STOCHASTIC PORTFOLIO THEORY

INTRODUCTION:

Stochastic Portfolio Theory (SPT), as we currently think of it, began in 1995 with the manuscript “On the diversity of equity markets”, which eventually appeared as Fernholz (1999) in the Journal of Mathematical Economics. Since then, SPT has evolved into a flexible framework for analyzing portfolio behavior and equity market structure that has both theoretical and practical applications.

As a theoretical methodology, this framework provides insight into questions of market behavior and arbitrage and can be used to construct portfolios with controlled behavior under quite general conditions.

ASSUMPTIONS:

- Finite number of companies in market, total number of shares are fixed, and companies do not merge or breakup.
- Trading is continuous in time with no transaction costs, tax or problems with indivisibility of share.
- Dividends are paid continuously and if are not necessary dividends are excluded.

STOCKS AND PORTFOLIOS:

The stock prices and portfolio value follow n-dimensional random Brownian motion process defined by W(t) = (W1(t), W2(t), …, Wn(t)), t = (0,∞),with probability space (Ω, F, P).

$$\boldsymbol{d \log X(t) = \gamma(t)\,\mathrm{d}t + \sum_{\nu=1}^{n} \xi_\nu(t) dW_\nu(t)}$$

or equivalently,

$$\boldsymbol{X(t) = x_0 \exp\left(\int_0^t \gamma(s)\,\mathrm{d}s + \int_0^t \sum_{\nu=1}^{n} \xi_\nu(s)\,\mathrm{d}w_\nu\,(s)\right)}$$

W_v: Brownian Motion

γ: Growth process, measure of growth of stock process, satisfies $\int_0^T |\gamma(t)|\,\mathrm{d}t < \infty$

ξ_v: $Volatlity\ process$, sensitivity of stock to v source of uncertainity W_v

Points to be noticed:

- Definition of our stock price/process ensure that the volatility of stock doesn't rise too quickly as to render meaningless the stock's growth rate.
- We have used growth rate unlike rate of return used in classical portfolio theory. The stochastic portfolio theory state that growth rate is a better measure of long term stock behaviour.

A portfolio in the market M is measurable-adapted vector process defined as:

$$\mathbf{d}\log \boldsymbol{z_\pi(t)} = \boldsymbol{\gamma_\pi(t)}\,\mathbf{d}\boldsymbol{t} + \sum_{i'v=1}^{n} \boldsymbol{\pi_i(t)\xi_{iv}(t)dw_v(t)}$$

or equivalently,

$$\boldsymbol{z_\pi(t)} = \boldsymbol{z_\pi(0)}\left(\int_0^t \boldsymbol{\gamma_\pi}\,\mathbf{d}\boldsymbol{S} + \int_0^t \sum_{iv=1}^{n} \boldsymbol{\pi_i(S)\xi_{iv}(S)}\,\mathbf{d}\boldsymbol{w_v}\,\boldsymbol{(S)}\right)$$

Where $\boldsymbol{\pi_i}$ represent the weight of corresponding stocks present in portfolio.

$$\boldsymbol{\pi_1(t)} + \boldsymbol{\pi_2(t)} + \boldsymbol{\pi_3(t)} \ldots + \boldsymbol{\pi_n(t)} = \mathbf{1}$$

If,

$\boldsymbol{\pi_i} > \mathbf{0}$ **:** X_i is long in the portfolio

$\boldsymbol{\pi_i} = \mathbf{0}$ **:** X_i is not present in the portfolio

$\boldsymbol{\pi_i} < \mathbf{0}$ **:** X_i is short in the portfolio

Growth rate of portfolio is given by,

$$\boldsymbol{\gamma_\pi}\,(\mathbf{t}) = \sum_{i=1}^{n} \boldsymbol{\pi_i(t)\gamma_i\,(t)} + \frac{\mathbf{1}}{\mathbf{2}}\left(\sum_{i=1}^{n} \boldsymbol{\pi_i\,(t)\sigma_{ii}(t)} - \sum_{i_1 j=1}^{n} \boldsymbol{\pi_i(t)\pi_j(t)\sigma_{ij}(t)}\right)$$

