EN1060 Signals and Systems: Fourier Transform Properties

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Section 1

Fourier Transform Properties

Fourier Transform: Recall

Synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \tag{1}$$

Analysis equation:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{2}$$

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega). \tag{3}$$

Linearity

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

and

$$y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(j\omega).$$

then

Time Shifting

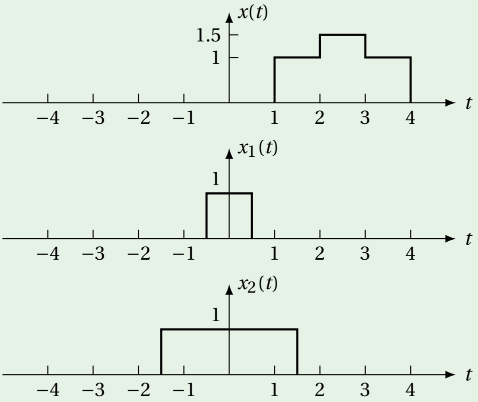
If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

Example

Evaluate the Fourier transform of $x(t)$.



Conjugation and Conjugate Symmetry

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega).$$

then

$$x^*(t) \xleftrightarrow{\mathscr{F}} X^*(-j\omega).$$

Using Conjugate Symmetry

Use the conjugate property to comment about the symmetry of Fourier transform of a signal $x(t)$ if

1. $x(t)$ is real,
2. $x(t)$ is real and even, and
3. $x(t)$ is real and odd.

Expressing $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathfrak{Re}\left\{X(j\omega)\right\} + j\mathfrak{Im}\left\{X(j\omega)\right\},$$

then if $x(t)$ is real $[x(t) = x^*(t)]$

$$\mathfrak{Re}\left\{X(j\omega)\right\} = \mathfrak{Re}\left\{X(-j\omega)\right\} \quad \text{and}$$
$$\mathfrak{Im}\left\{X(j\omega)\right\} = -\mathfrak{Im}\left\{X(-j\omega)\right\}$$

That is, the real part of the Fourier transform is an even function of frequency, and the imaginary part is an odd function of frequency. Considering

$$X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)},$$

we see that $|X(j\omega)|$ is an even function of frequency, and $\angle X(j\omega)$ is an odd function of frequency.

If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.
Proof:

Fourier Transforms of Odd and Even Parts

A real function $x(t)$ can be expressed as

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t) = \mathfrak{Ev}\{x(t)\}$ is the even part of $x(t)$ and $x_o(t) = \mathfrak{Od}\{x(t)\}$ is the odd part of $x(t)$. Express Fourier transforms of

1. $x_e(t) = \mathfrak{Ev}\{x(t)\}$, and
2. $x_o(t) = \mathfrak{Od}\{x(t)\}$.

in terms of $X(j\omega)$.

Example

Use the symmetry properties of the Fourier transform to evaluate the Fourier transform of

$$x(t) = e^{-a|t|}, \quad a > 0.$$

We have already found that

$$e^{-at} \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{a+j\omega}.$$

$$\begin{aligned} x(t) = e^{-a|t|} &= e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathfrak{Ev}\left\{e^{-at}u(t)\right\}. \end{aligned}$$

Since $e^{-at}u(t)$is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathfrak{Ev}\left\{e^{-at}u(t)\right\} \overset{\mathscr{F}}{\longleftrightarrow} \mathfrak{Re}\left\{\frac{1}{a+j\omega}\right\}.$$

$$X(j\omega) = 2\mathfrak{Re}\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}.$$

Differentiation and Integration

Synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$$

Differentiating both sides of the equation

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega X(j\omega).$$

Integration:

$$\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega).$$

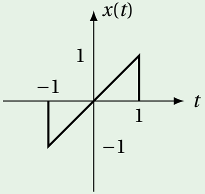
Example

Determine the Fourier transform of the unit step $x(t) = u(t)$ making use of the knowledge that

$$g(t) = \delta(t) \xleftrightarrow{\mathscr{F}} G(j\omega) = 1.$$

Example

Determine the Fourier transform of the signal $x(t)$ shown below:



Time and Frequency Scaling

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x(at) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

where a is a real constant.
Letting $a = -1$

$$x(-t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(-j\omega).$$

The scaling property is another example of the inverse relationship between time and frequency.

Duality

Because of the similarity between the synthesis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \tag{4}$$

and the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{5}$$

for any transform pair, there is a dual pair with the time and frequency variables interchanged.

