EN1060 Signals and Systems: Linear Time Invariant Systems

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Section 1

Laplace Transform

- Using the Fourier transform, we represented a signal as a linear combination of basic signals using the eigenfunctions $e^{j\omega t}$.
- Then we could represent a given LTI system as a spectrum of eigenvalues as a function of ω, which is the change in amplitude that the system applies to each of the basic inputs $e^{j\omega t}$.
- Now we study a generalization of the Fourier transform, referred to as the Laplace transform.
- The Laplace transform converges for a broader class of signals than does the Fourier transform.

- The general class of eigenfunctions for LTI systems consists of the complex exponential e^{st}, where s is a complex number.
- When s is purely imaginary, $s = j\omega$, the Laplace transform reduces to the Fourier transform.
- The Laplace transform is the Fourier transform of an exponentially weighted signal. Therefore, the Laplace transform can converge for signals for which the Fourier transform does not converge.
- The range of values of s for which the Laplace transform converges is the region of convergence (ROC).
- Two different signals can have Laplace transforms with identical algebraic expressions and differing only in the ROC.

Recall: Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega t} & \longrightarrow & H(\omega)e^{j\omega t} \\ & & \updownarrow \mathscr{F} \\ & & h(t) \end{array}$$

Laplace Transform: Eigenfunction Property

$$e^{st} \rightarrow \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$e^{st} \rightarrow e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$s = \sigma + j\omega$$

$$e^{st} \rightarrow H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$x(t) \xleftrightarrow{\mathscr{L}} X(s)$$

Laplace Transform and Fourier Transform Relationship

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$s = \sigma + j\omega$$

$$X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}$$

New notation:

$$\mathscr{F}\{x(t)\} = X(j\omega)$$

Laplace Transform: Convergence Comparison

$$
\begin{aligned}
X(s)|_{s=j\omega} &= X(j\omega) \\
X(s) &= \int_{-\infty}^{+\infty} x(t)e^{-st}dt \\
X(\sigma + j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}dt \\
&= \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt \\
X(s) &= \mathscr{F}\left\{x(t)e^{-\sigma t}\right\}
\end{aligned}
$$

LT may converge when FT does not.

Example

Find the LT of

$$x(t) = e^{-at}u(t).$$

Example

Find the LT of

$$x(t) = -e^{-at}u(-t).$$

Example

Find the LT of

$$x(t) = e^{-t}u(t) + e^{-2t}u(t).$$

Properties of the Region of Convergence

- The ROC contains no poles
- The ROC of $X(S)$ consists of s trip parallel to the $j\omega$ axis in the s-plane.
- $\mathscr{F}\{x(t)\}$ converges $\Leftrightarrow$ ROC includes the $j\omega$-axis in the s-plane.

Example

Sketch the choices of the ROC associated with

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

ROC of a Finite-Duration Signal

If $x(t)$ is a finite-duration signal, then the ROC is the entire s-plane.

ROC of a Right-Sided Signal

If $x(t)$ is right-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} > \sigma_0$ are in ROC.
If $x(t)$ is right-sided and $X(s)$ is rational, then ROC is the right of the rightmost pole.

ROC of a Left-Sided Signal

If $x(t)$ is left-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC.
If $x(t)$ is left-sided and $X(s)$ is rational, then ROC is the left of the leftmost pole.

ROC of a Two-Sided Signal

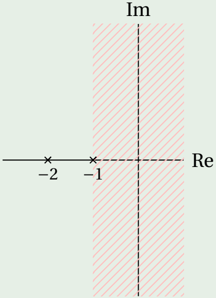
If $x(t)$ is two-sided and $\text{Re}\{s\} = \sigma_0$ is in ROC, then ROC is the strip in the s-plane.

Example

A Laplace transform is specified by

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1.$$

Find the inverse laplace transform.



Example

Find the inverse laplace transform of

1. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} > -1,$$

2. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} < -3,$$

3. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad -3 < \mathrm{Re}\{s\} < -1,$$

Instead of having to reevaluate the transform of a given signal, we can simply refer to the Laplace transform table and and read out the desired transform.

