EN1060 Signals and Systems: Discrete-Time Fourier Series

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

December 5, 2016

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- Consequently, as the period approaches infinity, this envelope function corresponds to a Fourier representation of the aperiodic signal corresponding to one period. This is, then, the Fourier transform of the aperiodic signal.

Discrete-Time Fourier Transform from DTFS

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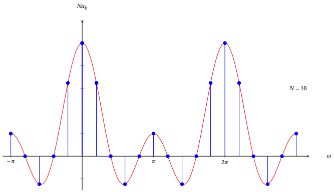
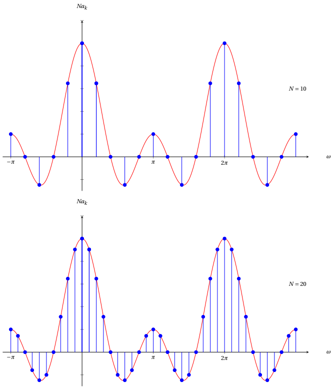
- Consequently, as the period approaches infinity, this envelope function corresponds to a Fourier representation of the aperiodic signal corresponding to one period. This is, then, the Fourier transform of the aperiodic signal.
- The discrete-time Fourier transform developed, as we have just described, corresponds to a decomposition of an aperiodic signal as a linear combination of a continuum of complex exponentials.

- The synthesis equation is then the limiting form of the Fourier series sum, specifically an integral. The analysis equation is the same one we used previously in obtaining the envelope of the Fourier series coefficients.
- While there was a duality in the expressions between the discrete-time Fourier series analysis and synthesis equations, the duality is lost in the discrete-time Fourier transform since the synthesis equation is now an integral and the analysis equation a summation. This is a difference compared to the continuous-time Fourier transform.
- Another important difference is that the discrete-time Fourier transform is always a periodic function of frequency.
- Consequently, it is completely defined by its behavior over a frequency range of 2π in contrast to the continuous-time Fourier transform, which extends over an infinite frequency range.

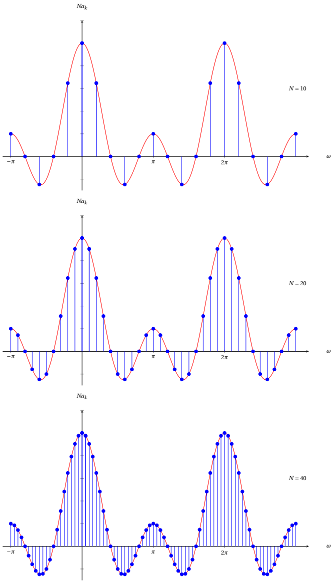
- Construct the periodic signal $\tilde{x}[n]$ for which one period is $x[n]$.
- $\tilde{x}[n]$ has a Fourier series.
- As the period of $\tilde{x}[n]$ increases,
  $\tilde{x}[n] \to x[n]$ and the Fourier series of $\tilde{x}[n]$ $\to$ Fourier transform of $x[n]$.



Na_k
$N = 10$
ω
$-\pi$
π
2π



Na_k
$N = 10$
ω
$-\pi$
π
2π
Na_k
$N = 20$
ω
$-\pi$
π
2π



Na_k
$N = 10$
ω
$-\pi$
π
2π
Na_k
$N = 20$
ω
$-\pi$
π
2π
Na_k
$N = 40$
ω
$-\pi$
π
2π

Fourier Representation of Aperiodic Signals

- $x[n]$ aperiodic
  - Construct periodic signals $\tilde{x}[n]$ for which one period is $x[n]$
  - $\tilde{x}[n]$ has a Fourier series
- As period of $\tilde{x}[n]$ increases
  - $\tilde{x}[n] \longrightarrow x[n]$
  - $\tilde{x}[n] \longrightarrow$ Fourier transform of $x[n]$.

