习题五补充讲解(1)

4 证用数学归纳法，设 $A=\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{pmatrix}$.

当 n=1 时 $A=O$；n=2 时，或 $A=O$，或 $E(2(1/a_{12}))^T AE(2(1/a_{12}))=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$，成立.

假设 n<m 时结论成立 .

当 n=m>2 时，若 $A=O$ 结论成立. 否则 $A=\begin{pmatrix} 0 & a_{12} & \cdots & a_{1m} \\ -a_{12} & 0 & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1m} & -a_{2m} & \cdots & 0 \end{pmatrix}$ 中有 $a_{ij}\neq 0, i<j$,

令 $P_1=\begin{cases} E(m(1/a_{1m})), & a_{1m}\neq 0 \\ E(j,m)E(1,i)E(m(1/a_{ij})), & a_{1m}=0 \end{cases}$，则有 $P_1^T AP_1=A_2=\begin{pmatrix} 0 & c_{12} & \cdots & 1 \\ -c_{12} & 0 & \cdots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -c_{2m} & \cdots & 0 \end{pmatrix}$,

再令 $P_2=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -c_{12} & -c_{13} & \cdots & 1 \end{pmatrix}\begin{pmatrix} 1 & -c_{2m} & -c_{3m} & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$,

则 $P_2^T A_2 P_2=A_3=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & d_{23} & \cdots & d_{2,m-1} & 0 \\ 0 & -d_{23} & 0 & \cdots & d_{3,m-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -d_{2,m-1} & -d_{3,m-1} & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & B & 0 \\ -1 & 0 & 0 \end{pmatrix}, B^T=-B$,

由归纳假设，存在可逆矩阵 Q，使得 $Q^T BQ=\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ -1 & \cdots & 0 \end{pmatrix}$,

令 $P=P_1P_2\begin{pmatrix} 1 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & 1 \end{pmatrix}$，则有 $P^T AP=\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ -1 & \cdots & 0 \end{pmatrix}$