Syllabus [Course Contents]

Unit- 1

Curve Tracing & Rectification

Tracing of curve in cartesian form:

1) Semi cubical parabola.
2) Cissiod of Diocles,
3) Strophoid.
4) Astroid
5) Witch of Agnesi
6) Common Catenary
7) Folium of Descartes

Tracing of curves in polar form:

1) Cardioid
2) Pascal's Limacon
3) Lemniscate of Bernoulli
4) Parabola
5) Hyperbola.
6) Rose curves
7) Rectification of plane curves (Cartesian and Polar form)

In this chapter we shall learn the methods of tracing a curve in general and the properties of some standard curves commonly met in engineering problems.

* Procedure for tracing curves given in Cartesian form.

Symmetary :

1) If power 'y' is even everywhere in the given eq^n then the curve is symmetry by x-axis.

2) If power 'x' is even everywhere in the given eq^n then curve is symmetry about y-axis.

3) If power of both x & y are even then curve is symmetry about both axis.

4) Symmetary in opposite co-ordinate if replace x by y & y by x eq^n remain unchanged then curve is symmetrical about the line $y = x$

5) If we replace x by $-y$ & y by $-x$ eq^n remains unchanged curve is symmetrical with line or about the line $y = -x$

6) If we replace x by $-x$ & y by $-y$ eq^n remains unchanged, curve is symmetric in opposite quadrant.

Origin

Put $x=0$ in given eq^n, get $y=0$ then the curve passing through origin.

Tangent at origin :

If curve is passing through origin tangent at origin is obtained by equating lowest degree term to origin.

Intersection with co-ordinate axis.

1) On x-axis put $y=0$ in given eq^n then we get point of x-axis.

2) On Y-axis put $x=0$ in given eq^n

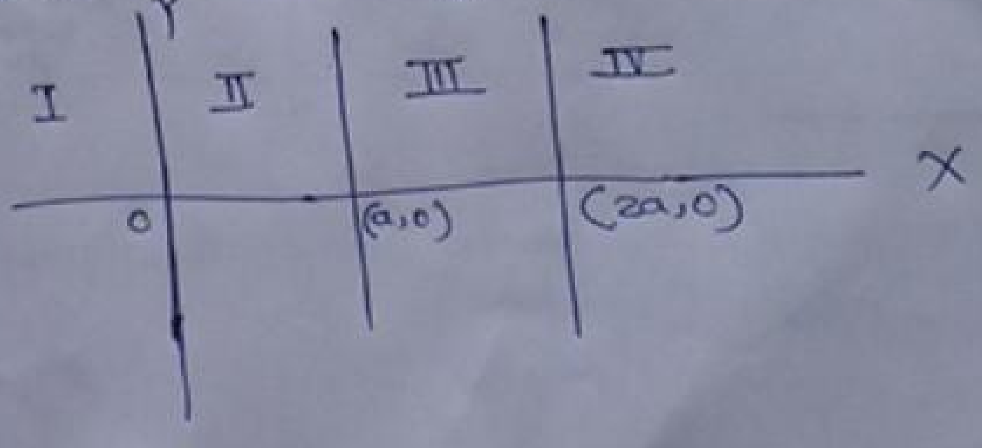
Asymptotes :

Asymptote is tangent which cut the curve at infinite $y^2 \to \infty$ at $x=a$ then $x=a$ is Asymptote parallel to y-axis.

Region of absences

Consider, If y^2 is possitive curve then it present in that region.

If y^2 is negative curve then it absent in that region.



Finding of $\frac{dy}{dx}$

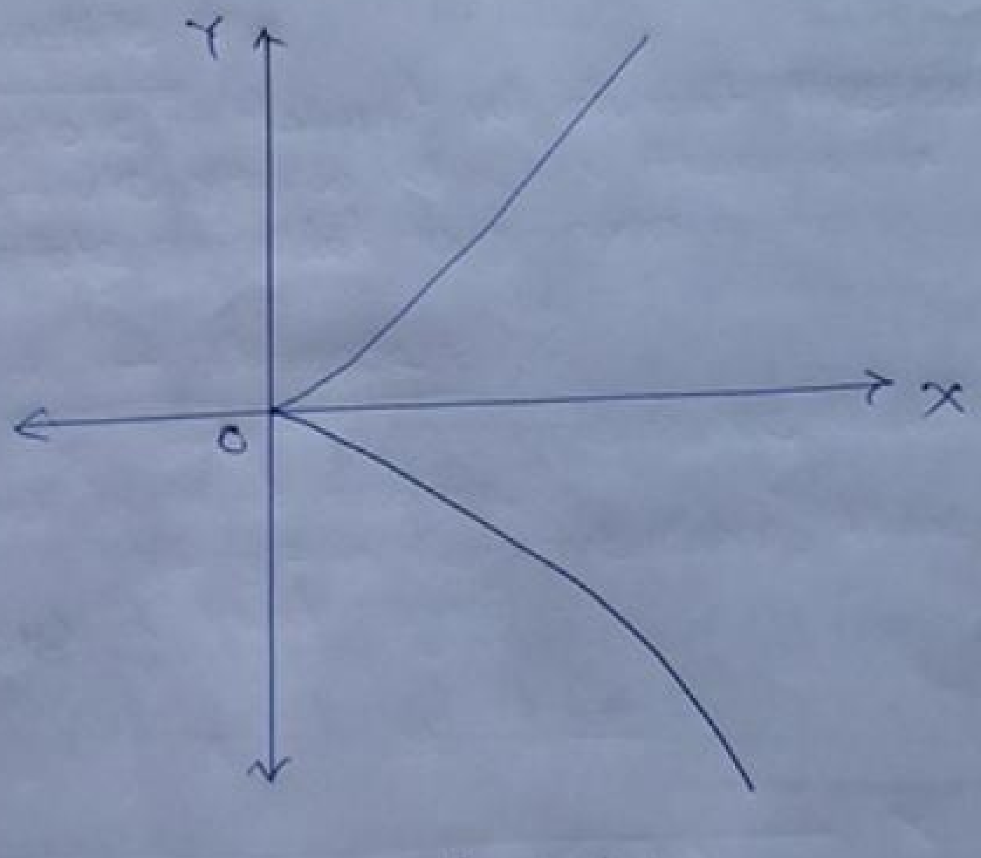
For this we consider intersection with co-ordinate axis.

1) If $\frac{dy}{dx} = 0$ then tangent is at point is parallel to x-axis.

2) If $\frac{dy}{dx} = \infty$ then tangent is at point is parallel to Y-axis.

Sketch:

In this we will draw the traced curve.



(Sample Diagram of sketchcurve or traced curve)

Ex. 1] Trace the curve for,
Semi-cubical parabola $y^2 = ax^3$

$\Rightarrow$ Let given eq^n of curve

$$y^2 = ax^3 \quad \ldots (1)$$

(i) Symmetry :

Power of y is even therefore curve is symmetrical about x-axis.

(ii) Origin :

Put $x = 0$ in eq^n (1)

$\therefore y^2 = a(0)^3 \Rightarrow y^2 = 0$

$\Rightarrow y = 0$

$\therefore$ Curve passing through origin.

(iii) Tangent at origin:

from eq^n (1)

$$\frac{y^2}{2} = \frac{ax^3}{3}$$

Equating lowest degree term to zero

$\therefore y^2 = 0 \Rightarrow y = 0$ (x-axis)

(iv) Intersection with co-ordinate axis.

1) On x-axis put $y=0$ in eq^n (1) we get $x = 0$ $\therefore$ point is $(0,0)$

2) On Y-axis put $x=0$ in eq^n (1) we get $y = 0$ $\therefore$ point is $(0,0)$

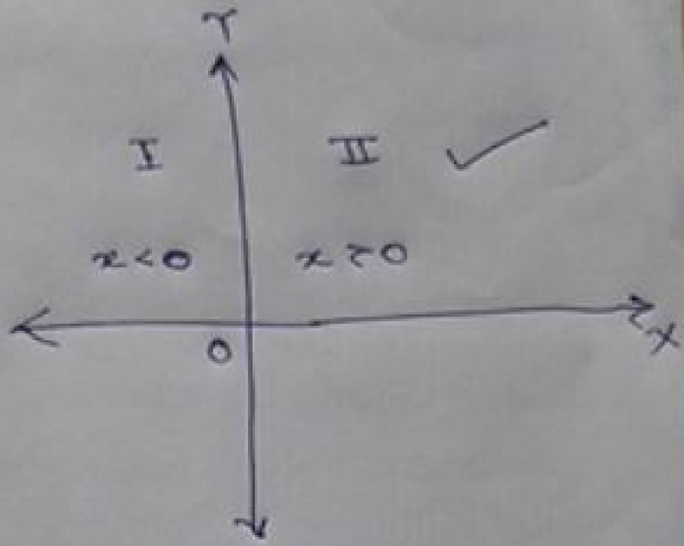
(v) Asymptote:

$$y^2 = ax^3$$

(There is no asymptote)

(vi) Region of absence

let curve eq^n $y^2 = ax^3$



Case I : $x < 0$

take $x = -a$

$\therefore y^2 = a(-a)^3$

$y^2 = -a^4 < 0$

curve is absent

Case II : $x > 0$

take $x = a$

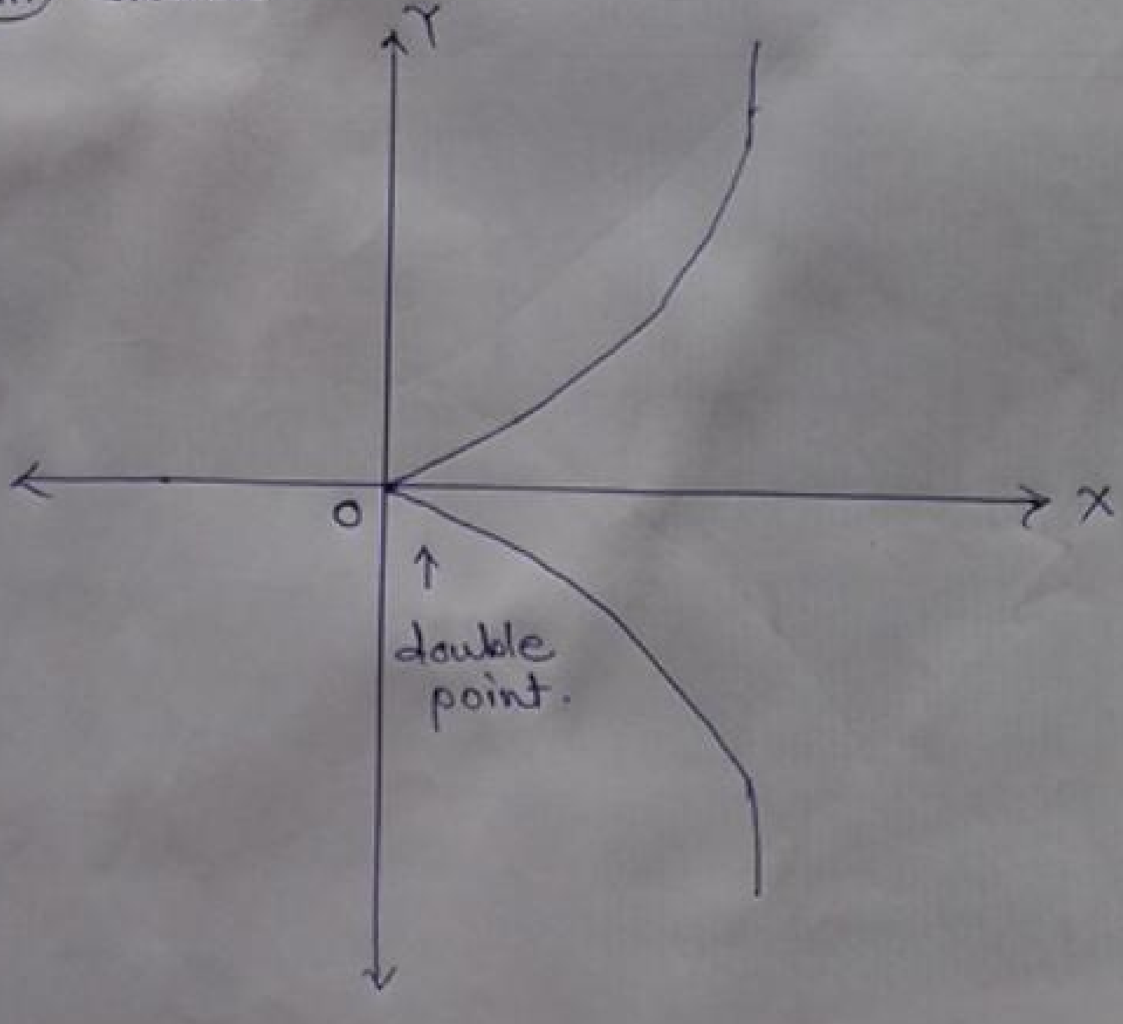
$\therefore y^2 = a(a)^3$

$y^2 = a^4 > 0$

Curve is present

(vii) $\left(\frac{dy}{dx}\right)_{(x,y)} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)}$ - not applicable

(viii) Sketch :



Ex. 2] Trace the curve for,

Cissoid of diocles $y^2(2a-x)=x^3$

$\Rightarrow$ let given eq^n of curve,

$$y^2(2a-x)=x^3 \quad \cdots (1)$$

(i) Symmetry:

Power of y is even

$\therefore$ curve is symmetric about x axis.

(ii) Origin:

Put $x=0$ in eq^n (1)

$y^2(2a-0)=0$

$2ay^2=0$

$\therefore y=0$

$\therefore$ curve passess through origin.

(iii) Tangent at origin:

$$\frac{2ay^2}{2} - \frac{xy^2}{3} = \frac{x^3}{3}$$

lowest degree term is $2ay^2=0$

$\therefore y^2=0$

$\Rightarrow y=0$

$\therefore$ (x-axis) tangent at origin.

(iv) Intersection with co-ordinate axis,

1) On x-axis put $y=0$ in eq^n (1)

$\therefore (0)^2(2a-x)=x^3 \Rightarrow 0=x^3$

$\Rightarrow x=0 \quad \therefore$ point is $(0,0)$

2) On y-axis put $x=0$ in eq^n (1)

$\therefore 2ay^2=0 \Rightarrow y=0$

$\therefore$ point is $(0,0)$

(v) Asymptote: let $y^2 = \dfrac{x^3}{2a-x}$

$y^2 \to \infty$ at $x = 2a$

$\therefore$ $x = 2a$ is an asymptot parallel to y-axis.

