EN1060 Signals and Systems: Laplace Transfrom

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Section 1

Laplace Transform

- Using the Fourier transform, we represented a signal as a linear combination of basic signals using the eigenfunctions $e^{j\omega t}$.
- Then we could represent a given LTI system as a spectrum of eigenvalues as a function of ω, which is the change in amplitude that the system applies to each of the basic inputs $e^{j\omega t}$.
- Now we study a generalization of the Fourier transform, referred to as the Laplace transform.
- The Laplace transform converges for a broader class of signals than does the Fourier transform.

- The general class of eigenfunctions for LTI systems consists of the complex exponential e^{st}, where s is a complex number.
- When s is purely imaginary, $s = j\omega$, the Laplace transform reduces to the Fourier transform.
- The Laplace transform is the Fourier transform of an exponentially weighted signal. Therefore, the Laplace transform can converge for signals for which the Fourier transform does not converge.
- The range of values of s for which the Laplace transform converges is the region of convergence (ROC).
- Two different signals can have Laplace transforms with identical algebraic expressions and differing only in the ROC.

Recall: Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$
$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega t} & \rightarrow & H(\omega)e^{j\omega t} \\ & & \updownarrow \mathscr{F} \\ & & h(t) \end{array}$$

Laplace Transform: Eigenfunction Property

$$e^{st} \rightarrow \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} \, d\tau$$

$$e^{st} \rightarrow e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} \, d\tau$$

$$s = \sigma + j\omega$$

$$e^{st} \rightarrow H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} \, d\tau$$

Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$$

Laplace Transform and Fourier Transform Relationship

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$s = \sigma + j\omega$$

$$X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}$$

New notation:

$$\mathscr{F}\{x(t)\} = X(j\omega)$$

Laplace Transform: Convergence Comparison

$$X(s)|_{s=j\omega} = X(j\omega)$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-(\sigma+j\omega)t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-\sigma t} e^{-j\omega t}\, dt$$

$$X(s) = \mathscr{F}\left\{x(t)\, e^{-\sigma t}\right\}$$

Laplace Transform: Convergence Comparison

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$$= \int_{-\infty}^{+\infty} x(t)\, e^{-\sigma t} e^{-j\omega t}\, dt$$

$$X(s) = \mathscr{F}\left\{x(t)\, e^{-\sigma t}\right\}$$

LT may converge when FT does not.

Example

Find the LT of

$$x(t) = e^{-at}u(t).$$

Solution

$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} x(t)e^{-st}dt \\ &= \int_{0}^{+\infty} e^{-at}e^{-st}dt \\ X(s) &= \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a \end{aligned}$$

$$e^{-at}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

Example

Find the LT of

$$x(t) = -e^{-at}u(-t).$$

Solution

$$-e^{-at}u(-t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

Note: Two time functions generate the same algebraic expression for the LT. The difference is only in the ROC.

Example

Find the LT of

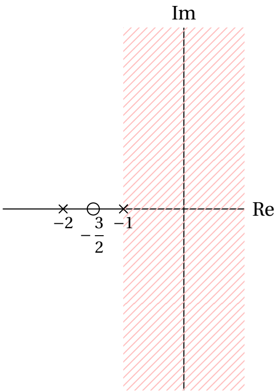
$$x(t) = e^{-t}u(t) + e^{-2t}u(t).$$

Solution

$$e^{-t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$$e^{-2t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2$$

$$e^{-t}u(t) + e^{-2t}u(t) \stackrel{\mathscr{L}}{\longleftrightarrow} \frac{2s+3}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$



$$X(s) = \frac{N(s)}{D(s)}$$

$N(s) = 0$ zeros of $X(s)$

$D(s) = 0$ poles of $X(s)$

- The ROC contains no poles
- The ROC of $X(S)$ consists of s trip parallel to the $j\omega$ axis in the s-plane.
- $\mathscr{F}\{x(t)\}$ converges $\Leftrightarrow$ ROC includes the $j\omega$-axis in the s-plane.

