EN1060 Signals and Systems: Part 1

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Section 1

Introduction to Signals and Systems

Introduction to the Course

- Signals and systems find many application in communications, automatic control, and form the basis for signal processing, machine vision, and pattern recognition.
- Electrical signals (voltages and currents in circuits, electromagnetic communication signals), acoustic signals, image and video signals, and biological signals are all example of signals that we encounter.
- They are functions of independent variables and carry information.

Introduction to Course Contd.

- We define a system as a mathematical relationship between an input signal and an output signal.
- We can use systems to analyze and modify signals.
- Signals and systems have brought about revolutionary changes.
- In this course we will study the fundamentals of signals and systems.
- Types of signals in continuous time and discrete time, linear time-invariant (LTI) systems, Fourier analysis, sampling, Laplace transform, z-transform, and stability of systems are the core components of the course.

Learning Outcomes

After completing this course you will be able to do the following:

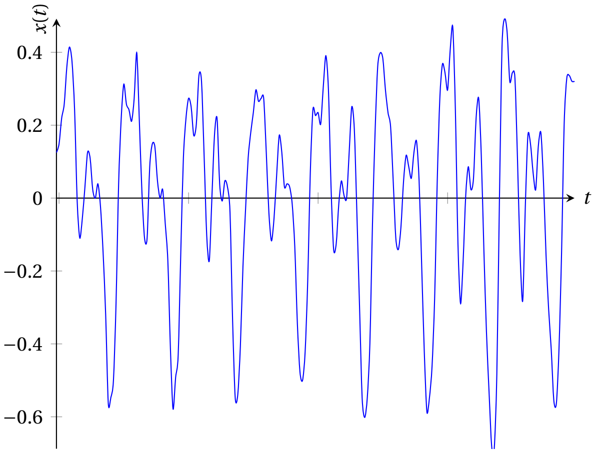
- Differentiate between continuous-time, discrete-time, and digital signals, and techniques applicable to the analysis of each type.
- Apply appropriate theoretical principles to characterize the behavior of linear time-invariant (LTI) Systems.
- Use Fourier techniques to understand frequency-domain characteristics of signals.
- Use appropriate theoretical principles for sampling and reconstruction of analog signals.
- Use the Laplace transform and the z-transform to treat a class of signals and systems broader than what Fourier techniques can handle.

Categories of Signals

- In this course we study signals and systems that process these signals.
- Categories of signals:
  - Continuous-time signals: independent variable is continuous, $x(t)$
  - Discrete-time signals:independent variable is an integer, $x[n]$
- There are some very strong similarities and also some very important differences between discrete-time signals and systems and continuous-time signals and systems.

Continuous-Time Signals $x(t)$

- The independent variable is continuous.
- E.g., sound pressure at a microphone as a function of time (one-dimensional signal).
- E.g., image brightness as a function of two spatial variables (two-dimensional signal).
- Con convenience, we refer to the independent variable as time.



A function of a continuous variable
A speech signal: a continuous-time, one-dimensional signal

An image on a film: a continuous-time, two-dimensional signal

- Function of an integer variable.
- Takes on values at integer values of the argument of $x[n]$.

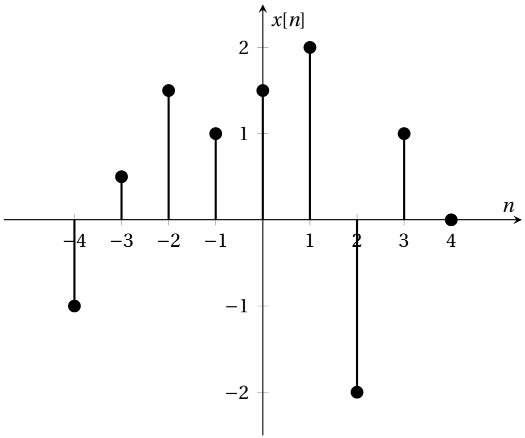


Figure: DT Signal

Digital Signals

- What is a digital signal?
  - A quantized discrete-time signal. I.e., $x[n,m]$ can take only a value from a finite set of values.
- What is a digital image?
  - A two-dimensional, quantized, discrete-time signal.
  - A 600×800 image: $n \in [0,599]$, $m \in [0,799]$, $x[n,m] \in [0,255]$. 8-bit image.

