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ESTIMATION IN THE CONSTANT ELASTICITY OF VARIANCE MODEL

By K. C. Yuen, H. Yang and K. L. Chu

ABSTRACT

The constant elasticity of variance (CEV) diffusion process can be used to model heteroscedasticity in returns of common stocks. In this diffusion process, the volatility is a function of the stock price and involves two parameters. Similar to the Black-Scholes analysis, the equilibrium price of a call option can be obtained for the CEV model. The purpose of this paper is to propose a new estimation procedure for the CEV model. A merit of our method is that no constraints are imposed on the elasticity parameter of the model. In addition, frequent adjustments of the parameter estimates are not required. Simulation studies indicate that the proposed method is suitable for practical use. As an illustration, real examples on the Hong Kong stock option market are carried out. Various aspects of the method are also discussed.

KEYWORDS

Constant Elasticity of Variance; Diffusion Process; Least Squares; Option Pricing; Volatility

CONTACT ADDRESS

Dr. K. C. Yuen, Ph.D., A.S.A., Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong. Tel: +852-2859-1915; Fax: +852-2858-9041.

1. Introduction

1.1 The valuation of options has been one of the main issues in the area of modern finance, and is of great interest to actuaries. In particular, stock options are often used in the study of option pricing. Suppose that the stock price process S_t is defined as the solution of a stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{1}$$

where μ is known as the expected rate of return; σ is the standard deviation of the percentage price return, and often called the stock price volatility; and B_t is a Wiener process. In option pricing theory, the risk-neutrality assumption allows us to replace the expected rate of return by the risk-free rate of interest; and hence the only unobservable value in (1) is the volatility. The parameter σ can be estimated from the history of stock prices, that is using the sample standard deviation of the return rate. Given model (1) and a certain set of assumptions, Black & Scholes (1973) obtained exact formulae for pricing European options. Through the Black-Scholes (BS) formulae, an observed option price in the market can be used to find an implied value of σ. This alternative estimate is termed an implied volatility.

1.2 Much attention has been paid to the constant volatility assumption in (1), which seems not very suitable in real cases. There is considerable evidence in the literature, indicating that stock returns are heteroscedastic. For example, see Black (1976), Blattberg & Gonedes (1974) and MacBeth & Merville (1979). In view of this property, Cox (1975) and Cox & Ross (1976) studied the constant elasticity of variance (CEV) diffusion process which takes the form:

$$dS_t = \mu S_t dt + \delta S_t^{\theta/2} dB_t \tag{2}$$

where δ and θ are constants. The volatility of returns in model (2) is $\delta S_t^{\theta/2-1}$. The variance rate of S_t is $\delta^2 S^\theta$, and the elasticity of this variance with respect to S_t is θ. It is easily seen that model (1) is equivalent to model (2) when $\theta = 2$, and that the volatility is an increasing (decreasing) function of S_t when $\theta > 2$ ($\theta < 2$). Under model (2) and the set of assumptions in the BS framework, Cox (1975) derived the equilibrium price of a call option for $\theta < 2$. Emanuel & MacBeth (1982) extended the pricing formula to the case of $\theta > 2$.

1.3 Models (1) and (2) are sometimes referred to as the BS and CEV models respectively. In this paper the number and abbreviation of the models are used interchangeably. Both the BS and CEV option pricing formulae are given in the Appendix. The option pricing formula for the CEV model certainly has a more complex form than that for the BS model.

1.4 Boyle & Tian (1999) considered the problems of pricing barrier and lookback options by using a trinomial lattice approach under the CEV model. Davydov & Linetsky (2000a, 2000b) also investigated the option pricing problems under the CEV model using analytical approaches. Moreover, these two papers contain very careful discussions on the boundary behaviour of the CEV process at the origin. For the CEV process (2) with $\theta > 2$, the origin is a natural boundary and infinity is an entrance boundary. For $\theta = 2$, both the origin and infinity are natural boundaries. For $1 \leq \theta < 2$, the origin is an exit boundary. For $\theta < 1$, the origin is a killing boundary. For $\theta < 2$, infinity is a natural boundary. Further details can be found in their original papers.

1.5 MacBeth & Merville (1980) investigated the problem of estimating δ and θ, and proposed a three-stage procedure. The first stage is to find a reasonable point estimate of θ. It relies on the fact that, given the true value of θ, the value of δ is the same for all options written on the same stock. For an integer value of θ and an observed option price, an application of a numerical search routine to the CEV option pricing formula yields an implied value of δ. Then, an arbitrary set of observed option prices from the same stock generates a set of implied values of δ for the same θ. The same steps are repeated for different integer values of θ. The final estimate of θ is the one for which the implied values of δ are most nearly constant. The second stage is based on their simulation study, indicating that the BS model with the correct variance rate of return will give approximately the correct price for at-the-money options, even if the underlying stock price process follows the CEV model with $\theta < 2$. Thus, the BS implied volatility, calculated using an at-the-money option price, is treated as a good estimate of the volatility of the CEV model. The final stage is to obtain an estimate of δ using the results in the first two stages. As the market option price changes, the value of δ needs to be re-estimated. Hence, daily adjustment of δ is possibly required.

