EN1060 Signals and Systems: Signals

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December 14, 2020

Section 1

Signals

Outline

Continuous-Time Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi). \tag{1}$$

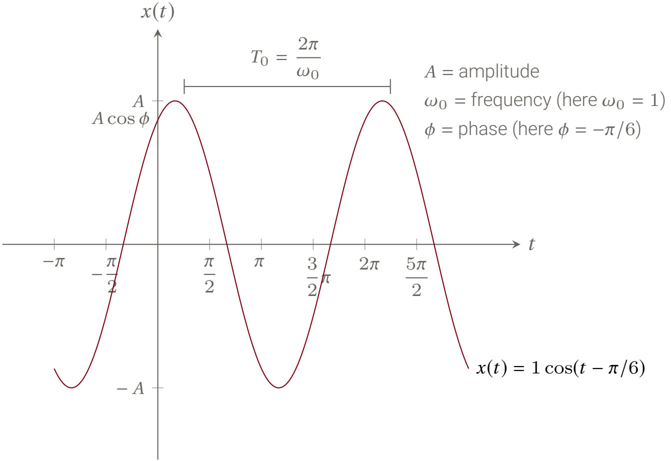

Figure: Continuous-time sinusoidal signal.

Periodicity of a Sinusoidal

Sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t+T) \tag{2}$$

for all values of t. Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t + \phi)$

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0(t+T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

$T = \dfrac{2\pi m}{\omega_0} \Rightarrow$ fundamental period $T_0 = \dfrac{2\pi}{\omega_0}$.

Phase of a Sinusoidal

A time-shift in a CT sinusoid is equivalent to a phase shift.
E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

$$A\cos[\omega_0(t+t_0)] = A\cos(\omega_0 t + \omega_0 t_0) = A\cos(\omega_0 t + \Delta\phi), \quad \Delta\phi \text{ is a change in phase.}$$

$$A\cos[\omega_0(t+t_0)+\phi] = A\cos(\omega_0 t + \omega_0 t_0 + \phi) = A\cos(\omega_0(t+t_1)), \quad t_1 = t_0 + \phi/\omega_0.$$

Even and Odd Signals

A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t)$$

$$x[-n] = x[n]$$

A is referred to as an odd if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

An odd signal must be) at $t = 0$ or $n = 0$.
A signal can be broken into a sum of two signals, one of which is even and one for which is odd. Even part of $x(t)$ is

$$\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

Odd part of $x(t)$ is

$$\mathfrak{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \dfrac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

$$x(t) = x_e(t) + x_o(t).$$

Even and Odd Signals Contd.

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$$x(-t) = x_e(-t) + x_o(-t).$$

$$x(-t) = x_e(t) - x_o(t).$$

Adding,

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t).$$

Even and Odd Signals Contd.

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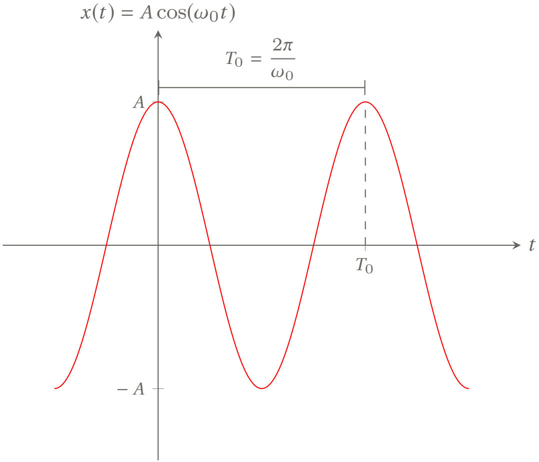
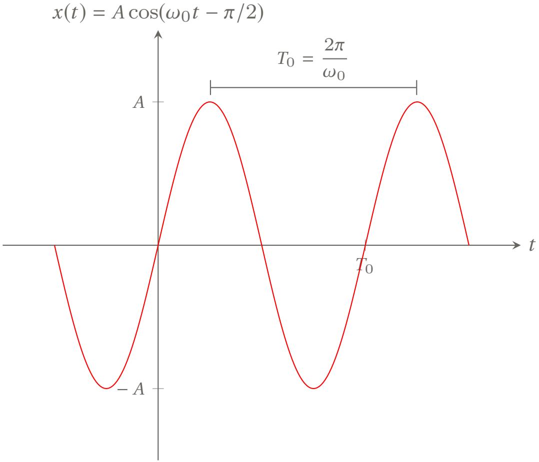
Phase of a Sinusoidal: $\phi = 0$



This signal is even. If we mirror an even signal about the time origin, it would look exactly the same.

