

# EN1060 Signals and Systems: Discrete-Time Fourier Series

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# Systems with and without Memory

A system is said to be **memoryless** if its output for each value of the independent variable at a given time is dependent only on the input at the same time.

## Examples of memoryless systems

$$y[n] = (2x[n] - x^2[n])^2,$$

$$y(t) = Rx(t),$$

where  $x(t)$  current through the resistor  $R$  and  $y(t)$  taken as the voltage across the resistor.

$$y(t) = x(t),$$

which is called the **identity system**. In DT

$$y[n] = x[n].$$

## Examples of systems with memory

Accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].$$

Delay:

$$y[n] = x[n-1].$$

Capacitor with current as the input and the output taken as the voltage:

$$y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau.$$

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## Examples of systems with memory

Accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].$$

Delay:

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n], \quad y[n] = y[n-1] + x[n]$$
$$y[n] \equiv x[n-1].$$

Capacitor with current as the input and the output taken as the voltage:

$$y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau.$$

# Invertibility and Inverse Systems

A system is **invertible** if different inputs lead to different outputs. If a system is invertible, then an **inverse system** exists, and when cascaded with the original system, yields an output equal to the input to the first system.



## Examples of invertible systems:

If  $y(t) = 2x(t)$ , the inverse system is

$$w(t) = \frac{1}{2}y(t),$$

If (accumulator)  $y[n] = \sum_{k=-\infty}^{\infty} x[k]$ , the inverse system is

$$w[n] = y[n] - y[n-1].$$

## Examples of non-invertible systems:

$$y[n] = 0.$$

$$y(t) = x^2(t).$$

A system is said to be causal if it only responds when you “kick it.” Its response at any time depends only on that input prior or equal to that time. The system cannot anticipate future inputs.

### Example

$$x_1(t) \rightarrow y_1(t)$$

$$x_2(t) \rightarrow y_2(t)$$

If

$$x_1(t) = x_2(t), \quad t < t_0,$$

then

$$y_1(t) = y_2(t), \quad t < t_0.$$

If inputs are identical until  $t_0$ , the outputs are identical until  $t_0$ . Same for DT.

Many forms. We choose Bounded Input Bounded Output (BIBO) stability.

If a system is stable in BIBO sense, for every bounded input the output is bounded.

**Example**

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

is not stable.

The system does not really care what we call the origin. If the input is shifted by any amount of time  $t_0$ , the output is also shifted by the same amount of time.

If

$$x(t) \rightarrow y(t),$$

then

$$x(t - t_0) \rightarrow y(t - t_0).$$

If

$$x[n] \rightarrow y[n],$$

then

$$x[n - n_0] \rightarrow y[n - n_0].$$

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## Example

Accumulator:

$$y[n] = \sum_{k=-\infty}^n x[k]$$

## Example

Modulator:

$$y(t) = (\sin(t)x(t))$$



If

$$x_1(t) \rightarrow y_1(t)$$

$$x_2(t) \rightarrow y_2(t)$$

then

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

If the system is linear, if we give a linear combination of inputs, the output will also be a similar linear combination of the original outputs.

- ① Systems that are both linear and time invariant are called Linear Time-Invariant (LTI) systems.
- ② With systems that are linear and time invariant, using the impulse function in CT and DT, produces an important and useful mechanism for characterizing those system.
- ③ In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems.
- ④ The resulting representation is referred to as convolution.

# Introduction (From Oppenheim)

- A linear system: the response to a linear combination of inputs is the same linear combination of the individual responses.
- Time invariance: the system is not sensitive to the time origin. If the input is shifted in time by some amount, then the output is simply shifted by the same amount.
- For a linear system, if the system inputs can be decomposed as a linear combination of some basic inputs and the system response is known for each of the basic inputs, then the response can be constructed as the same linear combination of the responses to each of the basic inputs.
- Signals can be decomposed as a linear combination of basic signals in a variety of ways (e.g., Taylor series expansion that expresses a function in polynomial form.) However, in the context of signals and systems, it is important to choose the basic signals in the expansion so that in some sense the response is easy to compute.
- For systems that are both linear and time-invariant, there are two particularly useful choices for these basic signals: delayed impulses and complex exponentials.

# Introduction (From Oppenheim)

- In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems. The resulting representation is referred to as convolution.
- Earlier, we developed in detail the decomposition of signals as linear combinations of complex exponentials (referred to as Fourier analysis) and the consequence of that representation for linear, time-invariant systems.

- Using the convolution we can express the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse.
- An LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.
- In discrete time, we have the convolution sum. In continuous time, we have the convolution integral.

# Strategy for Exploiting Linearity and Time Invariance

- Decompose the input signal to a linear combination of basic signals.
  - Chose basic signals so that the response is easy to compute (analytical convenience).
- ① Delayed impulses  $\rightarrow$  convolution
  - ② Complex exponentials  $\rightarrow$  Fourier analysis

## ① Basic System Properties

## ② Convolution

The Discrete-Time Unit Impulse Response and the Convolution Sum

Continuous-Time Systems: The Convolution Integral

Continuous-Time Systems: The Convolution Integral

## ③ Properties of Linear Time-Invariant Systems

## ④ Linear, Constant-Coefficient Differential and Difference Equations

Linear, Constant-Coefficient Differential Equations

Linear, Constant-Coefficient Difference Equations

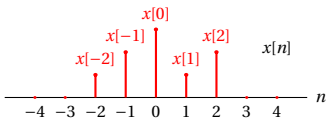
General block-Diagram Representation of the Recursive System

## ⑤ Revisiting Fourier Series

# A DT Signal as Superposition of Weighted Delayed Impulses

- We can express a DT signal as a linear combination of weighted delayed impulses.
- If we have a linear system, and a signal expressed as above as a linear combination of basic signals, the response would be the same linear combination of the responses for individual basic signals.





$$x[n] = x[-2]\delta[n+2]$$

$$x[-1]\delta[n+1]$$

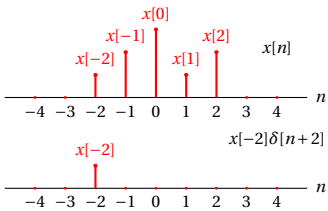
$$x[0]\delta[n]$$

$$x[1]\delta[n-1]$$

$$x[2]\delta[n-2]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

A linear combination of weighted delayed impulses.



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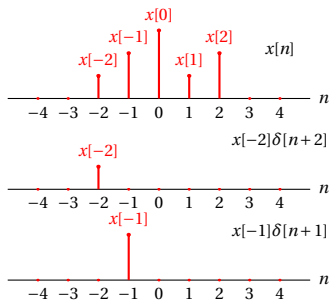
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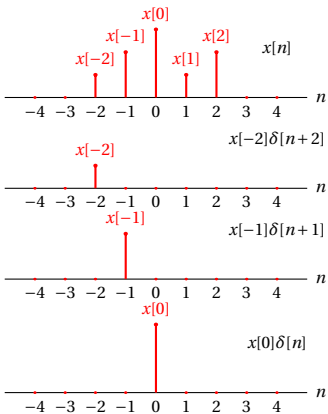
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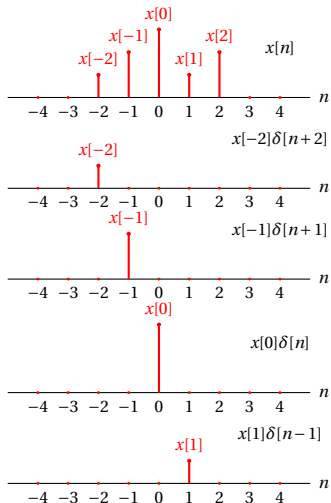
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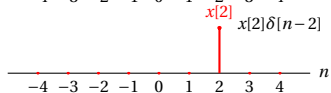
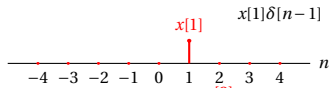
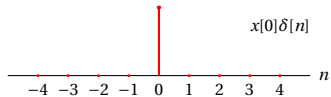
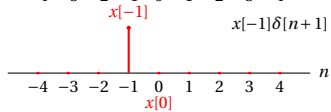
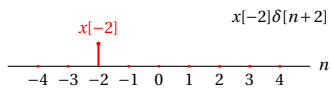
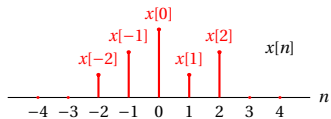
$$x[0]\delta[n]$$

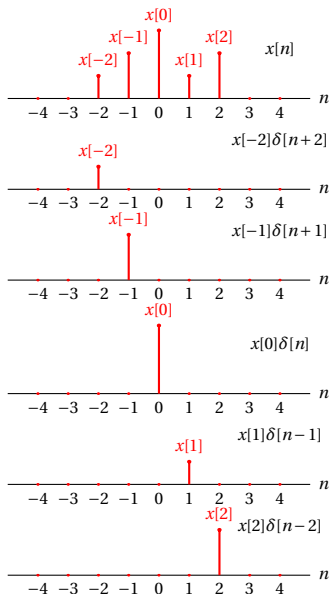
$$x[1]\delta[n-1]$$

$$x[2]\delta[n-2]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

A linear combination of weighted delayed impulses.





