

# $n$ 阶行列式性质的证明

定理1.2.1. 行列式与它的转置行列式的值相等.

证明思路:

先证明行列式可以按第1列展开

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

行列式按第1行展开

$$A = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{1+k}a_{1k}M_{1k} + \cdots + (-1)^{1+n}a_{1n}M_{1n}$$

行列式按第1列展开

$$B = a_{11}M_{11} - a_{21}M_{21} + \cdots + (-1)^{i+1}a_{i1}M_{i1} + \cdots + (-1)^{n+1}a_{n1}M_{n1}$$

只要比较后面  
 $n-1$ 项是否相等

按第1列展开

按第1行展开

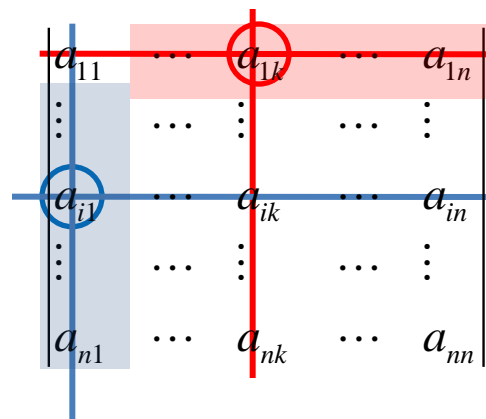
$$A = a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{1+k}a_{1k}M_{1k} + \cdots + (-1)^{1+n}a_{1n}M_{1n}$$

中的项：  $(-1)^{1+k}a_{1k}M_{1k}$

再按第1列展开后（归纳假设）的一般项：

$$(-1)^{1+k}a_{1k}(-1)^{i-1+1}a_{i1}M_{1k,i1} = (-1)^{i+k+1}a_{1k}a_{i1}M_{1k,i1}$$

二次展开后的  
每个小项相等



$$B = a_{11}M_{11} - a_{21}M_{21} + \cdots + (-1)^{i+1}a_{i1}M_{i1} + \cdots + (-1)^{n+1}a_{n1}M_{n1}$$

中的项：  $(-1)^{i+1}a_{i1}M_{i1}$

再按第1行展开后的一般项：

$$(-1)^{i+1}a_{i1}(-1)^{1+k-1}a_{1k}M_{i1,1k} = (-1)^{i+k+1}a_{i1}a_{1k}M_{i1,1k}$$

再证明转置后行列式的值不变

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$M_{ij} : \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$A' = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

$$N_{ji} : \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

易知:  $M_{ij}' = N_{ji}$

$$A = a_{11}M_{11} - a_{21}M_{21} + \cdots + (-1)^{k+1}a_{k1}M_{k1} + \cdots + (-1)^{n+1}a_{n1}M_{n1}$$

A按第1列展开

||

$M_{k1} = M_{k1}'$  (归纳假设)

$$a_{11}M_{11}' - a_{21}M_{21}' + \cdots + (-1)^{k+1}a_{k1}M_{k1}' + \cdots + (-1)^{n+1}a_{n1}M_{n1}'$$

||

$$A' = a_{11}N_{11} - a_{21}N_{12} + \cdots + (-1)^{1+k}a_{k1}N_{1k} + \cdots + (-1)^{1+n}a_{n1}N_{1n}$$

