n阶行列式性质的证明

定理1.2.1. 行列式与它的转置行列式的值相等.

证明思路:

先证明行列式可以按第1列展开

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

只要比较后面 n-1项是否相等

按第1列展开

行列式按第1行展开 $A=a_{11}M_{11}-a_{12}M_{12}+\cdots+(-1)^{1+k}a_{1k}M_{1k}+\cdots+(-1)^{1+n}a_{1n}M_{1n}$

行列式按第1列展开 $B=a_{11}M_{11}-a_{21}M_{21}+\cdots+(-1)^{i+1}a_{i1}M_{i1}+\cdots+(-1)^{n+1}a_{n1}M_{n1}$

按第1行展开

$$A=a_{11}M_{11}-a_{12}M_{12}+\cdots+(-1)^{1+k}a_{1k}M_{1k}+\cdots+(-1)^{1+n}a_{1n}M_{1n}$$

中的项： $(-1)^{1+k}a_{1k}M_{1k}$

再按第1列展开后（归纳假设）的一般项：

$$(-1)^{1+k}a_{1k}(-1)^{i-1+1}a_{i1}M_{1k,i1}=(-1)^{i+k+1}a_{1k}a_{i1}M_{1k,i1}$$

二次展开后的
每个小项相等

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix}$$

$$B=a_{11}M_{11}-a_{21}M_{21}+\cdots+(-1)^{i+1}a_{i1}M_{i1}+\cdots+(-1)^{n+1}a_{n1}M_{n1}$$

中的项： $(-1)^{i+1}a_{i1}M_{i1}$

再按第1行展开后的一般项：

$$(-1)^{i+1}a_{i1}(-1)^{1+k-1}a_{1k}M_{i1,1k}=(-1)^{i+k+1}a_{i1}a_{1k}M_{i1,1k}$$

再证明转置后行列式的值不变

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \qquad M_{ij}: \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

易知：$M_{ij}'=N_{ji}$

$$A'=\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} \qquad N_{ji}: \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

$$A=a_{11}M_{11}-a_{21}M_{21}+\cdots+(-1)^{k+1}a_{k1}M_{k1}+\cdots+(-1)^{n+1}a_{n1}M_{n1}$$ A按第1列展开

$$\|$$

$M_{k1}=M_{k1}'$ (归纳假设)

$$a_{11}M_{11}'-a_{21}M_{21}'+\cdots+(-1)^{k+1}a_{k1}M_{k1}'+\cdots+(-1)^{n+1}a_{n1}M_{n1}'$$

$$\|$$

$$A'=a_{11}N_{11}-a_{21}N_{12}+\cdots+(-1)^{1+k}a_{k1}N_{1k}+\cdots+(-1)^{1+n}a_{n1}N_{1n}$$ A' 按第1行展开

定理1.2.2. 对调两行(列)的位置，行列式的值相差一个负号，即

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

证明思路： 分三种情况：

第一第二两行交换；

任意相邻两行交换；

任意两行交换

情况1：第一第二两行交换

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3k} & \cdots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3k} & \cdots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}M_{11} + \cdots + (-1)^{1+k} a_{1k}M_{1k} + \cdots + (-1)^{1+n} a_{1n}M_{1n}$$

$$(-1)^{1+k} a_{1k}M_{1k} = (-1)^{1+k} a_{1k} \sum_{i=1}^{k-1} (-1)^{1+i} a_{2i} M_{1k,2i} + (-1)^{1+k} a_{1k} \sum_{i=k+1}^{n} (-1)^{i} a_{2i} M_{1k,2i}$$

$k<i$的项

$$B=\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3i} & \cdots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix} \qquad \begin{vmatrix} a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3i} & \cdots & a_{3n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{21}M_{21} + \cdots + (-1)^{1+i} a_{2i}M_{2i} + \cdots + (-1)^{1+n} a_{2n}M_{2n}$$

$$(-1)^{1+i} a_{2i}M_{2i} = (-1)^{1+i} a_{2i} \sum_{k=1}^{i-1} (-1)^{1+k} a_{1k} M_{2i,1k} + (-1)^{1+i} a_{2i} \sum_{k=i+1}^{n} (-1)^{k} a_{1k} M_{2i,1k}$$