(vi) Region of absence:

let curve eq^n is $y^2 = \dfrac{x^3}{2a-x}$

Case I: $x<0$

take $x = -2a$

$\therefore y^2 = \dfrac{(-2a)^3}{2a-(-a)} = \dfrac{(-2a)^3}{2a+a}$

$\therefore$ $x<0$ then y^2 become negative

$\therefore$ Curve is absent.

I	II	III
$x<0$	$0<x<2a$	$x>2a$
negative	✓	negative

Case II: $x>0$

take $x > 2a$

$\therefore$ y^2 become negative curve

$\therefore$ curve is absent

Case II: $0 < x < 2a$

take x as $0 < x < 2a$ for $x = a$

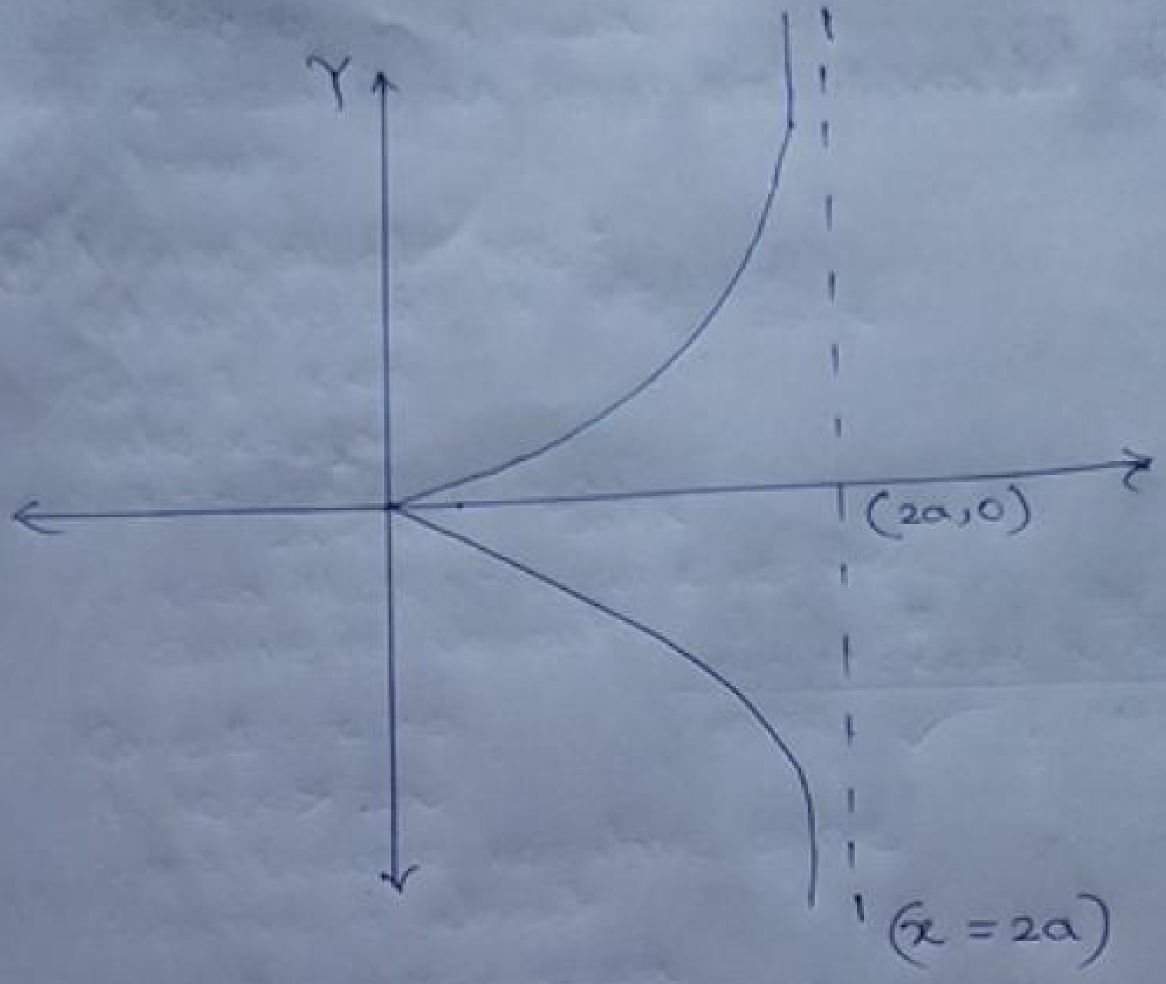
Then $y^2 = \dfrac{a^3}{2a-a} = \dfrac{a^3}{a} = a^2$

$\therefore$ Curve is positive.

$\therefore$ Curve is present.

(vii) $\left(\dfrac{dy}{dx}\right)_{(x,y)} \Rightarrow \left(\dfrac{dy}{dx}\right)_{(0,0)}$ is not applicable

(viii) Sketch



Lecture : 2

Ex. 3) Trace the curve for,

Strophoid $y^2(a+x) = x^2(b-x)$

$\Rightarrow$ let given eq^n of curve,

$$y^2(a+x) = x^2(b-x) \quad \cdots (1)$$

(i) Symmetry :

Powers of y is even so curve is symmetric about x-axis.

(ii) Origin :

Put $x=0$ in eq^n (1)

$\therefore \; y^2(a+0) = (0)^2(b-0)$

$y^2 a = 0$

$\Rightarrow y = 0$

$\therefore$ Curve passing through origin.

(iii) Tangent at origin :

~~Put~~ $y^2(a+x) = x^2(b-x)$

$ay^2 + xy^2 = bx^2 - x^3$

$\underbrace{ay^2}_{2} - \underbrace{bx^2}_{2} = \underbrace{-xy^2}_{3} - \underbrace{x^3}_{3}$

lowest degree term take as zero

$\therefore ay^2 - bx^2 = 0$

$y^2 = \dfrac{bx^2}{a}$

$y = \pm\sqrt{\dfrac{b}{a}}\, x$

This is tangent at origin.

(iv) Intersection with coordinate axis.

1) On x-axis $y=0$ put in eq^n (1)

then we get, $y^2(a+x) = x^2(b-x)$

$$0(a+x) = x^2(b-x)$$
$$0 = x^2(b-x)$$
$$0 = b-x$$
$$x = b$$

$\therefore$ we have point $(b, 0)$

2) On Y-axis $x=0$ put in eq^n (1)

then we get, $y^2(a+0) = (0)^2(b-0)$

$$y^2 a = 0$$
$$y = 0$$

$\therefore$ we have point $(0, 0)$

(v) Asymptote:

let eq^n $y^2(a+x) = x^2(b-x)$

$$y^2 = \frac{x^2(b-x)}{a+x}$$

here $y^2 = \infty$ at $x = -a$

$\therefore$ $x = -a$ is asymtote.

(vi) Region of absence

let curve eq^n $y^2 = \frac{x^2(b-x)}{a+x}$

Case I: when $-a > x$ then y^2 become negative hence curve is absent

Case II: $-a < x < 0$ then y^2 become positive hence curve is present

case III: $0 < x < b$ then y^2 become positive hence curve is present

case IV: $x > b$ then y^2 become negative hence curve is absent.

×	✓	✓	×
$x < -a$	$-a < x < 0$	$0 < x < b$	$x > b$

(vii) $\left(\frac{dy}{dx}\right)_{(x,y)}$

let $y^2(a+x) = x^2(b-x)$

diff. w.r.t. x

$\therefore \; y^2(1)+(a+x)2y\frac{dy}{dx} = x^2(-1)+(b-x)2x$

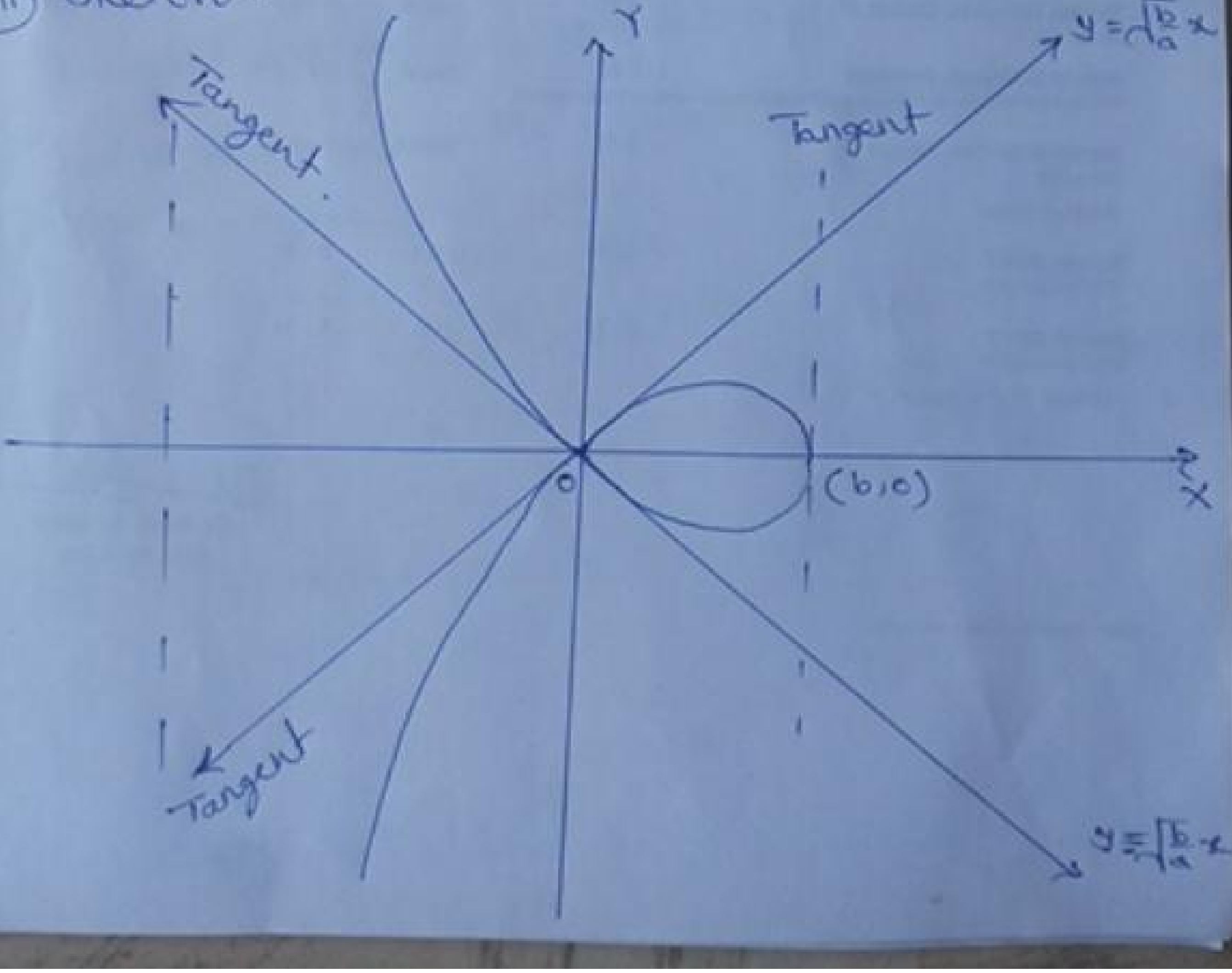
$$\therefore \frac{dy}{dx} = \frac{-x^2+2xb-2x^2-y^2}{2y(a+x)}$$

$$\frac{dy}{dx} = \frac{2xb-3x^2-y^2}{(a+x)2y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(b,0)} = \infty$$

$\therefore$ Tangent at point $(b,0)$ is parallel to y axis.

(viii) Sketch:



Ex. 1) Trace the curve for,

Astroid $x^{2/3} + y^{2/3} = a^{2/3}$

$\Rightarrow$ let given eq^n of curve,

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\Rightarrow (x^{1/3})^2 + (y^{1/3})^2 = a^{2/3}$$

(i) Symmetry:

Power of x & y are both even

$\therefore$ curve is symmetric about both x & y axis.

(ii) Origin:

Put $x = 0$ in eq^n (1)

$\therefore y^{2/3} = a^{2/3}$

$y \neq 0$

$\therefore$ Curve not passing through origin.

(iii) Tangent at origin:

tangent at origin is not applicable.

(iv) Intersection with co-ordinate axis.

1) On x-axis put $y = 0$ in eq^n (1)

$$\therefore x^{2/3} = a^{2/3}$$

$$\Rightarrow (x^2)^{1/3} = (a^2)^{1/3}$$

$$\Rightarrow x^2 = a^2$$

$$\Rightarrow x = \pm a$$

$\Rightarrow$ The points are $(a, 0)$, $(a, 0)$

2) On y-axis put $x = 0$ in eq^n (1)

$$\therefore y^{2/3} = a^{2/3}$$

$$\Rightarrow y^2 = a^2$$

$$\Rightarrow y = \pm a$$

$\therefore$ The points are $(0, a)$, $(0, -a)$

(v) Asymptote :
Not applicable.

(vi) Region of absence.
Neither x nor y can be greater than a.

