

EN1060 Signals and Systems: Fourier Transform

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Section 1

Continuous-Time Fourier Transform

① Continuous-Time Fourier Transform

Introduction

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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study linear, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

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- The synthesis integral itself, which uses these coefficients to represent the signal as a linear combination of complex exponentials, is called the inverse Fourier transform.

① Continuous-Time Fourier Transform

Introduction

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Fourier Series Representation for Square Wave

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodically repeats with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

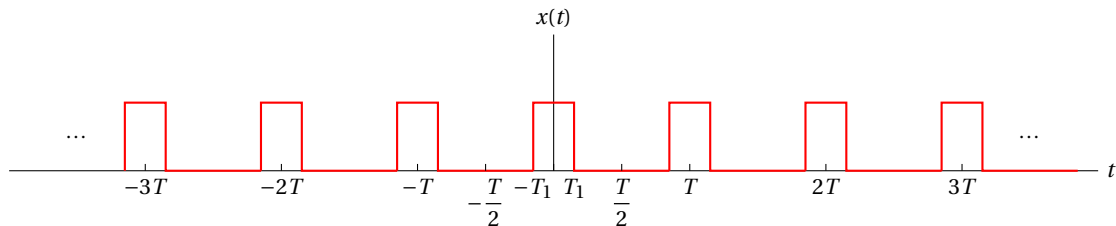


Figure: Periodic square wave

The Fourier series coefficients a_k of this wave are

$$a_k = \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T} \quad (1)$$

We plotted this for a fixed value of T_1 and several values of T (shown in the next slide). An alternative way of interpreting eq. 1 is as samples of an envelope function:

$$Ta_k = \left. \frac{2 \sin(\omega T_1)}{\omega} \right|_{\omega=k\omega_0}$$

For fixed T_1 , the envelope of Ta_k is independent of T .

Plots of scaled
Fourier series
coefficients Ta_k for
the periodic square
wave with T_1 fixed
and for several
values of T : $T = 4T_1$,
 $T = 8T_1$, $T = 16T_1$.

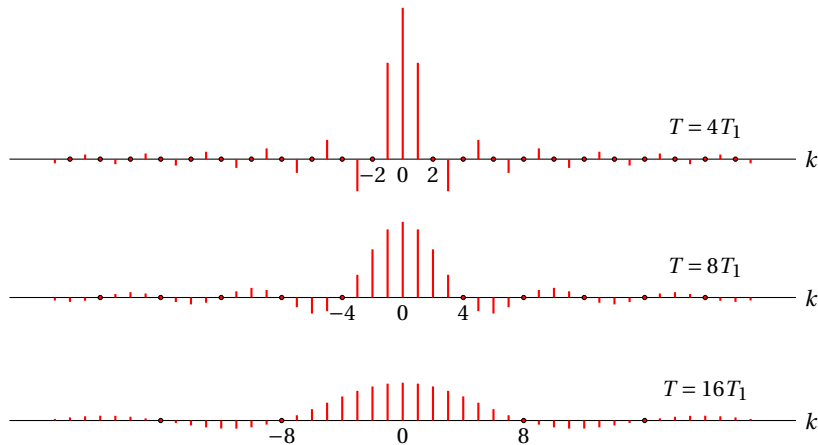
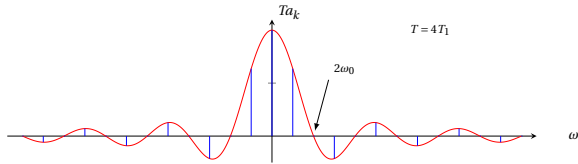
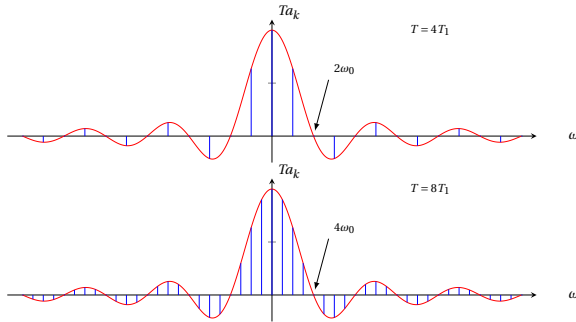