Systems

- A system processes signals.
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm for post-flight analysis of a space launch.
  - An edge detection algorithm for medical images.

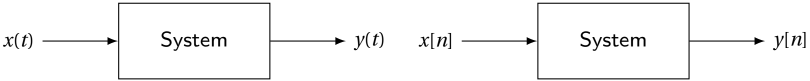


Figure: CT and DT Systems.

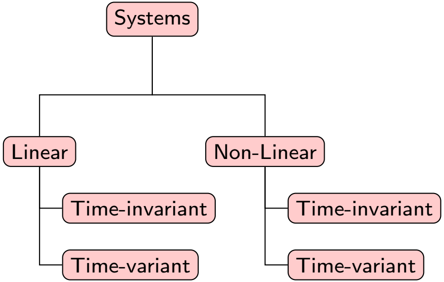


Figure: System types.

This course is focused on the class of linear, time-invariant (LTI) systems.

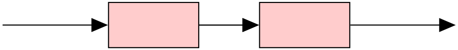
Examples of Systems

- Dynamics of an aircraft.
- An algorithm for analyzing financial and economic factors to predict bond prices.
- An algorithm for post-flight analysis of a space launch.
- An edge detection algorithm for medical images.

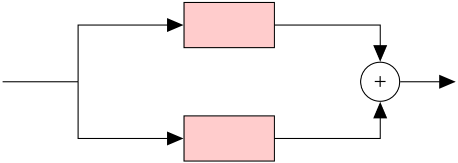
Systems Interconnections

- To build more complex systems by interconnecting simpler subsystems.
- To modify the response of a system.
- E.g.: amplifier design, stabilizing unstable systems.

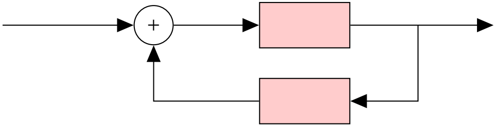
Signal-Flow (Block) Diagrams



Series (Cascade)



Parallel



Feedback

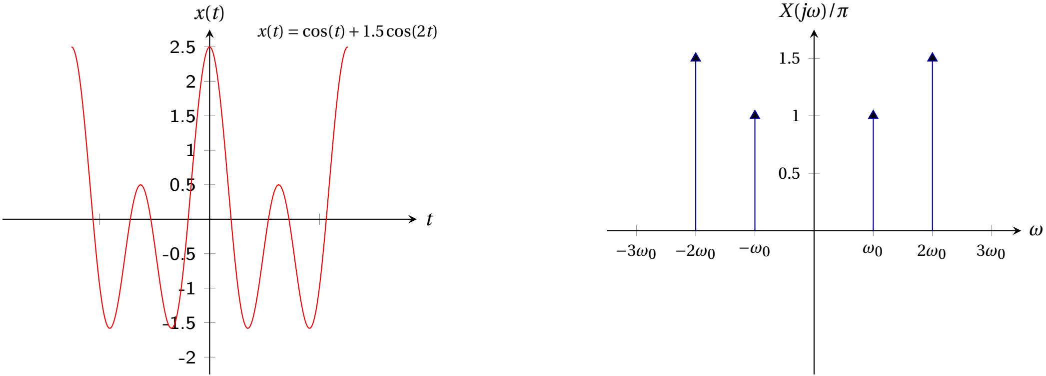


Figure: Domains.

