EN1060 Signals and Systems: Fourier Transform

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Section 1

Fourier Transform Properties

Synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \tag{1}$$

Analysis equation:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{2}$$

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega). \tag{3}$$

If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

and

$$y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega).$$

then

$$ax(t) + by(t) \overset{\mathscr{F}}{\longleftrightarrow}$$

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If

$$x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x(t-t_0) \overset{\mathscr{F}}{\longleftrightarrow}$$

Time Shifting

If

$$x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x(t - t_0) \stackrel{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega).$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$$

$$\begin{aligned} x(t - t_0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-t_0)} d\omega. \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right) e^{j\omega t} d\omega. \end{aligned}$$

Proof

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}d\omega.$$

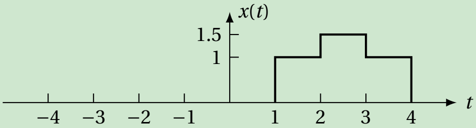
This is the synthesis equation for $x(t-t_0)$. Therefore,

$$\mathscr{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega).$$

Magnitude of the Fourier transform not altered. Time shift introduces a phase shift $-\omega t_0$, which is a linear function of ω.

Example

Evaluate the Fourier transform of $x(t)$.



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

$$X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2}\left[\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right].$$

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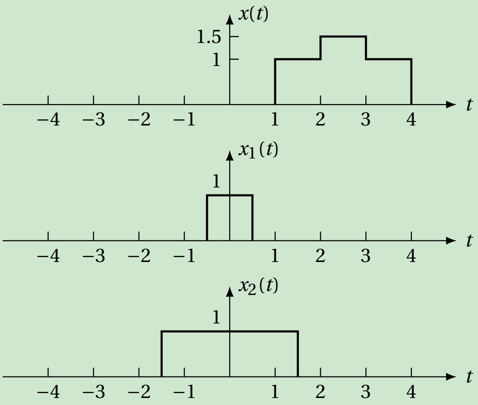

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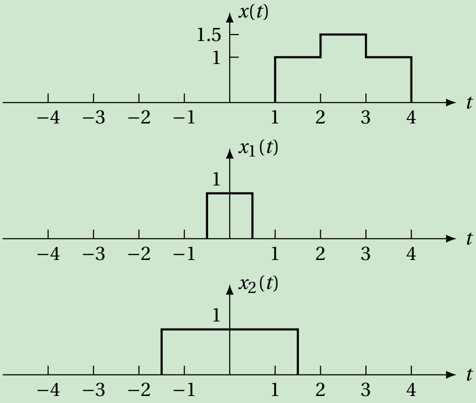

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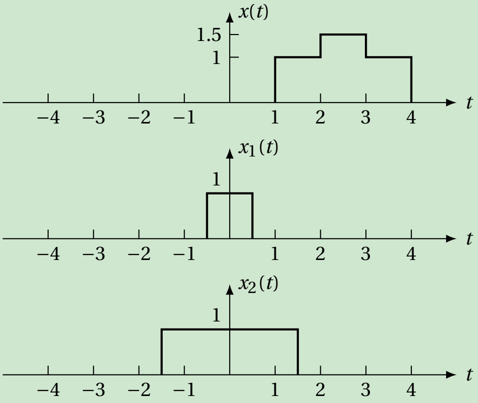

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Conjugation and Conjugate Symmetry

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega).$$

then

$$x^*(t) \xleftrightarrow{\mathscr{F}} X^*(-j\omega).$$

$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* \\ &= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt \end{aligned}$$

Replacing ω by $-\omega$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt$$

which is the analysis equation for $x^*(t)$.
If $x(t)$ is real, i.e., $x(t) = x^*(t)$, $X(j\omega)$ has conjugate symmetry.

$$X(-j\omega) = X^*(j\omega) \quad [x(t) \text{ real}]$$

Synthesis equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Differentiating both sides of the equation

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathscr{F}} j\omega X(j\omega).$$

Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathscr{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).$$

