EN1060 Signals and Systems: Part 2

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Section 1

Linear, Time-Invariant Systems

Systems

- A system processes signals.
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm for post-flight analysis of a space launch.
  - An edge detection algorithm for medical images.

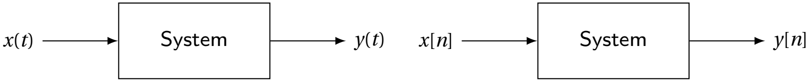


Figure: CT and DT Systems.

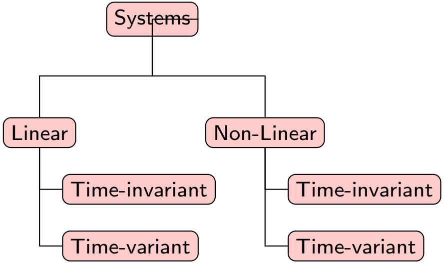


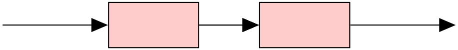
Figure: System types.

This course is focused on the class of linear, time-invariant (LTI) systems.

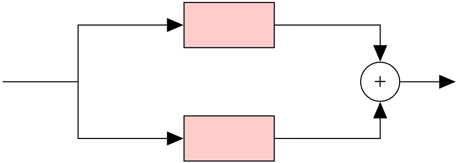
Systems Interconnections

- To build more complex systems by interconnecting simpler subsystems.
- To modify the response of a system.
- E.g.: amplifier design, stabilizing unstable systems.

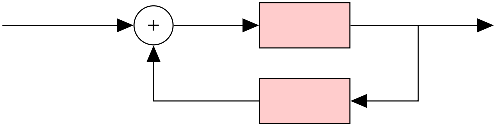
Signal-Flow (Block) Diagrams



Series (Cascade)



Parallel



Feedback

Systems with and without Memory

A systems is said to be memoryless if its output for each value of the independent variable at a given time is dependent only on the input at the same time.

Examples of memoryless systems

$$y[n] = (2x[n] - x^2[n])^2,$$

$$y(t) = Rx(t),$$

where $x(t)$ current through the resistor R and $y(t)$ taken as the voltage across the resistor.

$$y(t) = x(t),$$

which is called the identity system. In DT

$$y[n] = x[n].$$

Examples of systems with memory

Accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].$$

Delay:

$$y[n] = x[n-1].$$

Capacitor with current as the input and the output taken as the voltage:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau.$$

Invertibility and Inverse Systems

A system is invertible of different inputs lead to difference outputs. If a system is invertible, the an inverse system exists, and when cascaded with the original system, yields an output equal to the input to the first system.

Examples of invertible systems:

If $y(t) = 2x(t)$, the inverse system is

$$w(t) = \frac{1}{2}y(t),$$

If (accumulator) $y[n] = \sum_{k=-\infty}^{\infty} x[k]$. the inverse system is

$$w[n] = y[n] - y[n-1].$$

Examples of non-invertible systems:

$$y[n] = 0.$$

$$y(t) = x^2(t).$$

Causality

A system is said to be causal if it only responds when you "kick it." It response at any time depends only on that input prior or equal to that time. The system cannot anticipate future inputs.

Example

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

If

$$x_1(t) = x_2(t), \quad t < t_0,$$

then

$$y_1(t) = y_2(t), \quad t < t_0.$$

If inputs are identical until t_0, the outputs are identical until t_0. Same for DT.

Stability

Many forms. We choose Bounded Input Bounded Output (BIBO) stability.
If a system is stable in BIBO sense, for every bounded input the output is bounded.

Time Invariance

The system does not really care what we call the origin. If the input is shifted by any amount of time t_0, the output is also shifted by the same amount of time.

Linearity

If

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

then

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

If the system is linear, if we give a linear combination of inputs, the output will allose be a similar linear combination of the original outputs.

Linear Time-Invariant Systems

1. Systems that are both linear and time invariant are called Linear Time-Invariant (LTI) systems.
2. With systems that are linear and time invariant, using the impulse function in CT and DT, produces an important and useful mechanism for characterizing those system.
3. In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems.
4. The resulting representation is referred to as convolution.

Introduction (From Oppenheim)

- A linear system: the response to a linear combination of inputs is the same linear combination of the individual responses.
- Time invariance: the system is not sensitive to the time origin. If the input is shifted in time by some amount, then the output is simply shifted by the same amount.
- For a linear system, if the system inputs can be decomposed as a linear combination of some basic inputs and the system response is known for each of the basic inputs, then the response can be constructed as the same linear combination of the responses to each of the basic inputs.
- Signals can be decomposed as a linear combination of basic signals in a variety of ways (e.g., Taylor series expansion that expresses a function in polynomial form.) However, in the context of signals and systems, it is important to choose the basic signals in the expansion so that in some sense the response is easy to compute.
- For systems that are both linear and time-invariant, there are two particularly useful choices for these basic signals: delayed impulses and complex exponentials.

Introduction (From Oppenheim)

- In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems. The resulting representation is referred to as convolution.
- Earlier, we developed in detail the decomposition of signals as linear combinations of complex exponentials (referred to as Fourier analysis) and the consequence of that representation for linear, time-invariant systems.

Introduction

- Using the convolution we can express the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse.
- An LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.
- In discrete time, we have the convolution sum. In continuous time, we have the convolution integral.

Strategy for Exploiting Linearity and Time Invariance

A DT Signal as Superposition of Weighted Delayed Impulses

- We can express a DT signal as a linear combination of weighted delayed impulses.
- If we have a liner system, and a signal expressed as above as a linear combination of basic signals, the response would be the same linear combinaton fo the responses for individual basic signals.

Convolution Sum

Convolution Sum: Summary

The convolution of the sequence $x[n]$ and $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \tag{1}$$

which we represent symbolically as

$$y[n] = x[n] * h[n] \tag{2}$$

Example

Computer $y[n] = x[n] * h[n]$ for $x[n]$ and $h[n]$ as shown in Figure **??**.

Example

Consider and input $x[n]$ and a unit impulse response $h[n]$ given by

$$\begin{aligned} x[n] &= \alpha^n u[n] \\ h[n] &= u[n], \end{aligned} \tag{3}$$

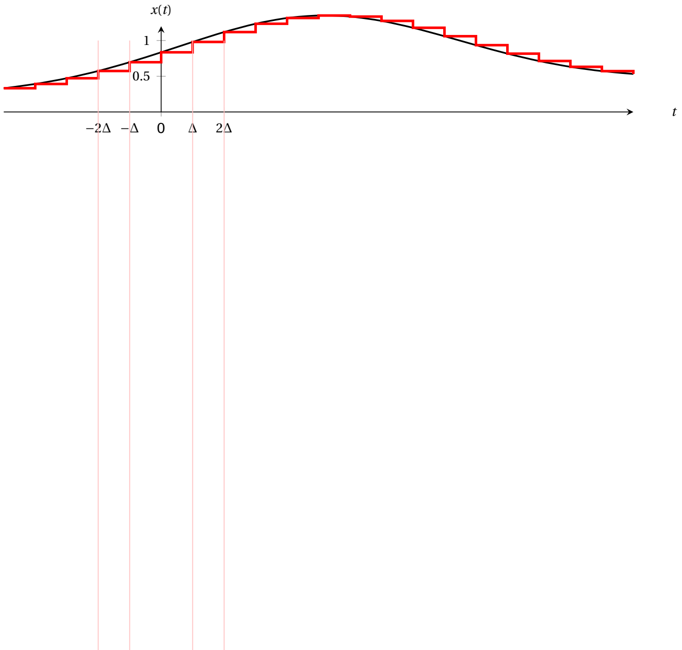
which $0 < \alpha < 1$. Find $y[n]$ and sketch.

Continuous-Time Systems: The Convolution Integral

1

Continuous-Time Systems: The Convolution Integral

1. Similar to what we did in DT, in this section we obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response.
2. In discrete time, the key to developing the convolution sum was the sifting property of the DT unit impulse—i.e., mathematical representation of a signal as a superposition of scaled and shifter unit impulse functions.
3. We begin by considering the staircase approximation $\hat{x}(t)$ of a CT signal $x(t0$.



x(t)
1
0.5
t
−2Δ
−Δ
0
Δ
2Δ

The approximation that we saw can be expressed as a linear combination of delayed impulses. Define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Since $\Delta\delta_\Delta(t)$ has unit amplitude, we have

As $\Delta \to 0$, the summation approaches an integral. Consequently,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This is known as the sifting property of the continuous time impulse

Example:

Use the sifting property to express $u(t)$ in terms of $\delta(t)$.

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

Let's define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t-k\Delta)$.

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta.$$

Since the pulse $\delta_\Delta(t-k\Delta)$ corresponds to a shifted unit impulse as $\Delta \to 0$, the response $\hat{h}_{k\Delta}(t)$ to this input pulse becomes the response to an impulse in the limit.If we let $h1_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ, then

$$y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_{k\Delta}(t)\Delta.$$

As a $\Delta \to 0$, the summation on the right-hand side becomes an integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

In addition to being linear, the system is time-invariant, the response of the LTI system to the unit impulse $\delta(t-\tau)$

$$h_\tau(t) = h_0(t-\tau).$$

Defining unit impulse response $h(t)$ as

$$h(t) = h_0(t),$$

we have

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

which is referred to as the convolution integral or the superposition integral. This corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse.

$$y(t) = x(t) * h(t).$$

As in discrete time, a continuous-time LTI system is completely characterized by its impulse response—i.e., by its response to a.single elementary signal, the unit impulse $\delta(t)$.

