EN1060 Signals and Systems: Fourier Series

Ranga Rodrigo
`ranga@uom.lk`

The University of Moratuwa, Sri Lanka

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Section 1

Continuous-Time Fourier Series

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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

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- Fourier series provides a way to represent a periodic signal as a sum of complex sinusoids.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \tag{1}$$

$$\omega_0 = \frac{2\pi}{T}$$

The set of coefficients $\{a_k\}$ is called the Fourier series coefficients or the spectral coefficients of $x(t)$. The coefficient a_0 is the dc or constant component of $x(t)$, given by Equation 1 with $k = 0$:

$$a_0 = \frac{1}{T} \int_T x(t) dt, \tag{2}$$

which is simply the average of $x(t)$ over one period.

Example

Let
$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing the right-hand side of this equation and Equation 1, we obtain

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$
$$a_k = 0, \qquad k \neq \pm 1.$$

Example

Let
$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos \left(2\omega_0 t + \frac{\pi}{4} \right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a liner combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \right.$$

<div style="border:1px solid blue; background:#c8c8ff; padding:4px">
Euler's formula
$e^{j\theta} = \cos\theta + j\sin\theta$
</div>

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

Collecting terms,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

The Fourier coefficients are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = \left(1 - \frac{j}{2}\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = \left(1 + \frac{j}{2}\right),$$

$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1 + j),$$
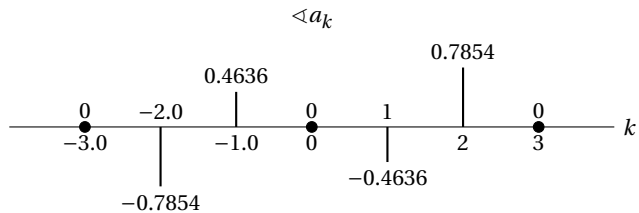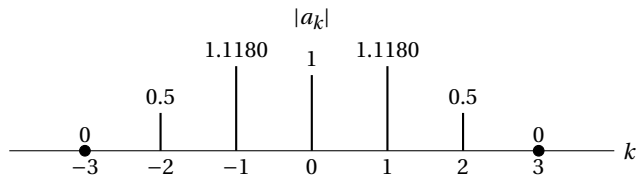$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, |k| > 2.$$

Figure: $|a_k|$, $\sphericalangle a_k$

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
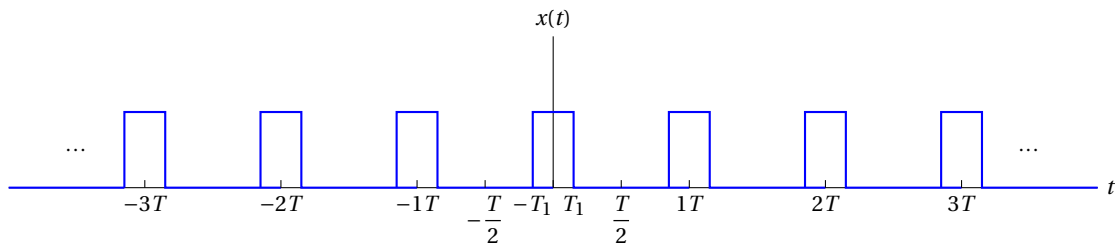2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

Figure: Periodic square wave

$$a_0 = \frac{1}{T} \int_T x(t)dt,$$
$$= \frac{1}{T} \int_{-T_1}^{T_1} 1dt,$$
$$= \frac{2T_1}{T}.$$

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$$a_k = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t}\,dt,$$
$$= \frac{1}{T} \int_{-T_1}^{T_1} e^{-jk\omega_0 t},$$
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$$a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j}\right]$$
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{2\sin(k\omega_0 T_1)}{k\pi}, k \neq 0.$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt,$$
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For $T = 4T_1$

$$a_k = 0, \quad k \text{ even.}$$

$$a_0 = \frac{1}{2}$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_3 = a_{-3} = \frac{1}{3\pi}$$

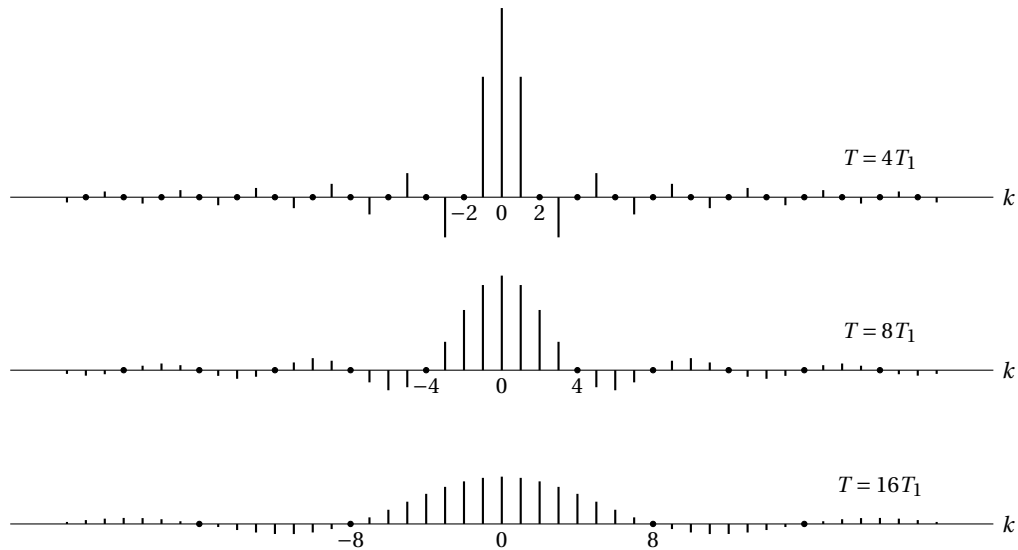$$a_5 = a_{-5} = \frac{1}{5\pi}$$

Figure: Plots of scaled Fourier series coefficients Ta_k

Section 2

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients are denoted by a_k, then