Example

Determine the Fourier transform of the unit step $x(t) = u(t)$ making use of the knowledge that

$$g(t) = \delta(t) \xleftrightarrow{\mathscr{F}} G(j\omega) = 1.$$

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we obtain that

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega).$$

Since $G(j\omega) = 1$

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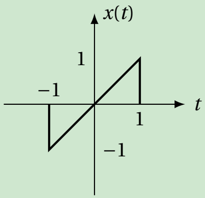
Observe that, we can apply the differentiation property to recover the transform of the impulse:

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{\mathscr{F}} j\omega \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] = 1.$$

Note: $\omega\delta(\omega) = 0$

Example

Determine the Fourier transform of the signal $x(t)$ shown below:



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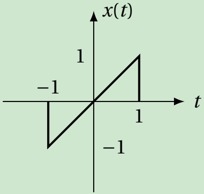
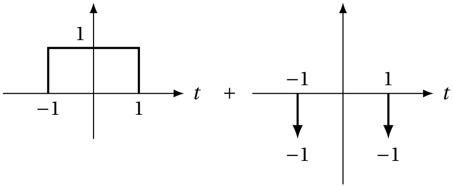


$$g(t) = \frac{dx(t)}{dt} =$$



$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

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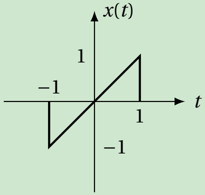

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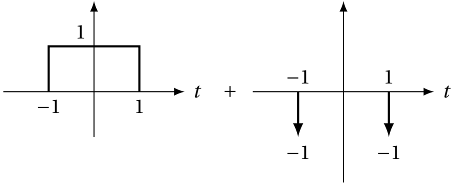

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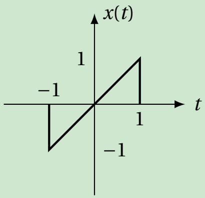

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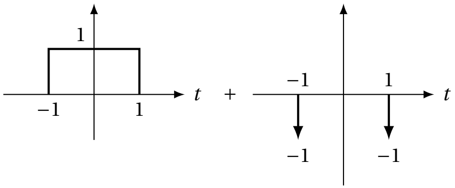

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$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega).$$

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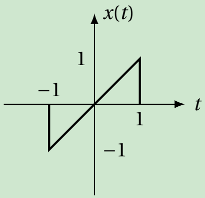

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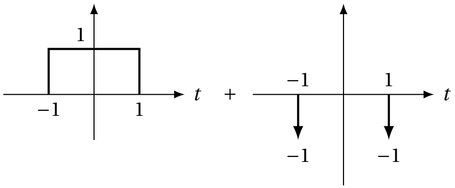

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$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega).$$

As $G(0) = 0$

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Note: $X(j\omega)$ is purely imaginary and odd.

If

$$x(t) \xleftrightarrow{\mathscr{F}} X(j\omega).$$

then

$$x(at) \xleftrightarrow{\mathscr{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

where a is a real constant.

Time and Frequency Scaling

If

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where a is a real constant.
Letting $a = -1$

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where a is a real constant.
Letting $a = -1$

$$x(-t) \overset{\mathscr{F}}{\longleftrightarrow} X(-j\omega).$$

The scaling property is another example of the inverse relationship between time and frequency.

Because of the similarity between the synthesis equation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega. \tag{4}$$

and the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{5}$$

for any transform pair, there is a dual pair with the time and frequency variables interchanged.

We determined the Fourier transform of the square pulse as

$$x_1(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1, \end{cases} \xleftrightarrow{\mathscr{F}} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

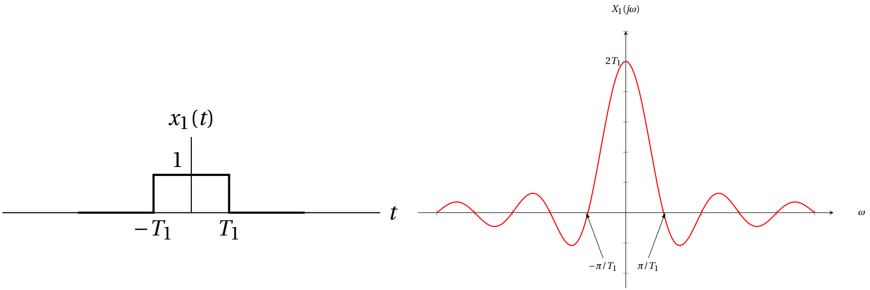


Figure: Rectangular pulse and the Fourier transform.