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}.$$

If $x[n]$ is aperiodic, for the periodic signal $\tilde{x}[n]$ whose one period is $x[n]$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Since $x[n] = \tilde{x}[n]$ over a period that includes $-N1 \leq n \leq N_2$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

we see that the coefficients a_k are proportional to the samples of $X(e^{j\omega})$,i.e.,

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

where $\omega_0 = 2\pi/N$ is the spacing of the samples in the frequency domain. Combining

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}.$$

Since $1/N = \omega_0/2\pi$,

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0.$$

As $N \longrightarrow \infty$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Synthesis:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Analysis

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$\begin{aligned} X(e^{j\omega}) &= \mathrm{Re}\{X(e^{j\omega})\} + j\mathrm{Im}\{X(e^{j\omega})\} \\ &= |X(e^{j\omega})| e^{\angle X(e^{j\omega})} \end{aligned}$$

Example

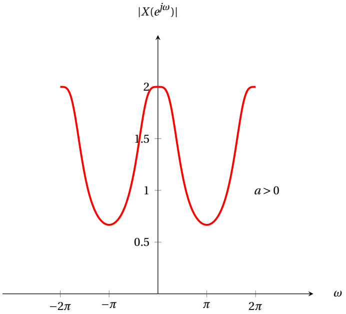
Obtain an expression for the DTFT of

$$x[n] = a^n u[n], |a| < 1.$$

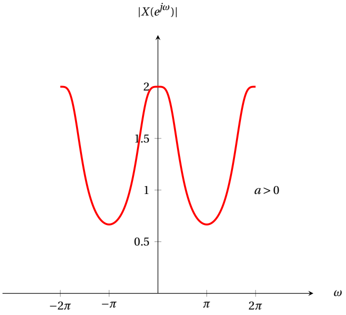
Sketch the magnitude and phase of $X(e^{j\omega})$ for

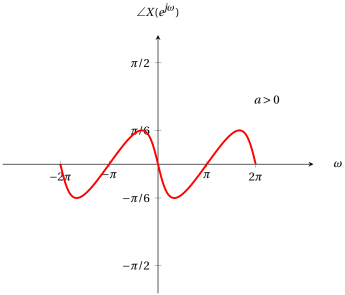
1. $a > 0$, $(a = 0.5)$ and
2. $a < 0$, $(a = -0.5)$.

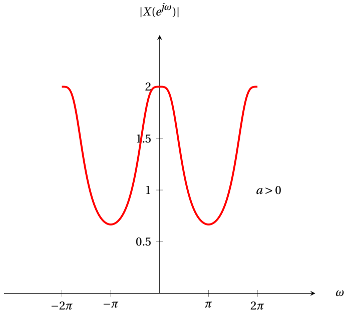
$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\
&= \frac{1}{1 - ae^{-j\omega}}.
\end{aligned}
$$

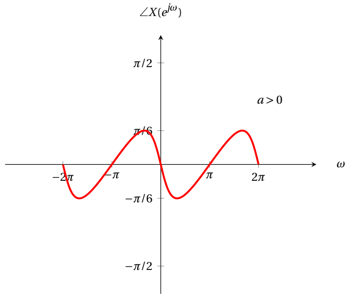


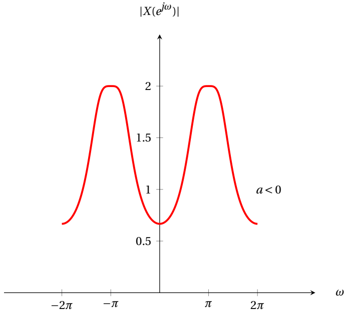
$|X(e^{j\omega})|$
2
1.5
1
0.5
$a > 0$
-2π
$-\pi$
π
2π
ω