1.6 In the method of MacBeth & Merville, the idea used in the first stage is intuitively clear and actually implied by model (2), while the third stage is straightforward. As pointed out by Manaster (1980), the validity of the second stage is in doubt. It is simply because the BS implied volatility is not the true volatility when the value of θ is other than 2. Their estimates of δ may, therefore, differ systematically from the true value. Furthermore, even though their second stage is supported by simulation results, it holds only for $\theta < 2$. The two parameters of the CEV model, in principle, should not require adjustments as frequently as the BS parameter. Because of the use of the BS implied volatility, this valuable feature of the CEV model cannot come into play in their method.

1.7 In the literature of finance, there are various estimation methods for parameters of continuous-time diffusions. Aït-Sahalia (1999) proposed a sequence of closed-form functions to approximate the true likelihood function, and proved that maximising the sequence results in an estimator which converges to the true maximum likelihood estimator. Singleton (2001) developed several estimation strategies for affine asset pricing models, based on the known functional form of the conditional characteristic function of affine diffusions. These estimation methods are applicable to a wider class of diffusion processes. Since the objective of this paper is to estimate the parameters of the CEV model, it is sensible to make use of the implied parameter calculated from the CEV option pricing formulae, so that valuable information from option prices can be extracted as well. The advantage of this approach is analogous to that of the BS implied volatility.

1.8 In this paper, we propose a new estimation procedure for the CEV model in which neither constraints on θ nor frequent adjustments of the parameter estimates are needed. Section 2 uses the log-linear property of the variance of the percentage price return and the results of Chesney *et al.* (1993) to estimate the parameters through the least-squares method. Although this simple idea allows us to jointly estimate δ and θ, the linearisation of the variance causes certain numerical problems. Section 3 introduces a two-stage approach as a remedy, and demonstrates its practicality through simulations. Real examples on the Hong Kong stock option market are carried out in Section 4. Finally, some remarks are given in Section 5.

2. Least-Squares Estimation

2.1 We now introduce a least-squares procedure, through which the two parameters of model (2) can be estimated jointly. To obtain point estimates of δ and θ, denoted by $\hat{\delta}$ and $\hat{\theta}$, we first find an estimate of $\sigma(S_t)^2 = \delta^2 S_t^{\theta-2}$, which is the square of the volatility at time t in model (2).

2.2 For a diffusion process, such as model (2), it is difficult to estimate the diffusion coefficient. Chesney *et al.* (1993) proposed a method to cope with this problem. Here, we follow their method to estimate $\sigma(S_t)^2$. Applying Ito's lemma to (2), we have:

$$df(S_t) = \left[f'(S_t)\mu S_t + \frac{1}{2}f''(S_t)\big(\sigma(S_t)S_t\big)^2\right]dt + f'(S_t)\sigma(S_t)S_t dB_t \tag{3}$$

where f is a twice differentiable function of S_t and $f'' \neq 0$. We then use Mihlstein (1974) approximation to (2) and (3). From the difference between the two approximations, we obtain:

$$\frac{f(S_{t+\Delta t}) - f(S_t)}{f'(S_t)S_t} - \frac{S_{t+\Delta t} - S_t}{S_t} \approx \frac{1}{2}\frac{f''(S_t)S_t}{f'(S_t)}\sigma(S_t)^2(\Delta B_t)^2$$

where Δt is the length of a small time interval. Since $\mathrm{E}[(\Delta B_t)^2] = \Delta t$, an estimate of $\sigma(S_t)^2$ is given by:

$$V_t = \frac{2}{\Delta t}\frac{f'(S_t)}{f''(S_t)S_t}\left[\frac{f(S_{t+\Delta t}) - f(S_t)}{f'(S_t)S_t} - \frac{S_{t+\Delta t} - S_t}{S_t}\right].$$

A power function $f(S_t) = S_t^{1+\alpha}$ yields:

$$V_t = \frac{2}{\alpha\Delta t}\left[\frac{S_{t+\Delta t}^{1+\alpha} - S_t^{1+\alpha}}{(1+\alpha)S_t^{1+\alpha}} - \frac{S_{t+\Delta t} - S_t}{S_t}\right] \tag{4}$$

where α is a constant. Chesney *et al.* (1993) have proved that the conditional expectation $\mathrm{E}(V_t|S_t)$ converges to $\sigma(S_t)^2$ as $\Delta t \to 0$, and that the conditional variance $\mathrm{Var}(V_t|S_t)$ is minimised when:

$$\alpha = -\frac{13}{11} - \frac{12}{11}\frac{\mu}{\sigma(S_t)^2}. \tag{5}$$

Computationally, we start with an initial guess of α, and then obtain a value of V_t from (4). The parameter μ can be estimated easily using the sample return denoted by $\hat{\mu}$. This can also be done by other more sophisticated estimation, say the method of moving average. Replacing $\sigma(S_t)^2$ by V_t and μ by $\hat{\mu}$ in (5), we have a revised value of α. We iterate the two steps until the last two values of α are within a certain tolerance. Empirical evidence indicates that the convergence of α only takes a few iterations in most cases.

