EN1060 Signals and Systems: Signals

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Continuous-Time Fourier Series

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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

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- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

Outline

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned}
\tag{1}
$$

The set of coefficients $\{a_k\}$ is called the Fourier series coefficients of the spectral coefficients of $x(t)$. The coefficient a_0 is the dc or constant component of $x(t)$, given by Equation 1 with $k=0$:

$$
a_0 = \frac{1}{T}\int_T x(t)dt, \tag{2}
$$

which is simply the average of $x(t)$ over one period.

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing the right-hand side of this equation and Equation 1, we obtain

$$\begin{aligned} a_1 &= \frac{1}{2j} \qquad & a_{-1} &= -\frac{1}{2j} \\ a_k &= 0, \qquad & k &\neq \pm 1. \end{aligned}$$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a liner combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

Collecting terms,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

The Fourier coefficients are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = \left(1 - \frac{j}{2}\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = \left(1 + \frac{j}{2}\right),$$

$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j),$$
$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j),$$
$$a_k = 0, |k| > 2.$$

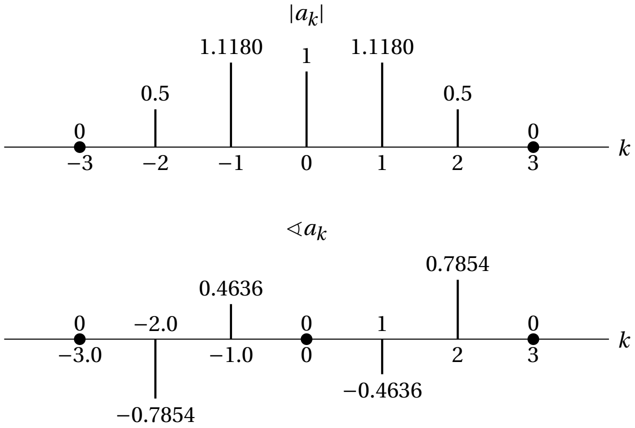

Figure: $|a_k|$, $\sphericalangle a_k$

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

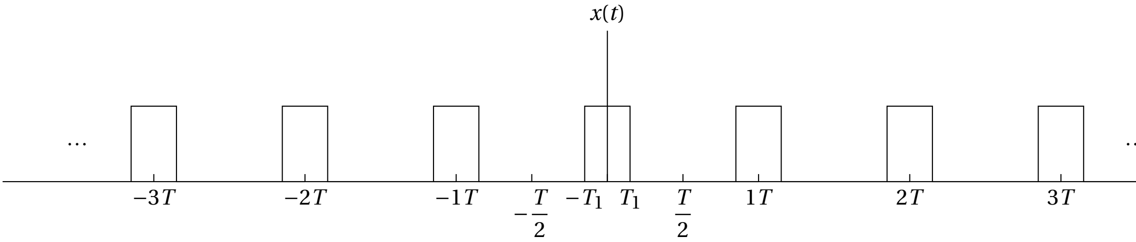

Figure: Periodic square wave

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)\,dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt, \\
&= \frac{2T_1}{T}.
\end{aligned}
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a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}, \\
&= -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}
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$$
a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j}\right]
$$

$$
a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{2\sin(k\omega_0 T_1)}{k\pi}, k \neq 0.
$$

For $T = 4T_1$

$$a_k = 0, \quad k \text{ even.}$$

$$a_0 = \frac{1}{2}$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_3 = a_{-3} = \frac{1}{3\pi}$$

$$a_5 = a_{-5} = \frac{1}{5\pi}$$

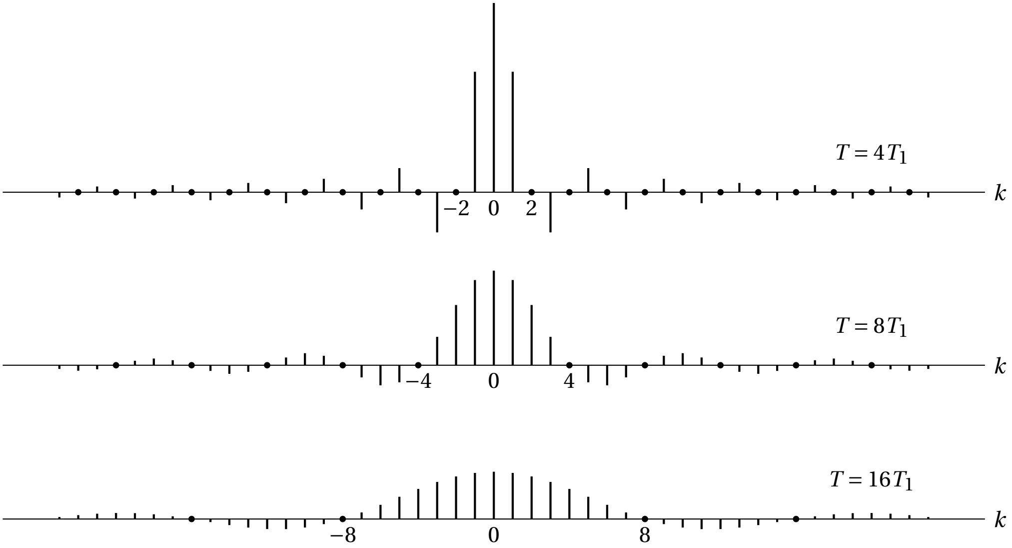

Figure: Plots of scaled Fourier series coefficients Ta_k

Section 2

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients are denoted by a_k, then

