EN1060 Signals and Systems: z Transforms

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November 16, 2017

Section 1

Introduction

Introduction

- We developed the Laplace transform as a generalization of the continuous-time Fourier transform.
- In this lecture, we introduce the corresponding generalization of the discrete-time Fourier transform.
- The resulting transform is referred to as the z-transform.

z-Transform Motivation

- The discrete-time Fourier transform developed out of choosing complex exponentials as basic building blocks for signals because they are eigenfunctions of discrete-time LTI systems.
- A more general class of eigenfunctions consists of signals of the form z^n, where z is a general complex number. A representation of discrete-time signals with these more general exponentials leads to the z-transform.

Relationship between the z-Transform and the Discrete-Time Fourier Transform

- We saw that the Laplace transform is a generalization of the continuous-time Fourier transform.
- A close relationship exists between the z-transform and the discrete-time Fourier transform.
- For $z = e^{j\omega}$ or, equivalently, for the magnitude of z equal to unity, the z-transform reduces to the Fourier transform.
- More generally, the z-transform can be viewed as the Fourier transform of an exponentially weighted sequence.
- Because of this, the z-transform may converge for a given sequence even if the Fourier transform does not: the z-transform offers the possibility of transform analysis for a broader class of signals and systems.

The Region of Convergence (ROC)

- The z-transform of a signal too has associated with it both a range of values of z, referred to as the region of convergence (ROC), for which this expression is valid.
- Two different sequences can have z-transforms with identical algebraic expressions such that their z-transforms differ only in the ROC.
- Consequently, the ROC is an important part of the specification of the z-transform.

z-Plane

- z-transforms of the form of a ratio of polynomials in z^{-1} are described by poles and zeros in the complex plane, referred to as the z-plane.
- The circle of radius 1, concentric with the origin in the z-plane, is referred to as the unit circle.
- Since this circle corresponds to the magnitude of z equal to unity, it is the contour in the z-plane on which the z-transform reduces to the Fourier transform.
- In contrast, for continuous time it is the imaginary axis in the s-plane on which the Laplace transform reduces to the Fourier transform.
- If the sequence is known to be right-sided, for example, then the ROC must be the portion of the z-plane outside the circle bounded by the outermost pole.

Section 2

The z-Transform

Recall: Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{-\infty}^{+\infty} x[n] e^{-j\omega n}$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega n} & \rightarrow & H(e^{j\omega}) e^{j\omega n} \\ & & \updownarrow \mathscr{F} \\ & & h[n] \end{array}$$

z-Transform: Eigenfunction Property

$$z^n \rightarrow \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$z^n \rightarrow z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$z = re^{j\omega}$$

$$z^n \rightarrow H(z) z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

z-Transform and Fourier Transform Relationship

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$z = re^{j\omega}$$

$$X(z)|_{z=e^{j\omega}} = \mathscr{F}\{x[n]\}$$

New notation:

$$\mathscr{F}\{x[n]\} = X(e^{j\omega})$$

z-Transform: Convergence Comparison

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega})$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] \left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] r^{-n} re^{-j\omega n}$$

$$X(z) = \mathscr{F}\left\{x[n] r^{-n}\right\}$$

z-Transform: Convergence Comparison

$$
\begin{aligned}
X(z)|_{z=e^{j\omega}} &= X(e^{j\omega}) \\
X(z) &= \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \\
X(re^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] \left(re^{j\omega}\right)^{-n} \\
&= \sum_{n=-\infty}^{+\infty} x[n] r^{-n} re^{-j\omega n} \\
X(z) &= \mathscr{F}\left\{x[n] r^{-n}\right\}
\end{aligned}
$$

ZT may converge when FT does not.

Example

Find the ZT of $x[n] = a^n u[n]$.