Example

Sketch the choices of the ROC associated with

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

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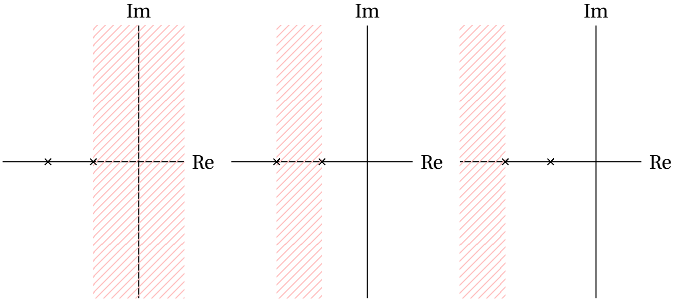
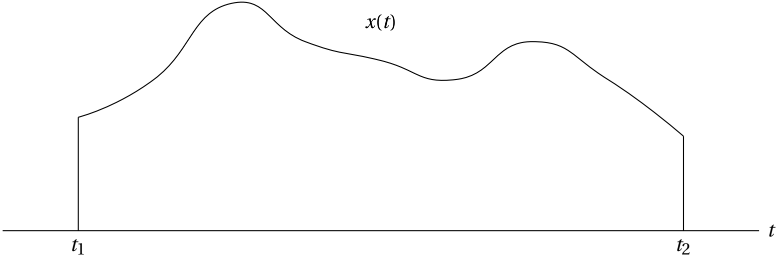
Solution



If $x(t)$ is a finite-duration signal, then the ROC is the entire s-plane.



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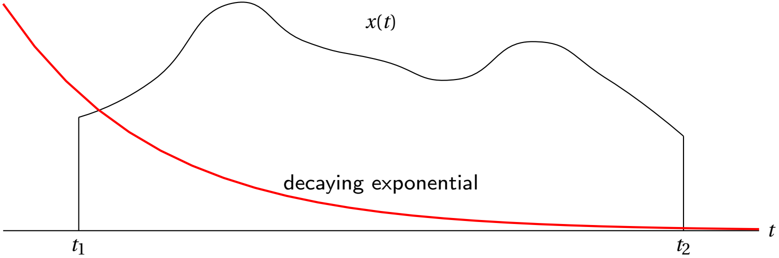

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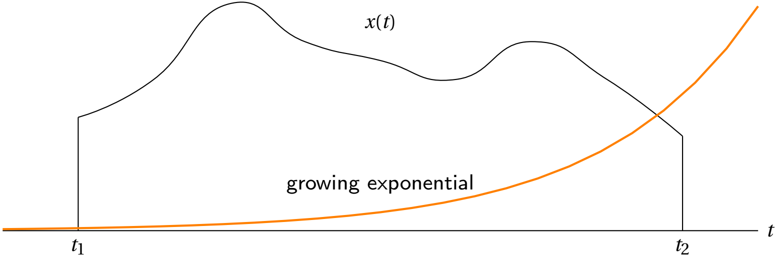
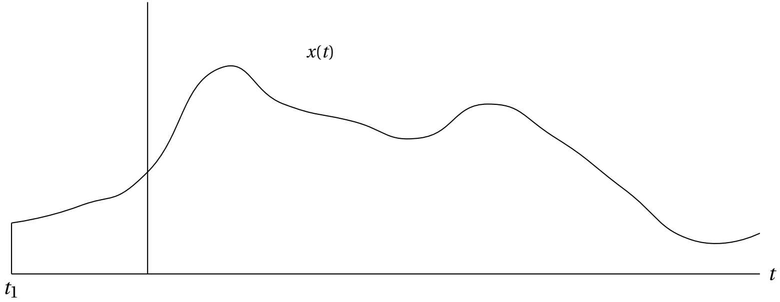


ROC of a Right-Sided Signal

If $x(t)$ is right-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} > \sigma_0$ are in ROC.



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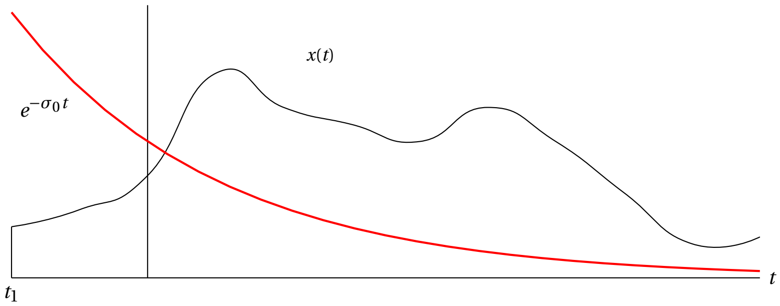

