

Equations

Fourier

	CT	DT
	Series (CT)	Series (DT)
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$
	Transform (CT)	Transform (DT)
Aperiodic	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t).$$

$$a_0 = \frac{1}{T} \int_T x(t) dt.$$

$$a_k = \frac{2}{T} \int_T x(t) \cos(k\omega_0 t) dt.$$

$$b_k = \frac{2}{T} \int_T x(t) \sin(k\omega_0 t) dt.$$

Table 1: Properties of Continuous Time Fourier Series

Property	Periodic signal	Fourier series coefficients
	$\left. \begin{matrix} x(t) \\ y(t) \end{matrix} \right\} \begin{matrix} \text{periodic with period} \\ T \text{ and fundamental} \\ \text{frequency } \omega_0 = 2\pi/T \end{matrix}$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency shifting	$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a_{-k}^*
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic convolution	$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{d}{dt} x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^t x(\tau) d\tau$ finite valued and periodic only if $a_0 = 0$	$\frac{1}{jk\omega_0} a_k$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and even signals	$x(t)$ real and even	a_k real and even
Real and odd signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t) = \mathfrak{E}\{x(t)\}, \quad [x(t) \text{ real}]$ $x_o(t) = \mathfrak{O}\{x(t)\}, \quad [x(t) \text{ real}]$	$\Re\{a_k\}$ $j\Im\{a_k\}$
Parseval's relation for aperiodic signals	$\frac{1}{T} \int_T x(t) ^2 dt = \sum_{k=-\infty}^{\infty} a_k ^2$	

Table 2: Properties of the Fourier Transform

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time reversal	$x(-t)$	$X(-j\omega)$
Time and frequency scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega) Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi} X(j\omega) * Y(j\omega)$
Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in frequency	$tx(t)$	$j \frac{d}{d\omega} X(j\omega)$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re\{X(j\omega)\} = \Re\{X(-j\omega)\} \\ \Im\{X(j\omega)\} = -\Im\{X(-j\omega)\} \\ X(j\omega) = X(-j\omega) \\ \angle X(j\omega) = -\angle X(-j\omega) \end{cases}$
Symmetry for real and even signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for real and odd signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t) = \mathfrak{E}\mathfrak{v}\{x(t)\}, \quad [x(t) \text{ real}]$	$\Re\{X(j\omega)\}$
	$x_o(t) = \mathfrak{O}\mathfrak{d}\{x(t)\}, \quad [x(t) \text{ real}]$	$j\Im\{X(j\omega)\}$
Parseval's relation for aperiodic signals $\int_{-\infty}^{\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) ^2 d\omega$		

Table 3: Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi \delta(\omega)$	$a_0 = 1, a_k = 0, k \neq 0$ (This is the Fourier series representation for any choice of $T > 0$)
Periodic square wave		
$x(t) = \begin{cases} 1, & t < T_1 \\ 0, & T_1 < t \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \omega < W \\ 0, & \omega > W \end{cases}$	
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at} u(t), \quad \Re\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$t e^{-at} u(t), \quad \Re\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t), \quad \Re\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Table 4: Properties of the Discrete-Time Fourier Series

Property	Periodic signal	Fourier series coefficients
$\left. \begin{matrix} x[t] \\ y[t] \end{matrix} \right\}$	Periodic with period N fundamental frequency $\omega_0 = 2\pi/N$	$\left. \begin{matrix} a_k \\ b_k \end{matrix} \right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a_{-k}^*
Time reversal	$x[-n]$	a_{-k}
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$	$\frac{1}{m} a_k$ (viewed as periodic with period mN)
Periodic convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running sum	$\sum_{k=-\infty}^{\infty} x[k] \left(\begin{matrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{matrix} \right)$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})} \right) a_k$
Conjugate symmetry for real signals	$x[n]$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and even signals	$x[n]$ real and even	a_k real and even
Real and odd signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-odd decomposition of real signals	$x_e[n] = \mathfrak{E}\{x[n]\}, \quad [x[n] \text{ real}]$ $x_o[n] = \mathfrak{O}\{x[n]\}, \quad [x[n] \text{ real}]$	$\Re\{a_k\}$ $j\Im\{a_k\}$
Parseval's relation for aperiodic signals		
$\frac{1}{N} \sum_{n=\langle N \rangle} x[n] ^2 = \sum_{k=\langle N \rangle} a_k ^2$		

Table 5: Properties of the z -Transform

Property	Aperiodic signal	z -transform	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting	$x[n - n_0]$	$z^{-n_0} X(z)$	R
Scaling in z domain	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$ z_0 R$
	$e^{j\omega n} x[n]$	$X(e^{-j\omega n} z)$	R
	$a^n x[n]$	$X(a^{-1} z)$	Scaled version of R (i.e., $ a R$, the set of points $\{a z \}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} , where z is in R).
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r .	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$	at least $R_1 \cap R_2$
Conjugation	$x^*[n]$	$X^*(z^*)$	R
First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $ z > 0$
Accumulation	$\sum_{k=-\infty}^{\infty} x[k]$	$\frac{1}{1-z^{-1}} X(z)$	At least the intersection of R and $ z > 1$
Differentiation in the z -domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	R
Initial value theorem:		If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \rightarrow \infty} X(z)$	

Table 6: Properties of the Laplace Transform

Property	Signal	Laplace transform	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
Shifting in s domain	$e^{s_0 t} x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$sX(s)$	At least R
Differentiation in the s -domain	$-tx(t)$	$\frac{dX(s)}{ds}$	R
Integration in the time domain	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{1}{s} X(s)$	At least $R \cap \{\Re\{s\} > 0\}$
Initial- and final-value theorems: If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then			
$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$			
$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$			