Periodic: $x(t) = x(t+T)$.
Even: $x(t) = x(-t)$.

Phase of a Sinusoidal: $\phi = -\pi/2$

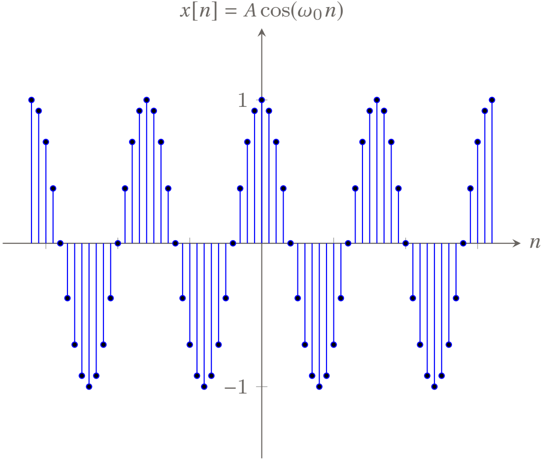


This signal is odd. If we flip an odd signal about the time origin, we also multiply it by a (–) sign to get the original signal.

Periodic: $x(t) = x(t+T)$.
Odd: $x(t) = -x(-t)$.

Outline

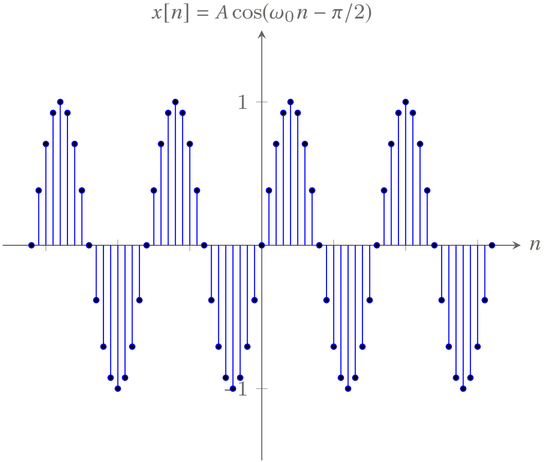
$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = 0$



The independent variable is an integer.
The sequence takes values only at integer values of the argument.
This signal is even.

Even: $x[n] = x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$
$\omega_0 = \dfrac{2\pi}{N} = \dfrac{\pi}{8}$.

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = -\pi/2$



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The sequence takes values only at inter values of he argument.
This signal is odd.

Odd: $x[n] = -x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$
$\omega_0 = \dfrac{2\pi}{N} = \dfrac{\pi}{8}$. $\phi = -\pi/2$, $x[n] = A\cos(\omega_0 n + \phi) = A\cos(\omega_0(n + n_0))$.
n_0 must be an integer.
$n_0 = \dfrac{\phi}{\omega_0} = \dfrac{\pi/2}{\pi/8} = 4$.

Phase Change and Time Shift in DT

Question

Does a phase change always correspond to a time shift in discrete-time signals?

Phase Change and Time Shift in DT

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Does a phase change always correspond to a time shift in discrete-time signals?

Answer: No.

$$A\cos[\omega_0 n + \phi)] \stackrel{?}{=} A\cos[\omega_0(n + n_0)]$$

$$\omega_0 n + \omega_0 n_0 = \omega_0 n + \phi$$

$$\omega_0 n_0 = \phi, \quad n_0 \text{ is an integer.}$$

- Depending on ϕ and ω_0, n_0 many not come out to be an integer.
- In discrete time, the amount of time shift must be an integer.

Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

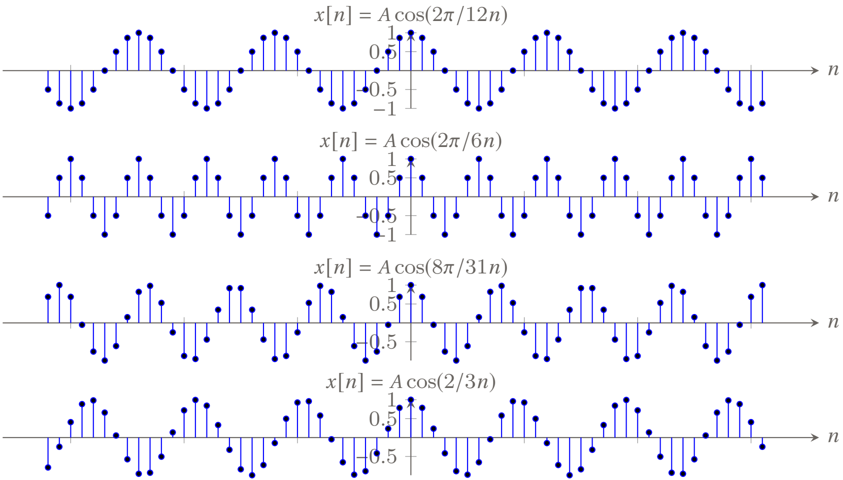
$$A\cos[\omega_0(n+N)+\phi] = A\cos[\omega_0 n + \omega_0 N + \phi]$$

$\omega_0 N$ must be an integer multiple of 2π.
Periodic $\Rightarrow \omega_0 N = 2\pi m$

$$N = \frac{2\pi m}{\omega_0} \tag{4}$$

N and m must be integers.
Smallest N, if any, is the fundamental period.
N may not be an integer. In this case, the signal is not periodic.

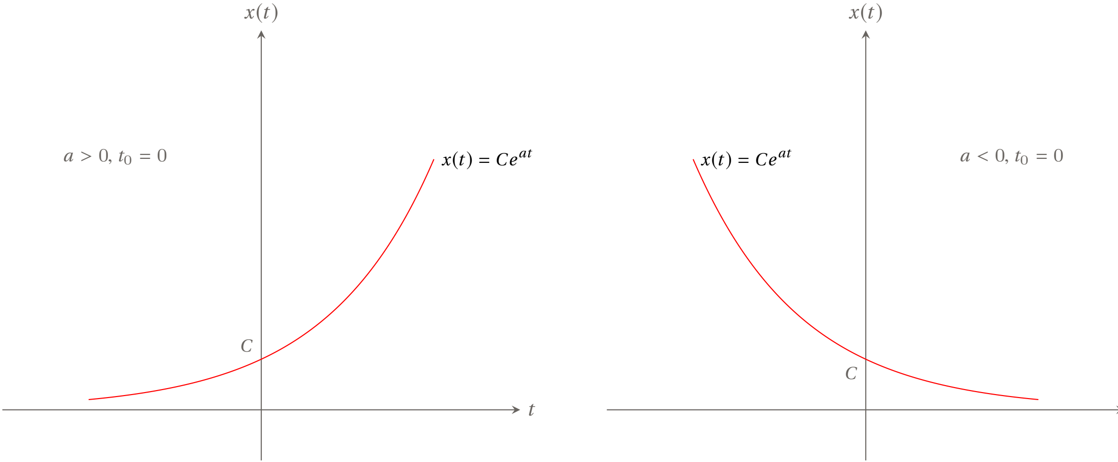
Periodicity of a DT Signal Cntd.



