EN1060 Signals and Systems: Fourier Series

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Section 1

Continuous-Time Fourier Series

Outline

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.

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- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.

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  3. Discrete-time Fourier series
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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

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- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned}
\tag{1}
$$

The set of coefficients $\{a_k\}$ is called the Fourier series coefficients or the spectral coefficients of $x(t)$. The coefficient a_0 is the dc or constant component of $x(t)$, given by Equation 1 with $k=0$:

$$
a_0 = \frac{1}{T}\int_T x(t)\,dt, \tag{2}
$$

which is simply the average of $x(t)$ over one period.

Example

Let

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$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing the right-hand side of this equation and Equation 1, we obtain

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$
$$a_k = 0, \qquad k \neq \pm 1.$$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a liner combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right]$$

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Collecting terms,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

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The Fourier coefficients are

$$\begin{aligned}
a_0 &= 1, \\
a_1 &= \left(1 + \frac{1}{2j}\right) = \left(1 - \frac{j}{2}\right), \\
a_{-1} &= \left(1 - \frac{1}{2j}\right) = \left(1 + \frac{j}{2}\right),
\end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1 + j), \\
a_{-2} &= \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1 - j), \\
a_k &= 0, |k| > 2.
\end{aligned}$$

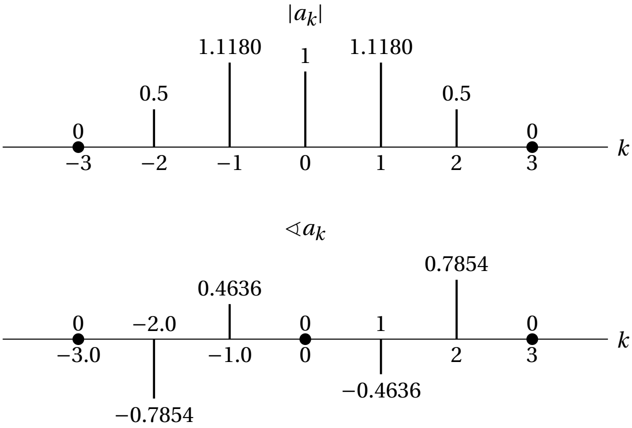

Figure: $|a_k|$, $\sphericalangle a_k$

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

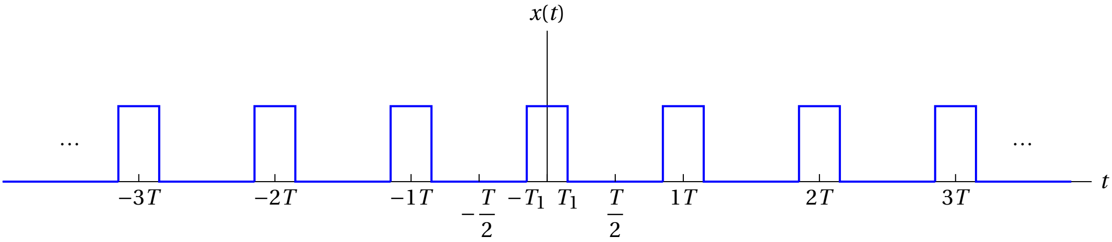

Figure: Periodic square wave

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)\,dt,\\
&= \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt,\\
&= \frac{2T_1}{T}.
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a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt, \\
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\end{aligned}
$$

$$
a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j}\right]
$$

$$
a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{2\sin(k\omega_0 T_1)}{k\pi}, k \neq 0.
$$

For $T = 4T_1$

$$a_k = 0, \quad k \text{ even.}$$

$$a_0 = \frac{1}{2}$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_3 = a_{-3} = \frac{1}{3\pi}$$