A'按第1行展开

定理1.2.2. 对调两行(列)的位置，行列式的值相差一个负号，即

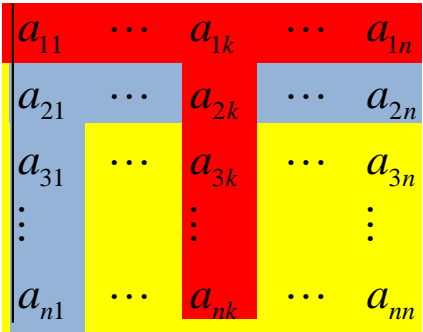
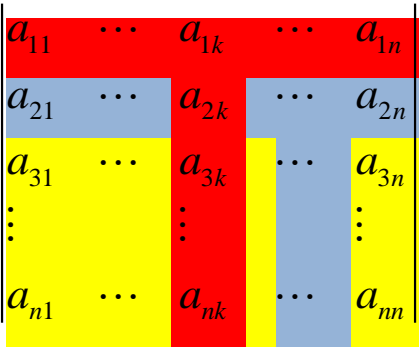
$$\begin{vmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{i1} & a_{i2} & \cdots & a_{in} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{j1} & a_{j2} & \cdots & a_{jn} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{n1} & a_{n2} & \cdots & a_{nn}
 \end{vmatrix}
 = -
 \begin{vmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{j1} & a_{j2} & \cdots & a_{jn} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{i1} & a_{i2} & \cdots & a_{in} \\
 \vdots & \vdots & \cdots & \vdots \\
 a_{n1} & a_{n2} & \cdots & a_{nn}
 \end{vmatrix}$$

证明思路：

分三种情况：

第一第二两行交换；  
 任意相邻两行交换；  
 任意两行交换

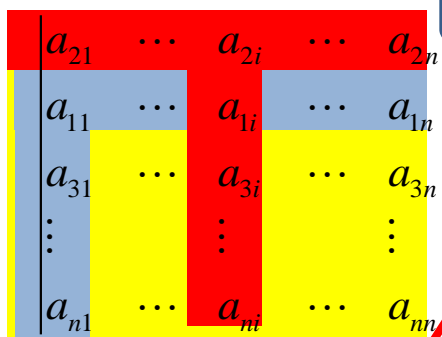
## 情况1：第一第二两行交换

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$



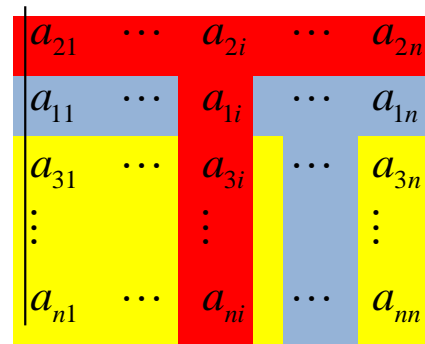
$$= a_{11}M_{11} + \cdots + (-1)^{1+k} a_{1k}M_{1k} + \cdots + (-1)^{1+n} a_{1n}M_{1n}$$

$$(-1)^{1+k} a_{1k}M_{1k} = (-1)^{1+k} a_{1k} \sum_{i=1}^{k-1} (-1)^{1+i} a_{2i}M_{1k,2i} + (-1)^{1+k} a_{1k} \sum_{i=k+1}^n (-1)^i a_{2i}M_{1k,2i}$$

$$B = \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$



$k < i$  的项



$$= a_{21}M_{21} + \cdots + (-1)^{1+i} a_{2i}M_{2i} + \cdots + (-1)^{1+n} a_{2n}M_{2n}$$

$$(-1)^{1+i} a_{2i}M_{2i} = (-1)^{1+i} a_{2i} \sum_{k=1}^{i-1} (-1)^{1+k} a_{1k}M_{2i,1k} + (-1)^{1+i} a_{2i} \sum_{k=i+1}^n (-1)^k a_{1k}M_{2i,1k}$$

## 情况2：任意相邻两行交换

考虑第2行到第*n*行之间的相邻两行交换

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}M_{11} + \cdots + (-1)^{1+k} a_{1k}M_{1k} + \cdots + (-1)^{1+n} a_{1n}M_{1n}$$

$$B = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}N_{11} + \cdots + (-1)^{1+k} a_{1k}N_{1k} + \cdots + (-1)^{1+n} a_{1n}N_{1n}$$

$N_{1k} = -M_{1k}$  (归纳假设)

$$\begin{aligned} &= -a_{11}M_{11} + \cdots - (-1)^{1+k} a_{1k}M_{1k} + \cdots - (-1)^{1+n} a_{1n}M_{1n} \\ &= -(a_{11}M_{11} + \cdots + (-1)^{1+k} a_{1k}M_{1k} + \cdots + (-1)^{1+n} a_{1n}M_{1n}) = -A \end{aligned}$$

### 情况3：任意两行交换

$$\begin{aligned}
 A = & \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \xrightarrow{\text{交换行 } i \text{ 和 } j} = (-1)^{j-i} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix} \xrightarrow{\text{交换行 } i+1 \text{ 和 } j} = (-1)^{j-i} (-1)^{j-(i+1)} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix} = -B
 \end{aligned}$$

推论1.2.3. 两行(列)相等的行列式的值为0.