2.3 Given V_t, the method of least squares can be employed. The least-squares estimates $\hat{\delta}$ and $\hat{\theta}$ minimise the sum of squares of deviations between $\ln V_t$ and $\ln \sigma(S_t)^2$:

$$\sum_{t=1}^{n-1}\left(\ln V_t - \ln \sigma(S_t)^2\right)^2 \tag{6}$$

where n is the number of data points. Taking logarithm of $\sigma(S_t)^2$ produces a linear function of $\ln S_t$, and hence simplifies the minimisation problem.

2.4 From model (2), we generate 100 stock prices over a three-month interval, so that the length of each subinterval Δt is 0.0025. The initial stock price is \$30, and the risk-free interest rate is 5%. Assuming the initial volatility to be around 0.25, we consider nine different pairs of δ and θ. For each of these nine cases, we repeat the simulation procedure 1000 times, and hence the means of $\hat{\delta}$ and $\hat{\theta}$, as well as their standard deviations, can be calculated. The key results are summarised in Table 1. This simulation study shows that the simple linear least-squares method produces reasonably good estimates for θ, but not for δ. The means of $\hat{\delta}$ are close to the true values

Table 1. Simulation results using (6)

True value		$\hat{\theta}$		$\hat{\delta}$	
θ	δ	Mean	Standard deviation	Mean	Standard deviation
−4	7000.00	−3.9963	0.7222	9881.2000	31003.0000
−3	1300.00	−3.0117	0.6993	1701.9000	4911.4000
−2	250.00	−1.9890	0.6483	283.9800	722.3400
−1	45.00	−1.0054	0.6221	47.6390	118.9300
0	8.00	0.0110	0.6205	8.3722	22.1250
1	1.50	1.0044	0.5741	1.4051	3.7028
2	0.25	1.9981	0.6319	0.2939	0.7888
3	0.05	3.0058	0.5838	0.0584	0.1079
4	0.01	3.9988	0.6453	0.0092	0.0311

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except for values of δ over 250, and all the standard errors appear very large relative to the means. The large standard deviations of $\hat{\delta}$ can be explained by the fact that a small deviation between $\hat{\theta}$ and θ yields a relatively large difference between $\hat{\delta}$ and δ. In fact, the variation of $\hat{\theta}$ has an exponential effect on that of $\hat{\delta}$. There is another problem of empirical nature. Theoretically, the values of S_t and $S_{t+\Delta t}$ are different, with probability 1. However, in real cases, two consecutive stock prices in a short period of time often have the same value. When applying (4) to real data, it is very likely that we have a large number of undefined values of $\ln V_t$. These numerical difficulties motivate us to consider a two-stage approach to estimate the parameters in the next section.

3. Two-Stage Estimation

3.1 Notice that model (2) is governed by the natural probability measure. In order to use the information from option prices, we need to work in the risk-neutral world. Under the risk-neutral probability measure, the stock price dynamic takes the form:

$$dS_t = rS_t dt + \delta S_t^{\theta/2} dW_t$$

where r is the risk-free interest rate and W_t is a standard Brownian motion under the risk-neutral probability measure. In our two-stage approach, we first obtain $\hat{\theta}$ using the implied parameter calculated from the CEV option pricing formulae, that is, under the risk-neutral probability measure. Given $\hat{\theta}$, we then estimate δ under the natural probability measure. Since the volatilities are the same under both probability measures, the two-stage method presented below should be valid.

3.2 Following the first stage of the MacBeth & Merville method, we make use of the fact that all options written on the same stock have the same values of δ and θ. We arbitrarily select m call option prices $(C_1, \ldots C_m)$ with the same underlying stock in the observation period. For each C_j and a given value of θ, we use a numerical search routine to calculate the implied value of δ, denoted by $\delta_j(\theta)$, from the CEV option pricing formula. To measure the degree of dispersion among δ_js, we consider the absolute relative error defined as:

$$U(\theta) = \sum_{j=1}^{m} \frac{|\delta_j(\theta) - \bar{\delta}_j(\theta)|}{\bar{\delta}_j(\theta)} \tag{7}$$

where $\bar{\delta}_j$ is the mean of δ_js. Then, our final point estimate $\hat{\theta}$ is the one that minimises U. After obtaining $\hat{\theta}$ in the first stage, we estimate δ by minimising the sum:

$$\sum_{t=1}^{n-1}\left(V_t - \delta^2 S_t^{\hat{\theta}-2}\right)^2 \tag{8}$$

where V_t is defined in (4). Hence, we have:

$$\hat{\delta} = \sqrt{\frac{\sum V_t S_t^{\hat{\theta}-2}}{\sum S_t^{2\hat{\theta}-4}}}. \tag{9}$$

3.3 As mentioned in ¶3.1, we are working under the risk-neutral probability measure when we use the implied volatility to get δ_js in (7). On the other hand, we are working under the natural probability measure when calculating $\hat{\delta}$ of (9), based on the historical data.