Outline

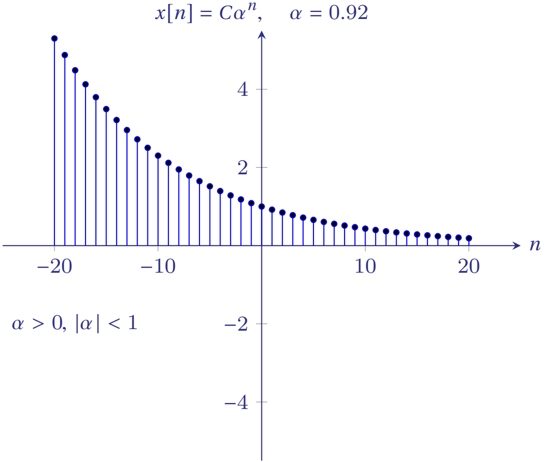
CT Real Exponentials

$$x(t) = Ce^{a(t+t_0)}, \qquad C \text{ and } a \text{ are real numbers}$$
$$= Ce^{at_0}e^{at}.$$

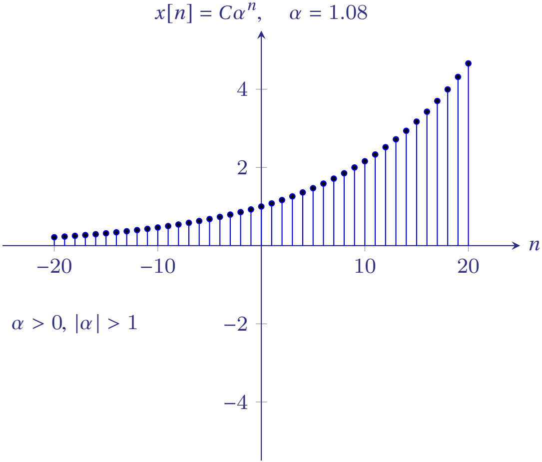


DT Real Exponentials

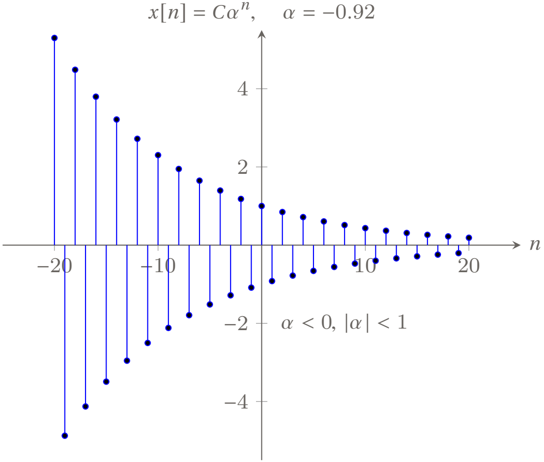
$x[n] = Ce^{\beta n} = C\alpha^n,$ $\quad C$ and α are real numbers



$x[n] = C\alpha^n, \quad \alpha = 0.92$
4
2
−2
−4
−20
−10
10
20
n
$\alpha > 0,\ |\alpha| < 1$



$x[n] = C\alpha^n, \quad \alpha = 1.08$
4
2
−2
−4
−20
−10
10
20
n
$\alpha > 0,\ |\alpha| > 1$



$x[n] = C\alpha^n, \quad \alpha = -0.92$
4
2
−2
−4
−20
−10
10
20
n
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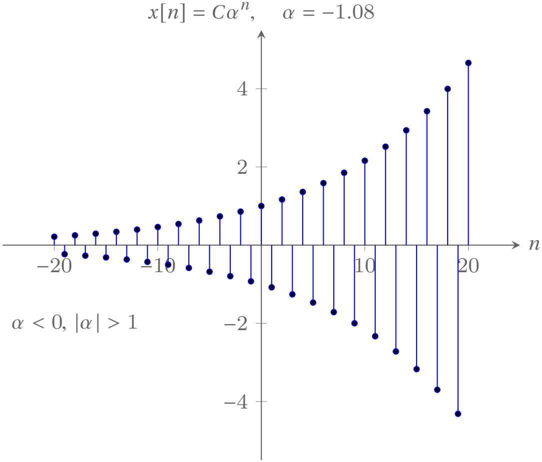

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Outline

CT Complex Exponentials

$x(t) = Ce^{at}$ C and a are complex numbers.