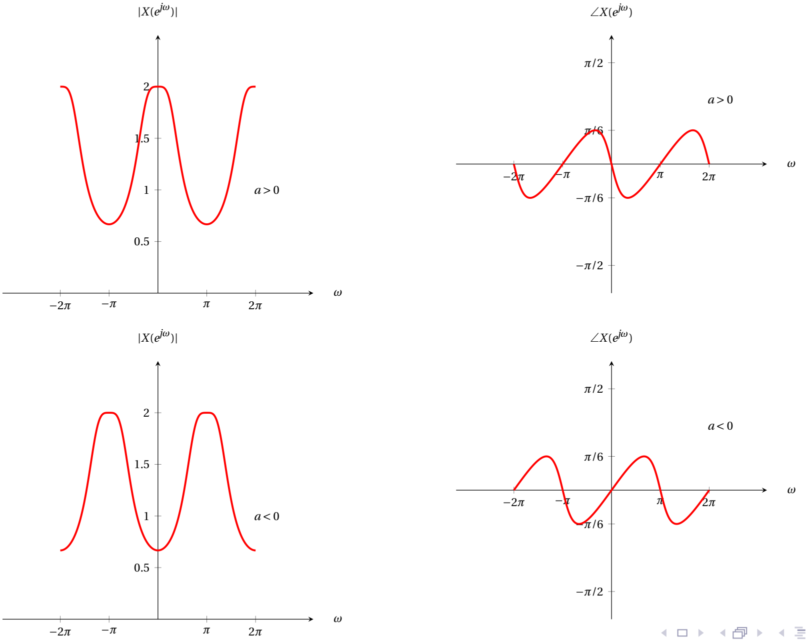

|X(e^{jω})|
2
1.5
1
0.5
a > 0
−2π
−π
π
2π
ω


∠X(e^{jω})
π/2
π/6
a > 0
−2π
−π
π
2π
ω
−π/6
−π/2


|X(e^{jω})|
2
1.5
1
0.5
a > 0
−2π
−π
π
2π
ω


∠X(e^{jω})
π/2
π/6
−π/6
−π/2
a > 0
−2π
−π
π
2π
ω


|X(e^{jω})|
2
1.5
1
0.5
a < 0
−2π
−π
π
2π
ω



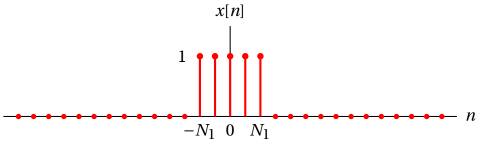
|X(e^{jω})|
2
1.5
1
0.5
a > 0
−2π
−π
π
2π
ω
∠X(e^{jω})
π/2
π/6
−π/6
−π/2
a > 0
−2π
−π
π
2π
ω
|X(e^{jω})|
2
1.5
1
0.5
a < 0
−2π
−π
π
2π
ω
∠X(e^{jω})
π/2
π/6
−π/6
−π/2
a < 0
−2π
−π
π
2π
ω

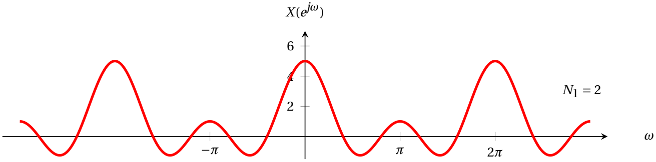
Example

Consider the rectangular pulse

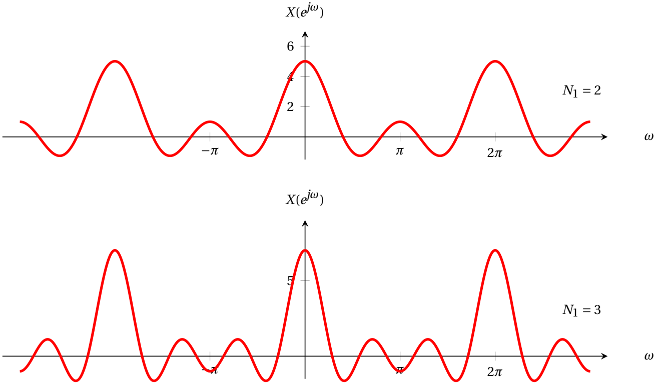
$$x[n] = \begin{cases} 1, & |n| \le N_1, \\ 0, & |n| > N_1. \end{cases}$$