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A linear combination of weighted delayed impulses.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \quad \text{input.}$$



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Linear system

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \quad \text{input.}$$

Linear system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \quad \text{where } h_k[n] \text{ is the output due to the delayed impulse.}$$

$$\delta[n-k] \rightarrow h_k[n].$$

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If time invariant

$$h_k[n] = h_0[n-k] \quad \text{where } h_0 \text{ is the response of the system for an impulse at 0.}$$

$$h_0[n] = h[n] \quad \text{define.}$$

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If LTI

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{convolution sum.}$$

The convolution of the sequence  $x[n]$  and  $h[n]$  is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k], \quad (1)$$

which we represent symbolically as

$$y[n] = x[n] * h[n] \quad (2)$$

Compute  $y[n] = x[n] * h[n]$  for  $x[n]$  and  $h[n]$  as shown in Figure 2.

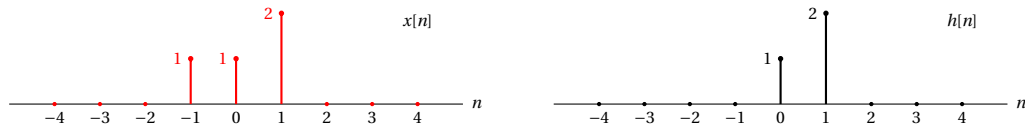


Figure: Computing convolution

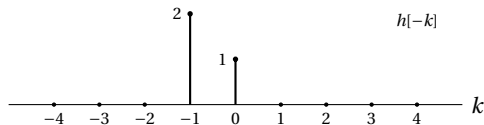
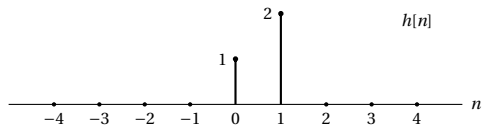
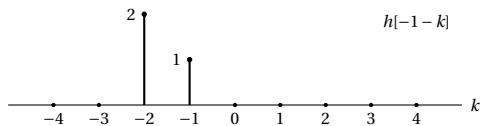
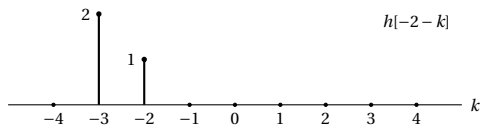
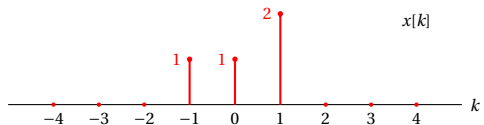
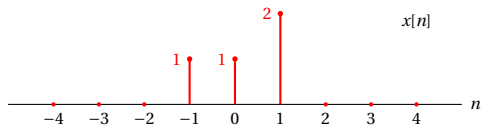


Figure: Computing convolution



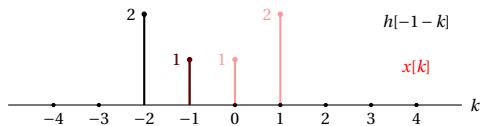
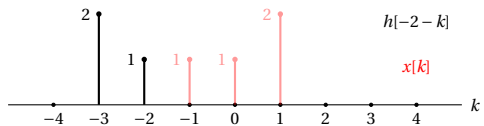
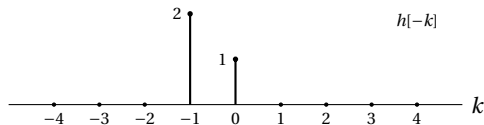
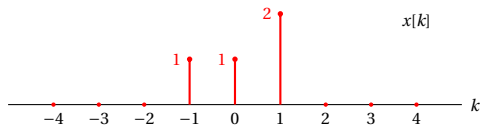
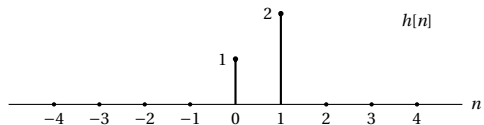
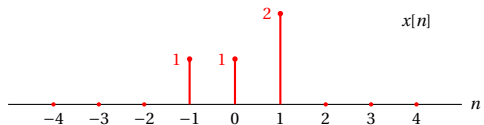
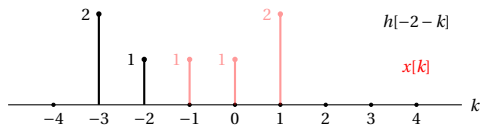
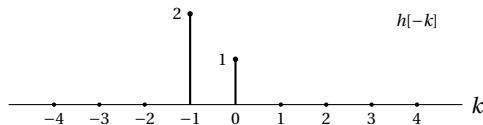
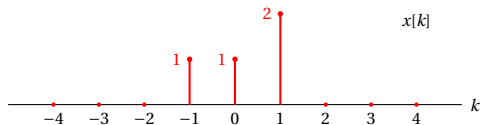
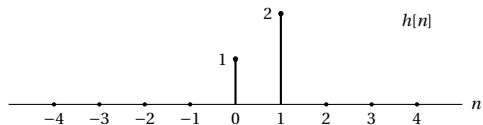
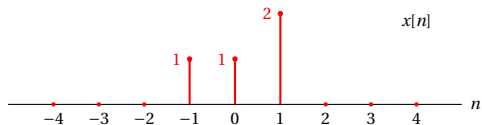
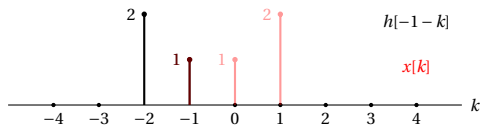


Figure: Computing convolution



$$y[-2] = 0$$



$$y[-1] = 1 \times 1 = 1$$

Figure: Computing convolution

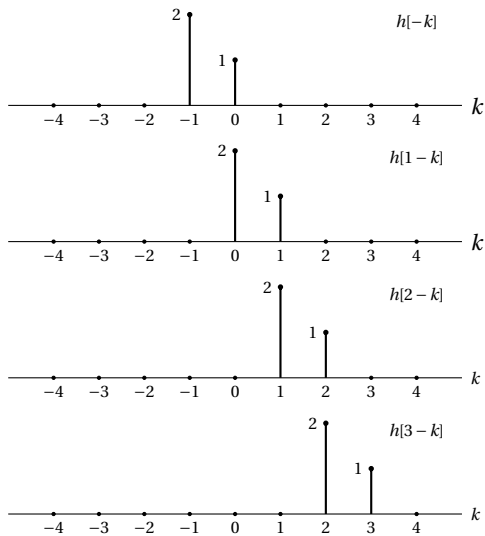


Figure: Computing convolution

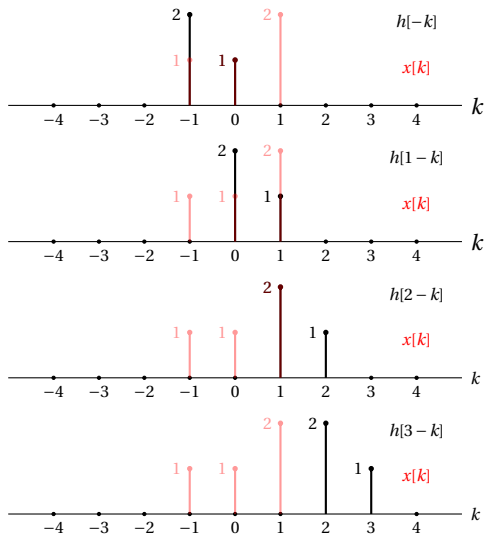
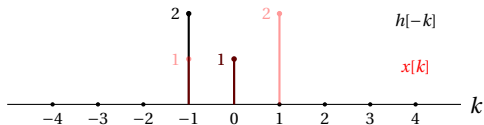
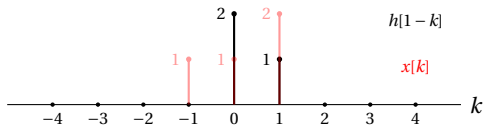