$x < a$	$x = -a$	$x = a$	$x > a$
✗	✓	✓	✗

(vii) $\left(\frac{dy}{dx}\right)_{(x,y)}$

let $x^{2/3} + y^{2/3} = a^{2/3}$

diff. w.r.t. x

$\frac{2}{3} x^{2/3 - 1} + \frac{2}{3} y^{2/3 - 1} \frac{dy}{dx} = 0$

$\frac{2}{3}\left(x^{-1/3} + y^{-1/3}\frac{dy}{dx}\right) = 0$

$\Rightarrow x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}}$

$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$

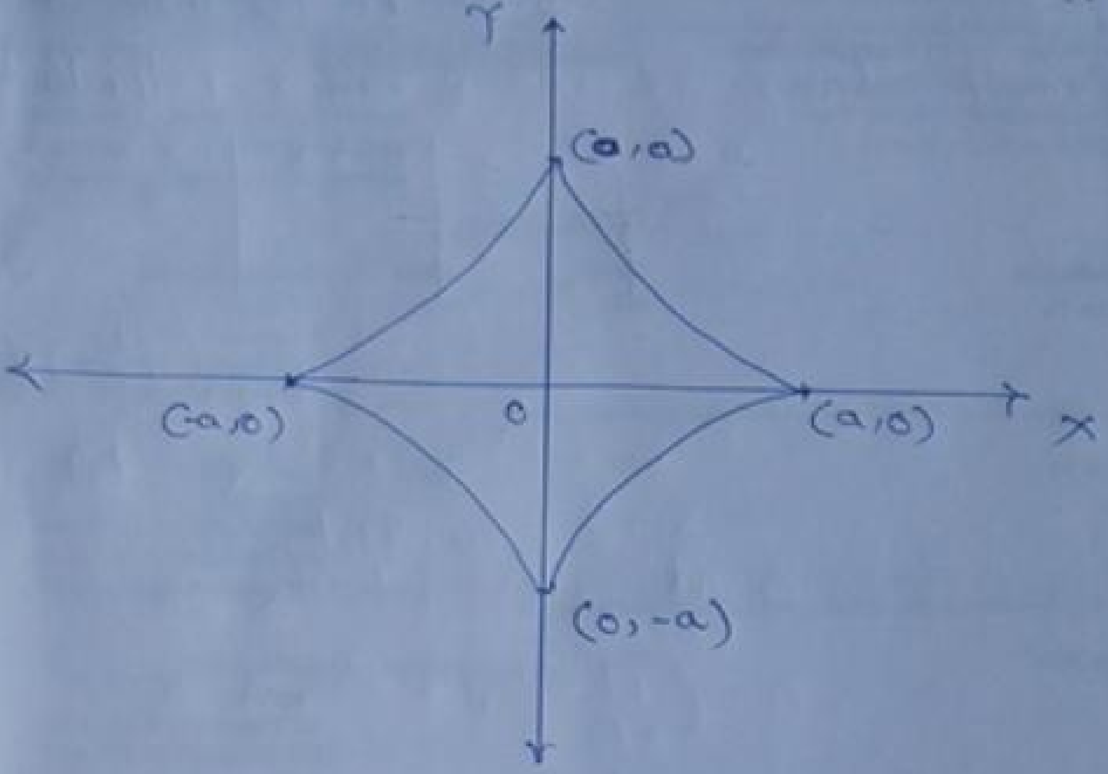
$\therefore \left(\frac{dy}{dx}\right)_{(\pm a, 0)} = \frac{0}{(\pm a)^{1/3}} = 0$

$\therefore$ tangent at point $(a,0), (-a,0)$ parallel to x-axis.

$\therefore \left(\frac{dy}{dx}\right)_{(0,\pm a)} = \frac{(\pm a)^{1/3}}{0} = \infty$

$\therefore$ tangent at point $(0,a), (0,-a)$ parallel to y-axis.

(viii) Sketch:



Ex.5) Trace the curve for,
Witch of Agnesi $xy^2 = a^2(a-x)$

$\Rightarrow$ let given eq^n of curve,

$$xy^2 = a^2(a-x) \quad \cdots ①$$

(i) Symmetry:
Power of y is even so cure is symmetric about x-axis.

(ii) Origin:
Put $x=0$ in eq^n ①

$\therefore y^2 = \infty \qquad \therefore y \neq 0$

Curve doesnt passes through origin.

(iii) Tangent at origin:
not applicable.

(iv) Intersection of Co-ordinate axis,

1) On x-axis put $y=0$ in eq^n ①

$$\therefore xy^2 = a^2(a-x)$$
$$y^2 = \frac{a^2(a-x)}{x}$$
$$0 = \frac{a^2(a-x)}{x}$$
$$0 = a^2(a-x)$$
$$\Rightarrow (a-x) = 0$$
$$\therefore a = x$$
$$\Rightarrow x = a$$

$\therefore$ The point is $(a,0)$

2) On Y-axis put $x=0$ in eq^n ①

$$\therefore y^2 = \frac{a^2(a-x)}{x}$$
$$y^2 = \infty$$

$\therefore$ The point is not on y axis.

(v) Asymptote :

let $xy^2 = a^2(a-x)$

$\Rightarrow \quad y^2 = \dfrac{a^2(a-x)}{x}$

$\therefore$ at $x = 0 \quad y^2 \to \infty$ (y-axis)

$\therefore$ Y-axis is Asymptote.

(vi) Region of absence. let $y^2 = \dfrac{a^2(a-x)}{x}$

Case I : $x < 0$

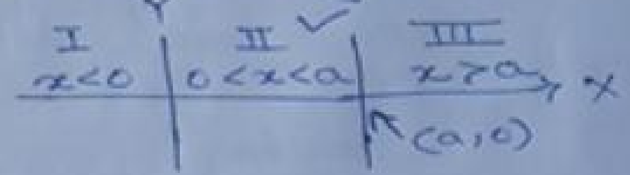
$\therefore$ The curve is negative $\therefore$ curve is absent

Case II : $0 < x < a$

$\therefore$ The curve is positive $\therefore$ curve is present

case III : $x > a$

$\therefore$ The curve is negative $\therefore$ curve is absent



(vii) $\left(\dfrac{dy}{dx}\right)_{(x,y)}$

let $xy^2 = a^2(a-x)$

diff. w.r.t. x

$x(2y)\dfrac{dy}{dx} + y^2 = a^2(0-1)$

$2xy\dfrac{dy}{dx} = -a^2 - y^2$

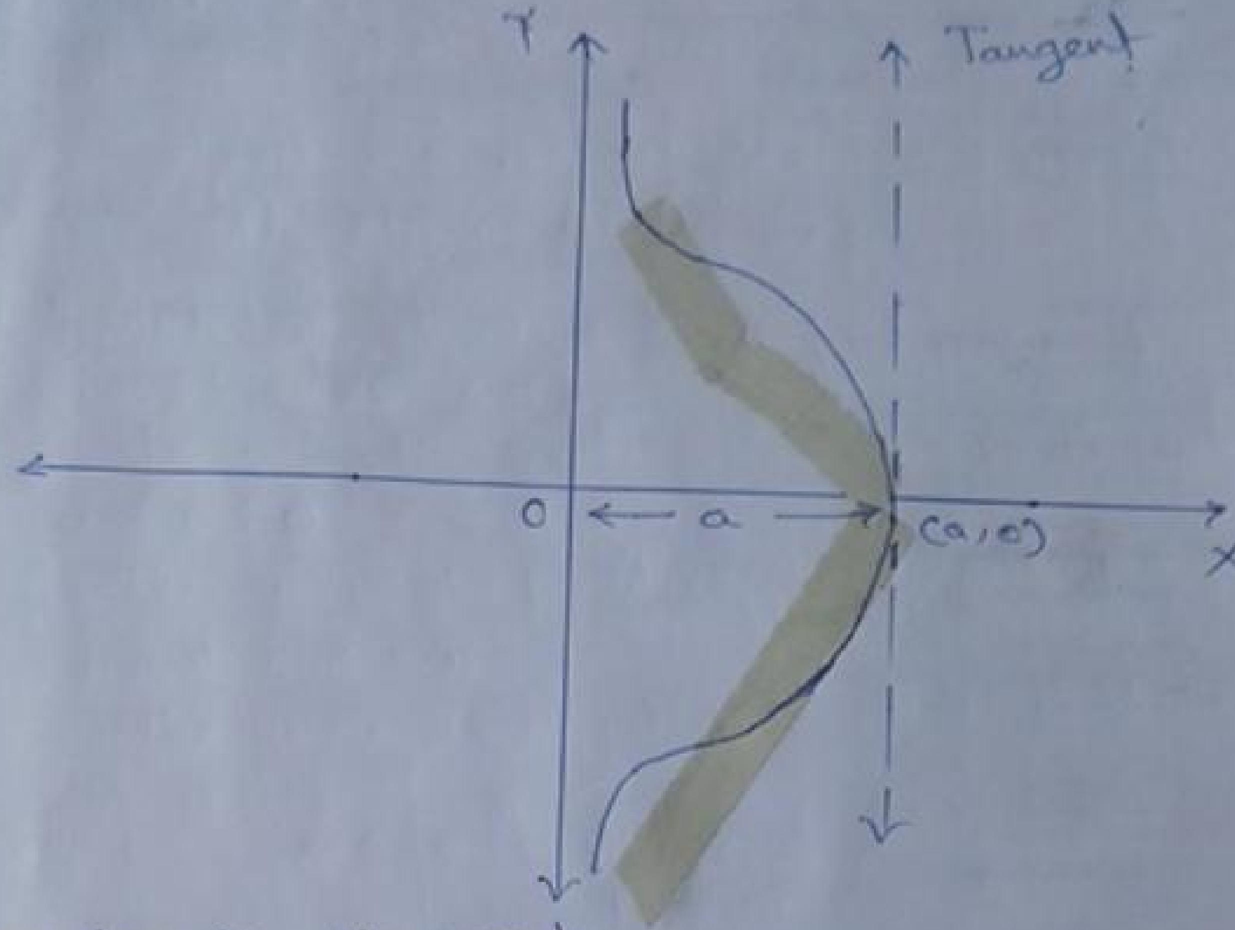
$\dfrac{dy}{dx} = \dfrac{-(a^2+y^2)}{2xy}$

$\left(\dfrac{dy}{dx}\right)_{(a,0)} = \dfrac{-(a^2+0)}{0}$

$= \infty$

$\therefore$ Tangent at point $(a,0)$ is parallel to y-axis

(viii) Sketch :



Examples For Homework.

1) Trace the curve (Strophoid)

$$y^2(a+x) = x^2(3a-x)$$

2) Trace the curve

$$a^2x^2 = y^3(2a-y)$$

3) Trace the curve (Astroid)

$$x = a\cos^3\theta \;;\; y = a\sin^3\theta$$

4) Stress the curve,

$$y(a+x) = x^2(3-x)$$

5) Stress the curve

$$a^2y^2 = x^2(a^2-x^2)$$

Lecture No: 3

Ex.6. Trace the curve for, $y^2(a^2-x^2)=a^3x$

$\Rightarrow$ Points: i) Symmetrical about x-axis

ii) Curve passes through origin.

iii) $y^2 = \dfrac{a^3x}{a^2-x^2} \rightarrow \infty$ if $x=a$ & $x=-a$

hence lines $x=a$ and $x=(-a)$ are asymptotes to the curve.

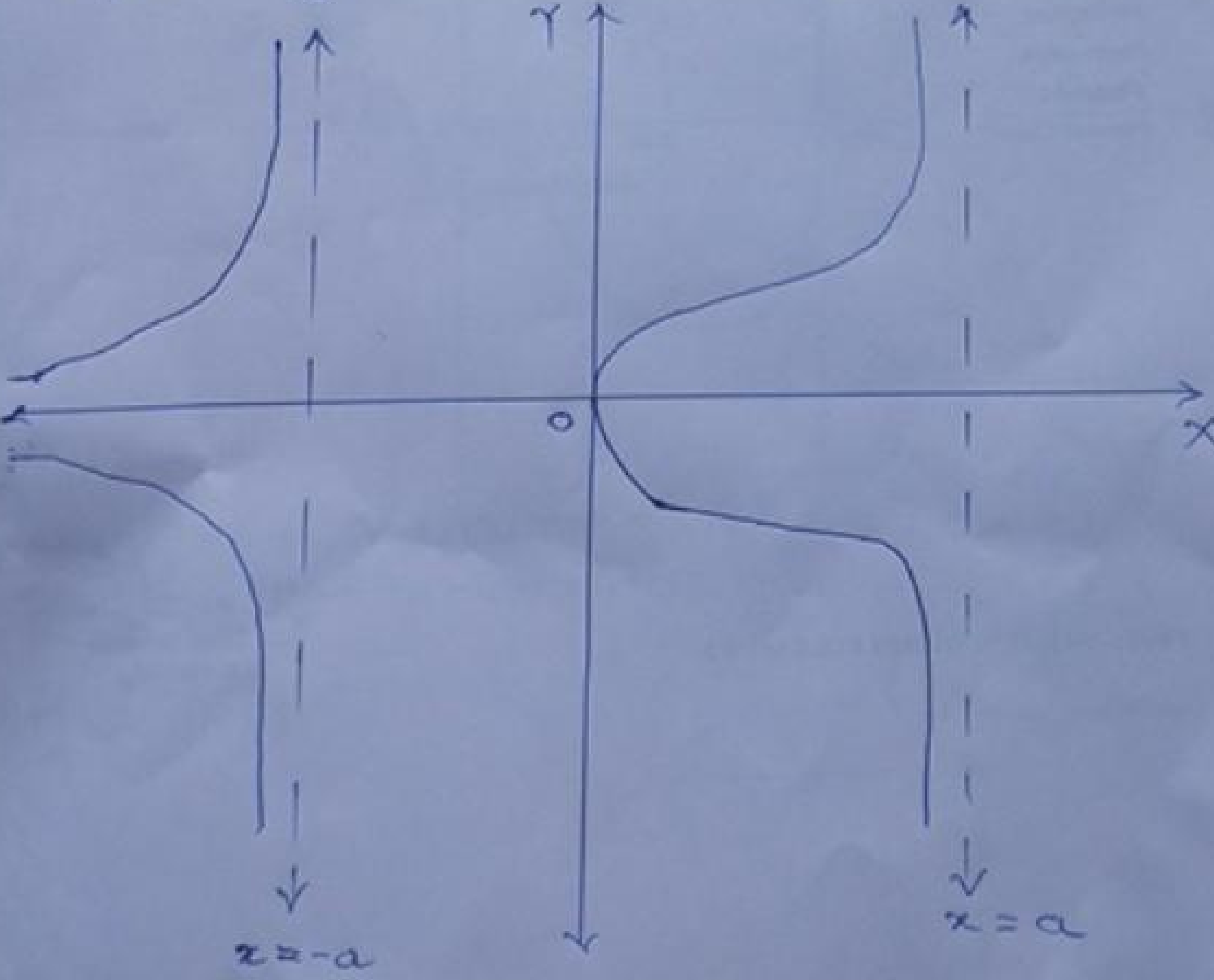
iv) If $x>0$ or $x>-a$ then y^2 is negative

$\Rightarrow y$ is imaginary

$\therefore$ Curve does not exist for $x>a$ & $x>-a$

v) If $x<(-a)$ then y^2 is positive and hence also it exist to infinity.

