EN1060 Signals and Systems: Discrete-Time Fourier Series

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Section 1

Discrete-Time Fourier Series

Introduction

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- In this lesson we will develop a similar tool for discrete time.
- Specifically, we consider the representation of discrete-time signals through a decomposition as a linear combination of complex exponentials.
  - DT periodic signals $\longrightarrow$ DT Fourier series
  - DT aperiodic signals $\longrightarrow$ DT Fourier transform

Philosophy

Decompose the input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots \quad \text{linear combination of basic inputs}$$

Then

$$y = a_1\psi_1 + a_2\psi_2 + \cdots \quad \text{linear combination of corresponding outputs}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ such that

- Broad class of signals can be constructed, and
- Response to ϕ_ks easy to compute.

Eigenfunction Property

Continuous-Time:
$\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(\omega_k)e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

“Discrete-Time”:

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"Discrete-Time": $\phi_k[n] = e^{j\omega_k n}$

$$e^{j\omega_k n} \longrightarrow \underset{\substack{\uparrow \\ \text{eigenfunction}}}{e^{j\omega_k n}} \underbrace{\sum_{r=-\infty}^{\infty} h[r] e^{-j\omega_k r}}_{\text{eigenvalue}}$$

Discrete-Time Fourier Series

Consider $x[n]$ to be periodic,
Period N,
Fundamental frequency $\omega_0 = \dfrac{2\pi}{N}$
$e^{jk\omega_0 n}$ are harmonically related, and periodic with the period N, although the fundamental period is different. $e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$

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Consider the complex exponential

$$e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n}$$

$\Rightarrow$ Only N distinct complex exponentials.

$$x[n] = \sum_k a_k e^{jk\omega_0 n}, \quad k = 0, 1, 2, \ldots, N-1.$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

N equations in N unknowns.

$$a_k = \frac{1}{N} \sum_{<N>} x[n] e^{-jk\omega_0 n}.$$

$k =< N >$: k ranges over one period (as a_k periodically repeats).

Discrete-Time Fourier Series

Continuous-Time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Discrete-Time Fourier Series

Synthesis

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}.$$

Analysis

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}.$$

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Note the duality.

Periodicity

$x[n]$	periodic in n,	true for CT
$e^{jk\omega_0 n}$	periodic in n,	true for CT
$e^{jk\omega_0 n}$	periodic in k,	not true for CT
a_k	periodic in k,	not true for CT

Convergence

Continuous-time:

- $x(t)$ square-integrable OR
- Dirichlet condition

Discrete-time

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$\hat{x}[n] = \sum_{p \text{ terms}} a_k e^{jk\omega_0 n}.$$

$p = N$

$$\hat{x}[n] \equiv x[n].$$

There is no issue of convergence in DT.

Example

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$x[n]$ is periodic with fundamental period N.
Expanding the signal as a sum of two complex exponentials,

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \tag{1}$$

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$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}.$$

Fourier Coefficients for $x[n] = \sin(2\pi/N)n$ for $N = 5$

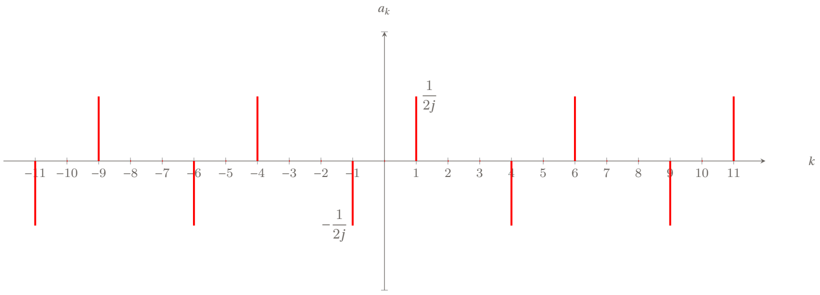

Figure: Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Example

Determine and sketch the DTFS of

$$x[n] = 1 + \sin \omega_0 n + 3 \cos \omega_0 n + \cos\left(2\omega_0 n + \frac{\pi}{2}\right).$$