证明思路:

交换*i, j*行

*i*行

*j*行

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1) \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = -A$$



推论1.2.4.行列式可以按任一行(列)展开.

证明思路:

$$\begin{aligned}
 A &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
 &= (-1)^{i-1} \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
 &= (-1)^{i-1} (a_{i1}M_{i1} - \cdots + (-1)^{1+k}a_{ik}M_{ik} + \cdots + (-1)^{1+n}a_{in}M_{in}) \\
 &= (-1)^{i+1}a_{i1}M_{i1} + \cdots + (-1)^{i+k}a_{ik}M_{ik} + \cdots + (-1)^{i+n}a_{in}M_{in} \\
 &= a_{i1}A_{i1} + \cdots + a_{ik}A_{ik} + \cdots + a_{in}A_{in}
 \end{aligned}$$

**定理1.2.5.** 行列式的任一行(列)元素的公因子可以提到行列式外面.

证明思路:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = ka_{i1}A_{i1} + \cdots + ka_{ik}A_{ik} + \cdots + ka_{in}A_{in}$$

$$\parallel$$

$$k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k(a_{i1}A_{i1} + \cdots + a_{ik}A_{ik} + \cdots + a_{in}A_{in})$$

**推论1.2.6.** 若行列式某两行(列)对应元素成比例，  
则行列式的值为零.

**证明思路：**

$$\begin{aligned}
 A &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
 &= k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \times 0 = 0
 \end{aligned}$$

**定理1.2.7.** 行列式的第*i*行(列) 的每一个元素都可以表示为两数的和, 则该行列式可以表示为两个行列式之和.

**证明思路:**

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & a_{in}+b_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= (a_{i1}+b_{i1})A_{i1} + \cdots + (a_{ik}+b_{ik})A_{ik} + \cdots + (a_{in}+b_{in})A_{in}$$

$$\begin{aligned} &= a_{i1}A_{i1} + \cdots + a_{ik}A_{ik} + \cdots + a_{in}A_{in} \\ &+ b_{i1}A_{i1} + \cdots + b_{ik}A_{ik} + \cdots + b_{in}A_{in} \end{aligned}$$

**定理1.2.8.** 将行列式的任意一行(列) 乘以数 $k$ 加到另一行(列)上去,  
行列式的值不变.

**证明思路:**

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

定理1.2.9. 行列式任一行(列) 的元素与另一行(列)元素的代数余子式对应乘积之和为零. 即, 若设  $A = |a_{ij}|_{n \times n}$  则有

$$\sum_{k=1}^n a_{ik} A_{jk} = a_{i1} A_{j1} + a_{i2} A_{j2} + \cdots + a_{in} A_{jn} = \begin{cases} A, & i=j, \\ 0, & i \neq j, \end{cases}$$

$$\left( \sum_{k=1}^n a_{ki} A_{kj} = a_{1i} A_{1j} + a_{2i} A_{2j} + \cdots + a_{ni} A_{nj} = \begin{cases} A, & i=j, \\ 0, & i \neq j. \end{cases} \right)$$

证明思路:

$i \neq j$  时:

$$\sum_{k=1}^n a_{ik} A_{jk} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

Diagram illustrating the proof for  $i \neq j$ . The determinant is shown with two identical rows (row  $i$  and row  $j$ ) highlighted in yellow. A red box labeled "i行" points to the  $i$ -th row, and a blue box labeled "j行" points to the  $j$ -th row. The sum  $\sum_{k=1}^n a_{ik} A_{jk}$  is shown to the left of the determinant.