Lecture 3: convex functions

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IE 539: Convex Optimization

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Outline

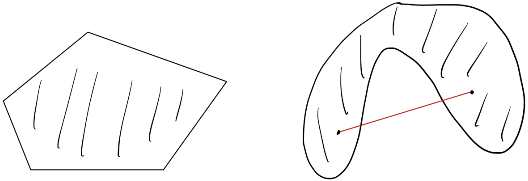
- (Review) Convex sets
- Convex functions
- Convex function examples
- Epigraph
- First-order and second-order characterizations of convex functions
- (If time allows) Operations preserving convexity

Convex set

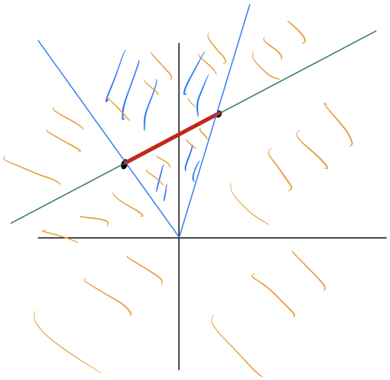
- A set $X \subseteq \mathbb{R}^d$ is convex if for any $u, v \in X$ and any $\lambda \in [0, 1]$,

$$\lambda u + (1 - \lambda)v \in X.$$

- In words, the line segment joining any two points is entirely contained the set.



Comparison



- $X = \{u, v\}$ consists of two points.
- RED: $\mathrm{conv}(X)$.
- BLUE: $\mathrm{cone}(X)$.
- Green: $\mathrm{aff}(X)$.
- ORANGE: $\mathrm{lin}(X)$.

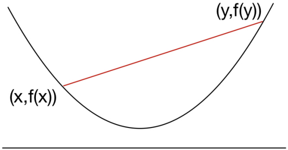
Convex set examples

- The convex hull and conic hull of a set are convex.
- The linear subspace and affine subspace spanned by a set are convex.
- Empty set $\emptyset$
- Norm ball.
- Ellipsoid.
- Half-space.
- Polyhedron and polytopes.
- Simplex
- Nonnegative and positive orthants
- Norm cone
- Positive semidefinite cone

Convex function

- A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if the domain $\mathrm{dom}(f)$ is convex and

- In words,



Concave, strictly convex, strongly convex functions

- We say that $f : \mathbb{R}^d \to \mathbb{R}$ is concave if
- A function $f : \mathbb{R}^d \to \mathbb{R}$ is strictly convex if $\text{dom}(f)$ is convex and
- A function $f : \mathbb{R}^d \to \mathbb{R}$ is strongly convex if $\text{dom}(f)$ is convex and
- Exercise: Show that Strongly convexity implies strict convexity.

Convex function examples

Univariate functions

- Exponential function
- Power function
- Logarithm
- Negative entropy

Convex function examples

- Linear function
- Quadratic function
- Least squares loss
- Norm

Convex function examples

- Maximum eigenvalue of a symmetric matrix

- Indicator function

- Support function

- Conjugate function

Epigraph

The epigraph of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as

Exercise: prove that f is convex if and only if its epigraph is convex

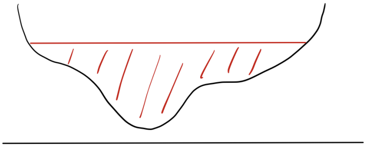
Example: norm cone = epigraph of the norm

Level set, convex level sets of a nonconvex function

A level set of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\{x \in \mathrm{dom}(f) : \; f(x) \leq \alpha\}$$

Level sets of a nonconvex set can be convex:



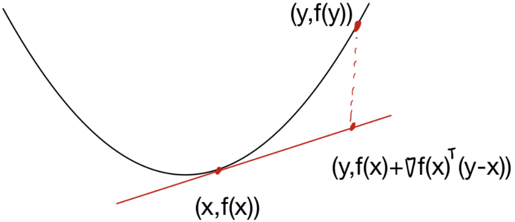
First-order characterization I

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if dom(f) is convex and

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

for all $x, y \in$ dom(f).



First-order characterization I

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if $dom(f)$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

for all $x, y \in dom(f)$.

Proof

First-order characterization I

First-order characterization II

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if dom(f) is convex and

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$$

for all $x, y \in$ dom(f).

Proof

First-order characterization II

Second-order characterization

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function[1]*. Then f is convex if and only if dom(f) is convex and*

$$\nabla^2 f(x) \succeq 0.$$

for all $x \in$ dom(f).

Proof

[1] $\nabla^2 f$ exists at any point in dom(f), and dom(f) is open.

Second-order characterization

Set operations preserving convexity

- Intersection
- Scaling
- Minkowski sum
- Product

Set operations preserving convexity

- Affine image

- Inverse affine image

Function operations preserving convexity

- Nonnegative weighted sum

- Maximum of arbitrary collection of convex functions

- Affine composition

Function operations preserving convexity

- Minimizing out variables

- Perspective function

Examples

Let C be an arbitrary set of locations. Note that

$$f_1(x) = \max_{y \in C} ||x - y||$$

measures the longest distance from x to a location in C, and

$$f_2(x) = \min_{y \in C} \|x - y\|$$

measures the shortest distance from x to a location in C.

Remark

f_1 is convex regardless of whether C is convex or not, and f_2 is convex if the set C is convex.