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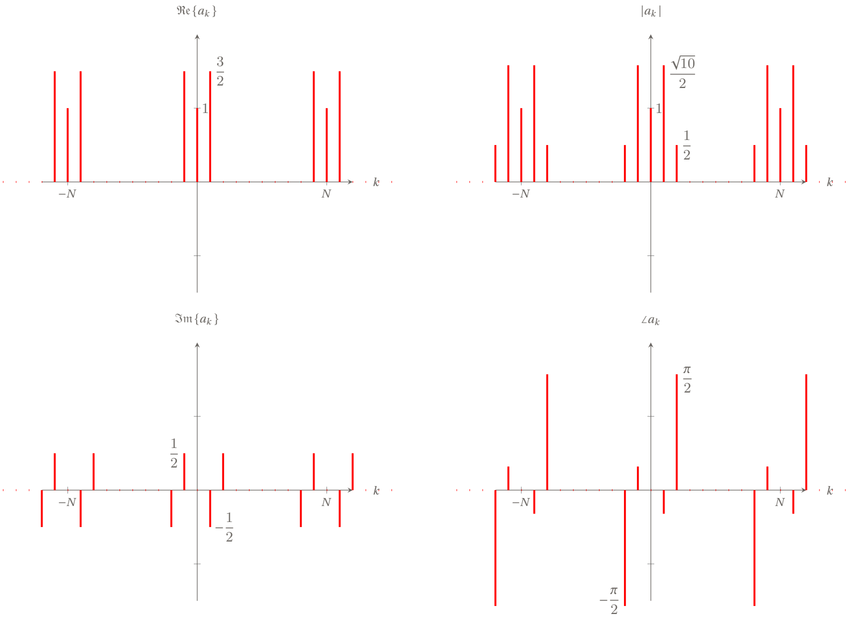
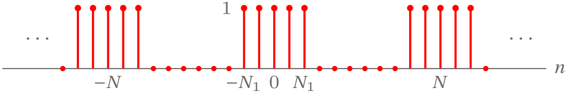


$\Re\mathfrak{e}\{a_k\}$
$\frac{3}{2}$
1
$-N$
N
k
$|a_k|$
$\frac{\sqrt{10}}{2}$
1
$\frac{1}{2}$
$-N$
N
k
$\Im\mathfrak{m}\{a_k\}$
$\frac{1}{2}$
$-\frac{1}{2}$
$-N$
N
k
$\angle a_k$
$\frac{\pi}{2}$
$-\frac{\pi}{2}$
$-N$
N
k

Example

Determine and sketch the DTFS of $x[n]$ of which is shown in the figure.



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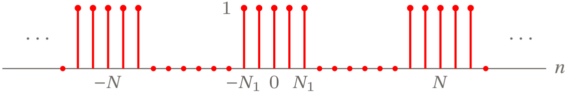

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

Letting $m = n + N_1$

$$= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)/(m-N_1)}$$

$2N_1 + 1$ terms in a geometric series

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \left[\frac{1 - e^{-jk(2\pi/N)(2N_1+1)}}{1 - e^{-jk(2\pi/N)}} \right]$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \cdot \frac{e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{-j\frac{k(2\pi/N)}{2}}} \left[\frac{e^{j\frac{k(2\pi/N)(2N_1+1)}{2}} - e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{j\frac{k(2\pi/N)}{2}} - e^{-j\frac{k(2\pi/N)}{2}}} \right]$$

$$= \frac{1}{N} \frac{\sin\left[\frac{k(2\pi/N)(2N_1+1)}{2}\right]}{\sin\left[\frac{k(2\pi/N)}{2}\right]}$$