3.4 We now perform a simulation study to assess the performance of the proposed method. For a given pair of values of δ and θ, we simulate five call option prices from model (2), with exercise prices \$26, \$28, \$30, \$32, \$34. We set the initial stock price, risk-free interest rate (r) and time to maturity (τ) as \$30, 5%, and 0.25 respectively. In this simulation study, we arbitrarily consider six cases characterised by integer values of θ ranging from -2 to 3. The corresponding δs are set in the way that the initial volatility σ is around 30%. The first step is to select the best θ according to (7). For simplicity, we just try integer values of θ from -3 to 4. Figure 1 displays the plots of U of (7) versus θ. We see that the measure U always attains its minimum at the true value of θ for all the six cases. If the CEV model is correct, this measure does give a clear indication where the true θ is located. The second step is to calculate the estimates of δ using (9). Table 2 shows that V_t of (4) together with (8) indeed produce a very good estimate of δ, given that $\hat{\theta}$ is close to θ.

3.5 To investigate the variability of $\hat{\delta}$ of (9), we perform another simulation similar to the one in the previous section. Here, we use the means of $\hat{\theta}$ in Table 1 as our final estimates of θ in the first stage, and the results are given in Table 3. Not surprisingly, the means and standard deviations of (9) look much better than those shown in Table 1. Although the first stage of our method is more or less the same as that of the MacBeth & Merville method, our second stage has a stronger theoretical basis. Moreover, instead of using an arbitrary at-the-money call price, it is more natural to estimate δ using historical data.

3.6 For further assessment of the performance of the proposed method, we repeat the simulation in ¶3.4 for different values of σ, r and τ. The values of τ used are 0.25 and 0.5. In both cases we can estimate the true θ correctly in the first stage. Since we do not use any option prices in the second stage, the parameter τ does not play a role in the estimation of δ. Simulation results for $\sigma = 0.2, 0.4$, and $r = 3\%, 6\%, 9\%$ are summarised in

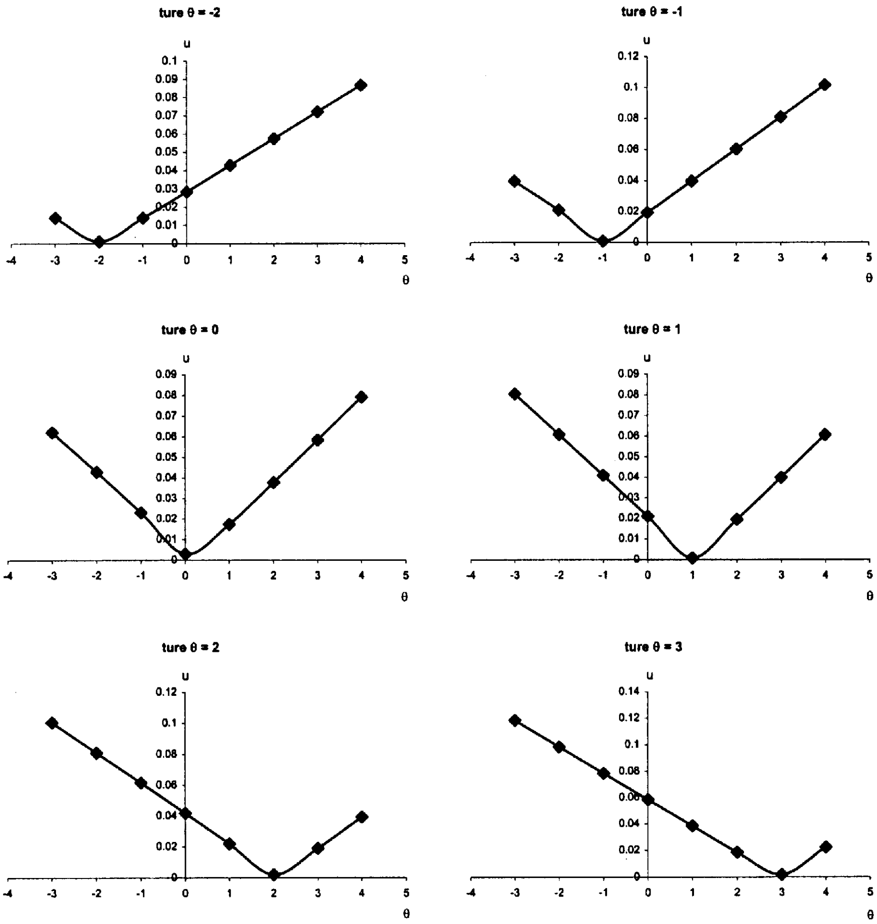

Figure 1. Plots of U

Table 2. Simulation results using the two-stage method

θ	δ	$\hat{\theta}$	$\hat{\delta}$
−2	270.00	−2	267.3800
−1	50.00	−1	51.1990
0	9.00	0	9.1553
1	1.65	1	1.6559
2	0.30	2	0.2999
3	0.06	3	0.0607

Table 3. Simulation results for $\hat{\hat{\delta}}$ of (9)

True value			$\hat{\hat{\delta}}$	
θ	δ	$\hat{\theta}$	Mean	Standard deviation
−4	7000.00	−3.9963	6995.1000	1137.7000
−3	1300.00	−3.0117	1318.2000	169.6800
−2	250.00	−1.9890	251.4500	27.3620
−1	45.00	−1.0054	45.2160	3.5961
0	8.00	0.0110	8.0173	0.2265
1	1.50	1.0044	1.4975	0.0349
2	0.25	1.9981	0.2483	0.0053
3	0.05	3.0058	0.0496	0.0011
4	0.01	3.9988	0.0098	0.0002

Table 4. All the estimates are very close to the true δ. In most cases, $\hat{\hat{\delta}}$ moves away from the true value as r increases. However, the magnitude seems negligible. On the other hand, σ does not have any effect on $\hat{\hat{\delta}}$. For the BS case, that is $\theta = 2$, the estimates are very stable.

