

EN1060 Signals and Systems: Signals

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

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Section 1

Signals

① Signals

Sinusoids

Discrete-Time Sinusoidal Signal

Exponentials

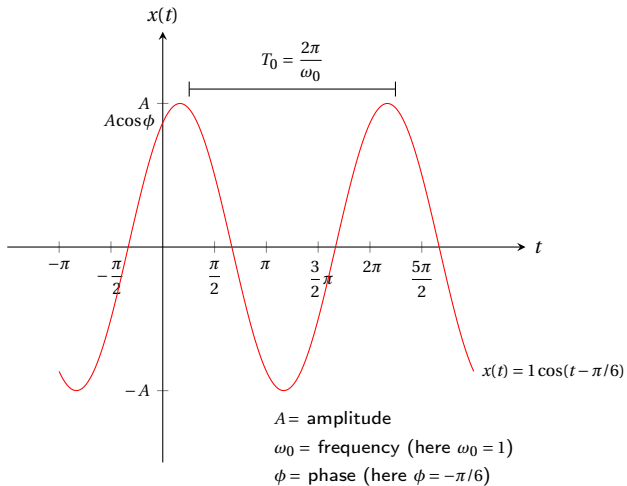
CT Complex Exponentials

Step and Impulse Functions

Signal Energy and Power

Continuous-Time Sinusoidal Signal

$$x(t) = A \cos(\omega_0 t + \phi). \quad (1)$$



Sinusoidal signal is **periodic**.

A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t + T) \quad (2)$$

for all values of t . Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T .

Fundamental period T_0 = smallest positive value of T for which 2 holds.

A signal that is not periodic is referred to as aperiodic.

E.g.: Consider $A\cos(\omega_0 t + \phi)$

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0(t + T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

$$T = \frac{2\pi m}{\omega_0} \Rightarrow \text{fundamental period } T_0 = \frac{2\pi}{\omega_0}.$$

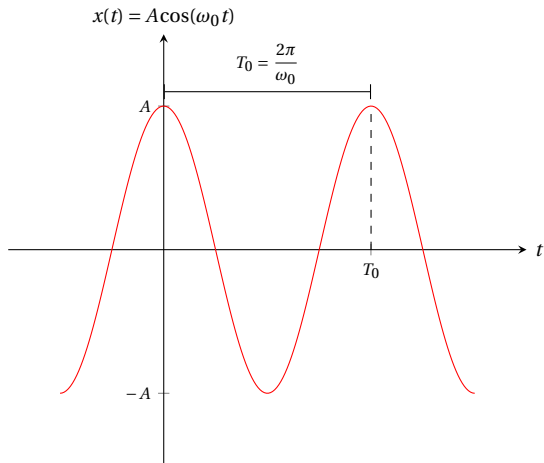
A time-shift in a CT sinusoid is equivalent to a phase shift.

E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

$$A\cos[\omega_0(t + t_0)] = A\cos(\omega_0 t + \omega_0 t_0) = A\cos(\omega_0 t + \Delta\phi), \quad \Delta\phi \text{ is a change in phase.}$$

$$A\cos[\omega_0(t + t_0) + \phi] = A\cos(\omega_0 t + \omega_0 t_0 + \phi) = A\cos(\omega_0(t + t_1)), \quad t_1 = t_0 + \phi/\omega_0.$$

Phase of a Sinusoidal: $\phi = 0$

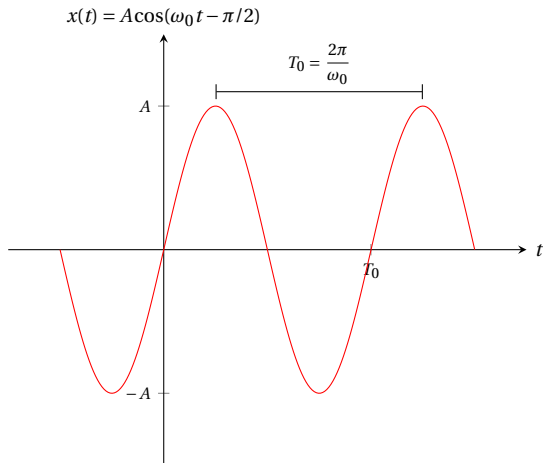


This signal is **even**. If we mirror an even signal about the time origin, it would look exactly the same.