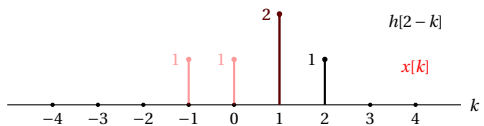
Figure: Computing convolution



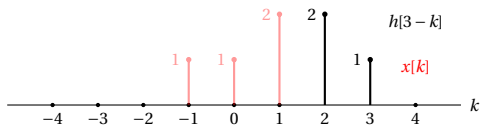
$$y[0] = 2 \times 1 + 1 \times 1 = 3$$



$$y[1] = 2 \times 1 + 1 \times 2 = 4$$

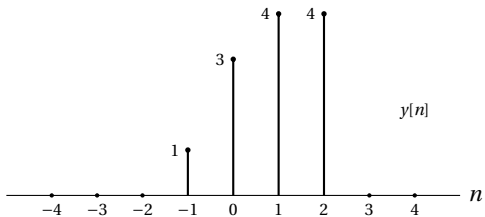


$$y[2] = 2 \times 2 = 4$$



$$y[3] = 0$$

Figure: Computing convolution



$$y[n] = \begin{cases} 0 & n \leq -2 \\ 1 & n = -1 \\ 3 & n = 0 \\ 4 & n = 1 \\ 4 & n = 2 \\ 0 & n \geq 3 \end{cases}$$

Figure: Computing convolution

Consider an input  $x[n]$  and a unit impulse response  $h[n]$  given by

$$\begin{aligned}x[n] &= \alpha^n u[n] \\ h[n] &= u[n],\end{aligned}\tag{3}$$

which  $0 < \alpha < 1$ . Find  $y[n]$  and sketch.

Figure: The signals  $x[n]$  and  $h[n]$  for the example.



## ① Basic System Properties

## ② Convolution

The Discrete-Time Unit Impulse Response and the Convolution Sum

**Continuous-Time Systems: The Convolution Integral**

Continuous-Time Systems: The Convolution Integral

## ③ Properties of Linear Time-Invariant Systems

## ④ Linear, Constant-Coefficient Differential and Difference Equations

Linear, Constant-Coefficient Differential Equations

Linear, Constant-Coefficient Difference Equations

General block-Diagram Representation of the Recursive System

## ⑤ Revisiting Fourier Series

# Continuous-Time Systems: The Convolution Integral

1

## ① Basic System Properties

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The Discrete-Time Unit Impulse Response and the Convolution Sum

Continuous-Time Systems: The Convolution Integral

**Continuous-Time Systems: The Convolution Integral**

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Linear, Constant-Coefficient Differential Equations

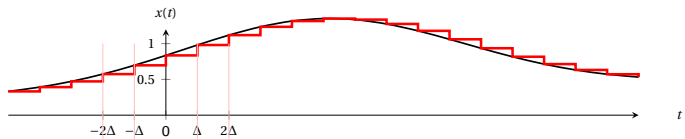
Linear, Constant-Coefficient Difference Equations

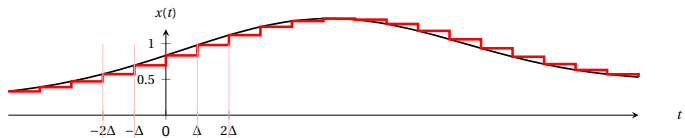
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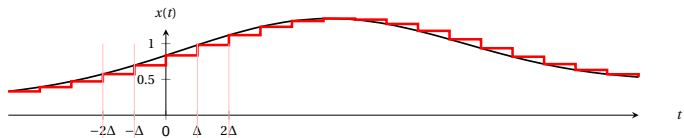
## ⑤ Revisiting Fourier Series

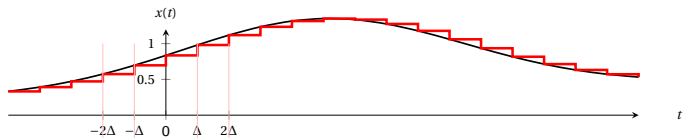
# Continuous-Time Systems: The Convolution Integral

- ① Similar to what we did in DT, in this section we obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response.
- ② In discrete time, the key to developing the convolution sum was the sifting property of the DT unit impulse—i.e., mathematical representation of a signal as a superposition of scaled and shifter unit impulse functions.
- ③ We begin by considering the staircase approximation  $\hat{x}(t)$  of a CT signal  $x(t)$ .

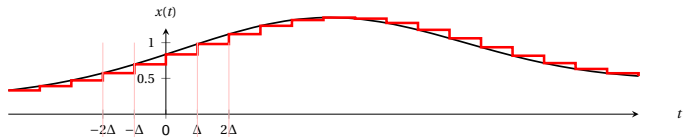












The approximation that we saw can be expressed as a linear combination of delayed impulses. Define

$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\Delta\delta_{\Delta}(t)$  has unit amplitude, we have

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta.$$

Here, for any value of  $t$ , only one term in the summation on the right hand side is nonzero.

$$x(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta.$$

As  $\Delta \rightarrow 0$ , the summation approaches an integral. Consequently,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This is known as the **sifting property** of the continuous time impulse.

**Example:**

Use the sifting property to express  $u(t)$  in terms of  $\delta(t)$ .

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$$u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t - \tau) d\tau = \int_0^{\infty} \delta(t - \tau) d\tau$$

# The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

Let's define  $\hat{h}_{k\Delta}(t)$  as the response of an LTI system to the input  $\delta_{\Delta}(t - k\Delta)$ .

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta.$$

Since the pulse  $\delta_{\Delta}(t - k\Delta)$  corresponds to a shifted unit impulse as  $\Delta \rightarrow 0$ , the response  $\hat{h}_{k\Delta}(t)$  to this input pulse becomes the response to an impulse in the limit. If we let  $h_{1\tau}(t)$  denote the response at time  $t$  to a unit impulse  $\delta(t - \tau)$  located at time  $\tau$ , then

$$y(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta) h_{k\Delta}(t) \Delta.$$

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As a  $\Delta \rightarrow 0$ , the summation on the right-hand side becomes an integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h_{\tau}(t) d\tau$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$

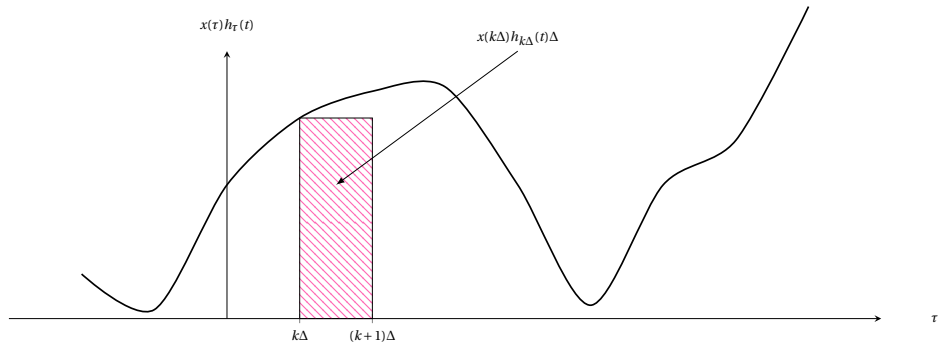


Figure: Graphical illustration

In addition to being linear, the system is time-invariant, the response of the LTI system to the unit impulse  $\delta(t - \tau)$

$$h_\tau(t) = h_0(t - \tau).$$

Defining unit impulse response  $h(t)$  as

$$h(t) = h_0(t),$$

we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

which is referred to as the **convolution integral** or the **superposition integral**. This corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse.

$$y(t) = x(t) * h(t).$$

As in discrete time, a continuous-time LTI system is completely characterized by its impulse response—i.e., by its response to a single elementary signal, the unit impulse  $\delta(t)$ .



