EN1060 Signals and Systems: Linear Time Invariant Systems

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Section 1

Introduction

Systems

- A system processes signals.

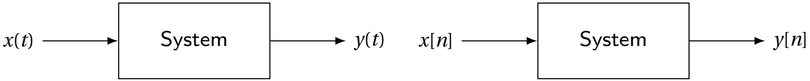
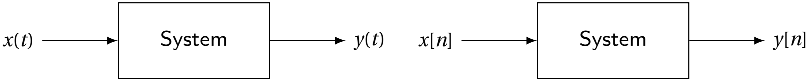


Figure: CT and DT Systems.

Systems

- A system processes signals.
- Examples of systems:



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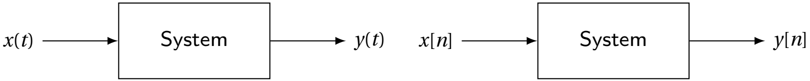
    - Dynamics of an aircraft.



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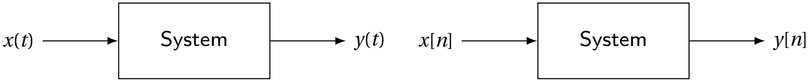
  - An algorithm for analyzing financial and economic factors to predict bond prices.



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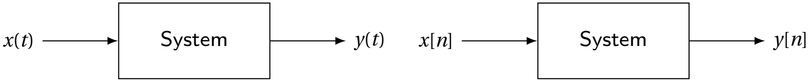
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm for post-flight analysis of a space launch.



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Systems

- A system processes signals.
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm for post-flight analysis of a space launch.
  - An edge detection algorithm for medical images.

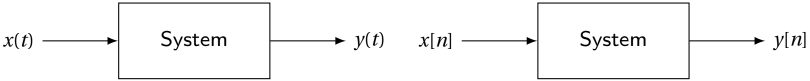


Figure: CT and DT Systems.

Types of Systems

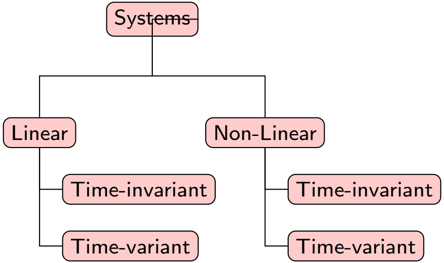


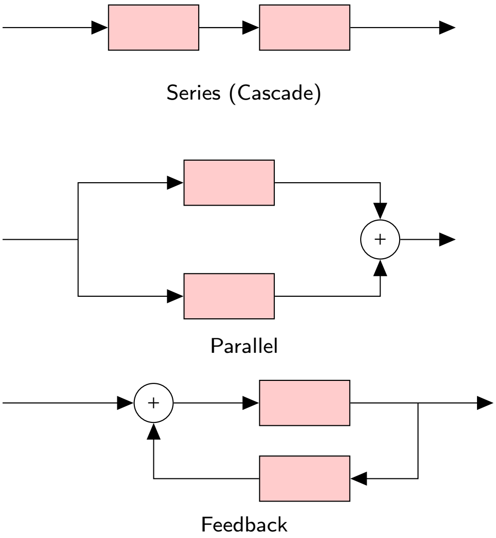
Figure: System types.

This course is focused on the class of linear, time-invariant (LTI) systems.

Systems Interconnections

- To build more complex systems by interconnecting simpler subsystems.
- To modify the response of a system.
- E.g.: amplifier design, stabilizing unstable systems.

Signal-Flow (Block) Diagrams



Section 2

Basic System Properties

Systems with and without Memory

A systems is said to be memoryless if its output for each value of the independent variable at a given time is dependent only on the input at the same time.

Examples of memoryless systems

$$y[n] = (2x[n] - x^2[n])^2,$$

$$y(t) = Rx(t),$$

where $x(t)$ current through the resistor R and $y(t)$ taken as the voltage across the resistor.

$$y(t) = x(t),$$

which is called the identity system. In DT

$$y[n] = x[n].$$

Examples of systems with memory

Accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].$$

Delay:

$$y[n] = x[n-1].$$

Capacitor with current as the input and the output taken as the voltage:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau.$$

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Examples of systems with memory

Accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].$$

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n], \quad y[n] = y[n-1] + x[n]$$

Delay:

$$y[n] = x[n-1].$$

Capacitor with current as the input and the output taken as the voltage:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau.$$

Invertibility and Inverse Systems

A system is invertible of different inputs lead to difference outputs. If a system is invertible, the an inverse system exists, and when cascaded with the original system, yields an output equal to the input to the first system.

$$x[n] \longrightarrow \boxed{\text{System}} \xrightarrow{y[n]} \boxed{\text{Inverse system}} \longrightarrow w[n] = x[n]$$

Examples of invertible systems:

If $y(t) = 2x(t)$, the inverse system is

$$w(t) = \frac{1}{2}y(t),$$

If (accumulator) $y[n] = \sum_{k=-\infty}^{\infty} x[k]$. the inverse system is

$$w[n] = y[n] - y[n-1].$$

Examples of non-invertible systems:

$$y[n] = 0.$$

$$y(t) = x^2(t).$$

Causality

A system is said to be causal if it only responds when you "kick it." It response at any time depends only on that input prior or equal to that time. The system cannot anticipate future inputs.

Example

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

If

$$x_1(t) = x_2(t), \quad t < t_0,$$

then

$$y_1(t) = y_2(t), \quad t < t_0.$$

If inputs are identical until t_0, the outputs are identical until t_0. Same for DT.

Stability

Many forms. We choose Bounded Input Bounded Output (BIBO) stability.
If a system is stable in BIBO sense, for every bounded input the output is bounded.

Example

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

is not stable.

Time Invariance

The system does not really care what we call the origin. If the input is shifted by any amount of time t_0, the output is also shifted by the same amount of time.

If

$$x(t) \to y(t),$$

then

$$x(t-t_0) \to y(t-t_0).$$

If

$$x[n] \to y[n],$$

then

$$x[n-n_0] \to y[n-n_0].$$

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If

$$x[n] \rightarrow y[n],$$

then

$$x[n-n_0] \rightarrow y[n-n_0].$$

Example

Accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Example

Modulator:

$$y(t) = (\sin(t)x(t)$$

Linearity

If

$$x_1(t) \rightarrow y_1(t)$$
$$x_2(t) \rightarrow y_2(t)$$

then

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

If the system is linear, if we give a linear combination of inputs, the output will allose be a similar linear combination of the original outputs.

Linear Time-Invariant Systems

1. Systems that are both linear and time invariant are called Linear Time-Invariant (LTI) systems.
2. With systems that are linear and time invariant, using the impulse function in CT and DT, produces an important and useful mechanism for characterizing those system.
3. In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems.
4. The resulting representation is referred to as convolution.

Section 3

Convolution

Introduction (From Oppenheim)

- A linear system: the response to a linear combination of inputs is the same linear combination of the individual responses.
- Time invariance: the system is not sensitive to the time origin. If the input is shifted in time by some amount, then the output is simply shifted by the same amount.
- For a linear system, if the system inputs can be decomposed as a linear combination of some basic inputs and the system response is known for each of the basic inputs, then the response can be constructed as the same linear combination of the responses to each of the basic inputs.
- Signals can be decomposed as a linear combination of basic signals in a variety of ways (e.g., Taylor series expansion that expresses a function in polynomial form.) However, in the context of signals and systems, it is important to choose the basic signals in the expansion so that in some sense the response is easy to compute.
- For systems that are both linear and time-invariant, there are two particularly useful choices for these basic signals: delayed impulses and complex exponentials.

Introduction (From Oppenheim)

- In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems. The resulting representation is referred to as convolution.
- Earlier, we developed in detail the decomposition of signals as linear combinations of complex exponentials (referred to as Fourier analysis) and the consequence of that representation for linear, time-invariant systems.

Introduction

- Using the convolution we can express the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse.
- An LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.
- In discrete time, we have the convolution sum. In continuous time, we have the convolution integral.

Strategy for Exploiting Linearity and Time Invariance

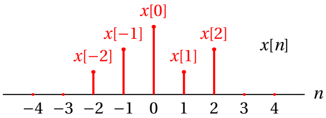
- Decompose the input signal to a linear combination of basic signals.
- Chose basic signals so that the response is easy to compute (analytical convenience).

1. Delayed impulses → convolution
2. Complex exponentials → Fourier analysis

Outline

A DT Signal as Superposition of Weighted Delayed Impulses

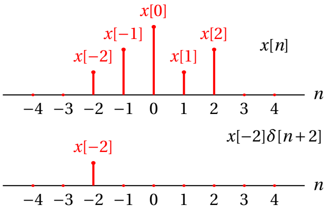
- We can express a DT signal as a linear combination of weighted delayed impulses.
- If we have a liner system, and a signal expressed as above as a linear combination of basic signals, the response would be the same linear combinaton fo the responses for individual basic signals.



$$
\begin{aligned}
x[n] = {} & x[-2]\delta[n+2] \\
& x[-1]\delta[n+1] \\
& x[0]\delta[n] \\
& x[1]\delta[n-1] \\
& x[2]\delta[n-2]
\end{aligned}
$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

A linear combination of weighted delayed impulses.