Example: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = e^{-at}u(t), a > 0$$

and

$$h(t) = u(t).$$

For $t < 0$, the product $x(\tau)$ and $h(t-\tau)$ is zero, consequently $y(t)$ is zero.
For $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau}\, d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t \\ &= \frac{1}{a}(1 - e^{-at}) \end{aligned}$$

Thus for all t,

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t)$$

Example: Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(t) = \begin{cases} t, & 0 < t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Find $y(t)$, the convolution of the following two signals:

$$x(t) = e^{2t}u(-t),$$

and

$$x(t) = u(t-3).$$

When $t-3 \leq 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < t-3$, and the convolving integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2} e^{2(t-3)}.$$

F0r $t-3 \geq 0$, the product of $x(\tau)h(t-\tau)$ is nonzero for $-\infty < \tau < 0$, and the convolving integral becomes

$$y(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2}.$$

Recapitulation

1. In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

2. Characteristics of an LTI system are completely determined by its impulse response.

The Commutative Property

DT

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

CT

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Verify the commutative property for DT.

The Distributive Property

Convolution distributes over addition.

DT

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

CT

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t).$$

The Associative Property

DT

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n].$$

CT

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t).$$

As a consequence,

$$y[n] = x[n] * h_1[n] * h_2[n]$$

and

$$y(t) = x(t) * h_1(t) * h_2(t).$$

are unambiguous.
Using the commutative property together with the associative property we can see that the order in which they are cascaded does not matter as far as the overall system impulse response is concerned.

LTI Systems with and without Memory

1. A system is memoryless if its output at any time depends only on the value of the input at that same time.
2. The only way that this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$.
3. In this case the impulse response has the form

$$h[n] = K\delta[n],$$

   where $K = h[0]$ is a constant.
4. The convolution sum reduces to the relation

$$y[n] = Kx[n]$$

5. If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory.
6. For CT:

$$h(t) = K\delta(t).$$

$$y(t) = Kx(t).$$

Invertibility of LTI Systems

1. An LTI system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.

Example

Consider the following relationship of a pure time shift:

$$y(t) = x(t - t_0)$$

Is the corresponding system memoryless? What is the inverse system of the system?

Example

Consider the LTI system with impulse response

$$h[n] = u[n].$$

Determine $y[n]$. Find the inverse system.

Causality for LTI Systems

1. The output of a causal system depends only on the present and past values of the input to the system.
2. For a DT LTI system, $y[n]$ must not depend on $x[k]$ for $k > n$.
3. For this to be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero.
4. This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0.$$

5. The impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality.
6. More generally, causality for a linear system is equivalent to the condition of initial rest; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time.
7. The equivalence of causality and the condition of initial rest applies only to linear systems.

Causality for LTI Systems

1. A continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0.$$

2. Causality of an LTI system is equivalent to its impulse response being a causal signal.

Stability for LTI Systems

A system is stable if every bounded input produces a bounded output. Consider an input $x[n]$ that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n.$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

If the impulse response is absolutely summable, then $y[n]$ is bounded in magnitude, and hence, the system is stable.

Stability for LTI Systems

In CT a system is stable if the impulse response is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

Examples

Determine whether the following systems are stable:

1. Pure time shift in DT.
2. Pure time shift in CT.
3. Accumulator in DT.
4. CT counterpart of the accumulator.

The Unit Step Response of an LTI System

There is another signal that is also used in describing the behavior of LTI systems: the unit step response, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$.

$$s[n] = u[n] * h[n]$$

Commutative property:

$$s[n] = h[n] * u[n]$$

$s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$.

$u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

$h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1].$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response.

Similarly, in CT, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

Zero-Input Response

For a linear system (time-invariant or not), if we put nothing into it, we get nothing out of it.

$$\begin{aligned} x(t) &= 0 \quad \text{for all } t, \quad \text{then} \\ y(t) &= 0 \quad \text{for all } t, \\ x[n] &= 0 \quad \text{for all } n, \quad \text{then} \\ y[n] &= 0 \quad \text{for all } n, \end{aligned}$$

"Proof": If the system is linear and

$$x(t) \to y(t), \quad \text{then if we scale}$$

$$ax(t) \to ay(t).$$

Select the sale factor $a = 0$.
Not all systems are like this, e.g., even if a battery is not connected to anything, the output is 1.5 V.

Implications for Causality

The system cannot anticipate the input.
I.e., If

$$x_1(t) = x_2(t), \quad \text{for } t < t_0,$$

then

$$y_1(t) = y_2(t), \quad \text{for } t < t_0,$$

Same for DT.

Implications for Causality for a Linear System

For linear systems, if

Initial rest: The system does not respond until an input is given.
For a linear system to be causal it must have the property of initial rest.
Why? For linear systems zero in $\rightarrow$ zero out.

Causality for Linear Time Invariant Systems

For LTI systems,

Causality $\Leftrightarrow$

$$
\begin{aligned}
h(t) &= 0, \quad t < 0 \\
h[n] &= 0, \quad n < 0
\end{aligned}
$$

“Proof": $\Rightarrow$: Why does causality imply the above? Ans:

$\Leftarrow$:Why does $h(t) = 0, \quad t < 0$ $(h[n] = 0, \quad n < 0$, imply the system is causal? Ans:

Example: Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

The accumulator is an LTI system. Also, we saw that its impulse response is

$$h[n] = u[n].$$

1. Does the accumulator have memory?
2. Is the accumulator causal?
3. Is accumulator stable in the BIBO sense?
4. If invertible, what is the inverse?

Example

$$y[n] - ay[n-1] = x[n]$$

under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

Example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

Impulses

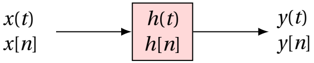


Figure:

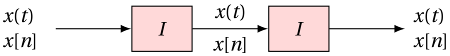


Figure:

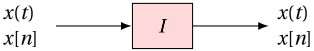
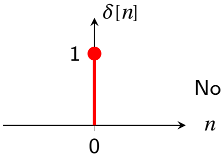


Figure:



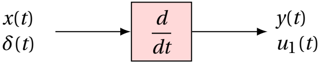
No problem in DT.

Operational Definition

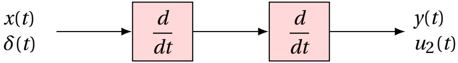
We use operational definitions through convolution to handle derivatives and integrals of impulse, which are badly behaved functions. This leads to a set of singularity functions. Impulse and step are examples of these.

$$x(t) * \delta(t) = x(t)$$

$$\frac{d}{dt}[\delta(t)]$$



$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$



$$u_2(t) = u_1(t) * u_1(t)$$

$$x(t) * u_2(t) = \frac{d^2 x(t)}{dt^2}$$

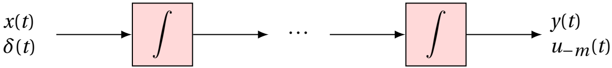
$$u_k(t) = u_1(t) * u_1(t) * \cdots k \text{ times}$$

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

$$u_0(t) = \delta(t)$$

$$u_{-1}(t) = \text{unit step}$$
$$u_{-2}(t) = \text{unit ramp}$$

$$u_0(t) = \delta(t)$$
$$u_{-1}(t) = u(t)$$



$$u_k(t) * u_l(t) = u_{k+l}(t).$$

$$x(t) * u_{-m}(t) = m^{\text{th}} \text{ running integral}$$

$u_k(t)$ defined by

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

Linear, Constant-Coefficient Differential and Difference Equations

- An important class of CT systems is that for which the input and output are related through a linear constant-coefficient differential equation.
- These arise in the description of a wide variety of systems and physical phenomena. E.g., the response of the RC circuit, the motion of a vehicle subject to acceleration inputs and frictional forces.
- Correspondingly, an important class of DT systems is that for which the input and output are related through a linear constant-coefficient difference equation.
- These are used to describe the sequential behavior of many different processes. E.g., accumulation of savings in a bank account, a digital simulation of a continuous-time system, DT designed to perform particular operations on the input signal such as a system that calculates the difference between successive input values, of computes the average value of the input over an interval.

Linear, Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{dx(t)}{dt^k} \tag{4}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \quad \text{Homogeneous equation.} \tag{5}$$

Given $x(t)$, if $y_p(t)$ satisfies 4, so does $y_p(t) + y_h(t)$ where $y_h(t)$ satisfies 5.

$$y_p(t) \triangleq \text{particular solution}$$

$$y_h(t) \triangleq \text{homogeneous solution}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0.$$

Guess a solution of the form

$$y_h(t) = Ae^{st}, \quad \text{a complex exponential}$$

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_N e^{s_N t}$$

Coefficients $A_1, A_2, \ldots, A_N$ are undetermined. We need N auxiliary conditions to determined them.

$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1} y(t)}{dt^{N-1}} \quad \text{at } t = t_0.$$

Linear system $\iff$ auxiliary conditions = 0
Linear system $\Rightarrow$ zero in, zero out.
Causal and LTI $\iff$ initial rest
If $x(t) = 0,\ t < t_0$ then
$y(t) = 0,\ t < t_0$ then

Example: First-Order Differential Equation

$$\frac{dy(t)}{dt} + ay(t) = x(t).$$

Homogeneous equation:

$$\frac{dy_h(t)}{dt} + ay_h(t) = 0.$$

Now guess a solution

$$y_h(t) = Ae^{st}.$$

Obtain the impulse response of the above system.

Linear, Constant-Coefficient Difference Equations

Consider the N^{th}-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{6}$$

$$\sum_{k=0}^{N} a_k y[n-k] = 0 : \quad \text{homogeneous equation.} \tag{7}$$

If $y_p[n]$ satisfies 6 so does $y_p[n] + y_h[n]$ where $y_h[n]$ satisfies 7.

$$y_p[n] \triangleq \text{ particular solution}$$
$$y_h[n] \triangleq \text{ homogeneous solution}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k y[n-k] = 0.$$

"Guess" a solution of the form

$$y_h[n] = Az^n.$$

$$\sum_{k=0}^{N} a_k A z^n z^{-k} = 0.$$

$$\sum_{k=0}^{N} a_k z^{-k} = 0. \quad N \text{ roots } , z_1, z_2, \dots, z_N.$$

$$y_h[n] = A_1 z_1^n + A_2 z_2^n + \cdots + A_N z_N^n.$$

$$y[n] = A_1 z_1^n + A_2 z_2^n + \cdots + A_N z_N^n + y_p[n]$$

The undetermined constants A_1 to A_N are to be found using the N auxiliary conditions, $y[n_0]$, $y[n_0-1]$, $y[n_0-N+1]$.
Linear system $\Longleftrightarrow$ auxiliary conditions 0
Causal, LTI $\Longleftrightarrow$ initial rest
If $x[n]=0,\ n<n_0$ then
$y[n]=0,\ n<n_0$ then

Explicit Solution to Difference Equations

Assume causality.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

This is said to be a recursive solution to linear constant-coefficient difference equations. To get the computation of $y[n_0]$ started we need the initial conditions or boundary conditions $y[n_0-1]$, $y[n_0-N]$. Then we compute $y[n_0+1]$ and so on.