It has two component

1. Sum of weighted growth rate of stocks

$$\textstyle\sum_{i=1}^{n} \boldsymbol{\pi_i(t)\gamma_i\,(t)}$$

2. Excess Growth rate

$$\frac{\mathbf{1}}{\mathbf{2}}\left(\sum_{i=1}^{n} \boldsymbol{\pi_i\,(t)\sigma_{ii}(t)} - \sum_{i_1 j=1}^{n} \boldsymbol{\pi_i(t)\pi_j(t)\sigma_{ij}(t)}\right)$$

Excess growth rate is half of difference between variance of stocks and the portfolio variance and can be considered as ability of portfolio diversification to reduce volatility as compare to its component stocks.

MARKET PORTFOLIO:

It is frequently on interest to measure the performance of stock or portfolio to a given benchmark portfolio or index.

A natural benchmark is a market portfolio consisting of all the shares of all stocks in the market.

Weights of stocks in Market portfolio,

$$\boldsymbol{\mu_i} = \frac{\boldsymbol{x_i(t)}}{\boldsymbol{x_1(t)^+} - \cdots + \boldsymbol{x_n(t)}}, \boldsymbol{t} \in (\mathbf{0}, \infty)$$

Value of Market portfolio is combined capitalization of all the stocks

$$\boldsymbol{z_\pi(t)} = \boldsymbol{x_1(t)} + \cdots + \boldsymbol{x_n(t)}, \boldsymbol{t} \in [\mathbf{0}, \infty)$$

Relative return of a portfolio with respect to market portfolio is given as,

$$\mathbf{d}\,\mathbf{log}\left(\frac{\boldsymbol{z_\pi(t)}}{\boldsymbol{z_\mu(t)}}\right) = \sum_{i=1}^{n} \boldsymbol{\pi_i(t)}\,\mathbf{d}\,\mathbf{log}\,\boldsymbol{u_i(t)} + \boldsymbol{\gamma^*(t)}\,\mathbf{d}\boldsymbol{t}$$

$\gamma^*(t)$ is excess growth rate of portfolio

MARKET BEHAVIOUR:

COHERENT MARKET:

Coherence is minimal stability condition for market , i.e. market is called coherent if none of the stock descends into nothingness too quickly.

Condition: $\lim_{t\to\infty} t^{-1} \log \mu_i(t) = 0,$ **i = 1,2,3,…n**

DIVERSITY:

Diversity is opposite of concentration, and a diverse market is one that avoids extreme concentration of capital into one stock.

A market is diverse if,

$$\mu_{\max}(t) = \max_{1\le i\le n} u_i(t) \le 1 - \delta,\ t \in [0, \infty), \delta > 0$$

and weakly diverse if,

$$\frac{1}{T}\int_0^T \mu_{\max}(t)\, \mathrm{d}(t) \le 1 - \delta, \delta > 0$$

ENTROPY:

A measure of diversity introduced by Shannon (1984), the market entropy process is defined as,

$$S(\mu(t)) = -\sum_{i=1}^{n} \mu_i(t) \log u_i(t)\,,\ t \in [0, \infty)$$

So, suppose if market is concentrated and total capital is invested in k^{th} stock then,

$$\mu_k(t) = 1 \text{ and } \mu_i(t) = 0, i = 1, 2, \dots, k-1, k+1, \dots, n$$

$$S(\mu(t)) = -\sum_{i=1}^{n} \mu_i(t) \log u_i(t) = 0$$

PORTFOLIO GENERATING FUNCTIONS:

Certain real-valued functions of the market weights $\mu_1(t), \ldots, \mu_n(t)$ can be used to construct dynamic portfolios that behave in a controlled manner. The portfolio generating functions that interest us most fall into two categories: smooth functions of the market weights, and smooth functions of the ranked market weights.