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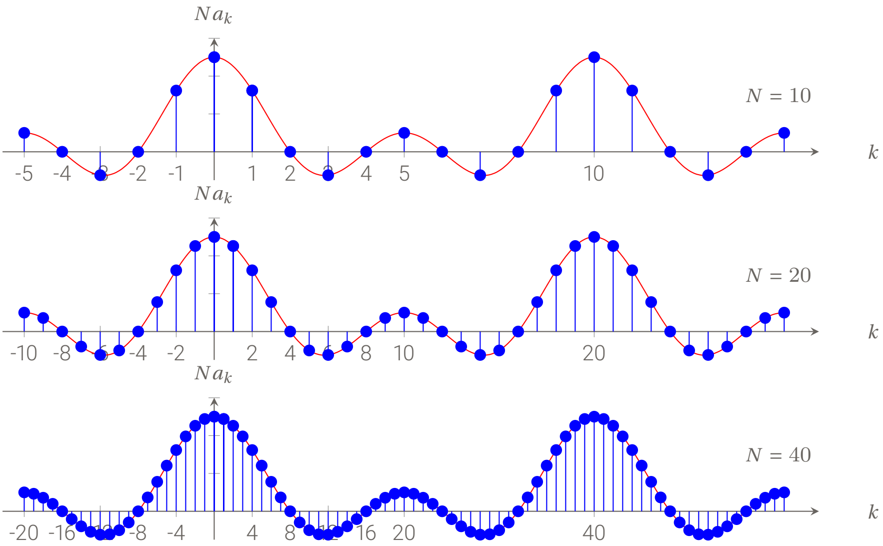
$$a_k = \frac{1}{N}\frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)} \quad k \neq 0, \pm N, \pm 2N, \ldots$$

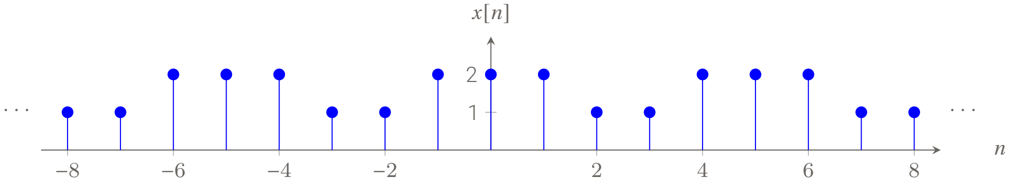
$$a_k = \frac{2N_1 + 1}{N} \quad k = 0, \pm N, \pm 2N, \ldots .$$


Na_k
-5 -4 -3 -2 -1 1 2 3 4 5 10
$N = 10$
k
Na_k
-10 -8 -6 -4 -2 2 4 6 8 10 20
$N = 20$
k
Na_k
-20 -16 -12 -8 -4 4 8 12 16 20 40
$N = 40$
k

Outline

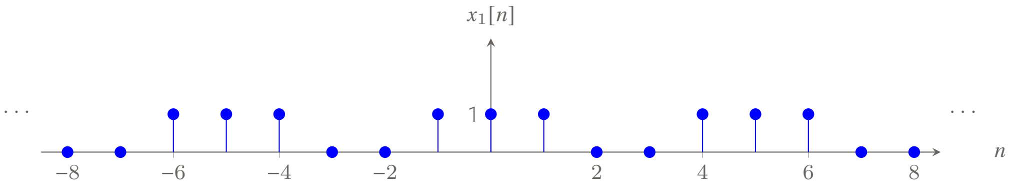
Example

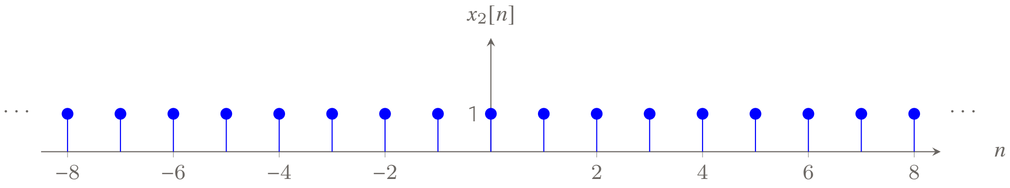
Find the Fourier series coefficients a_k of $x[n]$.



Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k. We use the linearity property of to conclude that

$$a_k = b_k + c_k.$$





From the previous work, (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients b_k corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

Example

Suppose that we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $n = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the proceeding three conditions.

Determine the sequence $x[n]$.

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From Parseval's relation, the average power in $x[n]$ is

$$P = \sum_{k=0}^{5} |a_k|^2.$$

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and $a3$ are pre-specified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a5 = 0$. It then follows that

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n.$$

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