Example: First-Order Difference Equation

$$y[n] - ay[n-1] = x[n].$$

Causal, LTI $\Longleftrightarrow$ initial rest

$$y[n] = x[n] + ay[n-1].$$

Set $x[n] = \delta[n]$.

$$h[n] = \delta[n] + ah[n-1].$$

$$h[n] = 0, n < 0 \quad \text{initial rest.}$$

$$h[0] = 1$$
$$h[1] = a$$
$$h[2] = a^2 \vdots \qquad = \vdots$$

$$h[n] = a^n u[n].$$

So if causal and LTI

$$\delta[n] \to a^n u[n].$$

Family of solutions

$$\delta[n] \to a^n u[n] + y_h[n].$$

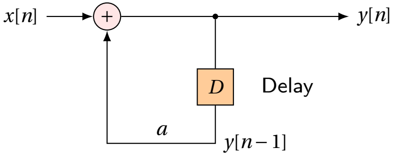
$$y_h[n] = Az^n.$$

$$Az^n - aAz^{n-1} = 0.$$

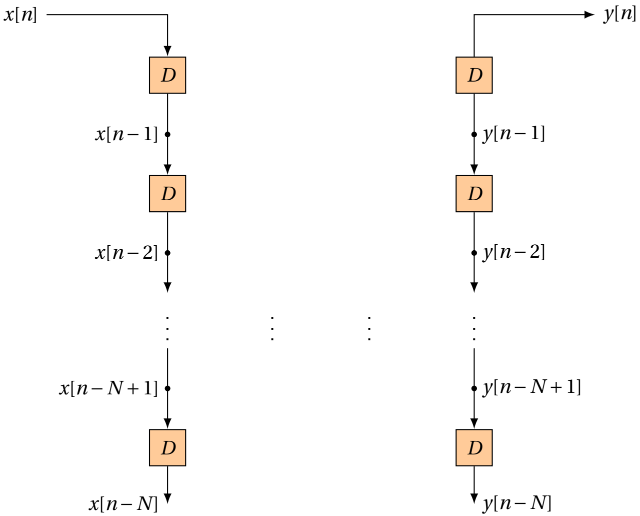
$$a - z^{-1} = 0 \Rightarrow z = a.$$

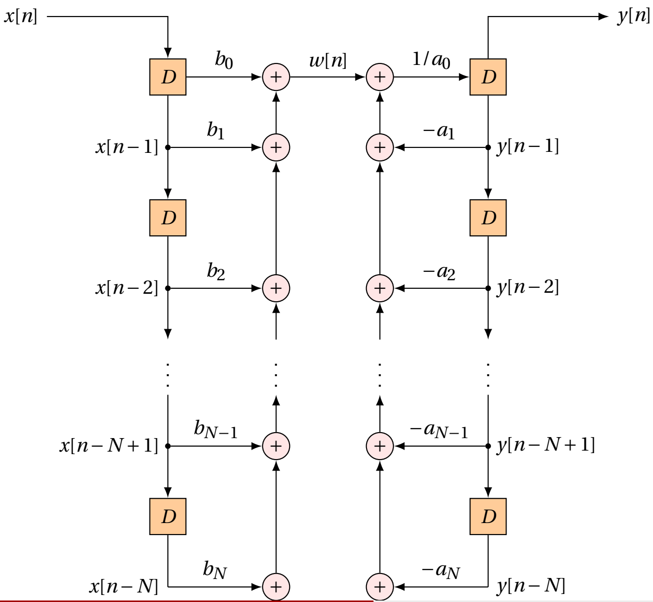
$$y_h[n] = Aa^n.$$

$$\delta[n] \to a^n u[n] + Aa^n.$$



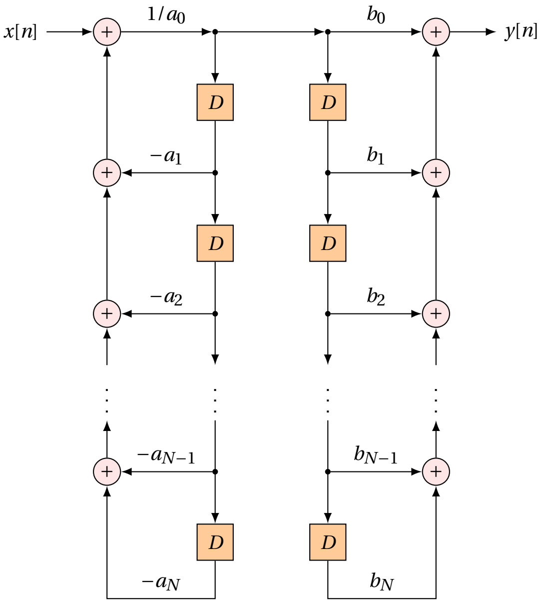
$x[n]$
$+$
$y[n]$
D
Delay
a
$y[n-1]$


$x[n]$
D
$x[n-1]$
D
$x[n-2]$
$x[n-N+1]$
D
$x[n-N]$
$y[n]$
D
$y[n-1]$
D
$y[n-2]$
$y[n-N+1]$
D
$y[n-N]$



Direct form I implementation

This is a cascade of two linear systems. We can interchange the order of the two segments.


$x[n]$
$1/a_0$
b_0
$y[n]$
D
D
$-a_1$
b_1
D
D
$-a_2$
b_2
$-a_{N-1}$
b_{N-1}
D
D
$-a_N$
b_N

Convolution

1. In representing and analyzing LTI systems, our approach has been to decompose the system inputs into a liner combination of basic signals and exploit the fact that for a linear system, the response is the same linear combination of the responses to the basic inputs.
2. The convolution sum and the convolution integral req out of the particular choice of the basic signals, delayed unit impulses.
3. This choice has the advantage that for systems that are time invariant in addition to being linear, once the response to an impulse at one time position is known, then the response id know at all time positions.

Complex Exponentials with Unity Magnitude as Basic Signals

1. When we select complex exponential with unity magnitude as the basic signals, the decomposition of this form of a periodic signal is the Fourier series.
2. For aperiodic signals, it becomes the Fourier transform.
3. In latter lectures, we will generalize this representation to Laplace tangram for continuous-time signals and z-transform for discrete-time signals.

Consider a linear system

$$\begin{matrix} x(t) \\ x[n] \end{matrix} \longrightarrow \boxed{\begin{matrix} h(t) \\ h[n] \end{matrix}} \longrightarrow \begin{matrix} y(t) \\ y[n] \end{matrix}$$

If

$$x(t) = a_1\phi_1(t) + a_2\phi_2(t) + \cdots$$

and

$$\phi_k(t) \longrightarrow \psi_k(t), \quad \text{(output due to } \phi_k(t)\text{)}$$

then

$$y(t) = a_1\psi_1(t) + a_2\psi_2(t) + \cdots$$

Identical for DT. So

$$\begin{aligned} \text{If} \quad & x(t) = a_1\phi_1 + a_2\phi_2 + \cdots, \\ \text{then} \quad & y(t) = a_1\psi_1 + a_2\psi_2 + \cdots. \end{aligned}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ so that

1. A broad class of signals can be constructed as a linear combination of ϕ_ks
2. Response to ϕ_ks easy to compute.

Choice of signals $\delta(t-k\Delta)$ and $\delta[n-k]$ lead to the convolution integral and convolution sum.

$$\begin{aligned} \text{CT} \quad & \phi_k(t) = \delta(t-k\Delta) \\ & \psi_k(t) = h(t-k\Delta) \quad \Rightarrow \text{ convolution integral} \\ \text{DT} \quad & \phi_k[n] = \delta[n-k] \\ & \psi_k[n] = h[n-k] \quad \Rightarrow \text{ convolution sum} \end{aligned}$$

Here, we choose complex exponentials as the set of basic signals.

$$\begin{aligned} \phi_k(t) &= e^{s_k t}, \quad s_k \text{ complex} \\ \phi_k[n] &= z_k^n, \quad z_k \text{ complex} \end{aligned}$$

Fourier Analysis

$$\begin{array}{llll} \text{CT} & s_k = j\omega k & \text{purely imaginary} & \phi_k(t) = e^{j\omega_k t} \\ \text{DT} & |z_k| = 1 & & \phi_k[n] = e^{j\omega_k n} \end{array}$$

s_k complex $\Rightarrow$ Laplace transfrom
z_k complex $\Rightarrow$ z-transform

Eigenfunction Property

Consider $\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(\omega_k) e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

“Proof”:

CT Fourier Series

$$x(t) = x(t+T), \quad T = \text{ period}, \quad \omega_0 = \frac{2\pi}{T}$$

Consider the periodic exponential signal with the same fundamental period:

$$e^{j\omega_0 t}, \quad T = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t}, \quad T = \frac{2\pi}{k\omega_0}, \quad \text{harmonically related}$$

$$\text{Fourier series:} \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

1. How do we find a_ks?
2. What are the signal that can be expressed in this form? [already answered, see

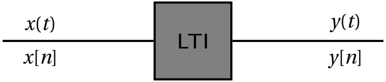
Section 2

Discrete-Time Fourier Series

Introduction

- Now, we have studied Fourier series and Fourier transform for CT signals.
- In this lesson we will develop a similar tool for discrete time.
- Specifically, we consider the representation of discrete-time signals through a decomposition as a linear combination of complex exponentials.
  - DT periodic signals → DT Fourier series
  - DT aperiodic signals → DT Fourier transform

Philosophy



Decompose the input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots \quad \text{linear combination of basic inputs}$$

Then

$$y = a_1\psi_1 + a_2\psi_2 + \cdots \quad \text{linear combination of corresponding outputs}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ such that

- Broad class of signals can be constructed, and
- Response to ϕ_ks easy to compute.

