EN1060 Signals and Systems: z-Transform and Sampling

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Section 1

Introduction

Introduction

- We developed the Laplace transform as a generalization of the continuous-time Fourier transform.
- In this lecture, we introduce the corresponding generalization of the discrete-time Fourier transform.
- The resulting transform is referred to as the z-transform.

z-Transform Motivation

- The discrete-time Fourier transform developed out of choosing complex exponentials as basic building blocks for signals because they are eigenfunctions of discrete-time LTI systems.
- A more general class of eigenfunctions consists of signals of the form z^n, where z is a general complex number. A representation of discrete-time signals with these more general exponentials leads to the z-transform.

Relationship between the z-Transform and the Discrete-Time Fourier transform

- We saw that the Laplace transform is a generalization of the continuous-time Fourier transform.
- A close relationship exists between the z-transform and the discrete-time Fourier transform.
- For $z = e^{j\omega}$ or, equivalently, for the magnitude of z equal to unity, the z-transform reduces to the Fourier transform.
- More generally, the z-transform can be viewed as the Fourier transform of an exponentially weighted sequence.
- Because of this, the z-transform may converge for a given sequence even if the Fourier transform does not: the z-transform offers the possibility of transform analysis for a broader class of signals and systems.

The Region of Convergence (ROC)

- The z-transform of a signal too has associated with it both a range of values of z, referred to as the region of convergence (ROC), for which this expression is valid.
- Two different sequences can have z-transforms with identical algebraic expressions such that their z-transforms differ only in the ROC.
- Consequently, the ROC is an important part of the specification of the z-transform.

z-Plane

- z-transforms of the form of a ration of polynomials in z^{-1} are described by poles and zeros in the complex plane, referred to as the z-plane.
- The circle of radius 1, concentric with the origin, in the z-plane, is referred to as the unit circle.
- Since this circle corresponds to the magnitude of z equal to unity, it is the contour in the z-plane on which the z-transform reduces to the Fourier transform.
- In contrast, for continuous time it is the imaginary axis in the s-plane on which the Laplace transform reduces to the Fourier transform.
- If the sequence is known to be right-sided, for example, then the ROC must be the portion of the z-plane outside the circle bounded by the outermost pole.

Section 2

The z-Transform

Recall: Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{-\infty}^{+\infty} x[n] e^{-j\omega n}$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega n} & \longrightarrow & H(e^{j\omega}) e^{j\omega n} \\ & & \updownarrow \mathscr{F} \\ & & h[n] \end{array}$$

z-Transform: Eigenfunction Property

$$z^n \to \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$z^n \to z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$z = re^{j\omega}$$

$$z^n \to H(z) z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

z-Transform and Fourier Transform Relationship

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$z = re^{j\omega}$$

$$X(z)|_{z=e^{j\omega}} = \mathscr{F}\{x[n]\}$$

New notation:

$$\mathscr{F}\{x[n]\} = X(e^{j\omega})$$

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega})$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]r^{-n}re^{-j\omega n}$$

$$X(z) = \mathscr{F}\left\{x[n]r^{-n}\right\}$$

ZT may converge when FT does not.

Example

Find the ZT of $x[n] = a^n u[n]$.

Example

Find the ZT of

$$x[n] = -a^n u[-n-1].$$

Pole-Zero Plot for a Right-Handed Sequence

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Pole-Zero Plot for a Left-Handed Sequence

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

First-Order Difference Equation

$$y[n] - ay[n-1] = x[n]$$

Pole-Zero Plot for a DT First-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{z}{z-a}, \quad |z| > |a|.$$

Second-Order Difference Equation

$$y[n] + 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

Pole-Zero Plot for a DT Under-Damped Second-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}}, \quad |z| > |a|.$$

Properties of the ROC of the z-Transform

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
- $\mathscr{F}\{x[n]\}$ converges $\Leftrightarrow$ ROC includes the unit circle in the z-plane
- $x[n]$ finite duration $\Rightarrow$ ROC is entire z-plane with the possible exception of $z = 0$ or $z = \infty$

Properties of the ROC for a Right-Sided Sequence

- $x[n]$ right-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all finite values of z for which $|z| > r_0$ are in ROC.
- $x[n]$ right-sided and $X(z)$ rational $\Rightarrow$ ROC is outside the outermost pole.