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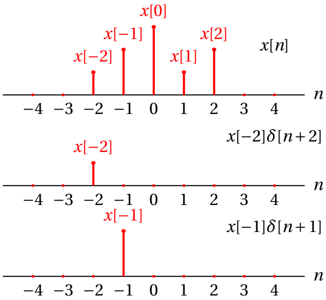

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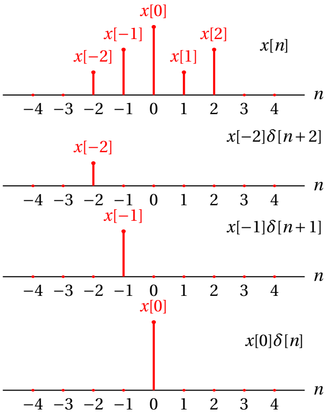

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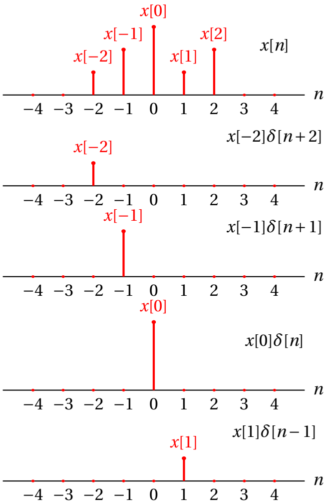

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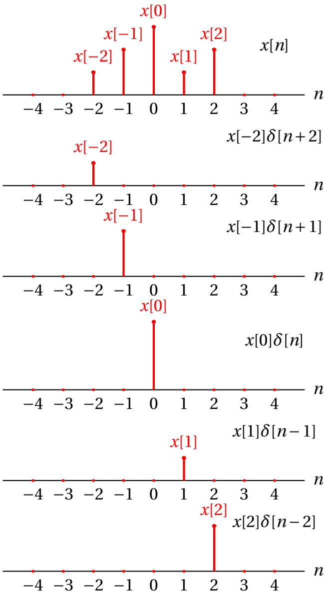
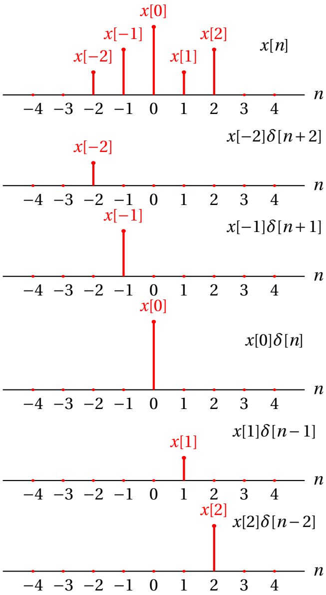
A linear combination of weighted delayed impulses.



x[n]
x[−2]
x[−1]
x[0]
x[1]
x[2]
n
−4 −3 −2 −1 0 1 2 3 4
x[−2]δ[n+2]
x[−1]δ[n+1]
x[0]δ[n]
x[1]δ[n−1]
x[2]δ[n−2]



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A linear combination of weighted delayed impulses.

Convolution Sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \quad \text{input.}$$

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Linear system

Convolution Sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \quad \text{input.}$$

Linear system

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \quad \text{where } h_k[n] \text{ is the output due to the delayed impulse.}$$

$$\delta[n-k] \to h_k[n].$$

Convolution Sum

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$$\delta[n-k] \rightarrow h_k[n].$$

If time invariant

$$h_k[n] = h_0[n-k] \quad \text{where } h_0 \text{ is the response of the system for an impulse at 0.}$$

$$h_0[n] = h[n] \quad \text{define.}$$

Convolution Sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k], \quad \text{input.}$$

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$$h_0[n] = h[n] \quad \text{define.}$$

If LTI

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]} \quad \text{convolution sum.}$$

Convolution Sum: Summary

The convolution of the sequence $x[n]$ and $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \tag{1}$$

which we represent symbolically as

$$y[n] = x[n] * h[n] \tag{2}$$

Example

Computer $y[n] = x[n] * h[n]$ for $x[n]$ and $h[n]$ as shown in Figure 5.

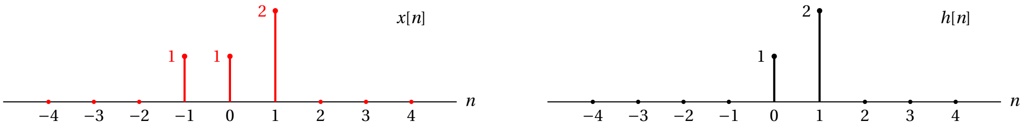
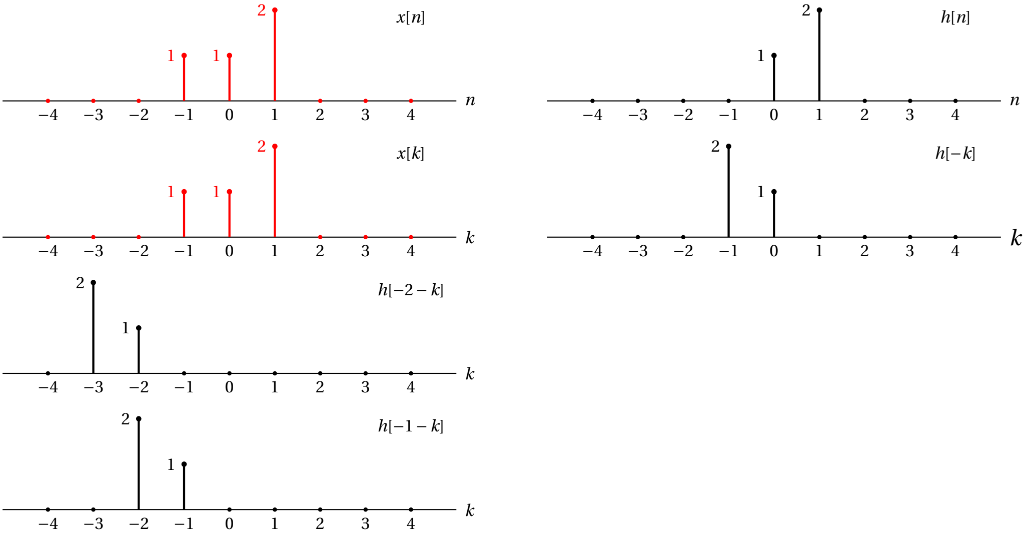


Figure: Computing convolution



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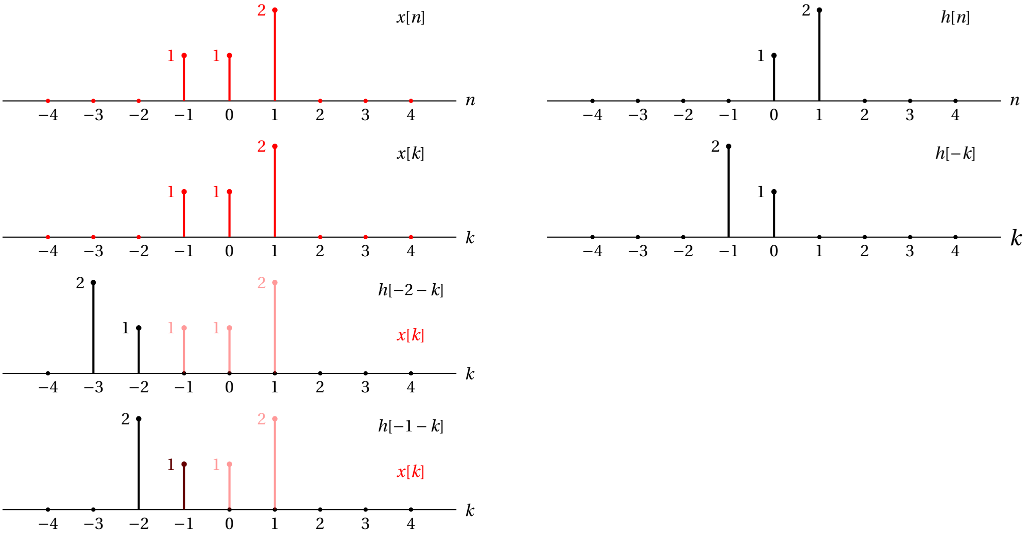

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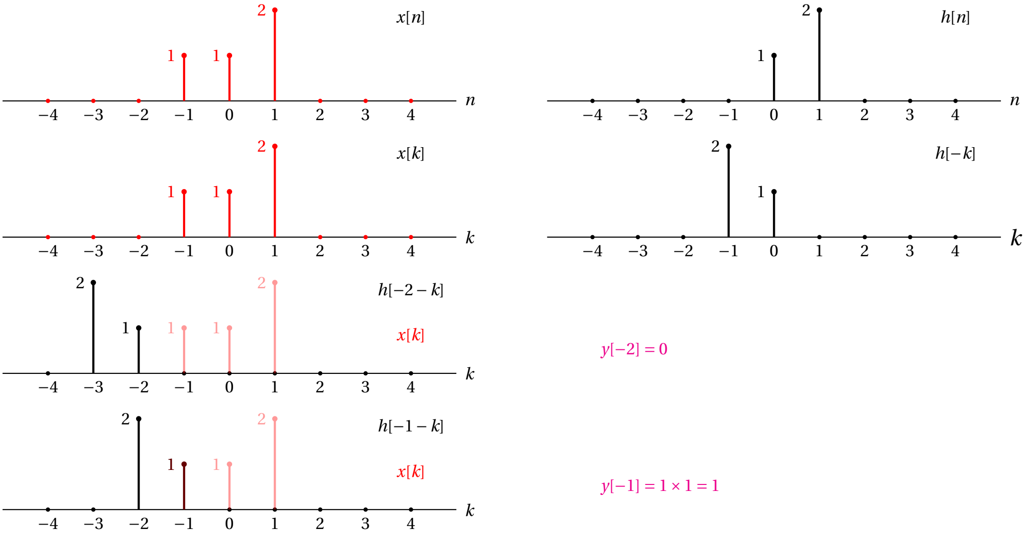