Eigenfunction Property

Continuous-Time:

$\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(\omega_k) e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

"Discrete-Time":

Discrete-Time Fourier Series

Discrete-Time Fourier Series

Continuous-Time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Discrete-Time Fourier Series

Example

Consider the signal

1. When is this signal periodic?
2. If it is periodic, what are discrete-time Fourier series coefficients?

Fourier Coefficients for $x[n] = \sin(2\pi/N)n$ for $N = 5$

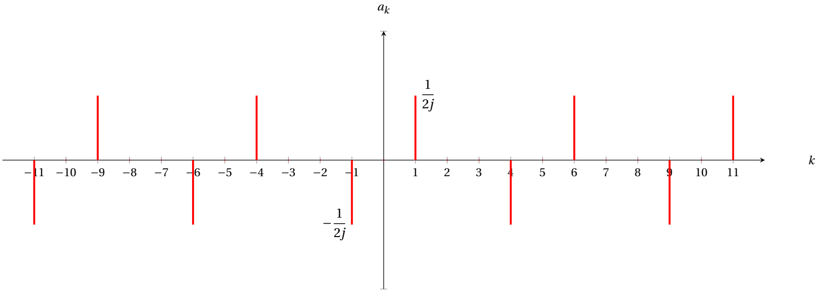
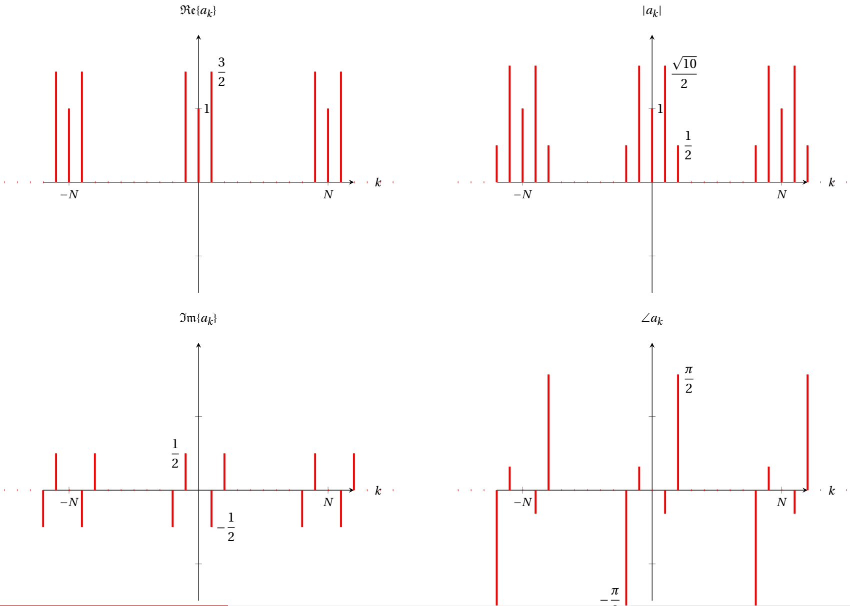


Figure: Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Example

Determine and sketch the DTFS of

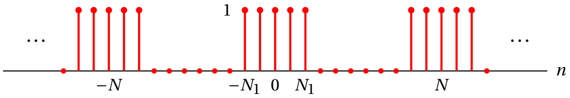
$$x[n] = 1 + \sin \omega_0 n + 3 \cos \omega_0 n + \cos\left(2\omega_0 n + \frac{\pi}{2}\right).$$

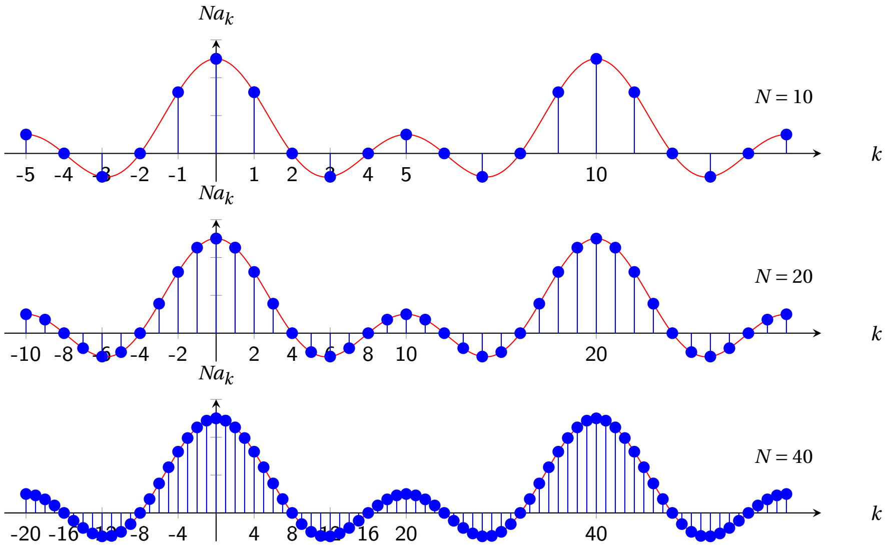


$\Re\mathfrak{e}\{a_k\}$
$\frac{3}{2}$
1
$-N$
N
k
$|a_k|$
$\frac{\sqrt{10}}{2}$
1
$\frac{1}{2}$
$-N$
N
k
$\Im\mathfrak{m}\{a_k\}$
$\frac{1}{2}$
$-\frac{1}{2}$
$-N$
N
k
$\angle a_k$
$\frac{\pi}{2}$
$-N$
N
k

Example

Determine and sketch the DTFS of $x[n]$ of which is shown in the figure.

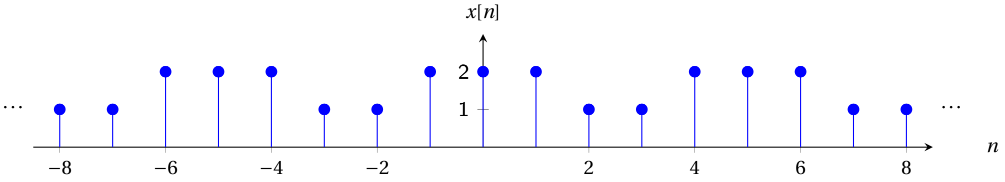




Na_k
$N = 10$
k
-5
-4
-3
-2
-1
1
2
3
4
5
10
Na_k
$N = 20$
k
-10
-8
-6
-4
-2
2
4
6
8
10
20
Na_k
$N = 40$
k
-20
-16
-12
-8
-4
4
8
12
16
20
40

Example

Find the Fourier series coefficients a_k of $x[n]$.



Example

Suppose that we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $n=6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the proceeding three conditions.

Determine the sequence $x[n]$.

Section 3

Discrete-Time Fourier Transform

Discrete-Time Fourier Transform from DTFS

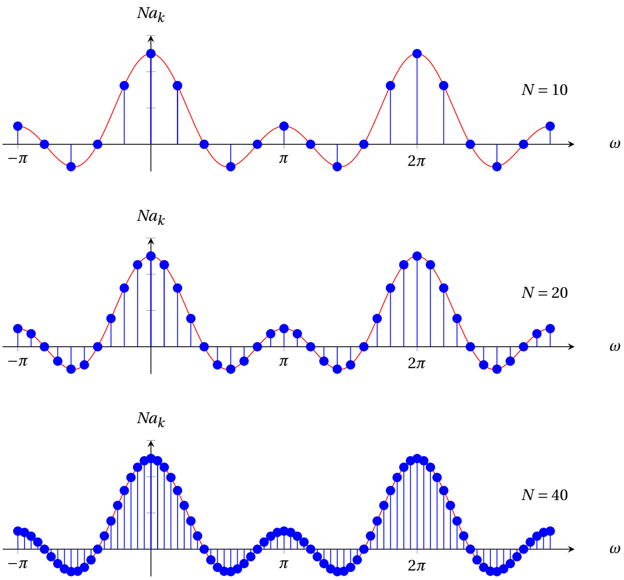
- We can represent the Fourier series coefficients as samples of an envelope. This envelope is determined by the behavior of the sequence over one period but is not dependent on the specific value of the period.
- As the period of the sequence increases, with the nonzero content in the period remaining the same, the Fourier series coefficients are samples of the same envelope function with increasingly finer spacing along the frequency axis (specifically, a spacing of $2\pi/N$ where N is the period).
- Consequently, as the period approaches infinity, this envelope function corresponds to a Fourier representation of the aperiodic signal corresponding to one period. This is, then, the Fourier transform of the aperiodic signal.
- The discrete-time Fourier transform developed, as we have just described, corresponds to a decomposition of an aperiodic signal as a linear combination of a continuum of complex exponentials.

Discrete-Time Fourier Transform from DTFS

- The synthesis equation is then the limiting form of the Fourier series sum, specifically an integral. The analysis equation is the same one we used previously in obtaining the envelope of the Fourier series coefficients.
- While there was a duality in the expressions between the discrete-time Fourier series analysis and synthesis equations, the duality is lost in the discrete-time Fourier transform since the synthesis equation is now an integral and the analysis equation a summation. This is a difference compared to the continuous-time Fourier transform.
- Another important difference is that the discrete-time Fourier transform is always a periodic function of frequency.
- Consequently, it is completely defined by its behavior over a frequency range of 2π in contrast to the continuous-time Fourier transform, which extends over an infinite frequency range.

Approach

- Construct the periodic signal $\tilde{x}[n]$ for which one period is $x[n]$.
- $\tilde{x}[n]$ has a Fourier series.
- As the period of $\tilde{x}[n]$ increases,
  $\tilde{x}[n] \to x[n]$ and the Fourier series of $\tilde{x}[n]$ $\to$ Fourier transform of $x[n]$.



