3.3 线性方程组的可解性

方程组$Ax=b$有解 ⇔ r(A)=r(A,b)

方程组$Ax=b$有无穷多组解 ⇔ r(A)=r(A,b)<(A的列数)

判断方程组有解性 $\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = -4, \\ 2x_1 - 2x_2 + x_3 = -5, \\ -x_1 + x_2 + x_3 - 3x_4 = 1, \\ x_1 - x_2 + x_4 = -2. \end{cases}$

解　对增广矩阵进行初等行变换

$$B=\left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & -4 \\ 2 & -2 & 1 & 0 & -5 \\ -1 & 1 & 1 & -3 & 1 \\ 1 & -1 & 0 & 1 & -2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & -4 \\ 0 & 0 & -3 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

由 r(A)=r(A,b)=2<4，知方程组有无穷多组解.

判断方程组有解性 $\begin{cases} x_1 + 2x_2 + x_3 = 0, \\ x_1 + 2x_2 + x_3 = 1, \\ x_1 + 2x_2 + x_3 = 2. \end{cases}$

解 $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$, $r(A)=1<r(A,b)=2$, 故方程组无解.

将增广矩阵进行行变换简化成行简化梯形，再分析是否有解.

考虑方程组 $\begin{cases} x_1 - x_2 + 2x_3 - 3x_4 = -4, \\ 2x_1 - 2x_2 + x_3 = -5, \\ -x_1 + x_2 + x_3 - 3x_4 = 1, \\ x_1 - x_2 + x_4 = -2. \end{cases}$

解

$$B=\left(\begin{array}{cccc|c} 1 & -1 & 2 & -3 & -4 \\ 2 & -2 & 1 & 0 & -5 \\ -1 & 1 & 1 & -3 & 1 \\ 1 & -1 & 0 & 1 & -2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \text{即} \begin{cases} x_1 - x_2 + x_4 = -2, \\ x_3 - 2x_4 = -1, \\ 0 = 0, \\ 0 = 0, \end{cases} \quad \text{即} \begin{cases} x_1 = -2 + x_2 - x_4, \\ x_3 = -1 + 2x_4. \end{cases}$$

进一步为了有解性分析的方便，简化行梯形进行适当列交换

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array} \xrightarrow{c_2 \leftrightarrow c_3} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x_1 & x_3 & x_2 & x_4 \end{array} \end{array}, \text{即} \begin{cases} y_1 - y_3 + y_4 = -2, \\ y_2 - 2y_4 = -1, \\ 0 = 0, \\ 0 = 0, \end{cases} \quad \text{即} \begin{cases} y_1 = -2 + y_3 - y_4, \\ y_2 = -1 + 2y_4. \end{cases}$$

该方程组有解
⇔原方程组有解

原方程组的变量x_1,x_2,x_3,x_4替换成y_1,y_2,y_3,y_4后方程组可解性不变.

故可以考虑矩阵先作适当的列变换，然后行变换将矩阵简化成每行的首元素都在最左边.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases} \quad (3.1)$$

即：$Ax=b$

定理3.3.1 线性方程组(3.1)有解的充要条件是系数矩阵的秩等于增广矩阵的秩。且当 r(*A*)=r(*B*)=*n*时，方程组有唯一解；而当 r(*A*)=r(*B*)<*n*时，方程组有无穷多组解。

证明思路：

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, B = (A,b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right),$$

$$(A,b) \to (C,d) = \widetilde{C} = \left(\begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & c_{1,r+1} & \cdots & c_{1n} & d_1 \\ 0 & 1 & \cdots & 0 & c_{2,r+2} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{r,r+1} & \cdots & c_{rn} & d_r \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d_{r+1} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{array}\right), \text{即} \begin{cases} x_1 = d_1 - c_{1,r+1}x_{r+1} - \cdots - c_{1n}x_n, \\ x_2 = d_2 - c_{2,r+1}x_{r+1} - \cdots - c_{2n}x_n, \\ \qquad\vdots \\ x_r = d_r - c_{r,r+1}x_{r+1} - \cdots - c_{rn}x_n, \\ 0 = d_{r+1}, \\ 0 = 0, \\ \qquad\vdots \\ 0 = 0. \end{cases}$$

- **r(*A*)=r(*C*)**
- **r(*B*)=r(*C*,*d*)**

- **方程组有解⇔d_{r+1}=0⇔r(*C*)=r(*C*,*d*)**
- **方程组有无穷多组解⇔r(A)=*r*<*n***

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \quad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases} \quad (3.1)$$

即：$\boldsymbol{Ax=b}$

定理3.3.1 线性方程组(3.1)有解的充要条件是系数矩阵的秩等于增广矩阵的秩。且当 r(*A*)=r(*B*)=*n*时，方程组有唯一解；而当 r(*A*)=r(*B*)<*n*时，方程组有无穷多组解。

证法二：利用向量组相关性

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = b, \text{其中：} \alpha_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \alpha_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \alpha_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

方程组有解⇔b可由$\alpha_1, \alpha_2, \ldots, \alpha_n$线性表示

$\Updownarrow$

向量组$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$和向量组$\{\alpha_1, \alpha_2, \ldots, \alpha_n, b\}$可相互表示，即向量组等价

$\Updownarrow$

$r\{\alpha_1, \alpha_2, \ldots, \alpha_n\} = r\{\alpha_1, \alpha_2, \ldots, \alpha_n, b\}$，即$r(A)=r(A,b)$

$r\{\alpha_1, \ldots, \alpha_n\} = r\{\alpha_1, \ldots, \alpha_n, b\} = n$，则$\alpha_1, \ldots, \alpha_n$无关，$b$可由$\alpha_1, \alpha_2, \ldots, \alpha_n$唯一表示