Periodic: $x(t) = x(t + T)$.

Even: $x(t) = x(-t)$.

Phase of a Sinusoidal: $\phi = -\pi/2$



This signal is **odd**. If we flip an odd signal about the time origin, we also multiply it by a $(-)$ sign to get the original signal.

Periodic: $x(t) = x(t + T)$.

Odd: $x(t) = -x(-t)$.

① Signals

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Discrete-Time Sinusoidal Signal

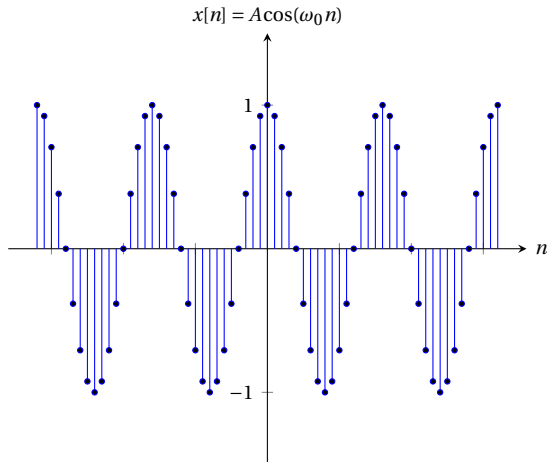
Exponentials

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$$x[n] = A \cos(\omega_0 n + \phi) \text{ with } \phi = 0$$



The independent variable is an integer.

The sequence takes values only at integer values of the argument.

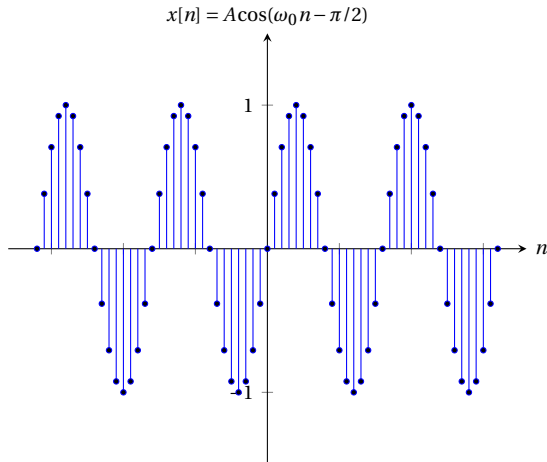
This signal is **even**.

Even: $x[n] = x[-n]$.

Periodic: $x[n] = x[n + N]$. Here, $N = 16$

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}.$$

$$x[n] = A \cos(\omega_0 n + \phi) \text{ with } \phi = -\pi/2$$



The independent variable is an integer.

The sequence takes values only at integer values of the argument.

This signal is **odd**.

Odd: $x[n] = -x[-n]$.

Periodic: $x[n] = x[n + N]$. Here, $N = 16$

$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}$. $\phi = -\pi/2$, $x[n] = A \cos(\omega_0 n + \phi) = A \cos(\omega_0(n + n_0))$. n_0 must be an integer.

$$n_0 = \frac{\phi}{\omega_0} = \frac{-\pi/2}{\pi/8} = -4.$$

Q

Does a phase change always correspond to a time shift in discrete-time signals?

Q

Does a phase change always correspond to a time shift in discrete-time signals?

Answer: No.

$$A\cos[\omega_0 n + \phi] = A\cos[\omega_0(n + n_0)]$$

$$\omega_0 n + \omega_0 n_0 = \omega_0 n + \phi$$

$$\omega_0 n_0 = \phi \quad n_0 \text{ is an integer.}$$

- Depending on ϕ and ω_0 , n_0 may not come out to be an **integer**.
- In discrete time, the amount of time shift must be an integer.