Properties of the ROC for a Left-Sided and for a Two-Sided Sequence

- $x[n]$ left-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all values of z for which $0 < |z| < r_0$ will also be in ROC.
- $x[n]$ left-sided and $X(z)$ rational $\Rightarrow$ ROC is inside the innermost pole.
- $x[n]$ two-sided and $|z| = r_0$ is in ROC $\Rightarrow$ ROC is a ring in the z-plane which includes the circle $|z| = r_0$.

Example

Show the choices of the ROC for

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Right-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}.$$

ROC If the Sequence Is Left-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Two-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

$$
\begin{aligned}
X(z) &= \mathscr{F}\left\{x[n]\,r^{-n}\right\} \\
x[n]\,r^{-n} &= \mathscr{F}^{-1}\left\{X(z)\right\} \\
&= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\,e^{-j\omega n}\,d\omega \\
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^{n} d\omega \\
z = re^{j\omega}, &\qquad dz = jre^{j\omega}\,d\omega \\
x[n] &= \frac{1}{2\pi j}\oint X(z)\,z^{n-1}\,dz
\end{aligned}
$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}, \qquad |z| > 2.$$

Example

Consider and LTI system for which

$$y[n] = h[n] * x[n],$$

where

$$h[n] = \delta[n] - \delta[n-1].$$

1. Find $H(z)$.
2. Find $y[n]$ in terms of $x[n]$.

Section 3

z-Transform properties

Recall: z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{-n} dz$$

$$X(z)|_{z=e^{j\Omega}} = \mathscr{F}\{x[n]\}$$

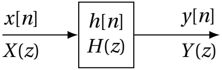
$$z = re^{j\Omega}$$

$$X(z) = \mathscr{F}\left\{x[n]\, r^{-n}\right\}$$

z-Transform Properties

Signal	Transform	ROC
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
$x[n-n_0]$	$z^{-n_0}X(z)$	R
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	at least $R_1 \cap R_2$

System Stability



$$Y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z)$$

$$\text{stable} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

$$\mathscr{F}\{h[n]\} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

The condition for stability and the existence of the Fourier transform are the same.

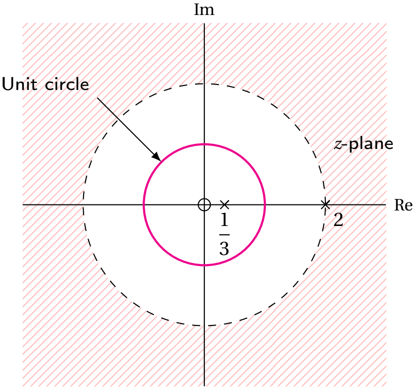
$$
\begin{aligned}
\text{stable} &\Leftrightarrow \text{ROC of } H(z) \text{ includes unit circle in } z\text{-plane} \\
\text{causal} &\Rightarrow h[n] \text{ is right-sided} \\
&\Rightarrow \text{ROC of } H(z) \text{ outside the outermost pole} \\
\text{causal and stable} &\Leftrightarrow \text{ All poles inside unit circle}
\end{aligned}
$$

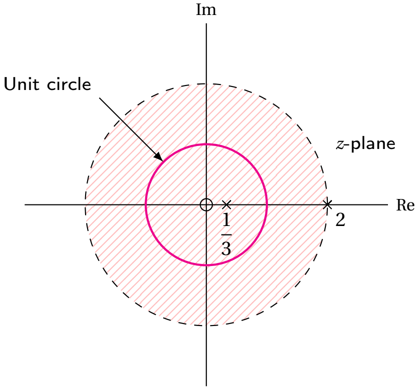
Example

Discuss the stability and causality of the system represented by the following system function with respect to different regions of convergence.

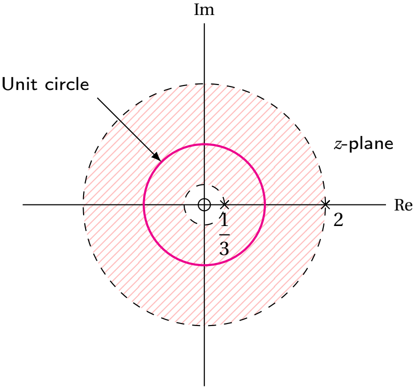
$$H(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

The system is causal and unstable.