Solution

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$
$$= \sum n = -\infty^{+\infty} a^n z^{-n} u[n]$$
$$X(s) = \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1$$

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$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

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Find the ZT of

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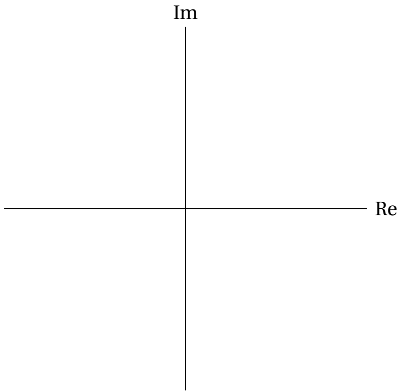
$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

z-Plane and the Unit Circle



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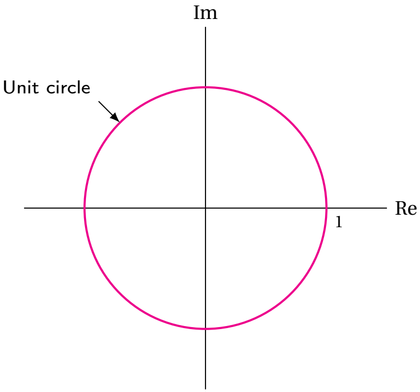

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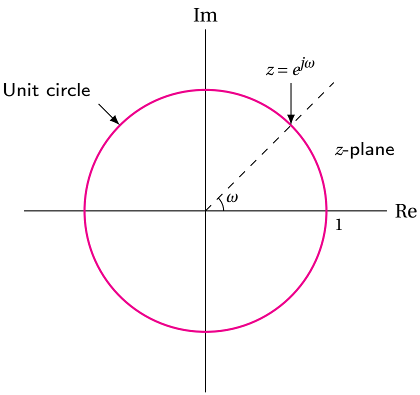
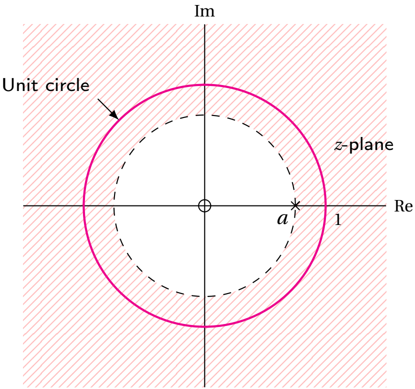


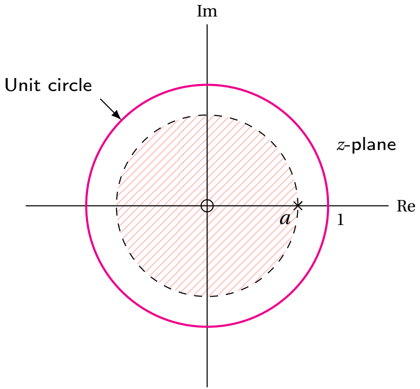
Pole-Zero Plot for a Right-Handed Sequence

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$



Pole-Zero Plot for a Left-Handed Sequence

$$-a^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$



First-Order Difference Equation

$$y[n] - ay[n-1] = x[n]$$

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$$H(z) = \frac{1}{1 - az^{-1}}$$

Causality:

First-Order Difference Equation

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Causality: $|z| > |a|$

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$$Y(z) - az^{-1}Y(z) = X(z)$$

$$Y(z) = \frac{1}{1 - az^{-1}} X(z)$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

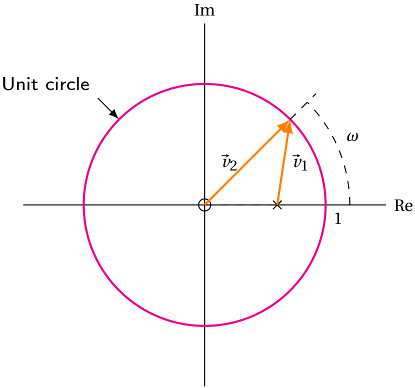
Causality: $|z| > |a|$

$$h[n] = a^n u[n]$$

Pole-Zero Plot for a DT First-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{z}{z-a}, \quad |z| > |a|.$$