Na_k
$N = 10$
ω
$-\pi$
π
2π
Na_k
$N = 20$
ω
$-\pi$
π
2π
Na_k
$N = 40$
ω
$-\pi$
π
2π

Fourier Representation of Aperiodic Signals

- $x[n]$ aperiodic
  - Construct periodic signals $\tilde{x}[n]$ for which one period is $x[n]$
  - $\tilde{x}[n]$ has a Fourier series
- As period of $\tilde{x}[n]$ increases
  - $\tilde{x}[n] \longrightarrow x[n]$
  - $\tilde{x}[n] \longrightarrow$ Fourier transform of $x[n]$.

DT Fourier series (from last lecture):

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}.$$

If $x[n]$ is aperiodic, for the periodic signal $\tilde{x}[n]$ whose one period is $x[n]$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Since $x[n] = \tilde{x}[n]$ over a period that includes $-N_1 \leq n \leq N_2$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

we see that the coefficients a_k are proportional to the samples of $X(e^{j\omega})$,i.e.,

$$a_k = \frac{1}{N}X(e^{jk\omega_0})$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}.$$

Since $1/N = \omega_0/2\pi$,

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0.$$

As $N \longrightarrow \infty$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete-Time Fourier Transform

Synthesis:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Analysis

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] \overset{\mathscr{F}}{\leftrightarrow} X(e^{j\omega})$$

$$\begin{aligned} X(e^{j\omega}) &= \mathrm{Re}\{X(e^{j\omega})\} + j\mathrm{Im}\{X(e^{j\omega})\} \\ &= |X(e^{j\omega})| e^{\angle X(e^{j\omega})} \end{aligned}$$

Example

Obtain an expression for the DTFT of

$$x[n] = a^n u[n], |a| < 1.$$

Sketch the magnitude and phase of $X(e^{j\omega})$ for

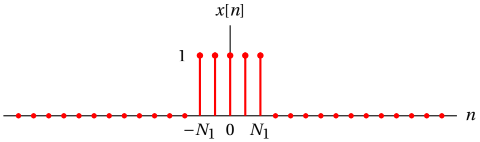
1. $a > 0$, $(a = 0.5)$ and
2. $a < 0$, $(a = -0.5)$.

Example

Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le N_1, \\ 0, & |n| > N_1. \end{cases}$$

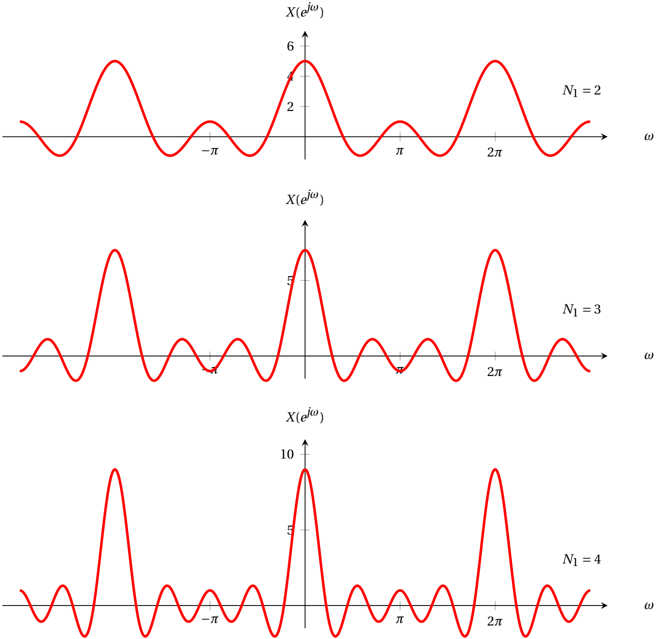
1. Obtain an expression for the DTFT $X(e^{j\omega})$ of this signal.
2. Sketch for $N_1 = 2$.



$$X(e^{j\omega}) = \sum_{-N_1}^{N_1} e^{-j\omega n}$$

As we did the simplification before for DTFS,

$$X(e^{j\omega}) = \frac{\sin\omega\left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}.$$



$X(e^{j\omega})$
6
4
2
$-\pi$
π
2π
$N_1 = 2$
ω
$X(e^{j\omega})$
5
$-\pi$
π
2π
$N_1 = 3$
ω
$X(e^{j\omega})$
10
5
$-\pi$
π
2π
$N_1 = 4$
ω

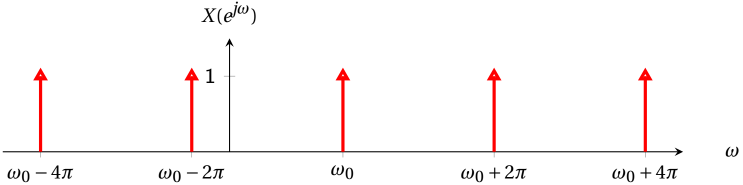
The Fourier Transform for Periodic Signals

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain.
Consider the signal

$$x[n] = e^{j\omega_0 n}.$$

In CT the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. However, the DT Fourier transform must be periodic in w with period π, requiring impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, ... So

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l).$$



Example

Consider the periodic signal

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}, \quad \text{with } \omega_0 = \frac{2\pi}{5}.$$

Example

The discrete-time counterpart of the periodic impulse train is the sequence

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

Find its discrete-time Fourier transform.

Periodicity

DTFT is always periodic in ω withe period 2π

$$x[n] \stackrel{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$$

Linearity

If

$$x_1[n] \overset{\mathscr{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \overset{\mathscr{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

then

$$ax_1[n] + bx_2[n] \overset{\mathscr{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time Shifting and Frequency Shifting

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

and

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Example

The frequency response of an ideal low-pass filter has the cutoff frequency of ω_c.

1. Obtain an expression for the frequency response of the corresponding high-pass filter (cutoff frequency $\pi - \omega_c$).
2. Obtain and expression for the impulse response of this high-pass filter in terms of the impulse response of the low-pass filter.

$H_{lp}(e^{j(\omega-\pi)})$ is the frequency response of $H_{lp}(e^{j\omega})$ shifted by one-half period, i.e., by π. Since high frequencies in discrete time are concentrated near π (and other odd multiples of π), the filter depicted in the second figure is an ideal highpass filter with cutoff frequency $\pi - \omega_c$.

$$\begin{aligned} h_{hp}[n] &= e^{j\pi n} h_{lp}[n] \\ &= (-1)^n h_{lp}[n]. \end{aligned}$$

Conjugation and Conjugate Symmetry

If

$$x[n] \stackrel{\mathscr{F}}{\leftrightarrow} X(e^{j\omega})$$

then

$$x^*[n] \stackrel{\mathscr{F}}{\leftrightarrow} X^*(e^{-j\omega}).$$

Also, if $x[n]$ is real-valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$X(e^{j\omega}) = X^*(e^{-j\omega}), \quad (x[n] \text{ real.})$$

$\mathrm{Re}\{X(e^{j\omega})\}$ is an even function of ω and $\mathrm{Im}\{X(e^{j\omega})\}$ is an odd function of ω.
The magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

Time Reversal

$$x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega}).$$

Example

Prove the time-reversal property.

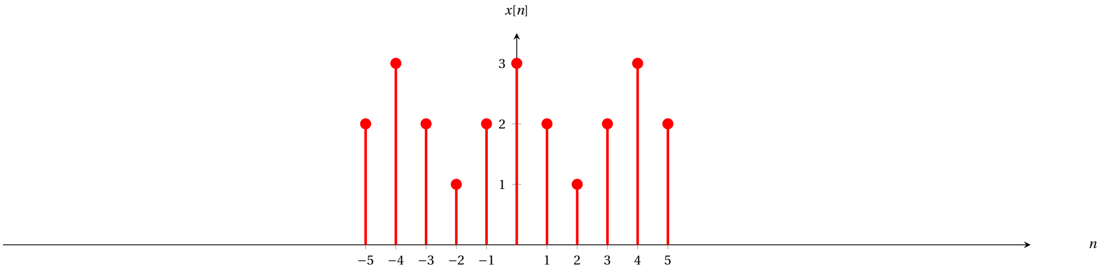
Time Expansion

Because of the discrete nature of the time index for discrete-time signals, the relation between time and frequency scaling in discrete time takes on a somewhat different form from its continuous-time counterpart. In CT

$$x(at) \overset{\mathscr{F}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

However, if we try to define the signal $x[an]$, we run into difficulties if a is not an integer. Therefore, we cannot slow down the signal by choosing $a<1$. On the other hand, if we let a be an integer other than ± 1—e.g., if we consider $x[2n]$—we do not merely speed up the original signal. That is, since n can take on only integer values, the signal $x[2n]$ consists of the even samples of $x[n]$ alone.
Let k be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k, \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$


$x[n]$
3
2
1
n
−5
−4
−3
−2
−1
1
2
3
4
5

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk]e^{-j\omega rk}$$

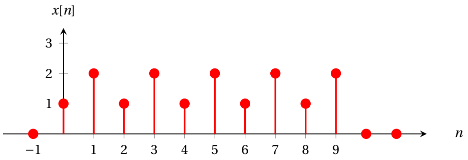
Furthermore, since $x_{(k)}[rk] = x[r]$,

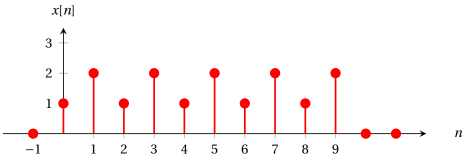
$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

$$x_{(k)}[n] \stackrel{\mathscr{F}}{\longleftrightarrow} X(e^{jk\omega})$$

Example

Find the DTFT of the signal $x[n]$ shown in the figure, expressing the signal in terms of a simpler signal.




$x[n]$
3
2
1
n
−1
1
2
3
4
5
6
7
8
9

Differentiation in Frequency

$$nx[n] \overset{\mathscr{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Example

Prove the differentiation property.

Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

The Convolution Property

If $x[n]$, $h[n]$ and $y[n]$ are the input, impulse response, and output respectively of an LTI system, so that

$$y[n] = x[n] * h[n]$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

Example

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0]$$

Obtain the output $y[n]$ for an input $x[n]$.

Example

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

and suppose that the input to this system is

$$x[n] = \beta^n u[n], \quad |\beta| < 1$$

Obtain the output $y[n]$ for $\alpha \neq \beta$ and $\alpha = \beta$.