情况2：任意相邻两行交换

考虑第2行到第n行之间的相邻两行交换

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}M_{11}+\cdots+(-1)^{1+k}a_{1k}M_{1k}+\cdots+(-1)^{1+n}a_{1n}M_{1n}$$

$$B=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad = a_{11}N_{11}+\cdots+(-1)^{1+k}a_{1k}N_{1k}+\cdots+(-1)^{1+n}a_{1n}N_{1n}$$

$\boxed{N_{1k}\text{=-}M_{1k}\ \text{(归纳假设)}}$

$$= -a_{11}M_{11}+\cdots-(-1)^{1+k}a_{1k}M_{1k}+\cdots-(-1)^{1+n}a_{1n}M_{1n}$$

$$= -(a_{11}M_{11}+\cdots+(-1)^{1+k}a_{1k}M_{1k}+\cdots+(-1)^{1+n}a_{1n}M_{1n})=-A$$

情况3：任意两行交换

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^{j-i} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix} = (-1)^{j-i}(-1)^{j-(i+1)} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix} = -B$$

推论1.2.3. 两行(列)相等的行列式的值为0.

证明思路:

交换i, j行

$$
\begin{array}{r} \\ \\ i\text{行} \\ \\ j\text{行} \\ \\ \\ \end{array}
A=\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \cdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \cdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \cdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
=(-1)\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & \cdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \cdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & \cdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
=-A
$$

推论1.2.4.行列式可以按任一行(列)展开.

证明思路:

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i-1}\begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$=(-1)^{i-1}(a_{i1}M_{i1}-\cdots+(-1)^{1+k}a_{ik}M_{ik}+\cdots+(-1)^{1+n}a_{in}M_{in})$$

$$=(-1)^{i+1}a_{i1}M_{i1}+\cdots+(-1)^{i+k}a_{ik}M_{ik}+\cdots+(-1)^{i+n}a_{in}M_{in}$$

$$=a_{i1}A_{i1}+\cdots+a_{ik}A_{ik}+\cdots+a_{in}A_{in}$$

定理1.2.5. 行列式的任一行(列)元素的公因子可以提到行列式外面.

证明思路:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = ka_{i1}A_{i1}+\cdots+ka_{ik}A_{ik}+\cdots+ka_{in}A_{in}$$

$$\Vert$$

$$k\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k(a_{i1}A_{i1}+\cdots+a_{ik}A_{ik}+\cdots+a_{in}A_{in})$$

推论1.2.6. 若行列式某两行(列)对应元素成比例，
则行列式的值为零.

证明思路：

$$A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\times 0 = 0$$

定理1.2.7. 行列式的第i行(列)的每一个元素都可以表示为两数的和，则该行列式可以表示为两个行列式之和.

证明思路:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & a_{in}+b_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (a_{i1}+b_{i1})A_{i1}+\cdots+(a_{ik}+b_{ik})A_{ik}+\cdots+(a_{in}+b_{in})A_{in}$$

$$= a_{i1}A_{i1}+\cdots+a_{ik}A_{ik}+\cdots+a_{in}A_{in} + b_{i1}A_{i1}+\cdots+b_{ik}A_{ik}+\cdots+b_{in}A_{in}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

定理1.2.8. 将行列式的任意一行(列) 乘以数k加到另一行(列)上去，行列式的值不变.

证明思路:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1}+ka_{j1} & a_{i2}+ka_{j2} & \cdots & a_{in}+ka_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{j1} & ka_{j2} & \cdots & ka_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

定理1.2.9. 行列式任一行(列) 的元素与另一行(列)元素的代数余子式对应乘积之和为零. 即，若设 $A=\left|a_{ij}\right|_{n\times n}$ 则有

$$\sum_{k=1}^{n} a_{ik}A_{jk}=a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=\begin{cases}A, & i=j,\\ 0, & i\neq j,\end{cases}$$

$$\left(\sum_{k=1}^{n} a_{ki}A_{kj}=a_{1i}A_{1j}+a_{2i}A_{2j}+\cdots+a_{ni}A_{nj}=\begin{cases}A, & i=j,\\ 0, & i\neq j.\end{cases}\right)$$

证明思路:

$i\neq j$ 时:

$$\begin{array}{r} \\ \\ i\text{行} \\ \\ j\text{行} \\ \\ \\ \end{array} \quad \sum_{k=1}^{n} a_{ik}A_{jk}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \vdots & \vdots & \cdots & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=0$$