EN1060 Signals and Systems: Sampling

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

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Section 1

Recall: Fourier Transform for Periodic Signals

Fourier Transform for Periodic Signals

- We studied the Fourier transform for aperiodic signals. We can also develop Fourier transform representations for periodic signals, thus allowing us to consider both periodic and aperiodic signals within a unified context.
- We can construct the Fourier transform of a periodic signal directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain, with the areas of impulses proportional to the Fourier series coefficients.

Let us consider the signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$:

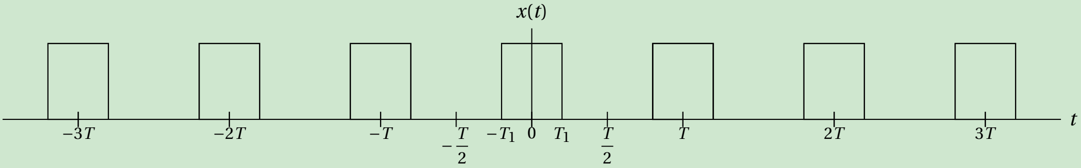
$$X(j\omega) = 2\pi\delta(\omega - \omega_0).$$

To determine the signal $x(t)$ we can apply the inverse Fourier transform relation

This corresponds exactly to the Fourier series representation of a periodic signal.
Thus the Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at harmonically related frequencies for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the $k$the Fourier series coefficient a_k.

Example

Consider the square wave.



1. Obtain the Fourier series coefficients $\{a_k\}$.
2. Obtain the Fourier transform of the signal.

Fourier series coefficients:

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}$$

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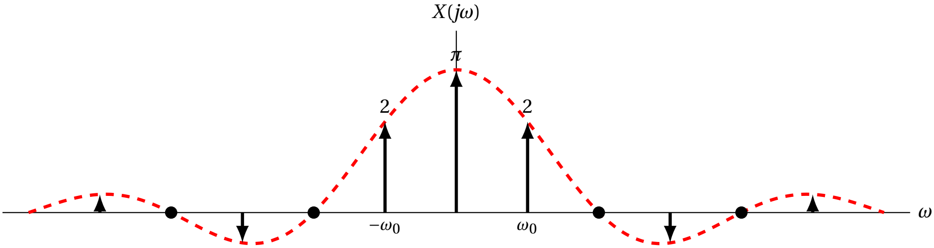
Fourier transform

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\frac{\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$$


$X(j\omega)$
π
2
2
$-\omega_0$
ω_0
ω

Example

Sketch the Fourier transform of

1. $x(t) = \sin \omega_0 t$
2. $x(t) = \cos \omega_0 t$

Example

Obtain the Fourier transform of the impulse train given by

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

which is periodic with period T. Sketch.

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt$$

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$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

Section 2

Sampling

Sampling

- The sampling theorem, which is a relatively straightforward consequence of the modulation theorem, is elegant in its simplicity.
- It states that a bandlimited time function can be exactly reconstructed from equally spaced samples provided that the sampling rate is sufficiently high—specifically, that it is greater than twice the highest frequency present in the signal.
- A similar result holds for both continuous time and discrete time.
- One of the important consequences of the sampling theorem is that it provides a mechanism for exactly representing a bandlimited continuous-time signal by a sequence of samples, that is, by a discrete-time signal.
- The reconstruction procedure consists of processing the impulse train of samples by an ideal lowpass filter.

Sampling Frequency and Aliasing

- Assumption: the sampling frequency is greater than twice the highest frequency in the signal.
- The reconstructing lowpass filter will always generate a reconstruction consistent with this constraint, even if the constraint was purposely or inadvertently violated in the sampling process.
- Said another way, the reconstruction process will always generate a signal that is bandlimited to less than half the sampling frequency and that matches the given set of samples.
- If the original signal met these constraints, the reconstructed signal will be identical to the original signal.
- On the other hand, if the conditions of the sampling theorem are violated, then frequencies in the original signal above half the sampling frequency become reflected down to frequencies less than half the sampling frequency.
- This distortion is commonly referred to as aliasing, a name suggestive of the fact that higher frequencies (above half the sampling frequency) take on the alias of lower frequencies.

Sampling Theorem

- Equally spaced samples of $x(t)$

$$x(nT), \quad n = 0, \pm 1, \pm 2, \dots$$

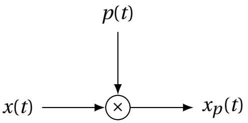
- $x(t)$ is band limited

$$X(\omega) = 0, \quad |\omega| > \omega_M$$

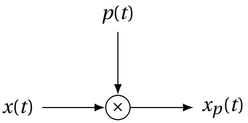
- If

$$\frac{2\pi}{T} \triangleq \omega_s > 2\omega_M$$

- then $x(t)$ is uniquely recoverable.



$$x_p(t) = x(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$
$$= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)$$