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n + N], \quad \text{smallest integer } N \text{ is the fundamental period.} \quad (3)$$

$$A \cos[\omega_0(n + N) + \phi] = A \cos[\omega_0 n + \omega_0 N + \phi]$$

$\omega_0 N$ must be an integer multiple of 2π .

$$\text{Periodic} \Rightarrow \omega_0 N = 2\pi m$$

$$N = \frac{2\pi m}{\omega_0} \quad (4)$$

N and m must be integers.

Smallest N , if any, is the fundamental period.

N may not be an integer. In this case, the signal is not periodic.

① Signals

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Discrete-Time Sinusoidal Signal

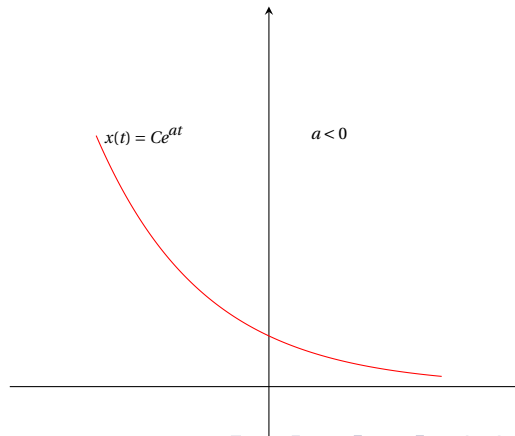
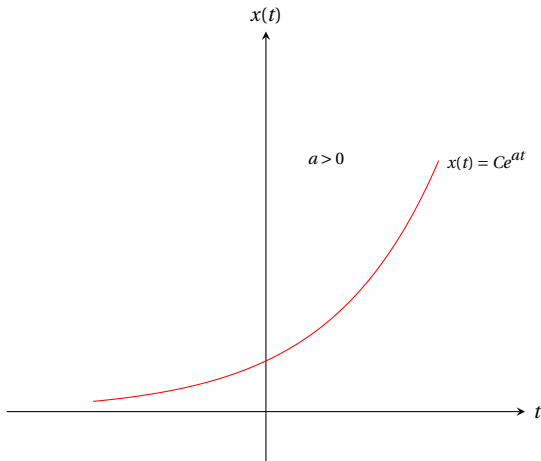
Exponentials

CT Complex Exponentials

Step and Impulse Functions

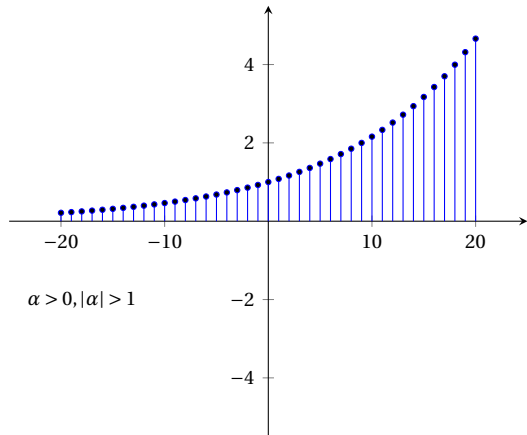
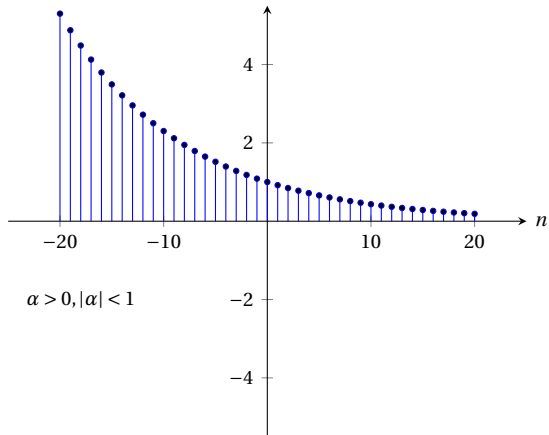
Signal Energy and Power