We also determined that for a time-domain signal that is similar in shape to the $X_1(j\omega)$ as

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{\mathscr{F}} X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

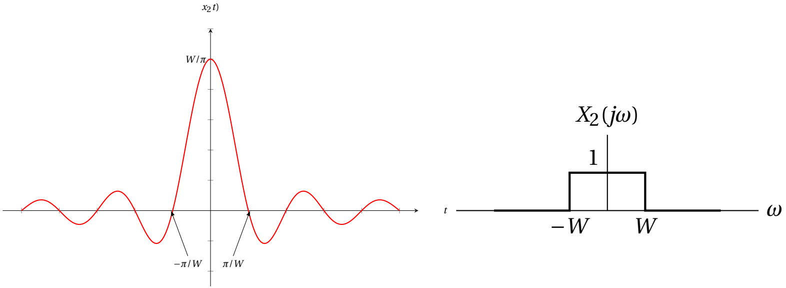


Figure: Fourier transform for $x(t)$.

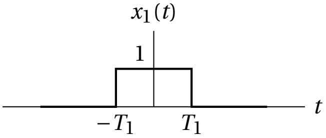
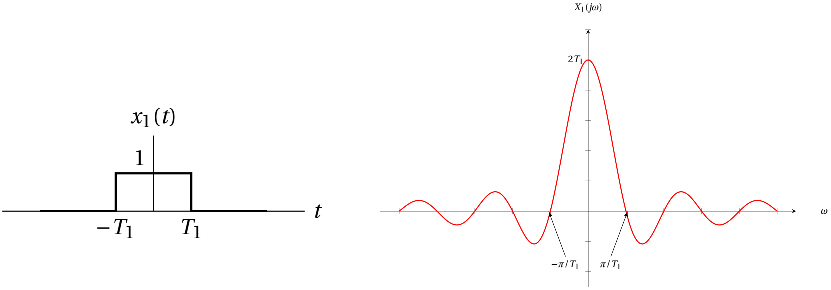


Figure: Duality.



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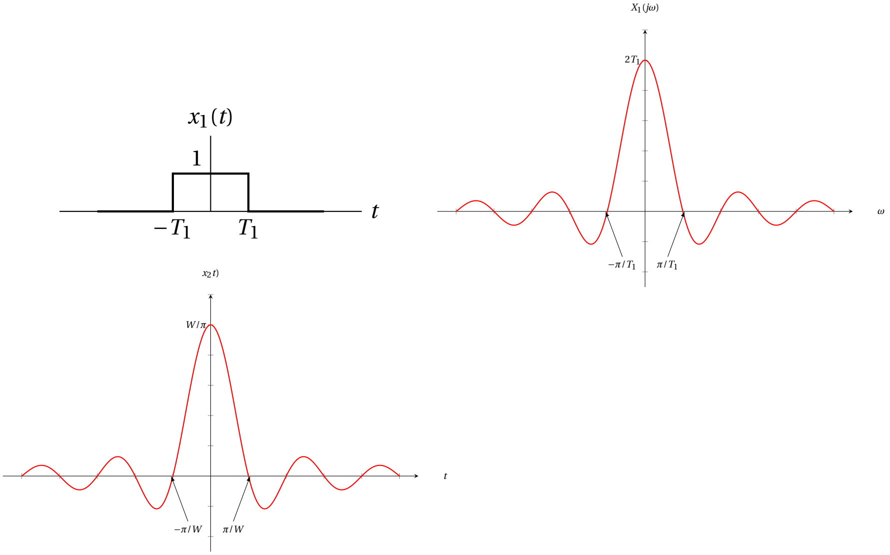