3.7 We then use the same simulations in ¶3.6 to estimate θ and δ through the MacBeth & Merville method. The way of estimating θ is exactly the same as that in the previous paragraph. Hence, all θ are estimated correctly, as shown in Table 4. In the second stage, we need the implied volatility of an at-the-money option for the calculation of $\hat{\hat{\delta}}$, and hence $\hat{\hat{\delta}}$ depends on the value of τ. Given all values of the associated parameters, we simply generate an arbitrary at-the-money call option price, and use it to get the implied volatility denoted by $\hat{\sigma}_{BS}$. The final step is to use the results of the first two stages to obtain an estimate of δ; that is to treat $\hat{\sigma}_{BS}$ as an estimate of $\sigma(S_t)$, and then to solve for $\hat{\hat{\delta}}$. Table 5 presents the numerical results for different values of the parameters using the MacBeth & Merville method.

3.8 From a empirical comparison between the two methods, we observe the following. The first stage, in both methods, indeed can provide very good

Table 4. Further simulation results using the two-stage method

		σ		0.2				0.4	
		r	3%	6%	9%		3%	6%	9%
θ	$\hat{\theta}$	δ		$\hat{\hat{\delta}}$		δ		$\hat{\hat{\delta}}$	
−2	−2	180.0	183.04	183.35	183.61	360.0	364.34	364.38	364.41
−1	−1	33.0	33.99	33.98	33.99	66.0	65.76	65.77	65.79
0	0	6.0	6.10	6.11	6.11	12.0	12.31	12.33	12.34
1	1	1.1	1.11	1.11	1.11	2.2	2.23	2.24	2.24
2	2	0.2	0.20	0.20	0.20	0.4	0.40	0.40	0.40

Table 5. Simulation results using the MacBeth & Merville method

		σ		0.2				0.4	
		r	3%	6%	9%		3%	6%	9%
τ	θ	δ		$\hat{\delta}$		δ		$\hat{\delta}$	
0.25	−2	180.0	169.65	168.94	168.13	360.0	350.63	347.95	346.54
	−1	33.0	31.05	30.92	30.77	66.0	62.44	62.33	62.20
	0	6.0	5.61	5.61	5.58	12.0	11.30	11.28	11.25
	1	1.1	1.03	1.03	1.03	2.2	2.07	2.06	2.06
	2	0.2	0.19	0.19	0.19	0.4	0.38	0.37	0.37
0.50	−2	180.0	178.52	177.72	176.87	360.0	368.86	377.04	381.96
	−1	33.0	32.73	32.58	32.43	66.0	66.86	66.65	66.28
	0	6.0	5.95	5.93	5.90	12.0	11.96	11.94	11.90
	1	1.1	1.09	1.09	1.08	2.2	2.19	2.19	2.19
	2	0.2	0.20	0.20	0.20	0.4	0.40	0.40	0.40

estimates of θ. Table 5 indicates that the performance of the MacBeth & Merville method does rely on the τ of the at-the-money option. It seems that the longer the time to maturity the better the estimates of δ. The volatility σ does not have an impact of $\hat{\delta}$ for both methods, while the risk-free interest rate r has a relatively larger effect on $\hat{\delta}$ in the MacBeth & Merville method, but the effect is still very small in magnitude. Finally, in the BS case, the two methods produce extremely good estimates of δ, and are stable with respect to all parameters.

3.9 On the whole, the numerical comparison implies that neither our method nor the MacBeth & Merville method always produces better estimates. Nevertheless, the proposed two-stage method, built on a more theoretical basis, may serve as an alternative to estimate the parameters of the CEV model.

4. Application to Hong Kong Stock Options

4.1 The Hong Kong stock option market began in late 1995, and most of the option contracts are on blue chips. Call option prices for three popular stocks, Cheung Kong Holdings Limited (CKH), Hong Kong Telecommunications Limited (HKT) and Swire Pacific Limited 'A' (SPA), from July 1, 1996 to June 30, 1997, are used in the analysis. The data are obtained from Research and Planning Division, the Stock Exchange of Hong Kong. The time series plots of S_t and V_t for the three stocks are shown in Figure 2. The plots of V_t suggest that the volatility for each stock does vary through time. Roughly speaking: for CKH and HKT, the volatility increases with the stock price; and the reverse is true for SPA.