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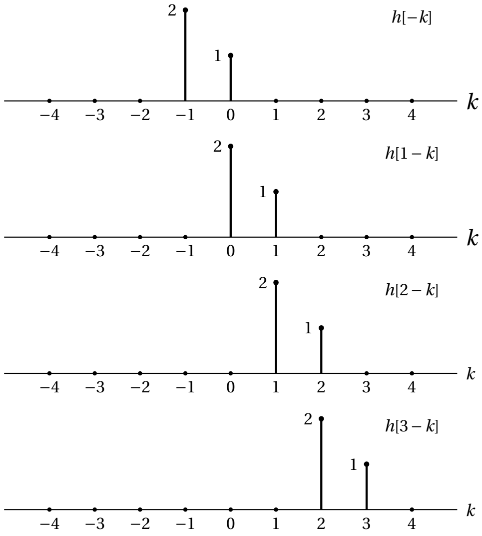

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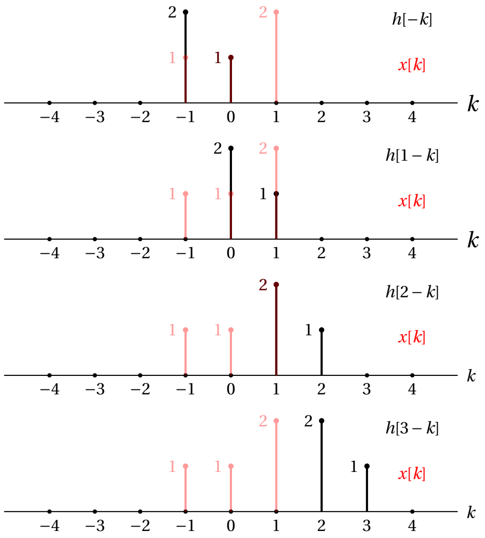

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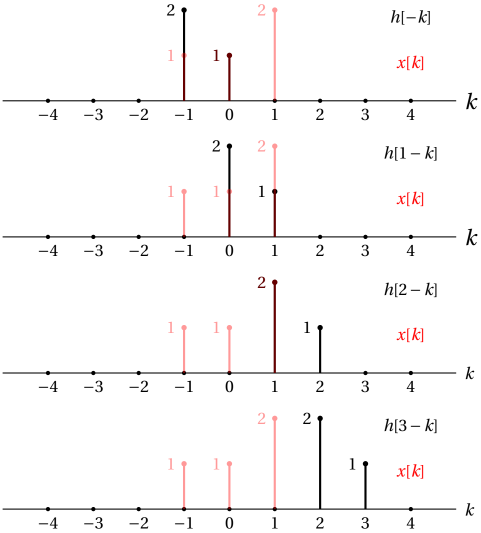


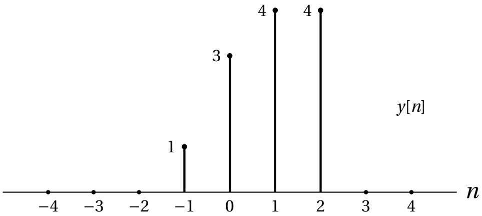
$y[0] = 2 \times 1 + 1 \times 1 = 3$

$y[1] = 2 \times 1 + 1 \times 2 = 4$

$y[2] = 2 \times 2 = 4$

$y[3] = 0$

Figure: Computing convolution



$$y[n] = \begin{cases} 0 & n \leq -2 \\ 1 & n = -1 \\ 3 & n = 0 \\ 4 & n = 1 \\ 4 & n = 2 \\ 0 & n \geq 3. \end{cases}$$

Figure: Computing convolution

Example

Consider and input $x[n]$ and a unit impulse response $h[n]$ given by

$$\begin{aligned} x[n] &= \alpha^n u[n] \\ h[n] &= u[n], \end{aligned} \tag{3}$$

which $0 < \alpha < 1$. Find $y[n]$ and sketch.

Figure: The signals $x[n]$ and $h[n]$ for the example.

Outline

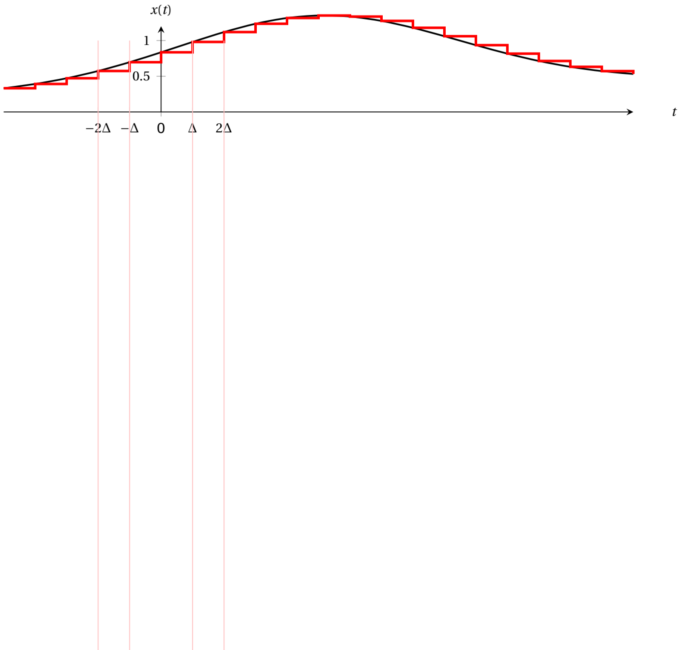
Continuous-Time Systems: The Convolution Integral

1.

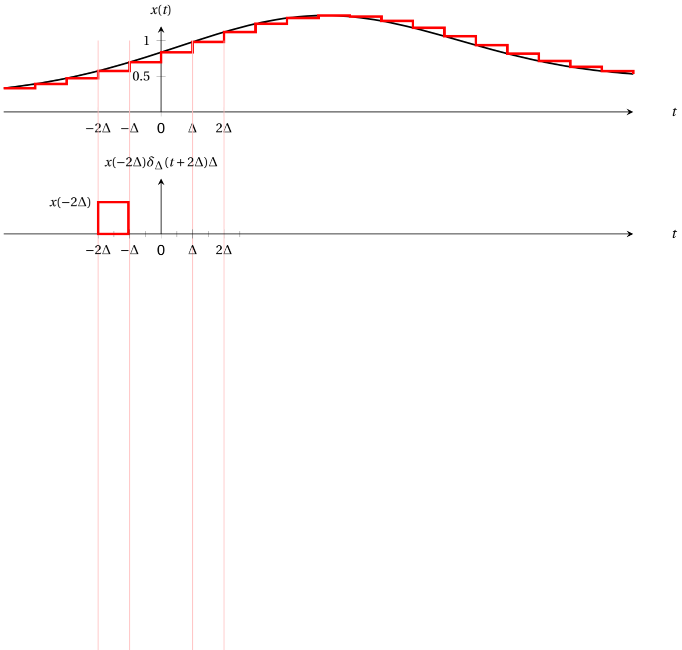
Outline

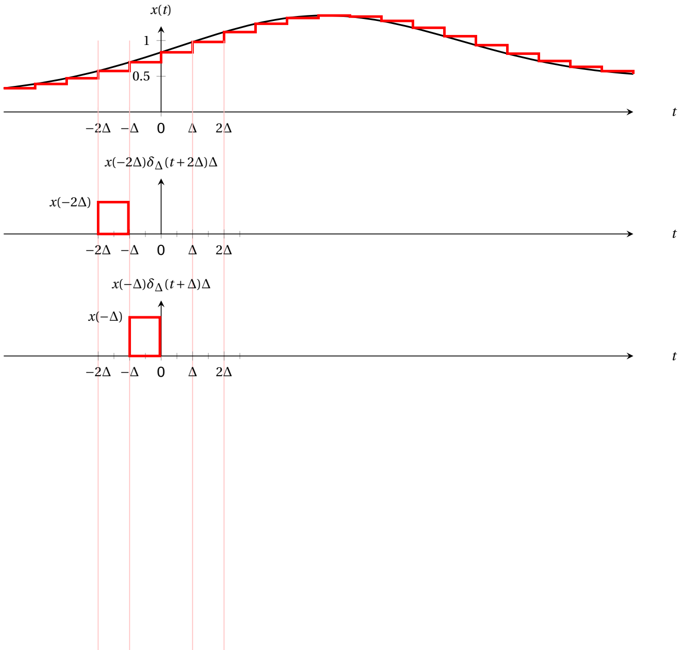
Continuous-Time Systems: The Convolution Integral

1. Similar to what we did in DT, in this section we obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response.
2. In discrete time, the key to developing the convolution sum was the sifting property of the DT unit impulse—i.e., mathematical representation of a signal as a superposition of scaled and shifter unit impulse functions.
3. We begin by considering the staircase approximation $\hat{x}(t)$ of a CT signal $x(t0$.

