Equations

Fourier

	CT	DT
	Series (CT)	Series (DT)
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$	$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(\frac{2\pi}{N})n}$
	Transform (CT)	Transform (DT)
Aperiodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

Trigonometric Fourier Series

$$
\begin{aligned}
x(t) &= a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t). \\
a_0 &= \frac{1}{T}\int_T x(t)dt. \\
a_k &= \frac{2}{T}\int_T x(t)\cos(k\omega_0 t)dt. \\
b_k &= \frac{2}{T}\int_T x(t)\sin(k\omega_0 t)dt.
\end{aligned}
$$

Table 1: Properties of Continuous Time Fourier Series

Property	Periodic signal	Fourier series coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency shifting	$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic convolution	$\int_T x(\tau) y(t-\tau) d\tau$	$T a_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{d}{dt} x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$ finite valued and periodic only if $a_0 = 0$	$\frac{1}{jk\omega_0} a_k$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and even signals	$x(t)$ real and even	a_k real and even
Real and odd signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t) = \mathfrak{Ev}\{x(t)\}$, $[x(t)$ real$]$ $x_o(t) = \mathfrak{Od}\{x(t)\}$, $[x(t)$ real$]$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's relation for aperiodic signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Table 2: Properties of the Fourier Transform

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time reversal	$x(-t)$	$X(-j\omega)$
Time and frequency scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathfrak{Re}\{X(j\omega)\}=\mathfrak{Re}\{X(-j\omega)\} \\ \mathfrak{Im}\{X(j\omega)\}=-\mathfrak{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
Symmetry for real and even signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for real and odd signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t)=\mathfrak{Ev}\{x(t)\}$, $[x(t)$ real$]$ $x_o(t)=\mathfrak{Od}\{x(t)\}$, $[x(t)$ real$]$	$\mathfrak{Re}\{X(j\omega)\}$ $j\mathfrak{Im}\{X(j\omega)\}$

Parseval's relation for aperiodic signals

$$\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2d\omega$$

Table 3: Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, a_k = 0, k \neq 0$ (This is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	–
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	
$\delta(t)$	1	–
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	–
$\delta(t-t_0)$	$e^{-j\omega t_0}$	–
$e^{-at}u(t), \quad \Re\mathfrak{e}\{a\} > 0$	$\frac{1}{a+j\omega}$	–
$te^{-at}u(t), \quad \Re\mathfrak{e}\{a\} > 0$	$\frac{1}{(a+j\omega)^2}$	–
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad \Re\mathfrak{e}\{a\} > 0$	$\frac{1}{(a+j\omega)^n}$	–

Table 4: Properties of the Discrete-Time Fourier Series

Property	Periodic signal	Fourier series coefficients
$\left.\begin{matrix} x[t] \\ y[t] \end{matrix}\right\}$	Periodic with period N fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time reversal	$x[-n]$	a_{-k}
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$	$\frac{1}{m}a_k \quad \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic convolution	$\sum_{r=<N>} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running sum	$\sum_{k=-\infty}^{\infty} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Conjugate symmetry for real signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and even signals	$x[n]$ real and even	a_k real and even
Real and odd signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-odd decomposition of real signals	$x_e[n] = \mathfrak{Ev}\{x[n]\}, \quad [x[n] \text{ real}]$ $x_o[n] = \mathfrak{Od}\{x[n]\}, \quad [x[n] \text{ real}]$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's relation for aperiodic signals

$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$