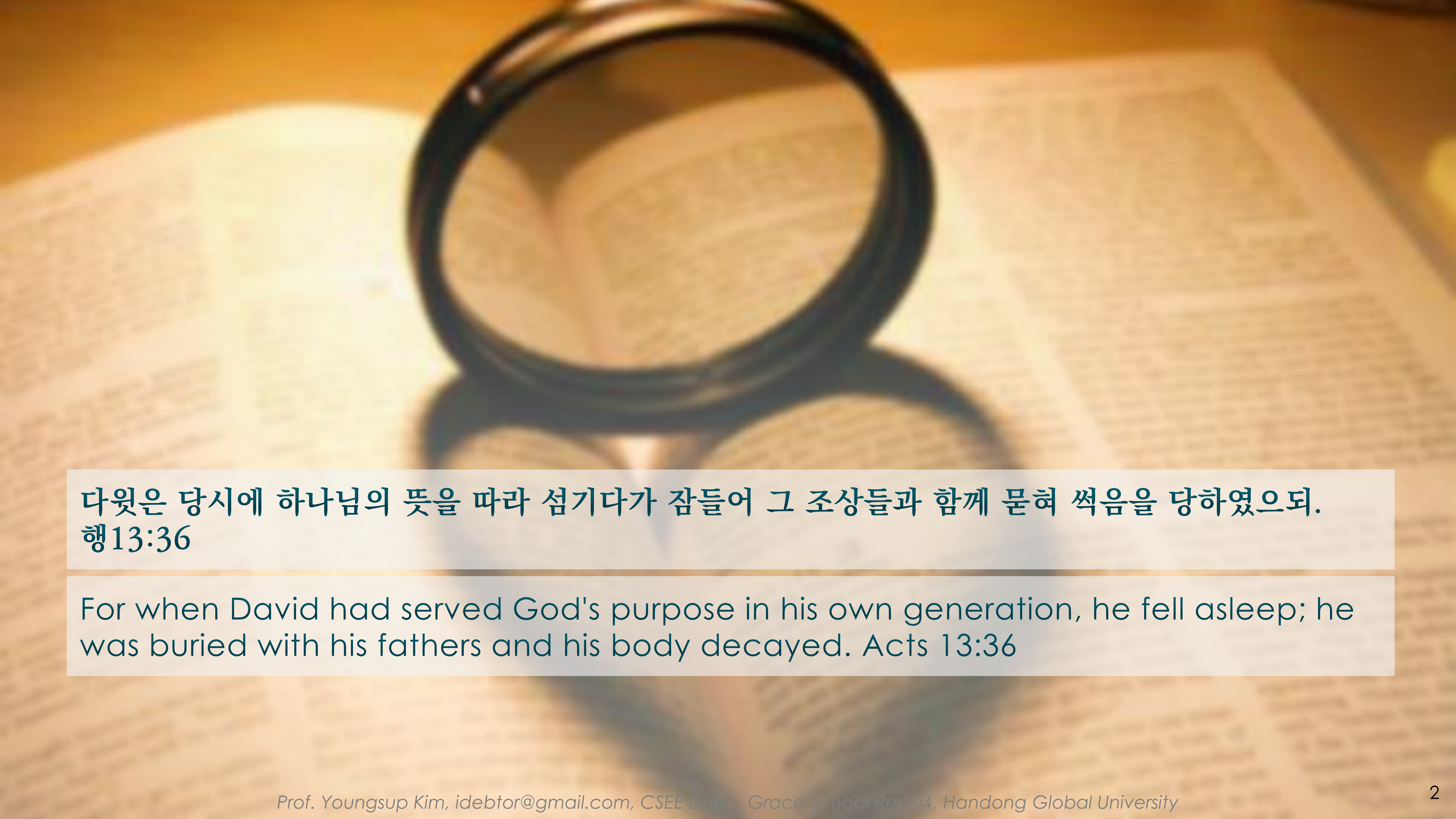
Data Structures
Chapter 5
Tree

1. **introduction**
2. **Binary tree**
3. Binary search tree
4. Tree balancing

Prof. Youngsup Kim, idebtor@gmail.com, CSEE Dept., Grace School Rm204, Handong Global University



다윗은 당시에 하나님의 뜻을 따라 섬기다가 잠들어 그 조상들과 함께 묻혀 썩음을 당하였으되. 행13:36

For when David had served God's purpose in his own generation, he fell asleep; he was buried with his fathers and his body decayed. Acts 13:36

다윗은 당시에 하나님의 뜻을 따라 섬기다가 잠들어 그 조상들과 함께 묻혀 썩음을 당하였으되. 행13:36

For when David had served God's purpose in his own generation, he fell asleep; he was buried with his fathers and his body decayed. Acts 13:36

하나님이 우리를 구원하사 거룩하신 소명으로 부르심은 우리의 행위대로 하심이 아니요 오직 자기의 뜻과 영원 전부터 그리스도 예수 안에서 우리에게 주신 은혜대로 하심이라 (딤후1:9)

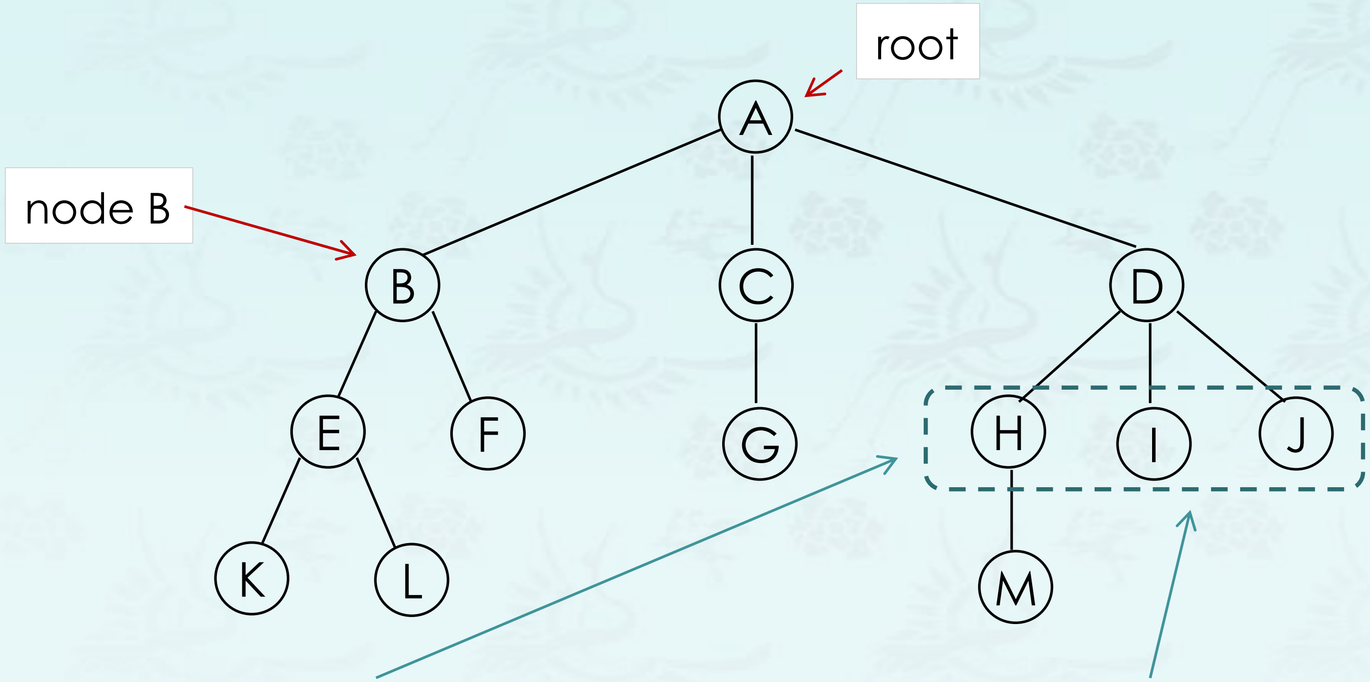
Data Structures
Chapter 5
Tree

1. **introduction**
   - **Definition and Terminology**
2. Binary tree
3. Binary search tree
4. Tree balancing

Prof. Youngsup Kim, idebtor@gmail.com, CSEE Dept., Grace School Rm204, Handong Global University

Introduction - Terminology

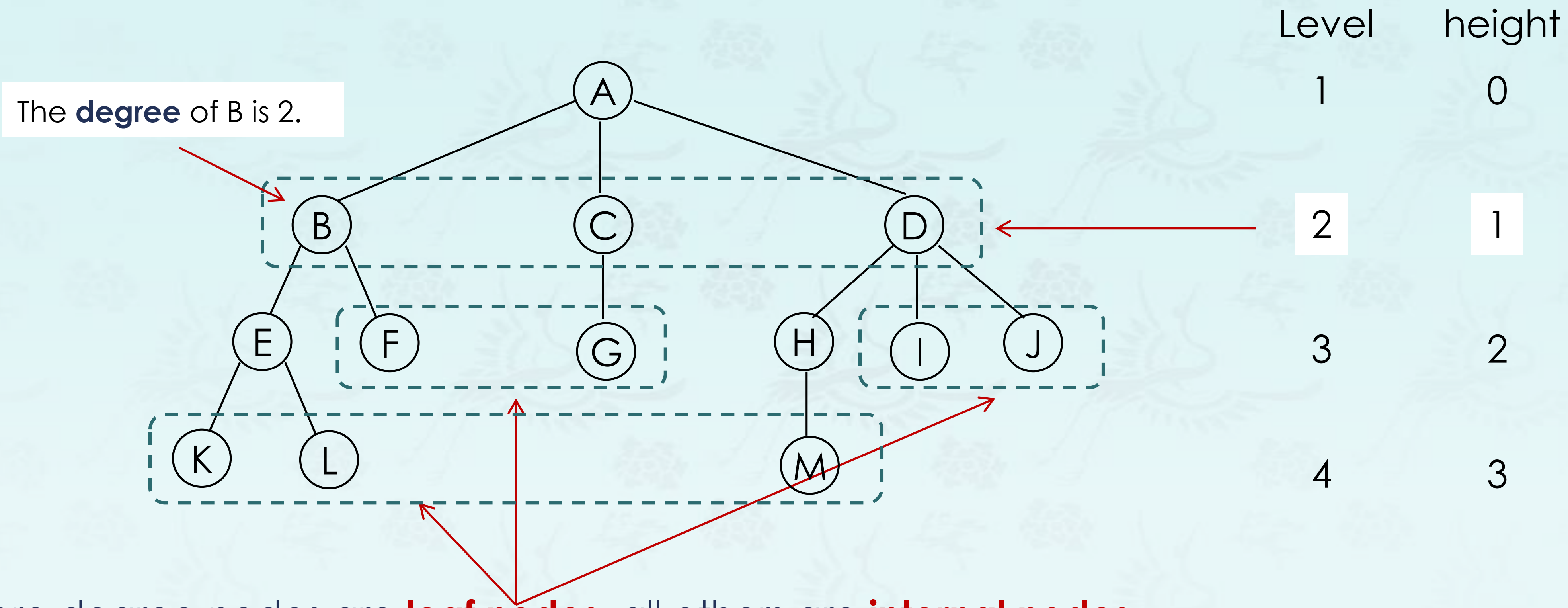
- **A tree data structure:** it is like a linked list that has a **first** node, this node is called as the **root** of the tree.
- **Example.** A **tree** with a root storing the value 'A'



- The **children** of **D** are **H, I, and J**; **H, I, and J** are **siblings**.
- The **parent** of **D** is **A**.