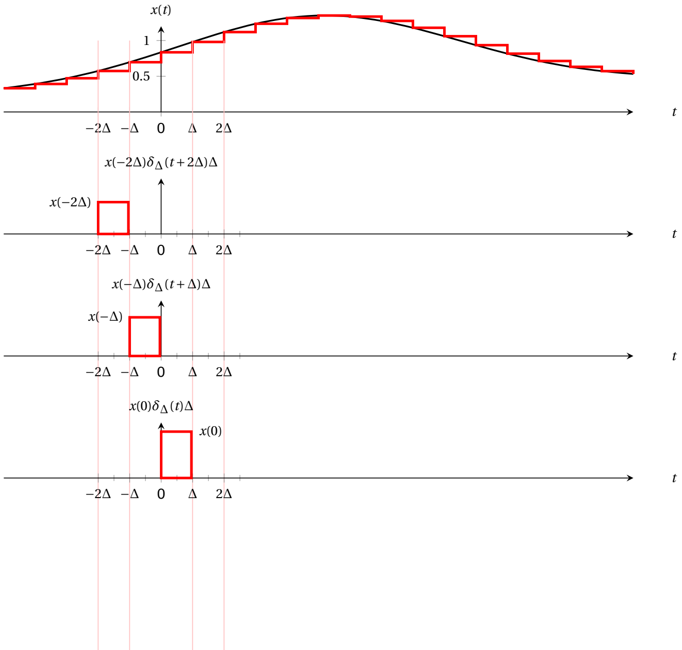


x(t)
1
0.5
t
−2Δ
−Δ
0
Δ
2Δ

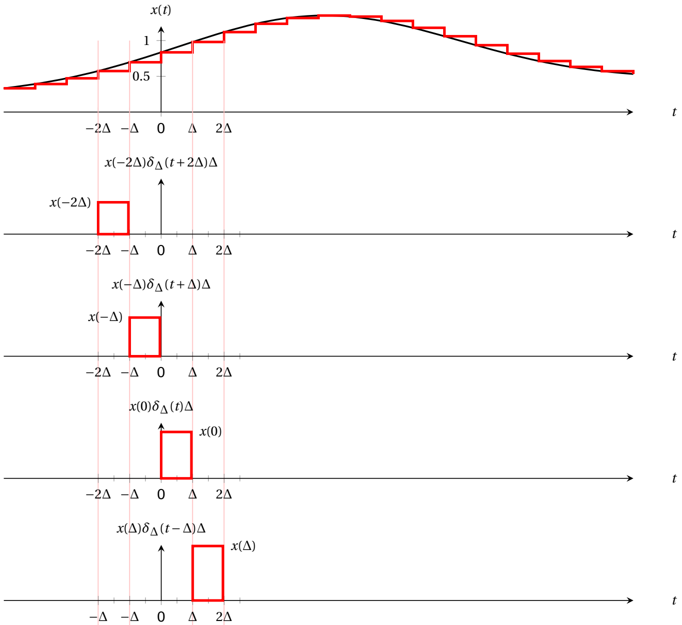

$x(t)$
1
0.5
t
-2Δ $-\Delta$ 0 Δ 2Δ
$x(-2\Delta)\delta_\Delta(t+2\Delta)\Delta$
$x(-2\Delta)$
t
-2Δ $-\Delta$ 0 Δ 2Δ



$x(t)$
1
0.5
t
-2Δ $-\Delta$ 0 Δ 2Δ
$x(-2\Delta)\delta_\Delta(t+2\Delta)\Delta$
$x(-2\Delta)$
t
-2Δ $-\Delta$ 0 Δ 2Δ
$x(-\Delta)\delta_\Delta(t+\Delta)\Delta$
$x(-\Delta)$
t
-2Δ $-\Delta$ 0 Δ 2Δ



$x(t)$
1
0.5
t
-2Δ $-\Delta$ 0 Δ 2Δ
$x(-2\Delta)\delta_\Delta(t+2\Delta)\Delta$
$x(-2\Delta)$
t
-2Δ $-\Delta$ 0 Δ 2Δ
$x(-\Delta)\delta_\Delta(t+\Delta)\Delta$
$x(-\Delta)$
t
-2Δ $-\Delta$ 0 Δ 2Δ
$x(0)\delta_\Delta(t)\Delta$
$x(0)$
t
-2Δ $-\Delta$ 0 Δ 2Δ



x(t)
1
0.5
t
−2Δ −Δ 0 Δ 2Δ
x(−2Δ)δ_Δ(t+2Δ)Δ
x(−2Δ)
t
−2Δ −Δ 0 Δ 2Δ
x(−Δ)δ_Δ(t+Δ)Δ
x(−Δ)
t
−2Δ −Δ 0 Δ 2Δ
x(0)δ_Δ(t)Δ
x(0)
t
−2Δ −Δ 0 Δ 2Δ
x(Δ)δ_Δ(t−Δ)Δ
x(Δ)
t
−Δ −Δ 0 Δ 2Δ

The approximation that we saw can be expressed as a linear combination of delayed impulses. Define

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Since $\Delta\delta_\Delta(t)$ has unit amplitude, we have

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

Here, for any value of t, only one term in the summation on the right hand side is nonzero.

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

As $\Delta \to 0$, the summation approaches an integral. Consequently,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This is known as the sifting property of the continuous time impulse.

Example:

Use the sifting property to express $u(t)$ in terms of $\delta(t)$.

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$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau$$

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

Let's define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t-k\Delta)$.

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta.$$

Since the pulse $\delta_\Delta(t-k\Delta)$ corresponds to a shifted unit impulse as $\Delta \to 0$, the response $\hat{h}_{k\Delta}(t)$ to this input pulse becomes the response to an impulse in the limit.If we let $h1_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ, then

$$y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_{k\Delta}(t)\Delta.$$

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$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

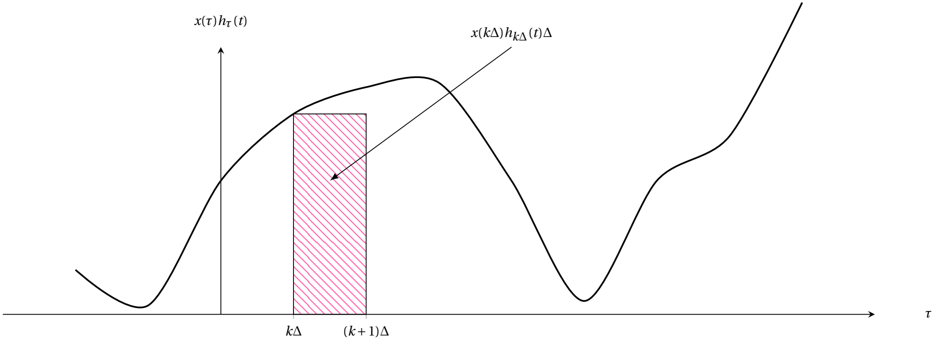


Figure: Graphical illustration

In addition to being linear, the system is time-invariant, the response of the LTI system to the unit impulse $\delta(t-\tau)$

$$h_\tau(t) = h_0(t-\tau).$$

Defining unit impulse response $h(t)$ as

$$h(t) = h_0(t),$$

we have

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

which is referred to as the convolution integral or the superposition integral. This corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse.

$$y(t) = x(t) * h(t).$$

As in discrete time, a continuous-time LTI system is completely characterized by its impulse response—i.e., by its response to a.single elementary signal, the unit impulse $\delta(t)$.

Example: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = e^{-at}u(t), a > 0$$

and

$$h(t) = u(t).$$

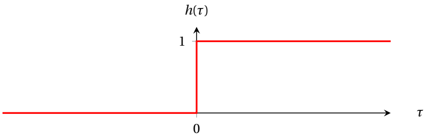
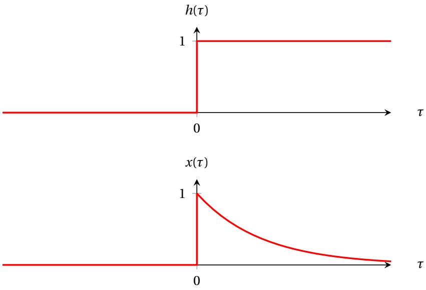


Figure: Calculation of convolution integral for the example



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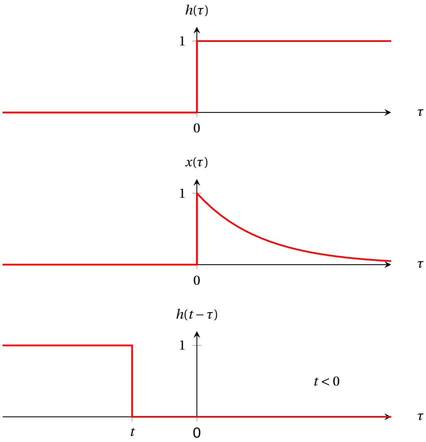

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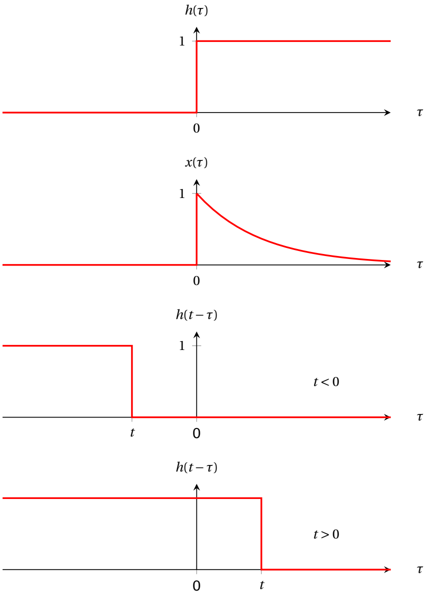

Figure: Calculation of convolution integral for the example

For $t < 0$, the product $x(\tau)$ and $h(t-\tau)$ is zero, consequently $y(t)$ is zero.
For $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau}\,d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t \\ &= \frac{1}{a}(1-e^{-at}) \end{aligned}$$

Thus for all t,

$$y(t) = \frac{1}{a}(1-e^{-at})u(t)$$

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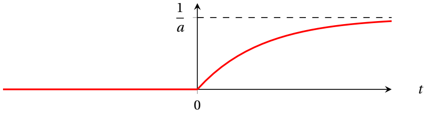


Figure: Response

Example: Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(t) = \begin{cases} t, & 0 < t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$

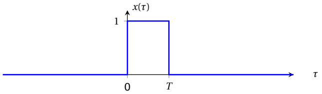
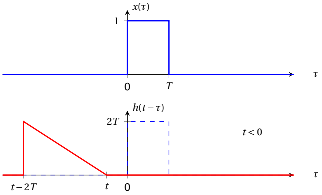