SMOOTH FUNCTIONS OF MARKET WEIGHTS:

Market portfolio defined on the set,

$$\boldsymbol{\Delta^n} = \{\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} : \boldsymbol{x_1} + \cdots + \boldsymbol{x_n} = \boldsymbol{1}; \boldsymbol{0} < \boldsymbol{x_i} < \boldsymbol{1}, \mathbf{i} = \boldsymbol{1}, \cdots, \boldsymbol{n}\}$$

let there be a continuous function S define on $\boldsymbol{\Delta^n}$ generate a portfolio π with weights,

$$\boldsymbol{\pi_i(t)} = \left(\boldsymbol{D_i} \log \boldsymbol{S}\left(\boldsymbol{\mu(t)}\right) + \boldsymbol{1} - \sum_{\boldsymbol{J=1}}^{\boldsymbol{n}} \boldsymbol{\mu_j(t) D_j} \log \boldsymbol{S}\left(\boldsymbol{u(t)}\right)\right) \boldsymbol{\mu_i(t)}$$

where,

$$\boldsymbol{D_x S} = \frac{\mathbf{d}\boldsymbol{s}}{\mathbf{d}\boldsymbol{x}}$$

some smooth functions of market weights:

1. **Constant function:** generate Market Portfolio

$$\boldsymbol{G} = \boldsymbol{w} > \boldsymbol{0}$$

$$\boldsymbol{\pi_i} = \boldsymbol{\mu_i}$$

2. **Geometric mean function:** generate equally weighted portfolio

$$\boldsymbol{s} = (\boldsymbol{x_1} \cdot \boldsymbol{x_2} \cdot \boldsymbol{x_3} \cdots \boldsymbol{x_n})^{\frac{\boldsymbol{1}}{\boldsymbol{n}}}$$

$$\boldsymbol{\pi_i} = \frac{\boldsymbol{1}}{\boldsymbol{n}}$$

3. **Entropy Function:** generate entropy-weighted portfolio

$$\boldsymbol{G} = -\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{\mu_i(t)} \log\left(\boldsymbol{\mu_i(t)}\right)$$

$$\pi_i = \frac{-u_i(t)\log(u_i(t))}{G}$$

4. **Diversity Function:** generate diversity weighted portfolio

$$D^{(P)}(x) = \left(\sum_{i=1}^{n} x_i^P\right)^{\frac{1}{P}}, 0 < P < 1$$

$$\pi_i = \frac{(\mu_i(t))^P}{\sum_{i=1}^{n}(\mu_i(t))^P}$$

SMOOTH FUNCTIONS OF RANKED MARKET WEIGHTS:

Normally, we identify stocks by their names (scripts), but when it comes to capitalization, it is better to identify stocks by their ranks.

for k= 1 ..., n the kth rank process is defined by,

$$x_{(k)}(t) = max_{1\leq i1<\cdots ik\leq n}\min(x_{i1}(t)\cdots x_{ik}(t)), t \in [0,\infty)$$

but we are particularly interested in ranked market weights,

$$\mu_{(1)}(t) \geq \mu_{(2)}(t) \geq \cdots \geq \mu_{(n)}(t), t \in [0,\infty)$$

these ranked weights are known as capital distribution of market

example,

Capitalization Weighted Portfolio:

if c_1, c_2,..., c_n are constants equal to 0 or 1, then weights are given by,

$$\pi_i(t) = \frac{c_i\mu_i(t)}{c_1\mu_1(t) + \cdots + c_n\mu_n(t)}$$

The value of these constants can be determined by any condition; for example, stocks with a capitalization greater than the average market capitalization of a stock has a constant value of 1

NO ARBITRAGE HYPOTHESIS:

An arbitrage opportunity is a combination of investments in portfolios such that the sum of the initial values of the investments is zero and such that at some given non-random future time T, the sum of the values will be non-negative with probability one and positive with positive probability.