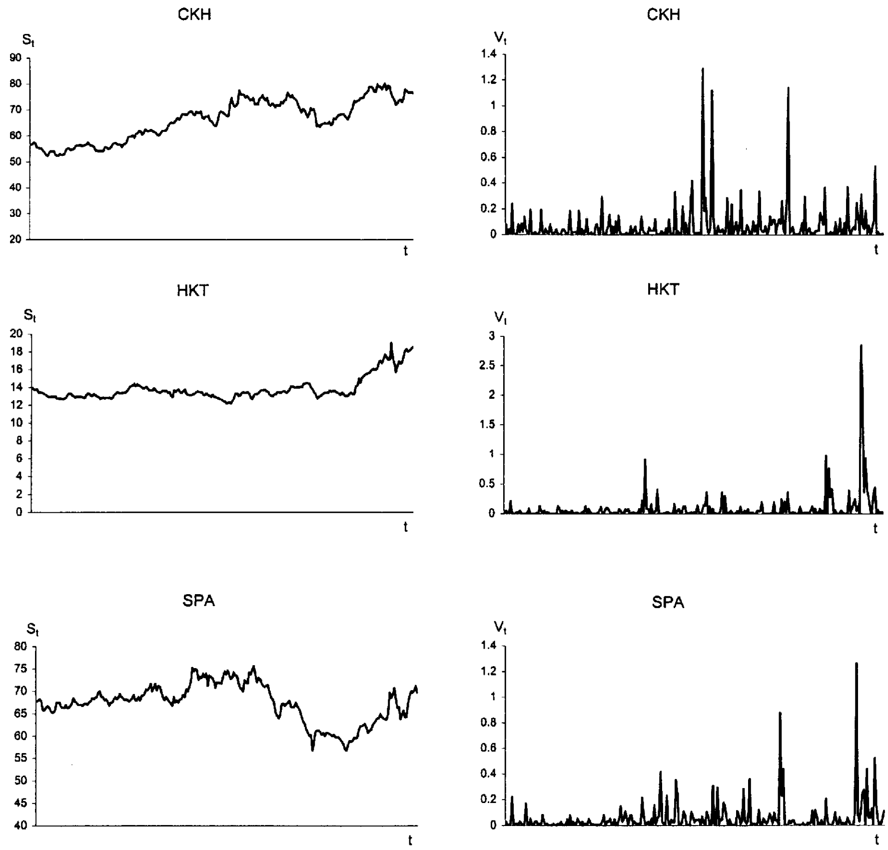

Figure 2. Plots of S_t and V_t

4.2 The objectives of this section are to apply our two-stage method to the Hong Kong data and to compare the performance of the CEV and BS models. In this application, we not only calculate the estimates of δ and θ,

Table 6. Parameter estimates

Stock	CEV		BS
	$\hat{\theta}$	$\hat{\delta}$	$\hat{\sigma}$
CKH	4	0.004	0.268
HKT	4	0.234	0.299
SPA	0	16.160	0.243

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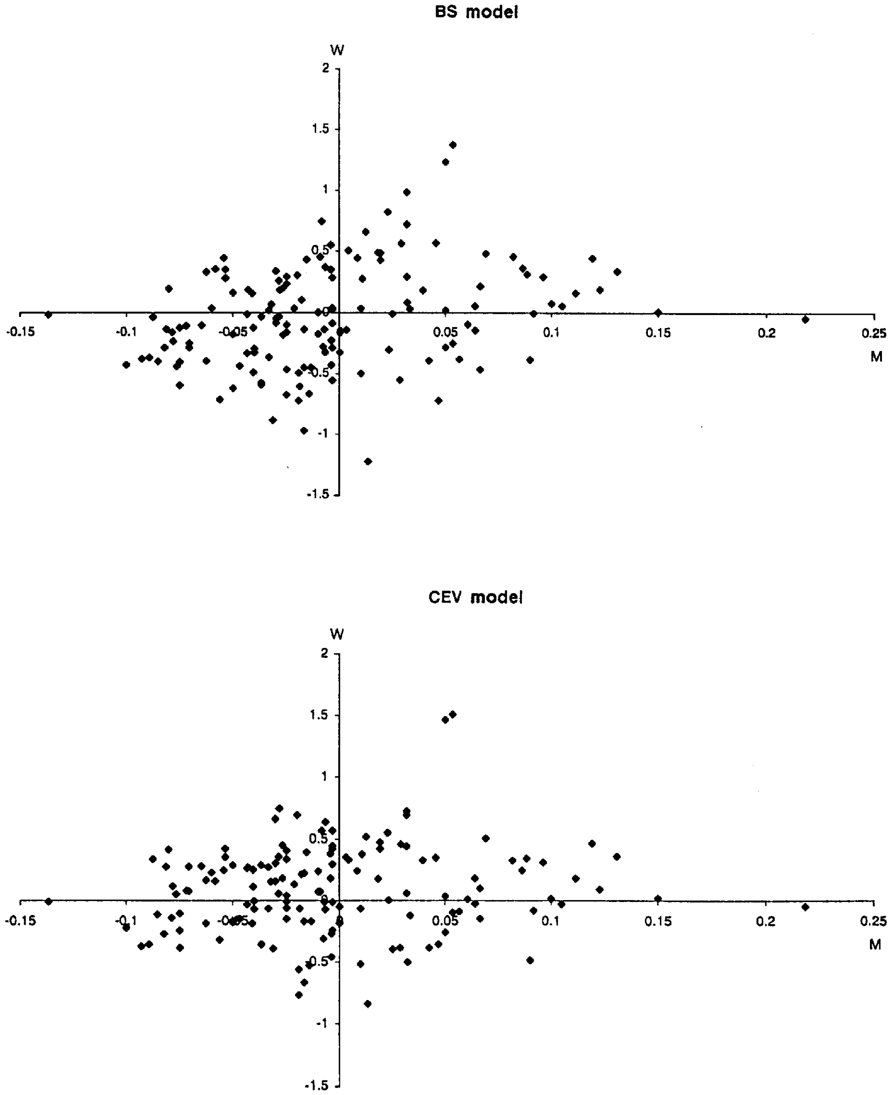

