Equations

Fourier

	CT	DT

	Series (CT)	**Series (DT)**
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$	$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$ $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$

	Transform (CT)	**Transform (DT)**
Aperiodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t).$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt.$$

$$a_k = \frac{2}{T}\int_T x(t)\cos(k\omega_0 t)\,dt.$$

$$b_k = \frac{2}{T}\int_T x(t)\sin(k\omega_0 t)\,dt.$$

Table 1: Properties of Continuous Time Fourier Series

Property	Periodic signal	Fourier series coefficients
	$x(t)$, $y(t)$ periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency shifting	$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\dfrac{d}{dt}x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$ finite valued and periodic only if $a_0 = 0$	$\dfrac{1}{jk\omega_0}a_k$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and even signals	$x(t)$ real and even	a_k real and even
Real and odd signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t) = \mathfrak{Ev}\{x(t)\}, \quad [x(t) \text{ real}]$ $x_o(t) = \mathfrak{Od}\{x(t)\}, \quad [x(t) \text{ real}]$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's relation for aperiodic signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Table 2: Properties of the Fourier Transform

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time reversal	$x(-t)$	$X(-j\omega)$
Time and frequency scaling	$x(at)$	$\dfrac{1}{\|a\|} X\left(\dfrac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\dfrac{1}{2\pi} X(j\omega) * Y(j\omega)$
Differentiation in time	$\dfrac{d}{dt} x(t)$	$j\omega X(j\omega)$
Integration	$\displaystyle\int_{-\infty}^{t} x(\tau)d\tau$	$\dfrac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in frequency	$tx(t)$	$j\dfrac{d}{d\omega} X(j\omega)$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathfrak{Re}\{X(j\omega)\} = \mathfrak{Re}\{X(-j\omega)\} \\ \mathfrak{Im}\{X(j\omega)\} = -\mathfrak{Im}\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for real and even signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for real and odd signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t) = \mathfrak{Ev}\{x(t)\}, \quad [x(t)\,\text{real}]$ $x_o(t) = \mathfrak{Od}\{x(t)\}, \quad [x(t)\,\text{real}]$	$\mathfrak{Re}\{X(j\omega)\}$ $j\mathfrak{Im}\{X(j\omega)\}$

Parseval's relation for aperiodic signals

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Table 3: Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients
$\displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\displaystyle 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \dfrac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\dfrac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \dfrac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, a_k = 0, k \neq 0$ $\left(\text{This is the Fourier series representation for any choice of } T > 0\right)$
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \dfrac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\displaystyle\sum_{k=-\infty}^{\infty} \dfrac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\dfrac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\dfrac{k\omega_0 T_1}{\pi}\right) = \dfrac{\sin k\omega_0 T_1}{k\pi}$
$\displaystyle\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\dfrac{2\pi}{T}\displaystyle\sum_{k=-\infty}^{\infty} \delta\left(\omega - \dfrac{2\pi k}{T}\right)$	$a_k = \dfrac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\dfrac{2\sin\omega T_1}{\omega}$	—
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	
$\delta(t)$	1	—
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \quad \mathfrak{Re}\{a\} > 0$	$\dfrac{1}{a + j\omega}$	—
$te^{-at}u(t), \quad \mathfrak{Re}\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$	—
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad \mathfrak{Re}\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$	—