Figure: $x(\tau)$ and $h(t-\tau)$



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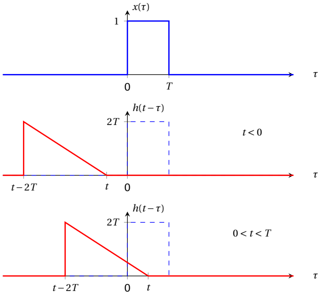

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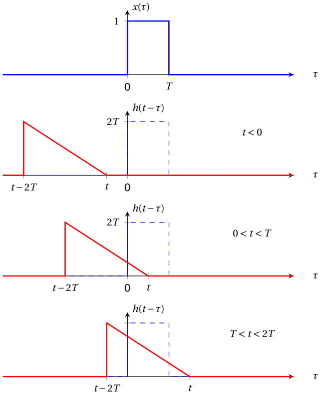

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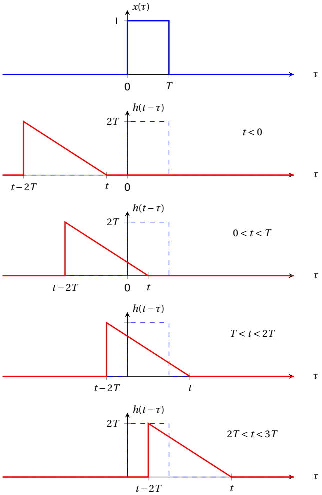

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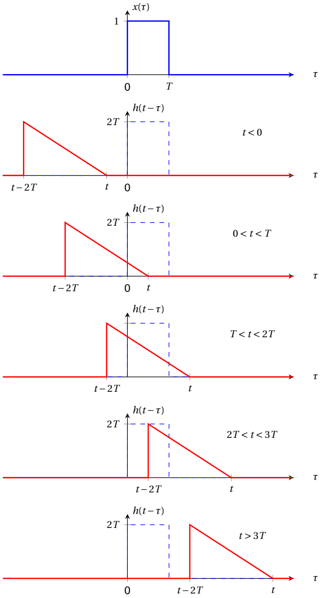

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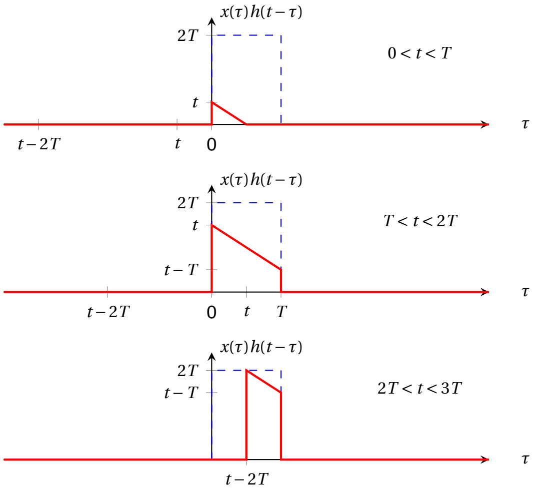


Figure: Product $x(\tau)h(t-\tau)$ for values of t for which this product is not identically zero.

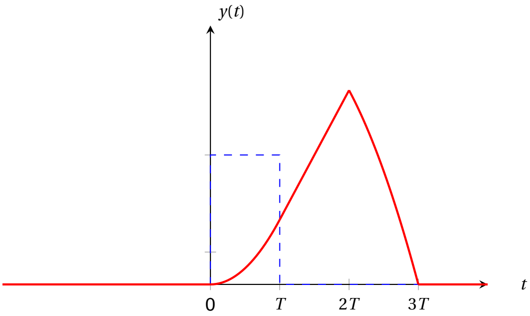


Figure: $y(t) = x(t) * h(t)$

Example: Find $y(t)$, the convolution of the following two signals:

$$x(t) = e^{2t}u(-t),$$

and

$$x(t) = u(t-3).$$

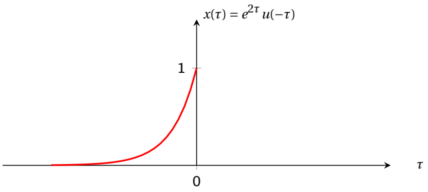
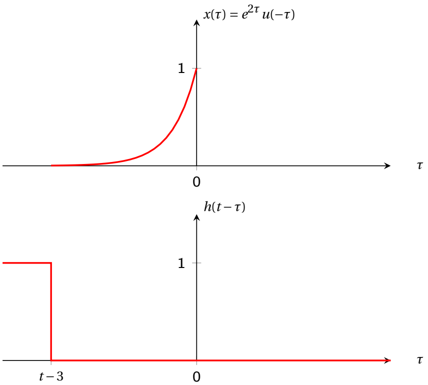


Figure: Convolution considered in the example.



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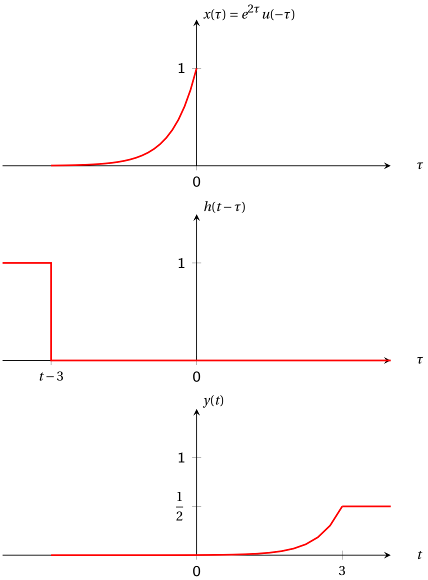

Figure: Convolution considered in the example.

When $t-3 \leq 0$, the product of $x(\tau)$ and $h(t-\tau)$ is nonzero for $-\infty < \tau < t-3$, and the convolving integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2} e^{2(t-3)}.$$

F0r $t-3 \geq 0$, the product of $x(\tau)h(t-\tau)$ is nonzero for $-\infty < \tau < 0$, and the convolving integral becomes

$$y(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2}.$$

Section 4

Properties of Linear Time-Invariant Systems

Recapitulation

1. In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

2. Characteristics of an LTI system are completely determined by its impulse response.

The Commutative Property

DT

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

CT

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Verify the commutative property for DT.

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$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-r]h[r] = h[n] * x[n].$$

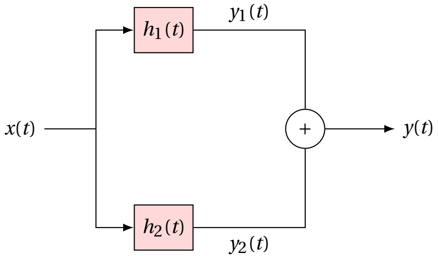
The Distributive Property

Convolution distributes over addition.
DT

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

CT

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t).$$



The Associative Property

DT

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n].$$

CT

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t).$$

As a consequence,

$$y[n] = x[n] * h_1[n] * h_2[n]$$

and

$$y(t) = x(t) * h_1(t) * h_2(t).$$

are unambiguous.
Using the commutative property together with the associative property we can see that the order in which they are cascaded does not matter as far as the overall system impulse response is concerned.

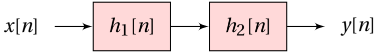
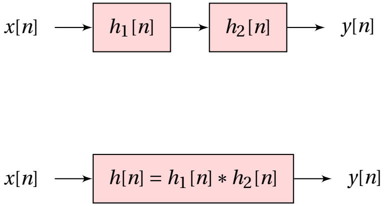


Figure: Associative property.



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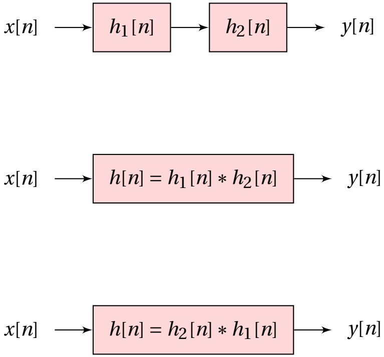

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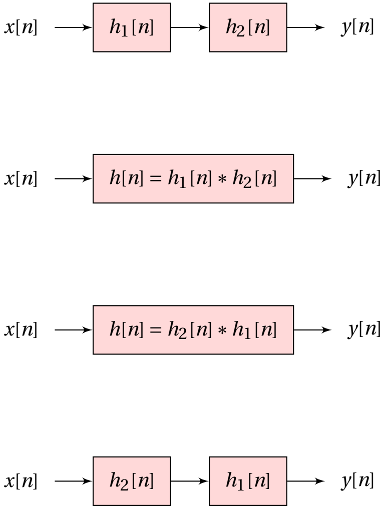

Figure: Associative property.

LTI Systems with and without Memory

1. A system is memoryless if its output at any time depends only on the value of the input at that same time.
2. The only way that this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$.
3. In this case the impulse response has the form

$$h[n] = K\delta[n],$$

   where $K = h[0]$ is a constant.
4. The convolution sum reduces to the relation

$$y[n] = Kx[n]$$

5. If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory.
6. For CT:

$$h(t) = K\delta(t).$$

$$y(t) = Kx(t).$$

Invertibility of LTI Systems

1. An LTI system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.

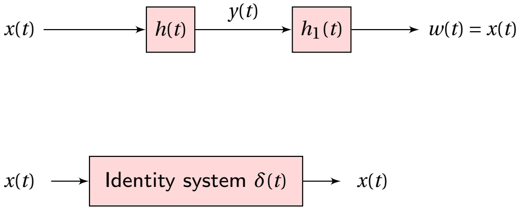

Figure: Inverse of a CT LTI system.