vi) At origin Y-axis is tangent to curve



Ex. 7 Trace the curve for $y^2 = x^5(2a-x)$

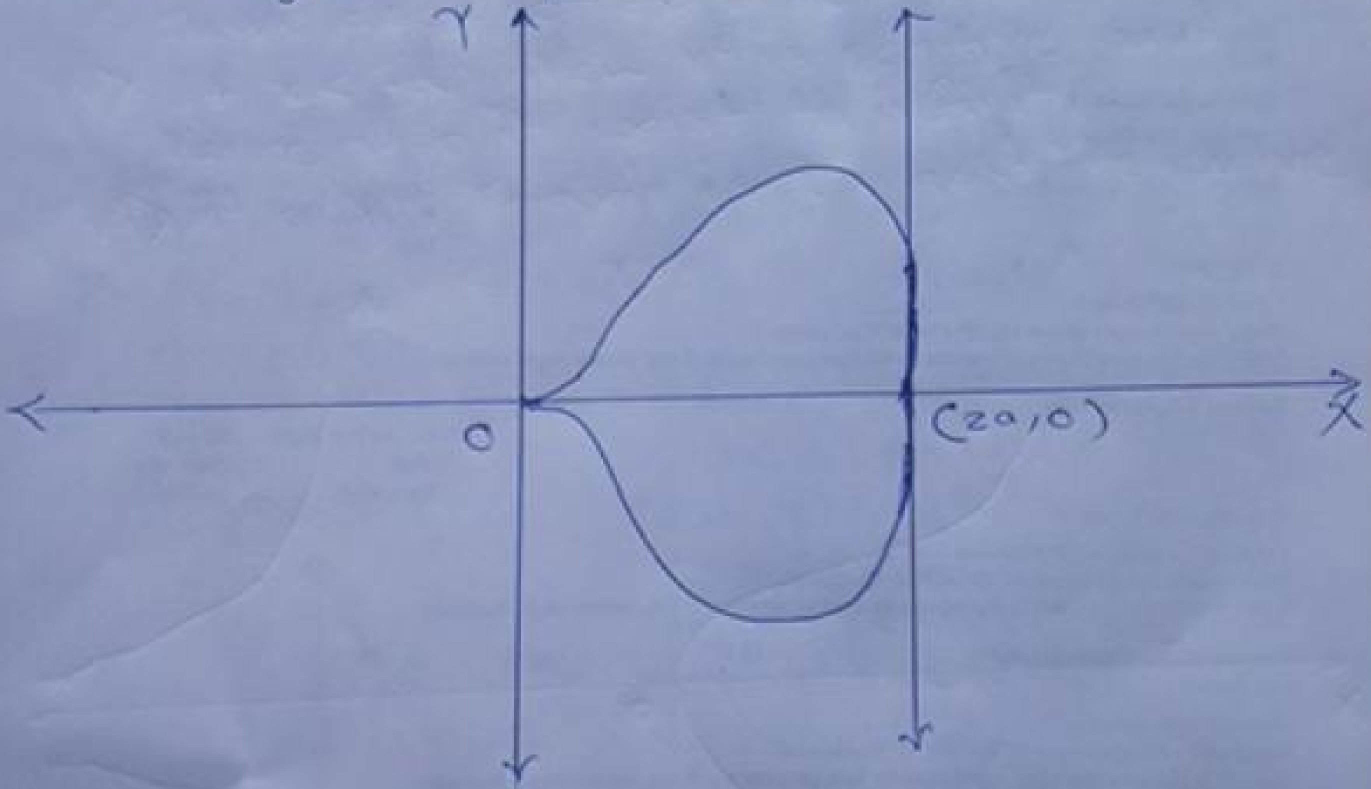
⇒ Points : i) Symmetric about x-axis.

ii) Curve passes through $(0,0)$ origin

iii) x-axis is tangent at origin

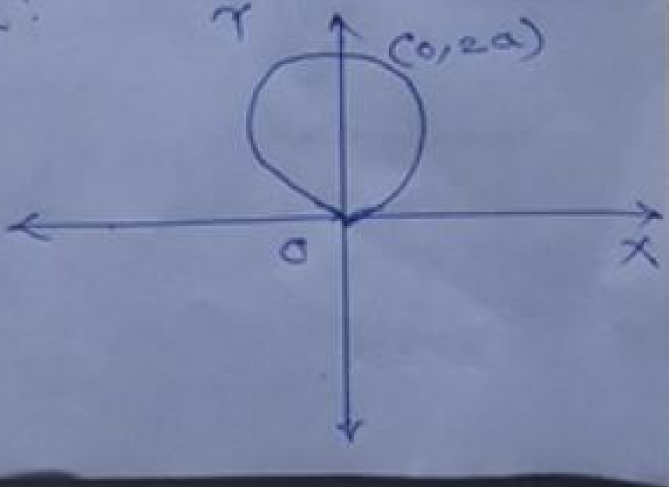
iv) Intersects x-axis at $(0,0)$ & $(2a,0)$

v) If $x<0$ and $x>2a$ then y^2 becomes negative hence curve does not exists before Y-axis and after the line $x=2a$ or beyond the point $(2a,0)$ on x-axis.



Ex. 8. Trace the curve $a^2x^2 = y^3(2a-y)$

⇒ Treat as homework :



Ex. 8 Trace the curve for,
Folium of Descartes $x^3 + y^3 = 3axy$.
$\Rightarrow$ where $a > 0$

Points : i) If we interchange x by y & y by x. curve remains same, ~~#~~ and hence its symmetrical about the line $y = x$

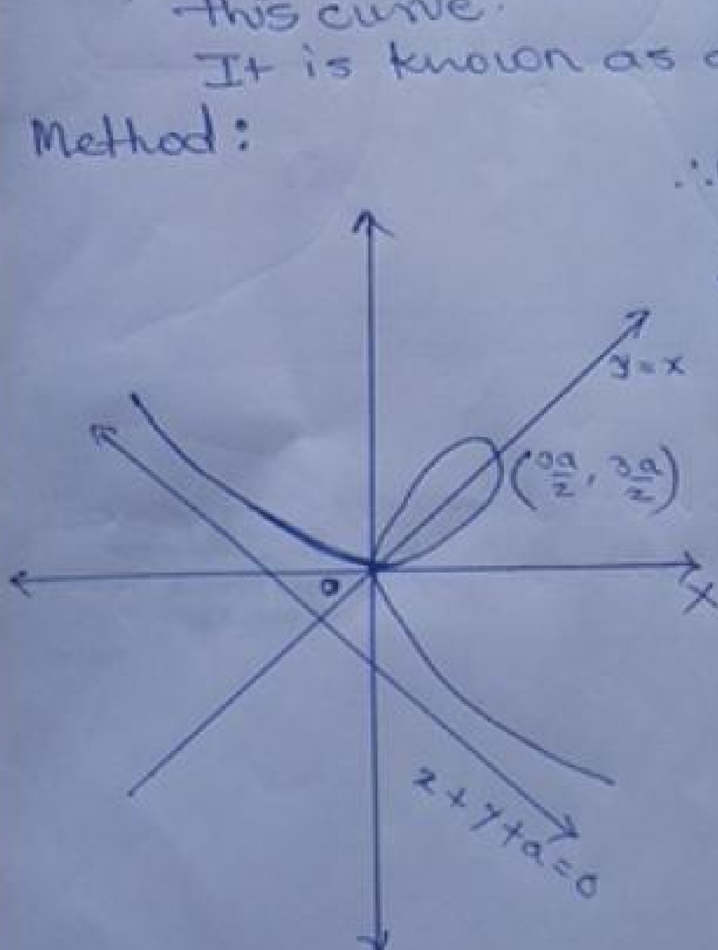
ii) Curve passes through $(0,0)$ and intersect the line ~~or~~ axis of symmetry at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

iii) Equating lowest degree term to zero
$3axy = 0$
$\Rightarrow x = 0$ as well as $y = 0$ ~~#~~ both X & Y axis are tangents to curve at $(0,0)$

iv) x and y both cannot be negative hence curve does not lies in third quadrant

v) The line $x + y + a = 0$ is an asymptote to this curve.
It is known as an oblique asymptote.

Method:



let $y = mx + c$ be the asymptote
$\therefore$ Put y in eq^n of curve
$x^3 + y^3 = 3axy$
$\Rightarrow x^3 + (mx + c)^3 = 3ax(mx+c)$
simpliting $\Rightarrow$
$(1+m^3)x^3 (3m^2c - 3am)x^2 + c^1 x + c^3 = 0$ $(3c^2m - 3ac)$
Equating coefficient of x^3 and x^2 to zero
$(1+m^2) = 0$ & $(3m^2c - 3am) = 0$
$\Rightarrow m = -1$ & $c = \frac{a}{m} \Rightarrow c = -a$
$\therefore y = mx + c$
$\Rightarrow y = -x - a$
$\Rightarrow x + y + a = 0$ is the line as an oblique asymptote.

Ex. 9. Trace the curve for, $y^2 = (x-a)(x-b)^2$

$\Rightarrow$ Point : i) Symmetrical about x - axis

ii) For $x = 0$, $y \neq 0$ hence curve does not passes through origin.

iii) For $x = a$ & $x = b$ $\Rightarrow y = 0$

$\therefore$ curve intersects x - axis at $(a,0)$ & $(b,0)$

iv) If $x < a$ then y^2 is negative

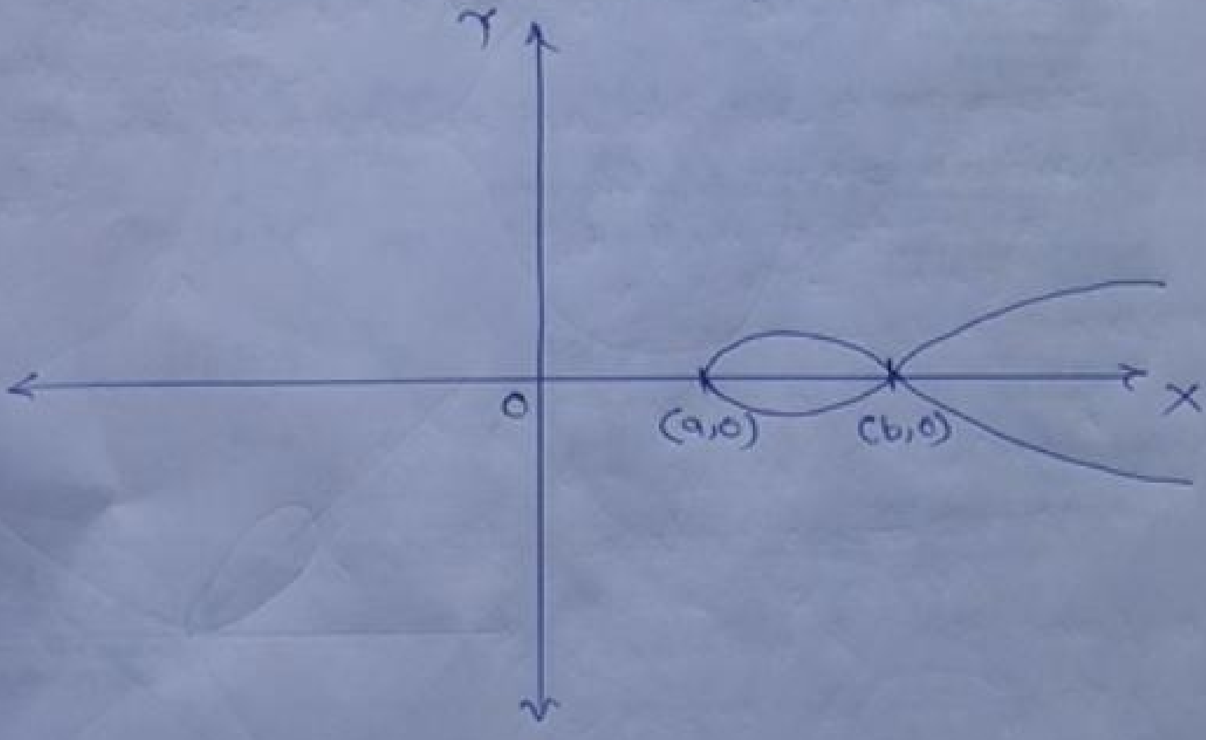
$\Rightarrow y$ is imaginary

$\therefore$ curve does not exist before point $(a,0)$

v) If $x > b$ then y increases

hence the curve is

vi) Sketch :



Ex. 10) Trace the curve for,

$\Rightarrow$ "Bernoulli's Lamniscate" $x^2(x^2+y^2) = a^2(x^2-y^2)$

Points: i) Symmetrical about both x & y axis.