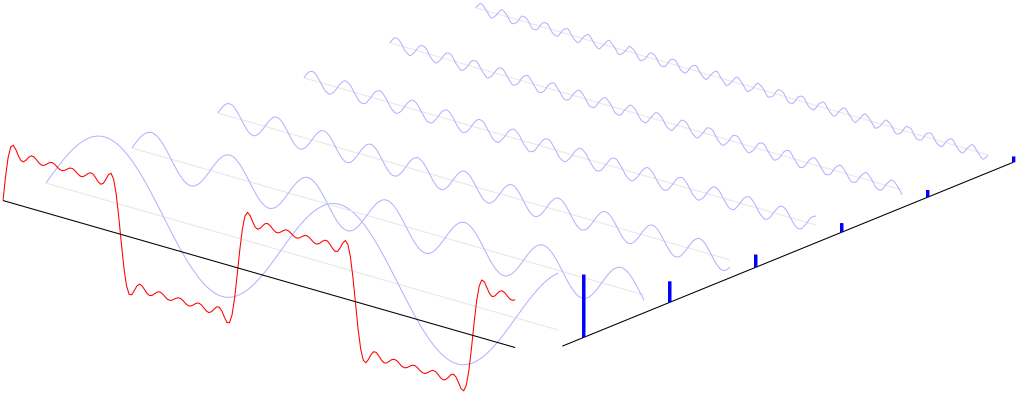


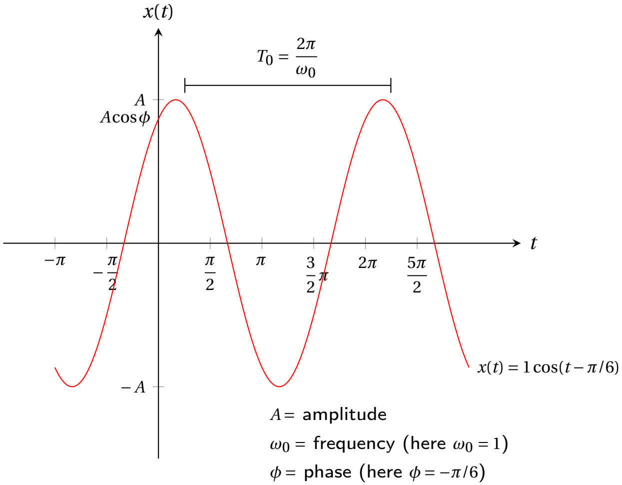
Figure: Square wave: time and frequency domains.

Section 2

Signals

Continuous-Time Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi). \tag{1}$$



Sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t+T) \tag{2}$$

for all values of t. Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t+\phi)$

$$\begin{aligned} A\cos(\omega_0 t+\phi) &= A\cos(\omega_0(t+T)+\phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t+\phi) \end{aligned}$$

$T = \dfrac{2\pi m}{\omega_0} \Rightarrow$ fundamental period $T_0 = \dfrac{2\pi}{\omega_0}$.

A time-shift in a CT sinusoid is equivalent to a phase shift.
E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

Even and Odd Signals

A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

A is referred to as an *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

An odd signal must be) at $t=0$ or $n=0$.
A signal can be broken into a sum of two signals, one of which is even adn one fo which is odd. Even part of $x(t)$ is

$$\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

Odd part of $x(t)$ is

$$\mathfrak{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Phase of a Sinusoidal: $\phi = 0$

Phase of a Sinusoidal: $\phi = -\pi/2$

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = 0$

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = -\pi/2$

Phase Change and Time Shift in DT

Q

Does a phase change always correspond to a time shift in discrete-time signals?

Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

$$\begin{aligned} x(t) &= Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers} \\ &= Ce^{at_0}\, e^{at}. \end{aligned}$$

DT Real Exponentials

$$x[n] = Ce^{\beta n} = C\alpha^n, \quad C \text{ and } \alpha \text{ are real numbers}$$

$x(t) = Ce^{at}$ $\quad C$ and a are complex numbers.

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

$$x(t) = |C|e^{j\theta}\, e^{(r+j\omega_0)t}$$

$$= |C|e^{rt}\, e^{j(\omega_0 t+\theta)}$$

$$= |C|e^{rt}\, [\cos(\omega_0 t + \theta) + j\, \sin(\omega_0 t + \theta)]$$

$$x[n] = C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \tag{4}$$

$$C = |C|e^{j\theta} \tag{5}$$

$$\alpha = |\alpha|e^{j\omega_0} \tag{6}$$

$$x[n] = |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \tag{7}$$

$$= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta) \tag{8}$$

$$\tag{9}$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

DT Complex Exponentials Plot

Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

- For the CT counterpart $e^{j\omega_0 t}$,
  1. The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal.
  2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.
- In DT, as
$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$
  the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0 .
- Although in CT $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In DT, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 + 2\pi$, $\omega_0 + 4\pi$, and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0.
- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{10}$$

Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{11}$$

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \tag{12}$$

The unit step sequence is the running sum of the unit impulse.