**Example:** Let  $x(t)$  be the input to an LTI system with unit impulse response  $h(t)$ , where

$$x(t) = e^{-at}u(t), a > 0$$

and

$$h(t) = u(t).$$

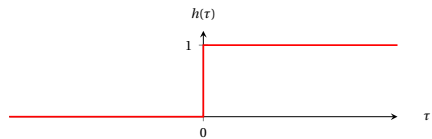


Figure: Calculation of convolution integral for the example

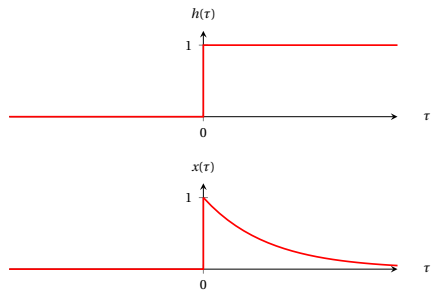


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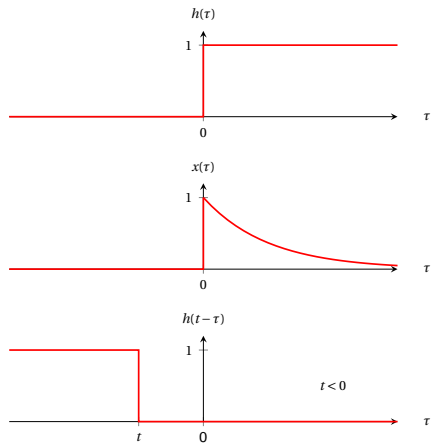


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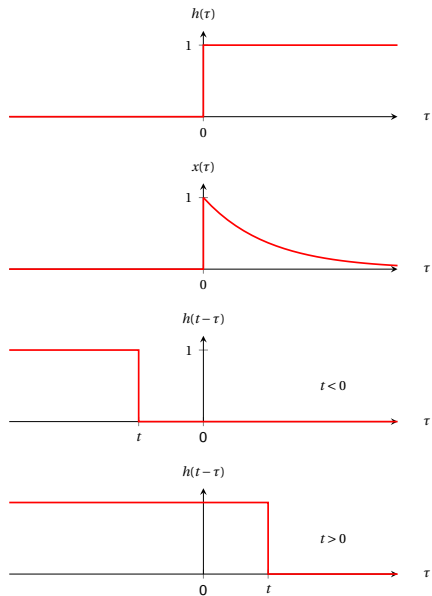


Figure: Calculation of convolution integral for the example

For  $t < 0$ , the product  $x(\tau)$  and  $h(t - \tau)$  is zero, consequently  $y(t)$  is zero.

For  $t > 0$ ,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau} \Big|_0^t \\ &= \frac{1}{a} (1 - e^{-at}) \end{aligned}$$

Thus for all  $t$ ,

$$y(t) = \frac{1}{a} (1 - e^{-at}) u(t)$$

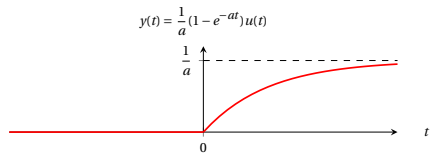


Figure: Response

**Example:** Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(t) = \begin{cases} t, & 0 < t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$



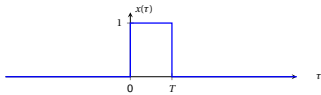


Figure:  $x(\tau)$  and  $h(t-\tau)$

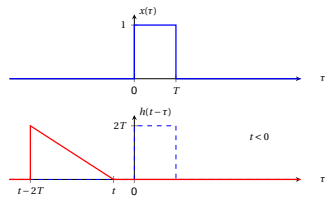


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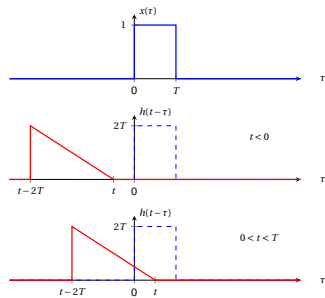


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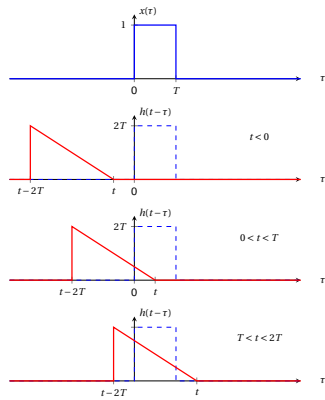


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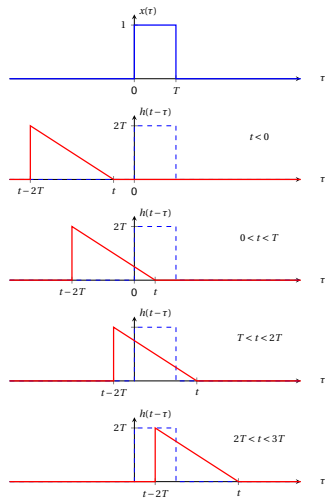


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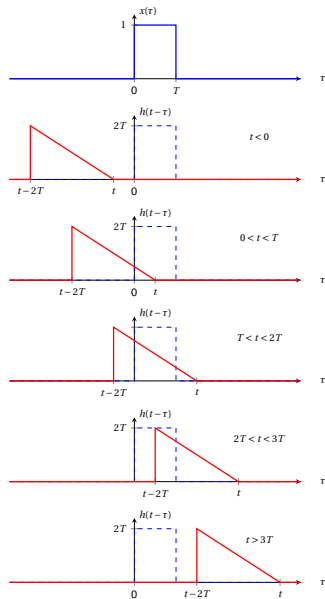


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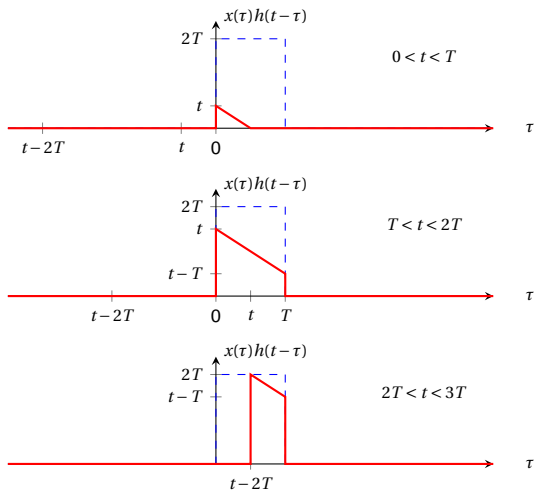


Figure: Product  $x(\tau)h(t-\tau)$  for values of  $t$  for which this product is not identically zero.

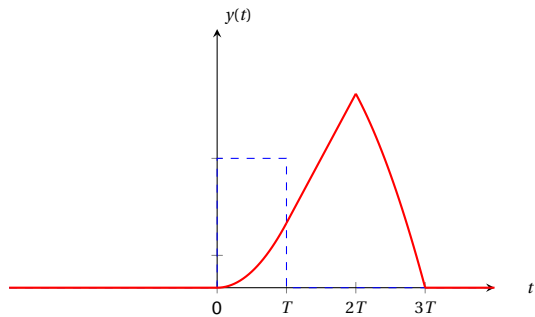


Figure:  $y(t) = x(t) * h(t)$



**Example:** Find  $y(t)$ , the convolution of the following two signals:

$$x(t) = e^{2t} u(-t),$$

and

$$x(t) = u(t-3).$$

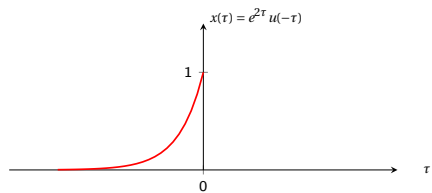


Figure: Convolution considered in the example.

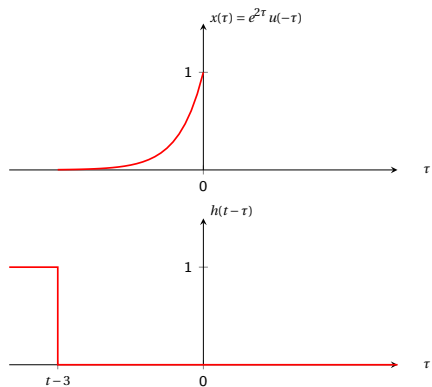


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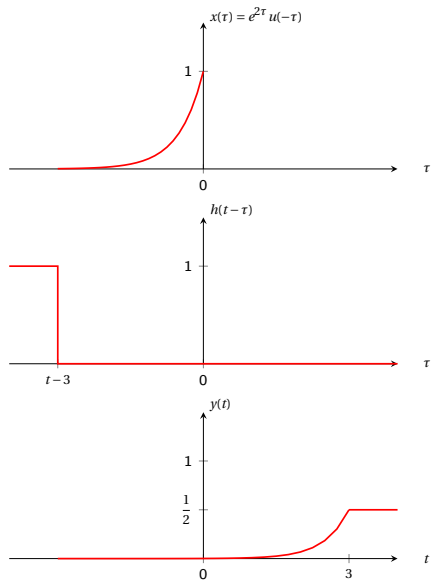


Figure: Convolution considered in the example.