Figure: Plots of scaled Fourier series coefficients Ta_k



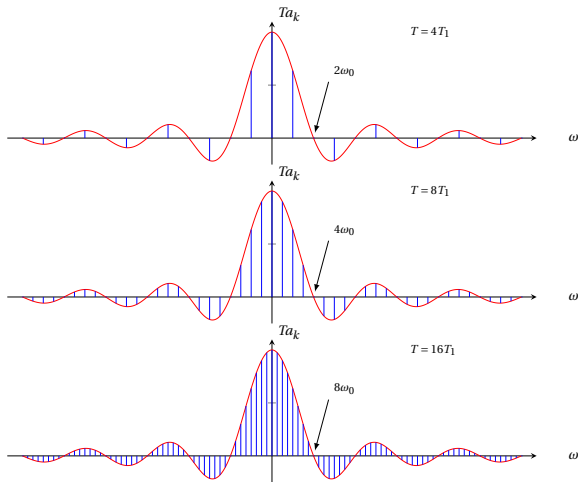
The Fourier series coefficients and their envelope for periodic square wave for several values of T (with T_1 fixed): $T = 4T_1$, $T = 8T_1$, $T = 16T_1$. The coefficients are regularly-spaced samples of the envelope $(2 \sin \omega T_1) / \omega$, where the spacing between samples, $2\pi / T$, decreases as T increases.

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Fourier Transform: Synthesis and Analysis Equations

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \quad (2)$$

Fourier transform or Fourier integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \quad (3)$$

The transform $X(j\omega)$ of an aperiodic signal $x(t)$ is referred to as the spectrum of $x(t)$.

Assume that the Fourier transform of $x(t)$ is $X(j\omega)$.

If we construct a periodic signal $\tilde{x}(t)$ by repeating the aperiodic signals $x(t)$ with period T , its Fourier series coefficients are

$$a_k = \frac{1}{T} X(j\omega) \Big|_{\omega=k\omega_0} \quad (4)$$

Convergence of Fourier Transform

Assume that we evaluated $X(j\omega)$ according to eq. 3, and let $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in 2:

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$$

When is $\hat{x}(t)$ a valid representation of the original signal $x(t)$? We define the error between $\hat{x}(t)$ and $x(t)$ as

$$e(t) = \hat{x}(t) - x(t).$$

If $x(t)$ has finite energy (square integrable), i.e.,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty, \quad (5)$$

$X(j\omega)$ is finite, and

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0, \quad (6)$$

If $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t , there is no energy in their difference.

Convergence of Fourier Transform: Dirichlet Conditions

There are alternative conditions sufficient to ensure that $\hat{x}(t)$ is equal to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

- ① $x(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty, \quad (7)$$

- ② $x(t)$ has a finite number of maxima and minima within any finite interval.
- ③ $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have finite number of discontinuities have a Fourier transform.

Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

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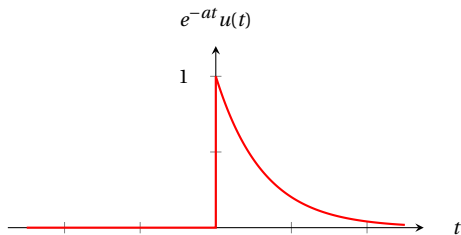


Figure: $e^{-at}u(t)$, $a > 0$.

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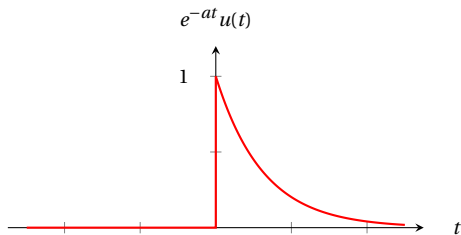


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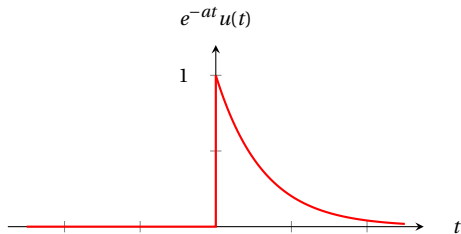