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{13}$$

The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{14}$$

Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{15}$$

Continuous-Time Unit Impulse Function $\delta(t)$

$$\delta(t) = \frac{du(t)}{dt}. \tag{16}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \tag{17}$$

Energy I

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{18}$$

Energy II

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{19}$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Power

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{20}$$

Total energy in a DT signal:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{21}$$

With these definitions, we can identify three important classes of signals:

1. Energy signals: Signals with finite total energy $E_\infty < \infty$. These have zero average power.
2. Power signals: Signals with finite average power $0 < P_\infty < \infty$. As $P_\infty > 0$, $E_\infty = \infty$.
3. Signals with neither E_∞ nor P_∞ are finite.

Section 3

Continuous-Time Fourier Series

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
  1. Continuous-time Fourier series
  2. Continuous-time Fourier transform
  3. Discrete-time Fourier series
  4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectra, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

Continuous-Time Fourier Series

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

Euler's formula
$e^{j\theta} = \cos\theta + j\sin\theta$
$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a liner combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

Suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients are denoted by a_k, then

$$x(t) \xleftrightarrow{\mathscr{F}\mathscr{S}} a_k \tag{22}$$

Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathscr{F}\mathscr{S}} a_k,$$
$$y(t) \xleftrightarrow{\mathscr{F}\mathscr{S}} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

$$x(t - t_0) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{23}$$

If

$$x(t) \xleftrightarrow{\mathscr{F}\mathscr{S}} a_k$$

then

$$x(-t) \xleftrightarrow{\mathscr{F}\mathscr{S}} a_{-k}.$$

Time Scaling

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{24}$$

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 11.

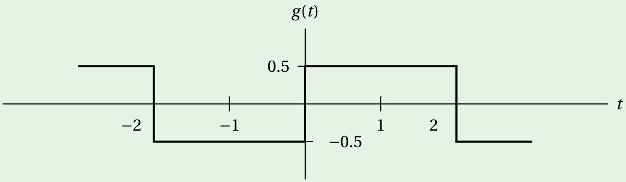


Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 12. The derivative signal is the signal $g(t)$ in Figure 11. Using this information, find the Fourier series coefficients of $x(t)$.

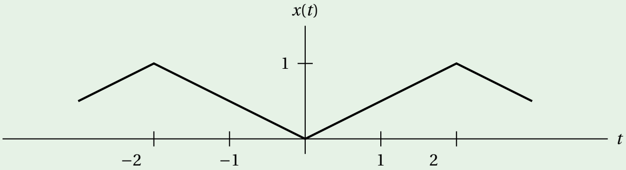


Figure: Figure for example

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{25}$$

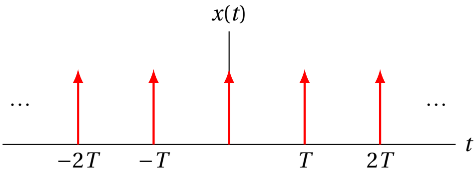


Figure: Impulse train

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

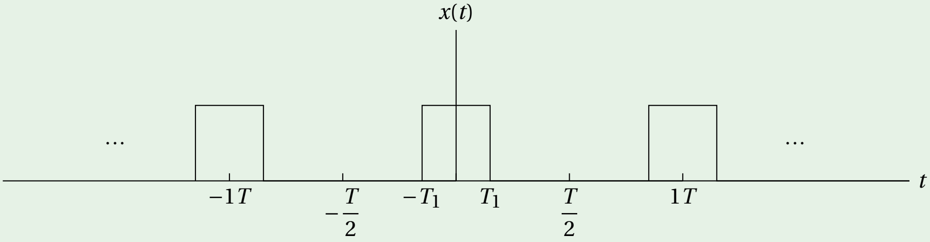


Figure: Figure for example

Complex Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \tag{26}$$

Trigonometric Fourier Series

$$x(t) = A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$
$$A_k = \frac{1}{T}\int_T x(t) A_k \cos k\omega_0 t dt \tag{27}$$
$$B_k = \frac{1}{T}\int_T x(t) A_k \sin k\omega_0 t dt$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$x(t) = C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k)$$
$$C_0 = A_0 \tag{28}$$
$$C_k = \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)$$

Relationship

$$A_0 = a_0$$
$$A_k = \frac{a_k + a_{-k}}{2}$$
$$B_k = j\frac{a_k - a_{-k}}{2} \tag{29}$$
$$\omega_0 = \frac{2\pi}{T}$$

Convergence of Fourier Series

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2 \, dt$$

Convergence of Fourier Series

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.
- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |e_N(t)|^2 \, dt < \infty \tag{30}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.
- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Alternative Conditions (Dirichlet Conditions)

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1

Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\, dt < \infty. \tag{31}$$

This guarantees that a_ks are finite.