Second-Order Difference Equation

$$y[n] + 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

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$$Y(z)\left[1 + 2r\cos\theta z^{-1} + r^2 z^{-2}\right] = X(z)$$

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$\cos\theta < 1 \Rightarrow$ complex poles
Poles are at

Second-Order Difference Equation

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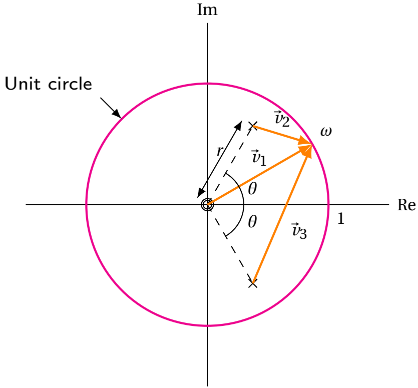
$\cos\theta < 1 \Rightarrow$ complex poles
Poles are at

$$re^{\pm j\theta}$$

Pole-Zero Plot for a DT Under-Damped Second-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}, \quad |z| > |a|.$$



Properties of the ROC of the z-Transform

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
- $\mathscr{F}\{x[n]\}$ converges $\Leftrightarrow$ ROC includes the unit circle in the z-plane
- $x[n]$ finite duration $\Rightarrow$ ROC is entire z-plane with the possible exception of $z = 0$ or $z = \infty$

Properties of the ROC for a Right-Sided Sequence

- $x[n]$ right-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all finite values of z for which $|z| > r_0$ are in ROC.
- $x[n]$ right-sided and $X(z)$ rational $\Rightarrow$ ROC is outside the outermost pole.

Properties of the ROC for a Left-Sided and for a Two-Sided Sequence

- $x[n]$ left-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all values of z for which $0 < |z| < r_0$ will also be in ROC.
- $x[n]$ left-sided and $X(z)$ rational $\Rightarrow$ ROC is inside the innermost pole.
- $x[n]$ two-sided and $|z| = r_0$ is in ROC $\Rightarrow$ ROC is a ring in the z-plane which includes the circle $|z| = r_0$.

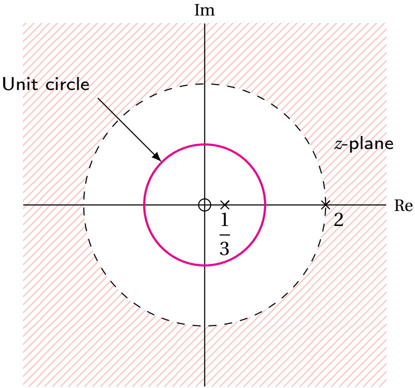
Example

Show the choices of the ROC for

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Right-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$



ROC If the Sequence Is Left-Sided.

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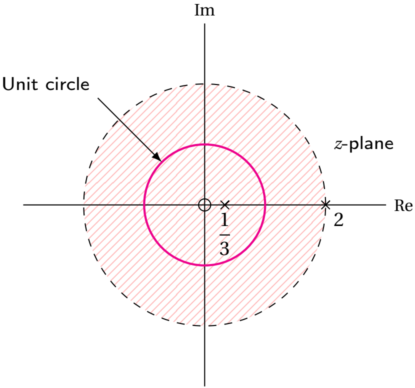
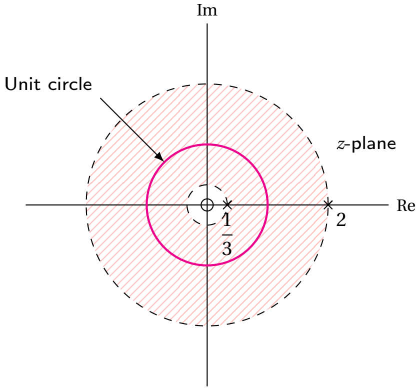


ROC If the Sequence Is Two-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$