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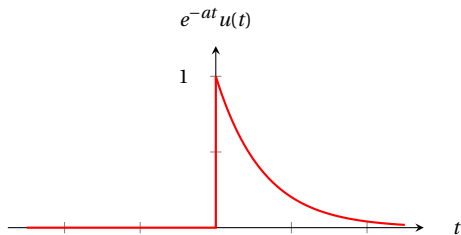


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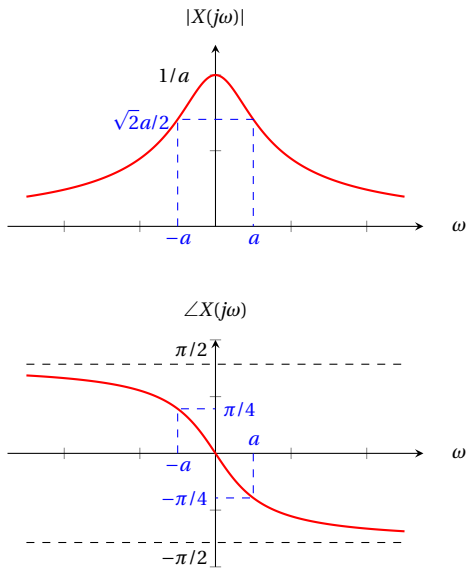


Figure: Fourier transform of the signal $e^{-at}u(t)$, $a > 0$.

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$$X(j\omega) = \frac{1}{a - j\omega} + \frac{1}{a + j\omega},$$

$$= \frac{2a}{a^2 + \omega^2}.$$

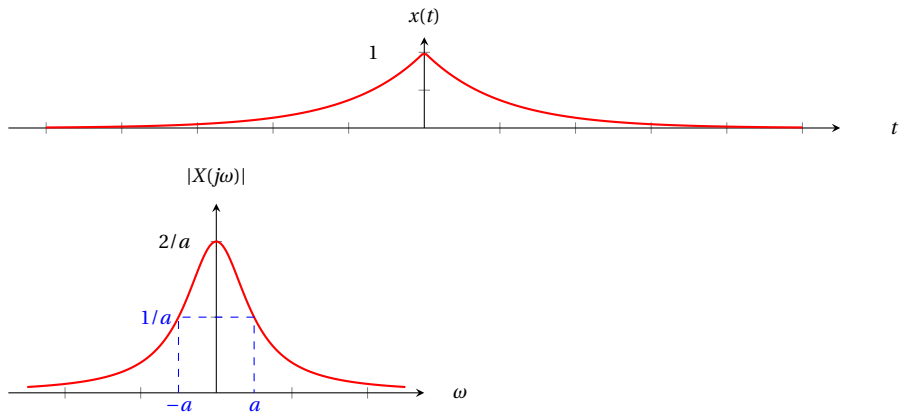


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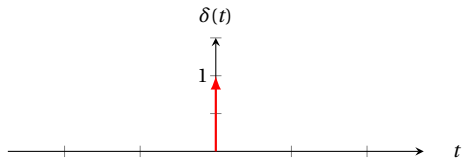


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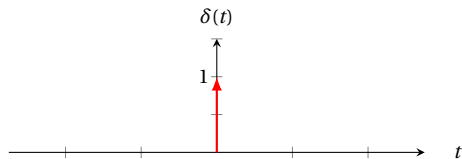


Figure: $\delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1.$$

The unit impulse has a Fourier transform consisting of equal contributions at all frequencies.