$$a_5 = a_{-5} = \frac{1}{5\pi}$$

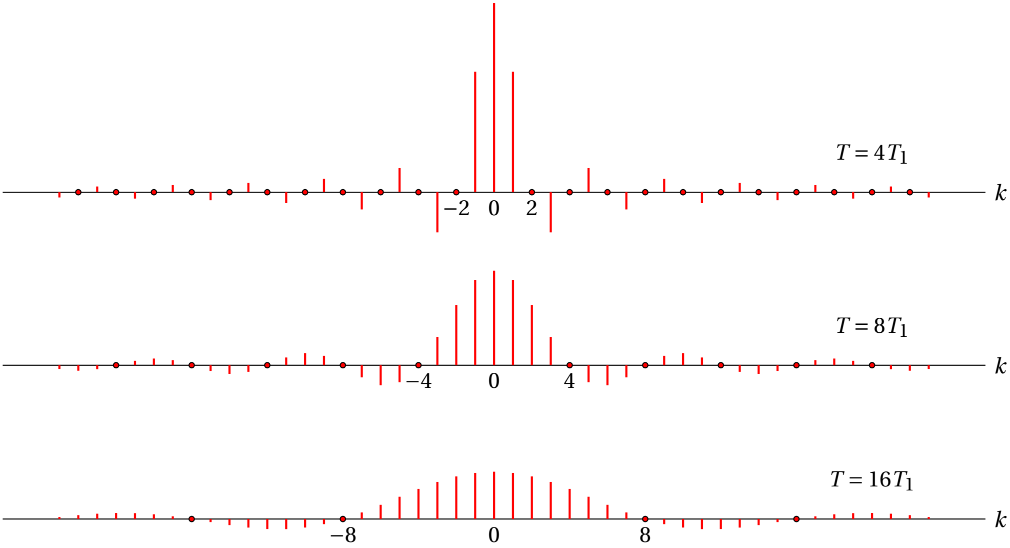

Figure: Plots of scaled Fourier series coefficients Ta_k

Section 2

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients are denoted by a_k, then

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \tag{3}$$

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathscr{F}\mathscr{S}} a_k,$$
$$y(t) \xleftrightarrow{\mathscr{F}\mathscr{S}} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $X(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

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$$z(t) = Ax(t) + By(t) \stackrel{\mathscr{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k. \tag{4}$$

$$x(t - t_0) \overset{\mathscr{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

$$\begin{aligned}
& x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, \quad x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} b_k, \\
b_k &= \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt, \\
&= \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 (\tau + t_0)} dt, \\
&= e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau, \\
&= e^{-jk\omega_0 t_0} a_k.
\end{aligned}$$

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k.$$

Note: $|a_k| = |b_k|$

If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \tag{6}$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}. \tag{7}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \tag{8}$$

Substitution: $k = -m$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{-jm2\pi t/T}.$$

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- If $x(t)$ is odd—$x(-t) = -x(t)$—then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

Time scaling, in general, changes the period.
If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{9}$$

While Fourier coefficients have not changes, the Fourier series representation has changed because of the change in the fundamental frequency.

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \stackrel{\mathscr{F}\mathscr{S}}{\longleftrightarrow} a_k,$$
$$y(t) \stackrel{\mathscr{F}\mathscr{S}}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, its Fourier series coefficients h_k are

$$x(t)y(t) \stackrel{\mathscr{F}\mathscr{S}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{10}$$

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- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.

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- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.

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- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{11}$$

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Note: Left-hand side of equation 11 is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$.

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 \, dt = |a_k|^2. \tag{12}$$

So, $|a_k|^2$ is the average power in the kth harmonic component of $x(k)$.
Thus, what Parseval's relation states is that the total power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 5.

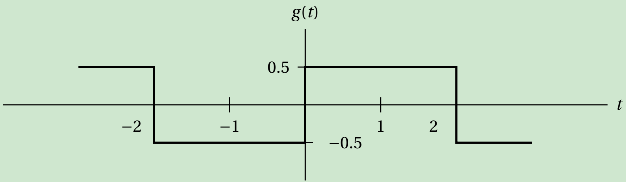

Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

We notice that

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$$b_k = a_k e^{-jk\pi/2},$$

The FS coefficients of the dc offset $-1/2$

$$c_k = \begin{cases} 0, & \text{for } k \neq 0, \\ -\dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

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$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0, \\ a_0 - \dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

yielding

$$d_k = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0, \\ 0 & \text{for } k = 0. \end{cases}$$

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 6. The derivative signal is the signal $g(t)$ in Figure 5. Using this information, find the Fourier series coefficients of $x(t)$.

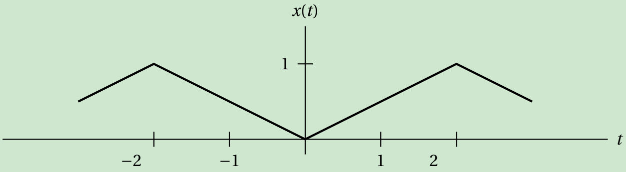

Figure: Figure for example

The derivative of this signal is the signal $g(t)$ in the previous example. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k we see that the differentiation property indicates that

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This equation can be used to express e_k in terms of d_k except when $k=0$. Specifically,

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$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2}, \quad k \neq 0.$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

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For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{13}$$

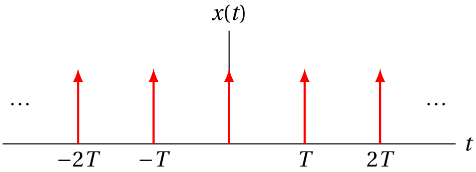

Figure: Impulse train

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$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi/T} dt = \frac{1}{T}.$$

In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index k).