Example

Consider the following relationship of a pure time shift:

$$y(t) = x(t - t_0)$$

Is the corresponding system memoryless? What is the inverse system of the system?

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Delay if $t_0 > 0$, and advance if $t_0 < 0$. E.g., if $t_0 > 0$ then the output at time t equals the input at the earlier time $t - t_0$. If $t_0 = 0$, the system is the identity system and that is memoryless. For any other value of t_0, the system has memory.

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Setting the input equal to $\delta(t)$, the impulse response can be obtained:

$$h(t) = \delta(t - t_0).$$

Therefore,

$$x(t - t_0) = x(t) * \delta(t - t_0).$$

That is, the convolution of a signal with a shifted impulse simply shifts the signal. To recover the input from the output, i.e., to invert the system, all that is required is to shift the output back.

$$h_1(t) = \delta(t + t_0).$$

Then

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t).$$

Example

Consider the LTI system with impulse response

$$h[n] = u[n].$$

Determine $y[n]$. Find the inverse system.

Causality for LTI Systems

1. The output of a causal system depends only on the present and past values of the input to the system.
2. For a DT LTI system, $y[n]$ must not depend on $x[k]$ for $k > n$.
3. For this to be true, all of the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero.
4. This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0.$$

5. The impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality.
6. More generally, causality for a linear system is equivalent to the condition of initial rest; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time.
7. The equivalence of causality and the condition of initial rest applies only to linear systems.

Causality for LTI Systems

1. A continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0.$$

2. Causality of an LTI system is equivalent to its impulse response being a causal signal.

Stability for LTI Systems

A system is stable if every bounded input produces a bounded output. Consider an input $x[n]$ that is bounded in magnitude:

$$
\begin{aligned}
|x[n]| &< B \quad \text{for all } n. \\
|y[n]| &= \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \\
|y[n]| &\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \\
|y[n]| &\leq B \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n
\end{aligned}
$$

$$
\sum_{k=-\infty}^{\infty} |h[k]| < \infty.
$$

If the impulse response is absolutely summable, then $y[n]$ is bounded in magnitude, and hence, the system is stable.

Stability for LTI Systems

In CT a system is stable if the impulse response is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

Examples

Determine whether the following systems are stable:

1. Pure time shift in DT.
2. Pure time shift in CT.
3. Accumulator in DT.
4. CT counterpart of the accumulator.

Pure time shift in DT:

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n-n_0]| = 1$$

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CT counterpart of the accumulator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

The impulse response of the integrator can be found by letting $x(t) = \delta(t)$:

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t).$$

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$$\int_{-\infty}^{\infty} |u(\tau)d\tau| = \int_{0}^{\infty} d\tau| = \infty$$

The Unit Step Response of an LTI System

There is another signal that is also used in describing the behavior of LTI systems: the unit step response, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$.

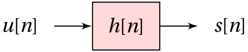

Figure: Unit step response.

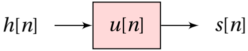

Figure: Unit step response.

$$s[n] = u[n] * h[n]$$

Commutative property:

$$s[n] = h[n] * u[n]$$

$s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit

$u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

$h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1].$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response. Similarly, in CT, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

the unit impulse response is the first derivative of the unit step response, or

$$h(t) = \frac{ds(t)}{d(t)} = s'(t).$$

Zero-Input Response

For a linear system (time-invariant or not), if we put nothing into it, we get nothing out of it.

$$
\begin{aligned}
x(t) &= 0 \quad \text{for all } t, \quad \text{then} \\
y(t) &= 0 \quad \text{for all } t, \\
x[n] &= 0 \quad \text{for all } n, \quad \text{then} \\
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\end{aligned}
$$

"Proof": If the system is linear and

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$$ax(t) \rightarrow ay(t).$$

Select the sale factor $a = 0$.
Not all systems are like this, e.g., even if a battery is not connected to anything, the output is 1.5 V.

Implications for Causality

The system cannot anticipate the input.
I.e., If

$$x_1(t) = x_2(t), \quad \text{for } t < t_0,$$

then

$$y_1(t) = y_2(t), \quad \text{for } t < t_0,$$

Same for DT.

Implications for Causality for a Linear System

For linear systems, if

$$x(t) = 0, \quad \text{for } t < t_0,$$

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Initial rest: The system does not respond until an input is given.
For a linear system to be causal it must have the property of initial rest.
Why?

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For a linear system to be causal it must have the property of initial rest.
Why? For linear systems zero in → zero out.

Causality for Linear Time Invariant Systems

For LTI systems,

Causality ⇔

$$h(t) = 0, \quad t < 0$$
$$h[n] = 0, \quad n < 0$$

"Proof": ⇒: Why does causality imply the above?

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$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)$$
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Only values of $x[n]$ up until n are used to compute $y[n]$ (causal). Similar in CT case.

Example: Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

The accumulator is an LTI system. Also, we saw that its impulse response is

$$h[n] = u[n].$$

1. Does the accumulator have memory?
2. Is the accumulator causal?
3. Is accumulator stable in the BIBO sense?
4. If invertible, what is the inverse?

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Inverse $h[n] = u[n]$, $h^{-1}[n] =?$:
Accumulator can be expressed as a recursive difference equation as

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n-1} x[k] + x[n] \\ &= y[n-1] + x[n]. \end{aligned}$$

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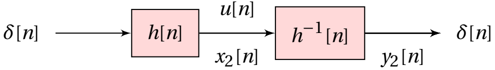

Figure:

We know that

$$u[n] - u[n-1] = \delta[n].$$

So

$$x_2[n] - x_2[n-1] = y_2[n].$$

$$\delta[n] - \delta[n-1] = h^{-1}[n].$$

Inverse of the accumulator:

1. Does it have memory? Yes.
2. Is the sysetm causal? Yes.
3. Is the system stable in the BIBO sense? Yes.

Example

$$y[n] - ay[n-1] = x[n]$$

under the assumption of initial rest ⇒ LTI. Memory? Causal? Stable?

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under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

$$x[n] = \delta[n]$$
$$h[n] = a^n u[n] \quad \text{(Why?)}$$

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under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

$$x[n] = \delta[n]$$
$$h[n] = a^n u[n] \quad \text{(Why?)}$$

1. Memory? Yes.
2. Causal? Yes.
3. Stable? If $|a| < 1$, yes.

Example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

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under the assumption of initial rest ⇒ LTI. Memory? Causal? Stable?

$$h(t) = e^{-2t}u(t) \quad \text{(Why?)}$$

1. Memory? Yes.
2. Causal? Yes.
3. Stable? Yes.

Impulses

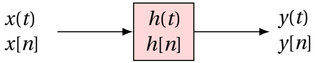


Figure:

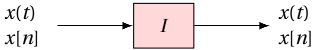


Figure:

Impulses

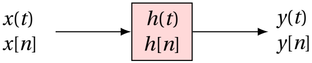


Figure:

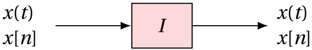
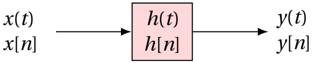


Figure:

$$h(t) = \delta(t)$$

$$h[n] = \delta[n]$$

Impulses



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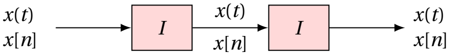

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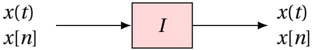

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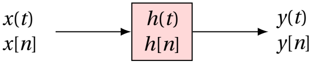

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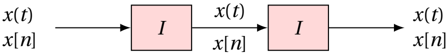

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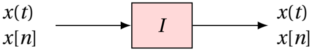
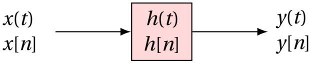


Figure:

$$\delta(t) * \delta(t) = \delta(t)$$
$$\delta[n] * \delta[n] = \delta[n]$$

$$h(t) = \delta(t)$$
$$h[n] = \delta[n]$$

Impulses



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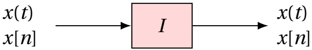

Figure:

$$h(t) = \delta(t)$$
$$h[n] = \delta[n]$$

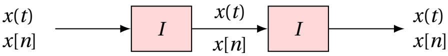

Figure:

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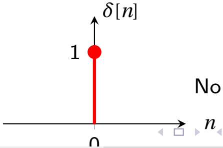
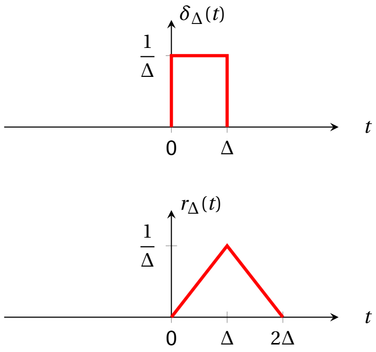


No problem in DT.