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

If $\alpha = \beta$,

The Multiplication Property

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

This equation corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluation over any given interval of length 2π.
See example 5.15.

Summary

	CT	DT
	Series (CT)	Series (DT)
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}$	$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$
	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$	$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$
	Transform (CT)	Transform (DT)
Aperiodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

CT: $\omega_0 = \frac{2\pi}{T}$, DT: $\omega_0 = \frac{2\pi}{N}$

Section 4

Laplace Transform

Introduction

- Using the Fourier transform, we represented a signal as a linear combination of basic signals using the eigenfunctions $e^{j\omega t}$.
- Then we could represent a given LTI system as a spectrum of eigenvalues as a function of ω, which is the change in amplitude that the system applies to each of the basic inputs $e^{j\omega t}$.
- Now we study a generalization of the Fourier transform, referred to as the Laplace transform.
- The Laplace transform converges for a broader class of signals than does the Fourier transform.

The Laplace Transform

- The general class of eigenfunctions for LTI systems consists of the complex exponential e^{st}, where s is a complex number.
- When s is purely imaginary, $s = j\omega$, the Laplace transform reduces to the Fourier transform.
- The Laplace transform is the Fourier transform of an exponentially weighted signal. Therefore, the Laplace transform can converge for signals for which the Fourier transform does not converge.
- The range of values of s for which the Laplace transform converges is the region of convergence (ROC).
- Two different signals can have Laplace transforms with identical algebraic expressions and differing only in the ROC.

Recall: Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega t} & \rightarrow & H(\omega) e^{j\omega t} \\ & & \updownarrow \mathscr{F} \\ & & h(t) \end{array}$$

Laplace Transform: Eigenfunction Property

$$e^{st} \rightarrow \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$e^{st} \rightarrow e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$s = \sigma + j\omega$$

$$e^{st} \rightarrow H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$$

Laplace Transform and Fourier Transform Relationship

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$s = \sigma + j\omega$$

$$X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}$$

New notation:

$$\mathscr{F}\{x(t)\} = X(j\omega)$$

Laplace Transform: Convergence Comparison

$$
\begin{aligned}
X(s)|_{s=j\omega} &= X(j\omega) \\
X(s) &= \int_{-\infty}^{+\infty} x(t) e^{-st} dt \\
X(\sigma + j\omega) &= \int_{-\infty}^{+\infty} x(t) e^{-(\sigma + j\omega)t} dt \\
&= \int_{-\infty}^{+\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt \\
X(s) &= \mathscr{F}\left\{ x(t) e^{-\sigma t} \right\}
\end{aligned}
$$

LT may converge when FT does not.

Example

Find the LT of

$$x(t) = e^{-at}u(t).$$

Example

Find the LT of

$$x(t) = -e^{-at}u(-t).$$

Example

Find the LT of

$$x(t) = e^{-t}u(t) + e^{-2t}u(t).$$

Properties of the Region of Convergence

- The ROC contains no poles
- The ROC of $X(S)$ consists of s trip parallel to the $j\omega$ axis in the s-plane.
- $\mathscr{F}\{x(t)\}$ converges $\Leftrightarrow$ ROC includes the $j\omega$-axis in the s-plane.

Example

Sketch the choices of the ROC associated with

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

ROC of a Finite-Duration Signal

If $x(t)$ is a finite-duration signal, then the ROC is the entire s-plane.

ROC of a Right-Sided Signal

If $x(t)$ is right-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} > \sigma_0$ are in ROC.
If $x(t)$ is right-sided and $X(s)$ is rational, then ROC is the right of the rightmost pole.

ROC of a Left-Sided Signal

If $x(t)$ is left-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC.
If $x(t)$ is left-sided and $X(s)$ is rational, then ROC is the left of the leftmost pole.

ROC of a Two-Sided Signal

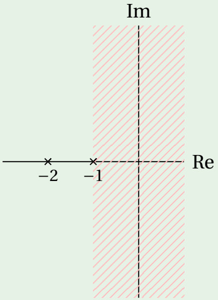
If $x(t)$ is two-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then ROC is the strip in the s-plane.

Example

A Laplace transform is specified by

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1.$$

Find the inverse laplace transform.



Example

Find the inverse laplace transform of

1. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} > -1,$$

2. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} < -3,$$

3. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad -3 < \mathrm{Re}\{s\} < -1,$$

Instead of having to reevaluate the transform of a given signal, we can simply refer to the Laplace transform table and and read out the desired transform.

$x(t)$	$X(s)$	ROC
$\delta(t)$		All s
$u(t)$		$\mathrm{Re}(s) > 0$
$-u(-t)$		$\mathrm{Re}(s) < 0$
$tu(t)$		$\mathrm{Re}(s) > 0$
$t^k u(t)$		$\mathrm{Re}(s) > 0$
$e^{-at}u(t)$		$\mathrm{Re}(s) > -\mathrm{Re}(s)$
$-e^{-at}u(-t)$		
$te^{-at}u(t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$

$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(s)$
$\cos\omega_0 tu(t)$	$\dfrac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
$\sin\omega_0 tu(t)$	$\dfrac{\omega_0}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos\omega_0 tu(t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$
$e^{-at}\sin\omega_0 tu(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(t) + a_2 X_2(t)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t - t_0)$		$R' = R$
Shifting in s	$e^{s_0 t} x(t)$		$R' = R + \mathrm{Re}(s_0)$
Time scaling	$x(at)$		$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\dfrac{dx(t)}{dt}$		$R' \supset R$

Differentiation in s	$-tx(t)$		$R' = R$
Integration	$\int_{-\infty}^{t} x(\tau)\tau$	$\frac{1}{s}X(s)$	$R' \supset R\{\mathrm{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$		$R' \supset R_1 \cap R_2$

Example

Verify the time-shifting property

$$x(t-t_0) \longleftrightarrow e^{-st_0}X(s), \quad R' = R.$$

$$\mathscr{L}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt$$

By the change of variables $\tau = t - t_0$, we obtain

$$\begin{aligned}\mathscr{L}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}dt \\ &= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}dt \\ &= e^{-st_0}X(s)\end{aligned}$$

with the same ROC as for $X(s)$ itself.

Example

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the Laplace transform of $u(t)$.

1. $\delta(t)$
2. $\delta'(t)$
3. $tu(t)$
4. $e^{-at}u(t)$
5. $te^{-at}u(t)$
6. $\cos\omega_0 tu(t)$
7. $e^{-at}\cos\omega_0 tu(t)$

Introduction

- The properties of the Laplace transform make it useful in analyzing LTI systems that are represented by linear constant-coefficient differential equations.
- Applying the Laplace transform to a differential equation converts it to an algebraic equation relating the Laplace transform of the system output to the product of the Laplace transform of the system input and the Laplace transform of the system impulse response, referred to as the system function.
- The system function is readily obtained by inspection of the differential equation, and the system impulse response can be obtained by evaluating the inverse Laplace transform of the system function.
- Alternatively, the response for any other input can be evaluated by first multiplying the Laplace transform of the input by the system function and then applying the inverse Laplace transform.

First- and Second-Order Systems

- Two particularly important classes of systems described by linear constant-coefficient differential equations are first-order and second-order systems.
- In implementing higher-order systems, it is very common to use first and second-order systems as building blocks.
- First-order systems are represented by a single pole in the s-plane, and second-order systems by a pair of poles. There may or may not also be zeros in the transfer function, depending on whether there are derivative terms on the right-hand side of the differential equation.
- From the differential equation, the system function can be written directly.
- If we assume that the systems are causal, so that the impulse response is right-sided, then the ROC of the system function is implicitly specified to be to the right of the rightmost pole in the s-plane.

Recall: Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$x(t) \leftrightarrow X(s)$$

$$X(s)|_{s=j\omega} = X(j\omega) = \mathscr{F}\{x(t)\}$$

$$s = \sigma + j\omega$$

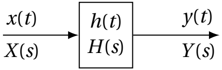
$$X(s) = \mathscr{F}\left\{x(t)e^{-\sigma t}\right\}$$

LT converges for some values of σ and not others: ROC.

Properties

$$ax_1(t) + bx_2(t) \stackrel{\mathscr{L}}{\longleftrightarrow} aX_1(s) + bX_2(s).$$

$$\frac{dx(t)}{dt} \stackrel{\mathscr{L}}{\longleftrightarrow} sX(s).$$



$$Y(s) = H(s)X(s).$$

Stable, causal $\Leftrightarrow$ all poles in left-half s-plane

First-Order System

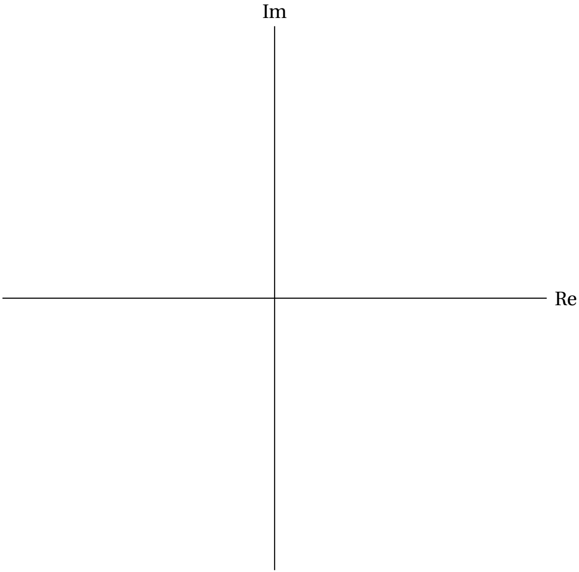
$$
\begin{array}{ccccc}
\dfrac{dy(t)}{dt} & + & ay(t) & = & x(t) \\
\downarrow & & \downarrow & & \downarrow \\
sY(s) & + & aY(s) & = & X(s)
\end{array}
$$