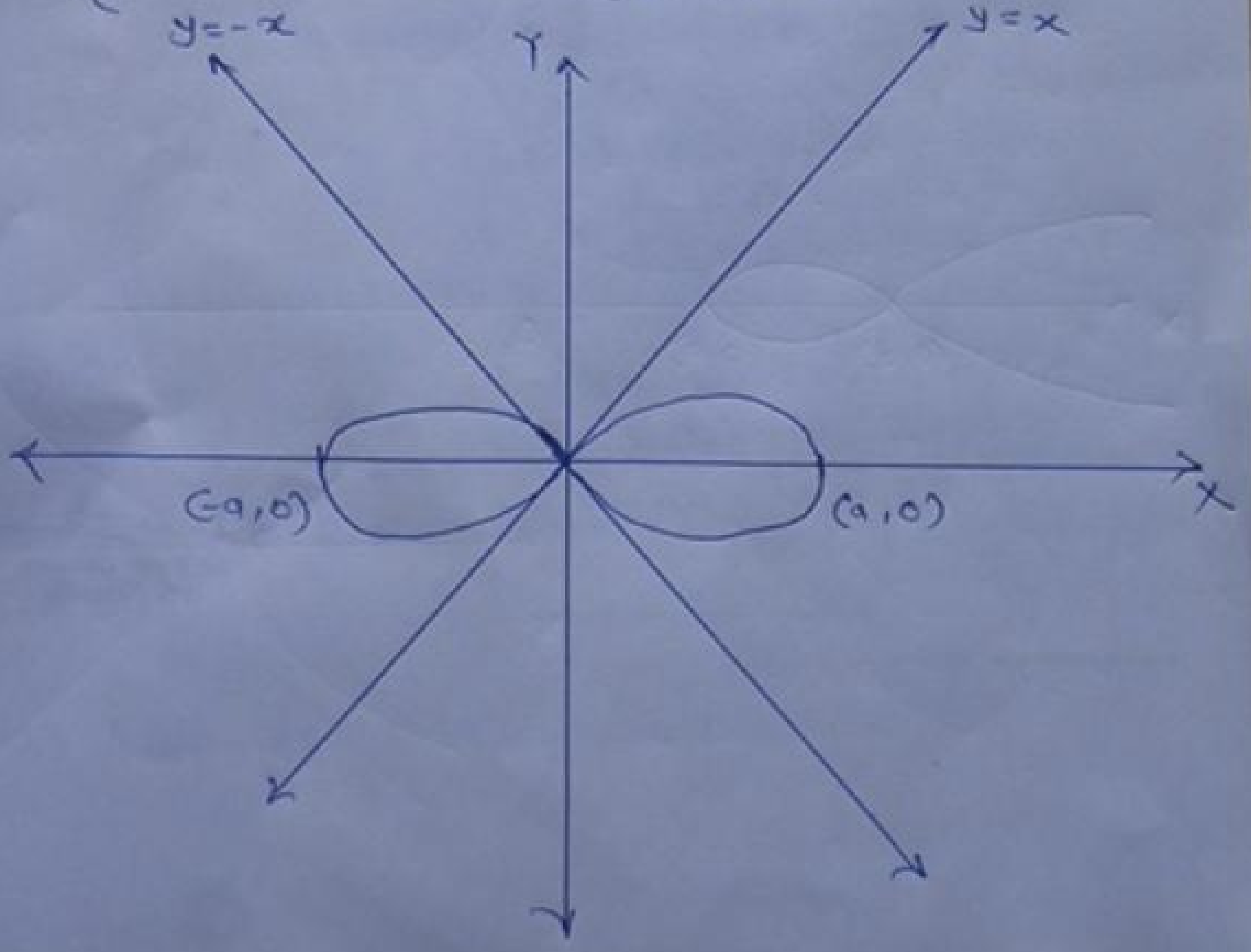
ii) Passes through origin.

iii) The lines $y = \pm x$ are tangent at point $(0,0)$

iv) For $x > a$ and $x < -a$ y^2 become negative Therefore curve does not exist before point $(-a,0)$ & after point $(a,0)$ on x axis

Note: Solving explicity for y^2 we get

$$\Rightarrow x^2(x^2+y^2) = a^2(x^2-y^2)$$
$$\Rightarrow x^2 \cdot x^2 + x^2y^2 = a^2x^2 - a^2y^2$$
$$\Rightarrow x^2y^2 + a^2y^2 = a^2x^2 - x^2x^2$$
$$\Rightarrow y^2(x^2+a^2) = x^2(a^2-x^2)$$
$$\Rightarrow y^2 = \frac{x^2(x^2-a^2)}{(x^2+a^2)}$$



Ex. 10) Trace the curve for,

Common Catenary $y = c \cdot \cosh\left(\frac{x}{c}\right)$

⇒ * for given eq^n put $x = -x$, then curve is symmetrical about opposite axis.

i) Symmetry put $x = -x$

$\therefore y = c \cdot \cosh\left(-\frac{x}{c}\right)$ ... ①

$y = c \cosh\left(\frac{x}{c}\right)$ ... property.

$\therefore$ curve is symmetric about y - axis.

ii) Origin: put $x = 0$ in ①

$y = c \cdot \cosh\left(\frac{0}{c}\right)$

$= c \cosh(0)$

$y = c \neq 0$

$\therefore$ Curve is not passes through origin.

iii) Tangent at origin not applicable.

iv) Intersection with co-ordinate axis.

1) On X-axis put $y = 0$ in ①

$\therefore 0 = c \cosh\left(\frac{x}{c}\right)$

$\Rightarrow \cosh\left(\frac{x}{c}\right) = 0$

$\frac{x}{c} = \cos^{-1} h(0)$

$x = c \cdot \cos^{-1} h(0)$

$\therefore$ No point on x - axis.

2) On Y - axis

put $x = 0$ in eq^n ①

$\therefore y = c \cdot \cosh\left(\frac{0}{c}\right)$

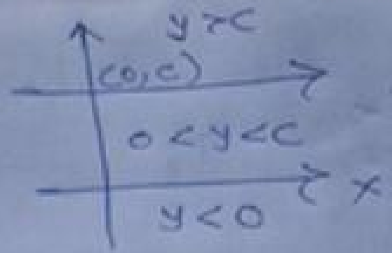
$y = c \cdot \cosh(0)$

$y = c$

Point is $(0, c)$

v) Asymptote: There is no asymptote.

vi) Region of absence:
when $y > c$ get L.H.S & R.H.S
curve is present.



vii) find $\left(\frac{dy}{dx}\right)_{(x,y)}$

$$\therefore \quad y = c \cdot \cosh\left(\frac{x}{c}\right)$$

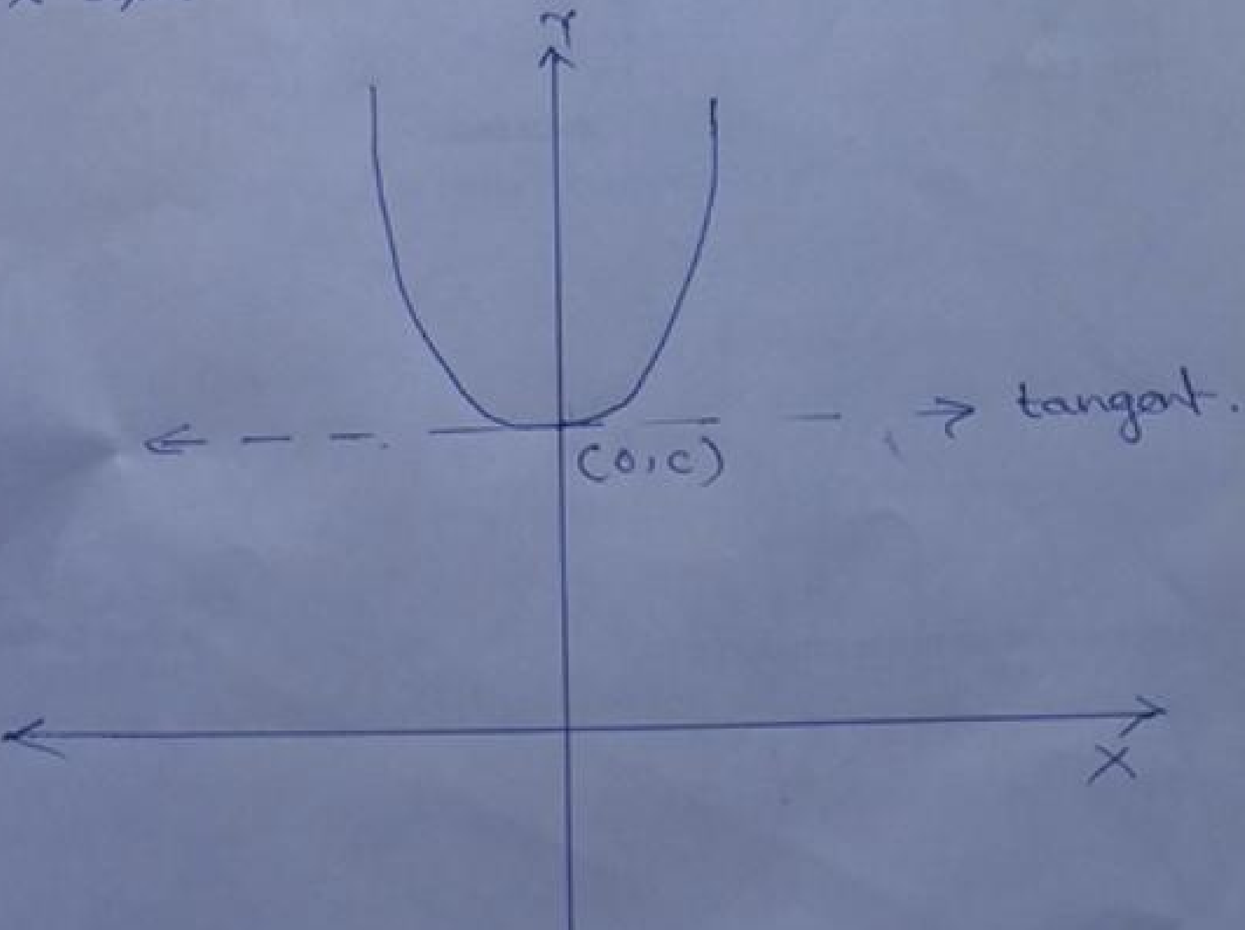
$$\frac{dy}{dx} = c \cdot \sinh\left(\frac{x}{c}\right) \times \frac{1}{c}$$

$$= \sinh\left(\frac{x}{c}\right)$$

$$\left(\frac{dy}{dx}\right)_{(0,c)} = \sinh\left(\frac{0}{c}\right)$$

$$= 0$$

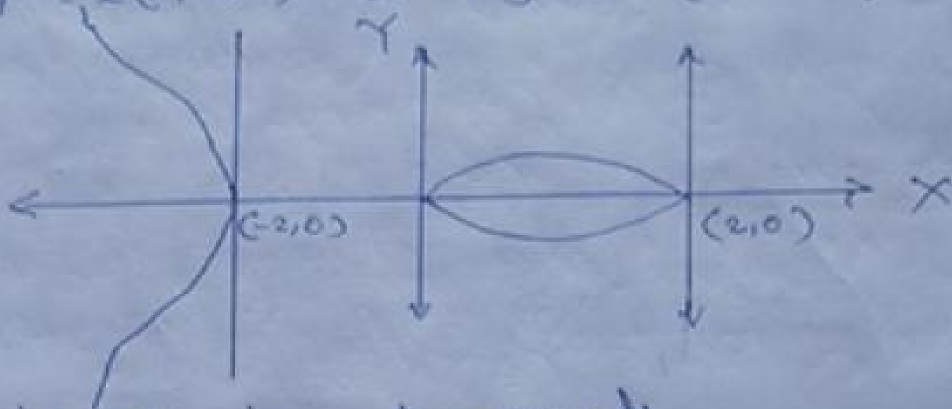
$\therefore$ tangent at point $(0,c)$ is parallel to x-axis.



Examples for homework.

Ex. Trace the curve,

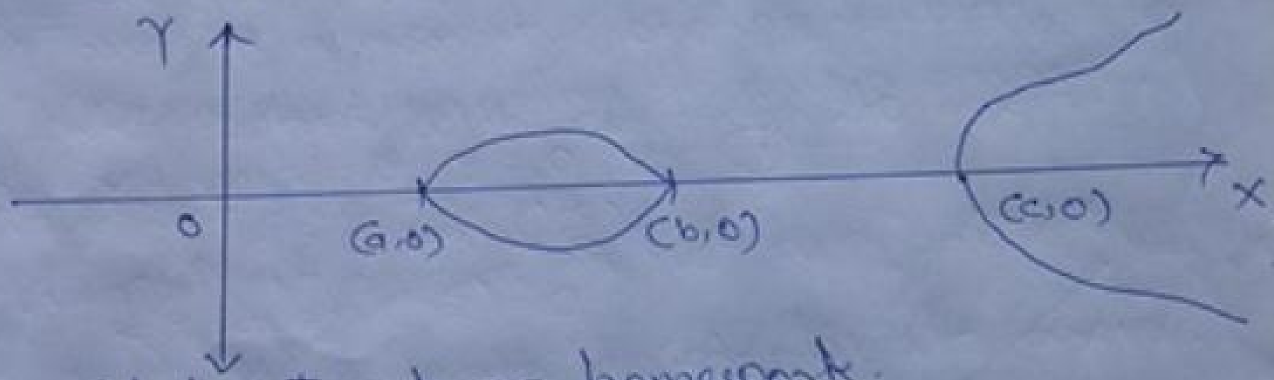
$2y^2 = x(4-x^2)$ or $ay^2 = x(a^2-x^2)$



Points : Treat as homework

Ex. Trace the curve

$y^2 = (x-a)(x-b)(x-c)$ if $a<b<c$



Points : treat as homework.

Ex. $y^2 = (x-a)^2(x-c)$; if $a=b$

Ex. $y^2 = (x-a)^3$; if $a=b=c$

Ex. $x^2+4x-4y+16=0$

$\Rightarrow (x^2+4x+4) = 4y-12$

$\Rightarrow (x+2)^2 = 4(y-3)$

Put $(x+2) = X$ & $(y-3) = Y$

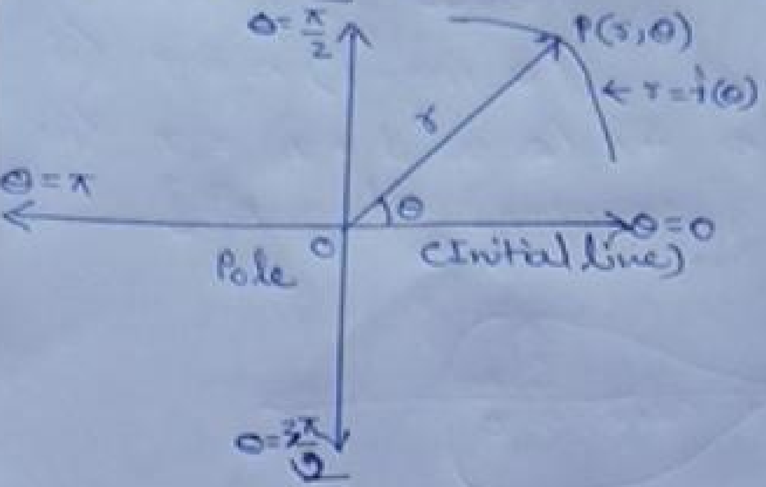
$\Rightarrow X^2 = 4Y \Rightarrow$ Parabola symmetry about y axis $(-2,3)$ point at vertex.

Ex. $y^2+4x-4y+8=0 \Rightarrow Y^2 = -4X$ (parabola)

Ex. $y^2 = x\left(1-\frac{x}{3}\right)^2$

Lecture No: 4

* Tracing of curves in ~~cartesian~~ Polar form:



Eq^n of polar curve is given by $r = f(\theta)$ where θ is angle made by radius vector $\vec{OP}$ with initial line $\theta = 0$; r is the length of $\vec{OP}$ & $P(r,\theta)$ be any point of the curve $r = f(\theta)$

* Rules of curve tracing in polar form:

\# Symmetry: i) After replacing θ by $(-\theta)$ if curve remains same then it is symmetric about Initial line $\theta = 0$

ii) If we replace θ by $(-\theta)$ and r by $(-r)$ simultaneously then eq^n of curves remains same then it is symmetrical about the line $\theta = \frac{\pi}{2}$ i.e. line $\perp^{lar}$ to initial line at pole.

iii) If we replace r by $(-r)$ and curve remains unchanged then it is symmetrical about the pole.