Introduction - Terminology

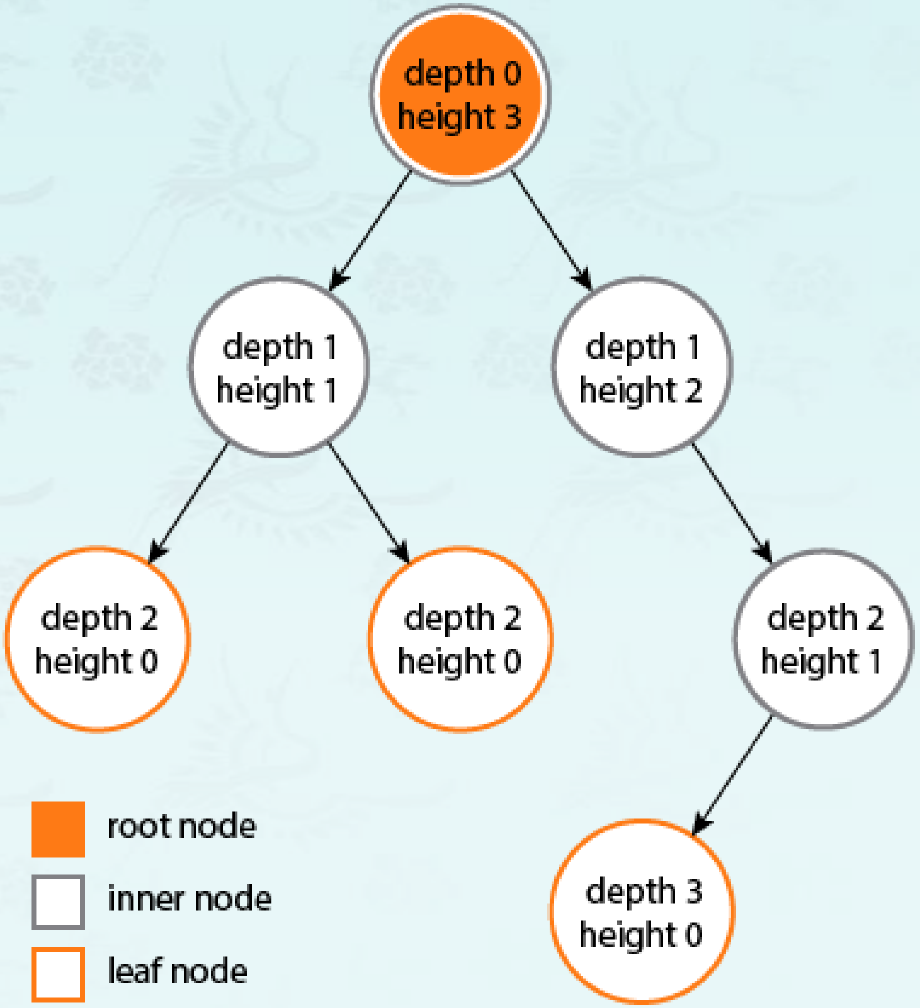
- **Definition.** child, parent, sibling, degree, leaf nodes, level, and internal node



- Zero degree nodes are **leaf nodes**, all others are **internal nodes.**
  - An ***internal node*** is any node that has at least one non-empty child.
- The **degree** of a node is the number of children.
- The **degree of a tree** is the maximum of the degree of the nodes in the tree.

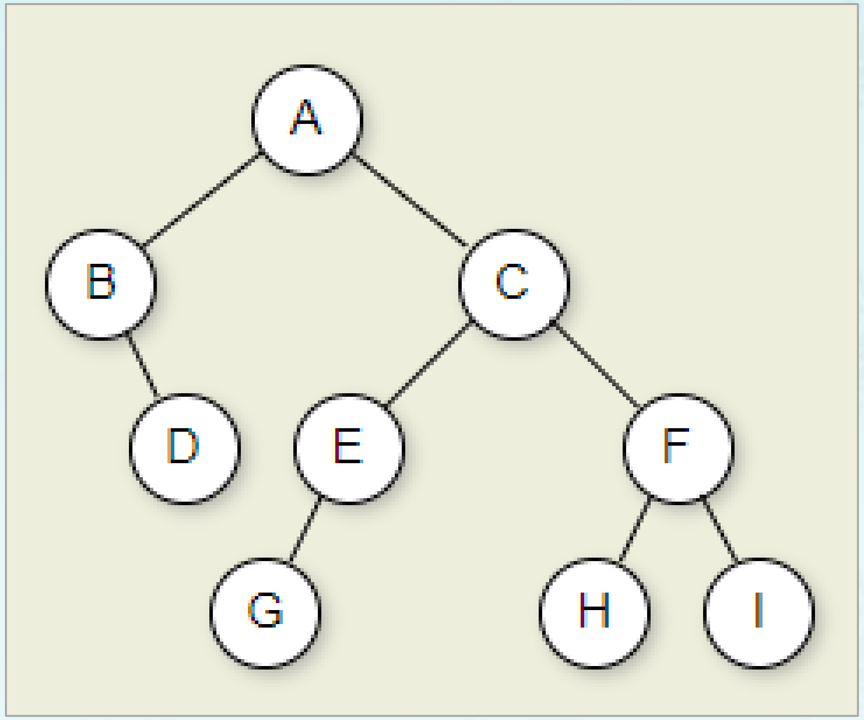
Introduction - Terminology

- **Definition.** height, depth, and level
- The **height** of a node is **the number of edges** on the *longest path* from the node to a leaf.
  - A leaf node will have a height of 0.
  - The height of a tree is the height of root.
  - The height of a tree with 1 node is 0.
  - The height of a tree is the maximum depth.
  - **The height of a tree is the depth of the deepest node in the tree.**
- The **depth** of a node is the number of edges from the node to the tree's root node.
  - A root node will have a depth of 0.
- The **level** of a node is defined by 1 + the number of connections between the node and the root.



Introduction - Terminology

- **Review**



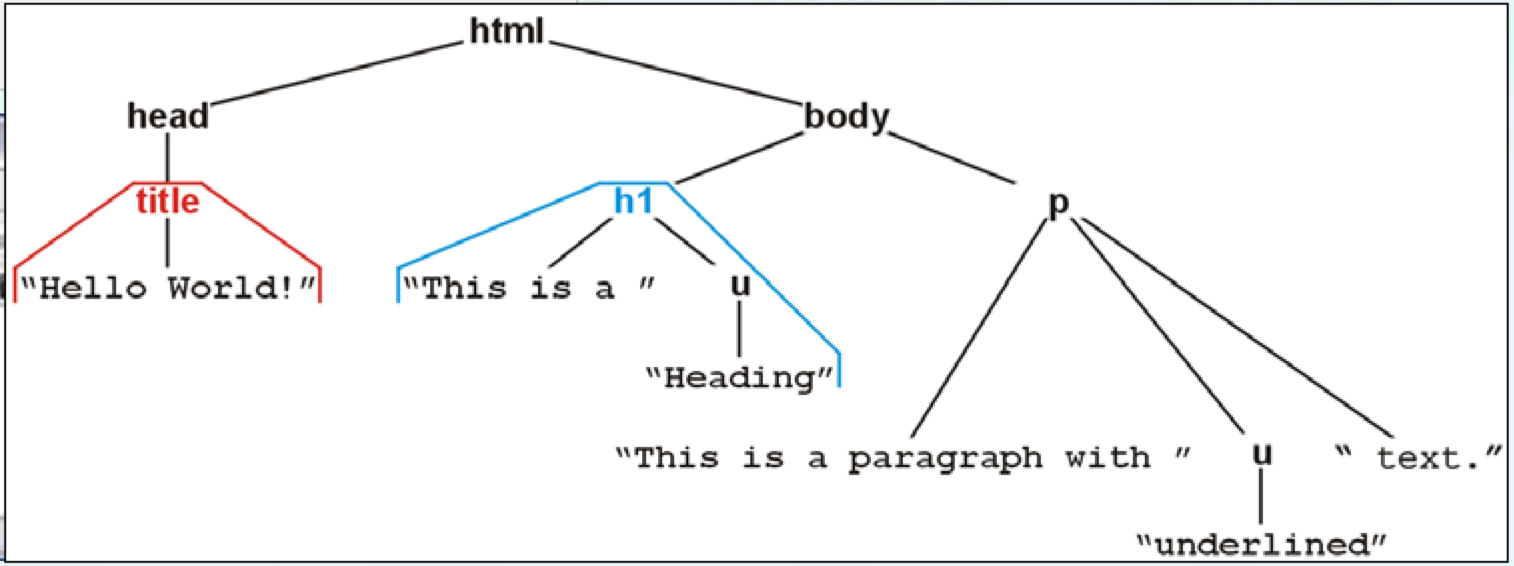
- A binary tree.
- Node A is the root.
- Nodes B and C are A's children.
- Nodes B and D together form a subtree.
- Node B has two children: Its left child is the empty tree and its right child is D.
- Nodes A, C, and E are ancestors of G.
- Nodes D, E, and F make up **level** 2 + 1 of the tree; node A is at **level** 0 + 1.
- The edges from A to C to E to G form a path of length 3.
- Nodes D, G, H, and I are leaves.
- Nodes A, B, C, E, and F are internal nodes.
- The depth of I is 3.
- **The height of this tree is 3.**

https://opendsa-server.cs.vt.edu/ODSA/Books/CS3/html/BinaryTree.html#definitions-and-properties

Introduction – Representation of trees

- **Exercise.** The tree representing the HTML document below:

```
<html>
    <head>
        <title>Hello World!</title>
    </head>
    <body>
        <h1>This is a <u>Heading</u></h1>
        <p>This is a paragraph with some
<u>underlined</u> text.</p>
    </body>
</html>
```



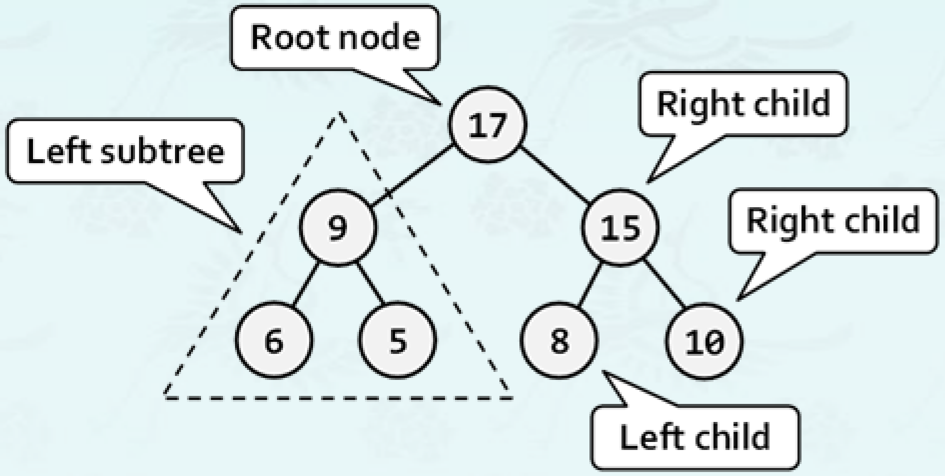
Data Structures
Chapter 5
Tree

1. introduction
2. **Binary tree**
   - **Definition and Properties**
   - **Traversal**
   - Coding
3. Binary search tree
4. Tree balancing

Prof. Youngsup Kim, idebtor@gmail.com, CSEE Dept., Grace School Rm204, Handong Global University

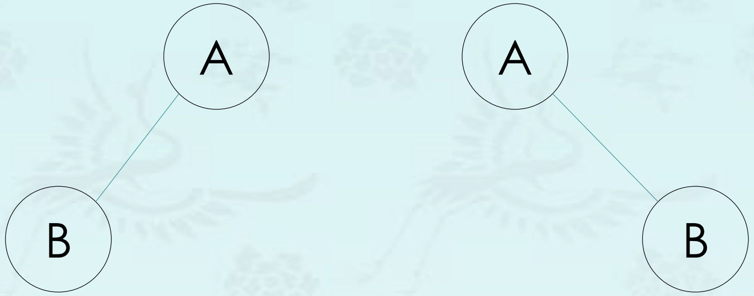
Binary trees

- **Definition:** A tree such that each node has *exactly* two children.
  - Notice, exactly two children - not up to two children! Because *exactly* two children means a left child **and/or** right child, no middle child.
  - Each child is either empty or another binary tree.
  - Given this constraint, we can label the two children as left and right nodes or subtrees.