Second-Order System

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

For $\zeta < 1$,

$$\begin{aligned} c_1 &= c_2^* \\ &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \end{aligned}$$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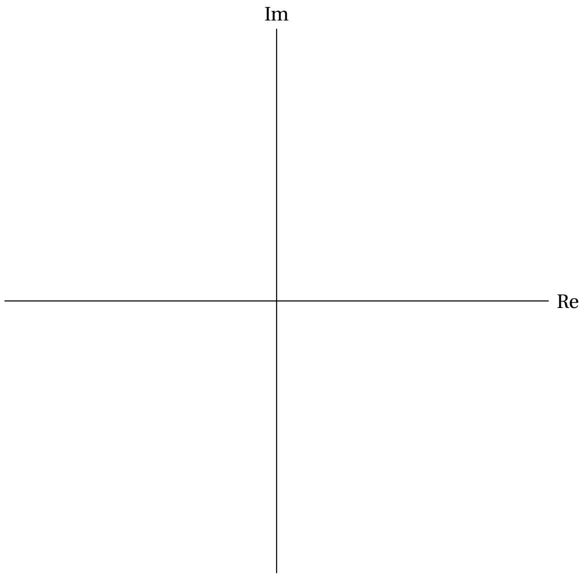
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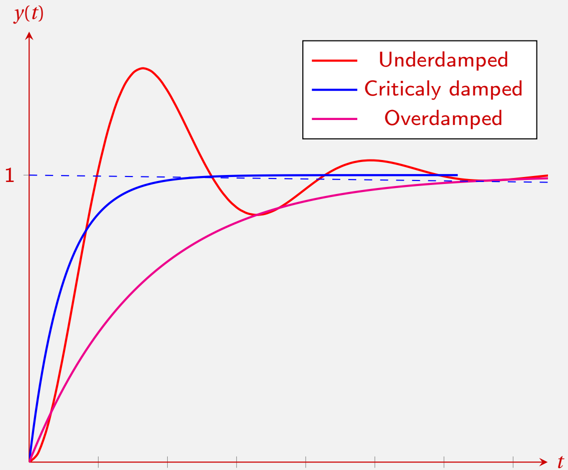


Figure: Second-order system responses.

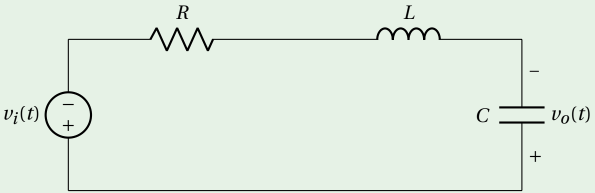
Example

The transfer function of a network is

$$H(s) = \frac{s+10}{s^2+4s+8}$$

Determine the pole-zero plot of $H(s)$, the type of damping exhibited by the network, and the unit step response of the.

Example



Consider an *RLC* series network.

1. Obtain the voltage transfer function.
2. If $\omega_n = 2000\,\mathrm{rad/s}$ and $zeta = 0.25, 0.50, 0.75$, and 1.0, sketch the pole-zero plots.
3. Sketch the step response for each case.

Introduction to The Unilateral Laplace Transform

- In the preceding sections, we have dealt with what is commonly called the bilateral Laplace transform.
- In this section, we briefly study the unilateral Laplace transform.
- It is of considerable value in analyzing causal systems and, particularly, systems specified by linear constant-coefficient differential equations with nonzero initial conditions (i.e., systems that are not initially at rest).

The Unilateral Laplace Transform

$$\mathscr{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt$$

where the lower limit of integration, 0^-, signifies that we include in the interval of integration any impulses or higher order singularity functions concentrated at $t = 0$.

$$x(t) \xleftrightarrow{\mathscr{UL}} \mathscr{X}(s)$$

The system analysis tools and system function algebra developed and used in this lecture apply without change to unilateral transforms, as long as we deal with causal LTI systems (for which the system function is both the bilateral and the unilateral transform of the impulse response) with inputs that are identically zero fort $t < 0$.

Example

A causal LTI system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Suppose that the system is at initial rest.

1. Find the system function $\mathscr{H}(s)$.
2. Find the Laplace transform of the output $\mathscr{Y}(s)$ if the input is $x(t) = \alpha u(t)$.
3. Find the output $y(t)$.

Section 5

z-Transform

Introduction

- We developed the Laplace transform as a generalization of the continuous-time Fourier transform.
- In this lecture, we introduce the corresponding generalization of the discrete-time Fourier transform.
- The resulting transform is referred to as the z-transform.

z-Transform Motivation

- The discrete-time Fourier transform developed out of choosing complex exponentials as basic building blocks for signals because they are eigenfunctions of discrete-time LTI systems.
- A more general class of eigenfunctions consists of signals of the form z^n, where z is a general complex number. A representation of discrete-time signals with these more general exponentials leads to the z-transform.

Relationship between the z-Transform and the Discrete-Time Fourier Transform

- We saw that the Laplace transform is a generalization of the continuous-time Fourier transform.
- A close relationship exists between the z-transform and the discrete-time Fourier transform.
- For $z = e^{j\omega}$ or, equivalently, for the magnitude of z equal to unity, the z-transform reduces to the Fourier transform.
- More generally, the z-transform can be viewed as the Fourier transform of an exponentially weighted sequence.
- Because of this, the z-transform may converge for a given sequence even if the Fourier transform does not: the z-transform offers the possibility of transform analysis for a broader class of signals and systems.

The Region of Convergence (ROC)

- The z-transform of a signal too has associated with it both a range of values of z, referred to as the region of convergence (ROC), for which this expression is valid.
- Two different sequences can have z-transforms with identical algebraic expressions such that their z-transforms differ only in the ROC.
- Consequently, the ROC is an important part of the specification of the z-transform.

z-Plane

- z-transforms of the form of a ratio of polynomials in z^{-1} are described by poles and zeros in the complex plane, referred to as the z-plane.
- The circle of radius 1, concentric with the origin in the z-plane, is referred to as the unit circle.
- Since this circle corresponds to the magnitude of z equal to unity, it is the contour in the z-plane on which the z-transform reduces to the Fourier transform.
- In contrast, for continuous time it is the imaginary axis in the s-plane on which the Laplace transform reduces to the Fourier transform.
- If the sequence is known to be right-sided, for example, then the ROC must be the portion of the z-plane outside the circle bounded by the outermost pole.

Recall: Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{-\infty}^{+\infty} x[n]e^{-j\omega n}$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega n} & \rightarrow & H(e^{j\omega})e^{j\omega n} \\ & & \updownarrow \mathscr{F} \\ & & h[n] \end{array}$$

z-Transform: Eigenfunction Property

$$z^n \rightarrow \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$z^n \rightarrow z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$z = re^{j\omega}$$

$$z^n \rightarrow H(z) z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

z-Transform and Fourier Transform Relationship

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$z = re^{j\omega}$$

$$X(z)|_{z=e^{j\omega}} = \mathscr{F}\{x[n]\}$$

New notation:

$$\mathscr{F}\{x[n]\} = X(e^{j\omega})$$

z-Transform: Convergence Comparison

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega})$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] \left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] r^{-n} re^{-j\omega n}$$

$$X(z) = \mathscr{F}\left\{x[n] r^{-n}\right\}$$

ZT may converge when FT does not.

Example

Find the ZT of $x[n] = a^n u[n]$.

Example

Find the ZT of

$$x[n] = -a^n u[-n-1].$$

z-Plane and the Unit Circle

Pole-Zero Plot for a Right-Handed Sequence

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

Pole-Zero Plot for a Left-Handed Sequence

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

First-Order Difference Equation

$$y[n] - ay[n-1] = x[n]$$

Pole-Zero Plot for a DT First-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{z}{z-a}, \quad |z| > |a|.$$

Second-Order Difference Equation

$$y[n] + 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

Pole-Zero Plot for a DT Under-Damped Second-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}, \quad |z| > |a|.$$

Properties of the ROC of the z-Transform

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
- $\mathscr{F}\{x[n]\}$ converges $\Leftrightarrow$ ROC includes the unit circle in the z-plane
- $x[n]$ finite duration $\Rightarrow$ ROC is entire z-plane with the possible exception of $z=0$ or $z=\infty$

Properties of the ROC for a Right-Sided Sequence

- $x[n]$ right-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all finite values of z for which $|z| > r_0$ are in ROC.
- $x[n]$ right-sided and $X(z)$ rational $\Rightarrow$ ROC is outside the outermost pole.

Properties of the ROC for a Left-Sided and for a Two-Sided Sequence

- $x[n]$ left-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all values of z for which $0 < |z| < r_0$ will also be in ROC.
- $x[n]$ left-sided and $X(z)$ rational $\Rightarrow$ ROC is inside the innermost pole.
- $x[n]$ two-sided and $|z| = r_0$ is in ROC $\Rightarrow$ ROC is a ring in the z-plane which includes the circle $|z| = r_0$.

Example

Show the choices of the ROC for

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Right-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Left-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}.$$

ROC If the Sequence Is Two-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

Inverse z-Transform

$$
\begin{aligned}
X(z) &= \mathscr{F}\left\{x[n]r^{-n}\right\} \\
x[n]r^{-n} &= \mathscr{F}^{-1}\{X(z)\} \\
&= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{-j\omega n}d\omega \\
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^{n}d\omega \\
z &= re^{j\omega}, \qquad dz = jre^{j\omega}d\omega \\
x[n] &= \frac{1}{2\pi j}\oint X(z)z^{n-1}dz
\end{aligned}
$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}, \quad |z| > 2.$$

Recall: z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{-n} dz$$

$$X(z)|_{z=e^{j\Omega}} = \mathscr{F}\{x[n]\}$$

$$z = re^{j\Omega}$$

$$X(z) = \mathscr{F}\left\{x[n] r^{-n}\right\}$$

z-Transform Properties

Property	Signal	Transform	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R
Scaling in z domain	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\|z_0\|R$
	$e^{j\omega n}x[n]$	$X(e^{-j\omega n}z)$	R
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$, the set of points $\{\|a\|z\|\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R).
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r.	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	at least $R_1 \cap R_2$

z-Transform Properties II

Property	Signal	Transform	ROC
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection R and $\lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{\infty}$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection R and $\lvert z\rvert > 1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial value theorem:
If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$

Example

Consider and LTI system for which

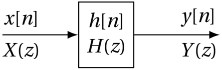
$$y[n] = h[n] * x[n],$$

where

$$h[n] = \delta[n] - \delta[n-1].$$

1. Find $H(z)$.
2. Find $y[n]$ in terms of $x[n]$.

System Stability



$$Y[n] = h[n] * x[n]$$
$$Y(z) = H(z)X(z)$$

$$\text{stable} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$
$$\mathscr{F}\{h[n]\} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

The condition for stability and the existence of the Fourier transform are the same.