Condition 2

In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3

In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Functions that Violate Dirichlet Conditions

ond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

ond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)|\, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

ond. 3 The signal, of period T = 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.

Section 4

Continuous-Time Fourier Transform

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
  1. Continuous-time Fourier series
  2. Continuous-time Fourier transform
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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

Fourier Transform

- In the last lecture, we represented a periodic signal as a linear combination of complex exponentials.
- We use Fourier transform to represent aperiodic signals. A larger class of signals, including all signals with finite energy, can be represented through a linear combination of complex exponentials.
- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an integral rather than a sum.
- The resulting spectrum of coefficients in this representation is called the Fourier transform.
- The synthesis integral itself, which uses these coefficients to represent the signal as a linear combination of complex exponentials, is called the inverse Fourier transform.

Fourier Series Representation for Square Wave

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodically repeats with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

The Fourier series coefficients a_k of this wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \tag{32}$$

We plotted this for a fixed value of T_1 and several values of T (shown in the next slide). An alternative wave of interpreting eq. 32 is as samples of an envelope function:

$$Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0}$$

For fixed T_1, the envelope of Ta_k is independent of T.

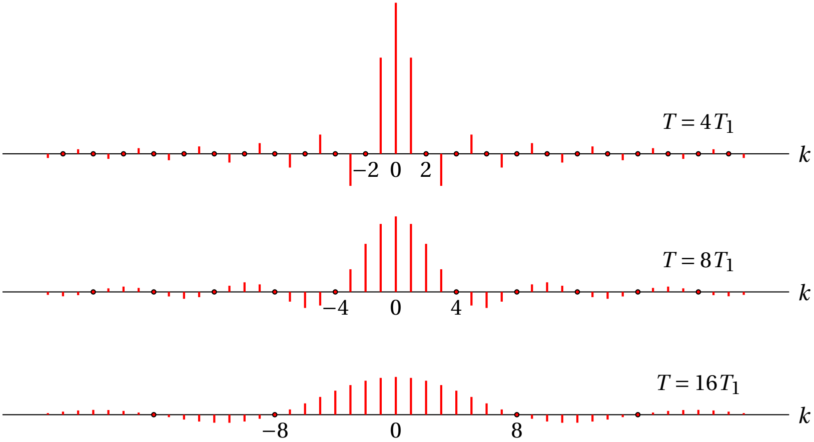


Figure: Plots of scaled Fourier series coefficients Ta_k

Plots of scaled Fourier series coefficients Ta_k for the periodic square wave with T_1 fixed and for several values of T: $T = 4T_1$, $T = 8T_1$, $T = 16T_1$.

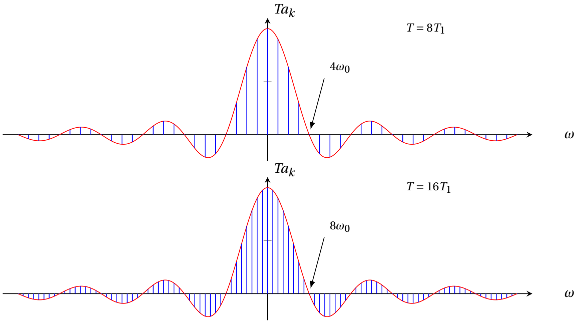


Figure: Fourier series coefficients and their envelope for periodic square wave.

The Fourier series coefficients and their envelope for periodic square wave for several values of T (with T_1 fixed): $T = 4T_1$, $T = 8T_1$, $T = 16T_1$. The coefficients are regularly-spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Fourier Transform: Synthesis and Analysis Equations

Relation with a_k

Assume that the Fourier transform of $x(t)$ is $X(j\omega)$.
If we construct a periodic signal $\tilde{x}(t)$ by repeating the aperiodic signals $x(t)$ with period T, its Fourier series coefficients are

Assume that we evaluated $X(j\omega)$ according to eq. **??**, and left $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in **??**:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

When is $\hat{x}(t)$ a valid representation of the original signal $x(t)$? We define the error between $\hat{x}(t)$ and $x(t)$ as

$$e(t) = \hat{x}(t) - x(t).$$

Convergence of Fourier Transform: Dirichlet Conditions

There are alternative conditions sufficient to ensure that $\hat{x}(t)$ is qual to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

Example

Find the Fourier transform of the signal

$$x(t) = e^{-a|t|}, \quad a > 0.$$

Example

Determine the Fourier transform of the unit impulse

$$x(t) = \delta(t).$$

Rectangular Pulse

Example

Determine the Fourier transform of the signal

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1. \end{cases}$$

Example

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine $x(t)$.