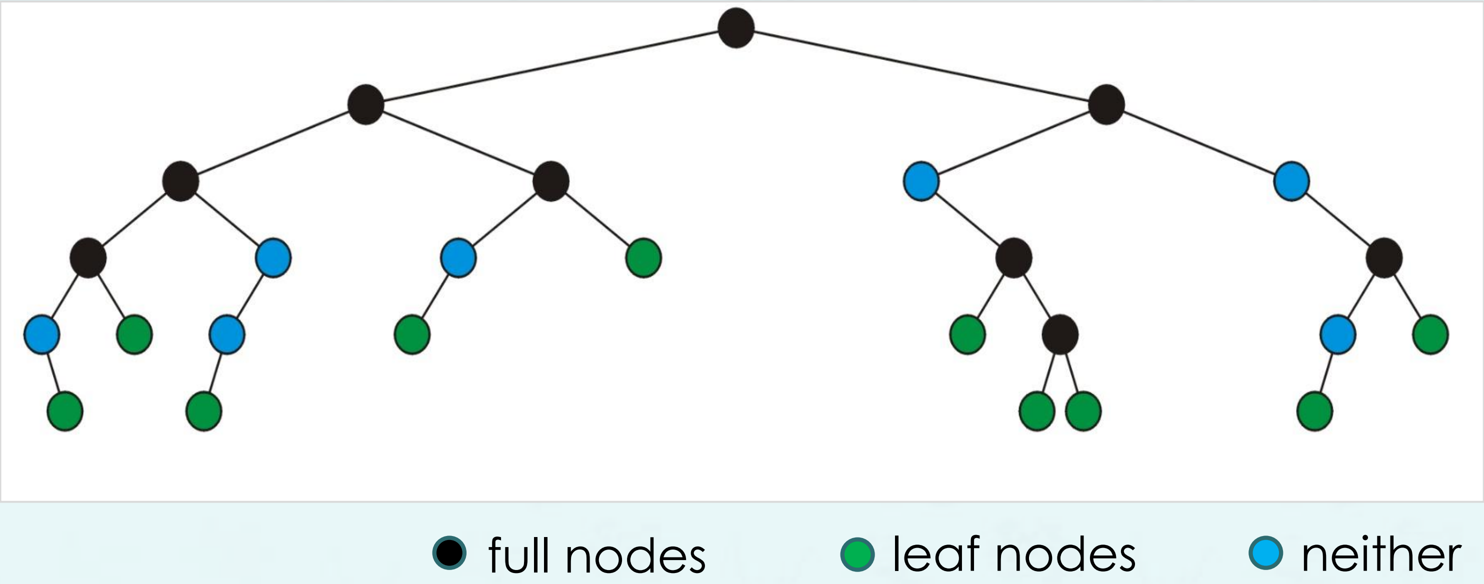
Binary trees

- **Example:** two binary trees with two nodes
  - Q: Are they two different **binary** trees?
  - A: Yes!



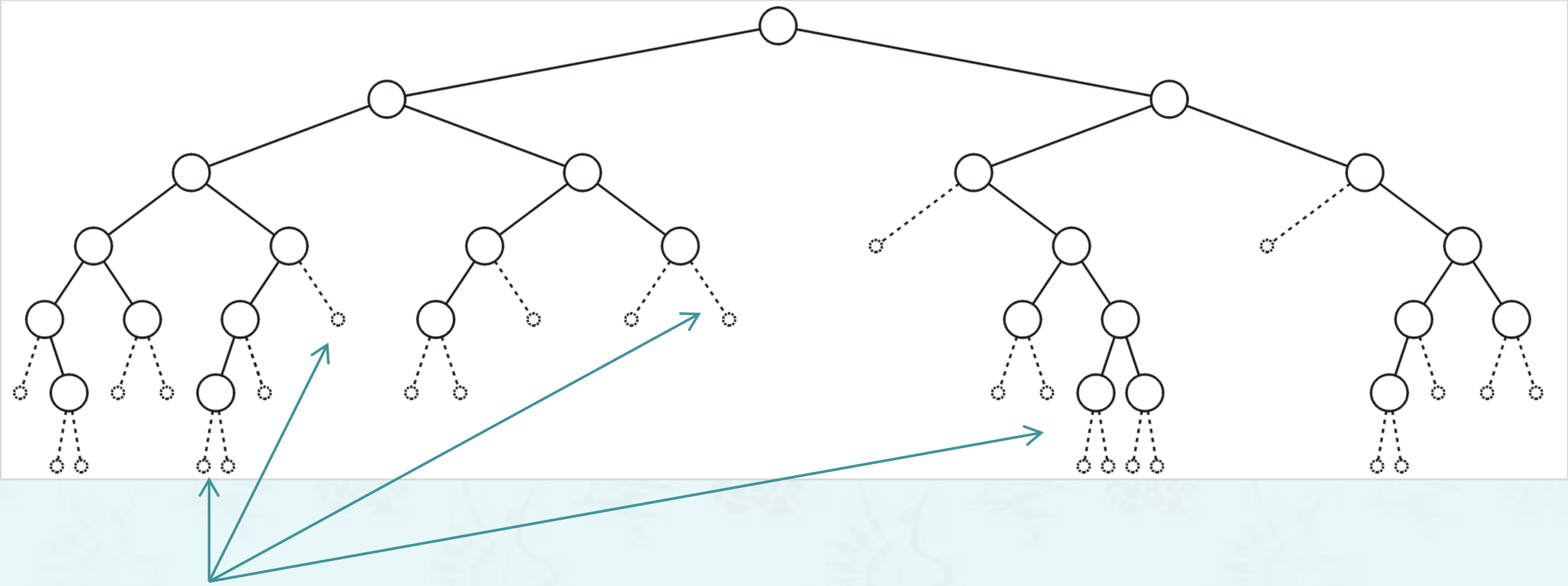
Binary trees

- **Definition:** A **full node** is a node where both left **and** right sub-trees are non-empty trees:
  - Q: How many full nodes are there?
  - Q: How many leaf nodes are there?
  - Q: What is the height of the tree?
  - Q: What is the degree of the tree?



Binary trees

- **Definition:** An ***empty node*** or ***null sub-tree*** is a location where a new leaf node (or a sub-tree) could be inserted.



null link, graphically, the missing branches

Binary trees - ADT

- Objects: a finite set of nodes either empty or consisting of a root node, leftBinaryTree, and rightBinaryTree.
- Functions:

    boolean empty(bt)

    binaryTree new Node{key, left, right}

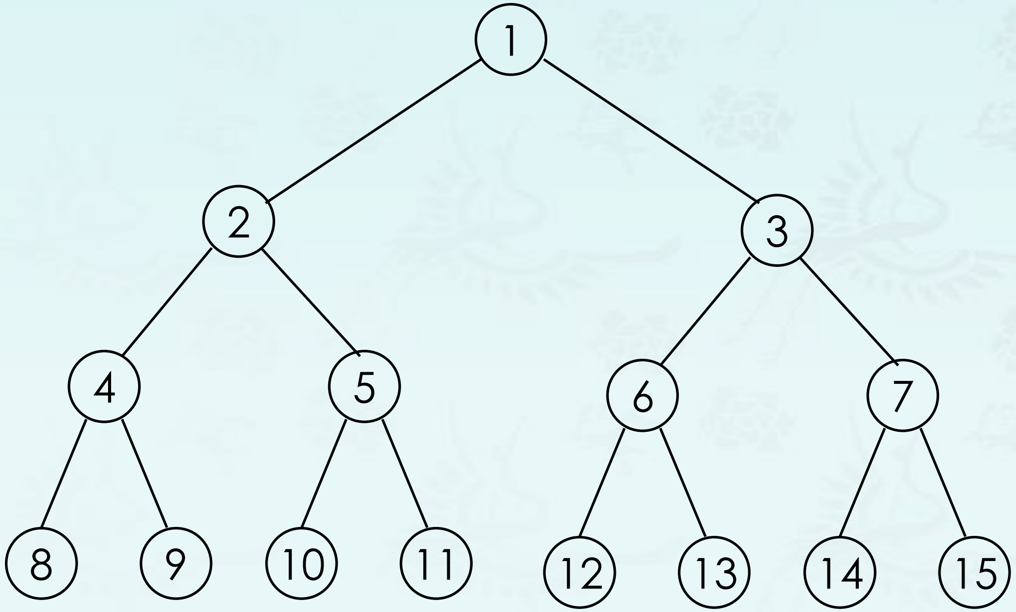
    binaryTree left(bt)

    element getKey(bt)

    binaryTree right(bt)

Binary trees - Properties

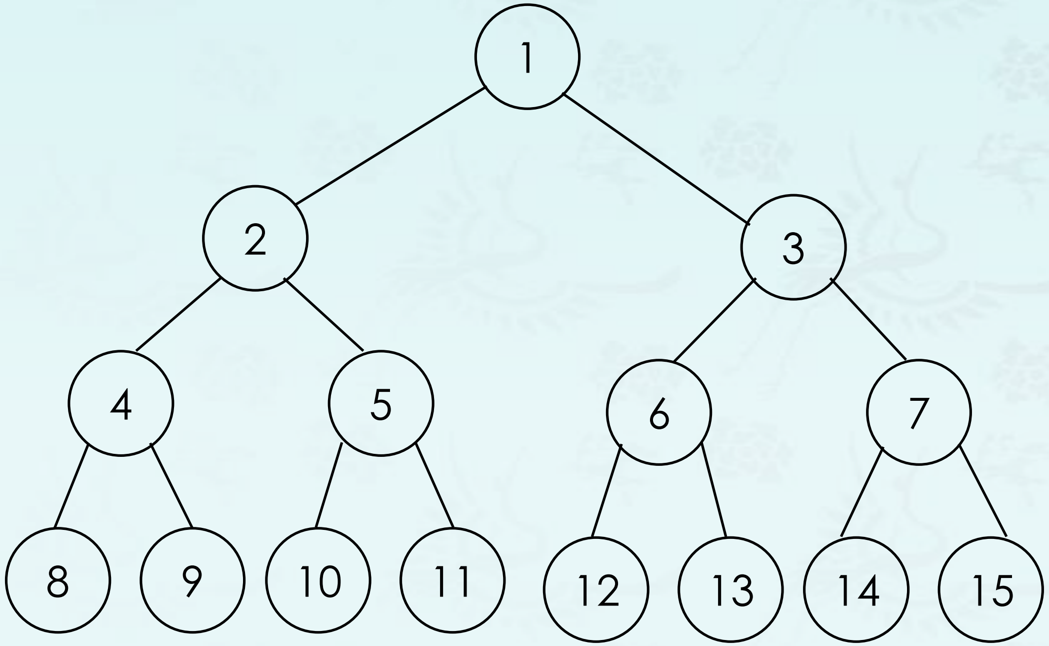
- **Observation:**
  - **Q:** Maximum number of nodes in binary trees in each level and all levels?
  - **Q:** What is the max level k if there are n nodes? k(n) = ?