We determined the Fourier transform of the square pulse as

$$x_1(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1, \end{cases} \xleftrightarrow{\mathscr{F}} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

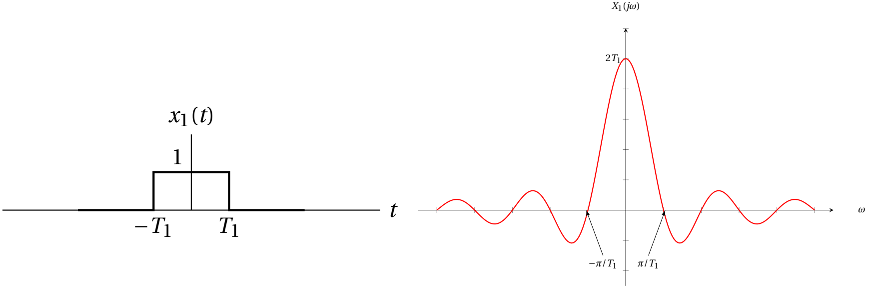


Figure: Rectangular pulse and the Fourier transform.

We also determined that for a time-domain signal that is similar in shape to the $X_1(j\omega)$ as

$$x_2(t) = \frac{\sin Wt}{\pi t} \overset{\mathscr{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

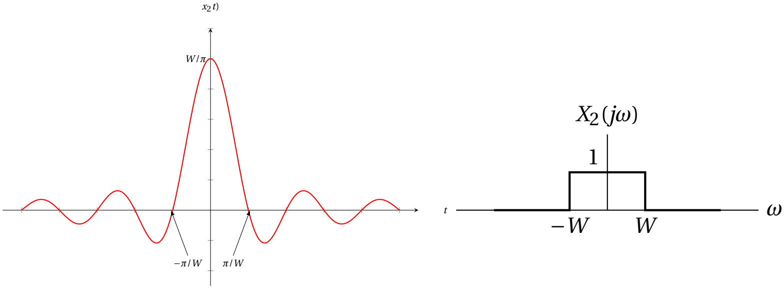


Figure: Fourier transform for $x(t)$.

Example

Use the duality property to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

More Properties Using Duality

$$-jtx(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.$$

$$e^{j\omega_0 t}x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)\, d\eta.$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

Section 2

The Convolution Property

Convolution Property

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathscr{F}} Y(j\omega) = H(j\omega)X(j\omega)$$

This equation is of major importance in signal and system analysis. This says that the Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

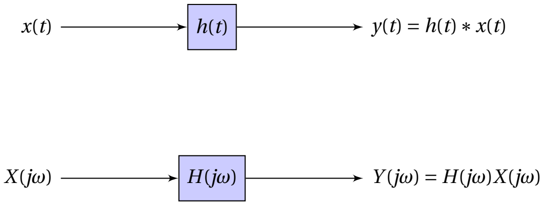


Figure: Convolution property.

Example

An LTI system has the impulse response

$$h(t) = \delta(t - t_0).$$

If the Fourier transform of the input signal $x(t)$ is $X(j\omega)$, what is the Fourier transform of the output?

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\begin{aligned}
h(t) &= \delta(t - t_0) \\
H(j\omega) &= e^{-j\omega t_0} \\
Y(j\omega) &= H(j\omega)X(j\omega) \\
&= e^{-j\omega t_0} X(j\omega)
\end{aligned}$$

Example

What is the frequency response of the differentiator?

The input output relationship of the differentiator is

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property

$$Y(j\omega) = j\omega X(j\omega).$$

Consequently, the frequency response of the differentiator is

$$H(j\omega) = j\omega.$$

Example

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, find $y(t)$ by transfroming the problem into the frequency domain.

Multiplication Property

The convolution property states that convolution in time domain corresponds to multiplication in frequency domain. Because of the duality between time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \xleftrightarrow{\mathscr{F}} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)].$$

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other. Consequently, the multiplication of two signals is often referred to as amplitude modulation. For this reason, this equation is sometime referred to as the modulation property.

Example

Let $s(t)$ be a signal whose spectrum is depicted in the figure below. Also consider the signal

$$p(t) = \cos\omega_0 t.$$

Show the spectrum of $r(t) = s(t)p(t)$.

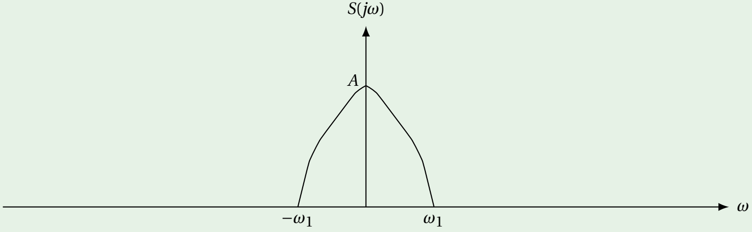


Figure: Spectrum of signal $s(t)$.