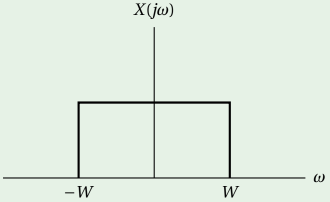


Figure: Fourier transform for $x(t)$.

The sinc Function

$$\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{33}$$

Express

$$\frac{2\sin\omega T_1}{\omega}$$

and

$$\frac{\sin Wt}{\pi t}$$

as sinc functions.

What Happens when W Increases?

The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transformof a periodic signal directly from its Fourier series representation.

Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$, i.e., Let's determine the signal $x(t)$:

Example

Find the Fourier transformof the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

Method: Multiply the Fourier seriescoefficients a_k by 2π, place them using the impulse function $\delta(\omega - k\omega_0)$, and sum.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example

Find the Fourier transformof

$$x(t) = \sin \omega_0 t,$$

and

$$x(t) = \cos \omega_0 t.$$

Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

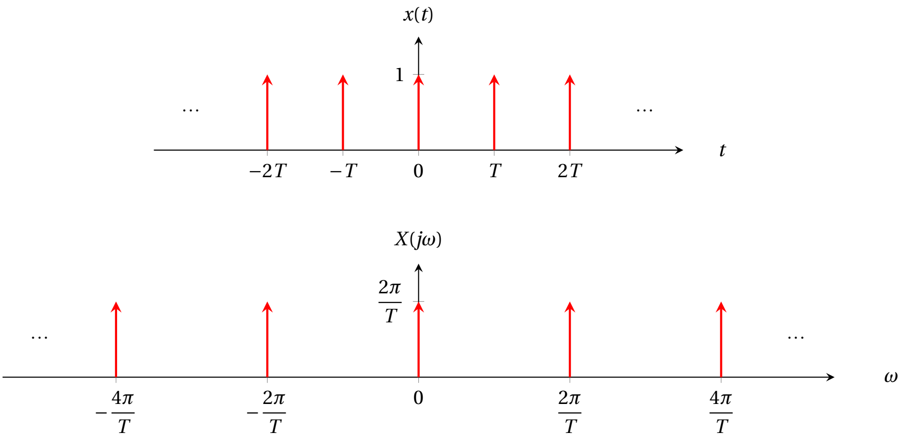


Figure: Periodic impulse train and its Fourier transform.

Section 5

Fourier Transform Properties

Fourier Transform: Recall

Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega. \tag{34}$$

Analysis equation:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{35}$$

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega). \tag{36}$$

Linearity

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

and

$$y(t) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(j\omega).$$

then

Time Shifting

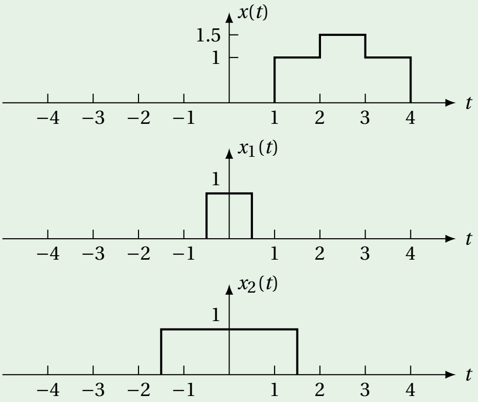
If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega).$$

then

Example

Evaluate the Fourier transform of $x(t)$.



Conjugation and Conjugate Symmetry

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x^*(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X^*(-j\omega).$$

Using Conjugate Symmetry

Use the conjugate property to comment about the symmetry of Fourier transform of a signal $x(t)$ if

1. $x(t)$ is real,
2. $x(t)$ is real and even, and
3. $x(t)$ is real and odd.

