分块矩阵

矩阵有时候是：对角、三角、O 等特殊形状

$$\begin{pmatrix} 1.33 & 0 & 0 \\ 0 & -2.4 & 0 \\ 0 & 0 & -0.3 \end{pmatrix}, \quad \begin{pmatrix} 7 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

矩阵有时候：没有任何形状上的特殊性

$$\begin{pmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -4 & 5 \\ 1 & 3 & -5 & 7 \\ 2 & 3 & -7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \end{pmatrix}$$

矩阵有时候是：有点对角、上三角等形状

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

有时候需要将矩阵分块来看待，如

$$\left(\begin{array}{cc|ccc} 3 & 1 & 2 & 0 & -3 \\ 1 & 3 & 1 & -2 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc} A_{11} & A_{12} \\ O_{3\times2} & E_3 \end{array}\right)$$

其中各块小矩阵为：

$$A_{11} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix},$$

$$O_{3\times2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

定义：对于一个$m \times n$矩阵，如果在行的方向分成s块，在列的方向分成t块，就得到A的一个$s \times t$分块矩阵. 记作

$$A = (A_{kl})_{s \times t} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{st} \end{pmatrix} \begin{matrix} m_1\text{行} \\ m_2\text{行} \\ \vdots \\ m_s\text{行} \end{matrix}$$

$$n_1\text{列} \quad n_2\text{列} \quad \cdots \quad n_t\text{列}$$

其中 $m_1+m_2+\cdots+m_s=m,\ n_1+n_2+\cdots+n_t=n$，
而A_{kl} (k=1,2, …,s;l=1,2,…,t)称为A的子块.

分块矩阵的运算

(1) 分块矩阵的加法．设分块矩阵

$$A=(\ A_{kl}\ )_{s\times t}\ ,\quad B=(\ B_{kl}\ )_{s\times t}\ ,$$

如果A与B对应的子块A_{kl}和B_{kl}都是同型矩阵，则

$$A+B=(\ A_{kl}+B_{kl}\ )_{s\times t}\ .$$

(2) 分块矩阵的数乘．设分块矩阵

$$A=(\ A_{kl}\ )_{s\times t}\ ,$$

λ是数，则

$$\lambda A=(\lambda A_{kl}\ )_{s\times t}\ .$$

(3) 分块矩阵的乘法．设分块矩阵

$$A=(\ a_{ij}\ )_{m\times n}\ ,\quad B=(\ b_{ij}\ )_{n\times p}\ ,$$

如果把A, B分别分块为$r\times s$和$s\times t$分块矩阵，且A的列的分块法与B的行的分块法完全相同，则

$$AB=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{st} \end{pmatrix}=C=\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rt} \end{pmatrix},$$

其中 C 是$r\times t$分块矩阵，且

$$C_{kl}=A_{k1}B_{1l}+A_{k2}B_{2l}+\cdots+A_{ks}B_{sl}=\sum_{i=1}^{s}A_{ki}B_{il},(k=1,2,\cdots,r;l=1,2,\cdots,t).$$

例**2.3.1** 设

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix},$$

求AB.

解 把A, B分成

$$A = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right) = \begin{pmatrix} A_{11} & O_2 \\ A_{21} & E_2 \end{pmatrix}, \quad B = \left(\begin{array}{cc} 1 & 0 \\ 1 & -2 \\ \hline 1 & 0 \\ -1 & 1 \end{array} \right) = \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix},$$

则

$$AB = \begin{pmatrix} A_{11} & O_2 \\ A_{21} & E_2 \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} \\ A_{21}B_{11} + B_{21} \end{pmatrix},$$

而

$$A_{11}B_{11} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & -4 \end{pmatrix}, A_{21}B_{11} + B_{21} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix},$$

于是

$$AB = \begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

(4) 分块矩阵的转置. $s \times t$ 分块矩阵 $A = (A_{kl})_{s \times t}$ 的转置矩阵 A^T 为 $t \times s$ 分块矩阵，如果记 $A^T = (B_{lk})_{t \times s}$，则

$$B_{lk} = A_{kl}{}^T, \quad (l=1,2,\ldots,t;\ k=1,2,\ldots,s).$$

例如，若 $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}$, 则 $A^T = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \\ A_{13}^T & A_{23}^T \end{pmatrix}.$

(5) 分块对角矩阵. 设 n 阶矩阵 A 的分块矩阵只有在对角线上有非零子块，其余子块都为零矩阵，且在对角线上的子块都是方阵，即

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix},$$

其中 A_i $(i=1,2,\ldots,s)$ 都是方阵，则称 A 为分块对角矩阵，也称准对角矩阵.

