EN1060 Signals and Systems: Discrete-Time Fourier Series

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Discrete-Time Fourier Transform from DTFS

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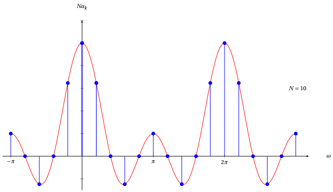
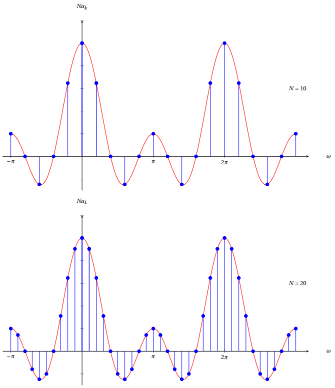
- Consequently, as the period approaches infinity, this envelope function corresponds to a Fourier representation of the aperiodic signal corresponding to one period. This is, then, the Fourier transform of the aperiodic signal.
- The discrete-time Fourier transform developed, as we have just described, corresponds to a decomposition of an aperiodic signal as a linear combination of a continuum of complex exponentials.

- The synthesis equation is then the limiting form of the Fourier series sum, specifically an integral. The analysis equation is the same one we used previously in obtaining the envelope of the Fourier series coefficients.
- While there was a duality in the expressions between the discrete-time Fourier series analysis and synthesis equations, the duality is lost in the discrete-time Fourier transform since the synthesis equation is now an integral and the analysis equation a summation. This is a difference compared to the continuous-time Fourier transform.
- Another important difference is that the discrete-time Fourier transform is always a periodic function of frequency.
- Consequently, it is completely defined by its behavior over a frequency range of 2π in contrast to the continuous-time Fourier transform, which extends over an infinite frequency range.

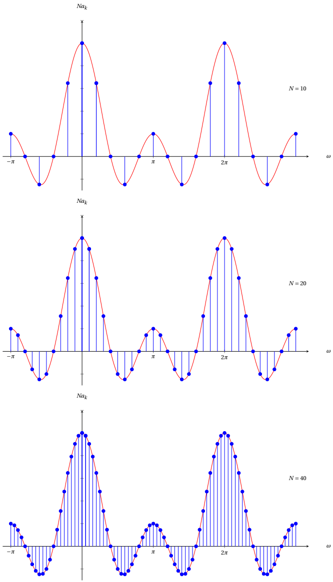
Approach

- Construct the periodic signal $\tilde{x}[n]$ for which one period is $x[n]$.
- $\tilde{x}[n]$ has a Fourier series.
- As the period of $\tilde{x}[n]$ increases,
  $\tilde{x}[n] \to x[n]$ and the Fourier series of $\tilde{x}[n]$ $\to$ Fourier transform of $x[n]$.


Na_k
$N = 10$
ω
$-\pi$
π
2π



Na_k
$N = 10$
$-\pi$
π
2π
ω
Na_k
$N = 20$
$-\pi$
π
2π
ω


Na_k
$N = 10$
ω
$-\pi$
π
2π
Na_k
$N = 20$
ω
$-\pi$
π
2π
Na_k
$N = 40$
ω
$-\pi$
π
2π

- $x[n]$ aperiodic
  - Construct periodic signals $\tilde{x}[n]$ for which one period is $x[n]$
  - $\tilde{x}[n]$ has a Fourier series
- As period of $\tilde{x}[n]$ increases
  - $\tilde{x}[n] \longrightarrow x[n]$
  - $\tilde{x}[n] \longrightarrow$ Fourier transform of $x[n]$.