Expressing $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathfrak{Re}\left\{X(j\omega)\right\} + j\mathfrak{Im}\left\{X(j\omega)\right\},$$

then if $x(t)$ is real $[x(t) = x^*(t)]$

$$\begin{aligned}\mathfrak{Re}\left\{X(j\omega)\right\} &= \mathfrak{Re}\left\{X(-j\omega)\right\} \quad \text{and} \\ \mathfrak{Im}\left\{X(j\omega)\right\} &= -\mathfrak{Im}\left\{X(-j\omega)\right\}\end{aligned}$$

That is, the real part of the Fourier transform is an even function of frequency, and the imaginary part is an odd function of frequency. Considering

$$X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)},$$

we see that $|X(j\omega)|$ is an even function of frequency, and $\angle X(j\omega)$ is an odd function of frequency.

If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.
Proof:

Fourier Transforms of Odd and Even Parts

A real function $x(t)$ can be expressed as

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t) = \mathfrak{Ev}\{x(t)\}$ is the even part of $x(t)$ and $x_o(t) = \mathfrak{Od}\{x(t)\}$ is the odd part of $x(t)$. Express Fourier transforms of

1. $x_e(t) = \mathfrak{Ev}\{x(t)\}$, and
2. $x_o(t) = \mathfrak{Od}\{x(t)\}$.

in terms of $X(j\omega)$.

Example

Use the symmetry properties of the Fourier transform to evaluate the Fourier transform of

$$x(t) = e^{-a|t|}, \quad a > 0.$$

We have already found that

$$e^{-at} \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{a+j\omega}.$$

$$\begin{aligned} x(t) = e^{-a|t|} &= e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathfrak{Ev}\left\{e^{-at}u(t)\right\}. \end{aligned}$$

Since $e^{-at}u(t)$is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathfrak{Ev}\left\{e^{-at}u(t)\right\} \overset{\mathscr{F}}{\longleftrightarrow} \mathfrak{Re}\left\{\frac{1}{a+j\omega}\right\}.$$

$$X(j\omega) = 2\mathfrak{Re}\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}.$$

Differentiation and Integration

Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

Differentiating both sides of the equation

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}\,d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \overset{\mathscr{F}}{\longleftrightarrow} j\omega X(j\omega).$$

Integration:

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).$$

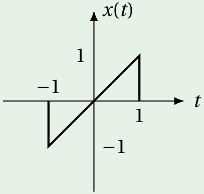
Example

Determine the Fourier transform of the unit step $x(t) = u(t)$ making use of the knowledge that

$$g(t) = \delta(t) \xleftrightarrow{\mathscr{F}} G(j\omega) = 1.$$

Example

Determine the Fourier transform of the signal $x(t)$ shown below:



Time and Frequency Scaling

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega).$$

then

$$x(at) \xleftrightarrow{\mathscr{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

where a is a real constant.
Letting $a = -1$

$$x(-t) \xleftrightarrow{\mathscr{F}} X(-j\omega).$$

The scaling property is another example of the inverse relationship between time and frequency.

Duality

Because of the similarity between the synthesis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \tag{37}$$

and the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{38}$$

for any transform pair, there is a dual pair with the time and frequency variables interchanged.

We determined the Fourier transform of the square pulse as

$$x_1(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1, \end{cases} \xleftrightarrow{\mathscr{F}} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

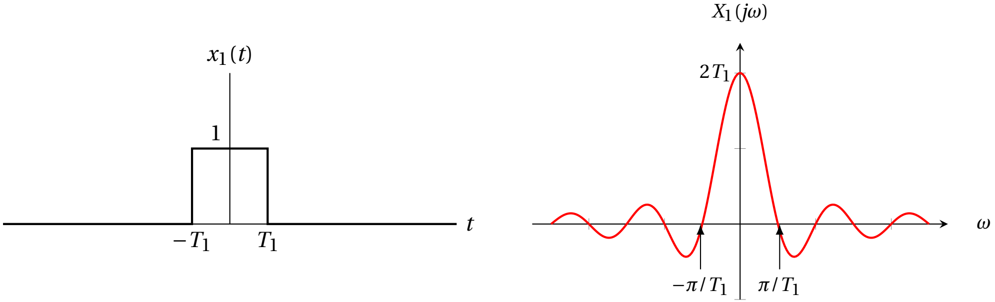


Figure: Rectangular pulse and the Fourier transform.

