EN1060 Signals and Systems: Discrete-Time Fourier Series

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- Specifically, we consider the representation of discrete-time signals through a decomposition as a linear combination of complex exponentials.
  - DT periodic signals $\rightarrow$ DT Fourier series
  - DT aperiodic signals $\rightarrow$ DT Fourier transform
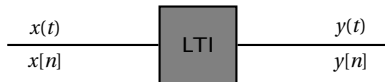
Decompose the input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots \quad \text{linear combination of basic inputs}$$

Then

$$y = a_1\psi_1 + a_2\psi_2 + \cdots \quad \text{linear combination of corresponding outputs}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ such that

- Broad class of signals can be constructed, and
- Response to ϕ_ks easy to compute.

Continuous-Time:

$\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(\omega_k)e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

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"Discrete-Time": $\phi_k[n] = e^{j\omega_k n}$

$$e^{j\omega_k n} \longrightarrow \underset{\uparrow}{e^{j\omega_k n}} \underbrace{\sum_{r=-\infty}^{\infty} h[r] e^{-j\omega_k r}}$$

eigenfunction eigenvalue

Consider $x[n]$ to be periodic,
Period N,
Fundamental frequency $\omega_0 = \dfrac{2\pi}{N}$
$e^{jk\omega_0 n}$ are harmonically related, and periodic
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Consider the complex exponential

$$e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$$

$\Rightarrow$ Only N distinct complex exponentials.

$$x[n] = \sum_k a_k e^{jk\omega_0 n}, \quad k = 0, 1, 2, \ldots, N-1.$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

N equations in N unknowns.

$$a_k = \frac{1}{N} \sum_{<N>} x[n] e^{-jk\omega_0 n}.$$

$k = <N>$: k ranges over one period (as a_k periodically repeats).

Continuous-Time

Discrete-Time Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Synthesis

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}.$$

Analysis

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}.$$

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Note the duality.

Periodicity

$x[n]$	periodic in n,	true for CT
$e^{jk\omega_0 n}$	periodic in n,	true for CT
$e^{jk\omega_0 n}$	periodic in k,	**not true for CT**
a_k	periodic in k,	**not true for CT**

Convergence

Continuous-time:

- $x(t)$ square-integrable OR
- Dirichlet condition

Discrete-time

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$\hat{x}[n] = \sum_{p \text{ terms}} a_k e^{jk\omega_0 n}.$$

$p = N$

$$\hat{x}[n] \equiv x[n].$$

There is no issue of convergence in DT.

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Expanding the signal as a sum of two complex exponentials,

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \tag{1}$$

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$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}.$$
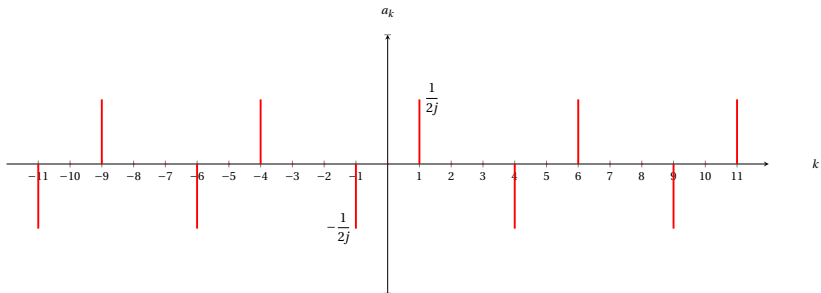
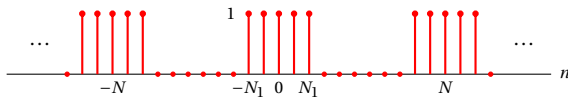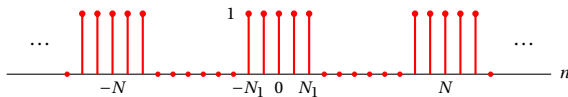Figure: Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Determine and sketch the DTFT of

$$x[n] = 1 + \sin \omega_0 n + 3 \cos \omega_0 n + \cos \left(2\omega_0 n + \frac{\pi}{2} \right).$$

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Letting $m = n + N_1$

$$= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)/(m-N_1)}$$

$2N_1 + 1$ terms in a geometric series

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \left[\frac{1 - e^{-jk(2\pi/N)(2N_1+1)}}{1 - e^{-jk(2\pi/N)}} \right]$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \cdot \frac{e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{-j\frac{k(2\pi/N)}{2}}} \left[\frac{e^{j\frac{k(2\pi/N)(2N_1+1)}{2}} - e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{j\frac{k(2\pi/N)}{2}} - e^{-j\frac{k(2\pi/N)}{2}}} \right]$$

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$$a_k = \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)} \quad k \neq 0, \pm N, \pm 2N, \ldots$$

$$a_k = \frac{2N_1 + 1}{N} \quad k = 0, \pm N, \pm 2N, \ldots.$$