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \tag{3}$$

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $X(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

$$z(t) = Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k. \tag{4}$$

$$x(t-t_0) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

$$x(t-t_0) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

$$\begin{aligned}
x(t) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} a_k, &\quad x(t-t_0) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} b_k, \\
b_k &= \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt, \\
&= \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} dt, \\
&= e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau)} d\tau, \\
&= e^{-jk\omega_0 t_0} a_k.
\end{aligned}$$

$$x(t-t_0) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k.$$

Note: $|a_k| = |b_k|$

If

$$x(t) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} a_k \tag{6}$$

then

$$x(-t) \overset{\mathscr{F}\mathscr{S}}{\longleftrightarrow} a_{-k}. \tag{7}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{8}$$

Substitution: $k = -m$

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{-jk2\pi t/T}.$$

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- If $x(t)$ is even—$x(-t) = x(t)$—then its Fourier series coefficients are also even, i.e., $a_{-k} = a_k$.
- If $x(t)$ is odd—$x(-t) = -x(t)$—then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

Time scaling, in general, changes the period.
If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{9}$$

While Fourier coefficients have not changes, the Fourier series representation has changed because of the change in the fundamental frequency.

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \overset{\mathscr{FS}}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, its Fourier series coefficients h_k are

$$x(t)y(t) \overset{\mathscr{FS}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{10}$$

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- If $x(t)$ is real—$x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric,i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.

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- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.

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- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \quad (11)$$

Note: Left-hand side of equation 11 is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$.

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2\, dt = |a_k|^2. \quad (12)$$

So, $|a_k|^2$ is the average power in the kth harmonic component of $x(k)$.
Thus, what Parseval's relation state is that the total power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 5.

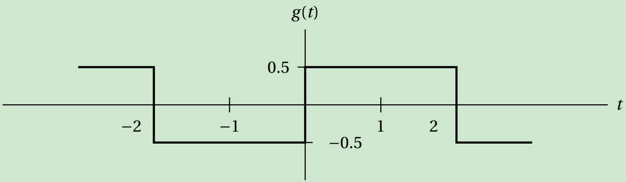

Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 6. The derivative signal is the signal $g(t)$ in Figure 5. Using this information, find the Fourier series coefficients of $x(t)$.

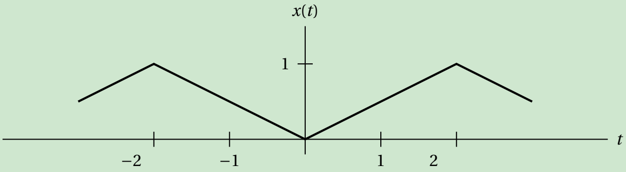

Figure: Figure for example

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{13}$$

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

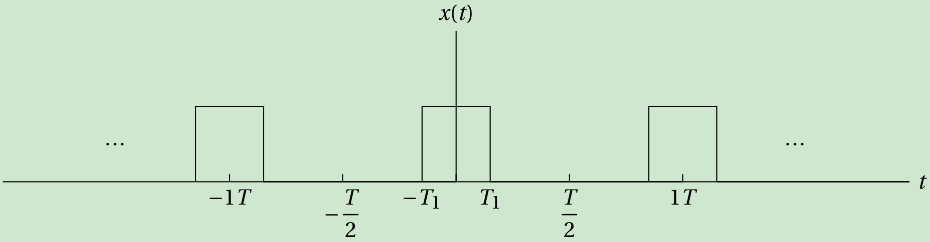

Figure: Figure for example

Complex Exponential Fourier Series

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt
\end{aligned} \tag{14}
$$

Trigonometric Fourier Series

$$
\begin{aligned}
x(t) &= A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t \\
A_k &= \frac{1}{T}\int_T x(t)A_k \cos k\omega_0 t dt \\
B_k &= \frac{1}{T}\int_T x(t)A_k \sin k\omega_0 t dt
\end{aligned} \tag{15}
$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$
\begin{aligned}
x(t) &= C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k) \\
C_0 &= A_0 \\
C_k &= \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)
\end{aligned} \tag{16}
$$

Relationship

$$
\begin{aligned}
A_0 &= a_0 \\
A_k &= \frac{a_k + a_{-k}}{2} \\
B_k &= j\frac{a_k - a_{-k}}{2} \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned} \tag{17}
$$