Inverse z-Transform

$$
\begin{aligned}
X(z) &= \mathscr{F}\left\{x[n]r^{-n}\right\} \\
x[n]r^{-n} &= \mathscr{F}^{-1}\left\{X(z)\right\} \\
&= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{-j\omega n}d\omega \\
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^{n}d\omega \\
z &= re^{j\omega}, \qquad dz = jre^{j\omega}d\omega \\
x[n] &= \frac{1}{2\pi j}\oint X(z)z^{n-1}dz
\end{aligned}
$$

Example

$$X(z) = \frac{z}{\left(z-\frac{1}{3}\right)(z-2)}, \quad |z| > 2.$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}, \quad |z| > 2.$$

$$\begin{aligned}
X(z) &= \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}, \quad |z| > 2, \\
&= \frac{z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)(1-2z^{-1})}, \quad |z| > 2, \\
&= \frac{-\frac{3}{5}}{\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{-\frac{3}{5}}{(1-2z^{-1})}, \quad |z| > 2. \\
x[n] &= -\frac{3}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{3}{5}(2)^n u[n].
\end{aligned}$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}, \quad |z| > 2.$$

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X(z) &= \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}, \quad |z| > 2, \\
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&= \frac{-\frac{3}{5}}{\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{-\frac{3}{5}}{(1 - 2z^{-1})}, \quad |z| > 2. \\
x[n] &= -\frac{3}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{3}{5}(2)^n u[n].
\end{aligned}$$

Section 3

z-Transform properties

Recall: z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{-n} dz$$

$$X(z)|_{z=e^{j\Omega}} = \mathscr{F}\{x[n]\}$$

$$z = re^{j\Omega}$$

$$X(z) = \mathscr{F}\left\{x[n] r^{-n}\right\}$$

z-Transform Properties

Property	Signal	Transform	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	R
Scaling in z domain	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\|z_0\|R$
	$e^{j\omega n}x[n]$	$X(e^{-j\omega n}z)$	R
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$, the set of points $\{a\|z\|\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R).
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r.	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	at least $R_1 \cap R_2$

z-Transform Properties II

Property	Signal	Transform	ROC
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersectio R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z)$	At least the intersectio R and $\lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{\infty}$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersectio R and $\lvert z\rvert > 1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial value theorem:

If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$

Example

Consider and LTI system for which

$$y[n] = h[n] * x[n],$$

where

$$h[n] = \delta[n] - \delta[n-1].$$

1. Find $H(z)$.
2. Find $y[n]$ in terms of $x[n]$.

Note that

$$\delta[n]-\delta[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} 1-z^{-1},$$

with ROC equal to the entire z-plane, except the origin. Also, this z-transfrom has a zero at $z=1$. If

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad \text{with ROC } = R. \tag{1}$$

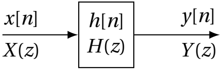
then

$$y[n] \overset{\mathcal{Z}}{\longleftrightarrow} (1-z^{-1})X(z), \tag{2}$$

with ROC equal to R with the possible deletion of $z=0$ and or additon of $z=1$.
Note for this system

$$y[n] = [\delta[n]-\delta[n-1]] * x[n] = x[n]-x[n-1]. \tag{3}$$

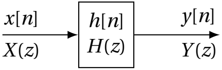
System Stability



$$Y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z)$$

System Stability



$$Y[n] = h[n] * x[n]$$
$$Y(z) = H(z)X(z)$$

$$\text{stable} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$
$$\mathscr{F}\{h[n]\} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

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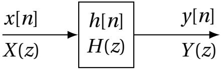

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The condition for stability and the existence of the Fourier transform are the same.