例如：$A = \left(\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & -2 \end{array} \right) = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{pmatrix},$ 其中：$A_1 = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}, A_2 = (2),$

$A_3 = \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}.$

矩阵按行分块和按列分块是两种常见的分块法 (设$A=(a_{ij})_{m \times n}$)；

(1) 按行分块
$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}, 其中 \alpha_i = (a_{i1}, a_{i2}, \cdots, a_{in}), i = 1, 2, \cdots, m.$$

(2) 按列分块 $\quad A = (\beta_1, \beta_2, \cdots, \beta_n), 其中 \beta_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, j = 1, 2, \cdots, n.$

矩阵乘法可以将矩阵行列分块后再进行相乘，以例**2.3.1**为例。

A按行B按列：
$$AB = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2 \\ \alpha_3\beta_1 & \alpha_3\beta_2 \\ \alpha_4\beta_1 & \alpha_4\beta_2 \end{pmatrix},$$

A按列B按行：
$$AB = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 & \tilde{\alpha}_3 & \tilde{\alpha}_4 \end{pmatrix} \begin{pmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \tilde{\beta}_3 \\ \tilde{\beta}_4 \end{pmatrix} = \tilde{\alpha}_1\tilde{\beta}_1 + \tilde{\alpha}_2\tilde{\beta}_2 + \tilde{\alpha}_3\tilde{\beta}_3 + \tilde{\alpha}_4\tilde{\beta}_4.$$

例**2.3.2** 设
$$A = (a_{ij})_{m \times s} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}, B = (b_{ij})_{s \times n} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix},$$

则
$$AB = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \cdots & \alpha_1\beta_n \\ \alpha_2\beta_1 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_n \\ \vdots & \vdots & & \vdots \\ \alpha_m\beta_1 & \alpha_m\beta_2 & \cdots & \alpha_m\beta_n \end{pmatrix} = (c_{ij})_{m \times n},$$

其中
$$c_{ij} = \alpha_i \beta_j = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{is} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix} = \sum_{k=1}^{s} a_{ik} b_{kj},$$

又设对角矩阵

$$\Lambda_m = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}, \quad \Lambda_n = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}, \quad 其中的 \lambda_i 为数,$$

则有：

$$\Lambda_m A_{m \times n} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ \vdots \\ \lambda_m \alpha_m \end{pmatrix},$$

$$A_{m \times n} \Lambda_n = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \beta_1 & \lambda_2 \beta_2 & \cdots & \lambda_n \beta_n \end{pmatrix}.$$

使用分块思想的实例：

习题一 5(2)中行列式的计算：

$$D = \begin{vmatrix} x & a & b & c \\ a & x & c & b \\ b & c & x & a \\ c & b & a & x \end{vmatrix} = \begin{vmatrix} S & T \\ T & S \end{vmatrix} = \begin{vmatrix} S+T & T+S \\ T & S \end{vmatrix} = \begin{vmatrix} S+T & O \\ T & S-T \end{vmatrix} = |S+T| \cdot |S-T|$$

例2.2.7 中行列式的计算：

$$D_{2n} = \begin{vmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 0 \\ -1 & & 0 & b_{11} & \cdots & b_{1n} \\ & \ddots & & \vdots & & \vdots \\ 0 & & -1 & b_{n1} & \cdots & b_{nn} \end{vmatrix} = \begin{vmatrix} A & O \\ -E & B \end{vmatrix} = \begin{vmatrix} \begin{pmatrix} E & A \\ O & E \end{pmatrix} \begin{pmatrix} A & O \\ -E & B \end{pmatrix} \end{vmatrix} = \begin{vmatrix} O & AB \\ -E & B \end{vmatrix} = \cdots$$