$$x(t) \xleftrightarrow{\mathscr{FS}} a_k \tag{3}$$

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathscr{FS}} a_k,$$
$$y(t) \xleftrightarrow{\mathscr{FS}} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $X(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathscr{FS}} c_k = Aa_k + Bb_k. \tag{4}$$

$$x(t - t_0) \overset{\mathscr{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

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Proof:

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} a_k, \quad x(t - t_0) \overset{\mathscr{FS}}{\longleftrightarrow} b_k,$$

$$b_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t} \, dt,$$

$$= \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 (\tau + t_0)} \, dt,$$

$$= e^{-jk\omega_0 t_0} \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 \tau} \, d\tau,$$

$$= e^{-jk\omega_0 t_0} a_k.$$

$$x(t - t_0) \overset{\mathscr{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k.$$

Note: $|a_k| = |b_k|$

If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad (6)$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}. \qquad (7)$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \qquad (8)$$

Substitution: $k = -m$

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{-jk2\pi t/T}.$$

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- If $x(t)$ is odd—$x(-t) = -x(t)$—then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

Time scaling, in general, changes the period.

If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{9}$$

While Fourier coefficients have not changes, the Fourier series representation has changed because of the change in the fundamental frequency.

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathscr{FS}} a_k,$$
$$y(t) \xleftrightarrow{\mathscr{FS}} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, its Fourier series coefficients h_k are

$$x(t)y(t) \xleftrightarrow{\mathscr{FS}} \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{10}$$

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- If $x(t)$ is real—$x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.

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- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.

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- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{11}$$

Note: Left-hand side of equation 11 is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$.

$$\frac{1}{T}\int_T \left| a_k e^{jk\omega_0 t} \right|^2 \, dt = \frac{1}{T}\int_T |a_k|^2 \, dt = |a_k|^2. \tag{12}$$

So, $|a_k|^2$ is the average power in the kth harmonic component of $x(k)$.

Thus, what Parseval's relation states is that the total power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example

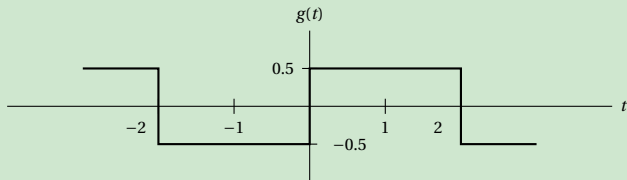Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 5.

Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 6. The derivative signal is the signal $g(t)$ in Figure 5. Using this information, find the Fourier series coefficients of $x(t)$.
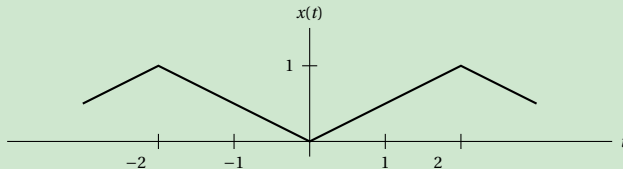

Figure: Figure for example

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{13}$$

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.
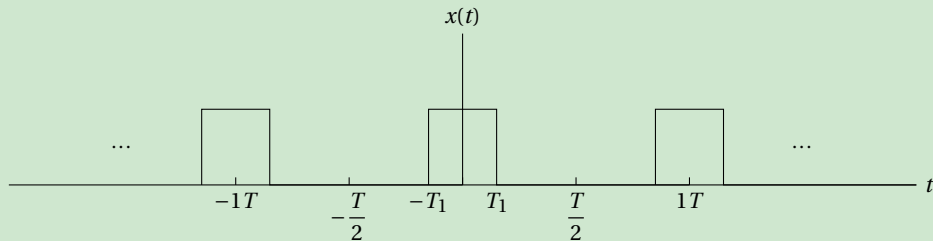

Figure: Figure for example

Complex Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \tag{14}$$

Trigonometric Fourier Series

$$x(t) = A_0 + 2 \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$
$$A_k = \frac{1}{T} \int_T x(t) A_k \cos k\omega_0 t dt \tag{15}$$
$$B_k = \frac{1}{T} \int_T x(t) A_k \sin k\omega_0 t dt$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$x(t) = C_0 + 2 \sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k)$$
$$C_0 = A_0 \tag{16}$$
$$C_k = \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)$$

Relationship

$$A_0 = a_0$$
$$A_k = \frac{a_k + a_{-k}}{2}$$
$$B_k = j\frac{a_k - a_{-k}}{2} \tag{17}$$
$$\omega_0 = \frac{2\pi}{T}$$

Section 3

Convergence of Fourier Series

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2 \, dt$$

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.

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- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.

Fourier series representation:

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One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

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- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1

Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\, dt < \infty. \tag{19}$$

This guarantees that a_ks are finite.

Condition 2

In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3

In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \le 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \le 1.$$

For this

$$\int_0^1 |x(t)| \, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period $T = 8$, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.