EN1060 Signals and Systems: The z Transform

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Section 1

Introduction

Introduction

- We developed the Laplace transform as a generalization of the continuous-time Fourier transform.
- In this lecture, we introduce the corresponding generalization of the discrete-time Fourier transform.
- The resulting transform is referred to as the z-transform.

z-Transform Motivation

- The discrete-time Fourier transform developed out of choosing complex exponentials as basic building blocks for signals because they are eigenfunctions of discrete-time LTI systems.
- A more general class of eigenfunctions consists of signals of the form z^n, where z is a general complex number. A representation of discrete-time signals with these more general exponentials leads to the z-transform.

Relationship between the z-Transform and the Discrete-Time Fourier Transform

- We saw that the Laplace transform is a generalization of the continuous-time Fourier transform.
- A close relationship exists between the z-transform and the discrete-time Fourier transform.
- For $z = e^{j\omega}$ or, equivalently, for the magnitude of z equal to unity, the z-transform reduces to the Fourier transform.
- More generally, the z-transform can be viewed as the Fourier transform of an exponentially weighted sequence.
- Because of this, the z-transform may converge for a given sequence even if the Fourier transform does not: the z-transform offers the possibility of transform analysis for a broader class of signals and systems.

The Region of Convergence (ROC)

- The z-transform of a signal too has associated with it both a range of values of z, referred to as the region of convergence (ROC), for which this expression is valid.
- Two different sequences can have z-transforms with identical algebraic expressions such that their z-transforms differ only in the ROC.
- Consequently, the ROC is an important part of the specification of the z-transform.

z-Plane

- z-transforms of the form of a ratio of polynomials in z^{-1} are described by poles and zeros in the complex plane, referred to as the z-plane.
- The circle of radius 1, concentric with the origin in the z-plane, is referred to as the unit circle.
- Since this circle corresponds to the magnitude of z equal to unity, it is the contour in the z-plane on which the z-transform reduces to the Fourier transform.
- In contrast, for continuous time it is the imaginary axis in the s-plane on which the Laplace transform reduces to the Fourier transform.
- If the sequence is known to be right-sided, for example, then the ROC must be the portion of the z-plane outside the circle bounded by the outermost pole.

Section 2

The z-Transform

Recall: Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{-\infty}^{+\infty} x[n]e^{-j\omega n}$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega n} & \rightarrow & H(e^{j\omega})e^{j\omega n} \\ & & \updownarrow \mathscr{F} \\ & & h[n] \end{array}$$

z-Transform: Eigenfunction Property

$$z^n \rightarrow \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$z^n \rightarrow z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$z = re^{j\omega}$$

$$z^n \rightarrow H(z) z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

z-Transform and Fourier Transform Relationship

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$z = re^{j\omega}$$

$$X(z)|_{z=e^{j\omega}} = \mathscr{F}\{x[n]\}$$

New notation:

$$\mathscr{F}\{x[n]\} = X(e^{j\omega})$$

z-Transform: Convergence Comparison

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega})$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] \left(re^{j\omega}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] r^{-n} re^{-j\omega n}$$

$$X(z) = \mathscr{F}\left\{x[n] r^{-n}\right\}$$

ZT may converge when FT does not.

Example

Find the ZT of $x[n] = a^n u[n]$.

Example

Find the ZT of

$$x[n] = -a^n u[-n-1].$$

z-Plane and the Unit Circle

Pole-Zero Plot for a Right-Handed Sequence

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

Pole-Zero Plot for a Left-Handed Sequence

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|$$

First-Order Difference Equation

$$y[n] - ay[n-1] = x[n]$$

Pole-Zero Plot for a DT First-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{z}{z-a}, \quad |z| > |a|.$$

Second-Order Difference Equation

$$y[n] + 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

Pole-Zero Plot for a DT Under-Damped Second-Order System

This illustrates the determination of the Fourier transform form the pole-zero plot.

$$H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}, \quad |z| > |a|.$$

Properties of the ROC of the z-Transform

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin
- $\mathscr{F}\{x[n]\}$ converges $\Leftrightarrow$ ROC includes the unit circle in the z-plane
- $x[n]$ finite duration $\Rightarrow$ ROC is entire z-plane with the possible exception of $z=0$ or $z=\infty$

Properties of the ROC for a Right-Sided Sequence

- $x[n]$ right-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all finite values of z for which $|z| > r_0$ are in ROC.
- $x[n]$ right-sided and $X(z)$ rational $\Rightarrow$ ROC is outside the outermost pole.