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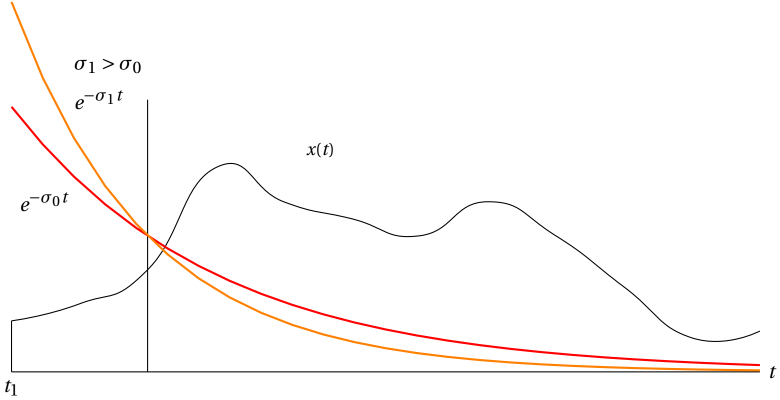

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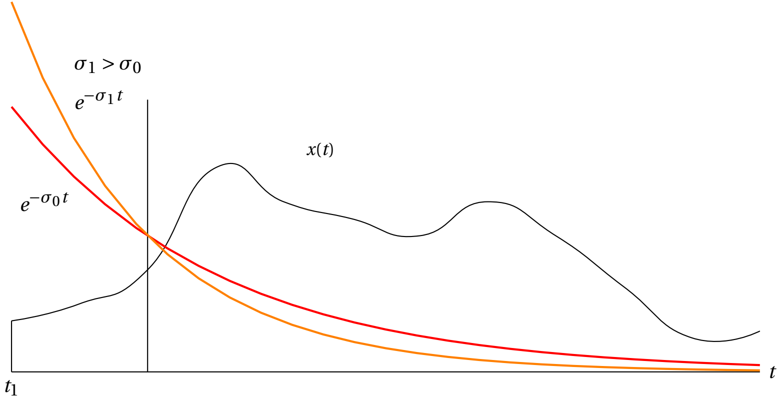


If $x(t)$ is right-sided and $X(s)$ is rational, then ROC is the right of the rightmost pole.

ROC of a Left-Sided Signal

If $x(t)$ is left-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC.

ROC of a Left-Sided Signal

If $x(t)$ is left-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC.
If $x(t)$ is left-sided and $X(s)$ is rational, then ROC is the left of the leftmost pole.

ROC of a Two-Sided Signal

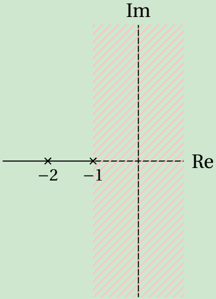
If $x(t)$ is two-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then ROC is the strip in the s-plane.

Example

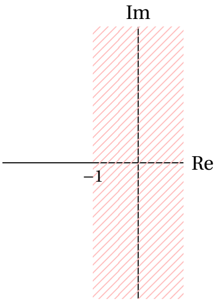
A Laplace transform is specified by

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1.$$

Find the inverse laplace transform.

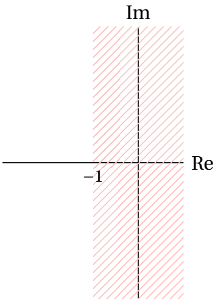


Solution



$$
\begin{aligned}
X(s) &= \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1, \\
&= \frac{1}{(s+1)} - \frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -1.
\end{aligned}
$$

Solution



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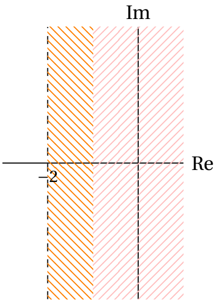
$$= \frac{1}{(s+1)} - \frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -1.$$

Consider

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathrm{Re}\{s\} > -1,$$
$$x_1(t) = e^{-t}u(t).$$

Solution



Consider

$$X_2(s) = -\frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -1,$$
$$x_2(t) = -e^{-2t}u(t).$$

So

$$x(t) = (e^{-t} - e^{-2t})u(t).$$

Example

Find the inverse laplace transform of

1. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} > -1,$$

2. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} < -3,$$

3. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad -3 < \mathrm{Re}\{s\} < -1,$$

