

EN1060 Signals and Systems: Discrete-Time Fourier Series

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Section 1

Discrete-Time Fourier Series

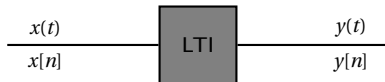
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- Specifically, we consider the representation of discrete-time signals through a decomposition as a linear combination of complex exponentials.
 - DT periodic signals \rightarrow DT Fourier series
 - DT aperiodic signals \rightarrow DT Fourier transform



Decompose the input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots \quad \text{linear combination of basic inputs}$$

Then

$$y = a_1\psi_1 + a_2\psi_2 + \cdots \quad \text{linear combination of corresponding outputs}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ such that

- Broad class of signals can be constructed, and
- Response to ϕ_k s easy to compute.

Continuous-Time:

$$\phi_k(t) = e^{j\omega_k t}:$$

$$e^{j\omega_k t} \longrightarrow H(\omega_k) e^{j\omega_k t} \quad (\text{a scaled-version of the input})$$

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“Discrete-Time”: $\phi_k[n] = e^{j\omega_k n}$

$$e^{j\omega_k n} \longrightarrow e^{j\omega_k n} \underbrace{\sum_{r=-\infty}^{\infty} h[r] e^{-j\omega_k r}}_{\text{eigenvalue}}$$

↑
eigenfunction

Consider $x[n]$ to be periodic,

Period N ,

Fundamental frequency $\omega_0 = \frac{2\pi}{N}$

$e^{jk\omega_0 n}$ are harmonically related, and periodic with the period N , although the fundamental period is different. $e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$

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Consider the complex exponential

$$e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$$

\Rightarrow Only N distinct complex exponentials.

$$x[n] = \sum_k a_k e^{jk\omega_0 n}, \quad k = 0, 1, 2, \dots, N-1.$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}.$$

N equations in N unknowns.

$$a_k = \frac{1}{N} \sum_{\langle N \rangle} x[n] e^{-jk\omega_0 n}.$$

$k = \langle N \rangle$: k ranges over one period (as a_k periodically repeats).

Continuous-Time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Discrete-Time Fourier Series

Synthesis

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

Analysis

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}.$$

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Note the duality.

Periodicity

$x[n]$	periodic in n ,	true for CT
$e^{jk\omega_0 n}$	periodic in n ,	true for CT
$e^{jk\omega_0 n}$	periodic in k ,	not true for CT
a_k	periodic in k ,	not true for CT

Convergence

Continuous-time:

- $x(t)$ square-integrable OR
- Dirichlet condition

Discrete-time

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}.$$

$$\hat{x}[n] = \sum_{p \text{ terms}} a_k e^{jk\omega_0 n}.$$

$$p = N$$

$$\hat{x}[n] \equiv x[n].$$

There is no issue of convergence in DT.

Example

Consider the signal $x[n] = \sin \omega_0 n$.

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Expanding the signal as a sum of two complex exponentials,

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \quad (1)$$

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$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}.$$

Fourier Coefficients for $x[n] = \sin(2\pi/N)n$ for $N = 5$

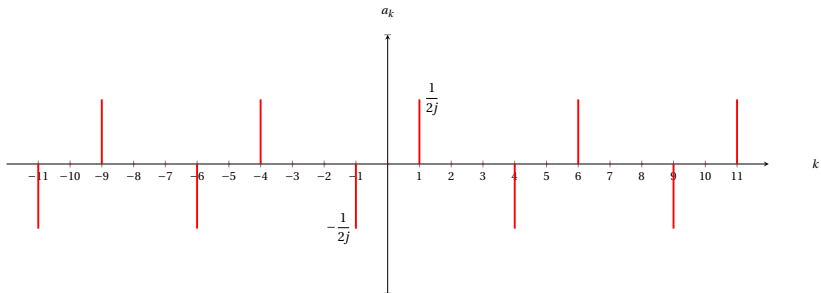


Figure: Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

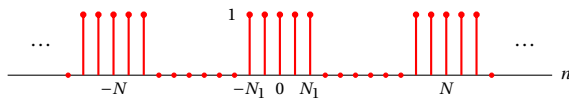
Determine and sketch the DTFS of

$$x[n] = 1 + \sin \omega_0 n + 3 \cos \omega_0 n + \cos \left(2\omega_0 n + \frac{\pi}{2} \right).$$

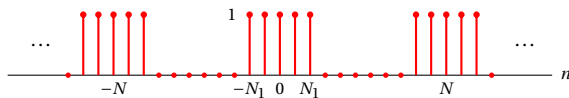
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Letting $m = n + N_1$

$$= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)/(m-N_1)}$$

$2N_1 + 1$ terms in a geometric series

$$\begin{aligned} &= \frac{1}{N} e^{jk(2\pi/N)N_1} \left[\frac{1 - e^{-jk(2\pi/N)(2N_1+1)}}{1 - e^{-jk(2\pi/N)}} \right] \\ &= \frac{1}{N} e^{jk(2\pi/N)N_1} \cdot \frac{e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{-j\frac{k(2\pi/N)}{2}}} \left[\frac{e^{j\frac{k(2\pi/N)(2N_1+1)}{2}} - e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{j\frac{k(2\pi/N)}{2}} - e^{-j\frac{k(2\pi/N)}{2}}} \right] \\ &= \frac{1}{N} \frac{\sin \left[\frac{k(2\pi/N)(2N_1+1)}{2} \right]}{\sin \left[\frac{k(2\pi/N)}{2} \right]} \end{aligned}$$