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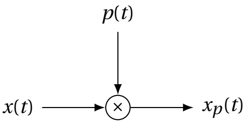
$$X_p(\omega) = \frac{1}{2\pi} [X(\omega) * P(\omega)]$$
$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$



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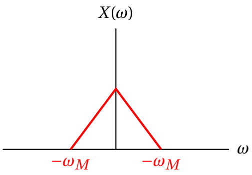
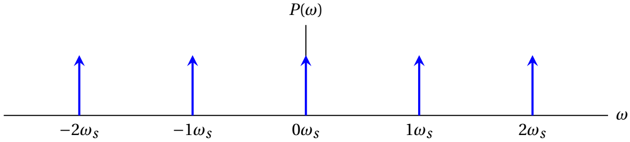
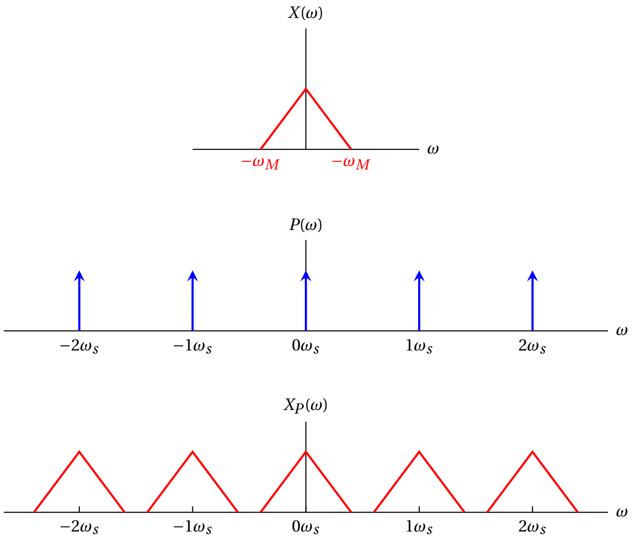
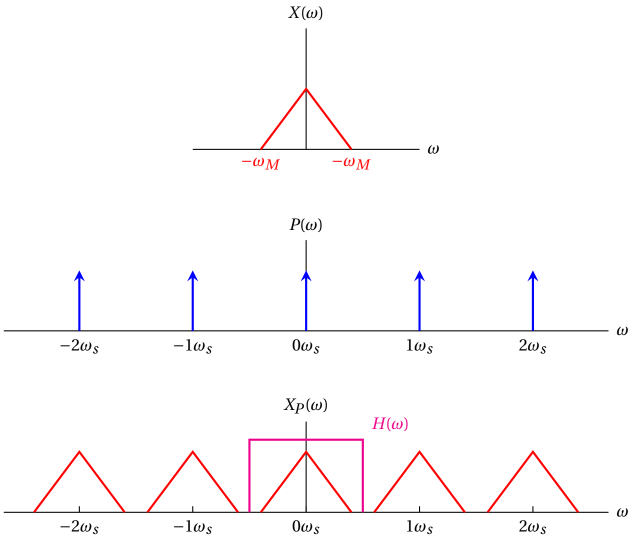
$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$