Solution

$$X(s) = \frac{2s+4}{s^2+4s+3} = \frac{1}{s+1} + \frac{1}{s+3}$$

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$$X(s) = \frac{2s+4}{s^2+4s+3} = \frac{1}{s+1} + \frac{1}{s+3}$$

1. The ROC of $X(s)$ is $\mathrm{Re}\{s\} > -1$. Thus, $x(t)$ is a right-sides signal. We obtain

$$x(t) = e^{-t}u(t) + e^{-3t}u(t) = (e^{-t} + e^{-3t})u(t).$$

Solution

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$$x(t) = e^{-t}u(t) + e^{-3t}u(t) = (e^{-t} + e^{-3t})u(t).$$

2. The ROC of $X(s)$ is $\mathrm{Re}\{s\} < -3$. Thus, $x(t)$ is a left-sides signal. We obtain

$$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t) = -(e^{-t} + e^{-3t})u(-t).$$

Solution

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1. The ROC of $X(s)$ is $\mathrm{Re}\{s\} > -1$. Thus, $x(t)$ is a right-sides signal. We obtain

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$$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t) = -(e^{-t} + e^{-3t})u(-t).$$

3. The ROC of $X(s)$ is $-3 < \mathrm{Re}\{s\} < -1$. Thus, $x(t)$ is a double-sided signal. We obtain

$$x(t) = -e^{-t}u(-t) + e^{-3t}u(t).$$

Instead of having to reevaluate the transform of a given signal, we can simply refer to the Laplace transform table and and read out the desired transform.

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\mathrm{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -\mathrm{Re}(s)$
$te^{-at} u(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$

$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -\mathrm{Re}(s)$
$\cos\omega_0 tu(t)$	$\dfrac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
$\sin\omega_0 tu(t)$	$\dfrac{\omega_0}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
$e^{-at}\cos\omega_0 tu(t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$
$e^{-at}\sin\omega_0 tu(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega^2}$	$\mathrm{Re}(s) > -\mathrm{Re}(s)$

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(t) + a_2 X_2(t)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	$R' = R$
Shifting in s	$e^{s_0 t} x(t)$	$X(s - s_0)$	$R' = R + \mathrm{Re}(s_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X(s)$	$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
Differentiation in s	$-tx(t)$	$\frac{dX(s)}{ds}$	$R' = R$

Integration	$\int_{-\infty}^{t} x(\tau)\tau$	$\frac{1}{s}X(s)$	$R' \supset R\{\mathrm{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	$R' \supset R_1 \cap R_2$

Example

Verify the time-shifting property

$$x(t-t_0) \longleftrightarrow e^{-st_0}X(s), \quad R' = R.$$

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$$x(t - t_0) \longleftrightarrow e^{-st_0} X(s), \quad R' = R.$$

$$\mathcal{L}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-st} dt$$

By the change of variables $\tau = t - t_0$, we obtain

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By the change of variables $\tau = t - t_0$, we obtain

$$\begin{aligned}\mathscr{L}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}dt \\ &= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}dt \\ &= e^{-st_0}X(s)\end{aligned}$$

with the same ROC as for $X(s)$ itself.

Example

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\mathscr{L}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)}\,dt \\
&= e^{-st_0}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau}\,dt \\
&= e^{-st_0}X(s)
\end{aligned}$$

with the same ROC as for $X(s)$ itself.

Example

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the Laplace transform of $u(t)$.

1. $\delta(t)$
2. $\delta'(t)$
3. $tu(t)$
4. $e^{-at}u(t)$
5. $te^{-at}u(t)$
6. $\cos\omega_0 tu(t)$
7. $e^{-at}\cos\omega_0 tu(t)$

❶

$$
\begin{aligned}
\mathscr{L}\{u(t)\} &= \int_{-\infty}^{\infty} u(t)e^{-st}dt \\
&= \int_{0}^{\infty} e^{-st}dt \\
&= -\frac{1}{s}e^{-st}\Big|_{0}^{\infty} \\
&= \frac{1}{s}, \quad \mathrm{Re}(s) > 0.
\end{aligned}
$$

$$
\delta(t) = \frac{du(t)}{dt}
$$

Thus, using time differentiation property, we obtain

$$
\delta(t) \longleftrightarrow s\frac{1}{s} = 1, \quad \text{all } s
$$

2. Again applying the time-differentiation property to the result above), we obtain

$$\delta'(t) \longleftrightarrow s, \quad \text{all } s$$

3. Using the differentiation in s property, we have

$$tu(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}, \quad \mathrm{Re}(s) > 0$$