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}.$$

If $x[n]$ is aperiodic, for the periodic signal $\tilde{x}[n]$ whose one period is $x[n]$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Since $x[n] = \tilde{x}[n]$ over a period that includes $-N1 \leq n \leq N_2$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

we see that the coefficients a_k are proportional to the samples of $X(e^{j\omega})$,i.e.,

$$a_k = \frac{1}{N}X(e^{jk\omega_0})$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}.$$

Since $1/N = \omega_0/2\pi$,

$$\tilde{x}[n] = \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0.$$

As $N \longrightarrow \infty$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Synthesis:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Analysis

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$\begin{aligned} X(e^{j\omega}) &= \mathrm{Re}\{X(e^{j\omega})\} + j\mathrm{Im}\{X(e^{j\omega})\} \\ &= |X(e^{j\omega})|e^{\angle X(e^{j\omega})} \end{aligned}$$

Example

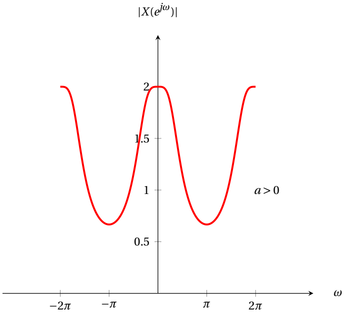
Obtain an expression for the DTFT of

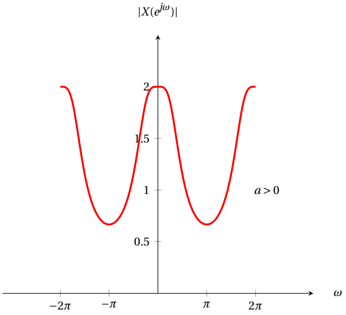
$$x[n] = a^n u[n], |a| < 1.$$

Sketch the magnitude and phase of $X(e^{j\omega})$ for

1. $a > 0$, $(a = 0.5)$ and
2. $a < 0$, $(a = -0.5)$.

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\
&= \frac{1}{1 - ae^{-j\omega}}.
\end{aligned}
$$


$|X(e^{j\omega})|$
2
1.5
1
0.5
$a > 0$
ω
-2π
$-\pi$
π
2π



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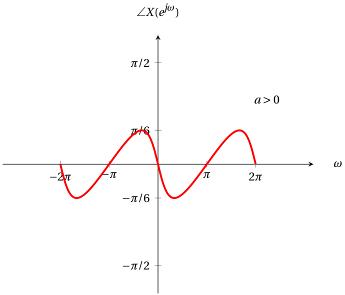
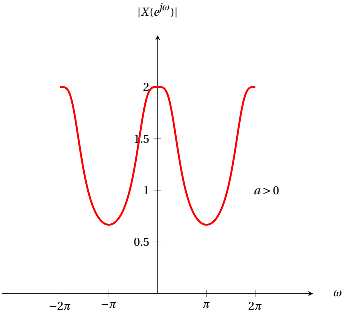


