EN1060 Signals and Systems: Signals

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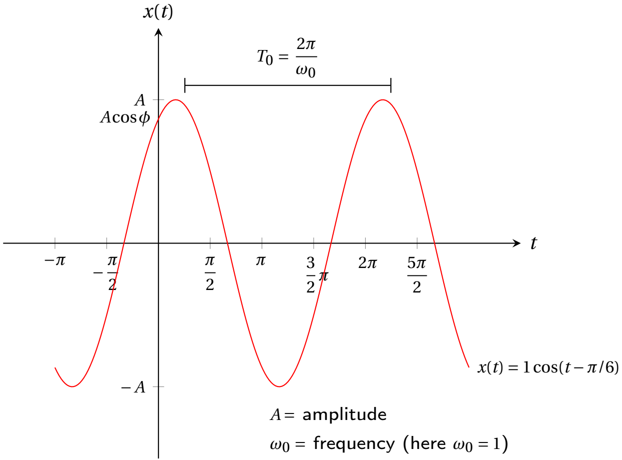
August 23, 2018

Section 1

Signals

$$x(t) = A\cos(\omega_0 t + \phi). \qquad (1)$$



$A =$ amplitude

$\omega_0 =$ frequency (here $\omega_0 = 1$)

$\phi =$ phase (here $\phi = -\pi/6$)

Sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t+T) \tag{2}$$

for all values of t. Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t + \phi)$

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0 (t+T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

$T = \dfrac{2\pi m}{\omega_0} \Rightarrow$ fundamental period $T_0 = \dfrac{2\pi}{\omega_0}$.

A time-shift in a CT sinusoid is equivalent to a phase shift.

E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

A signal $x(t)$ or $x[n]$ is referred to as an *even* signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

A is referred to as an *odd* if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

An odd signal must be) at $t = 0$ or $n = 0$.
A signal can be broken into a sum of two signals, one of which is even adn one fo which is odd.
Even part of $x(t)$ is

$$\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

Odd part of $x(t)$ is

$$\mathfrak{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Phase of a Sinusoidal: $\phi = 0$

Phase of a Sinusoidal: $\phi = -\pi/2$

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = 0$

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = -\pi/2$

Q

Does a phase change always correspond to a time shift in discrete-time signals?

Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

CT Real Exponentials

$$
\begin{aligned}
x(t) &= Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers} \\
&= Ce^{at_0} e^{at}.
\end{aligned}
$$

DT Real Exponentials

$x[n] = Ce^{\beta n} = C\alpha^n, \quad C$ and α are real numbers

$x(t) = Ce^{at}$ $\quad$ C and a are complex numbers.

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

$$\begin{aligned}
x(t) &= |C|e^{j\theta}\, e^{(r+j\omega_0)t} \\
&= |C|e^{rt}\, e^{j(\omega_0 t+\theta)} \\
&= |C|e^{rt}\, [\cos(\omega_0 t+\theta) + j \sin(\omega_0 t+\theta)]
\end{aligned}$$

DT Complex Exponentials

$$x[n] = C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \tag{4}$$

$$C = |C|e^{j\theta} \tag{5}$$

$$\alpha = |\alpha|e^{j\omega_0} \tag{6}$$

$$x[n] = |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \tag{7}$$

$$= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta) \tag{8}$$

$$\tag{9}$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

DT Complex Exponentials Plot

Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

- For the CT counterpart $e^{j\omega_0 t}$,
  1. The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal.
  2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.
- In DT, as
$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$
  the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0 .
- Although in CT $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0. In DT, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 + 2\pi$, $\omega_0 + 4\pi$, and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0.
- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{10}$$

Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{11}$$

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \tag{12}$$

The unit step sequence is the running sum of the unit impulse.

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{13}$$

The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{14}$$

Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{15}$$

Continuous-Time Unit Impulse Function $\delta(t)$

$$\delta(t) = \frac{du(t)}{dt}. \tag{16}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau. \tag{17}$$

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{18}$$

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N \to \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{19}$$

Energy II

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt. \tag{20}$$

Total energy in a DT signal:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{21}$$

With these definitions, we can identify three important classes of signals:

1. Energy signals: Signals with finite total energy $E_\infty < \infty$. These have zero average power.
2. Power signals: Signals with finite average power $0 < P_\infty < \infty$. As $P_\infty > 0$, $E_\infty = \infty$.
3. Signals with neither E_∞ nor P_∞ are finite.