1. Obtain an expression for the DTFT $X(e^{j\omega})$ of this signal.
2. Sketch for $N_1 = 2$.

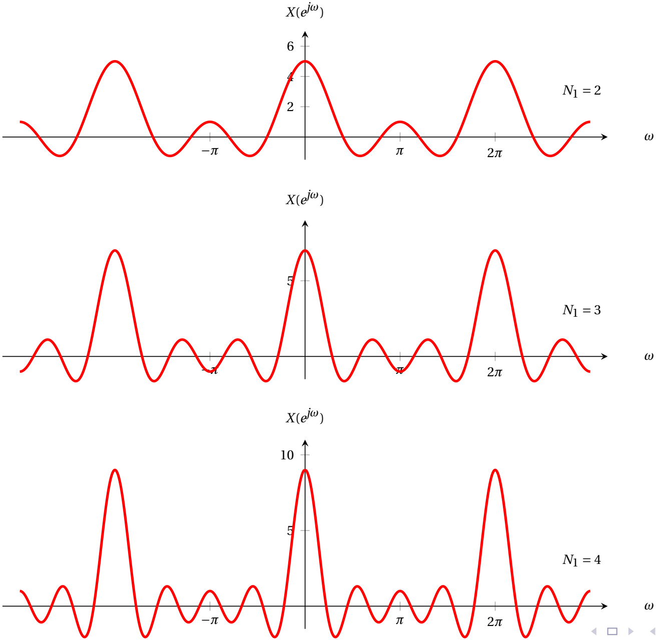




$X(e^{j\omega})$
6
4
2
$N_1 = 2$
$-\pi$
π
2π
ω



$X(e^{j\omega})$
6
4
2
$N_1 = 2$
ω
$-\pi$
π
2π
$X(e^{j\omega})$
5
$N_1 = 3$
ω
$-\pi$
π
2π

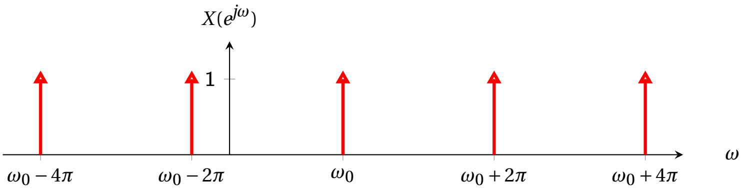

$X(e^{j\omega})$
6
4
2
$N_1 = 2$
ω
$-\pi$
π
2π
$X(e^{j\omega})$
5
$N_1 = 3$
ω
$-\pi$
π
2π
$X(e^{j\omega})$
10
5
$N_1 = 4$
ω
$-\pi$
π
2π

As in the continuous-time case, discrete-time periodic signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an impulse train in the frequency domain.
Consider the signal

$$x[n] = e^{j\omega_0 n}.$$

In CT the Fourier transform of $e^{j\omega_0 t}$ can be interpreted as an impulse at $\omega = \omega_0$. However, the DT Fourier transform must be periodic in w with period π, requiring impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, ... So

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l).$$



For a periodic sequence $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

the Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

Example

Consider the periodic signal

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad \text{with}\,\omega_0 = \frac{2\pi}{5}.$$

Periodicity

DTFT is always periodic in ω withe period 2π

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$$

Linearity

If

$$x_1[n] \overset{\mathscr{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \overset{\mathscr{F}}{\longleftrightarrow} X_2(e^{j\omega})$$

then

$$ax_1[n] + bx_2[n] \overset{\mathscr{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time Shifting and Frequency Shifting

If

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x[n-n_0] \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