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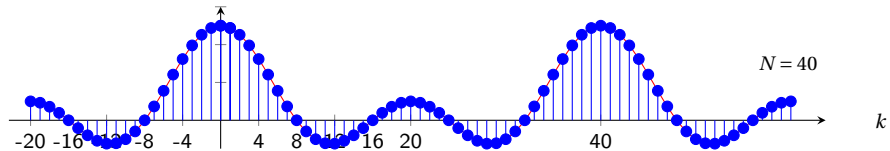
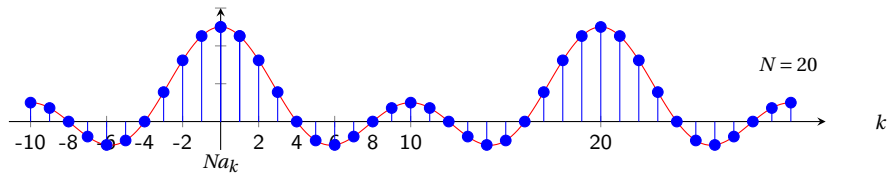
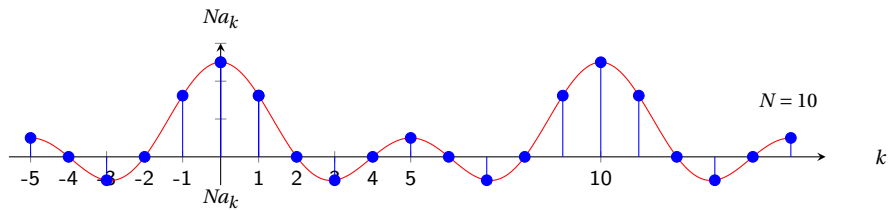
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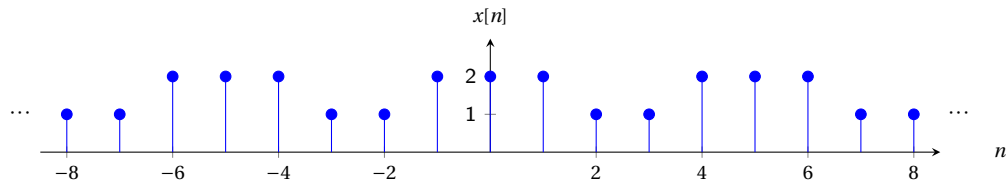
$$a_k = \frac{1}{N} \frac{\sin [2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)} \quad k \neq 0, \pm N, \pm 2N, \dots$$

$$a_k = \frac{2N_1 + 1}{N} \quad k = 0, \pm N, \pm 2N, \dots$$



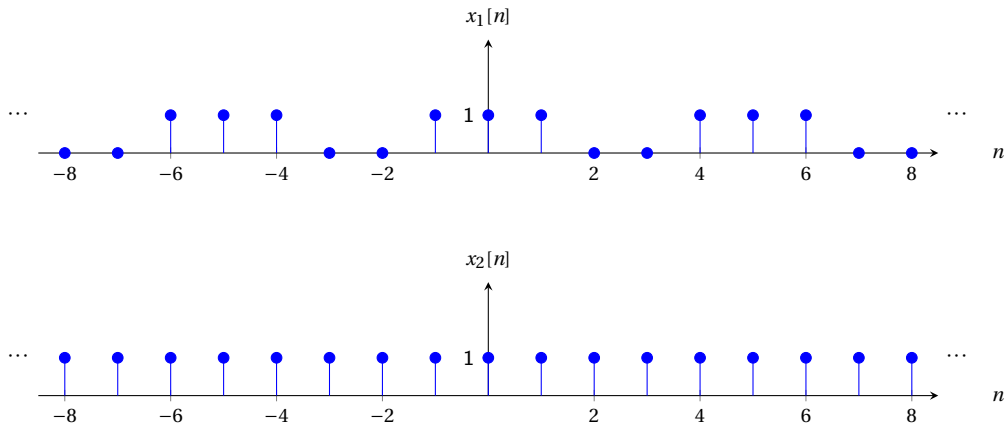
- 1 Discrete-Time Fourier Series
Properties of Discrete-Time Fourier Series

Find the Fourier series coefficients a_k of $x[n]$.



Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k . We use the linearity property of to conclude that

$$a_k = b_k + c_k.$$



From the previous work, (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients b_k corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \frac{1}{5} \frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases}$$

Suppose that we are given the following facts about a sequence $x[n]$:

① $x[n]$ is periodic with period $n = 6$.

② $\sum_{n=0}^5 x[n] = 2.$

③ $\sum_{n=2}^7 (-1)^n x[n] = 1.$

④ $x[n]$ has the minimum power per period among the set of signals satisfying the proceeding three conditions.

Determine the sequence $x[n]$.

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$$P = \sum_{k=0}^5 |a_k|^2.$$

Since each nonzero coefficient contributes a positive amount to P , and since the values of a_0 and a_3 are pre-specified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a_5 = 0$. It then follows that

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n.$$

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