Properties of the ROC for a Left-Sided and for a Two-Sided Sequence

- $x[n]$ left-sided and $|z| = r_0$ is in ROC $\Rightarrow$ all values of z for which $0 < |z| < r_0$ will also be in ROC.
- $x[n]$ left-sided and $X(z)$ rational $\Rightarrow$ ROC is inside the innermost pole.
- $x[n]$ two-sided and $|z| = r_0$ is in ROC $\Rightarrow$ ROC is a ring in the z-plane which includes the circle $|z| = r_0$.

Example

Show the choices of the ROC for

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Right-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Left-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

ROC If the Sequence Is Two-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

Inverse z-Transform

$$
\begin{aligned}
X(z) &= \mathscr{F}\left\{x[n]r^{-n}\right\} \\
x[n]r^{-n} &= \mathscr{F}^{-1}\{X(z)\} \\
&= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{-j\omega n}d\omega \\
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})\left(re^{j\omega}\right)^{n} d\omega \\
z = re^{j\omega}&, \qquad dz = jre^{j\omega}d\omega \\
x[n] &= \frac{1}{2\pi j}\oint X(z)z^{n-1}dz
\end{aligned}
$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}, \quad |z| > 2.$$

Section 3

z-Transform properties

Recall: z-Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{-n}dz$$

$$X(z)|_{z=e^{j\Omega}} = \mathscr{F}\{x[n]\}$$

$$z = re^{j\Omega}$$

$$X(z) = \mathscr{F}\left\{x[n]r^{-n}\right\}$$

z-Transform Properties

Property	Signal	Transform	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting	$x[n - n_0]$	$z^{-n_0} X(z)$	R
Scaling in z domain	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\lvert z_0 \rvert R$
Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$	at least $R_1 \cap R_2$

Example

Consider and LTI system for which

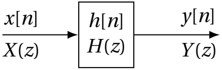
$$y[n] = h[n] * x[n],$$

where

$$h[n] = \delta[n] - \delta[n-1].$$

1. Find $H(z)$.
2. Find $y[n]$ in terms of $x[n]$.

System Stability



$$Y[n] = h[n] * x[n]$$
$$Y(z) = H(z)X(z)$$

$$\text{stable} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$
$$\mathscr{F}\{h[n]\} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

The condition for stability and the existence of the Fourier transform are the same.

Stability, Causality, and ROC

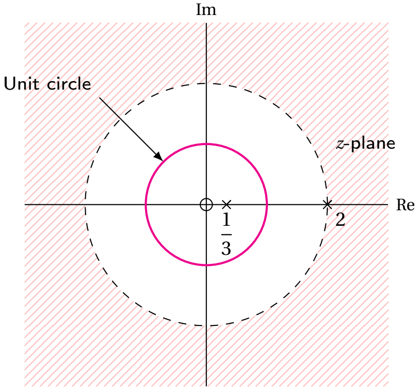
$$
\begin{aligned}
\text{stable} &\Leftrightarrow \text{ROC of } H(z) \text{ includes unit circle in } z\text{-plane} \\
\text{causal} &\Rightarrow h[n] \text{ is right-sided} \\
&\Rightarrow \text{ROC of } H(z) \text{ outside the outermost pole} \\
\text{causal and stable} &\Leftrightarrow \text{ All poles inside unit circle}
\end{aligned}
$$

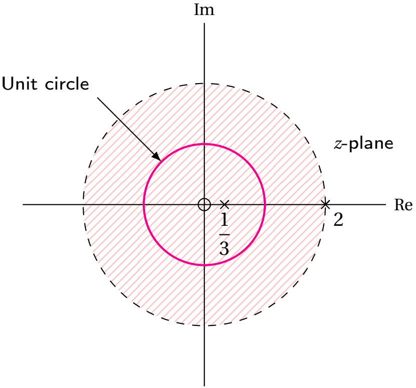
Example

Discuss the stability and causality of the system represented by the following system function with respect to different regions of convergence.

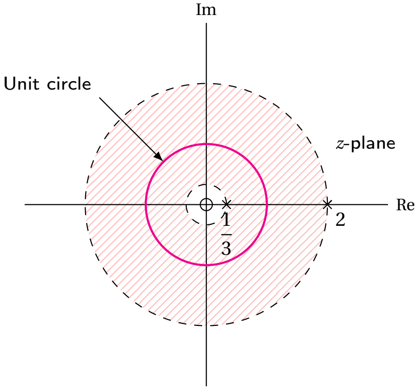
$$H(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z-2)}.$$

The system is causal and unstable.





The system is unstable and not causal.



The system is stable and not causal.

Example

Consider the LTI system for which the input $x[n]$ and the output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

1. Obtain an expression for the system function $H(z)$.
2. What are the two choices for the region of convergence?
3. Obtain $h[n]$ for each of these cases and comment on the stability and causality.