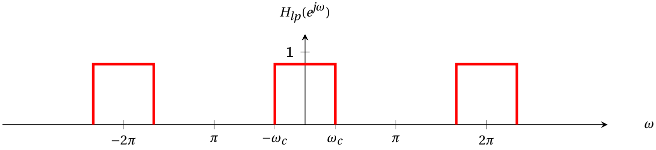
and

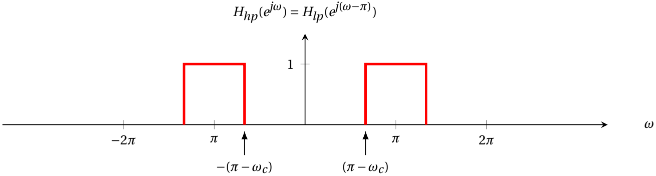
$$e^{j\omega_0 n} x[n] \overset{\mathscr{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Example

The frequency response of an ideal low-pass filter has the cutoff frequency of ω_c.

1. Obtain an expression for the frequency response of the corresponding high-pass filter (cutoff frequency $\pi - \omega_c$).
2. Obtain and expression for the impulse response of this high-pass filter in terms of the impulse response of the low-pass filter.





$H_{lp}(e^{j(\omega-\pi)})$ is the frequency response of $H_{lp}(e^{j\omega})$ shifted by one-half period, i.e., by π. Since high frequencies in discrete time are concentrated near π (and other odd multiples of π), the filter depicted in the second figure is an ideal highpass filter with cutoff frequency $\pi - \omega_c$.

$$
\begin{aligned}
h_{hp}[n] &= e^{j\pi n} h_{lp}[n] \\
&= (-1)^n h_{lp}[n].
\end{aligned}
$$

If

$$x[n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{j\omega})$$

then

$$x^*[n] \overset{\mathscr{F}}{\longleftrightarrow} X^*(e^{-j\omega}).$$

Also, if $x[n]$ is real-valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$X(e^{j\omega}) = X^*(e^{-j\omega}), \quad (x[n] \text{ real.})$$

$\mathrm{Re}\{X(e^{j\omega})\}$ is an even function of ω and $\mathrm{Im}\{X(e^{j\omega})\}$ is an odd function of ω.
The magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function.

$$x[-n] \overset{\mathscr{F}}{\longleftrightarrow} X(e^{-j\omega}).$$

Example

Prove the time-reversal property.

$$x[-n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{-j\omega}).$$

Example

Prove the time-reversal property.

Let $x[n]$ be a signal with spectrum $X(e^{j\omega})$, and consider the transform $Y(e^{j\omega})$ of $y[n] = x[-n]$.

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$$

Substituting $m = -n$

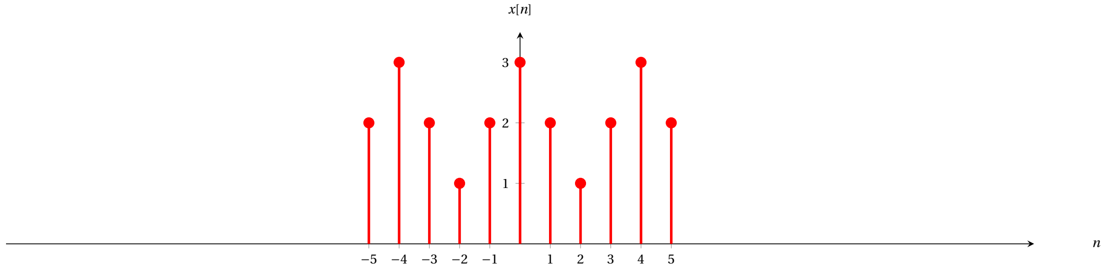
$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega}).$$

Because of the discrete nature of the time index for discrete-time signals, the relation between time and frequency scaling in discrete time takes on a somewhat different form from its continuous-time counterpart. In CT