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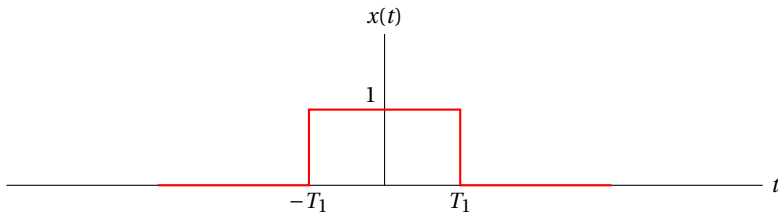


Figure: Rectangular pulse and the Fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt.$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \\ &= \int_{-T_1}^{T_1} e^{-j\omega t} dt. \end{aligned}$$

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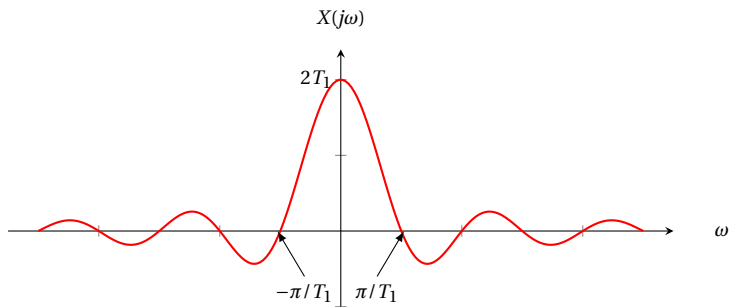


Figure: Fourier transform of the rectangular pulse.

Example

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine $x(t)$.

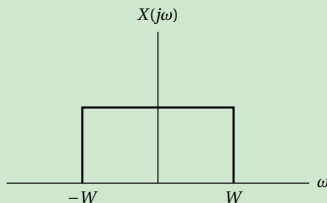


Figure: Fourier transform for $x(t)$.

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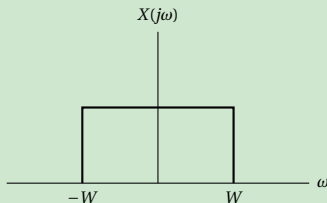


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Using the synthesis equation:

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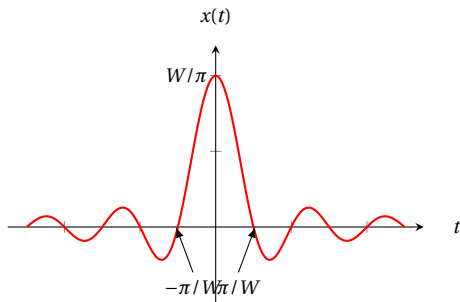


Figure: Time function.

$$\text{sinc}(\theta) = \frac{\sin \pi \theta}{\pi \theta}. \quad (8)$$

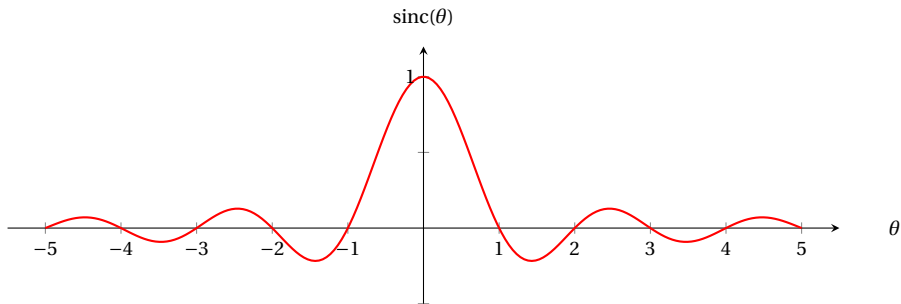


Figure: Fourier transform for $x(t)$.

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The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transform of a periodic signal directly from its Fourier series representation. Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$, i.e.,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \quad (9)$$

Let's determine the signal $x(t)$:

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$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega, \\ &= e^{j\omega_0 t}. \end{aligned}$$

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More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, i.e.,

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \quad (10)$$

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then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \quad (11)$$

which is exactly the Fourier series representation of a periodic signal.

Thus, the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the k th harmonic frequency $k\omega_0$ is 2π times the k th Fourier series coefficient a_k .

Example

Find the Fourier transform of the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0).$$

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Figure: Fourier transform of a symmetric periodic square wave.

Example

Find the Fourier transform of

$$x(t) = \sin \omega_0 t.$$

and

$$x(t) = \cos \omega_0 t.$$

Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$