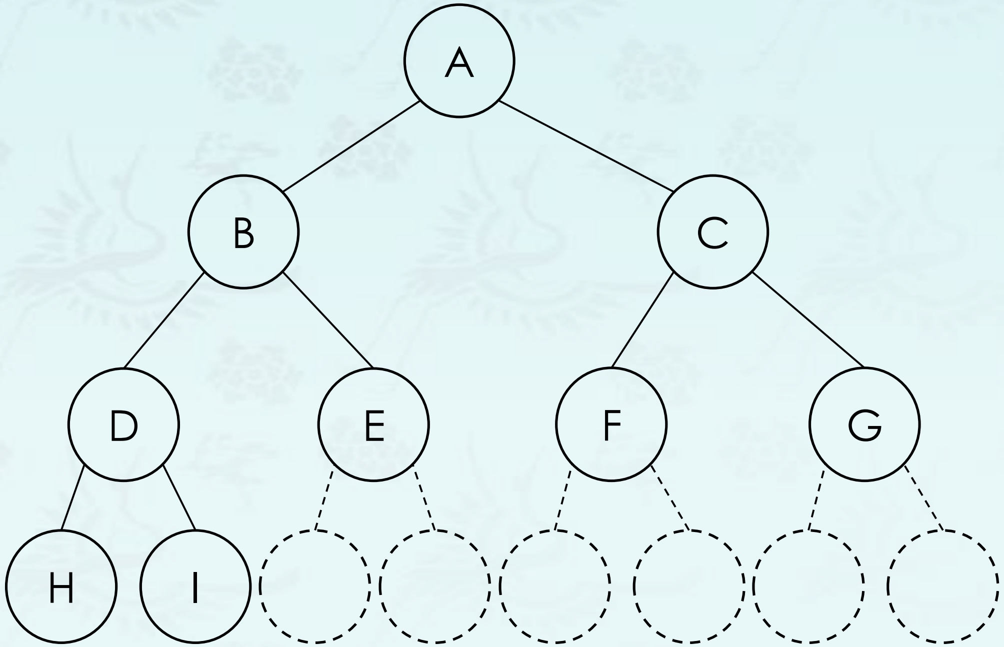
Level	Node Numbers at Each Level	Total Numbers of Nodes
1	$1 = 2^0$	$1 = 2^1 - 1$
2	$2 = 2^1$	$3 = 2^2 - 1$
3	$4 = 2^2$	$7 = 2^3 - 1$
4	$8 = 2^3$	$15 = 2^4 - 1$
.	.	.
11	$1024 = 2^{10}$	$2047 = 2^{11} - 1$
.	.	
k	$\mathbf{2^{k-1}}$	$\mathbf{2^{k} - 1}$
h	$\mathbf{2^{h}}$	$\mathbf{2^{h+1} - 1}$

Binary trees - Properties

- **Definition:** *A **full** binary tree* of level k is a binary tree having **2^k – 1 nodes**, $k \geq 0$.
- **Definition**: A binary tree with n nodes and level k is **complete** iff its nodes correspond to the nodes numbered from 1 to n in the full binary tree of level k .



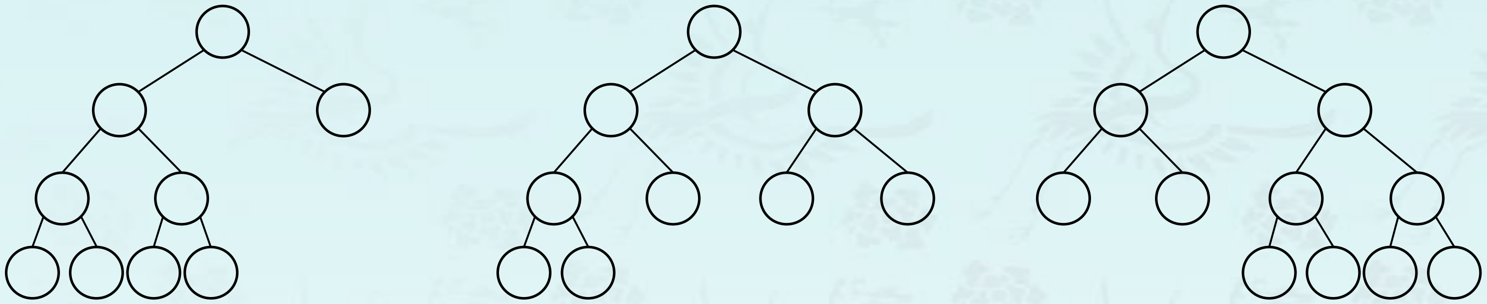
*A **full** binary tree*



*A **complete** binary tree*

Binary trees - Properties

- **Q:** Identify a **complete** binary tree.

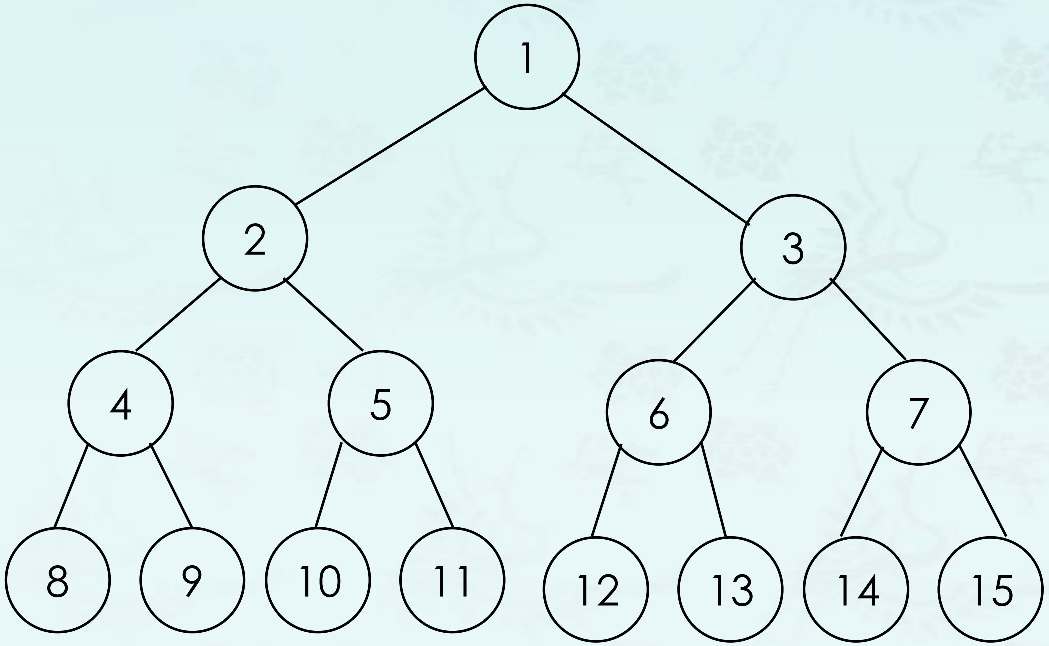


(1) (2) (3)

- **Q: Meanings of a complete tree in terms of ADT?**
  A: Removals of a node are only allowed from the "last" position.
  There is one position available to insert a node every time!

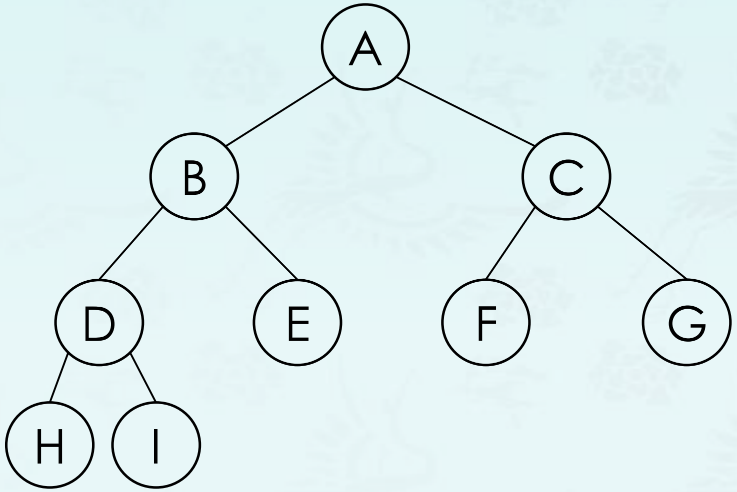
Binary trees - Array representation

- **Q:** What is the potential problem when you use an one-dimensional array to represent a binary tree in memory?
- A: **It is** good for a full binary tree, but not good memory usage for a skewed or complete binary tree.



Binary trees - Array representation

- **Q:** Let's suppose that you have a **complete binary tree** in an array. Find its parent, left child and right child at node D.



[0]	-
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Solution:

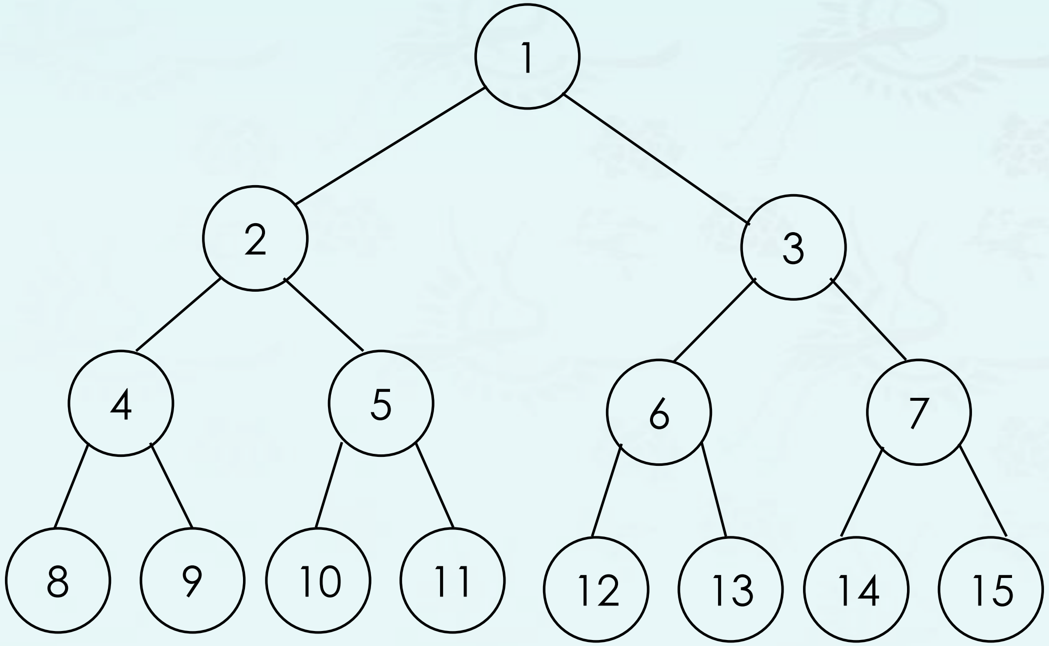
$parent(x = 4)$ is at 4/2 = 2

$leftChild(4)$ is at 2x4 = 8

$rightChild(4)$ is at 2x4 + 1 = 9

Binary trees - Array representation

- **Q:** Let's suppose that you have a **complete binary tree** in an array, how can we locate node x's parent or child?
- A **complete** binary tree with n nodes, any node index i, $1 \leq i \leq n$, we have
  - `parent(i) is at` $\lfloor i/2 \rfloor$ `If i = 1, i is at the root and has no parent`
  - `leftChild(i) is at 2i  if 2i <= n. If 2i > n, then i has no left child.`
  - `rightChild(i) is at 2i+1 if 2i+1 <= n. If 2i+1 > n, then i has no right child.`