\# Pole: If for some value of θ, $r = 0$ then curve passes through pole or Pole lies on curve.

\# Tangent at pole: The value of θ for which $r = 0$ is the eq^n of tangent to curve at pole.

Limitations : Find out maximum value of 'r' for some value of θ and minimum value of 'r' also. This indicates curve lies entirely within the circle of maximum value of r and out side the circle of minimum value of r.

Find Φ , this is angle between radius vector and tangent to the curve. by using $\tan \Phi = \left(r \frac{d\theta}{dr}\right)$

If possible transform polar curve to cartesian curve and then trace the curve

sketch:

Ex. 1) Trace the curve for,

$\Rightarrow$ "Cardioid" $\boxed{r = a(1+\cos\theta)}$

Points: i) If θ is replaced by $(-\theta)$ then curve remains unchanged.

$\therefore$ It's symmetrical about the line initial line $\theta = 0$

ii) For $\theta = \cancel{0} \Rightarrow r = 2a$

$\theta = 0 \Rightarrow r = 2a$ and when $\theta = \pi$ then $r = 0$

Also when $\theta = \frac{\pi}{2}$ then $r = a$

Also when $\theta = \frac{3\pi}{2}$ then $r = -a$

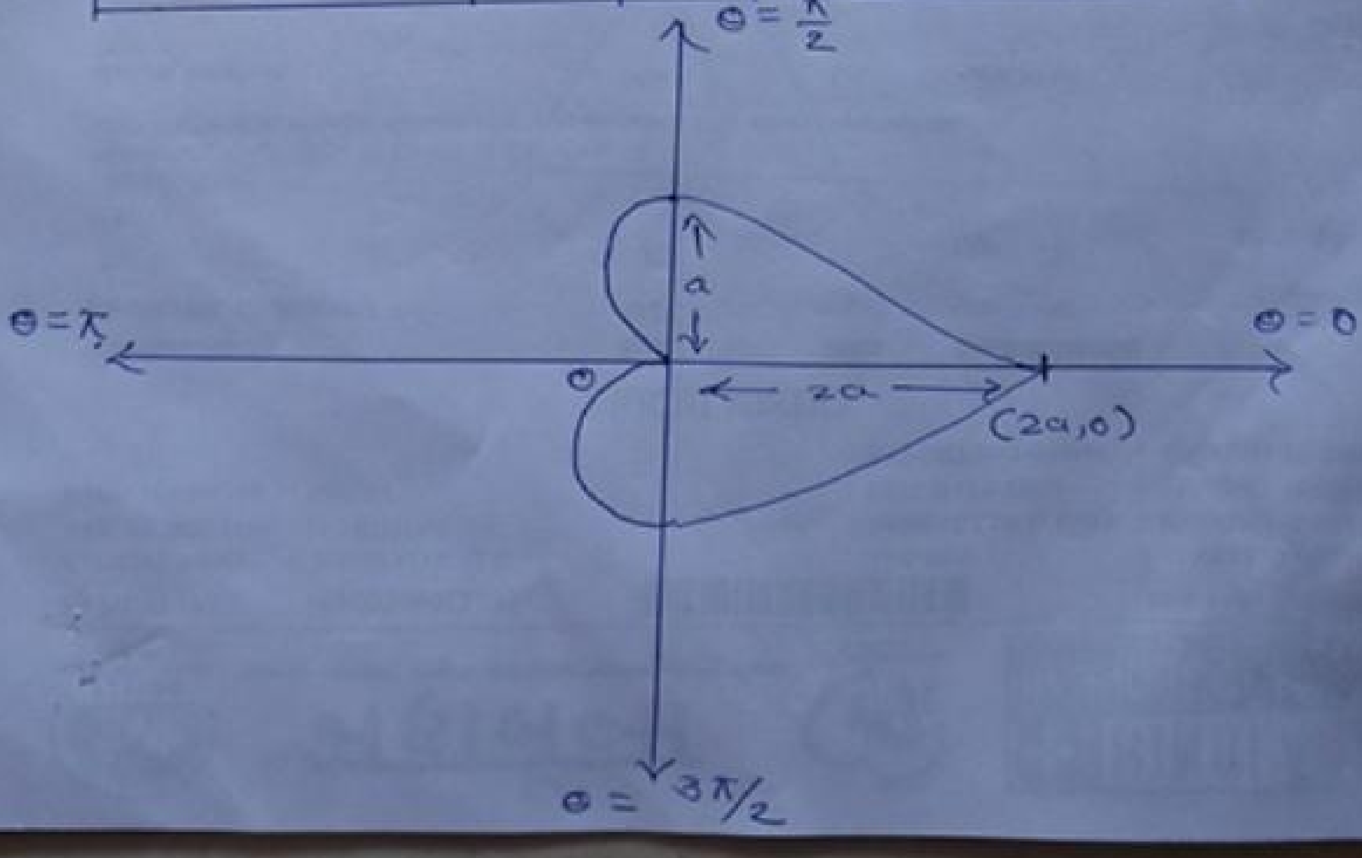
iii) $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$

$\therefore \tan\Phi = -\cot$

For $\theta = 0 \Rightarrow r = 2a$ & $\theta = \pi \Rightarrow r = 0$ hence curve entirely lies within a circle of radius $2a$

iv) Table of values of r & θ

θ	0	$\pi/2$	π	$3\pi/2$	2π
$r = a(1+\cos\theta)$	$2a$	a	0	$-a$	$2a$



Ex. 2 Trace the curve for, $\boxed{r = a(1+\sin\theta)}$

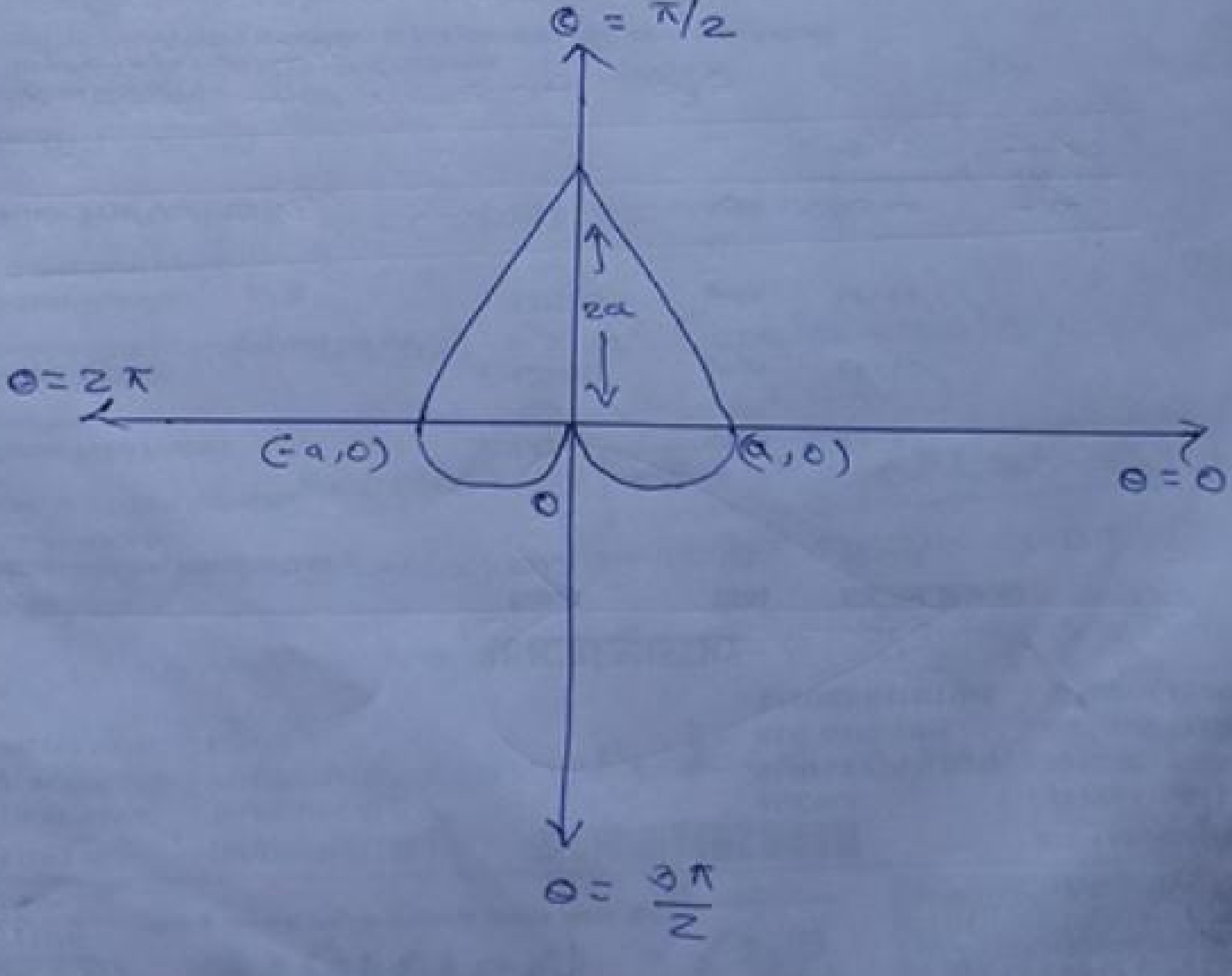
⟹ Points: i) If we replace θ by $-\theta$ eqn of curve changes its form hence its symmetrical about the line $\theta = \frac{\pi}{2}$

ii) Table of values of r & θ

θ	0	$\pi/2$	π	$3\pi/2$	2π
$r = a(1+\sin\theta)$	a	$2a$	$-a$	0	a

Minimum value of $r = 0$ & maximum value of $r = 2a$

iii) The line $\theta = \frac{3\pi}{2}$ is tangent to the curve at pole and pole is at tangent.



Ex. ⇒ Trace the curve for,

"Bernoulli's Lamniscate" $\boxed{r^2 = a^2\cos(2\theta)}$

Points: i) If we replace r by $(-r)$ curve remains unchanged hence its symmetrical about pole

Also it is symmetrical about initial line as well as about line $\perp^{lar}$ to initial line.

ii) If $\theta = 0$ & $\pi \Rightarrow r = a$, $r = a$ resp.

If $\theta = \frac{\pi}{4}$ then $r = 0$

$\therefore$ Pole lies non curve.

at $\theta = \frac{\pi}{4}$ & $\theta = \frac{3\pi}{4} \Rightarrow r = 0$

$\therefore$ These are tangents to the curve

iii) Table of values of r & θ

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$r^2 = a^2\cos(2\theta)$ $r = a\sqrt{\cos(2\theta)}$	a	0	img	0	a

iv) Put $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$

and $a^2\cos(2\theta) \Rightarrow a^2(\cos^2\theta - \sin^2\theta)$

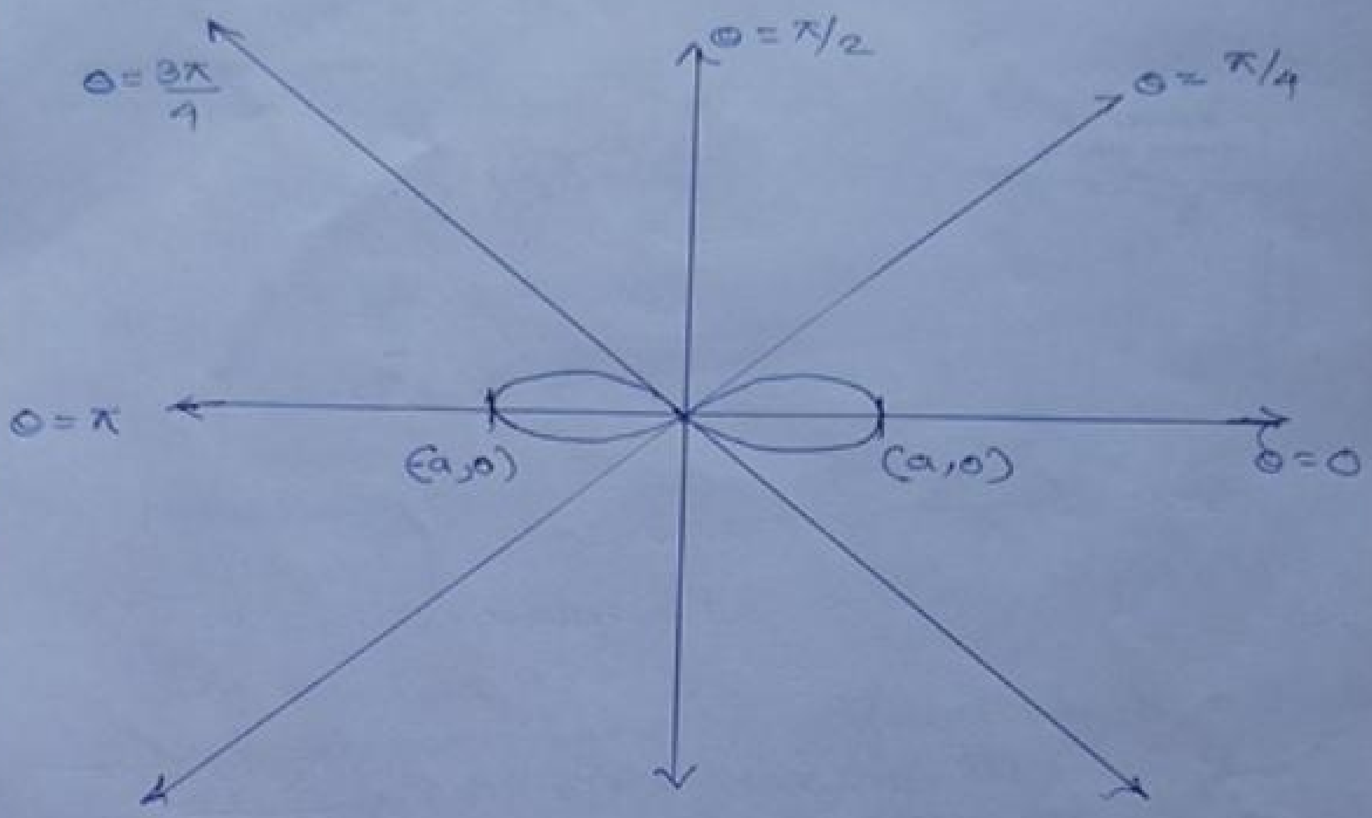
$\Rightarrow a^2(x^2 - y^2)$

$\therefore r^2 = a^2\cos(2\theta) \Rightarrow \boxed{x^2 + y^2 = a^2(x^2 - y^2)}$