$\angle X(e^{j\omega})$
$\pi/2$
$a > 0$
$\pi/6$
-2π
$-\pi$
π
2π
ω
$-\pi/6$
$-\pi/2$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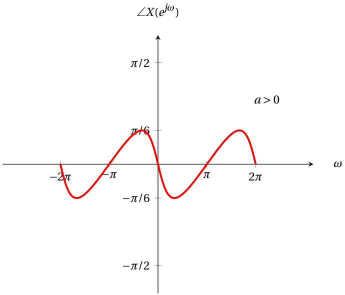
$\angle X(e^{j\omega})$
$\pi/2$
$a > 0$
$\pi/6$
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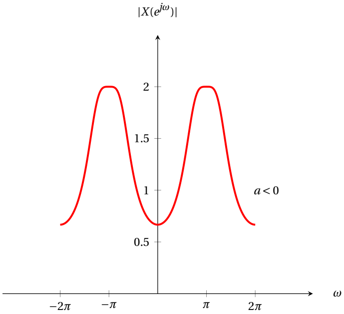
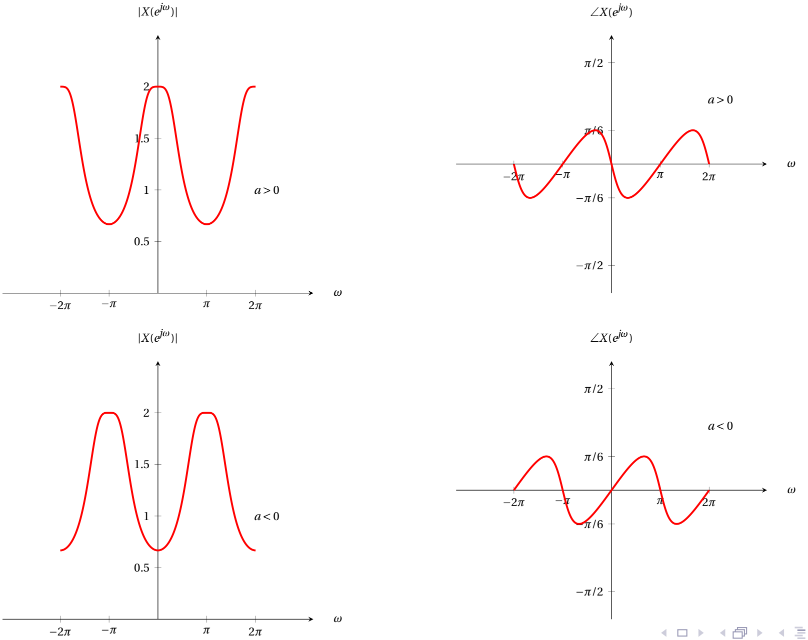
$-\pi$
π
2π
ω
$-\pi/6$
$-\pi/2$


$|X(e^{j\omega})|$
2
1.5
1
0.5
$a < 0$
ω
-2π
$-\pi$
π
2π



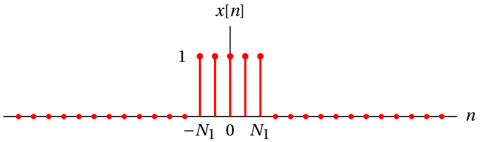
$|X(e^{j\omega})|$
$a > 0$
$\angle X(e^{j\omega})$
$a > 0$
$|X(e^{j\omega})|$
$a < 0$
$\angle X(e^{j\omega})$
$a < 0$

Example

Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \leq N_1, \\ 0, & |n| > N_1. \end{cases}$$

1. Obtain an expression for the DTFT $X(e^{j\omega})$ of this signal.
2. Sketch for $N_1 = 2$.



DTFT is always periodic in ω withe period 2π

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$$

If

$$x_1[n] \overset{\mathscr{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \overset{\mathscr{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

then

$$ax_1[n] + bx_2[n] \overset{\mathscr{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time Shifting and Frequency Shifting

If

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x[n-n_0] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

and

$$e^{j\omega_0 n} x[n] \overset{\mathscr{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Example

The frequency response of an ideal low-pass filter has the cutoff frequency of ω_c.

1. Obtain an expression for the frequency response of the corresponding high-pass filter (cutoff frequency $\pi - \omega_c$).
2. Obtain and expression for the impulse response of this high-pass filter in terms of the impulse response of the low-pass filter.

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega}).$$

Also, if $x[n]$ is real-valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$X(e^{j\omega}) = X^*(e^{-j\omega}), \quad (x[n] \text{ real.})$$

$\mathrm{Re}\{X(e^{j\omega})\}$ is an even function of ω and $\mathrm{Im}\{X(e^{j\omega})\}$ is an odd function of ω.
The magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

$$x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega}).$$

Example

Prove the time-reversal property.

$$nx[n] \overset{\mathscr{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Example

Prove the differentiation property.

Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

If $x[n]$, $h[n]$ and $y[n]$ are the input, impulse response, and output respectively of an LTI system, so that

$$y[n] = x[n] * h[n]$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

Example

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0]$$

Obtain the output $y[n]$ for an input $x[n]$.