分块矩阵运算规律的说明

(1) 分块矩阵的加法

$$\left(\begin{array}{c|c|c} & & \\ \hline & \begin{matrix} a_{i_1 j_1} & \cdots & a_{i_1 j_2} \\ \vdots & & \vdots \\ a_{i_2 j_1} & \cdots & a_{i_2 j_2} \end{matrix} & \\ \hline & & \end{array} \right) + \left(\begin{array}{c|c|c} & & \\ \hline & \begin{matrix} b_{i_1 j_1} & \cdots & b_{i_1 j_2} \\ \vdots & & \vdots \\ b_{i_2 j_1} & \cdots & b_{i_2 j_2} \end{matrix} & \\ \hline & & \end{array} \right) = \left(\begin{array}{c|c|c} & & \\ \hline & \begin{matrix} a_{i_1 j_1}+b_{i_1 j_1} & \cdots & a_{i_1 j_2}+b_{i_1 j_2} \\ \vdots & & \vdots \\ a_{i_2 j_1}+b_{i_2 j_1} & \cdots & a_{i_2 j_2}+b_{i_2 j_2} \end{matrix} & \\ \hline & & \end{array} \right)$$

(2) 分块矩阵的数乘

$$k \left(\begin{array}{c|c|c} & & \\ \hline & \begin{matrix} a_{i_1 j_1} & \cdots & a_{i_1 j_2} \\ \vdots & & \vdots \\ a_{i_2 j_1} & \cdots & a_{i_2 j_2} \end{matrix} & \\ \hline & & \end{array} \right) = \left(\begin{array}{c|c|c} & & \\ \hline & \begin{matrix} ka_{i_1 j_1} & \cdots & ka_{i_1 j_2} \\ \vdots & & \vdots \\ ka_{i_2 j_1} & \cdots & ka_{i_2 j_2} \end{matrix} & \\ \hline & & \end{array} \right)$$

(3) 分块矩阵的乘法

$$\begin{pmatrix} & & & & \\ \begin{matrix} a_{k_11_1} \cdots a_{k_11_2} \\ \vdots \quad A_{k1} \quad \vdots \\ a_{k_21_1} \cdots a_{k_21_2} \end{matrix} & \cdots & \cdots & \begin{matrix} a_{k_1i_1} \cdots a_{k_1i_2} \\ \vdots \quad A_{ki} \quad \vdots \\ a_{k_2i_1} \cdots a_{k_2i_2} \end{matrix} & \cdots & \cdots & \begin{matrix} a_{k_1s_1} \cdots a_{k_1s_2} \\ \vdots \quad A_{ks} \quad \vdots \\ a_{k_2s_1} \cdots a_{k_2s_2} \end{matrix} \\ & & & & \end{pmatrix} \begin{pmatrix} \begin{matrix} b_{1_1l_1} \cdots b_{1_1l_2} \\ \vdots \quad B_{1l} \quad \vdots \\ b_{1_2l_1} \cdots b_{1_2l_2} \end{matrix} \\ \vdots \\ \begin{matrix} b_{i_1l_1} \cdots b_{i_1l_2} \\ \vdots \quad B_{il} \quad \vdots \\ b_{i_2l_1} \cdots b_{i_2l_2} \end{matrix} \\ \vdots \\ \begin{matrix} b_{s_1l_1} \cdots b_{s_1l_2} \\ \vdots \quad B_{sl} \quad \vdots \\ b_{s_2l_1} \cdots b_{s_2l_2} \end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{matrix} \displaystyle\sum_{j=1}^{n} a_{k_1j}b_{jl_1} & \cdots & \displaystyle\sum_{j=1}^{n} a_{k_1j}b_{jl_2} \\ \vdots & C_{kl} & \vdots \\ \displaystyle\sum_{j=1}^{n} a_{k_2j}b_{jl_1} & \cdots & \displaystyle\sum_{j=1}^{n} a_{k_2j}b_{jl_2} \end{matrix} \end{pmatrix}$$