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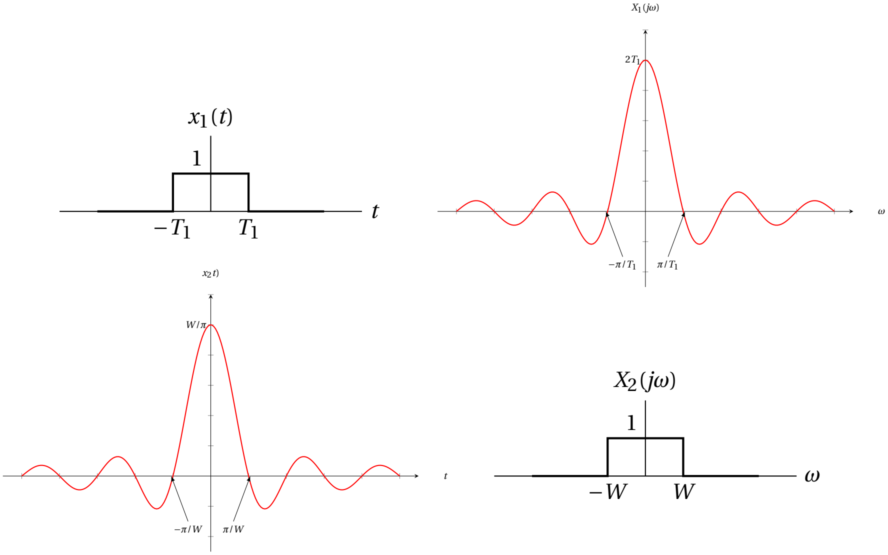

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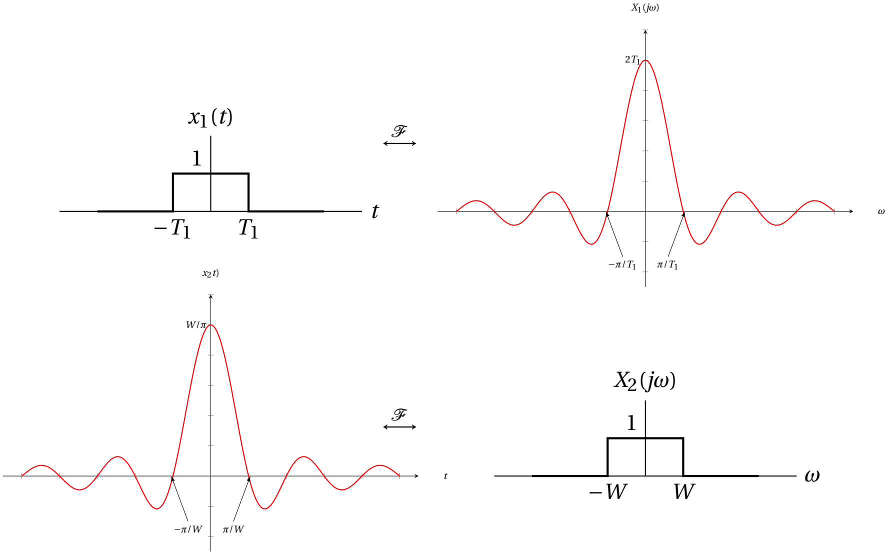

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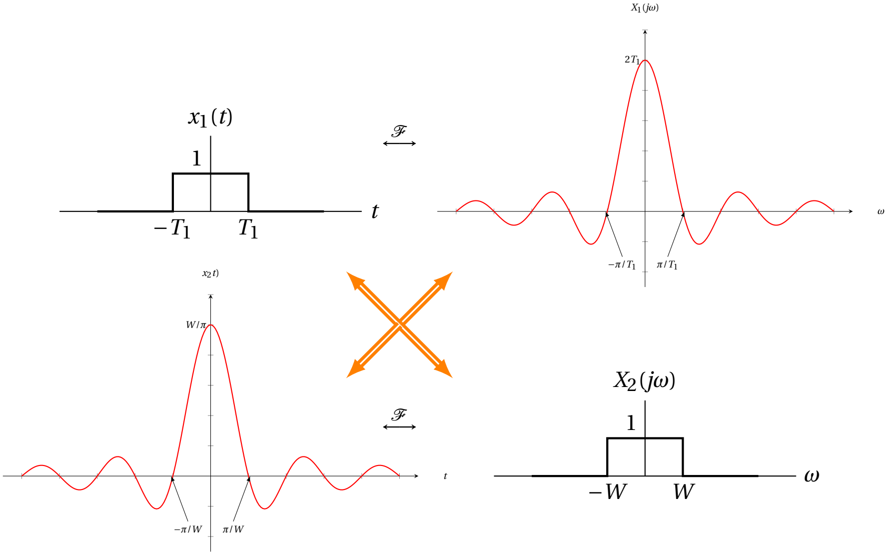