4. Using the shifting in the s-domain property, we have

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad \mathrm{Re}(s) > -a$$

Section 2

Continuous-Time Second-Order Systems

- The properties of the Laplace transform make it useful in analyzing LTI systems that are represented by linear constant-coefficient differential equations.
- Applying the Laplace transform to a differential equation converts it to an algebraic equation relating the Laplace transform of the system output to the product of the Laplace transform of the system input and the Laplace transform of the system impulse response, referred to as the system function.
- The system function is readily obtained by inspection of the differential equation, and the system impulse response can be obtained by evaluating the inverse Laplace transform of the system function.
- Alternatively, the response for any other input can be evaluated by first multiplying the Laplace transform of the input by the system function and then applying the inverse Laplace transform.

- Two particularly important classes of systems described by linear constant-coefficient differential equations are first-order and second-order systems.
- In implementing higher-order systems, it is very common to use first and second-order systems as building blocks.
- First-order systems are represented by a single pole in the s-plane, and second-order systems by a pair of poles. There may or may not also be zeros in the transfer function, depending on whether there are derivative terms on the right-hand side of the differential equation.
- From the differential equation, the system function can be written directly.
- If we assume that the systems are causal, so that the impulse response is right-sided, then the ROC of the system function is implicitly specified to be to the right of the rightmost pole in the s-plane.

Recall: Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$$x(t) \leftrightarrow X(s)$$

$$X(s)|_{s=j\omega} = X(j\omega) = \mathscr{F}\{x(t)\}$$

$$s = \sigma + j\omega$$

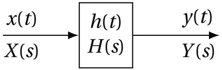
$$X(s) = \mathscr{F}\left\{x(t) e^{-\sigma t}\right\}$$

LT converges for some values of σ and not others: ROC.

Properties

$$ax_1(t) + bx_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s).$$

$$\frac{dx(t)}{dt} \stackrel{\mathcal{L}}{\longleftrightarrow} sX(s).$$



$$Y(s) = H(s)X(s).$$

Stable, causal ⇔ all poles in left-half s-plane

First-Order System

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

$$\frac{dy(t)}{dt} \quad + \quad ay(t) \quad = \quad x(t)$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

First-Order System

$$
\begin{array}{ccccc}
\dfrac{dy(t)}{dt} & + & ay(t) & = & x(t) \\
\downarrow & & \downarrow & & \downarrow \\
sY(s) & + & aY(s) & = & X(s)
\end{array}
$$

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$$Y(s) = \frac{1}{s+a} X(s), \quad \mathrm{Re}\{s\} > a$$

$$h(t) \overset{\mathscr{L}}{\longleftrightarrow} H(s)$$

First-Order System

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$$Y(s) = \frac{1}{s+a} X(s), \quad \mathrm{Re}\{s\} > a$$

$$h(t) \overset{\mathscr{L}}{\longleftrightarrow} H(s)$$

$$e^{-at}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

$$e^{-at}u(-t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

Section 3

Continuous Time Second Order Systems

Second-Order System

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

Second-Order System

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \omega_n^2 X(s).$$

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$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

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$c_1 =$

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$$c_2 =$$

Second-Order System

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \omega_n^2 X(s).$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

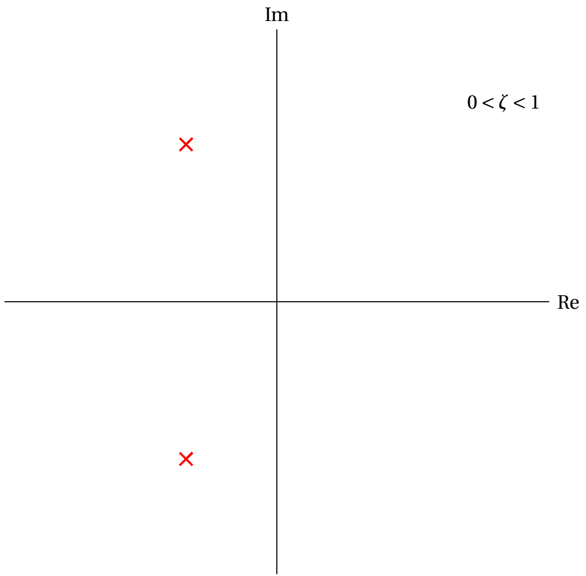
$$H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_2)}.$$