We also determined that for a time-domain signal that is similar in shape to the $X_1(j\omega)$ as

$$x_2(t) = \frac{\sin Wt}{\pi t} \overset{\mathscr{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

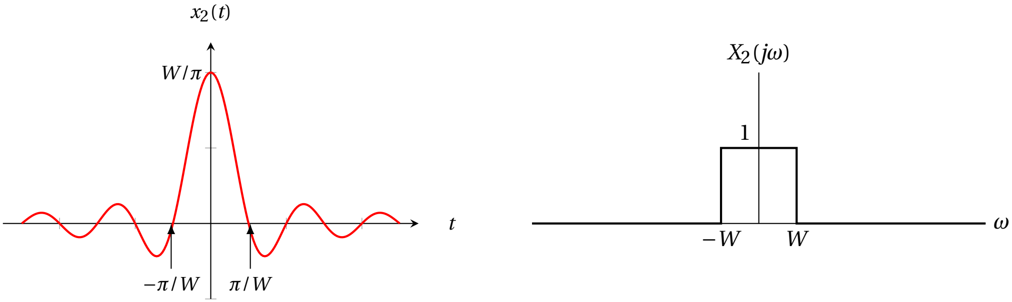


Figure: Fourier transform for $x(t)$.

Example

Use the duality property to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

More Properties Using Duality

$$-jtx(t) \xleftrightarrow{\mathscr{F}} \frac{dX(j\omega)}{d\omega}.$$

$$e^{j\omega_0 t}x(t) \xleftrightarrow{\mathscr{F}} X(j(\omega-\omega_0)).$$

$$-\frac{1}{jt}x(t)+\pi x(0)\delta(t) \xleftrightarrow{\mathscr{F}} \int_{-\infty}^{\omega} x(\eta)\,d\eta.$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

Convolution Property

$$y(t) = h(t) * x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

This equation is of major importance in signal and system analysis. This says that the Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

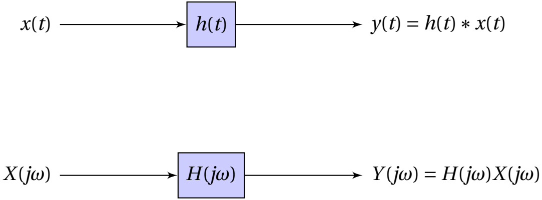


Figure: Convolution property.

Example

An LTI system has the impulse response

$$h(t) = \delta(t - t_0).$$

If the Fourier transform of the input signal $x(t)$ is $X(j\omega)$, what is the Fourier transform of the output?

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\begin{aligned}
h(t) &= \delta(t - t_0) \\
H(j\omega) &= e^{-j\omega t_0} \\
Y(j\omega) &= H(j\omega)X(j\omega) \\
&= e^{-j\omega t_0} X(j\omega)
\end{aligned}$$

Example

What is the frequency response of the differentiator?

The input output relationship of the differentiator is

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property

$$Y(j\omega) = j\omega X(j\omega).$$

Consequently, the frequency response of the differentiator is

$$H(j\omega) = j\omega.$$

Example

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, find $y(t)$ by transfroming the problem into the frequency domain.

Multiplication Property

The convolution property states that convolution in time domain corresponds to multiplication in frequency domain. Because of the duality between time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \overset{\mathscr{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)].$$

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other. Consequently, the multiplication of two signals is often referred to as amplitude modulation. For this reason, this equation is sometime referred to as the modulation property.

Example

Let $s(t)$ be a signal whose spectrum is depicted in the figure below. Also consider the signal

$$p(t) = \cos \omega_0 t.$$

Show the spectrum of $r(t) = s(t)p(t)$.

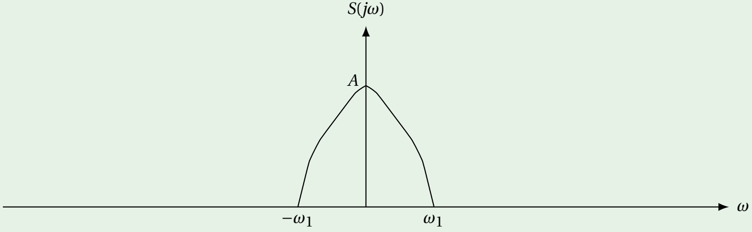


Figure: Spectrum of signal $s(t)$.