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

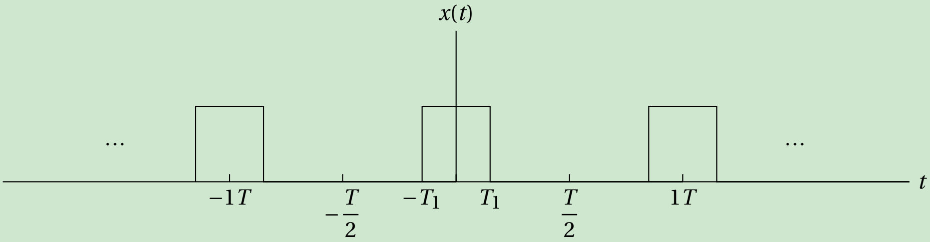

Figure: Figure for example

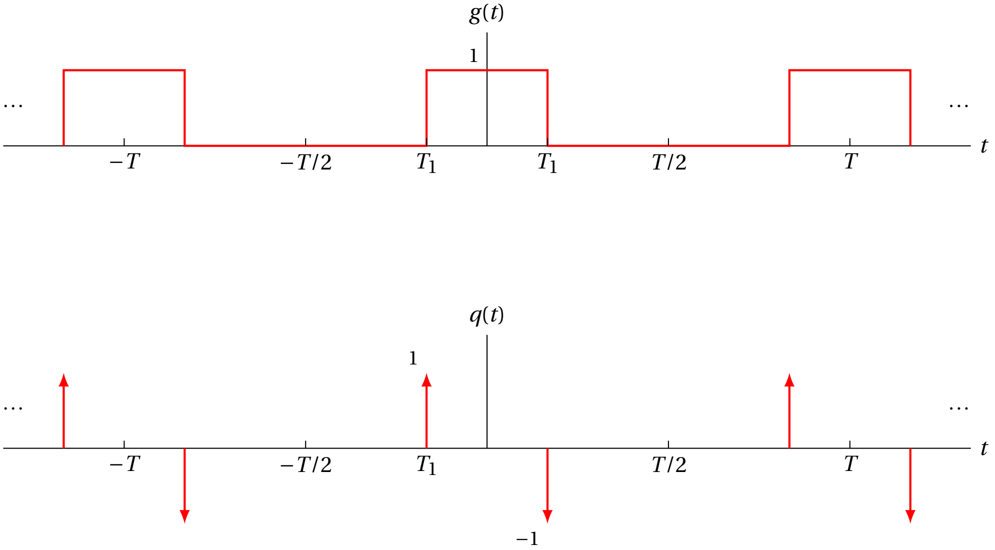

Figure: Periodic square wave and its derivative.

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$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k, \quad \omega_0 = 2\pi / T,$$
$$= \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] = \frac{2j \sin(k\omega_0 T_1}{T}).$$

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Since $q(t)$ is the derivative of $g(t)$, we can use the differentiation property:

$$b_k = jk\omega_0 c_k,$$

where the c_k are the Fourier series coefficients of $g(t)$. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

Since c_0 is just the average value of $g(t)$ over one period,

$$c_0 = \frac{2T_1}{T}.$$

Complex Exponential Fourier Series

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt
\end{aligned} \tag{14}
$$

Trigonometric Fourier Series

$$
\begin{aligned}
x(t) &= A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t \\
A_k &= \frac{1}{T}\int_T x(t) A_k \cos k\omega_0 t dt \\
B_k &= \frac{1}{T}\int_T x(t) A_k \sin k\omega_0 t dt
\end{aligned} \tag{15}
$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$
\begin{aligned}
x(t) &= C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k) \\
C_0 &= A_0 \\
C_k &= \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)
\end{aligned} \tag{16}
$$

Relationship

$$
\begin{aligned}
A_0 &= a_0 \\
A_k &= \frac{a_k + a_{-k}}{2} \\
B_k &= j\frac{a_k - a_{-k}}{2} \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned} \tag{17}
$$

Section 3

Convergence of Fourier Series

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2\, dt$$

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- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.
- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |e_N(t)|^2 \, dt < \infty \tag{18}$$

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- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1
Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\, dt < \infty. \tag{19}$$

This guarantees that a_ks are finite.

Condition 2
In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3
In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)|\, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period T = 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.