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$$
$$\delta(t) = \lim_{\Delta \to 0} r_\Delta(t)?$$

Figure:

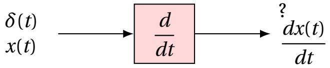


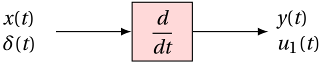
Figure:

Operational Definition

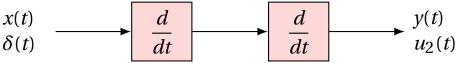
We use operational definitions through convolution to handle derivatives and integrals of impulse, which are badly behaved functions. This leads to a set of singularity functions. Impulse and step are examples of these.

$$x(t) * \delta(t) = x(t)$$

$$\frac{d}{dt}[\delta(t)]$$



$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$



$$u_2(t) = u_1(t) * u_1(t)$$

$$x(t) * u_2(t) = \frac{d^2 x(t)}{dt^2}$$

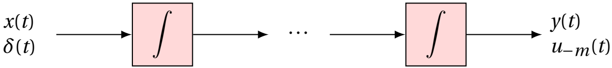
$$u_k(t) = u_1(t) * u_1(t) * \cdots k \text{ times}$$

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

$$u_0(t) = \delta(t)$$

$$u_{-1}(t) = \text{unit step}$$
$$u_{-2}(t) = \text{unit ramp}$$

$$u_0(t) = \delta(t)$$
$$u_{-1}(t) = u(t)$$



$$u_k(t) * u_l(t) = u_{k+l}(t).$$

$$x(t) * u_{-m}(t) = m^{\text{th}} \text{ running integral}$$

$u_k(t)$ defined by

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

Section 5

Linear, Constant-Coefficient Differential and Difference Equations

Linear, Constant-Coefficient Differential and Difference Equations

- An important class of CT systems is that for which the input and output are related through a linear constant-coefficient differential equation.
- These arise in the description of a wide variety of systems and physical phenomena. E.g., the response of the RC circuit, the motion of a vehicle subject to acceleration inputs and frictional forces.
- Correspondingly, an important class of DT systems is that for which the input and output are related through a linear constant-coefficient difference equation.
- These are used to describe the sequential behavior of many different processes. E.g., accumulation of savings in a bank account, a digital simulation of a continuous-time system, DT designed to perform particular operations on the input signal such as a system that calculates the difference between successive input values, of computes the average value of the input over an interval.

Outline

Linear, Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{dx(t)}{dt^k} \tag{4}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \quad \text{Homogeneous equation.} \tag{5}$$

Given $x(t)$, if $y_p(t)$ satisfies 4, so does $y_p(t) + y_h(t)$ where $y_h(t)$ satisfies 5.

$$y_p(t) \triangleq \text{particular solution}$$
$$y_h(t) \triangleq \text{homogeneous solution}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0.$$

Guess a solution of the form

$$y_h(t) = Ae^{st}, \quad \text{a complex exponential}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0.$$

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$$\sum_{k=0}^{N} a_k s^k e^{st} = 0.$$

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$$\sum_{k=0}^{N} a_k s^k = 0, \quad N \text{ roots } s_i, i = 1, \ldots, N$$

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$$\sum_{k=0}^{N} a_k s^k = 0, \quad N \text{ roots } s_i, i = 1, \ldots, N$$

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_N e^{s_N t}$$

Coefficients $A_1, A_2, \ldots, A_N$ are undetermined. We

$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1} y(t)}{dt^{N-1}} \quad \text{at } t = t_0.$$

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$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1} y(t)}{dt^{N-1}} \quad \text{at } t = t_0.$$

Linear system $\Longleftrightarrow$ auxiliary conditions = 0
Linear system $\Rightarrow$ zero in, zero out.
Causal and LTI $\Longleftrightarrow$ initial rest
If $x(t) = 0, \ t < t_0$ then
$y(t) = 0, \ t < t_0$ then

Example: First-Order Differential Equation

$$\frac{dy(t)}{dt} + ay(t) = x(t).$$

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Homogeneous equation:

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Now guess a solution

$$y_h(t) = Ae^{st}.$$

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$$Ase^{st} + aAe^{st} = 0$$
$$s + a = 0.$$

Specific input

$$x(t) = ku(t).$$

A solution is

$$y_1(t) = \frac{k}{a}\left[1 - e^{-at}\right] u(t)$$

Family of solutions

$$y(t) = y_1(t) + y_h(t).$$

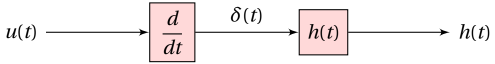
$$y_1(t) = \frac{k}{a}\left[1 - e^{-at}\right] u(t) + Ae^{-at}.$$

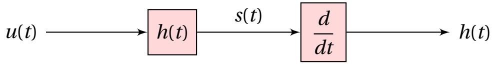
Causal, LTI $\Longleftrightarrow$ initial rest

$$ku(t) \rightarrow \frac{k}{a}\left[1 - e^{-at}\right] u(t)$$

Obtain the impulse response of the above system.

Obtain the impulse response of the above system. Differentiating the step response, we get the impulse response.





$$s(t) = \frac{1}{a}\left[1 - e^{-at}\right] u(t).$$

$$\begin{aligned}
h(t) &= \frac{d}{dt} s(t) = \frac{d}{dt}\frac{1}{a}\left[1 - e^{-at}\right] u(t), \\
&= u(t)\frac{d}{dt}\frac{k}{a}\left[1 - e^{-at}\right] + \frac{1}{a}\left[1 - e^{-at}\right]\frac{d}{dt}u(t) \\
&= e^{-at}u(t) + \frac{1}{a}\left[1 - e^{-at}\right]\delta(t) \\
&= e^{-at}u(t)
\end{aligned}$$

Outline

Linear, Constant-Coefficient Difference Equations

Consider the N^{th}-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{6}$$

$$\sum_{k=0}^{N} a_k y[n-k] = 0: \quad \text{homogeneous equation.} \tag{7}$$

If $y_p[n]$ satisfies 6 so does $y_p[n] + y_h[n]$ where $y_h[n]$ satisfies 7.

$$y_p[n] \triangleq \text{particular solution}$$
$$y_h[n] \triangleq \text{homogeneous solution}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k y[n-k] = 0.$$

"Guess" a solution of the form

$$y_h[n] = Az^n.$$

$$\sum_{k=0}^{N} a_k A z^n z^{-k} = 0.$$

$$\sum_{k=0}^{N} a_k z^{-k} = 0. \quad N \text{ roots } , z_1, z_2, \ldots, z_N.$$

$$y_h[n] = A_1 z_1^n + A_2 z_2^n + \cdots + A_N z_N^n.$$

$$y[n] = A_1 z_1^n + A_2 z_2^n + \cdots + A_N z_N^n + y_p[n]$$

The undetermined constants A_1 to A_N are to be found using the N auxiliary conditions, $y[n_0]$, $y[n_0 - 1]$, $y[n_0 - N + 1]$.

Linear system $\iff$ auxiliary conditions 0

Causal, LTI $\iff$ initial rest

If $x[n] = 0,\ n < n_0$ then

$y[n] = 0,\ n < n_0$ then

Explicit Solution to Difference Equations

Assume causality.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

Explicit Solution to Difference Equations

Assume causality.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

This is said to be a recursive solution to linear constant-coefficient difference equations. To get the computation of $y[n_0]$ started we need the initial conditions or boundary conditions $y[n_0-1]$, $y[n_0-N]$. Then we compute $y[n_0+1]$ and so on.

Example: First-Order Difference Equation

$$y[n] - ay[n-1] = x[n].$$

Causal, LTI $\Longleftrightarrow$ initial rest

$$y[n] = x[n] + ay[n-1].$$

Set $x[n] = \delta[n]$.

$$h[n] = \delta[n] + ah[n-1].$$

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Set $x[n] = \delta[n]$.