相关子块的关系

$$A_{k1}B_{1l} + \cdots + A_{ki}B_{il} + \cdots + A_{ks}B_{sl}$$

$$= \begin{pmatrix} a_{k_1 1_1} & \cdots & a_{k_1 1_2} \\ \vdots & \vdots & \vdots \\ a_{k_2 1_1} & \cdots & a_{k_2 1_2} \end{pmatrix} \begin{pmatrix} b_{1_1 l_1} & \cdots & b_{1_1 l_2} \\ \vdots & \vdots & \vdots \\ b_{1_2 l_1} & \cdots & b_{1_2 l_2} \end{pmatrix} + \cdots + \begin{pmatrix} a_{k_1 i} & \cdots & a_{k_1 i_2} \\ \vdots & \vdots & \vdots \\ a_{k_2 i_1} & \cdots & a_{k_2 i_2} \end{pmatrix} \begin{pmatrix} b_{i_1 l_1} & \cdots & b_{i_1 l_2} \\ \vdots & \vdots & \vdots \\ b_{i_2 l_1} & \cdots & b_{i_2 l_2} \end{pmatrix} + \cdots + \begin{pmatrix} a_{k_1 s_1} & \cdots & a_{k_1 s_2} \\ \vdots & \vdots & \vdots \\ a_{k_2 s_1} & \cdots & a_{k_2 s_2} \end{pmatrix} \begin{pmatrix} b_{s_1 l_1} & \cdots & b_{s_1 l_2} \\ \vdots & \vdots & \vdots \\ b_{s_2 l_1} & \cdots & b_{s_2 l_2} \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{j=1_1}^{1_2} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=1_1}^{1_2} a_{k_1 j} b_{j l_2} \\ \vdots & & \vdots \\ \sum\limits_{j=1_1}^{1_2} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=1_1}^{1_2} a_{k_1 j} b_{j l_2} \end{pmatrix} + \cdots + \begin{pmatrix} \sum\limits_{j=i_1}^{i_2} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=i_1}^{i_2} a_{k_1 j} b_{j l_2} \\ \vdots & & \vdots \\ \sum\limits_{j=i_1}^{i_2} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=i_1}^{i_2} a_{k_1 j} b_{j l_2} \end{pmatrix} + \cdots + \begin{pmatrix} \sum\limits_{j=s_1}^{s_2} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=s_1}^{s_2} a_{k_1 j} b_{j l_2} \\ \vdots & & \vdots \\ \sum\limits_{j=s_1}^{s_2} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=s_1}^{s_2} a_{k_1 j} b_{j l_2} \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{j=1}^{n} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=1}^{n} a_{k_1 j} b_{j l_2} \\ \vdots & & \vdots \\ \sum\limits_{j=1}^{n} a_{k_1 j} b_{j l_1} & \cdots & \sum\limits_{j=1}^{n} a_{k_1 j} b_{j l_2} \end{pmatrix} = \begin{pmatrix} c_{k_1 l_1} & \cdots & c_{k_1 l_2} \\ \vdots & & \vdots \\ c_{k_2 l_1} & \cdots & c_{k_2 l_2} \end{pmatrix} = C_{kl} \quad .$$

(4) 分块矩阵的转置

$$\left(\begin{array}{ccc|c} & & & \\ \hline & \begin{matrix} a_{k_1 l_1} & \cdots & a_{k_1 l_2} \\ \vdots & \boldsymbol{A}_{kl} & \vdots \\ a_{k_2 l_1} & \cdots & a_{k_2 l_2} \end{matrix} & \\ \hline & & & \end{array} \right)^{T} = \left(\begin{array}{ccc|c} & & & \\ \hline & \begin{matrix} a_{k_1 l_1} & \cdots & a_{k_2 l_1} \\ \vdots & \boldsymbol{B}_{lk} & \vdots \\ a_{k_1 l_2} & \cdots & a_{k_2 l_2} \end{matrix} & \\ \hline & & & \end{array} \right) = \boldsymbol{A}_{kl}^{\mathbf{T}}$$