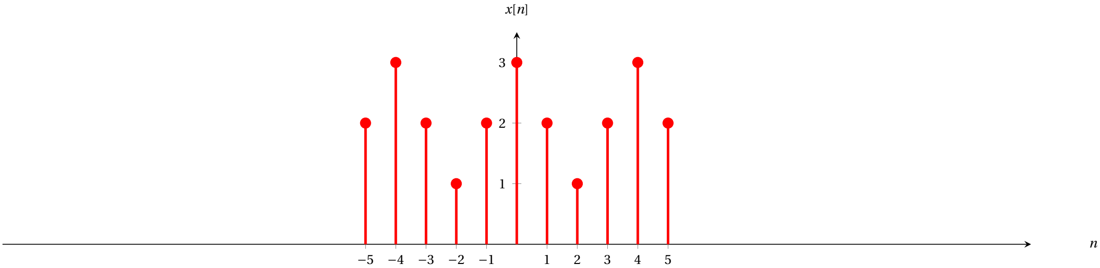
$$x(at) \overset{\mathscr{F}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

However, if we try to define the signal $x[an]$, we run into difficulties if a is not an integer. Therefore, we cannot slow down the signal by choosing $a < 1$. On the other hand, if we let a be an integer other than ± 1—e.g., if we consider $x[2n]$—we do not merely speed up the original signal. That is, since n can take on only integer values, the signal $x[2n]$ consists of the even samples of $x[n]$ alone.
Let k be a positive integer, and define the signal

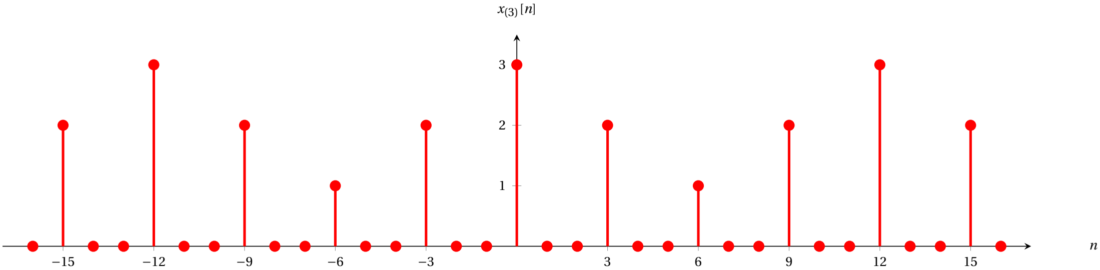
$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k, \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases}$$



x[n]
3
2
1
n
−5
−4
−3
−2
−1
1
2
3
4
5



$x[n]$
3
2
1
n
−5 −4 −3 −2 −1 1 2 3 4 5



$x_{(3)}[n]$
3
2
1
n
−15 −12 −9 −6 −3 3 6 9 12 15

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_{(k)}[rk]e^{-j\omega rk}$$

Furthermore, since $x_{(k)}[rk] = x[r]$,

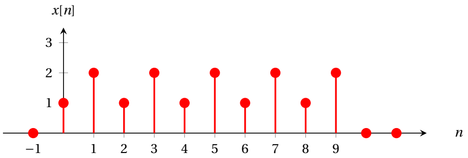
$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

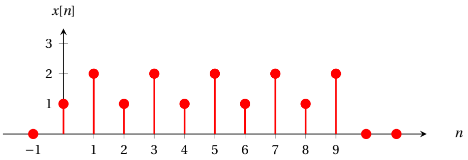
$$x_{(k)}[n] \overset{\mathscr{F}}{\leftrightarrow} X(e^{jk\omega})$$

For figure 5.14

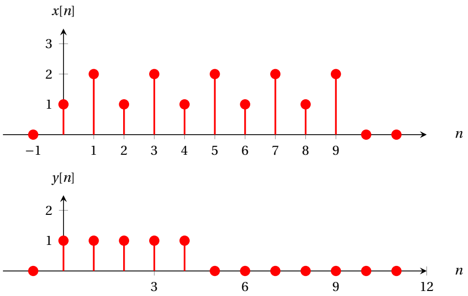
Example

Find the DTFT of the signal $x[n]$ shown in the figure, expressing the signal in terms of a simpler signal.