When  $t-3 \leq 0$ , the product of  $x(\tau)$  and  $h(t-\tau)$  is nonzero for  $-\infty < \tau < t-3$ , and the convolving integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}.$$

For  $t-3 \geq 0$ , the product of  $x(\tau)h(t-\tau)$  is nonzero for  $-\infty < \tau < 0$ , and the convolving integral becomes

$$y(t) = \int_{-\infty}^0 e^{2\tau} d\tau = \frac{1}{2}.$$

- ① In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t)$$

- ② Characteristics of an LTI system are completely determined by its impulse response.

# The Commutative Property

DT

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

CT

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Verify the commutative property for DT.

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Substituting  $r = n - k$ , or equivalently  $k = n - r$



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# The Distributive Property

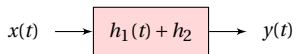
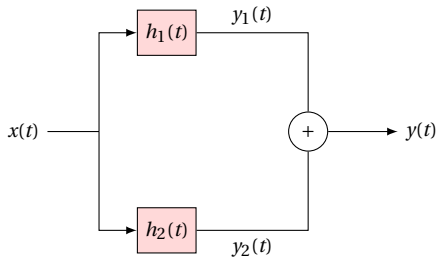
Convolution distributes over addition.

DT

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

CT

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t).$$



DT

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n].$$

CT

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t).$$

As a consequence,

$$y[n] = x[n] * h_1[n] * h_2[n]$$

and

$$y(t) = x(t) * h_1(t) * h_2(t).$$

are unambiguous.

Using the commutative property together with the associative property we can see that the order in which they are cascaded does not matter as far as the overall system impulse response is concerned.

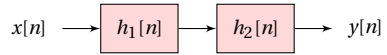


Figure: Associative property.

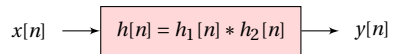
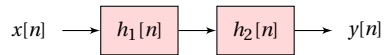


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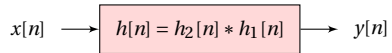
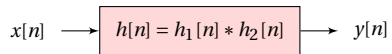
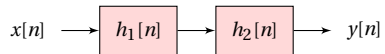


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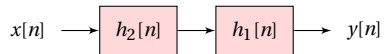
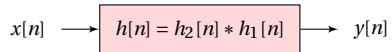
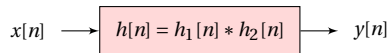
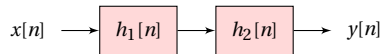


Figure: Associative property.

# LTI Systems with and without Memory

- ① A system is memoryless if its output at any time depends only on the value of the input at that same time.
- ② The only way that this can be true for a discrete-time LTI system is if  $h[n] = 0$  for  $n \neq 0$ .
- ③ In this case the impulse response has the form

$$h[n] = K\delta[n],$$

where  $K = h[0]$  is a constant.

- ④ The convolution sum reduces to the relation

$$y[n] = Kx[n]$$

- ⑤ If a discrete-time LTI system has an impulse response  $h[n]$  that is not identically zero for  $n \neq 0$ , then the system has memory.
- ⑥ For CT:

$$h(t) = K\delta(t).$$

$$y(t) = Kx(t).$$



- 1 An LTI system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.

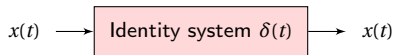
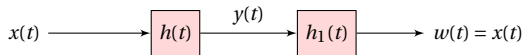


Figure: Inverse of a CT LTI system.

### Example

Consider the following relationship of a pure time shift:

$$y(t) = x(t - t_0)$$

Is the corresponding system memoryless? What is the inverse system of the system?

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Delay if  $t_0 > 0$ , and advance if  $t_0 < 0$ . E.g., if  $t_0 > 0$  then the output at time  $t$  equals the input at the earlier time  $t - t_0$ . If  $t_0 = 0$ , the system is the identity system and that is memoryless. For any other value of  $t_0$ , the system has memory.

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Setting the input equal to  $\delta(t)$ , the impulse response can be obtained:

$$h(t) = \delta(t - t_0).$$

Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0).$$

That is, the convolution of a signal with a shifted impulse simply shifts the signal. To recover the input from the output, i.e., to invert the system, all that is required is to shift the output back.

$$h_1(t) = \delta(t + t_0).$$

Then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

### Example

Consider the LTI system with impulse response

$$h[n] = u[n].$$

Determine  $y[n]$ . Find the inverse system.

- ① The output of a causal system depends only on the present and past values of the input to the system.
- ② For a DT LTI system,  $y[n]$  must not depend on  $x[k]$  for  $k > n$ .
- ③ For this to be true, all of the coefficients  $h[n-k]$  that multiply values of  $x[k]$  for  $k > n$  must be zero.
- ④ This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0.$$

- ⑤ The impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality.
- ⑥ More generally, causality for a linear system is equivalent to the condition of initial rest; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time.
- ⑦ The equivalence of causality and the condition of initial rest applies only to linear systems.

- ① A continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0.$$

- ② Causality of an LTI system is equivalent to its impulse response being a causal signal.

A system is stable if every bounded input produces a bounded output. Consider an input  $x[n]$  that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n.$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

If the impulse response is absolutely summable, then  $y[n]$  is bounded in magnitude, and hence, the system is stable.



In CT a system is stable if the impulse response is **absolutely integrable**.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

## Examples

Determine whether the following systems are stable:

- ① Pure time shift in DT.
- ② Pure time shift in CT.
- ③ Accumulator in DT.
- ④ CT counterpart of the accumulator.

Pure time shift in DT:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_0]| = 1$$

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CT counterpart of the accumulator:

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

The impulse response of the integrator can be found by letting  $x(t) = \delta(t)$ :

$$h(t) = \int_{-\infty}^t \delta(\tau) d\tau = u(t).$$

$$\int_{-\infty}^{\infty} |u(\tau)| d\tau = \int_0^{\infty} d\tau = \infty$$

## Examples

Determine whether the following systems are stable:

- ① Pure time shift in DT.
- ② Pure time shift in CT.
- ③ Accumulator in DT.
- ④ CT counterpart of the accumulator.

Pure time shift in DT:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_0]| = 1$$

Ans: **stable**.

Pure time shift in CT:

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} |\delta(\tau - t_0)| d\tau = 1$$

Ans: **stable**.

Accumulator in DT:  $h[n] = u[n]$

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=-\infty}^{\infty} |u[n]| = \infty$$

CT counterpart of the accumulator:

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Ans: **unstable**.



# The Unit Step Response of an LTI System

There is another signal that is also used in describing the behavior of LTI systems: the unit step response,  $s[n]$  or  $s(t)$ , corresponding to the output when  $x[n] = u[n]$  or  $x(t) = u(t)$ .

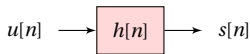


Figure: Unit step response.

$$s[n] = u[n] * h[n]$$

Commutative property:

$$s[n] = h[n] * u[n]$$

$s[n]$  can be viewed as the response to the input  $h[n]$  of a discrete-time LTI system with unit impulse response  $u[n]$ .

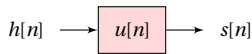


Figure: Unit step response.

$u[n]$  is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

$h[n]$  can be recovered from  $s[n]$  using the relation

$$h[n] = s[n] - s[n-1].$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response.

Similarly, in CT, the step response of an LTI system with impulse response  $h(t)$  is given by  $s(t) = u(t) * h(t)$ , which also equals the response of an integrator [with impulse response  $u(t)$ ] to the input  $h(t)$ . That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^t h(\tau) d\tau$$

the unit impulse response is the first derivative of the unit step response, or

$$h(t) = \frac{ds(t)}{dt} = s'(t).$$

For a linear system (time-invariant or not), if we put nothing into it, we get nothing out of it.

$x(t) = 0$  for all  $t$ , then

$y(t) = 0$  for all  $t$ ,

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$$ax(t) \rightarrow ay(t).$$

Select the scale factor  $a = 0$ .

Not all systems are like this, e.g., even if a battery is not connected to anything, the output is 1.5 V.

The system cannot anticipate the input.

I.e., If

$$x_1(t) = x_2(t), \quad \text{for } t < t_0,$$

then

$$y_1(t) = y_2(t), \quad \text{for } t < t_0,$$

Same for DT.

# Implications for Causality for a Linear System

For linear systems, if

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Initial rest: The system does not respond until an input is given.

For a linear system to be causal it must have the property of initial rest.

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Why? For linear systems zero in  $\rightarrow$  zero out.

# Causality for Linear Time Invariant Systems

For LTI systems,

Causality  $\Leftrightarrow$

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$$y[n] = \sum_{k=-\infty}^n x[k]$$

The accumulator is an LTI system. Also, we saw that its impulse response is

$$h[n] = u[n].$$

- 1 Does the accumulator have memory?
- 2 Is the accumulator causal?
- 3 Is accumulator stable in the BIBO sense?
- 4 If invertible, what is the inverse?