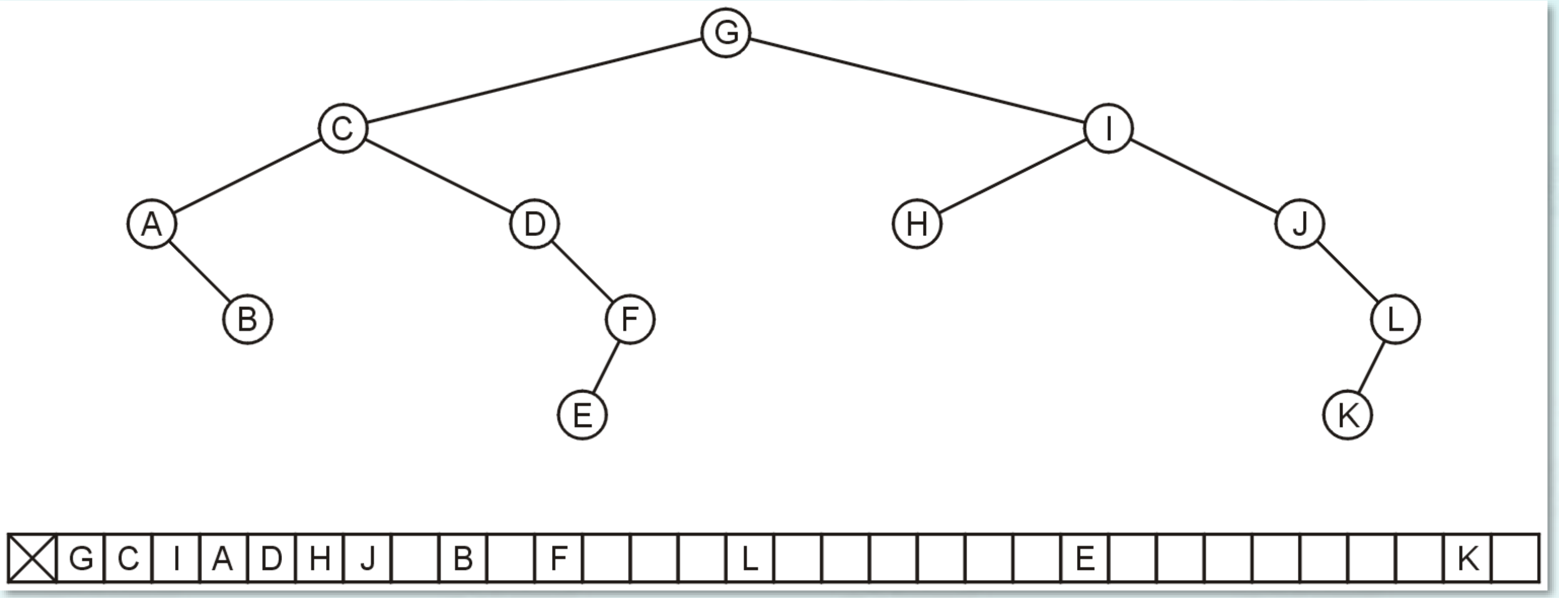


Wow! Can we use this to all binary trees?
Why not?

Problem remains:
The problem with storing an arbitrary binary tree using an array is the inefficiency *in memory usage*.

Binary trees - Array representation

- **Q:** Can we use this array rep. to store all binary trees? Why not?
- **A: For example,** the tree has 12 nodes, and requires an array of 32 elements. Adding one extra node, as a child of node K or E **doubles** the required memory for the array!



A. In the worst case a skewed tree of level k will require **$2^k - 1$** space which is O(**2^k**).
Of these, **only k** will be used.

Q. What happens when k = n? (Is there such a tree?)

Binary trees - Properties

(1) The maximum number of **nodes on level k** of a binary tree is

$$2^{k-1}, \qquad k \geq 1$$

(2) The maximum number of **nodes in a binary tree of level k** is

$$2^k - 1, \quad k \geq 1$$

(3) The maximum level of a **complete binary tree** with **n** nodes is

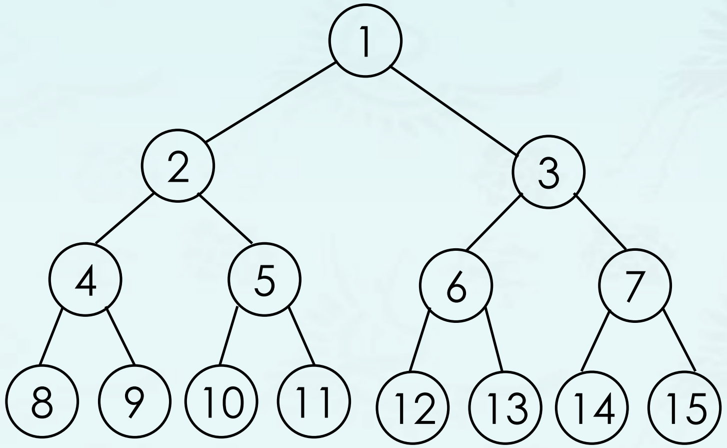
$k(n) = \lceil log_2 (n+1) \rceil$, $\quad \lceil x \rceil$ is the smallest integer $\geq x$.

$$
\begin{aligned}
n &= 2^k - 1 \\
n + 1 &= 2^k \\
\log(n+1) &= log2^k \\
\log(n+1) &= k \\
k(n) &= \lceil \log(n+1) \rceil \\
k(n) &= \lfloor \log(n) \rfloor + 1
\end{aligned}
$$

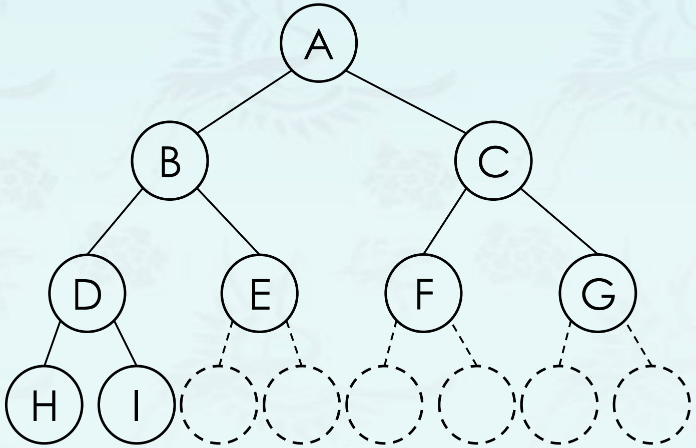
since k is an integer, and includes
the max level of complete binary tree.

Binary trees - Properties

- **Observation:** The max level of a full binary tree of *n* nodes is $k = \lfloor \log(n) \rfloor + 1$:
  - Many operations with trees have a run time that goes with the max level of some path within the tree;
  - If we have a full binary tree (or something *close* to it), we know that those operations run in $\boldsymbol{O}(\log \boldsymbol{n})$.



*A **full** binary tree*



*A **complete** binary tree*

Binary trees – Linked representation

- **Node representations:**

```
struct TreeNode{
    int       key;
    TreeNode* left;
    TreeNode* right;
};
using tree = TreeNode*;
```

```
TreeNode* t = new TreeNode(9);
tree t = new TreeNode(9);
```

Recursion & Tree Structure

```
struct TreeNode{
    int       key;
    TreeNode* left;
    TreeNode* right;
};
using tree = TreeNode*;
```

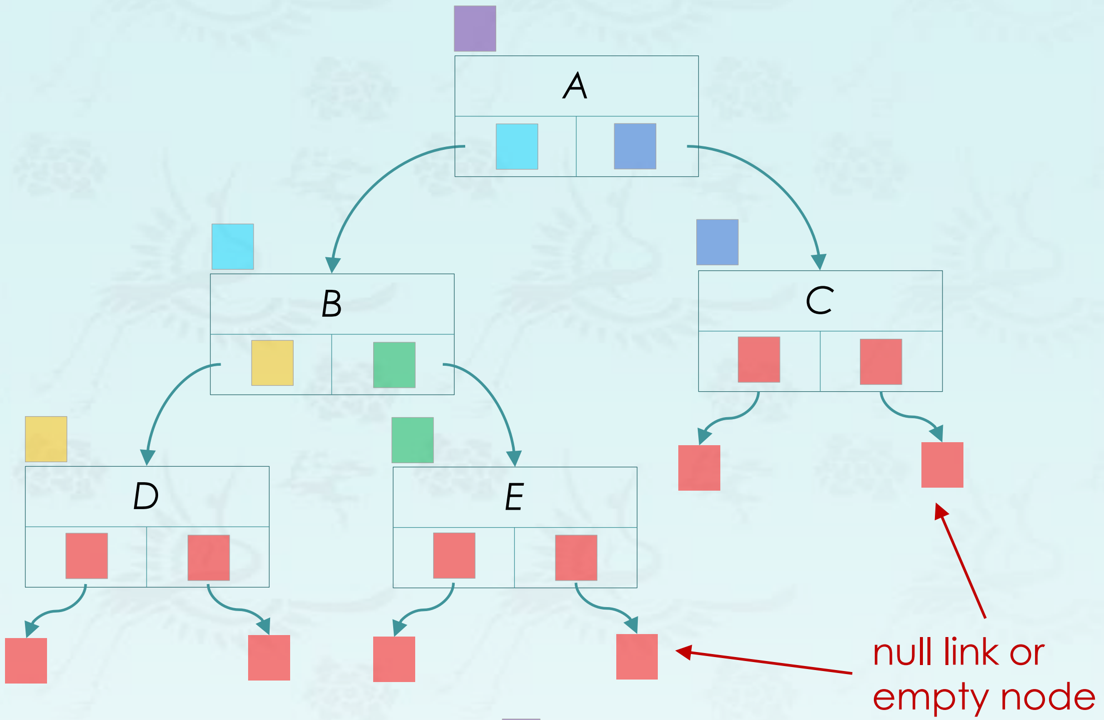
```
struct TreeNode{
  int       key;
  TreeNode* left;
  TreeNode* right;

  TreeNode(int k, TreeNode* l, TreeNode* r) {
    key = k; left = l; right = r;
  }
  TreeNode(int k) : key(k), left(nullptr), right(nullptr) {}

  ~TreeNode(){}
};
using tree = TreeNode*;
```

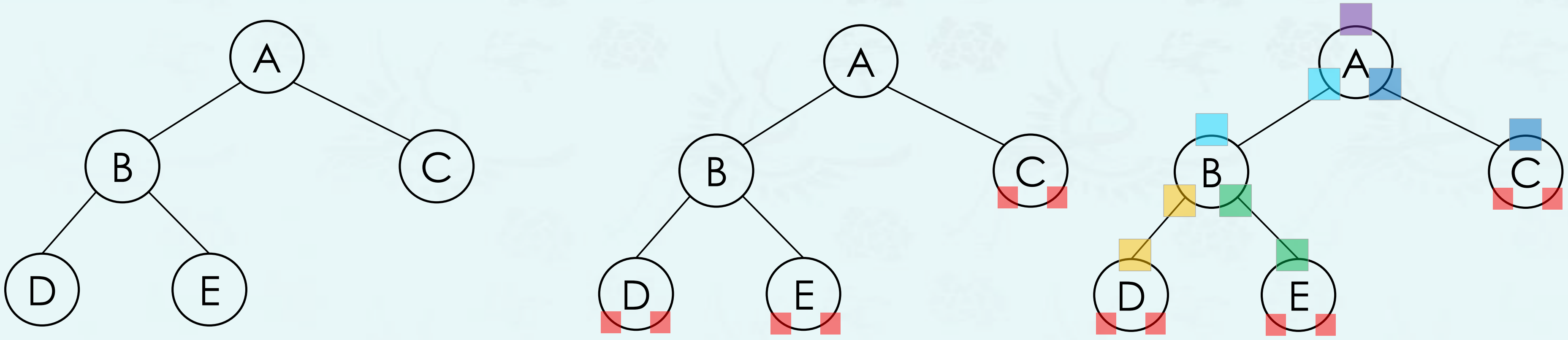
Binary trees – Linked representation

- **Node representations:**

```
struct TreeNode{
    int       key;
    TreeNode* left;
    TreeNode* right;
};
using tree = TreeNode*;
```

```
TreeNode* t = new TreeNode(9);
tree t = new TreeNode(9);
```