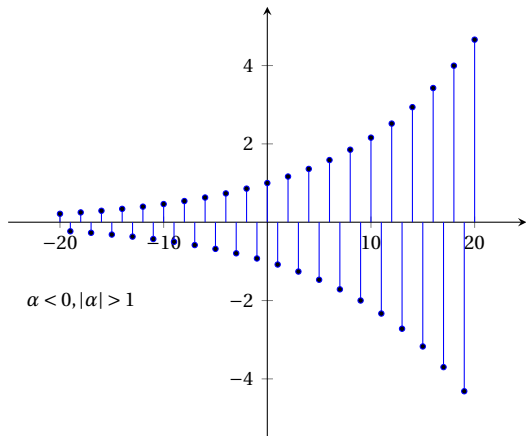
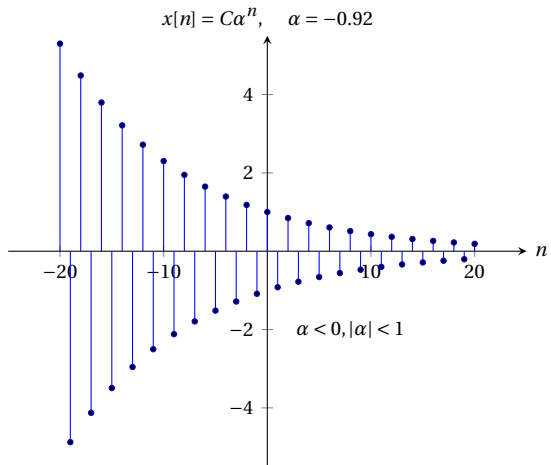
$$\begin{aligned}x(t) &= Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers} \\&= Ce^{at_0} e^{at}.\end{aligned}$$



$$x[n] = Ce^{\beta n} = C\alpha^n, \quad C \text{ and } \alpha \text{ are real numbers}$$

$$x[n] = C\alpha^n, \quad \alpha = 0.92$$





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$$x(t) = Ce^{at} \quad C \text{ and } a \text{ are complex numbers.}$$

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

$$x(t) = |C|e^{j\theta} e^{(r+j\omega_0)t}$$

$$= |C|e^{rt} e^{j(\omega_0 t + \theta)}$$

$$= |C|e^{rt} [\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)]$$

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- $e^{j(\omega_0 t + \theta)} = \cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)$

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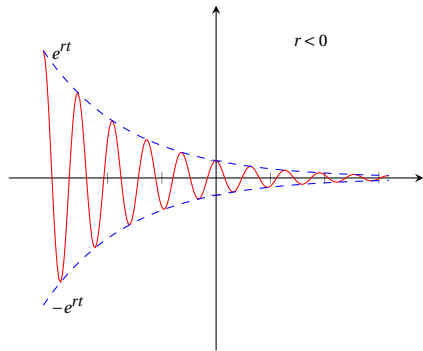
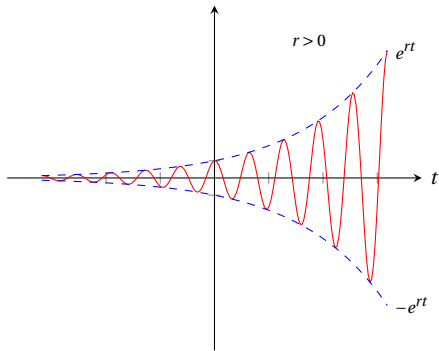
$$= |C|e^{rt} [\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)]$$

- $e^{j(\omega_0 t + \theta)} = \cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)$

- Real

- 90° out of phase

$$x(t) = |C|e^{rt} \cos(\omega_0 t + \phi)$$



$$x[n] = C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \quad (5)$$

$$C = |C|e^{j\theta} \quad (6)$$

$$\alpha = |\alpha|e^{j\omega_0} \quad (7)$$

$$x[n] = |C|e^{j\theta} \left(|\alpha|e^{j\omega_0} \right)^n \quad (8)$$

$$= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta) \quad (9)$$

$$(10)$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0 .
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

- For the CT counterpart $e^{j\omega_0 t}$,

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- In DT, as

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}$$

the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0 .