The system is unstable and not causal.



The system is stable and not causal.

Example

Consider the LTI system for which the input $x[n]$ and the output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

1. Obtain an expression for the system function $H(z)$.
2. What are the two choices for the region of convergence?
3. Obtain $h[n]$ for each of these cases and comment on the stability and causality.

Section 4

Recall: Fourier Transform for Periodic Signals

Fourier Transform for Periodic Signals

- We studied the Fourier transform for aperiodic signals. We can also develop Fourier transform representations for periodic signals, thus allowing us to consider both periodic and aperiodic signals within a unified context.
- We can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of impulses proportional to the Fourier series coefficients.

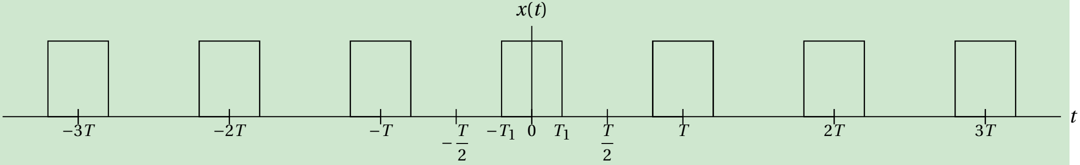
Let us consider the signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0).$$

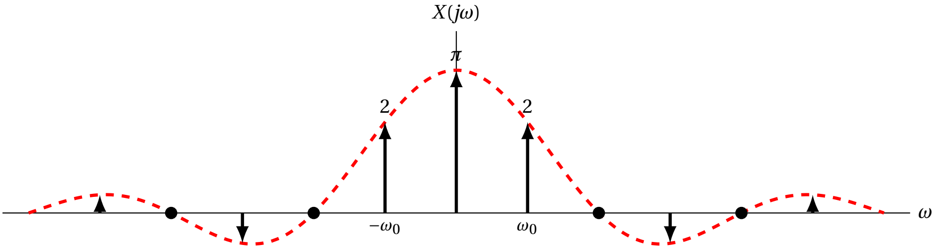
To determine the signal $x(t)$ we can apply the inverse Fourier transform relation
This corresponds exactly to the Fourier series representation of a periodic signal.
Thus the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at harmonically related frequencies for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the $k$the Fourier series coefficient a_k.

Example

Consider the square wave.



1. Obtain the Fourier series coefficients $\{a_k\}$.
2. Obtain the Fourier transform of the signal.



$X(j\omega)$
π
2
2
$-\omega_0$
ω_0
ω

Example

Sketch the Fourier transform of

1. $x(t) = \sin \omega_0 t$
2. $x(t) = \cos \omega_0 t$

Example

Obtain the Fourier transform of the impulse train given by

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

which is periodic with period T. Sketch.

Section 5

Sampling

- The sampling theorem, which is a relatively straightforward consequence of the modulation theorem, is elegant in its simplicity.
- It states that a bandlimited time function can be exactly reconstructed from equally spaced samples provided that the sampling rate is sufficiently high—specifically, that it is greater than twice the highest frequency present in the signal.
- A similar result holds for both continuous time and discrete time.
- One of the important consequences of the sampling theorem is that it provides a mechanism for exactly representing a bandlimited continuous-time signal by a sequence of samples, that is, by a discrete-time signal.
- The reconstruction procedure consists of processing the impulse train of samples by an ideal lowpass filter.

Sampling Frequency and Aliasing

- Assumption: the sampling frequency is greater than twice the highest frequency in the signal.
- The reconstructing lowpass filter will always generate a reconstruction consistent with this constraint, even if the constraint was purposely or inadvertently violated in the sampling process.
- Said another way, the reconstruction process will always generate a signal that is bandlimited to less than half the sampling frequency and that matches the given set of samples.
- If the original signal met these constraints, the reconstructed signal will be identical to the original signal.
- On the other hand, if the conditions of the sampling theorem are violated, then frequencies in the original signal above half the sampling frequency become reflected down to frequencies less than half the sampling frequency.
- This distortion is commonly referred to as aliasing, a name suggestive of the fact that higher frequencies (above half the sampling frequency) take on the alias of lower frequencies.