This is cartesian form of given curve.

v) minimum value of $r = 0$ & max value of $r = a$

Hence curve entirely lies within a circle of radius a



"Bernoulli's Lamnisates"

Ex. 4) Trace the curve $\boxed{r = a\sin^3(\theta/3)}$

$\Rightarrow$ Points: i) If r replaced by $(-r)$ and θ by $(-\theta)$ then eq^n of curve remains same

Hence curve is symmetrical about the line $\pi/2$

ii) If $\theta = 0 \Rightarrow r = 0$; $\theta = \frac{\pi}{2} \Rightarrow r = \left(\frac{a}{8}\right)$

$\theta = \pi \Rightarrow r = \left(\frac{3\sqrt{3}}{8}\right)a$

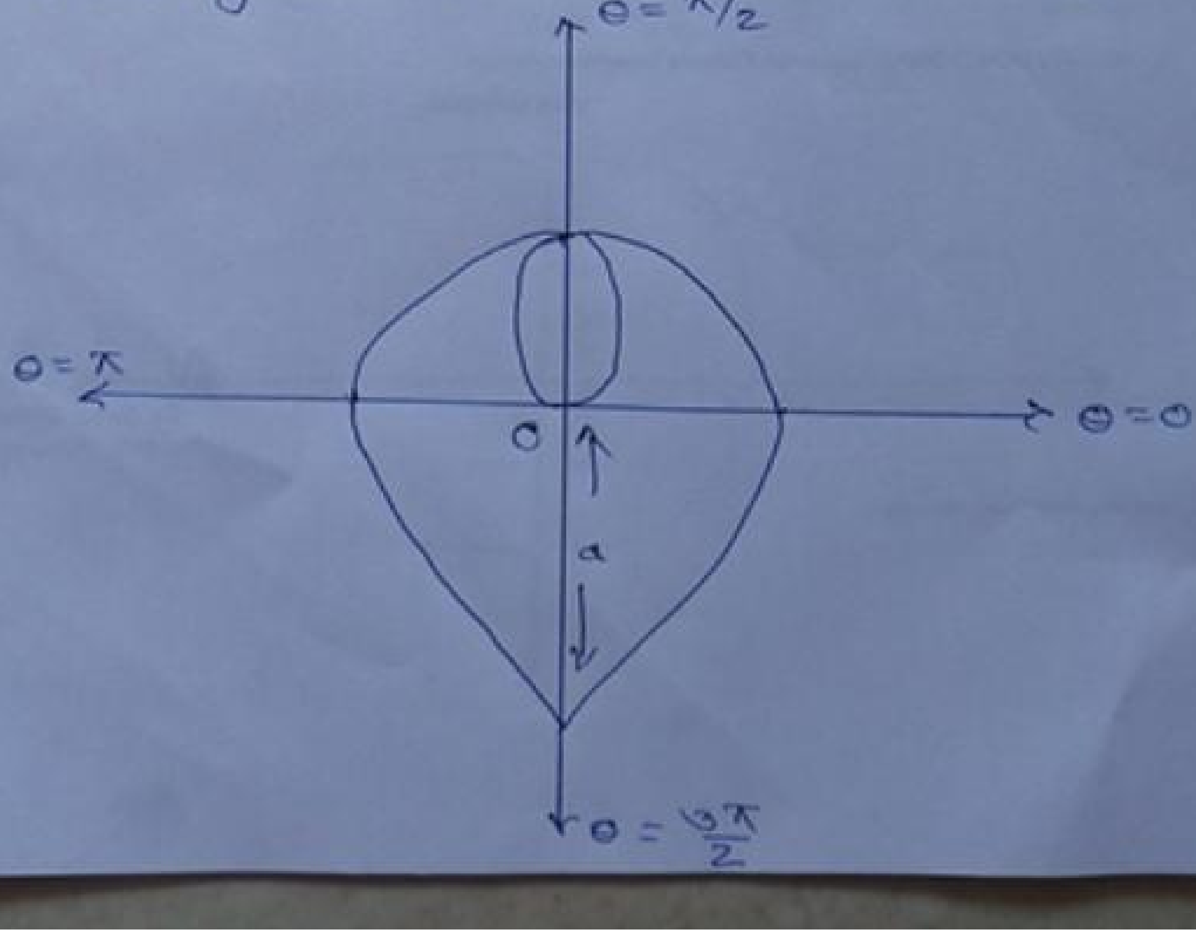
$\theta = \frac{3\pi}{2} \Rightarrow r = a$

$\theta = 2\pi \Rightarrow r = \left(\frac{3\sqrt{3}}{8}\right)a$

Hence minimum value of $r = 0$ & maximum value of $r = a$

$\therefore$ Curve entirely lies within a circle of radius a.

iii) We observe that $\theta = 0$ is tangent to curve at Pole. Also its opposite to line $\theta = \pi$ will be a tangent to curve at pole as its symmetric about the line $\theta = \frac{\pi}{2}$



Ex. 5] Trace the curve for,
"Parabola in Polar form"

$\boxed{\frac{2a}{r} = 1 + \cos\theta}$ or $\boxed{r = \frac{2a}{1+\cos\theta}}$

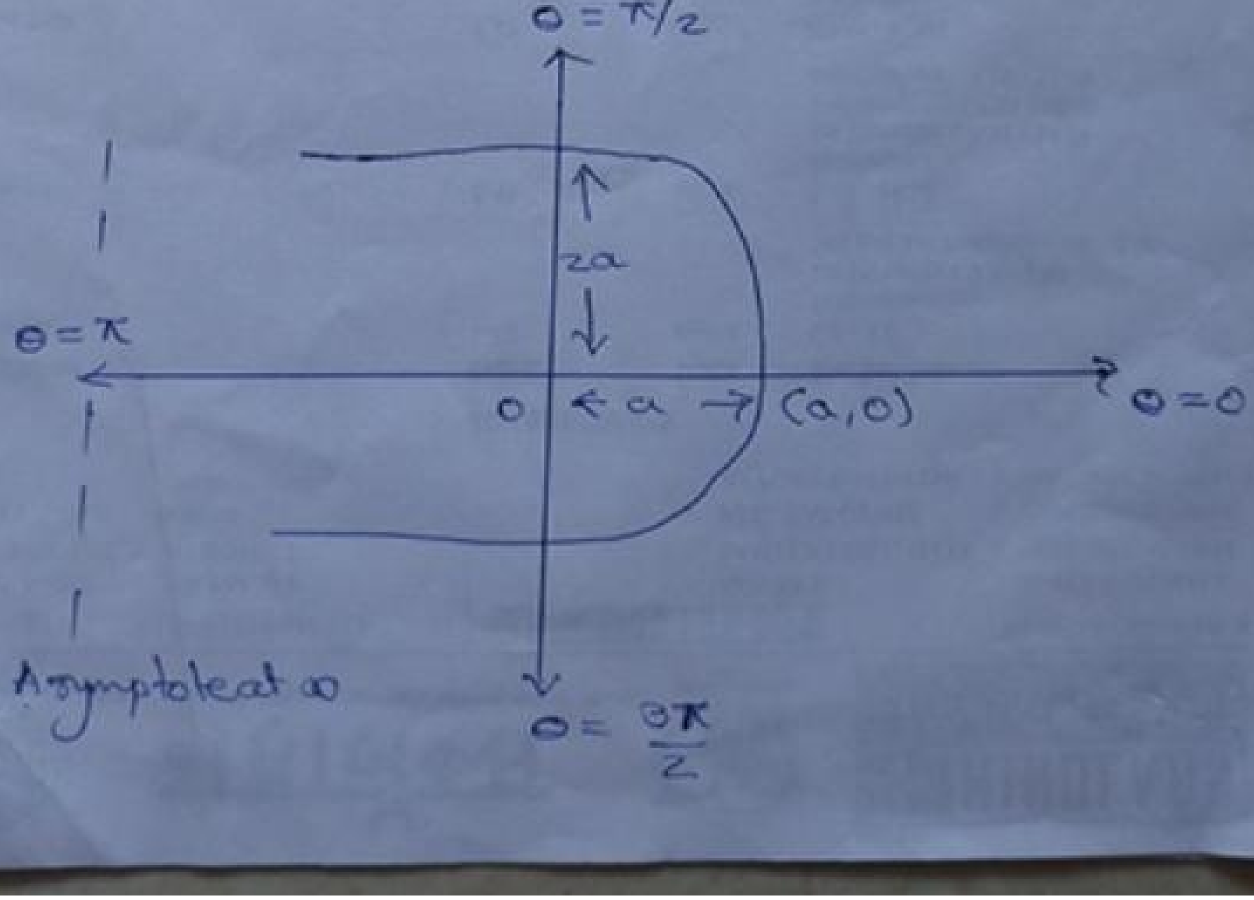
⇒ Points: i> If $\theta = (-\theta)$ curve remains same
Hence symmetrical about intial line.

ii> If $\theta = 0 \Rightarrow r = a$

$\theta = \frac{\pi}{2} \Rightarrow r = 2a$

$\theta = \pi \Rightarrow r = \infty$

iii> Curve does not passes through pole minimum value of $r = a$ & maximum value of r is infinity for $\theta = \pi$ line is asymptote to curve.



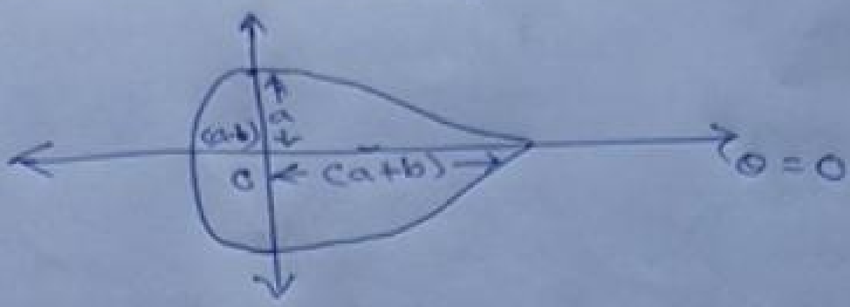
Examples for homework.

i) $r = a(1-\cos\theta)$

ii) $r = a(1-\sin\theta)$

iii) $r = (a + b\cos\theta)$ if $a > b$

[Pascal's Limacon]



iv) $r = a\cos^3(\theta/3)$

v) $r = (a + b\cos\theta)$ for $a = b$

vi) $r = (a + b\cos\theta)$ for $b = -a$

vii) $\frac{2a}{r} = (1-\cos\theta)$

Lecture - 5

* Tracing of Rose curves:

Equations of rose curves are given by

$\boxed{r = a\sin(n\theta)}$ or $\boxed{r = a\cos(n\theta)}$

Remark: i) If n is <u>even</u> then curves has <u>"2n"</u> no. of leaves

ii) If n is odd then curve has <u>"n"</u> no. of leaves.

Ex. i) Trace the curve for, $\boxed{r = a\sin(3\theta)}$

[Three leved Rose curve]

$\Rightarrow$ Points : i) If we replace θ by $(-\theta)$ and r by $(-r)$ then eq^n of curve remains same hence symmetrical about $\theta = \frac{\pi}{2}$ line.

ii) If $\theta = 0 \Rightarrow r = 0$ Hence pole lies on curve

iii) for $3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots$

$\Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \ldots$

are tangent to the curve at pole.

Draw these lines and place loops in alternate divisions.

first loop lies bet^n $\theta = 0$ to $\theta = \frac{\pi}{3}$

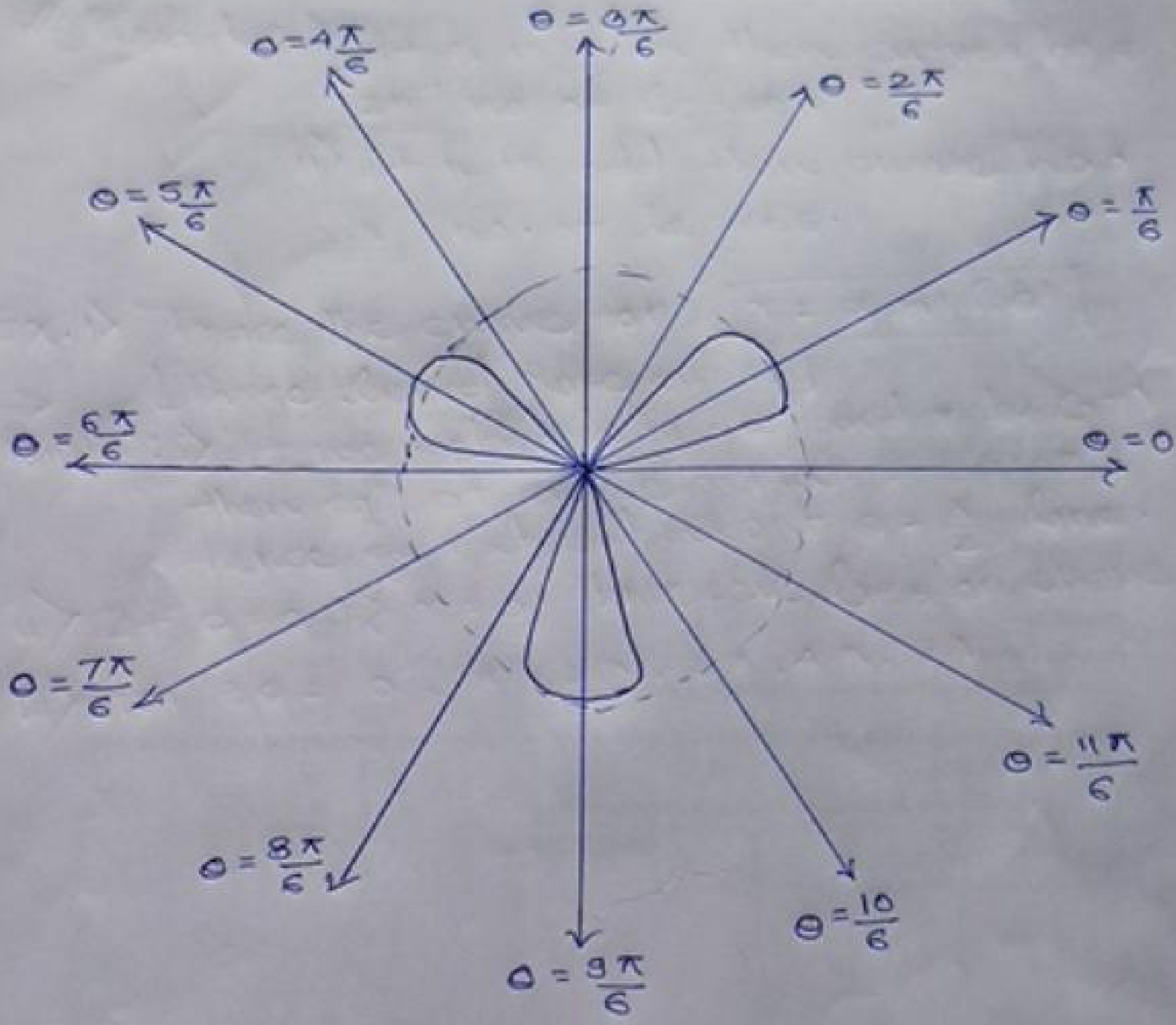
2nd loop lies bet^n $\theta = \frac{2\pi}{3}$ to $\theta = \pi$

3rd loop lies bet^n $\theta = \frac{4\pi}{3}$ to $\theta = \frac{5\pi}{3}$

iv) Divide each quadrant into 3 parts as $n = 3$ and prepare table as follows.

