克莱姆法则

定理1.2.10. 对于n元线性方程组

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1, \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=b_n, \end{cases}$$

若系数行列式

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}\neq 0$$

则原方程组有解，且解是唯一的，这个解可用公式表示为：

$$x_j=\frac{D_j}{D}, \quad (j=1,2,\cdots,n)$$

其中D_j (j=1,2, …,n)为：

$$D_j=\begin{array}{c} \begin{array}{ccccccc} \phantom{a_{11}} & & \phantom{a_{1,j-1}} & \text{第j列} & \phantom{a_{1,j+1}} & & \phantom{a_{1n}} \end{array} \\ \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} \end{array}\neq 0$$

证明思路： $D \neq 0$

先证明解的唯一性

a_{1j}的代数余子式

各行乘以一个数

$$\begin{cases} a_{11}x_1+\cdots+a_{1j}x_j+\cdots+a_{1n}x_n=b_1, & \times D_{1j} \\ a_{21}x_1+\cdots+a_{2j}x_j+\cdots+a_{2n}x_n=b_2, & \times D_{2j} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots \\ a_{n1}x_1+\cdots+a_{nj}x_j+\cdots+a_{nn}x_n=b_n, & \times D_{nj} \end{cases}$$

方程组即为

$$(+)\begin{cases} a_{11}D_{1j}x_1+\cdots+a_{1j}D_{1j}x_j+\cdots+a_{1n}D_{1j}x_n=b_1D_{1j}, \\ a_{21}D_{2j}x_1+\cdots+a_{2j}D_{2j}x_j+\cdots+a_{2n}D_{2j}x_n=b_2D_{2j}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}D_{nj}x_1+\cdots+a_{nj}D_{nj}x_j+\cdots+a_{nn}D_{nj}x_n=b_nD_{nj}, \end{cases}$$

各行对应项相加

$$\underbrace{\sum_{i=1}^{n}a_{i1}D_{ij}}_{0}x_1+\cdots+\underbrace{\sum_{i=1}^{n}a_{ij}D_{ij}}_{D}x_j+\cdots+\underbrace{\sum_{i=1}^{n}a_{in}D_{ij}}_{0}x_n=\underbrace{\sum_{i=1}^{n}b_iD_{ij}}_{D_j}$$

$Dx_j=D_j$ 即 $x_j=\dfrac{D_j}{D}$

再证明解的存在性

将 $x_j = \dfrac{D_j}{D}$ 代入第i个方程得：

$$a_{i1}\frac{D_1}{D} + a_{i2}\frac{D_2}{D} + \cdots + a_{in}\frac{D_n}{D}$$

$$= \frac{1}{D}(a_{i1}D_1 + a_{i2}D_2 + \cdots + a_{in}D_n)$$

因为：

$$\begin{aligned}
& a_{i1}D_1 + a_{i2}D_2 + \cdots + a_{in}D_n \\
&= a_{i1}b_1D_{11} + a_{i2}b_1D_{12} + \cdots + a_{in}b_1D_{1n} \\
&+ a_{i1}b_2D_{21} + a_{i2}b_2D_{22} + \cdots + a_{in}b_2D_{2n} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&+ a_{i1}b_nD_{n1} + a_{i2}b_nD_{n2} + \cdots + a_{in}b_nD_{nn}
\end{aligned}$$

$$\left.\begin{aligned}
&= b_1(a_{i1}D_{11} + a_{i2}D_{12} + \cdots + a_{in}D_{1n}) && = b_1 \times 0 \\
&+ b_2(a_{i1}D_{21} + a_{i2}D_{22} + \cdots + a_{in}D_{2n}) && = b_2 \times 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && \cdots\cdots \\
&+ b_i\boxed{(a_{i1}D_{i1} + a_{i2}D_{i2} + \cdots + a_{in}D_{in})} && = b_i \times D \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && \cdots\cdots \\
&+ b_n(a_{i1}D_{n1} + a_{i2}D_{n2} + \cdots + a_{in}D_{nn}) && = b_n \times 0
\end{aligned}\right\} = b_i \times D$$

故有：

$$a_{i1}\frac{D_1}{D} + a_{i2}\frac{D_2}{D} + \cdots + a_{in}\frac{D_n}{D} = b_i$$

故有解： $x_j = \dfrac{D_j}{D}$