$$h[n] = \delta[n] + ah[n-1].$$

$$h[n] = 0, n < 0 \quad \text{initial rest.}$$

$$h[0] = 1$$
$$h[1] = a$$
$$h[2] = a^2 \vdots$$

$$h[n] = a^n u[n].$$

So if causal and LTI

$$\delta[n] \to a^n u[n].$$

Family of solutions

$$\delta[n] \to a^n u[n] + y_h[n].$$

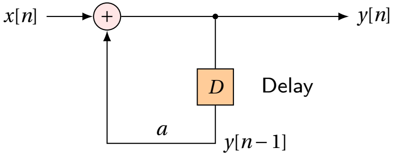
$$y_h[n] = Az^n.$$

$$Az^n - aAz^{n-1} = 0.$$

$$a - z^{-1} = 0 \Rightarrow z = a.$$

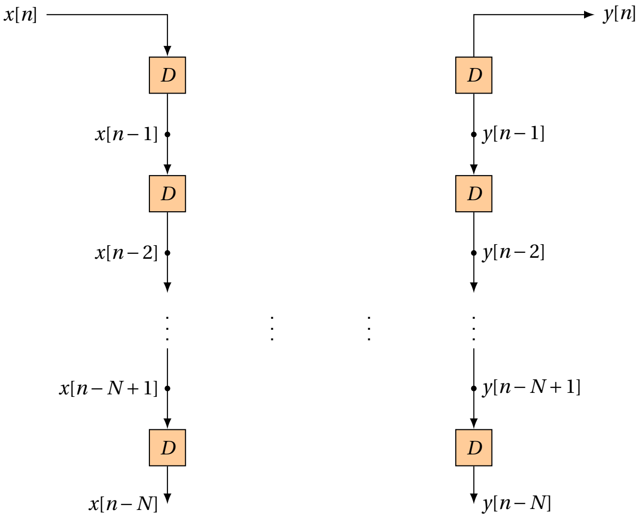
$$y_h[n] = Aa^n.$$

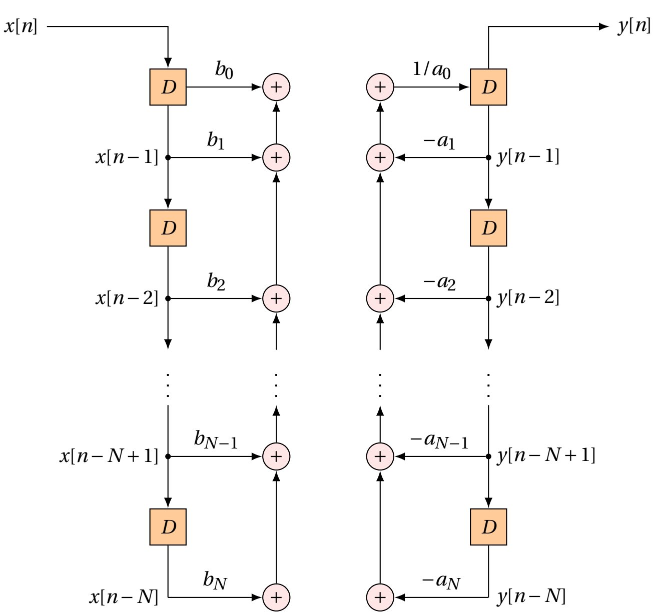
$$\delta[n] \to a^n u[n] + Aa^n.$$



$x[n]$
+
$y[n]$
D
Delay
a
$y[n-1]$

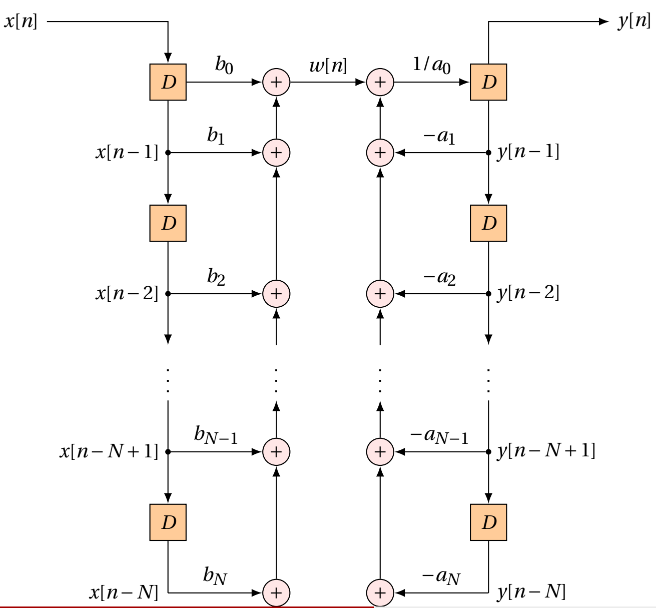
Outline


x[n]
y[n]
D
D
x[n−1]
y[n−1]
D
D
x[n−2]
y[n−2]
x[n−N+1]
y[n−N+1]
D
D
x[n−N]
y[n−N]



Direct form I implementation

This is a cascade of two linear systems. We can interchange the order of the two segments.



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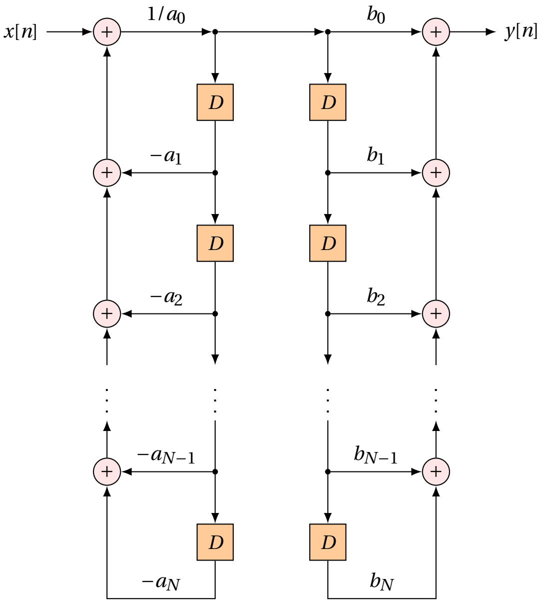
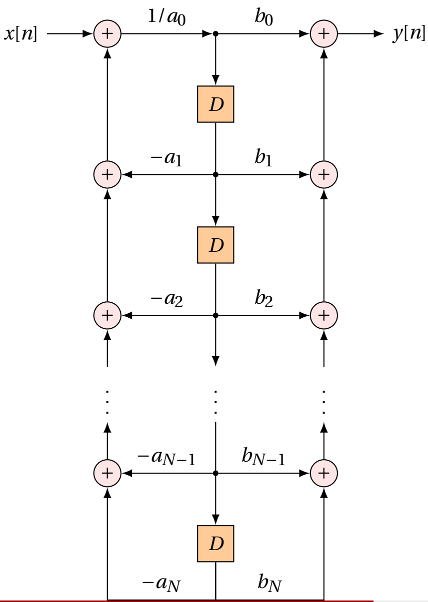

x[n]
1/a0
b0
y[n]
D
D
−a1
b1
D
D
−a2
b2
−aN−1
bN−1
D
D
−aN
bN



Result of combining the two chains of delays.
Direct form II implementation

Section 6

Revisiting Fourier Series

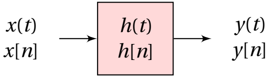
Convolution

1. In representing and analyzing LTI systems, our approach has been to decompose the system inputs into a liner combination of basic signals and exploit the fact that for a linear system, the response is the same linear combination of the responses to the basic inputs.
2. The convolution sum and the convolution integral req out of the particular choice of the basic signals, delayed unit impulses.
3. This choice has the advantage that for systems that are time invariant in addition to being linear, once the response to an impulse at one time position is known, then the response id know at all time positions.

Complex Exponentials with Unity Magnitude as Basic Signals

1. When we select complex exponential with unity magnitude as the basic signals, the decomposition of this form of a periodic signal is the Fourier series.
2. For aperiodic signals, it becomes the Fourier transform.
3. In latter lectures, we will generalize this representation to Laplace tangram for continuous-time signals and z-transform for discrete-time signals.

Consider a linear system



If

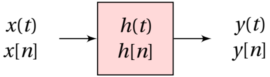
$$x(t) = a_1\phi_1(t) + a_2\phi_2(t) + \cdots$$

and

$$\phi_k(t) \longrightarrow \psi_k(t), \quad \text{(output due to } \phi_k(t)\text{)}$$

then

Consider a linear system



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then

$$y(t) = a_1\psi_1(t) + a_2\psi_2(t) + \cdots$$

Identical for DT. So

Consider a linear system

$$\begin{matrix} x(t) \\ x[n] \end{matrix} \longrightarrow \boxed{\begin{matrix} h(t) \\ h[n] \end{matrix}} \longrightarrow \begin{matrix} y(t) \\ y[n] \end{matrix}$$

If

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$$\begin{aligned} \text{If} \quad & x(t) = a_1\phi_1 + a_2\phi_2 + \cdots, \\ \text{then} \quad & y(t) = a_1\psi_1 + a_2\psi_2 + \cdots. \end{aligned}$$

Consider a linear system

$$x(t),\ x[n] \longrightarrow \boxed{\begin{matrix} h(t) \\ h[n] \end{matrix}} \longrightarrow y(t),\ y[n]$$

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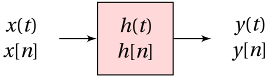
Choose $\phi_k(t)$ or $\phi_k[n]$ so that

1. A broad class of signals can be constructed as a linear combination of ϕ_ks
2. Response to ϕ_ks easy to compute.

Consider a linear system



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Choice of signals $\delta(t-k\Delta)$ and $\delta[n-k]$ lead to the convolution integral and convolution sum.

$$\begin{aligned} \text{CT} \quad & \phi_k(t) = \delta(t-k\Delta) \\ & \psi_k(t) = h(t-k\Delta) \quad \Rightarrow \text{ convolution integral} \\ \text{DT} \quad & \phi_k[n] = \delta[n-k] \\ & \psi_k[n] = h[n-k] \quad \Rightarrow \text{ convolution sum} \end{aligned}$$

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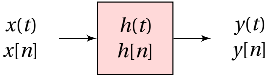

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Here, we choose complex exponentials as the set of basic signals.

$$\begin{aligned} \phi_k(t) &= e^{s_k t}, \quad s_k \text{ complex} \\ \phi_k[n] &= z_k^n, \quad z_k \text{ complex} \end{aligned}$$

Fourier Analysis

CT	$s_k = j\omega k$	purely imaginary	$\phi_k(t)$	$= e^{j\omega_k t}$
DT	$\lvert z_k \rvert = 1$		$\phi_k[n]$	$= e^{j\omega_k n}$

s_k complex $\Rightarrow$ Laplace transfrom

z_k complex $\Rightarrow$ z-transform

Eigenfunction Property

Consider $\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(\omega_k) e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

"Proof":

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"Proof":

$$e^{j\omega_k t} \longrightarrow \int_{-\infty}^{\infty} h(\tau) e^{j\omega_k (t-\tau)} d\tau$$

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$$e^{j\omega_k t} \longrightarrow \underset{\substack{\uparrow \\ \text{eigenfunction}}}{e^{j\omega_k t}} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_k \tau} d\tau}_{\text{eigenvalue}}$$

CT Fourier Series

$$x(t) = x(t+T), \quad T = \text{ period}, \quad \omega_0 = \frac{2\pi}{T}$$

Consider the periodic exponential signal with the same fundamental period:

$$e^{j\omega_0 t}, \quad T = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t}, \quad T = \frac{2\pi}{k\omega_0}, \quad \text{harmonically related}$$

$$\text{Fourier series:} \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

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2. What are the signal that can be expressed in this form? [already answered, see

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$$\begin{array}{cccc} \int_T e^{jm\omega_0 t} dt & = \int_T \cos m\omega_0 t dt & + j \int_T \sin m\omega_0 t dt \\ m \neq 0 & 0 & 0 \\ m = 0 & T & 0 \end{array}$$

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Start with the Fourier series equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Multiplying both sides by $e^{-jn\omega_0 t}$

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

Integrating over one period

$$\int_T x(t)e^{-jn\omega_0 t}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

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$$\int_T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_T \left[e^{j(k-n)\omega_0 t} \right] dt$$

Integrating over one period

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$$a_k = \frac{1}{T} \int_T x(t)e^{-jk\omega_0 t} dt$$