$r\{\alpha_1, \alpha_2, \ldots, \alpha_n\} < n \Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_n$线性相关$\Rightarrow \theta = k_1\alpha_1 + k_2\alpha_2 + \ldots + k_n\alpha_n$有无穷多非零解$\Rightarrow b$有$\alpha_1, \alpha_2, \ldots, \alpha_n$无穷多种表示方式

例3.1.1 解方程组 $\begin{cases} 3x_1+\lambda x_2+x_3=4, \\ x_1+2x_2-4x_3=\mu, \\ x_1-x_2+9x_3=19. \end{cases}$

解

$$B=\left(\begin{array}{ccc|c} 3 & \lambda & 1 & 4 \\ 1 & 2 & -4 & \mu \\ 1 & -1 & 9 & 19 \end{array}\right) \xrightarrow{r_1\leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -1 & 9 & 19 \\ 1 & 2 & -4 & \mu \\ 3 & \lambda & 1 & 4 \end{array}\right) \xrightarrow[r_3-3r_1]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & -1 & 9 & 19 \\ 0 & 3 & -13 & \mu-19 \\ 0 & \lambda+3 & -26 & -53 \end{array}\right) \xrightarrow{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & -1 & 9 & 19 \\ 0 & 3 & -13 & \mu-19 \\ 0 & \lambda-3 & 0 & -2\mu-15 \end{array}\right)=B_1.$$

当$\lambda\neq 3$时，r(A)=r(B)=3，方程组有唯一解. 进一步化简B，

$$B_1 \xrightarrow[r_3\div(\lambda-3)]{\substack{r_1+\frac{9}{13}r_2 \\ r_2\div(-13)}} \left(\begin{array}{ccc|c} 1 & \frac{14}{13} & 0 & 19+\frac{9}{13}(\mu-19) \\ 0 & -\frac{3}{13} & 1 & -\frac{1}{13}(\mu-19) \\ 0 & 1 & 0 & -\frac{2\mu+15}{\lambda-3} \end{array}\right) \xrightarrow[r_2\leftrightarrow r_3]{\substack{r_1-\frac{14}{13}r_3 \\ r_2+\frac{3}{13}r_3}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{9\mu+76}{13}+\frac{14}{13}\times\frac{2\mu+15}{\lambda-3} \\ 0 & 1 & 0 & -\frac{2\mu+15}{\lambda-3} \\ 0 & 0 & 1 & -\frac{\mu-19}{13}-\frac{3}{13}\times\frac{2\mu+15}{\lambda-3} \end{array}\right),$$

得方程组的唯一解为

$$x_1=\frac{9\mu+76}{13}+\frac{14}{13}\times\frac{2\mu+15}{\lambda-3}, x_2=-\frac{2\mu+15}{\lambda-3}, x_3=-\frac{\mu-19}{13}-\frac{3}{13}\times\frac{2\mu+15}{\lambda-3}.$$

当λ=3 , μ= -5/2时，r(A)=r(B)=2，方程组有无穷多解. 进一步化简B，

$$B_1=\left(\begin{array}{ccc|c} 1 & -1 & 9 & 19 \\ 0 & 3 & -13 & -53/2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_2\div 3]{r_1+\frac{1}{3}r_2} \left(\begin{array}{ccc|c} 1 & 0 & 14/3 & 61/6 \\ 0 & 1 & -13/3 & -53/6 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

得方程组的解为：x_1=61/6-14t , x_2=-53/6+13t , x_3=3t , $t\in$ R .

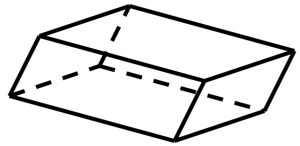
当λ=3 , $\mu\neq$ -5/2时，2=r(A)<r(B)=3，方程组无解.

*分析三元方程组行列式求解

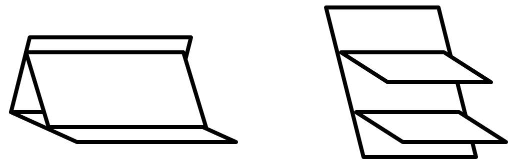
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{cases}$$

即：$\boldsymbol{Ax=b}$

$\Delta \neq 0$时有唯一解

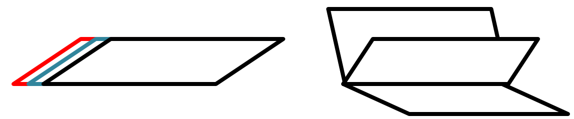


$\Delta = 0$,但有$\Delta_i \neq 0$时无解

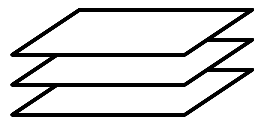


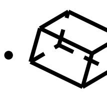
$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

无穷多组解



无解



$\Delta \neq 0$表示 r(A)=3，即有3个独立方程(3个独立方向平面)，故交于一点. 

Δ =0表示 r(A)=1或r(A)=2：

r(A)=1时只有1个独立方程（1个独立方向平面）：

 **重合成一个平面则有无穷多组解：r(A)=r(A,b)=1,则$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$**

 **成为平行平面则无解：1=r(A)<r(A,b)=2，则$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$**

r(A)=2时有2个独立方程（2个独立方向平面）：

 **平面有交线则有无穷多组解：r(A)=r(A,b)=2，则$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$**

 **无解时平面没有公共交点：2=r(A)<r(A,b)=3，则$\Delta = 0$,但有$\Delta_i \neq 0$**