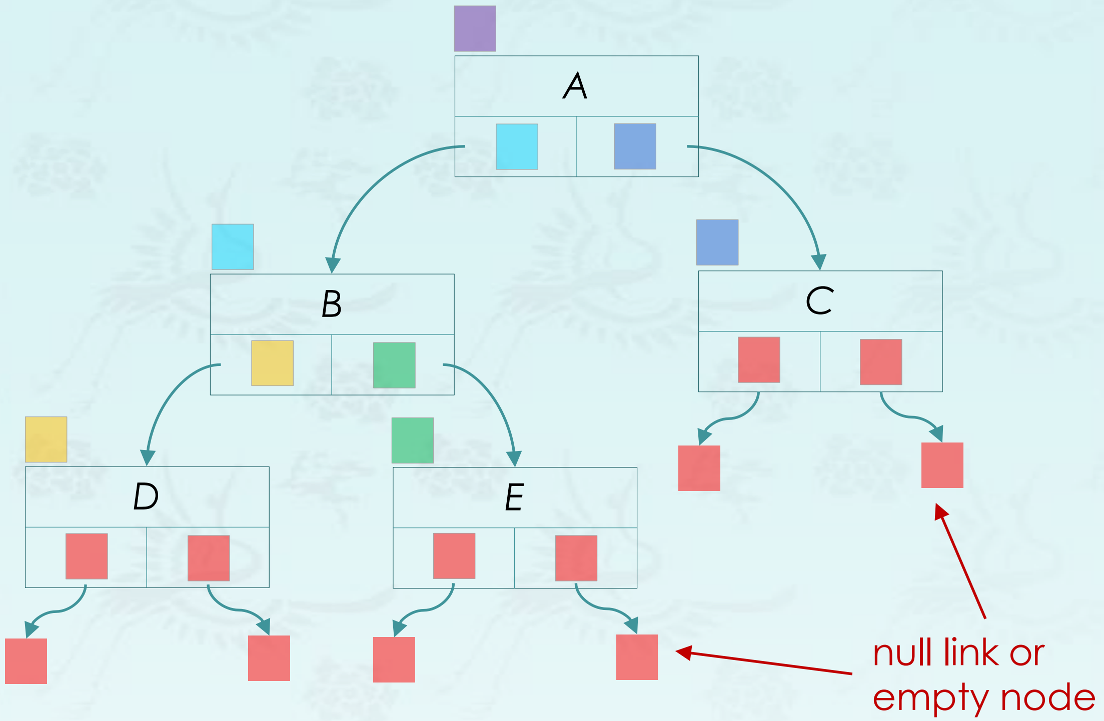


Binary trees – Linked representation

- **Node representations:**

```
struct TreeNode{
    int       key;
    TreeNode* left;
    TreeNode* right;
};
using tree = TreeNode*;
```

```
TreeNode* t = new TreeNode(9);
tree t = new TreeNode(9);
```



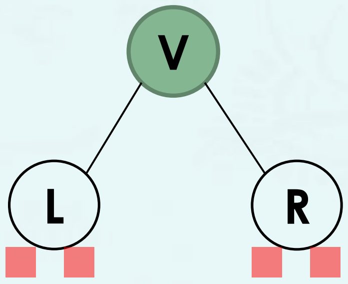
- **Q.** Is this node structure good enough?
  - Not easy to find its parent node. Parent field could be added if necessary.
- **Q.** It is similar to a doubly-linked list(DLL). What is different?
  - One head, but many tails. Null points empty node conceptually.

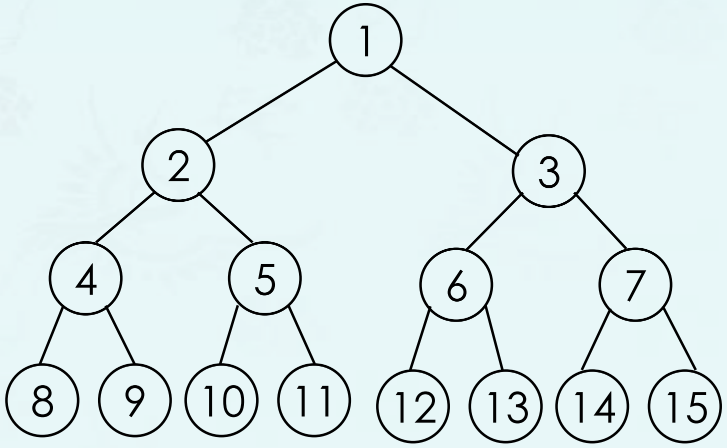
Data Structures
Chapter 5
Tree

1. introduction
   - Definition and Terminology
2. Binary tree
   - Definition and Properties
   - **Traversal**
   - Coding
3. Binary search tree
4. Tree balancing

Binary tree traversals

- **Tree traversal** (known as **tree search**) refers to the process of visiting each node in a tree, exactly once, in a systematic way.
- DFS (Depth-first Search)
  - There are three possible moves if we traverse left before right:
    LVR – inorder
    LRV – postorder
    VLR – preorder
  - They are named according to the position of **V**(the visiting node) with respect to the L and R.
  - These searches are referred to as **depth-first search(DFS)** since the search tree is deepened as much as possible on each child before going to the next sibling.



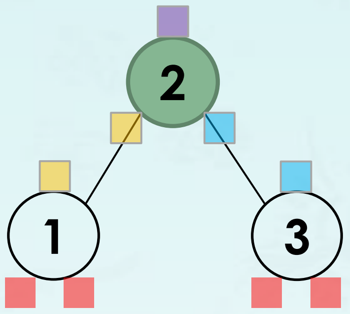


Binary tree traversals

- **Tree traversal** (known as **tree search**) refers to the process of visiting each node in a tree, exactly once, in a systematic way.
- DFS (Depth-first Search)
  - There are three possible moves if we traverse left before right:
    LVR – inorder
    LRV – postorder
    VLR – preorder
  - They are named according to the position of **V**(the visiting node) with respect to the L and R.
  - These searches are referred to as **depth-first search(DFS)** since the search tree is deepened as much as possible on each child before going to the next sibling.
- BFS (Breadth-first search)
  - The **level order traversal** traverses in level-order, where it visit every node one a level before going to a lower level.
  - This search is referred as breadth-first search(DFS), as the search tree broadened as much as possible on each depth before going to the next depth.

Binary tree traversals

- **Example: Inorder traversal(LVR)**
  - Step 1 – Recursively traverse left subtree.
  - Step 2 – Visit root node. (print or save it.)
  - Step 3 – Recursively traverse right subtree.




```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);      L
   cout << root->key;        V
   inorder(root->right);     R
}
```

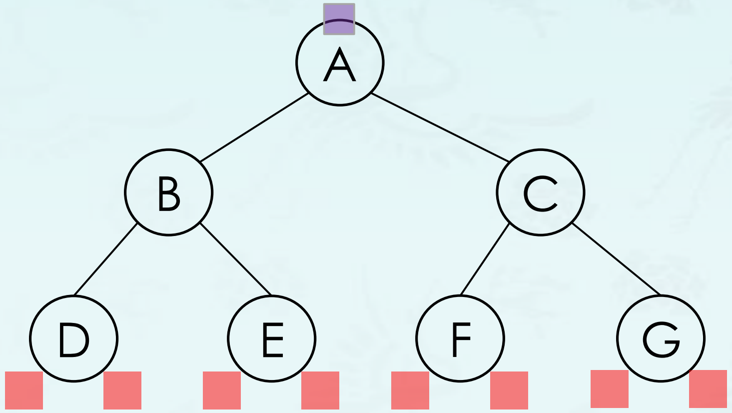
Output(LVR) : 1 2 3

Binary tree traversals

- **Example: Inorder traversal(LVR)**
  - Step 1 – Recursively traverse left subtree.
  - Step 2 – Visit root node. (print or save it.)
  - Step 3 – Recursively traverse right subtree.

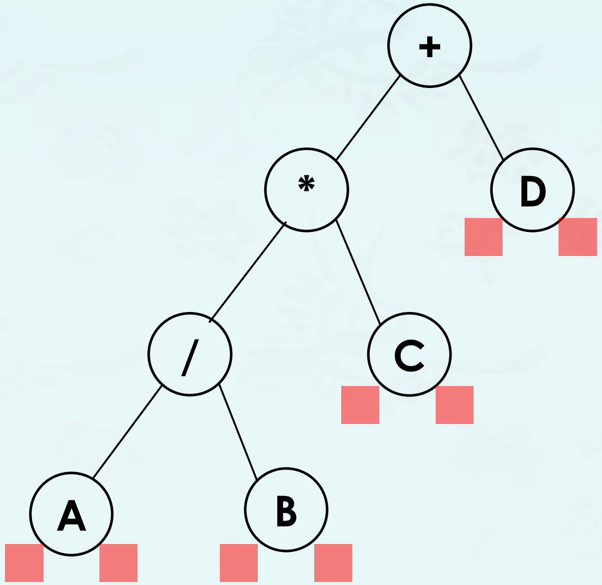
```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);      L
   cout << root->key;        V
   inorder(root->right);     R
}
```

Output(LVR) : D B E A F C G



Binary tree traversals

- **Q1: Inorder Traversal(LVR):**
- **Q2: How many times is** inorder( ) invoked for the complete traversal?
- **A2:** Every leaf node must visit (call the function) its left child and right child to make sure they don't have the child. 7 + 4 * 2 = 15

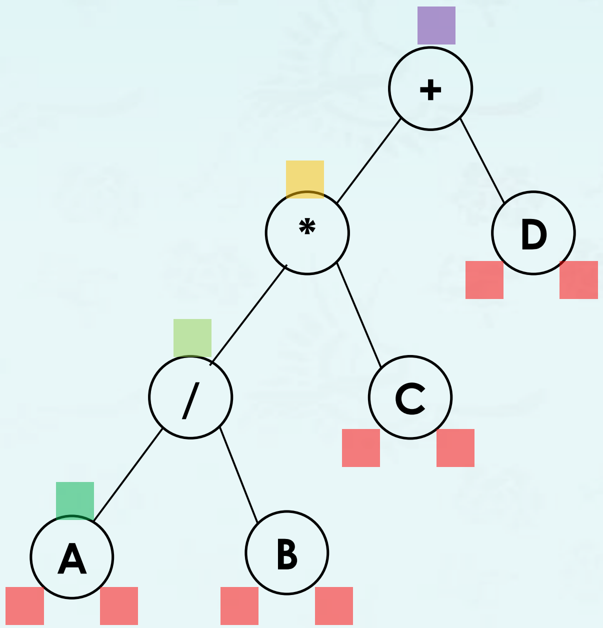



```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);          L
   cout << root->key;            V
   inorder(root->right);         R
}
```

Output(LVR) : A / B * C + D

Binary tree traversals

```
void inorder(tree root) {
    if (root == nullptr) return;
    inorder(root->left);
    cout << root->key;
    inorder(root->right);
}
```



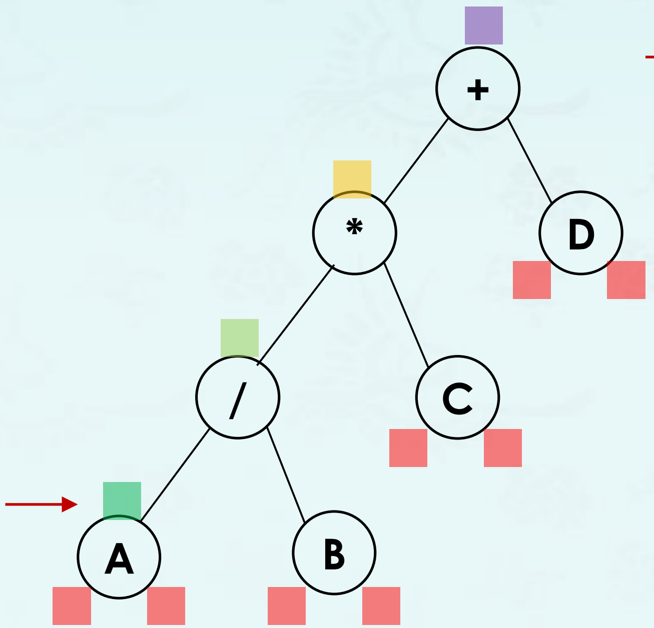
Call of inorder	**root** or **root→key**	Action
1	+	
2	*	
3	/	
4	A	
5	NULL	
4	A	**cout**
6	NULL	
3	/	
7	B	
8	NULL	
7	B	**cout**
9	NULL	
2	*	**cout**
10	C	

inorder	**root** or **root→key**	Value Action
11	NULL	
10	C	**cout**
12	NULL	
1	+	**cout**
13	D	
14	NULL	
13	D	**cout**
15	NULL	

system stack

Binary tree traversals

```
void inorder(tree root) {
    if (root == nullptr) return;
    inorder(root->left);
    cout << root->key;
    inorder(root->right);
}
```