$$
\begin{aligned}
C &= |C|e^{j\theta} \\
a &= r + j\omega_0 \\
x(t) &= |C|e^{j\theta}e^{(r+j\omega_0)t} \\
&= |C|e^{rt}\, e^{j(\omega_0 t+\theta)} \\
&= |C|e^{rt}\, [\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)]
\end{aligned}
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- $e^{j(\omega_0 t+\theta)} = \cos(\omega_0 t + \theta) + j\sin(\omega_0 t + \theta)$

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\end{aligned}$$

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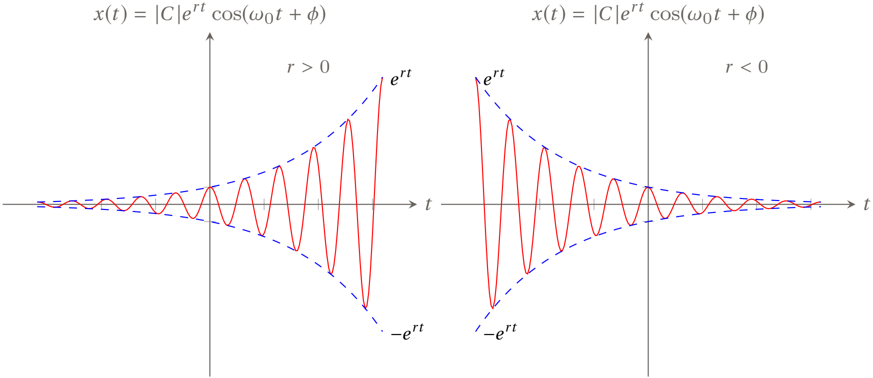
$$

- $e^{j(\omega_0 t+\theta)} = \cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)$
- Real
- 90° out of phase



$x(t) = |C|e^{rt}\cos(\omega_0 t + \phi)$
$r > 0$
e^{rt}
$-e^{rt}$
t
$x(t) = |C|e^{rt}\cos(\omega_0 t + \phi)$
$r < 0$
e^{rt}
$-e^{rt}$
t

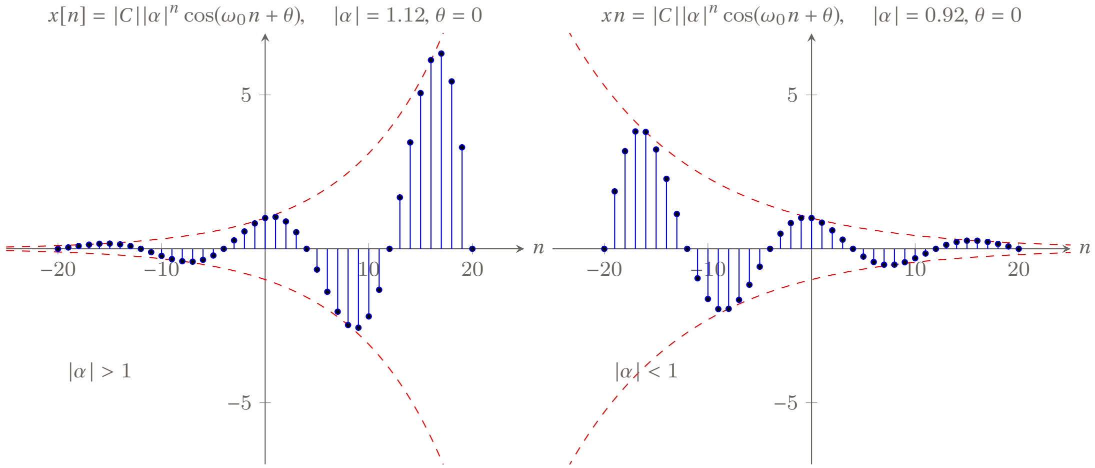
DT Complex Exponentials

$$
\begin{aligned}
x[n] &= C\alpha^n, \qquad C \text{ and } \alpha \text{ are complex numbers.} \\
C &= |C|e^{j\theta} \\
\alpha &= |\alpha|e^{j\omega_0} \\
x[n] &= |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \\
&= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)
\end{aligned}
$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

DT Complex Exponentials Plot



Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

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- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