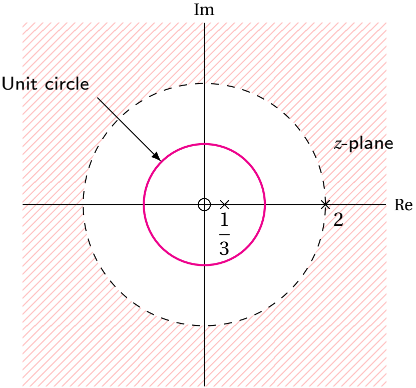
Stability, Causality, and ROC

$$\begin{aligned}
\text{stable} &\Leftrightarrow \text{ROC of } H(z) \text{ includes unit circle in } z\text{-plane} \\
\text{causal} &\Rightarrow h[n] \text{ is right-sided} \\
&\Rightarrow \text{ROC of } H(z) \text{ outside the outermost pole} \\
\text{causal and stable} &\Leftrightarrow \text{ All poles inside unit circle}
\end{aligned}$$

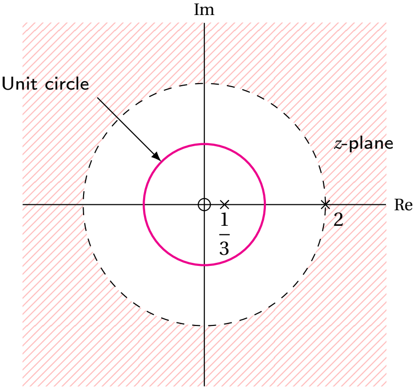
Example

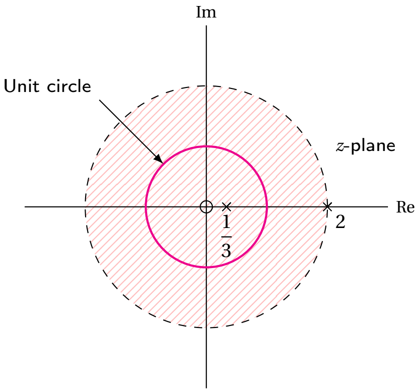
Discuss the stability and causality of the system represented by the following system function with respect to different regions of convergence.

$$H(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$


Im
Unit circle
z-plane
Re
$\frac{1}{3}$
2

The system is causal and unstable.

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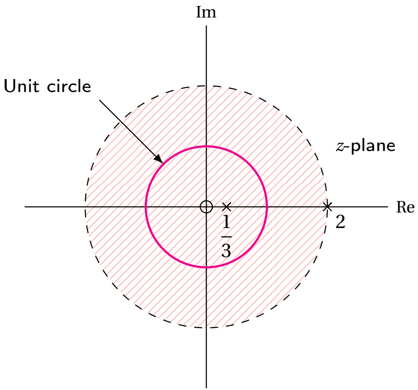
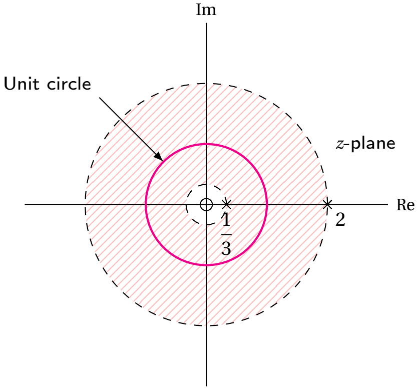


The system is unstable and not causal.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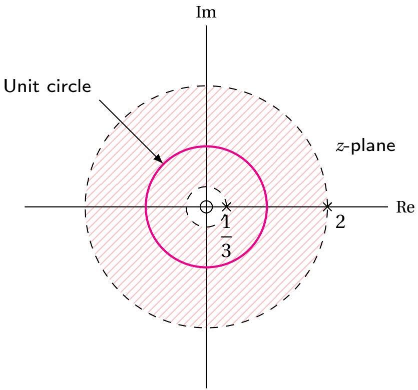
2



The system is stable and not causal.

Example

Consider the LTI system for which the input $x[n]$ and the output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

1. Obtain an expression for the system function $H(z)$.
2. What are the two choices for the region of convergence?
3. Obtain $h[n]$ for each of these cases and comment on the stability and causality.