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- Although in CT $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0 . In DT, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 + 2\pi$, $\omega_0 + 4\pi$, and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0 .

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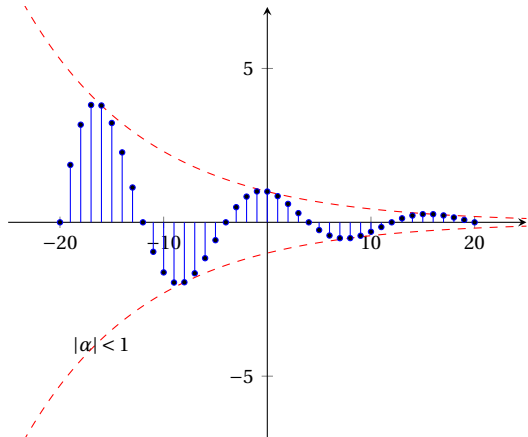
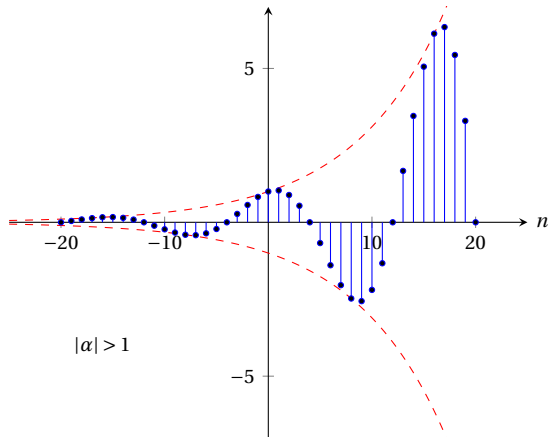
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- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0 , we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = (e^{j\pi})^n = (-1)^n$.

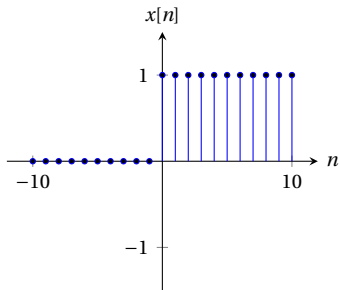
DT Complex Exponentials Plot

$$x[n] = |C|\alpha^n \cos(\omega_0 n + \theta), \quad |\alpha| = 1.12, \theta = 0$$



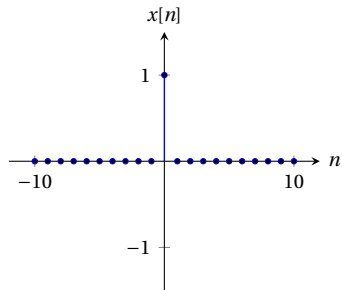
Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \quad (11)$$



Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

$$u[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \quad (12)$$



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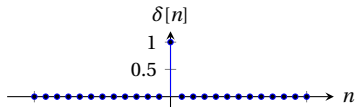
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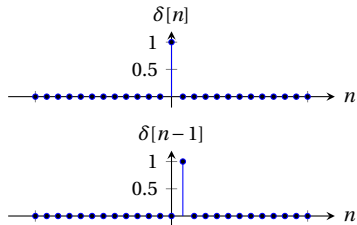
Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \quad (13)$$



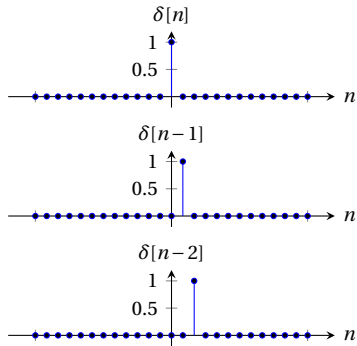
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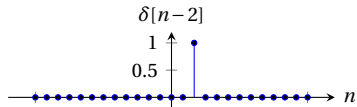
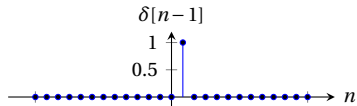
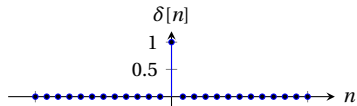
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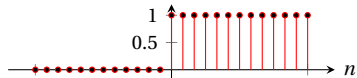


Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \quad (13)$$



$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \dots$$

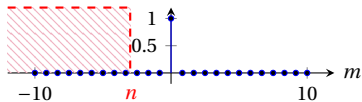


The unit step sequence is the running sum of the unit impulse.

$$u[n] = \sum_{m=-\infty}^n \delta[m]. \quad (14)$$

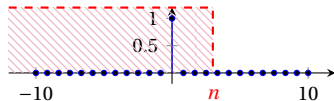
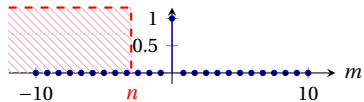
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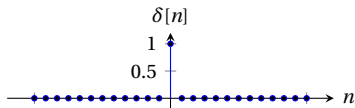


The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \quad (15)$$

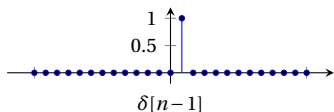
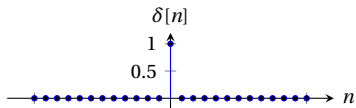
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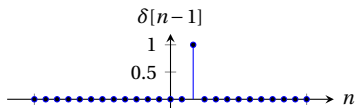
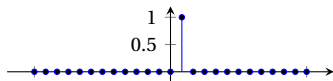
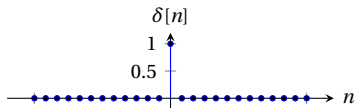
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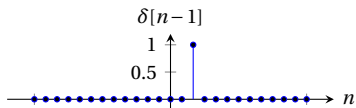
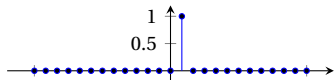
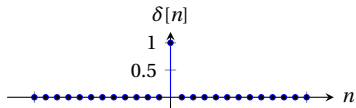
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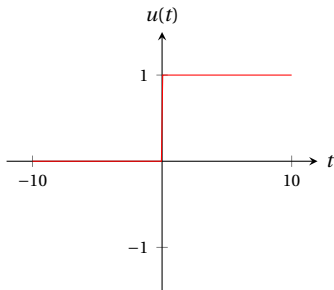
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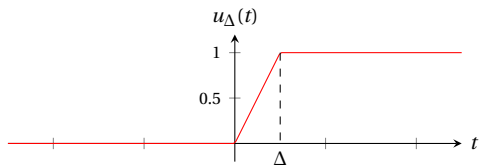


Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \quad (16)$$

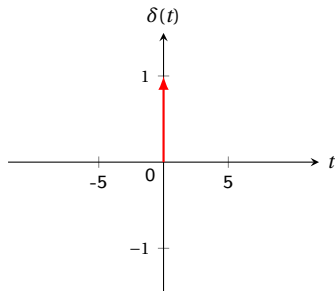


Continuous-Time Unit Impulse Function $\delta(t)$

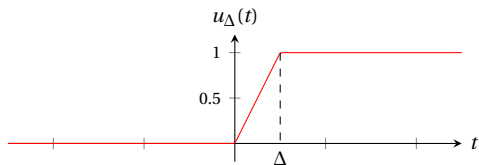


$u_\Delta(t) \rightarrow u(t)$ as $\Delta \rightarrow 0$.

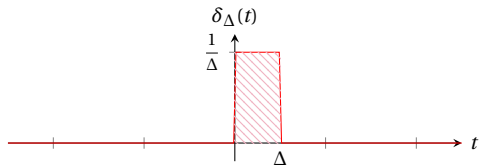
$$\delta(t) = \frac{du(t)}{dt}. \quad (17)$$



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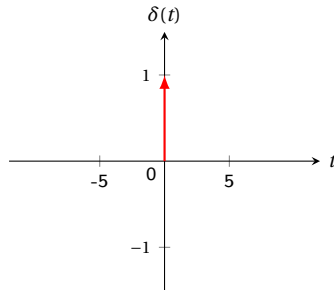


$u_\Delta(t) \rightarrow u(t)$ as $\Delta \rightarrow 0$.