No arbitrage is a common hypothesis in current financial theory, we can think of at least two good reasons. First, over the short term, no-arbitrage appears to be an accurate representation of actual equity markets. Second arbitrage opportunity are probability one-events, and outside mathematics there are no probability-one events (except for death and taxes, off-course ☺).

Let there be two portfolios η and ξ define on $t \in [0, T]$ then,

η dominates ξ if:

$$\frac{\boldsymbol{z_\eta(T)}}{\boldsymbol{z_\eta(0)}} \geq \frac{\boldsymbol{z_\xi(T)}}{\boldsymbol{z_\xi(0)}}$$

η strictly dominates ξ if:

$$\frac{\boldsymbol{z_\eta(T)}}{\boldsymbol{z_\eta(0)}} > \frac{\boldsymbol{z_\xi(T)}}{\boldsymbol{z_\xi(0)}}$$

so, we can buy one dollar’s worth on η at time 0, and finance it buy selling one dollar’s worth of ξ short at the same time, keeping initial portfolio value equals zero. At time T the dollar value of holding in η will be greater than the dollar we owe on the short sale of ξ.

Total holding at time T is non-negative with probability one and positive with positive probability. Hence, this combination is an arbitrage opportunity and refutes the no-arbitrage hypothesis.

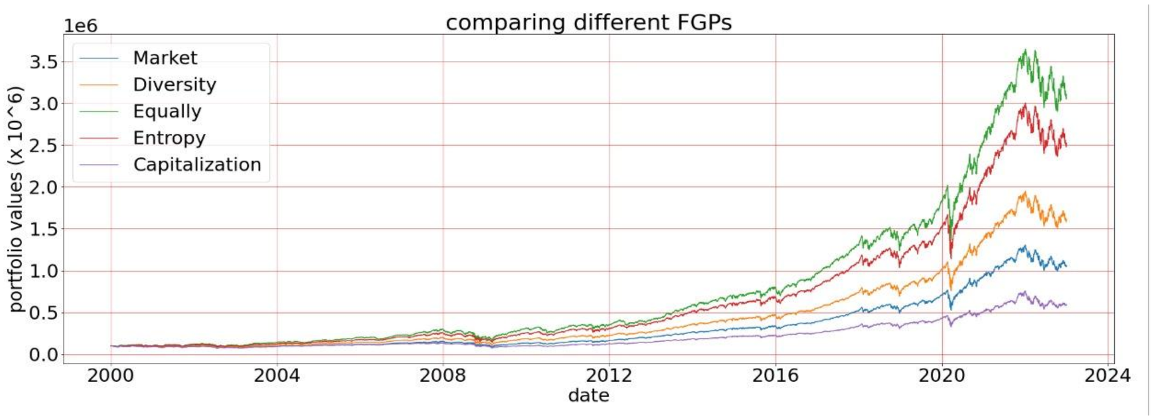
Simply, arbitrage can be possible with weak diversity in market, and for no-arbitrage condition we need equivalent martingale measure, which is a complicated mathematical construct.

Hence, the no-arbitrage hypothesis is an empirically undecidable statement.

IMPLEMENTATION:

Comparing different Functionally Generated Portfolios on a 20 stocks universe.

- Code: SPT_Implementation
- Reference: SPT_ML_Perspective
- Result:



APPLICATION:

- Diversity weighted Indexing
- Change in Diversity
- Functionally generated weighting of stock or functionally generated distribution of capital

DRAWBACKS OF SPT:

- SPT is entirely based on capitalization and takes no account of any risk measure.
- SPT disregards the possibility of a company's insolvency.
- The Inverse Problem:
  How to design investment strategies that achieve targeted financial goals?

Note: for detailed theory please refer STOCHASTIC_PORTFOLIO_THEORY