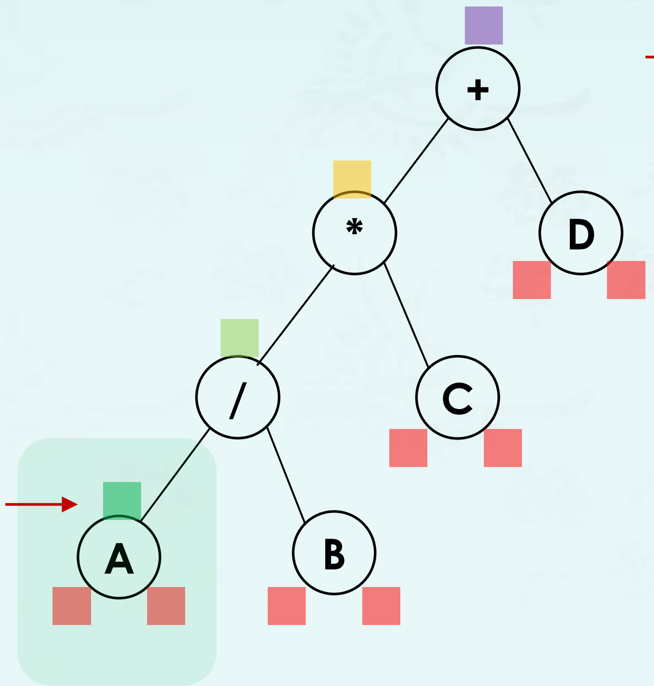


Call of inorder	root or root→key	Action
1	+	
2	*	
3	/	
4	A	
5	NULL	
4	A	**cout**
6	NULL	
3	/	
7	B	
8	NULL	
7	B	**cout**
9	NULL	
2	*	**cout**
10	C	

inorder	root or root→key	Value Action
11	NULL	
10	C	**cout**
12	NULL	
1	+	**cout**
13	D	
14	NULL	
13	D	**cout**
15	NULL	

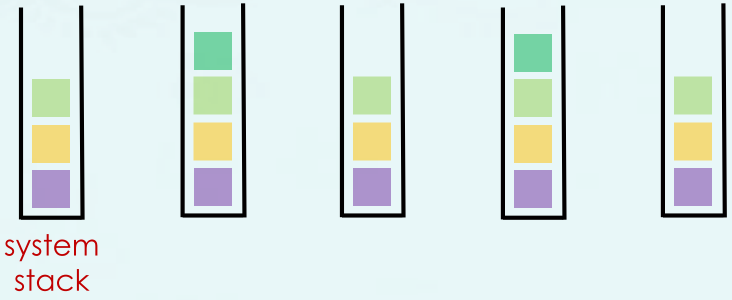
Binary tree traversals

```
void inorder(tree root) {
    if (root == nullptr) return;
    inorder(root->left);
    cout << root->key;
    inorder(root->right);
}
```



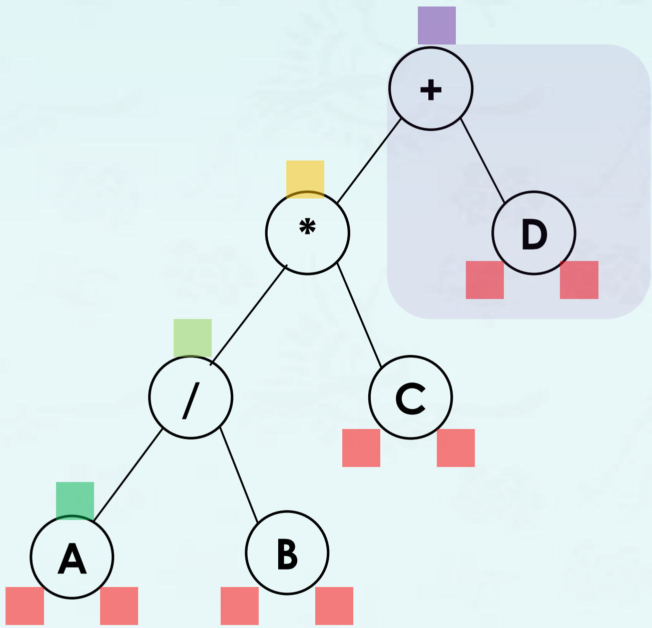
Call of inorder	root or root→key	Action
1	+	
2	*	
3	/	
4	A	push 4
5	NULL	pop 4
4	A	cout
6	NULL	push 4
3	/	pop 4, A done
7	B	
8	NULL	
7	B	cout
9	NULL	
2	*	cout
10	C	

inorder	root or root→key	Value Action
11	NULL	
10	C	cout
12	NULL	
1	+	cout
13	D	
14	NULL	
13	D	cout
15	NULL	



Binary tree traversals

```
void inorder(tree root) {
    if (root == nullptr) return;
    inorder(root->left);
    cout << root->key;
    inorder(root->right);
}
```

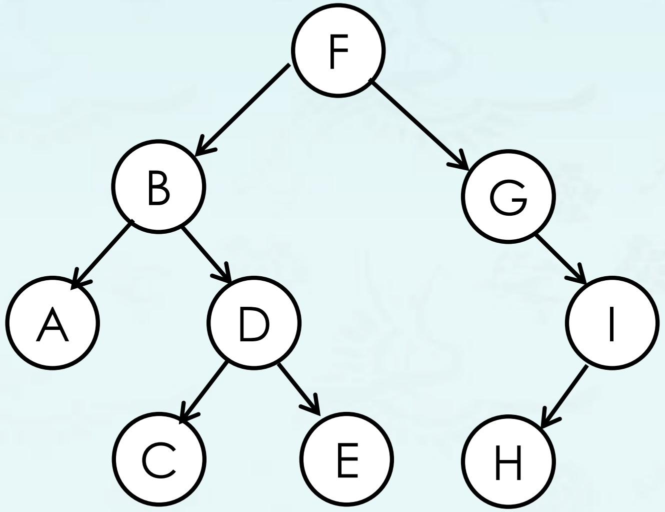


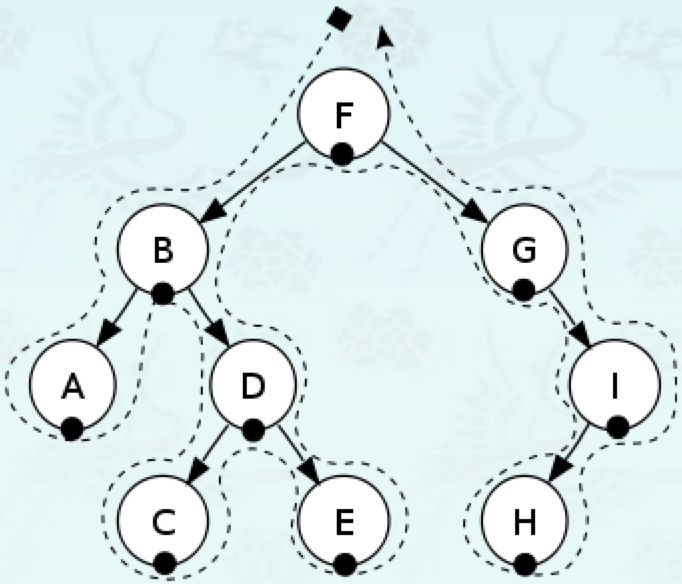
Call of inorder	root or root→key	Action
1	+	
2	*	
3	/	
4	A	
5	NULL	
4	A	**cout**
6	NULL	
3	/	
7	B	
8	NULL	
7	B	**cout**
9	NULL	
2	*	**cout**
10	C	

inorder	root or root→key		Value Action
11	NULL		
10	C		**cout**
12	NULL		
1	+	push 1	**cout**
13	D	push 13	
14	NULL	pop 13	
13	D		**cout**
15	NULL	push 13 pop 13	
			+ done

Binary tree traversals

- **Q: Inorder traversal(LVR)**
  - Traverse the left subtree.
  - Visit the root.
  - Traverse the right subtree.



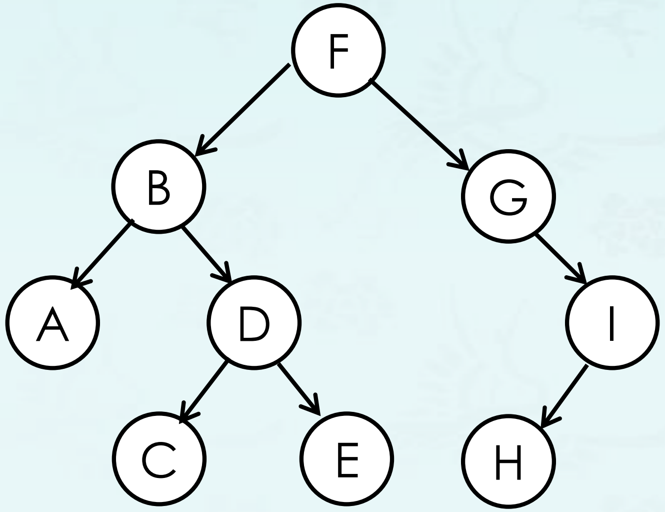


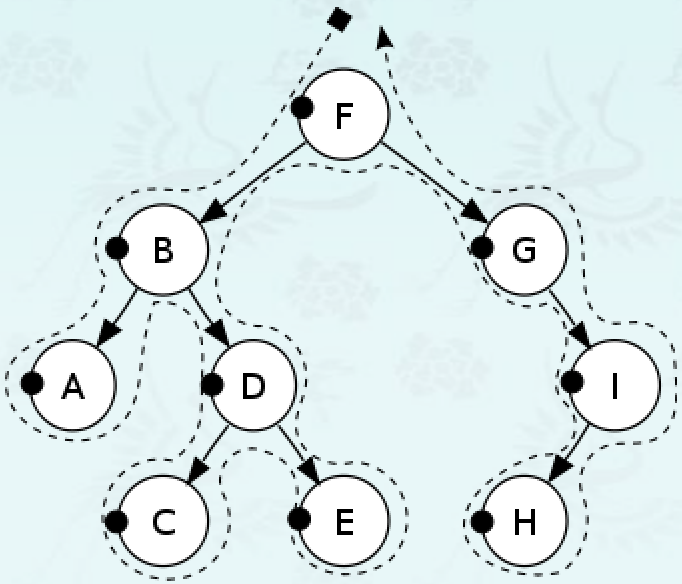

```
void inorder(tree root) {
   if (root == nullptr) return;

   inorder(root->left);
   cout << root->key;
   inorder(root->right);
}
```

Output: A, B, C, D, E, F, G, H, I

Binary tree traversals

- **Q: Preorder traversal(VLR)**
  - Step 1 – Visit root node.
  - Step 2 – Recursively traverse left subtree.
  - Step 3 – Recursively traverse right subtree.



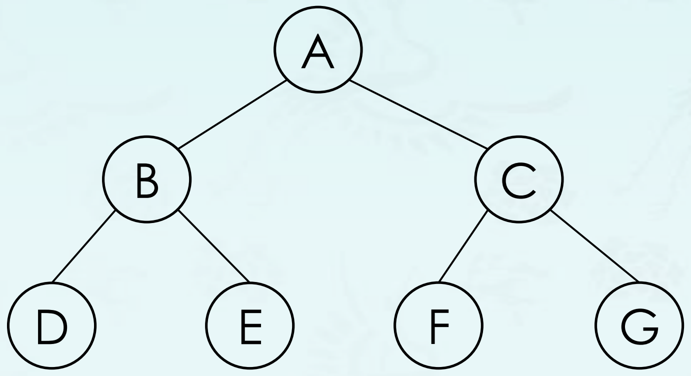



```
void preorder(tree root) {
    if (root == nullptr) return;

    cout << root->key;          V
    preorder(root->left);       L
    preorder(root->right);      R
}
```

Output: (VLR) : F, B, A, D, C, E, G, I, H

Binary tree traversals

- **Q: Preorder traversal(VLR)**
  - Step 1 – Visit root node.
  - Step 2 – Recursively traverse left subtree.
  - Step 3 – Recursively traverse right subtree.