Figure 3. Plots of W vs M for Cheung Kong Holdings Limited

but also the estimate of the volatility of model (1), denoted by $\hat{\sigma}$. Instead of using the implied volatility or the sample standard deviation, we use the method of least squares, together with V_t of (4), to obtain $\hat{\sigma}$. The results are

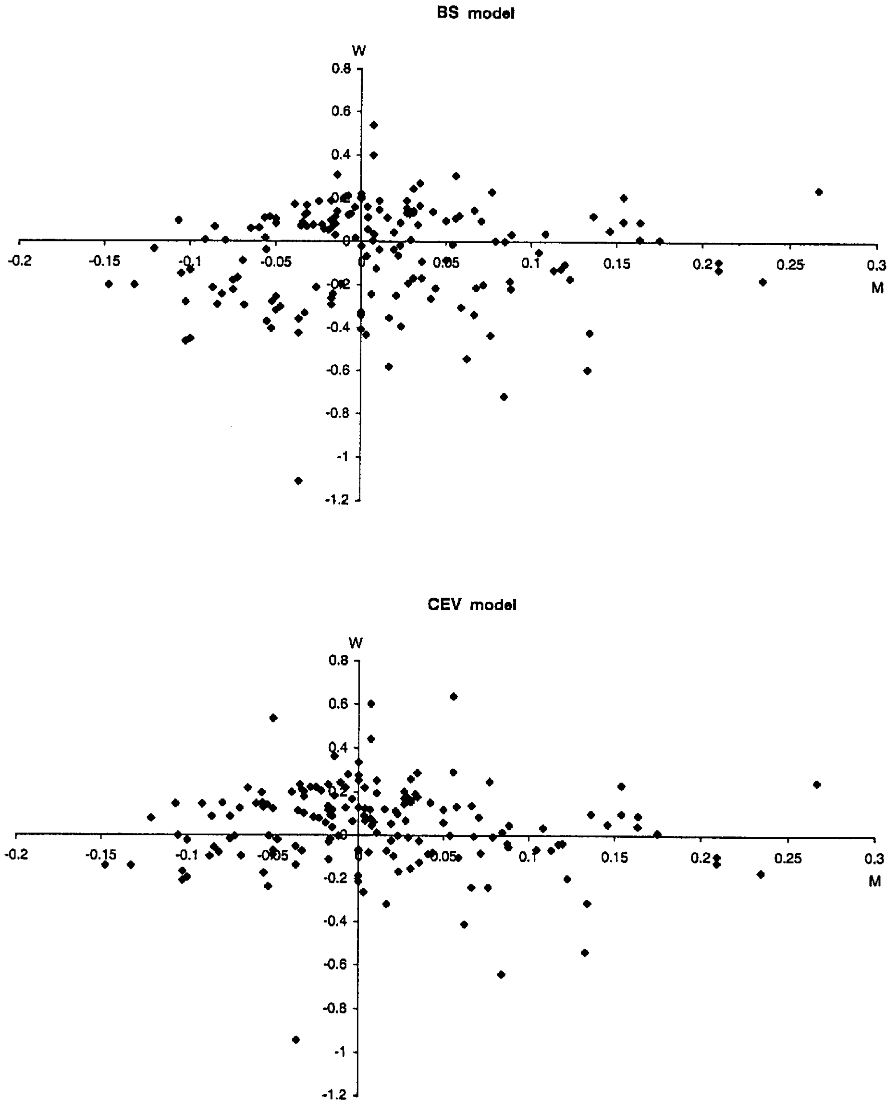

Figure 4. Plots of W vs M for Hong Kong Telecommunications Limited

summarised in Table 6. As expected, the value of $\hat{\theta}$ for CKH and HKT is greater than 2 and that for SPA is less than 2.

4.3 We have mentioned, in ¶1.5, that the BS model works quite well for

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pricing at-the-money calls under the CEV model with $\theta < 2$. The simulation study of MacBeth & Merville (1980) also manifests that the BS model systematically underprices in-the-money call options and overprices out-of-the-money call options if the underlying stock process follows the CEV model with $\theta < 2$. It is interesting to see if there are any systematic pricing biases of the BS model for $\theta > 2$ in our examples.