Sampling Theorem

- Equally spaced samples of $x(t)$

$$x(nT), \quad n = 0, \pm 1, \pm 2, \ldots$$

- $x(t)$ is band limited

$$X(\omega) = 0, \quad |\omega| > \omega_M$$

- If

$$\frac{2\pi}{T} \triangleq \omega_s > 2\omega_M$$

- then $x(t)$ is uniquely recoverable.

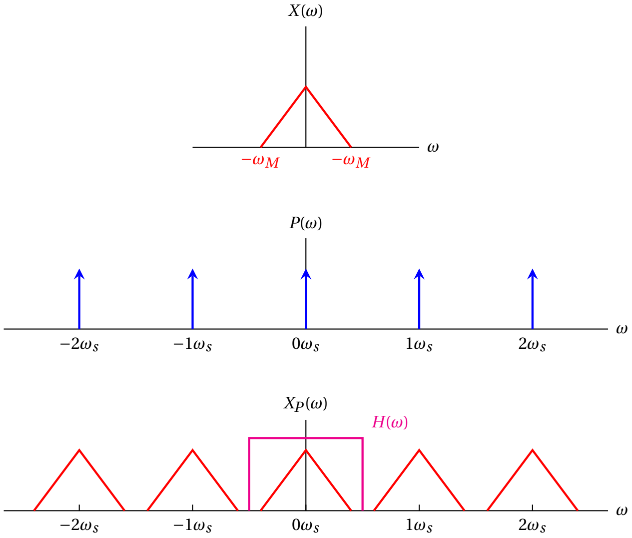
$$x_p(t) = x(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)$$

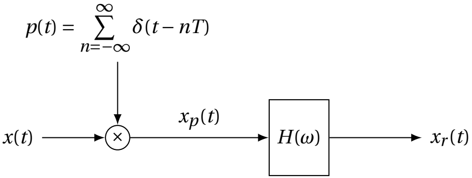
$$X_p(\omega) = \frac{1}{2\pi}\left[X(\omega) * P(\omega)\right]$$

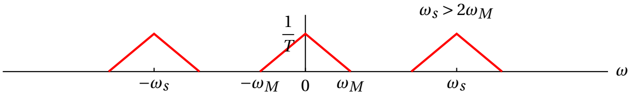
$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - k\frac{2\pi}{T}\right)$$

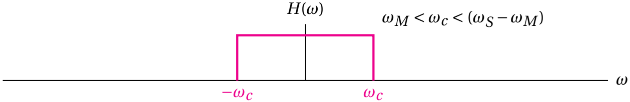
$$X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$



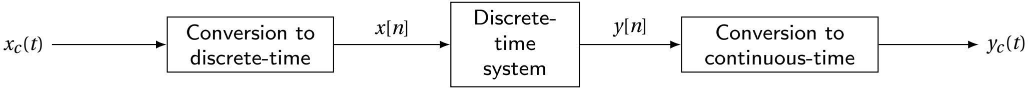
$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
$H(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

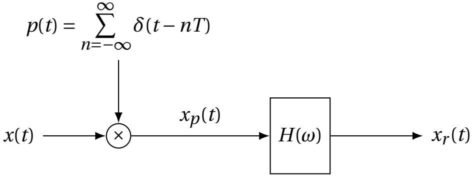

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
$x(t)$
$x_p(t)$
$H(\omega)$
$x_r(t)$


$\omega_s > 2\omega_M$
$\frac{1}{T}$
ω
$-\omega_s$
$-\omega_M$
0
ω_M
ω_s


$H(\omega)$
$\omega_M < \omega_c < (\omega_S - \omega_M)$
ω
$-\omega_c$
ω_c

Discrete-Time Processing of Continuous-Time Signals





$$
\begin{aligned}
x_p(t) &= x(t)p(t) \\
&= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)
\end{aligned}
$$

$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)
\end{aligned}
$$