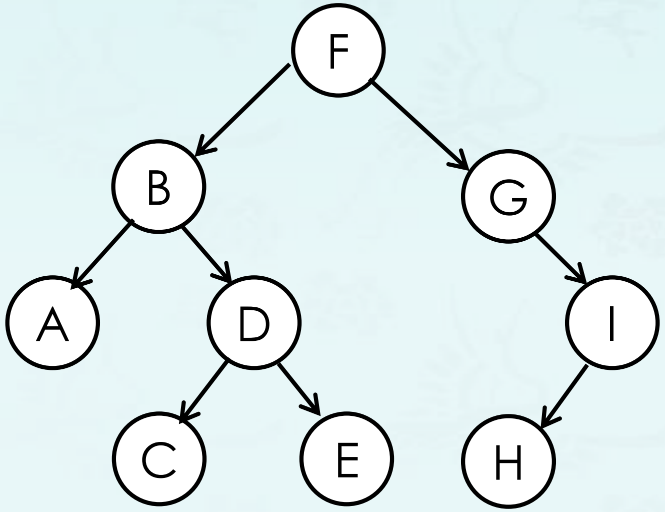


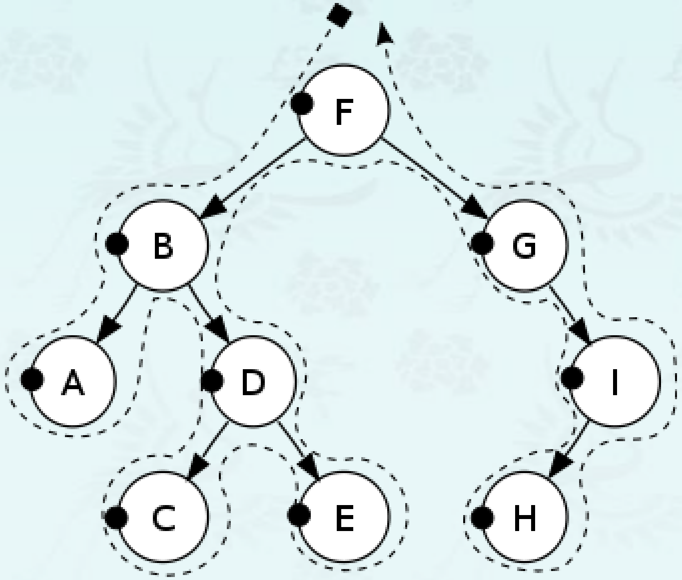

```
void preorder(tree root) {
   if (root == nullptr) return;

   cout << root->key;         V
   preorder(root->left);      L
   preorder(root->right);     R
}
```

Output: (VLR) : A B D E C F G

Binary tree traversals

- **Q: Postorder traversal(LRV)**
  - Step 1 – Recursively traverse left subtree.
  - Step 2 – Recursively traverse right subtree.
  - Step 3 – Visit root node.



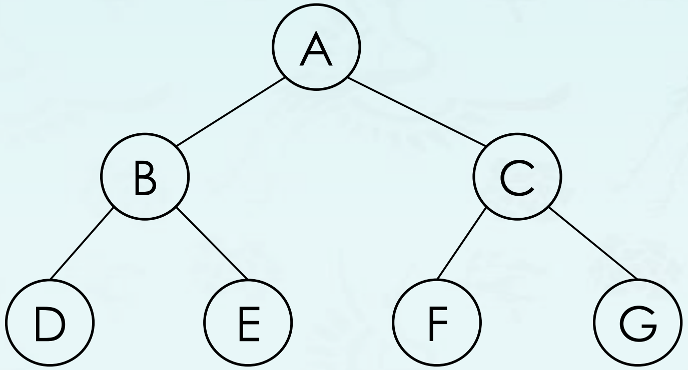



```
void postorder(tree root) {
    if (root == nullptr) return;

    postorder(root->left);       L
    postorder(root->right);      R
    cout << root->key;           V
}
```

Output(LRV): A C E D B H I G F

Binary tree traversals

- **Q: Postorder traversal(LRV)**
  - Step 1 – Recursively traverse left subtree.
  - Step 2 – Recursively traverse right subtree.
  - Step 3 – Visit root node.

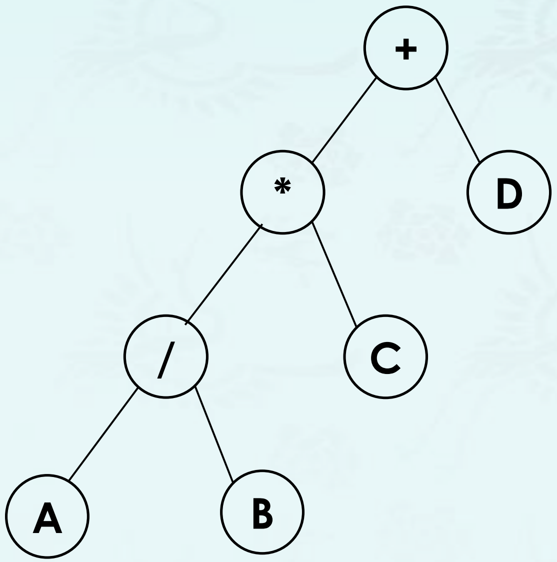



```
void postorder(tree root) {
   if (root == nullptr) return;

   postorder(root->left);
   postorder(root->right);
   cout << root->key;
}
```

Output(LRV): D E B F G C A

Binary tree traversals

- **Q: Postorder traversal(LRV)**
  - Step 1 – Recursively traverse left subtree.
  - Step 2 – Recursively traverse right subtree.
  - Step 3 – Visit root node.

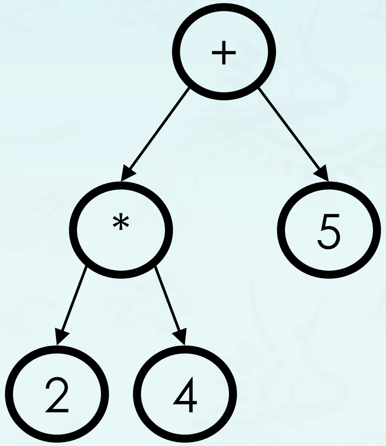



```
void postorder(tree root) {
   if (root == nullptr) return;

   postorder(root->left);
   postorder(root->right);
   cout << root->key;
}
```

Output(LRV): A B / C * D +

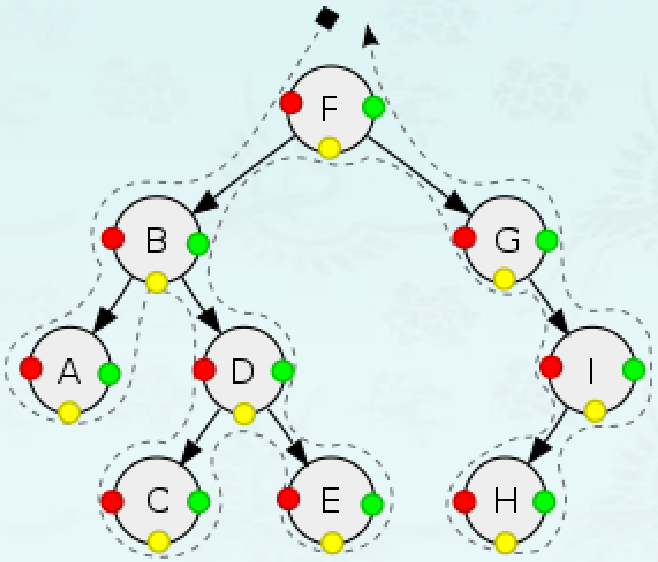
Binary tree traversals

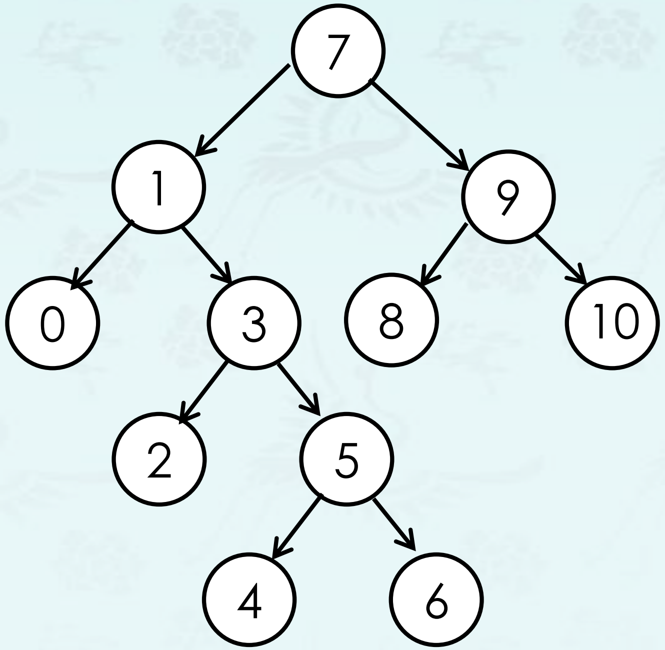
- **Exercise 1:**
  - preorder Traversal(VLR):
  - inorder traversal(LVR)   :
  - postorder traversal(LRV):
  - Level Order traversal :

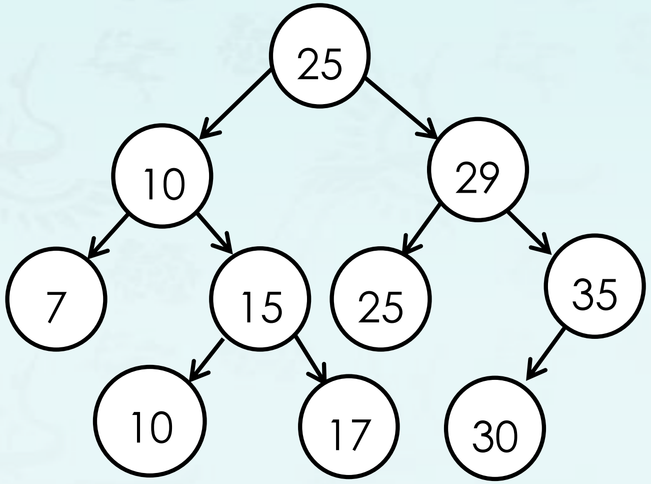


Binary tree traversals

- **Exercise 2:**
  - preorder Traversal(VLR)
  - inorder traversal(LVR)
  - postorder traversal(LRV)







Binary tree traversals

- **Observations:**
  1. If you know you need to **explore the roots** before inspecting any leaves, you pick **preorder** because you will encounter all the roots before all of the leaves.

  2. If you know you need to **explore all the leaves** before any nodes, you select **postorder** because you don't waste any time inspecting roots in search for leaves.

  3. If you know that the tree has an inherent sequence in the nodes, and you want to flatten the tree back into its original sequence, than an **inorder** traversal should be used. The tree would be flattened in the same way it was created. A pre-order or post-order traversal might not unwind the tree back into the sequence which was used to create it.

  4. In a binary search tree ordered such that in each node the key is greater than all keys in its left subtree and less than all keys in its right subtree, in-order traversal retrieves the keys in *ascending* sorted order.

Summary &
quaestio quaestio qo ? ? ? ?

Data Structures
Chapter 5
Tree

1. introduction
2. **Binary tree**
   - **Definition and Properties**
   - **Traversal**
   - Coding
3. Binary search tree
4. Tree balancing

Prof. Youngsup Kim, idebtor@gmail.com, CSEE Dept., Grace School Rm204, Handong Global University