Example

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

and suppose that the input to this system is

$$x[n] = \beta^n u[n], \quad |\beta| < 1$$

Obtain the output $y[n]$ for $\alpha \neq \beta$ and $\alpha = \beta$.

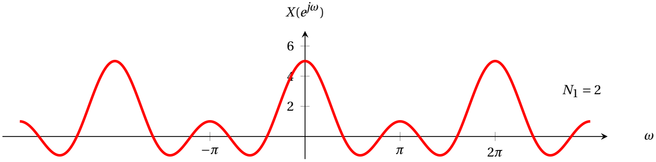
Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, then

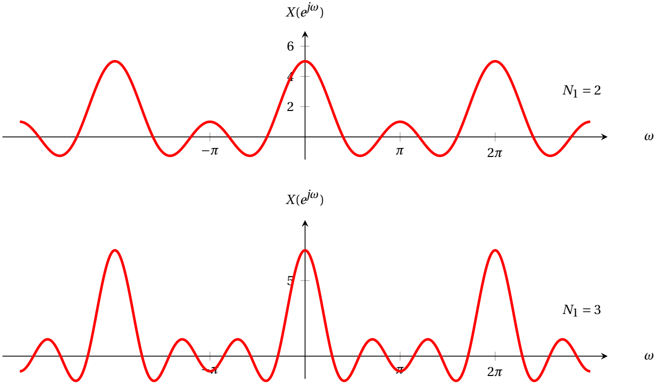
$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

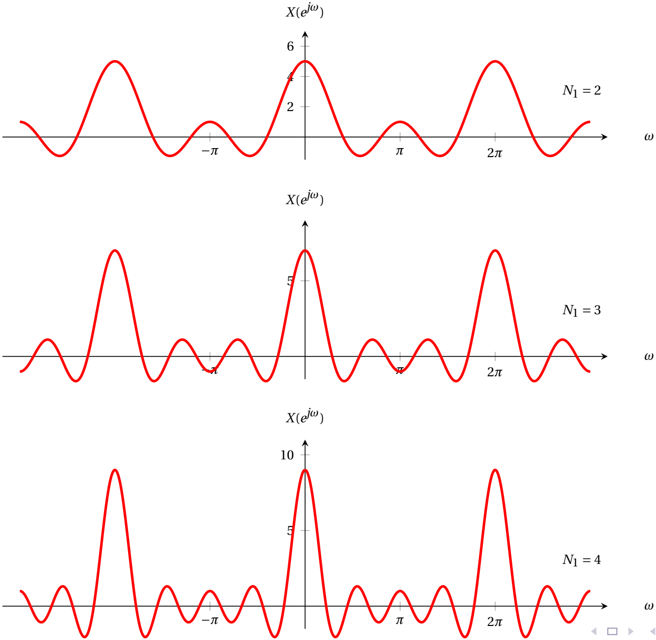
This equation corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluation over any given interval of length 2π.
See example 5.15.

	CT	DT
	Series (CT)	Series (DT)
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\frac{2\pi}{T})t}$	$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$
	$a_k = \frac{1}{T}\int_T x(t) e^{-jk(\frac{2\pi}{T})t} dt$	$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk(\frac{2\pi}{N})n}$
	Transform (CT)	Transform (DT)
Aperiodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

CT: $\omega_0 = \frac{2\pi}{T}$, DT: $\omega_0 = \frac{2\pi}{N}$


$X(e^{j\omega})$
6
4
2
$N_1 = 2$
$-\pi$
π
2π
ω


$X(e^{j\omega})$
6
4
2
$-\pi$
π
2π
ω
$N_1 = 2$
$X(e^{j\omega})$
5
$-\pi$
π
2π
ω
$N_1 = 3$


$X(e^{j\omega})$
6
4
2
$N_1 = 2$
ω
$-\pi$
π
2π
$X(e^{j\omega})$
5
$N_1 = 3$
ω
$-\pi$
π
2π
$X(e^{j\omega})$
10
5
$N_1 = 4$
ω
$-\pi$
π
2π