4.4 We compute the BS and CEV model prices for CKH and HKT, using the estimates shown in Table 6. For having more reasonably quoted market prices, we only include the options with high trading volume. Otherwise, the estimates will be distorted, since the prices of the low-trading ones do not truly reflect their intrinsic values. We use a risk-free interest rate of 5% in our calculations. We now define $M = (S - K)/K$ as a measure of how far the option is in or out of the money, where K represents the exercise price of the option. Furthermore, we use $W = (C_{model} - C_{market})/C_{model}$ to measure the percentage difference between model price C_{model} and market price C_{market}. The model prices are calculated using the option pricing formulae shown in the Appendix. Figures 3 and 4 plot W versus M for CKH and HKT respectively. In Figure 3, the BS model tends to overprice in-the-money calls and underprice out-of-the-money calls. This phenomenon is exactly opposite to the systematic pricing biases of the BS model for $\theta < 2$, mentioned in ¶4.3. On the other hand, the measure W for the CEV model looks more randomly scattered around the horizontal line. As for HKT, the underpricing problem can also be seen to a certain degree. Although Figure 4 does not exhibit much overpricing of in-the-money options for the BS model, we see that the points for the BS model are distributed farther away from zero than those for the CEV model.

5. Remarks

5.1 At the first glance, it is natural to consider the sum of the squared deviations between σ_t^2 and V_t, and to use the non-linear least-squares method to estimate the two parameters jointly. This can be done easily using some statistical software package. However, the non-linear method does not result in acceptable estimates of the parameters. The resulting correlation matrix indicates that $\hat{\delta}$ and $\hat{\theta}$ are highly correlated.

5.2 In our real examples, it is computationally fast to get the estimates, since we restrict our search to integer values of θ only for simplicity. A more detailed search for $\hat{\theta}$ surely produces better results, but it certainly takes a longer time to do so. The simulation results in Section 2 suggest that we can make use of (6) to obtain a good initial estimate of θ. To do this, we simply ignore those points with zero V_t in the estimation. The true θ is likely to be somewhere near the initial estimate. It is expected that a more accurate estimate can be obtained within a few tries.

5.3 For checking model adequacy, we may examine the distribution of the following random variables:

$$X_t = \frac{dS_t - \hat{\mu} S_t dt}{\hat{\delta} S_t^{\hat{\theta}/2}}. \tag{10}$$

If the CEV model is correct, the histogram of X_t should look roughly like a normal curve with zero mean. This simple graphical method can also check informally whether the estimates are plausible. Since our method assumes that the CEV model is correct, it is rational to have some preliminary check before applying our results.

5.4 In this paper, we use V_t of (4) and the least-squares method to develop an estimation procedure for the CEV model. It is worth mentioning that this idea can be extended to any general diffusion processes for stock prices. As shown in the real examples, the simplest case is the BS model; and the constant volatility is estimated by the square root of the mean of V_ts.

5.5 This paper proposes a new estimation procedure for the parameters of the CEV model, and provides potential users with useful insights into various aspects of the procedure. Based on the numerical results, the proposed method is a good alternative to other existing ones, and should be suitable for real applications.

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APPENDIX

BS AND CEV OPTION PRICING FORMULAE

A.1 Consider a European call option with an exercise price of K, maturing at time T. Denote the risk-free interest rate and time to maturity by r and τ respectively. Then, from the risk-neutral valuation argument, the call option price is the discounted expected payoff, that is:

$$C = e^{-r\tau}\mathrm{E}[\max(S_T - K, 0)].$$

A.2 The BS option pricing formula is given by:

$$C = S\mathrm{N}(d_1) - Ke^{-r\tau}\mathrm{N}(d_2)$$

where:

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= d_1 - \sigma\sqrt{\tau} \\ \mathrm{N} &= \text{standard normal cdf} \\ \sigma &= \text{constant volatility in model (1).} \end{aligned}$$

A.3 The CEV option pricing formula is given by:

$$C = SM_1 - Ke^{-r\tau}M_2$$

where:

$$\begin{aligned} M_1 &= \sum_{n=0}^{\infty} g(S'|n+1)G(K'|n+p) & \theta < 2 \\ &= 1 - \sum_{n=0}^{\infty} g(S'|n+p)G(K'|n+1) & \theta > 2 \\ M_2 &= \sum_{n=0}^{\infty} g(S'|n+p)G(K'|n+1) & \theta < 2 \\ &= 1 - \sum_{n=0}^{\infty} g(S'|n+1)G(K'|n+p) & \theta > 2 \end{aligned}$$

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$$S' = \frac{2re^{r\tau(2-\theta)}S^{2-\theta}}{\delta^2(2-\theta)(e^{r\tau(2-\theta)}-1)}$$

$$K' = \frac{2rK^{2-\theta}}{\delta^2(2-\theta)(e^{r\tau(2-\theta)}-1)}$$

$$g(x|m) = \frac{e^{-x}x^{m-1}}{\Gamma(m)}$$

$$G(x|m) = \int_x^{\infty} g(y|m)dy$$

$$p = 1 + \frac{1}{|2-\theta|}$$

$\theta =$ the elasticity of variance in model (2).