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Accumulator can be expressed as a **recursive difference equation** as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n]. \end{aligned}$$

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Inverse of the accumulator:

- ① Does it have memory? Yes.
- ② Is the system causal? Yes.
- ③ Is the system stable in the BIBO sense? Yes.

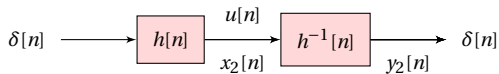


Figure:

We know that

$$u[n] - u[n-1] = \delta[n].$$

So

$$x_2[n] - x_2[n-1] = y_2[n].$$

$$\delta[n] - \delta[n-1] = h^{-1}[n].$$

$$y[n] - ay[n-1] = x[n]$$

under the assumption of initial rest  $\Rightarrow$  LTI. Memory? Causal? Stable?

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$$h[n] = a^n u[n] \quad (\text{Why?})$$

- ① Memory? Yes.
- ② Causal? Yes.
- ③ Stable? If  $|a| < 1$ , yes.

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

under the assumption of initial rest  $\Rightarrow$  LTI. Memory? Causal? Stable?



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$$h(t) = e^{-2t}u(t) \quad (\text{Why?})$$

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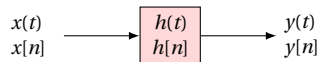


Figure:

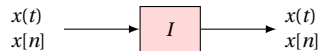


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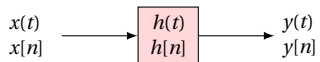


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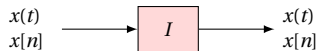


Figure:

$$h(t) = \delta(t)$$

$$h[n] = \delta[n]$$

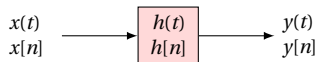


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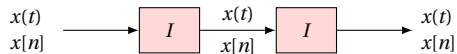


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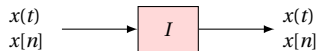


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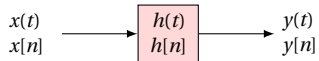


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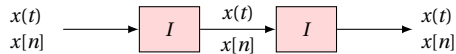


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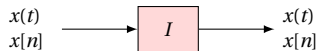


Figure:

$$\delta(t) * \delta(t) = \delta(t)$$

$$\delta[n] * \delta[n] = \delta[n]$$

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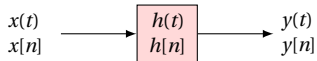


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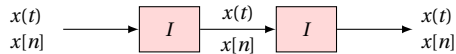


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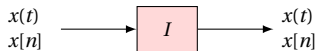


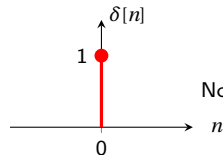
Figure:

$$\delta(t) * \delta(t) = \delta(t)$$

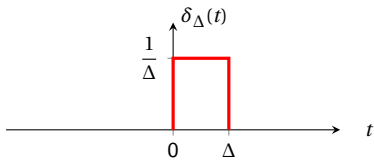
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$$h(t) = \delta(t)$$

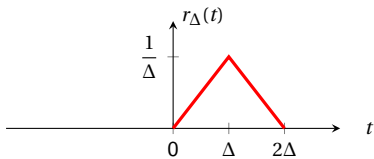
$$h[n] = \delta[n]$$



No problem in DT.



$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$



$$r_{\Delta}(t) = \delta_{\Delta}(t) * \delta_{\Delta}(t)$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} r_{\Delta}(t)?$$

Figure:

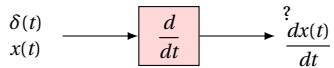


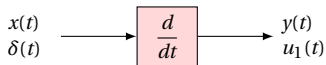
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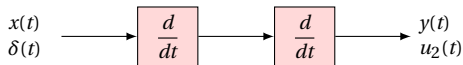
We use operational definitions through convolution to handle derivatives and integrals of impulse, which are badly behaved functions. This leads to a set of singularity functions. Impulse and step are examples of these.

$$x(t) * \delta(t) = x(t)$$

$$\frac{d}{dt} [\delta(t)]$$



$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$



$$u_2(t) = u_1(t) * u_1(t)$$

$$x(t) * u_2(t) = \frac{d^2 x(t)}{dt^2}$$

$$u_k(t) = u_1(t) * u_1(t) * \dots k \text{ times}$$

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

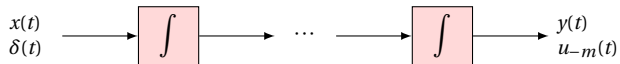
$$u_0(t) = \delta(t)$$

$u_{-1}(t)$  = unit step

$u_{-2}(t)$  = unit ramp

$u_0(t) = \delta(t)$

$u_{-1}(t) = u(t)$



$$u_k(t) * u_l(t) = u_{k+l}(t).$$

$x(t) * u_{-m}(t) = m^{\text{th}}$  running integral

$u_k(t)$  defined by

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

# Linear, Constant-Coefficient Differential and Difference Equations

- An important class of CT systems is that for which the input and output are related through a linear constant-coefficient differential equation.
- These arise in the description of a wide variety of systems and physical phenomena. E.g., the response of the RC circuit, the motion of a vehicle subject to acceleration inputs and frictional forces.
- Correspondingly, an important class of DT systems is that for which the input and output are related through a linear constant-coefficient difference equation.
- These are used to describe the sequential behavior of many different processes. E.g., accumulation of savings in a bank account, a digital simulation of a continuous-time system, DT designed to perform particular operations on the input signal such as a system that calculates the difference between successive input values, or computes the average value of the input over an interval.

## ① Basic System Properties

## ② Convolution

The Discrete-Time Unit Impulse Response and the Convolution Sum

Continuous-Time Systems: The Convolution Integral

Continuous-Time Systems: The Convolution Integral

## ③ Properties of Linear Time-Invariant Systems

## ④ Linear, Constant-Coefficient Differential and Difference Equations

Linear, Constant-Coefficient Differential Equations

Linear, Constant-Coefficient Difference Equations

General block-Diagram Representation of the Recursive System

## ⑤ Revisiting Fourier Series

# Linear, Constant-Coefficient Differential Equations

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{dx(t)}{dt^k} \quad (4)$$

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0. \quad \text{Homogeneous equation.} \quad (5)$$

Given  $x(t)$ , if  $y_p(t)$  satisfies 4, so does  $y_p(t) + y_h(t)$  where  $y_h(t)$  satisfies 5.

$y_p(t) \triangleq$  particular solution

$y_h(t) \triangleq$  homogeneous solution

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0.$$

Guess a solution of the form

$$y_h(t) = Ae^{st}, \quad \text{a complex exponential}$$

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$$\sum_{k=0}^N a_k s^k e^{st} = 0.$$

$$\sum_{k=0}^N a_k s^k = 0, \quad N \text{ roots } s_i, i = 1, \dots, N$$



$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0.$$

$$y(t), \frac{dy(t)}{dt}, \dots, \frac{d^{N-1}y(t)}{dt^{N-1}} \quad \text{at } t = t_0.$$

Guess a solution of the form

$$y_h(t) = Ae^{st}, \quad \text{a complex exponential}$$

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$$\sum_{k=0}^N a_k s^k = 0, \quad N \text{ roots } s_i, i = 1, \dots, N$$

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_N e^{s_N t}$$

Coefficients  $A_1, A_2, \dots, A_N$  are undetermined.

We need  $N$  auxiliary conditions to determine them

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0.$$

Guess a solution of the form

$$y_h(t) = Ae^{st}, \quad \text{a complex exponential}$$

$$\sum_{k=0}^N a_k s^k e^{st} = 0.$$

$$\sum_{k=0}^N a_k s^k = 0, \quad N \text{ roots } s_i, i = 1, \dots, N$$

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_N e^{s_N t}$$

Coefficients  $A_1, A_2, \dots, A_N$  are undetermined.

We need  $N$  auxiliary conditions to determine them

$$y(t), \frac{dy(t)}{dt}, \dots, \frac{d^{N-1}y(t)}{dt^{N-1}} \quad \text{at } t = t_0.$$

Linear system  $\iff$  auxiliary conditions = 0

Linear system  $\Rightarrow$  zero in, zero out.