$x(t)$	$X(s)$	ROC
$\delta(t)$		All s
$u(t)$		$\mathrm{Re}(s) > 0$
$-u(-t)$		$\mathrm{Re}(s) < 0$
$tu(t)$		$\mathrm{Re}(s) > 0$
$t^k u(t)$		$\mathrm{Re}(s) > 0$
$e^{-at}u(t)$		$\mathrm{Re}(s) > -\mathrm{Re}(s)$
$-e^{-at}u(-t)$		
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$

$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) < -\text{Re}(s)$
$\cos\omega_0 tu(t)$	$\frac{s}{s^2+\omega^2}$	$\text{Re}(s) > 0$
$\sin\omega_0 tu(t)$	$\frac{\omega_0}{s^2+\omega^2}$	$\text{Re}(s) > 0$
$e^{-at}\cos\omega_0 tu(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\text{Re}(s) > -\text{Re}(s)$
$e^{-at}\sin\omega_0 tu(t)$	$\frac{\omega_0}{(s+a)^2+\omega^2}$	$\text{Re}(s) > -\text{Re}(s)$

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(t) + a_2 X_2(t)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t - t_0)$		$R' = R$
Shifting in s	$e^{s_0 t} x(t)$		$R' = R + \mathrm{Re}(s_0)$
Time scaling	$x(at)$		$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\dfrac{dx(t)}{dt}$		$R' \supset R$
Differentiation in s	$-tx(t)$		$R' = R$

Integration	$\int_{-\infty}^{t} x(\tau)\tau$	$\frac{1}{s}X(s)$	$R' \supset R\{\mathrm{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$		$R' \supset R_1 \cap R_2$

Example

Verify the time-shifting property

$$x(t-t_0) \longleftrightarrow e^{-st_0}X(s), \quad R' = R.$$

$$\mathscr{L}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt$$

By the change of variables $\tau = t - t_0$, we obtain

$$\begin{aligned}\mathscr{L}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}dt \\ &= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}dt \\ &= e^{-st_0}X(s)\end{aligned}$$

with the same ROC as for $X(s)$ itself.

Example

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the Laplace transform of $u(t)$.

1. $\delta(t)$
2. $\delta'(t)$
3. $tu(t)$
4. $e^{-at}u(t)$
5. $te^{-at}u(t)$
6. $\cos\omega_0 tu(t)$
7. $e^{-at}\cos\omega_0 tu(t)$

Section 2

Analysis of LTI Systems Using the Laplace Transform

- The properties of the Laplace transform make it useful in analyzing LTI systems that are represented by linear constant-coefficient differential equations.
- Applying the Laplace transform to a differential equation converts it to an algebraic equation relating the Laplace transform of the system output to the product of the Laplace transform of the system input and the Laplace transform of the system impulse response, referred to as the system function.
- The system function is readily obtained by inspection of the differential equation, and the system impulse response can be obtained by evaluating the inverse Laplace transform of the system function.
- Alternatively, the response for any other input can be evaluated by first multiplying the Laplace transform of the input by the system function and then applying the inverse Laplace transform.

First- and Second-Order Systems

- Two particularly important classes of systems described by linear constant-coefficient differential equations are first-order and second-order systems.
- In implementing higher-order systems, it is very common to use first and second-order systems as building blocks.
- First-order systems are represented by a single pole in the s-plane, and second-order systems by a pair of poles. There may or may not also be zeros in the transfer function, depending on whether there are derivative terms on the right-hand side of the differential equation.
- From the differential equation, the system function can be written directly.
- If we assume that the systems are causal, so that the impulse response is right-sided, then the ROC of the system function is implicitly specified to be to the right of the rightmost pole in the s-plane.

Recall: Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$x(t) \leftrightarrow X(s)$$

$$X(s)|_{s=j\omega} = X(j\omega) = \mathscr{F}\{x(t)\}$$

$$s = \sigma + j\omega$$

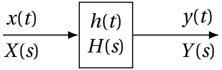
$$X(s) = \mathscr{F}\left\{x(t) e^{-\sigma t}\right\}$$

LT converges for some values of σ and not others: ROC.

Properties

$$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s).$$

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s).$$



$$Y(s) = H(s)X(s).$$

Stable, causal ⇔ all poles in left-half s-plane

First-Order System

$$\begin{array}{ccccc} \dfrac{dy(t)}{dt} & + & ay(t) & = & x(t) \\ \downarrow & & \downarrow & & \downarrow \\ sY(s) & + & aY(s) & = & X(s) \end{array}$$