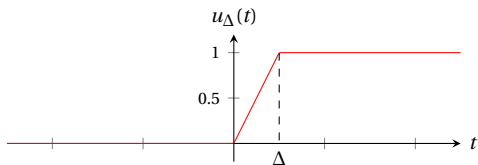


$\delta_\Delta(t) \rightarrow \delta(t)$ as $\Delta \rightarrow 0$.
area = 1

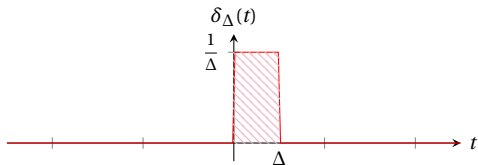
$$\delta(t) = \frac{du(t)}{dt}. \quad (17)$$



Continuous-Time Unit Impulse Function $\delta(t)$

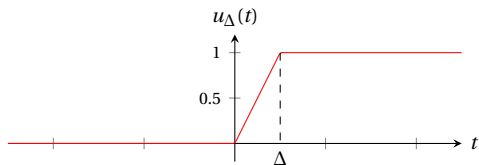


$u_\Delta(t) \rightarrow u(t)$ as $\Delta \rightarrow 0$.

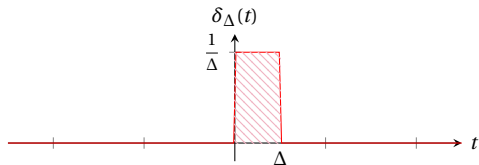


$\delta_\Delta(t) \rightarrow \delta(t)$ as $\Delta \rightarrow 0$.
area = 1

Continuous-Time Unit Impulse Function $\delta(t)$

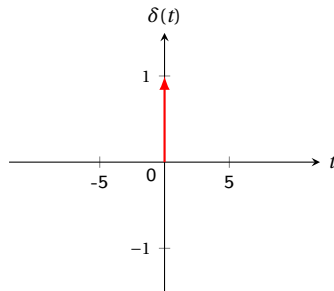


$u_\Delta(t) \rightarrow u(t)$ as $\Delta \rightarrow 0$.



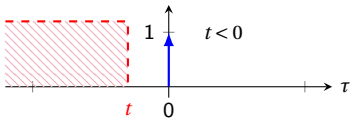
$\delta_\Delta(t) \rightarrow \delta(t)$ as $\Delta \rightarrow 0$.
area = 1

$$\delta(t) = \frac{du(t)}{dt}. \quad (17)$$



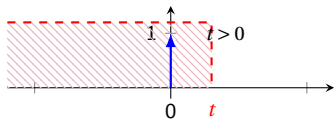
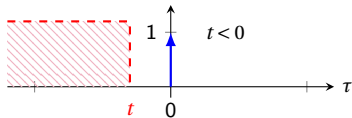
CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau. \quad (18)$$



CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau. \quad (18)$$



① Signals

Sinusoids

Discrete-Time Sinusoidal Signal

Exponentials

CT Complex Exponentials

Step and Impulse Functions

Signal Energy and Power

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

Total energy over an infinite interval in a CT signal:

$$E_{\infty} \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \quad (19)$$

Total energy over an infinite interval in a DT signal:

$$E_{\infty} \triangleq \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \quad (20)$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Time-averaged power over an infinite interval in a CT signal:

$$P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt. \quad (21)$$

Total energy in a DT signal:

$$P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \quad (22)$$

With these definitions, we can identify three important classes of signals:

- ① Energy signals: Signals with finite total energy $E_{\infty} < \infty$. These have zero average power.
- ② Power signals: Signals with finite average power $0 < P_{\infty} < \infty$. As $P_{\infty} > 0$, $E_{\infty} = \infty$.
- ③ Signals with neither E_{∞} nor P_{∞} are finite.