

# EN1060 Signals and Systems: Discrete-Time Fourier Series

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November 5, 2017

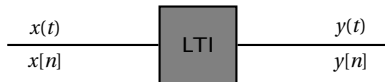
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  - DT periodic signals  $\rightarrow$  DT Fourier series
  - DT aperiodic signals  $\rightarrow$  DT Fourier transform



Decompose the input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots \quad \text{linear combination of basic inputs}$$

Then

$$y = a_1\psi_1 + a_2\psi_2 + \cdots \quad \text{linear combination of corresponding outputs}$$

Choose  $\phi_k(t)$  or  $\phi_k[n]$  such that

- Broad class of signals can be constructed, and
- Response to  $\phi_k$ s easy to compute.

## Continuous-Time:

$$\phi_k(t) = e^{j\omega_k t}:$$

$$e^{j\omega_k t} \longrightarrow H(\omega_k) e^{j\omega_k t} \quad (\text{a scaled-version of the input})$$

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“Discrete-Time”:  $\phi_k[n] = e^{j\omega_k n}$

$$e^{j\omega_k n} \longrightarrow e^{j\omega_k n} \underbrace{\sum_{r=-\infty}^{\infty} h[r] e^{-j\omega_k r}}_{\text{eigenvalue}}$$

↑  
eigenfunction

Consider  $x[n]$  to be periodic,

Period  $N$ ,

Fundamental frequency  $\omega_0 = \frac{2\pi}{N}$

$e^{jk\omega_0 n}$  are harmonically related, and periodic with the period  $N$ , although the fundamental period is different.  $e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$

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Consider the complex exponential

$$e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$$

$\Rightarrow$  Only  $N$  distinct complex exponentials.

$$x[n] = \sum_k a_k e^{jk\omega_0 n}, \quad k = 0, 1, 2, \dots, N-1.$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}.$$

$N$  equations in  $N$  unknowns.

$$a_k = \frac{1}{N} \sum_{\langle N \rangle} x[n] e^{-jk\omega_0 n}.$$

$k = \langle N \rangle$ :  $k$  ranges over one period (as  $a_k$  periodically repeats).

## Continuous-Time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

## Discrete-Time Fourier Series

Synthesis

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

Analysis

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}.$$

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Note the duality.

## Periodicity

$x[n]$	periodic in $n$ ,	true for CT
$e^{jk\omega_0 n}$	periodic in $n$ ,	true for CT
$e^{jk\omega_0 n}$	periodic in $k$ ,	<b>not true for CT</b>
$a_k$	periodic in $k$ ,	<b>not true for CT</b>

## Convergence

Continuous-time:

- $x(t)$  square-integrable OR
- Dirichlet condition

Discrete-time

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}.$$

$$\hat{x}[n] = \sum_{p \text{ terms}} a_k e^{jk\omega_0 n}.$$

$$p = N$$

$$\hat{x}[n] \equiv x[n].$$

There is no issue of convergence in DT.

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Expanding the signal as a sum of two complex exponentials,

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \quad (1)$$

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$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}.$$

# Fourier Coefficients for $x[n] = \sin(2\pi/N)n$ for $N = 5$

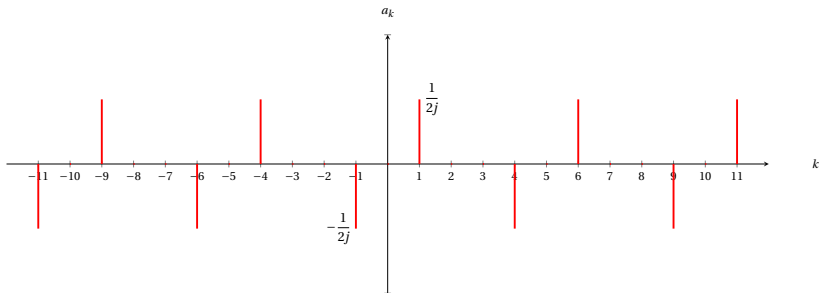


Figure: Fourier coefficients for  $x[n] = \sin(2\pi/5)n$ .

Determine and sketch the DTFT of

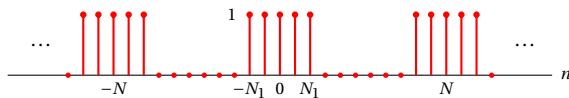
$$x[n] = 1 + \sin \omega_0 n + 3 \cos \omega_0 n + \cos \left( 2\omega_0 n + \frac{\pi}{2} \right).$$

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