Stability, Causality, and ROC

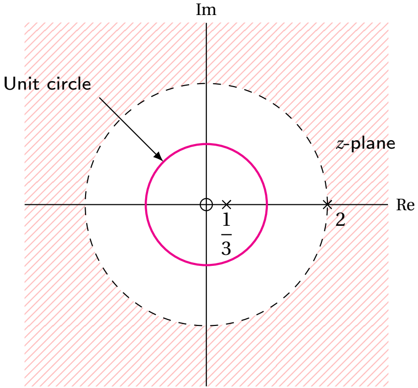
$$
\begin{aligned}
\text{stable} &\Leftrightarrow \text{ROC of } H(z) \text{ includes unit circle in } z\text{-plane} \\
\text{causal} &\Rightarrow h[n] \text{ is right-sided} \\
&\Rightarrow \text{ROC of } H(z) \text{ outside the outermost pole} \\
\text{causal and stable} &\Leftrightarrow \text{ All poles inside unit circle}
\end{aligned}
$$

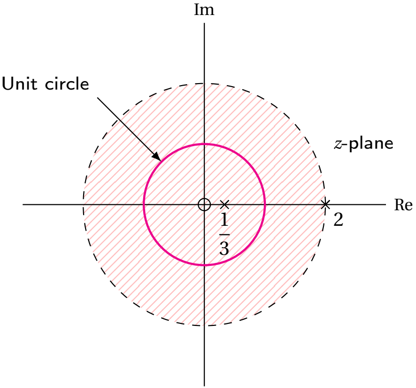
Example

Discuss the stability and causality of the system represented by the following system function with respect to different regions of convergence.

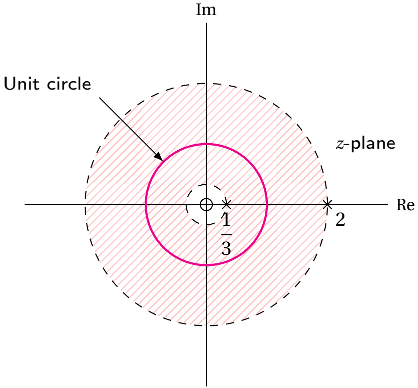
$$H(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

The system is causal and unstable.





The system is unstable and not causal.



The system is stable and not causal.

Example

Consider the LTI system for which the input $x[n]$ and the output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

1. Obtain an expression for the system function $H(z)$.
2. What are the two choices for the region of convergence?
3. Obtain $h[n]$ for each of these cases and comment on the stability and causality.

Section 6

Sampling and Reconstruction

Fourier Transform for Periodic Signals

- We studied the Fourier transform for aperiodic signals. We can also develop Fourier transform representations for periodic signals, thus allowing us to consider both periodic and aperiodic signals within a unified context.
- We can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of impulses proportional to the Fourier series coefficients.

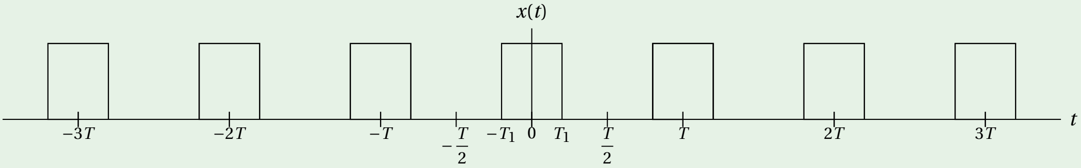
Let us consider the signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0).$$

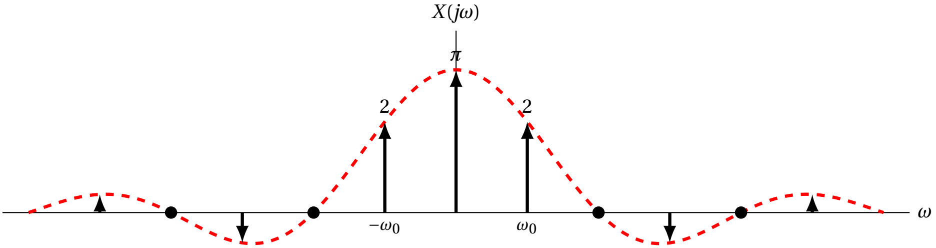
To determine the signal $x(t)$ we can apply the inverse Fourier transform relation
This corresponds exactly to the Fourier series representation of a periodic signal.
Thus the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at harmonically related frequencies for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the $k$the Fourier series coefficient a_k.

Example

Consider the square wave.



1. Obtain the Fourier series coefficients $\{a_k\}$.
2. Obtain the Fourier transform of the signal.



$X(j\omega)$
π
2
2
$-\omega_0$
ω_0
ω

Example

Sketch the Fourier transform of

1. $x(t) = \sin \omega_0 t$
2. $x(t) = \cos \omega_0 t$

Example

Obtain the Fourier transform of the impulse train given by

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

which is periodic with period T. Sketch.

Sampling

- The sampling theorem, which is a relatively straightforward consequence of the modulation theorem, is elegant in its simplicity.
- It states that a bandlimited time function can be exactly reconstructed from equally spaced samples provided that the sampling rate is sufficiently high—specifically, that it is greater than twice the highest frequency present in the signal.
- A similar result holds for both continuous time and discrete time.
- One of the important consequences of the sampling theorem is that it provides a mechanism for exactly representing a bandlimited continuous-time signal by a sequence of samples, that is, by a discrete-time signal.
- The reconstruction procedure consists of processing the impulse train of samples by an ideal lowpass filter.

Sampling Frequency and Aliasing

- Assumption: the sampling frequency is greater than twice the highest frequency in the signal.
- The reconstructing lowpass filter will always generate a reconstruction consistent with this constraint, even if the constraint was purposely or inadvertently violated in the sampling process.
- Said another way, the reconstruction process will always generate a signal that is bandlimited to less than half the sampling frequency and that matches the given set of samples.
- If the original signal met these constraints, the reconstructed signal will be identical to the original signal.
- On the other hand, if the conditions of the sampling theorem are violated, then frequencies in the original signal above half the sampling frequency become reflected down to frequencies less than half the sampling frequency.
- This distortion is commonly referred to as aliasing, a name suggestive of the fact that higher frequencies (above half the sampling frequency) take on the alias of lower frequencies.

Sampling Theorem

- Equally spaced samples of $x(t)$

$$x(nT), \quad n = 0, \pm 1, \pm 2, \ldots$$

- $x(t)$ is band limited

$$X(\omega) = 0, \quad |\omega| > \omega_M$$

- If

$$\frac{2\pi}{T} \triangleq \omega_s > 2\omega_M$$

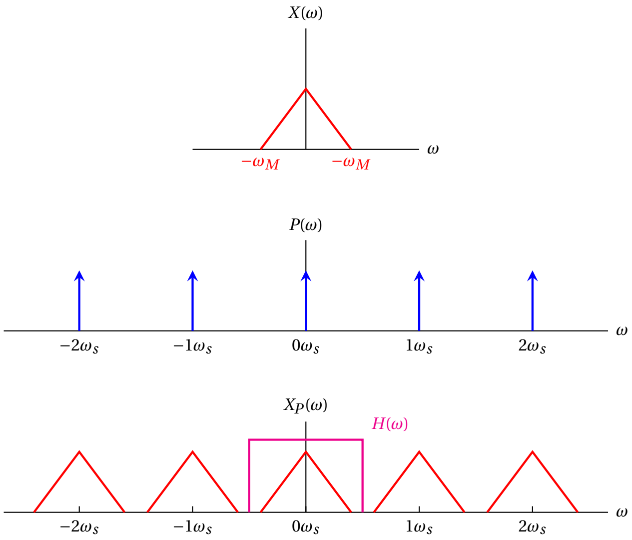
- then $x(t)$ is uniquely recoverable.

$$
\begin{aligned}
x_p(t) &= x(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)
\end{aligned}
$$

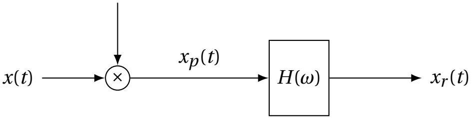
$$X_p(\omega) = \frac{1}{2\pi} [X(\omega) * P(\omega)]$$

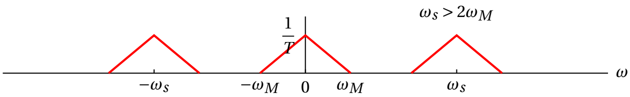
$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

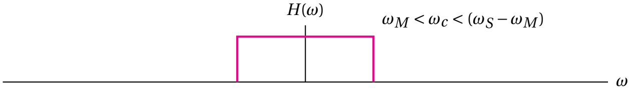
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$


$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
$H(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

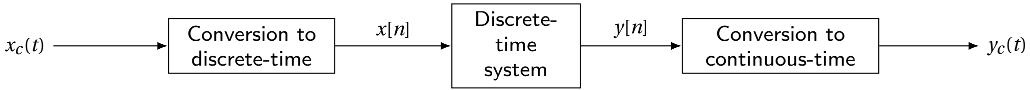
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

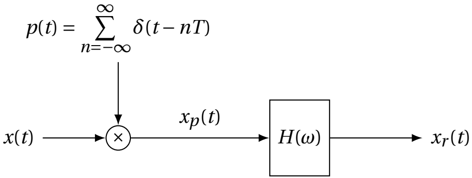






Discrete-Time Processing of Continuous-Time Signals





$$
\begin{aligned}
x_p(t) &= x(t)p(t) \\
&= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)
\end{aligned}
$$

$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)
\end{aligned}
$$