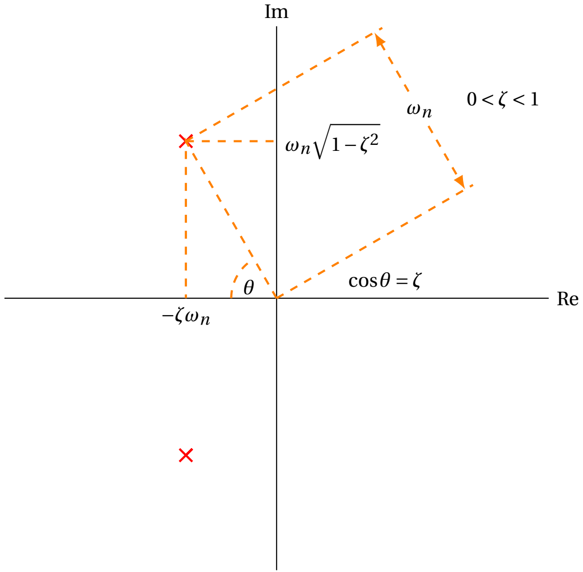
$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

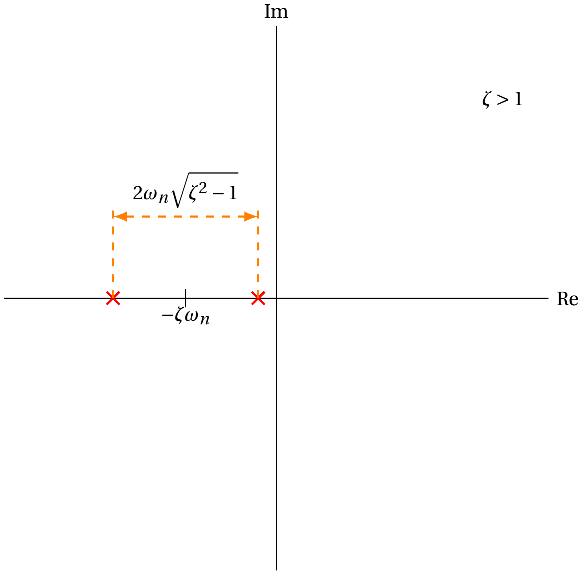
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

For $\zeta < 1$,

$$\begin{aligned} c_1 &= c_2^* \\ &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \end{aligned}$$


Im
$0 < \zeta < 1$
Re


Im
$0 < \zeta < 1$
ω_n
$\omega_n\sqrt{1-\zeta^2}$
θ
$\cos\theta = \zeta$
Re
$-\zeta\omega_n$



Im
$\zeta > 1$
$2\omega_n\sqrt{\zeta^2 - 1}$
Re
$-\zeta\omega_n$

Example

The transfer function of a network is

$$H(s) = \frac{s+10}{s^2+4s+8}$$

Determine the pole-zero plot of $H(s)$, the type of damping exhibited by the network, and the unit step response of the.

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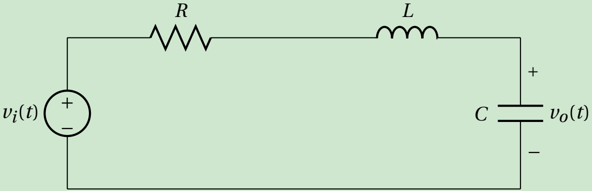
$$y(t) = \left[\frac{10}{8} + 1.4e^{-2t}\cos(2t - 210.96^\circ)\right] u(t)$$

Example



Consider an RLC series network.

1. Obtain the voltage transfer function.
2. If $\omega_n = 2000\,\mathrm{rad/s}$ and $zeta = 0.25, 0.50, 0.75$, and 1.0, sketch the pole-zero plots.
3. Sketch the step response for each case.

Example

A causal LTI system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Suppose that the system is at initial rest.

1. Find the system function $H(s)$.
2. Find the Laplace transform of the output $Y(s)$ if the input is $x(t) = \alpha u(t)$.
3. Find the output $y(t)$.