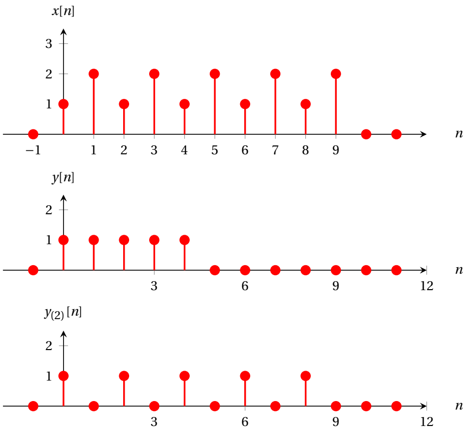




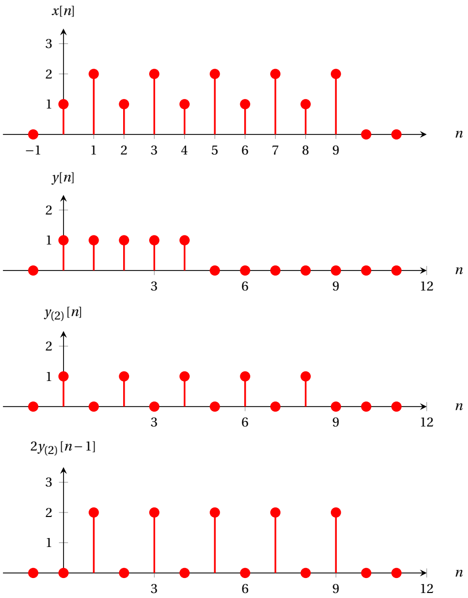
x[n]
3
2
1
n
−1
1
2
3
4
5
6
7
8
9



x[n]
3
2
1
n
−1
1
2
3
4
5
6
7
8
9
y[n]
2
1
n
3
6
9
12



x[n]
3
2
1
−1
1
2
3
4
5
6
7
8
9
n
y[n]
2
1
3
6
9
12
n
y(2)[n]
2
1
3
6
9
12
n


$x[n]$
3
2
1
n
-1
1
2
3
4
5
6
7
8
9
$y[n]$
2
1
n
3
6
9
12
$y_{(2)}[n]$
2
1
n
3
6
9
12
$2y_{(2)}[n-1]$
3
2
1
n
3
6
9
12

$$nx[n] \overset{\mathscr{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Example

Prove the differentiation property.

$$nx[n] \overset{\mathscr{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Example

Prove the differentiation property.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Differentiating

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}.$$

Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

If $x[n]$, $h[n]$ and $y[n]$ are the input, impulse response, and output respectively of an LTI system, so that

$$y[n] = x[n] * h[n]$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

Example

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0]$$

Obtain the output $y[n]$ for an input $x[n]$.

Example

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

and suppose that the input to this system is

$$x[n] = \beta^n u[n], \quad |\beta| < 1$$

Obtain the output $y[n]$ for $\alpha \neq \beta$ and $\alpha = \beta$.

Consider $y[n]$ equal to the product of $x_1[n]$ and $x_2[n]$, then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta$$

This equation corresponds to the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluation over any given interval of length 2π.
See example 5.15.

Summary

	CT	DT
	Series (CT)	Series (DT)
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\frac{2\pi}{T})t}$	$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$
	$a_k = \frac{1}{T}\int_T x(t)e^{-jk(\frac{2\pi}{T})t}dt$	$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(\frac{2\pi}{N})n}$
	Transform (CT)	Transform (DT)
Aperiodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$
	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

CT: $\omega_0 = \frac{2\pi}{T}$, DT: $\omega_0 = \frac{2\pi}{N}$