Causal and LTI  $\iff$  initial rest

If  $x(t) = 0, t < t_0$  then

$y(t) = 0, t < t_0$  then

## Example: First-Order Differential Equation

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$$y_h(t) = Ae^{st}.$$

$$\begin{aligned} Ase^{st} + aAe^{st} &= 0 \\ s + a &= 0. \end{aligned}$$

$$y_h(t) = Ae^{-at}.$$

Specific input

$$x(t) = ku(t).$$

A solution is

$$y_1(t) = \frac{k}{a} [1 - e^{-at}] u(t)$$

Family of solutions

$$y(t) = y_1(t) + y_h(t).$$

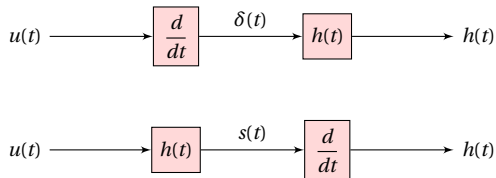
$$y_1(t) = \frac{k}{a} [1 - e^{-at}] u(t) + Ae^{-at}.$$

Causal, LTI  $\iff$  initial rest

$$ku(t) \rightarrow \frac{k}{a} [1 - e^{-at}] u(t)$$

Obtain the impulse response of the above system.

Obtain the impulse response of the above system. Differentiating the step response, we get the impulse response.



$$s(t) = \frac{1}{a} [1 - e^{-at}] u(t).$$

$$\begin{aligned} h(t) &= \frac{d}{dt} s(t) = \frac{d}{dt} \frac{1}{a} [1 - e^{-at}] u(t), \\ &= u(t) \frac{d}{dt} \frac{1}{a} [1 - e^{-at}] + \frac{1}{a} [1 - e^{-at}] \frac{d}{dt} u(t) \\ &= e^{-at} u(t) + \frac{1}{a} [1 - e^{-at}] \delta(t) \\ &= e^{-at} u(t) \end{aligned}$$



## ① Basic System Properties

## ② Convolution

The Discrete-Time Unit Impulse Response and the Convolution Sum

Continuous-Time Systems: The Convolution Integral

Continuous-Time Systems: The Convolution Integral

## ③ Properties of Linear Time-Invariant Systems

## ④ Linear, Constant-Coefficient Differential and Difference Equations

Linear, Constant-Coefficient Differential Equations

Linear, Constant-Coefficient Difference Equations

General block-Diagram Representation of the Recursive System

## ⑤ Revisiting Fourier Series

# Linear, Constant-Coefficient Difference Equations

Consider the  $N^{\text{th}}$ -order difference equation

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]. \quad (6)$$

$$\sum_{k=0}^N a_k y[n-k] = 0: \quad \text{homogeneous equation.} \quad (7)$$

If  $y_p[n]$  satisfies 6 so does  $y_p[n] + y_h[n]$  where  $y_h[n]$  satisfies 7.

$y_p[n] \triangleq$  particular solution

$y_h[n] \triangleq$  homogeneous solution

$$\sum_{k=0}^N a_k y[n-k] = 0.$$

“Guess” a solution of the form

$$y_h[n] = Az^n.$$

$$\sum_{k=0}^N a_k Az^n z^{-k} = 0.$$

$$\sum_{k=0}^N a_k z^{-k} = 0. \quad N \text{ roots } z_1, z_2, \dots, z_N.$$

$$y_h[n] = A_1 z_1^n + A_2 z_2^n + \dots + A_N z_N^n.$$

$$y[n] = A_1 z_1^n + A_2 z_2^n + \dots + A_N z_N^n + y_p[n]$$

The undetermined constants  $A_1$  to  $A_N$  are to be found using the  $N$  auxiliary conditions,  $y[n_0]$ ,  $y[n_0 - 1]$ ,  $y[n_0 - N + 1]$ .

Linear system  $\iff$  auxiliary conditions 0

Causal, LTI  $\iff$  initial rest

If  $x[n] = 0$ ,  $n < n_0$  then

$y[n] = 0$ ,  $n < n_0$  then

# Explicit Solution to Difference Equations

Assume causality.

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k].$$

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Assume causality.

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k].$$

$$y[n] = \frac{1}{a_0} \left[ \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k] \right]$$

This is said to be a recursive solution to linear constant-coefficient difference equations. To get the computation of  $y[n_0]$  started we need the initial conditions or boundary conditions  $y[n_0 - 1]$ ,  $y[n_0 - N]$ . Then we compute  $y[n_0 + 1]$  and so on.

## Example: First-Order Difference Equation

$$y[n] - ay[n-1] = x[n].$$

Causal, LTI  $\iff$  initial rest

$$y[n] = x[n] + ay[n-1].$$

Set  $x[n] = \delta[n]$ .

$$h[n] = \delta[n] + ah[n-1].$$

## Example: First-Order Difference Equation

$$y[n] - ay[n-1] = x[n].$$

Causal, LTI  $\iff$  initial rest

$$y[n] = x[n] + ay[n-1].$$

Set  $x[n] = \delta[n]$ .

$$h[n] = \delta[n] + ah[n-1].$$

$h[n] = 0, n < 0$  initial rest.

$$h[0] = 1$$

$$h[1] = a$$

$$h[2] = a^2 \vdots \quad = \vdots$$

$$h[n] = a^n u[n].$$

So if causal and LTI

$$\delta[n] \rightarrow a^n u[n].$$

Family of solutions

$$\delta[n] \rightarrow a^n u[n] + y_h[n].$$

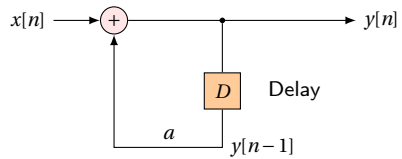
$$y_h[n] = Az^n.$$

$$Az^n - aAz^{n-1} = 0.$$

$$a - z^{-1} = 0 \Rightarrow z = a.$$

$$y_h[n] = Aa^n.$$

$$\delta[n] \rightarrow a^n u[n] + Aa^n.$$





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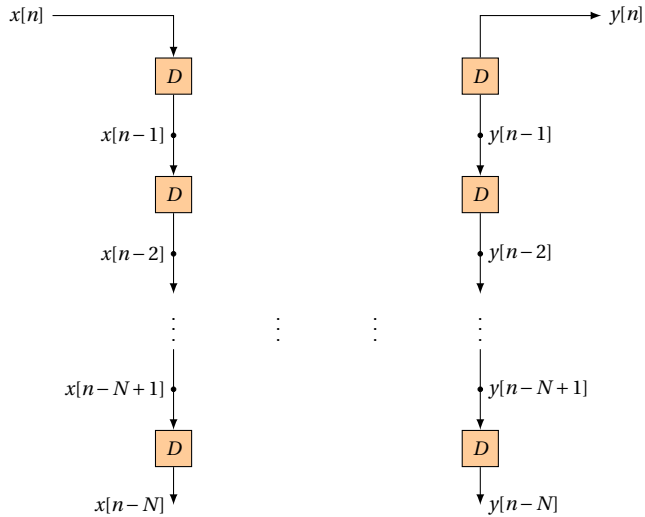
## ④ Linear, Constant-Coefficient Differential and Difference Equations

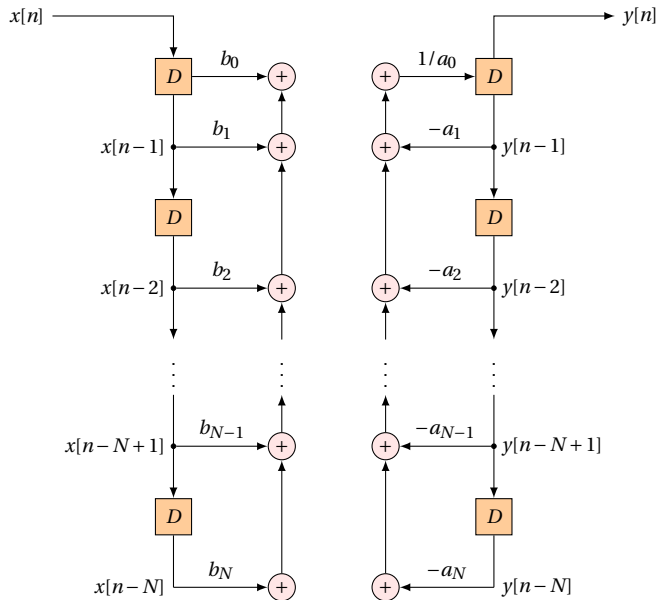
Linear, Constant-Coefficient Differential Equations

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General block-Diagram Representation of the Recursive System