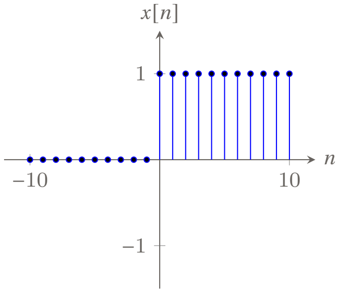
Outline

Signals

- Sinusoids
- Discrete-Time Sinusoidal Signal
- Exponentials
- CT Complex Exponentials
- Step and Impulse Functions
- Signal Energy and Power

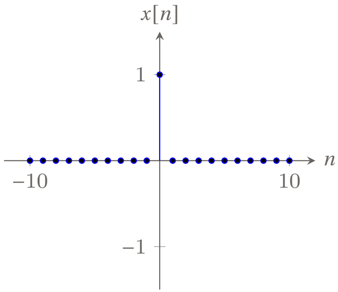
Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{5}$$



Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

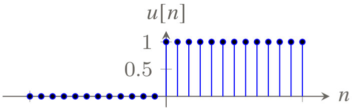
$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{6}$$



DT Step and Impulse

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \tag{7}$$



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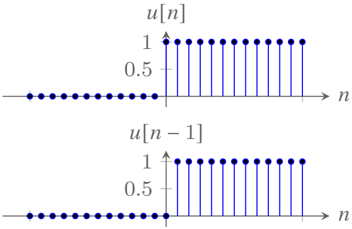

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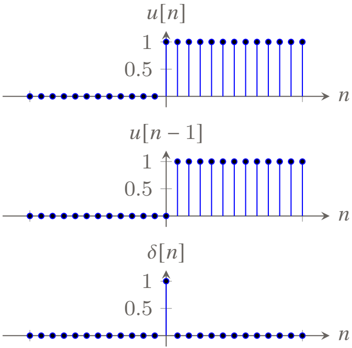

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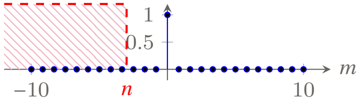

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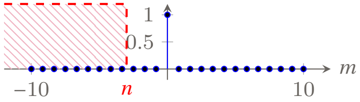

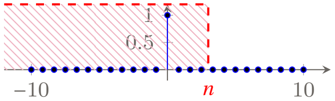

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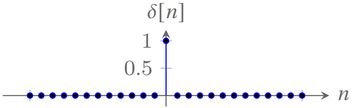

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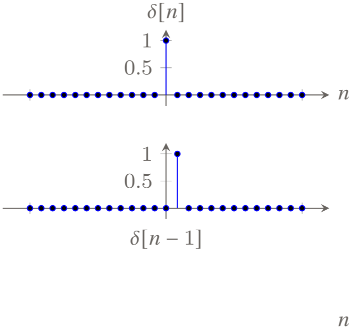

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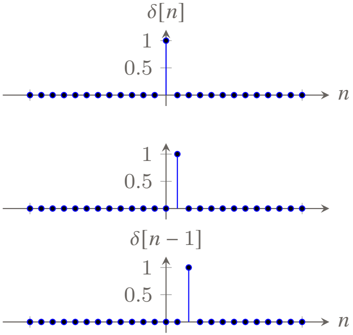

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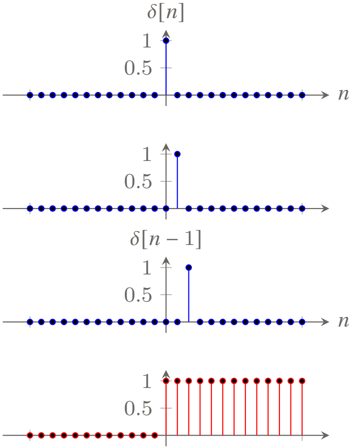
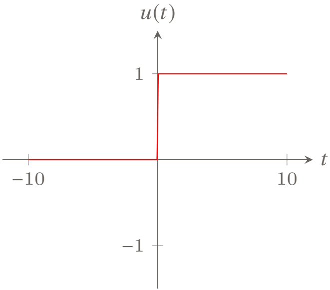
The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{9}$$

