return _binarySearch(a,k,mid+1,hi);
    else return _binarySearch(a,k,lo,mid-1);
}
```

## Performance Analysis – Binary search

**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + T(\frac{n}{2})$  where  $n$  is *hi – lo*

- $O(\log n)$  where  $n$  is *array.length*

1. Determine the recurrence relation. What is the base case?

$$T(n) = 1 + T(n/2)$$

telescoping  $\longrightarrow$

$$= T(1)$$

2. Sum up the left and right sides of the equations above:

$$T(n) += (\text{red underline}) + T(1)$$

3. Cross out the equal terms to simplify. How many 1's on the right side?

$$T(n) =$$
$$=$$

Therefore the time complexity of binary search is  $T(n)$  is  $O(\log n)$

## Performance Analysis – Binary search

**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + T(\frac{n}{2})$  where  $n$  is  $hi - lo$

- $O(\log n)$  where  $n$  is *array.length*

1. Determine the recurrence relation. What is the base case?

$$T(n) = 1 + T(n/2)$$

$$T(n/2) = 1 + T(n/4)$$

$$T(n/4) = 1 + T(n/8)$$

telescoping  $\longrightarrow$

...

$$T(4) = 1 + T(2)$$

$$T(2) = 1 + T(1)$$

2. Sum up the left and right sides of the equations above:

$$T(n) += (\text{red underline}) + T(1)$$

3. Cross out the equal terms to simplify. How many 1's on the right side?

$$T(n) =$$
$$=$$

Therefore the time complexity of binary search is  $T(n)$  is  $O(\log n)$



## Performance Analysis – Binary search

**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + T(\frac{n}{2})$  where  $n$  is *hi – lo*

- $O(\log n)$  where  $n$  is *array.length*

1. Determine the recurrence relation. What is the base case?

$$T(n) = 1 + T(n/2)$$

$$T(n/2) = 1 + T(n/4)$$

$$T(n/4) = 1 + T(n/8)$$

...

$$T(4) = 1 + T(2)$$

$$T(2) = 1 + T(1)$$

telescoping  $\longrightarrow$

2. Sum up the left and right sides of the equations above:

$$T(n) = (1 + 1 + \dots + 1) + T(1)$$

3. Cross out the equal terms to simplify. How many 1's on the right side?

$$T(n) = \log_2 n + T(1)$$

$$= \log_2 n + 1$$

Therefore the time complexity of binary search is  $T(n)$  is  $O(\log n)$



## Performance Analysis – Binary search

**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + T(\frac{n}{2})$  where  $n$  is *hi – lo*

- $O(\log n)$  where  $n$  is *array.length*

1. Determine the recurrence relation. What is the base case?

$$T(n) = 1 + T\left(\frac{n}{2}\right) \quad T(1) = 1$$

2. "**Unfolding**" the original relation to find an equivalent general expression in terms of the number of expansions.

$$\begin{aligned} T(n) &= 1 + 1 + T(n/4) \\ &= 1 + 1 + 1 + T(n/8) \\ &= 1 + 1 + 1 + 1 + T(n/16) \\ &= 1 + \dots + 1 + T(n/n) \end{aligned}$$

← How many 1's here?

## Performance Analysis – Binary search

**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + T(\frac{n}{2})$  where  $n$  is *hi – lo*

- $O(\log n)$  where  $n$  is *array.length*

1. Determine the recurrence relation. What is the base case?

$$T(n) = 1 + T\left(\frac{n}{2}\right) \quad T(1) = 1$$

2. "**Unfolding**" the original relation to find an equivalent general expression in terms of the number of expansions.

$$\begin{aligned} T(n) &= 1 + 1 + T(n/4) \\ &= 1 + 1 + 1 + T(n/8) \\ &= 1 + 1 + 1 + 1 + T(n/8) \\ &= 1 + \dots + 1 + T(n/n) \\ &= 1k + T\left(\frac{n}{2^k}\right) \end{aligned}$$

$2^2$   
 $2^3$   
 $2^4$   
 $2^n$   
 $2^k$

number of 1's

## Performance Analysis – Binary search

**Base case:**  $T(1) = O(1) = 1$

**Recurrence:** Let suppose that  $T(n) = 1 + T(\frac{n}{2})$  where  $n$  is *hi – lo*

- $O(\log n)$  where  $n$  is *array.length*

1. Determine the recurrence relation. What is the base case?

$$T(n) = 1 + T\left(\frac{n}{2}\right) \quad T(1) = 1$$

2. "**Unfolding**" the original relation to find an equivalent general expression *in terms of the number of expansions*.

$$\begin{aligned} T(n) &= 1 + 1 + T(n/4) \\ &= 1 + 1 + 1 + T(n/8) \\ &= 1 + \dots + 1 + T(n/n) \\ &= \mathbf{1k} + T\left(\frac{n}{2^k}\right) \end{aligned}$$

Find a closed-form expression by setting the number of expansions to a value which reduces the problem to a base case

$$n/(2^k) = 1 \text{ means } n = 2^k \rightarrow \mathbf{k = \log_2 n}$$

So  $T(n) = 1 \log_2 n + 1$  (get to base case and do it)

So  $T(n)$  is  $\mathbf{O(\log n)}$

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

### Asymptotic Analysis:

Suppose that two algorithms, A and B, solving the same problem have the running time of  $O(n)$  and  $O(n^2)$ , respectively. Then this implies that algorithm A is **asymptotically better** than algorithm B.

We can use the **big-Oh** notation to order classes of functions by **asymptotic growth rate**.

Seven functions below are often used and ordered by increasing growth rate.

※  $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$

n	log n	n	n log n	$n^2$	$n^3$	$2^n$
1	0	1	0	1	1	2
2	1	2	2	4	8	4
4	2	4	8	16	64	16
8	3	8	24	64	512	256
16	4	16	64	256	4,096	65,536
32	5	32	160	1,024	32,768	4,294,967,296
64	6	64	384	4,096	262,144	$1.84 \times 10^{19}$
128	7	128	896	16,384	2,097,152	$3.40 \times 10^{38}$
256	8	256	2,048	65,536	16,777,216	$1.15 \times 10^{77}$

※ Even if we achieve a dramatic speed-up in hardware, we still cannot overcome the handicap of an asymptotically slow program.

## Asymptotic notation ( $O, \Omega, \Theta$ ) - 점금표기법

### Example: Running time estimates - empirical analysis

- Laptop executes  $10^8$  compares/second
- Supercomputer executes  $10^{12}$  compares/second

use a reasonable time unit

	Insertion sort ( $N^2$ )			Merge sort ( $N \log_2 N$ )		
N	Thousand	Million	Billion	Thousand	Million	Billion
Laptop	Instant	2.8 hours		Instant	1 sec	
Super Com	Instant	1 sec		Instant	Instant	Instant

$$\log_{10} 2 \cong 0.3$$
$$86,400 \text{ sec/day}$$

※ **Bottom line:** Good algorithms are better than supercomputers.

## **Data Structures**

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- *performance analysis - time complexity*