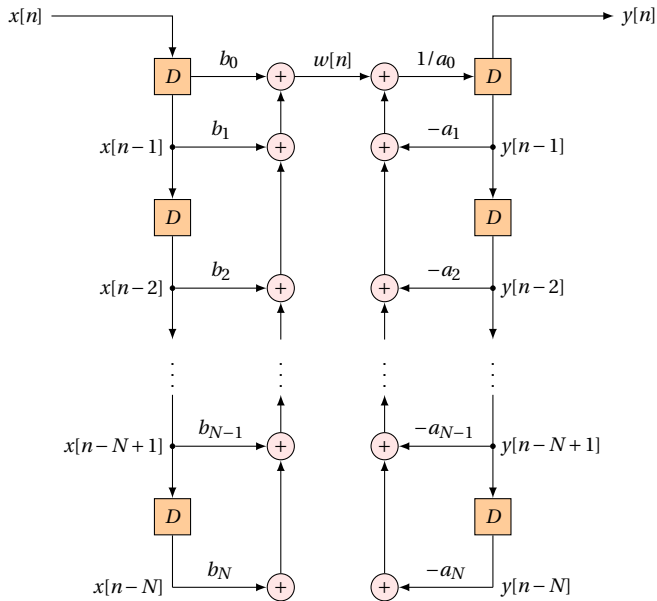
## ⑤ Revisiting Fourier Series





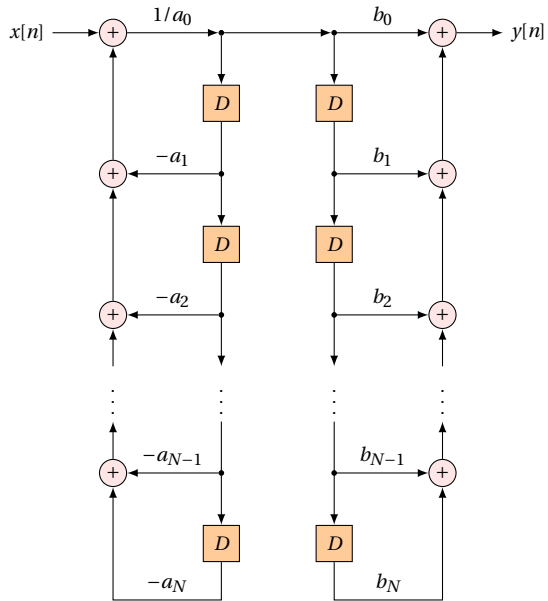
## Direct form I implementation

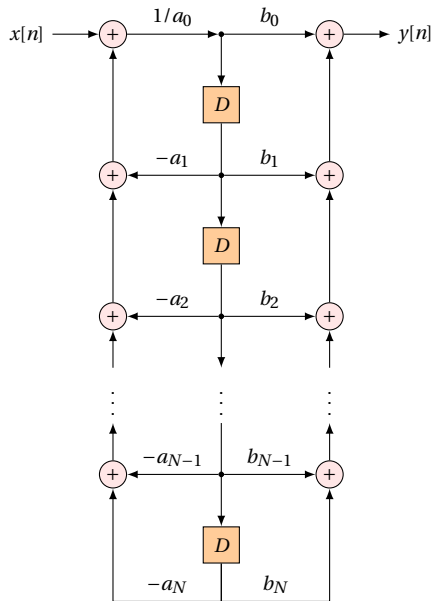
This is a cascade of two linear systems. We can interchange the order of the two segments.



## Direct form I implementation

This is a cascade of two linear systems. We can interchange the order of the two segments.





Result of combining the two chains of delays.

**Direct form II  
implementation**

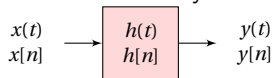
- ① In representing and analyzing LTI systems, our approach has been to decompose the system inputs into a linear combination of basic signals and exploit the fact that for a linear system, the response is the same linear combination of the responses to the basic inputs.
- ② The convolution sum and the convolution integral req out of the particular choice of the basic signals, delayed unit impulses.
- ③ This choice has the advantage that for systems that are time invariant in addition to being linear, once the response to an impulse at one time position is known, then the response id know at all time positions.

# Complex Exponentials with Unity Magnitude as Basic Signals

- ① When we select complex exponential with unity magnitude as the basic signals, the decomposition of this form of a periodic signal is the Fourier series.
- ② For aperiodic signals, it becomes the Fourier transform.
- ③ In latter lectures, we will generalize this representation to Laplace transform for continuous-time signals and  $z$ -transform for discrete-time signals.



Consider a linear system



If

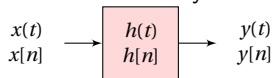
$$x(t) = a_1 \phi_1(t) + a_2 \phi_2(t) + \dots$$

and

$$\phi_k(t) \longrightarrow \psi_k(t), \quad (\text{output due to } \phi_k(t))$$

then

Consider a linear system



If

$$x(t) = a_1 \phi_1(t) + a_2 \phi_2(t) + \cdots$$

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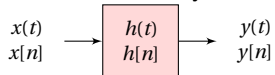
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then

$$y(t) = a_1 \psi_1(t) + a_2 \psi_2(t) + \cdots$$

Identical for DT. So

Consider a linear system



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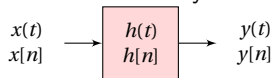
$$y(t) = a_1 \psi_1(t) + a_2 \psi_2(t) + \cdots$$

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$$\text{If } x(t) = a_1 \phi_1 + a_2 \phi_2 + \cdots,$$

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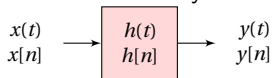
$$\text{If } x(t) = a_1 \phi_1 + a_2 \phi_2 + \dots,$$

$$\text{then } y(t) = a_1 \psi_1 + a_2 \psi_2 + \dots.$$

Choose  $\phi_k(t)$  or  $\phi_k[n]$  so that

- 1 A broad class of signals can be constructed as a linear combination of  $\phi_k$ s
- 2 Response to  $\phi_k$ s easy to compute.

Consider a linear system



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$$\begin{aligned} \text{If } x(t) &= a_1 \phi_1 + a_2 \phi_2 + \dots, \\ \text{then } y(t) &= a_1 \psi_1 + a_2 \psi_2 + \dots. \end{aligned}$$

Choose  $\phi_k(t)$  or  $\phi_k[n]$  so that

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Choice of signals  $\delta(t - k\Delta)$  and  $\delta[n - k]$  lead to the convolution integral and convolution sum.

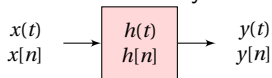
$$\text{CT } \phi_k(t) = \delta(t - k\Delta)$$

$$\psi_k(t) = h(t - k\Delta) \Rightarrow \text{convolution integral}$$

$$\text{DT } \phi_k[n] = \delta[n - k]$$

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Here, we choose complex exponentials as the set of basic signals.

$$\phi_k(t) = e^{s_k t}, \quad s_k \text{ complex}$$

$$\phi_k[n] = z_k^n, \quad z_k \text{ complex}$$

$$\begin{array}{llll}
 \text{CT} & s_k = j\omega_k & \text{purely imaginary} & \phi_k(t) = e^{j\omega_k t} \\
 \text{DT} & |z_k| = 1 & \phi_k[n] & = e^{j\omega_k n}
 \end{array}$$

$s_k$  complex  $\Rightarrow$  Laplace transform

$z_k$  complex  $\Rightarrow$   $z$ -transform

Consider  $\phi_k(t) = e^{j\omega_k t}$ :

$$e^{j\omega_k t} \longrightarrow H(\omega_k) e^{j\omega_k t} \quad (\text{a scaled-version of the input})$$

**“Proof”:**



Consider  $\phi_k(t) = e^{j\omega_k t}$ :

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$$e^{j\omega_k t} \longrightarrow \underbrace{e^{j\omega_k t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_k \tau} d\tau}_{\substack{\uparrow \\ \text{eigenfunction} \quad \text{eigenvalue}}}$$

$$x(t) = x(t + T), \quad T = \text{period}, \quad \omega_0 = \frac{2\pi}{T}$$

Consider the periodic exponential signal with the same fundamental period:

$$e^{j\omega_0 t}, \quad T = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t}, \quad T = \frac{2\pi}{k\omega_0}, \quad \text{harmonically related}$$

Fourier series: 
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

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- ② What are the signals that can be expressed in this form? [already answered,

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$$\int_T e^{jm\omega_0 t} dt = \int_T \cos m\omega_0 t dt + j \int_T \sin m\omega_0 t dt$$

$m \neq 0$	0	0
$m = 0$	$T$	0

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$m \neq 0$	0	0
$m = 0$	$T$	0

Start with the Fourier series equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Multiplying both sides by  $e^{-jn\omega_0 t}$

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

Integrating over one period

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So

Integrating over one period

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So

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$