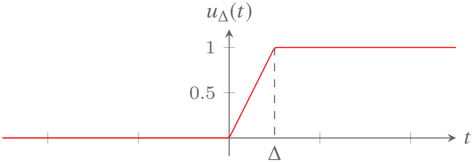


Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{10}$$

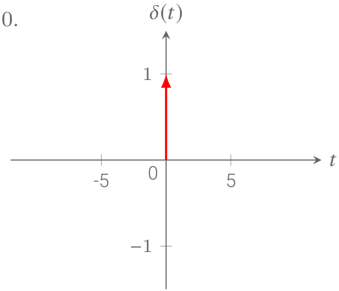


Continuous-Time Unit Impulse Function $\delta(t)$



$u_\Delta(t) \to u(t)$ as $\Delta \to 0$.

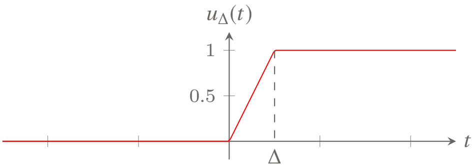
$$\delta(t) = \frac{du(t)}{dt}. \quad (11)$$



$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Continuous-Time Unit Impulse Function $\delta(t)$



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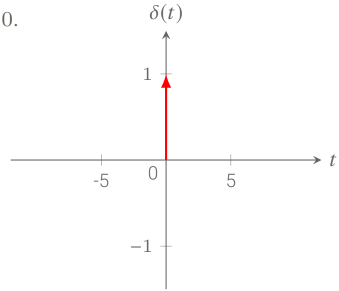
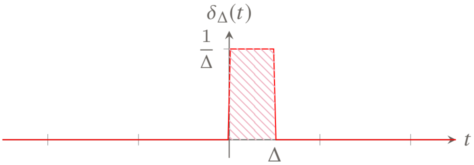


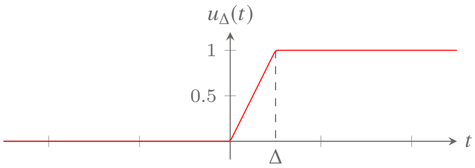


$\delta_\Delta(t) \to \delta(t)$ as $\Delta \to 0$.
area = 1

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Continuous-Time Unit Impulse Function $\delta(t)$



$u_\Delta(t) \to u(t)$ as $\Delta \to 0$.

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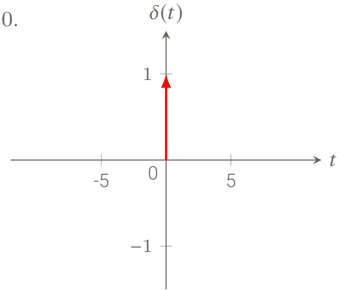
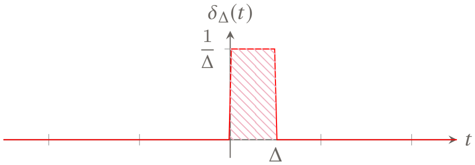




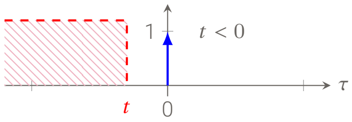
$\delta_\Delta(t) \to \delta(t)$ as $\Delta \to 0$.
area = 1

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

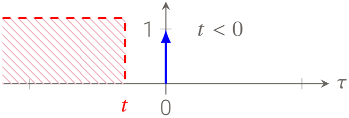
CT Unit Step Function and Unit Impulse Function

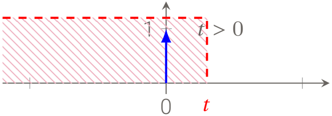
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau. \tag{12}$$



CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau. \tag{12}$$





Outline

Energy I

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{13}$$

Energy II

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{14}$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Power

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{15}$$

Total energy in a DT signal:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{16}$$

With these definitions, we can identify three important classes of signals:

1. Energy signals: Signals with finite total energy $E_\infty < \infty$. These have zero average power.
2. Power signals: Signals with finite average power $0 < P_\infty < \infty$. As $P_\infty > 0$, $E_\infty = \infty$.
3. Signals with neither E_∞ nor P_∞ are finite.