$X(\omega)$
ω
$-\omega_M$
$-\omega_M$

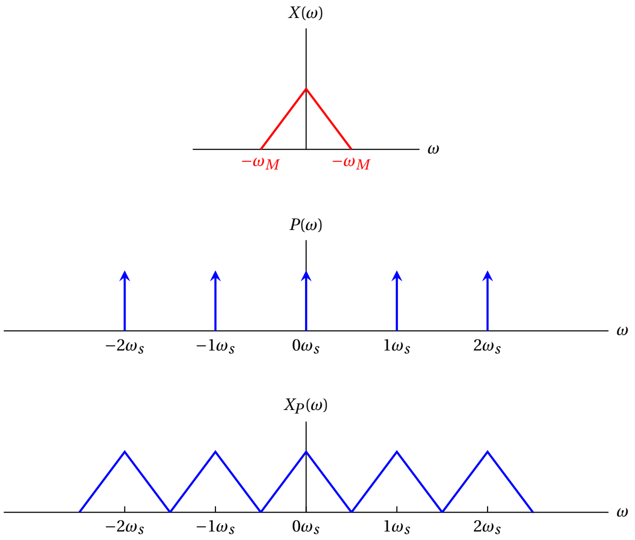

$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

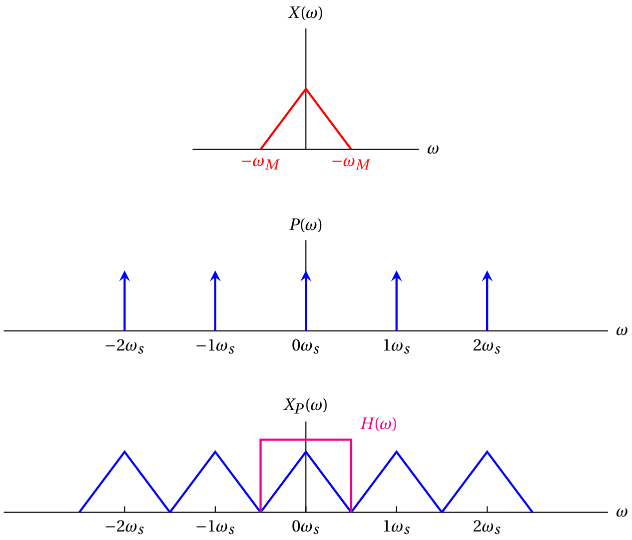


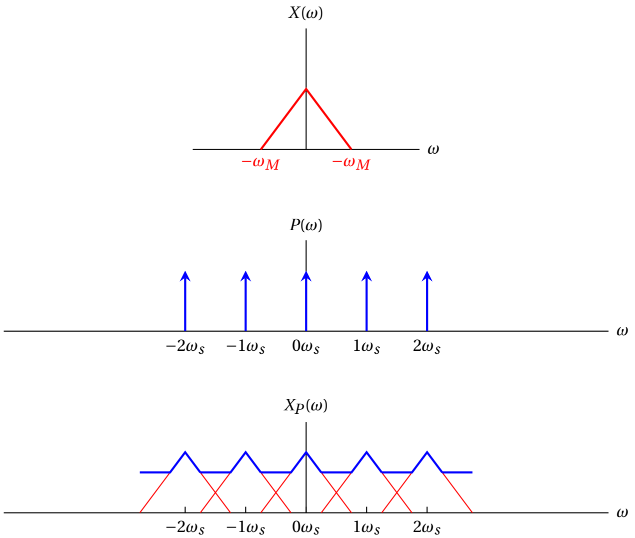
$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

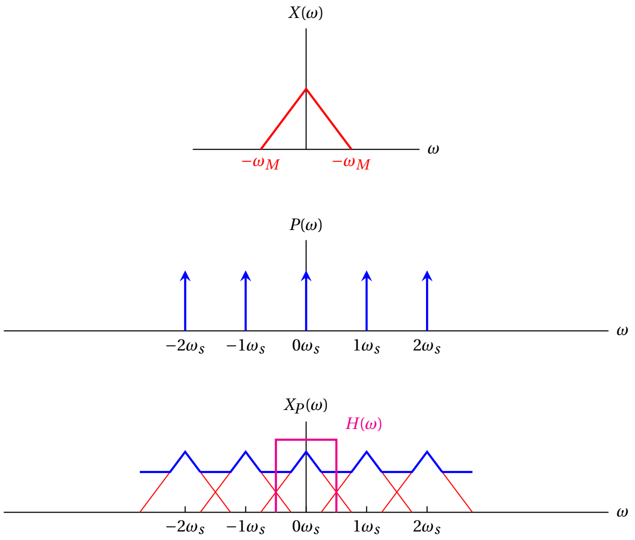


$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
$H(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

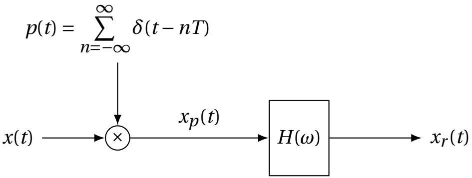

$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

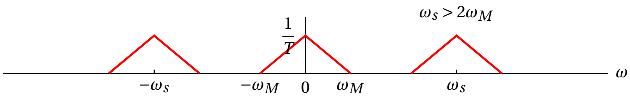

$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
$H(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

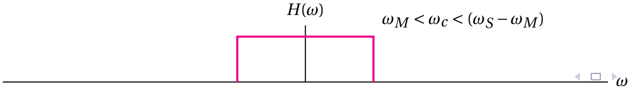

$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$



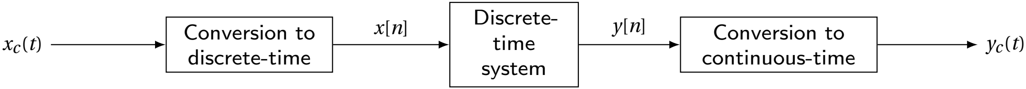
$X(\omega)$
ω
$-\omega_M$
$-\omega_M$
$P(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
$H(\omega)$
ω
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$

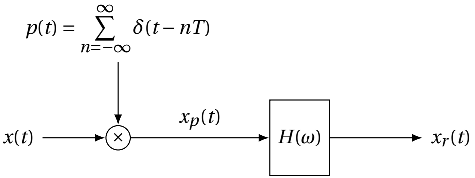

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
$x(t)$
$\times$
$x_p(t)$
$H(\omega)$
$x_r(t)$


$\omega_s > 2\omega_M$
$\frac{1}{T}$
$-\omega_s$
$-\omega_M$
0
ω_M
ω_s
ω


$H(\omega)$
$\omega_M < \omega_c < (\omega_S - \omega_M)$
ω

Discrete-Time Processing of Continuous-Time Signals





$$
\begin{aligned}
x_p(t) &= x(t)p(t) \\
&= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)
\end{aligned}
$$

$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)
\end{aligned}
$$