Section 3

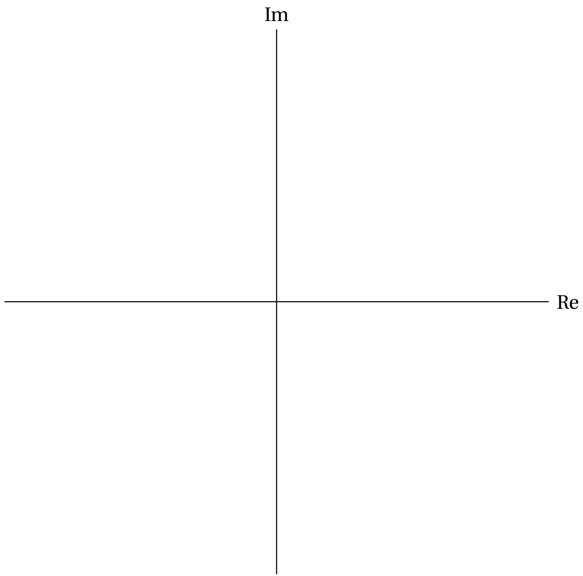
Continuous Time Second Order Systems

Second-Order System

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

For $\zeta < 1$,

$$\begin{aligned} c_1 &= c_2^* \\ &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \end{aligned}$$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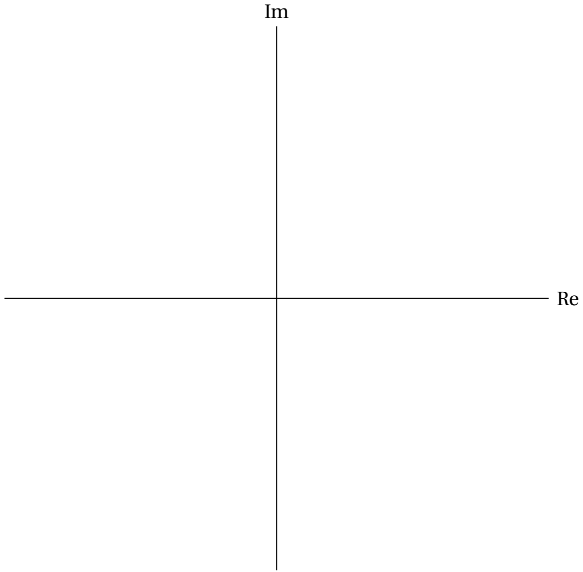
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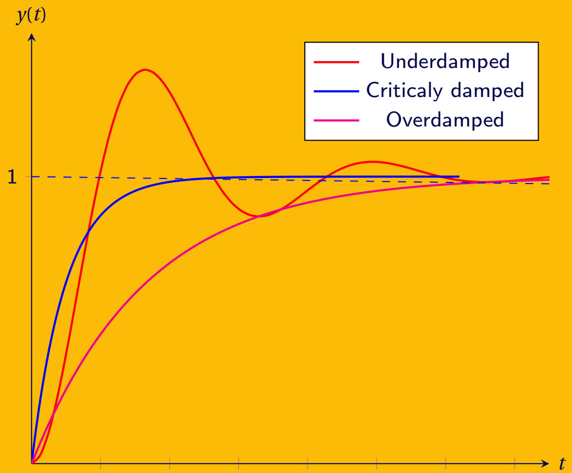


Figure: Second-order system responses.

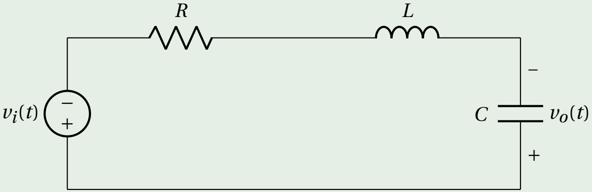
Example

The transfer function of a network is

$$H(s) = \frac{s+10}{s^2+4s+8}$$

Determine the pole-zero plot of $H(s)$, the type of damping exhibited by the network, and the unit step response of the.

Example



Consider an RLC series network.

1. Obtain the voltage transfer function.
2. If $\omega_n = 2000\,\text{rad/s}$ and $zeta = 0.25, 0.50, 0.75$, and 1.0, sketch the pole-zero plots.
3. Sketch the step response for each case.

Section 4

The Unilateral Laplace Transform

Introduction to The Unilateral Laplace Transform

- In the preceding sections, we have dealt with what is commonly called the bilateral Laplace transform.
- In this section, we briefly study the unilateral Laplace transform.
- It is of considerable value in analyzing causal systems and, particularly, systems specified by linear constant-coefficient differential equations with nonzero initial conditions (i.e., systems that are not initially at rest).

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t) e^{-st} dt$$

where the lower limit of integration, 0^-, signifies that we include in the interval of integration any impulses or higher order singularity functions concentrated at $t = 0$.

$$x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s)$$

The system analysis tools and system function algebra developed and used in this lecture apply without change to unilateral transforms, as long as we deal with causal LTI systems (for which the system function is both the bilateral and the unilateral transform of the impulse response) with inputs that are identically zero fort $t < 0$.

Example

A causal LTI system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Suppose that the system is at initial rest.

1. Find the system function $\mathscr{H}(s)$.
2. Find the Laplace transform of the output $\mathscr{Y}(s)$ if the input is $x(t) = \alpha u(t)$.
3. Find the output $y(t)$.