Figure: Duality.

Example

Use the duality property to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

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Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

$$x(t) = e^{-2|t|} \overset{\mathscr{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

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$$2\pi e^{-2|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2} \right) e^{-j\omega t} dt.$$

The right-hand side of this expression is the Fourier transform analysis equation for $2/(1+t^2)$. Thus

$$\mathscr{F}\left\{ \frac{2}{1+t^2} \right\} = 2\pi e^{-2|\omega|}.$$

More Properties Using Duality

$$-jtx(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.$$

$$e^{j\omega_0 t}x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(j(\omega-\omega_0)).$$

$$-\frac{1}{jt}x(t)+\pi x(0)\delta(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)\,d\eta.$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

Convolution Property

$$y(t) = h(t) * x(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

This equation is of major importance in signal and system analysis. This says that the Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

Example

An LTI system has the impulse response

$$h(t) = \delta(t - t_0).$$

If the Fourier transform of the input signal $x(t)$ is $X(j\omega)$, what is the Fourier transform of the output? $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

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$$\begin{aligned} h(t) &= \delta(t - t_0) \\ H(j\omega) &= e^{-j\omega t_0} \\ Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0} X(j\omega) \end{aligned}$$

The convolution property states that convolution in time domain corresponds to multiplication in frequency domain. Because of the duality between time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \xleftrightarrow{\mathscr{F}} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)].$$

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other. Consequently, the multiplication of two signals is often referred to as amplitude modulation. For this reason, this equation is sometime referred to as the modulation property.

Example

Let $s(t)$ be a signal whose spectrum is depicted in the figure below. Also consider the signal

$$p(t) = \cos \omega_0 t.$$

Show the spectrum of $r(t) = s(t)p(t)$.

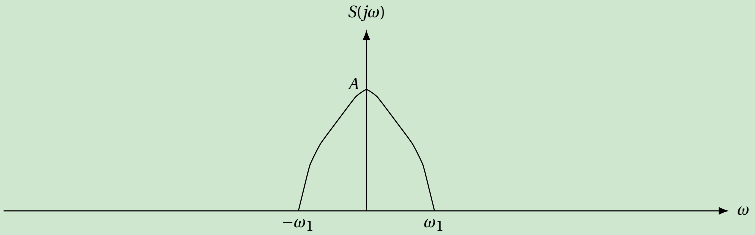

Figure: Spectrum of signal $s(t)$.

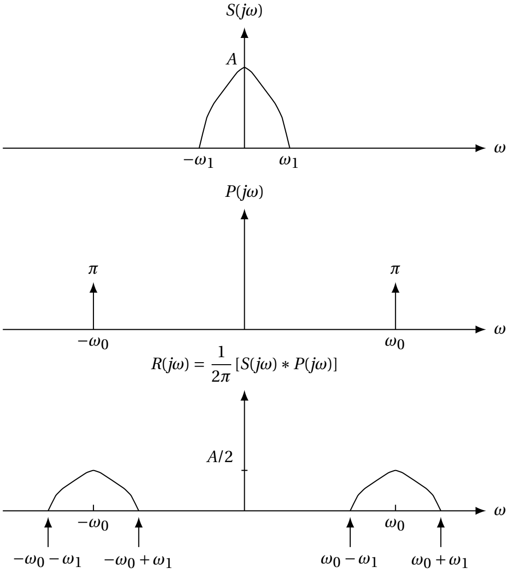

Figure: Fourier transform of $r(t) = s(t)p(t)$.