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{\pi}{2}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	$\frac{12\pi}{6}$
r	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

Note: $r = -a$ means radial distance to be taken in opposite direction.



Three Leaved Rose Curve

Ex. 2) Trace the curve $\boxed{r = a\cos(2\theta)}$

$\Rightarrow$ Points: i) for $\theta = (-\theta)$ curve remains same hence its symmetrical about the line $\theta = 0$ (Initial line)

ii) As $r = a\cos(n\theta) = a\cos(2\theta)$

here $\underline{n = 2}$ even number

This curve has $2n$ no. of loops

i.e. four loops.

iii) for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ we get $r = 0$

Hence curve passes through pole and at pole all above lines are tangents to curve.

iv) Divide each quadrants into two equal parts as $n = 2$

Hence $\theta = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$

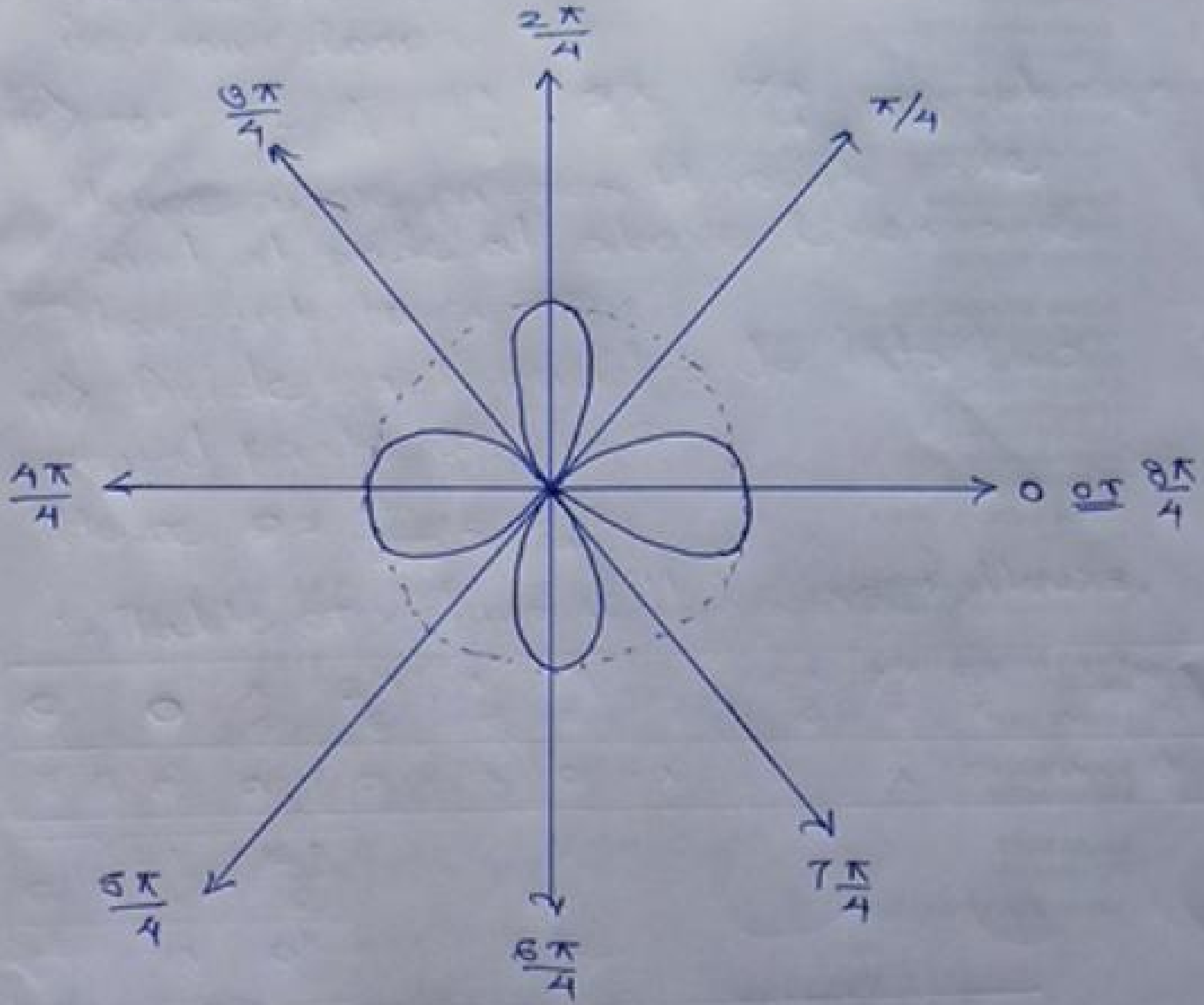
$\therefore$ Table of values will be as follows,

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{\pi}{2}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{6\pi}{6}$	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	$\frac{12\pi}{6}$
r	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

θ	0	$\frac{\pi}{4}$	$\frac{2\pi}{4}$	$\frac{3\pi}{4}$	$\frac{4\pi}{4}$	$\frac{5\pi}{4}$	$\frac{6\pi}{4}$	$\frac{7\pi}{4}$	$\frac{8\pi}{4}$
r	a	0	$-a$	0	a	0	$-a$	0	a

v) Min. value of r is $\underline{0}$ and max. value of r is $\underline{a}$

$\therefore$ Curve entirely lies bet^n a circle of radius a.



Four leaved Rose Curve

Ex. 3) Trace the curve $\boxed{r^m = a^m \cos(m\theta)}$

$\Rightarrow$ Cases for values of "m"

i) If $m = 1 \Rightarrow r = a\cos\theta$ ... circle.

($\because r^2 = a r\cos\theta \Rightarrow x^2 + y^2 = ax$

for $x = a\cos\theta$, $y = a\sin\theta$)

$\Rightarrow \boxed{x^2 + y^2 - ax = 0}$

[This is circle with $(a/2, 0)$ on centre on x-axis]

ii) If $m = -1 \Rightarrow r^{-1} = a^{-1}\cos(-\theta)$

$\Rightarrow r\cos\theta = a \Rightarrow \boxed{x = a}$

for $x = r\cos\theta$ i.e. asymptote line parallel to Y-axis.

iii) If $m = 2 \Rightarrow r^2 = a^2\cos(2\theta)$

$\Rightarrow$ This is Bernoulli's Lamniscate.

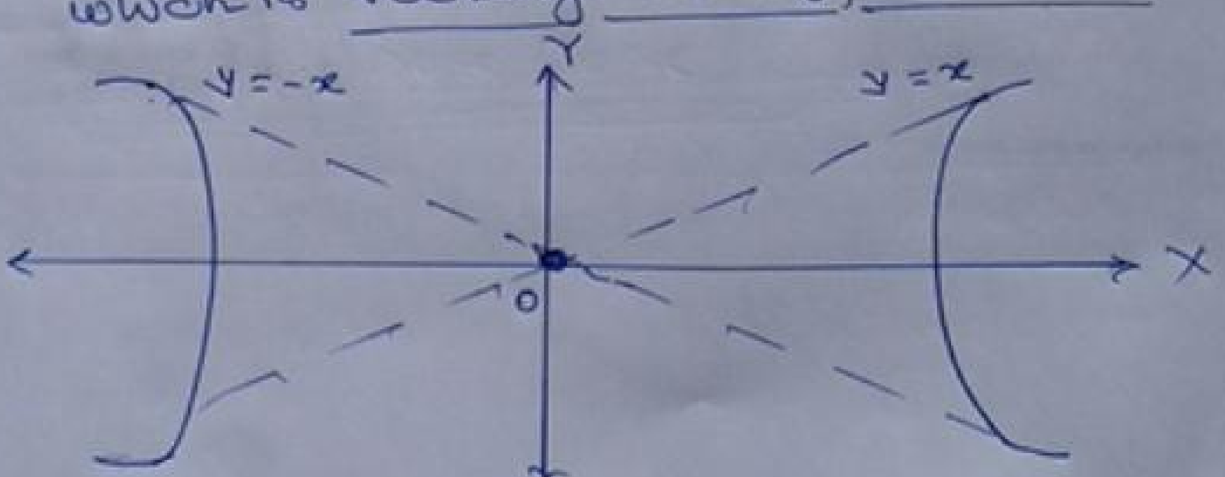
iv) If $m = -2 \Rightarrow r^{-2} = a^{-2}\cos(-2\theta)$

$\Rightarrow a^2 = r^2\cos 2\theta$

i.e. $r^2(\cos^2\theta - \sin^2\theta) = a^2$

$\Rightarrow x^2 - y^2 = a^2$

which is rectangular hyperbola.



vi) $m = \frac{1}{2} \Rightarrow \sqrt{r} = \sqrt{a}\cos\left(\frac{\theta}{2}\right)$

$\Rightarrow r = a\cos^2\left(\frac{\theta}{2}\right) \Rightarrow \boxed{r = a(1+\cos\theta)}$

$\Rightarrow$ This is <u>Cardioid</u>

vii) If $m = \left(-\frac{1}{2}\right)$ then

$r^{-1/2} = a^{-1/2}\cos\frac{\theta}{2} \Rightarrow r = \frac{a}{\cos^2\left(\frac{\theta}{2}\right)}$

$\therefore r = \left(\frac{2a}{1+\cos\theta}\right)$

$\Rightarrow$ <u>Parabola</u>

<u>Examples for homework</u>

i) $r = a\sin(2\theta)$

ii) $r = a\cos(3\theta)$

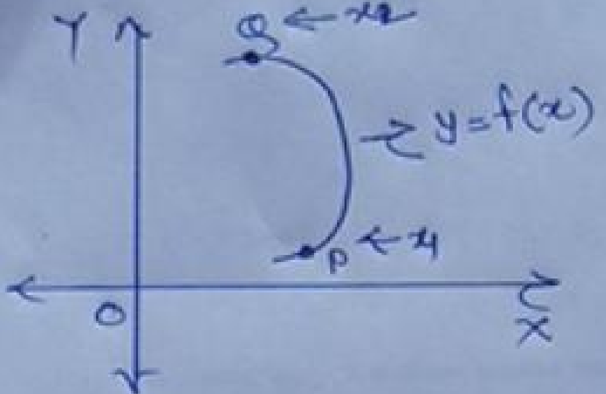
iii) $r = a\sin(5\theta)$

iv) $r = a\cos(5\theta)$

<u>Lecture - 6</u>

* Rectification of Curve

Rectification of curve means to find length of curve bet^n any two points on it.



Rectification of curve in Cartesian form

i) Length of curve $y = f(x)$ bet^n $x = x_1$ and $x = x_2$ is given by,

$$s = \int_{x_1}^{x_2} \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx$$

ii) length of curve $x = f(y)$ bet^n $y = y_1$ and $y = y_2$ is given by

$$s = \int_{y_1}^{y_2} \left(\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right) dy$$

* Imp formulae:

i) length of curve $y = f(x)$

$$s = \int_{x_1}^{x_2} \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx$$

ii length of curve $x = f(y)$

$$s = \int_{y_1}^{y_2} \left(\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right) dy$$

Ex. 1) find the length of loop of curve $3ay^2 = x(x-a)^2$

$\Rightarrow$ i) The curve is symmetrical about x-axis

ii) For $x = 0 \Rightarrow y = 0$
i.e. curve passes through origin

iii) Also for $x = a \Rightarrow y = 0$ i.e. curve intersects x-axis (at) point $(a,0)$ on it

$\therefore$ Curve has a loop bet^n the points $(0,0)$ and $(a,0)$ as shown in figure.

$\therefore$ length of loop of curve bet^n.
$x=0$ to $x=a$ is given by

$$S = 2\int_0^a \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)dx \quad \cdots (1)$$

Now, $3ay^2 = x(x-a)^2$

Diff. w.r.t. x

$$6ay\frac{dy}{dx} = x \cdot 2(x-a) + (x-a)^2$$

$$6ay\frac{dy}{dx} = (x-a)(3x-a)$$

$$\frac{dy}{dx} = \left[\frac{(x-a)(3x-a)}{6ay}\right]$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x-a)^2(3x-a)^2}{36a^2y^2}$$

$$= 1 + \frac{(x-a)^2(3x-a)^2}{36a^2\left[\frac{x(x-a)^2}{3a}\right]}$$

$$= \left[\frac{12ax + (3x-a)^2}{12ax}\right]$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = \frac{(3x+a)^2}{12ax}$$

from eq^n ①

Taking square root

$$\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) = \left(\sqrt{\frac{(3x+a)^2}{12ax}}\right)$$

$\therefore$ length of loop of curve

$$s = 2\int_0^a \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)dx$$

$$= 2\int_0^a \left(\frac{(3x+a)}{\sqrt{12ax}}\right)dx$$

$$= 2\int_0^a \frac{(3x+a)}{2\sqrt{3a}\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \left(3\sqrt{x}+\frac{a}{\sqrt{x}}\right)dx$$

$$= \frac{1}{\sqrt{3a}}\left[\frac{3x^{3/2}}{3/2}+\frac{ax^{1/2}}{1/2}\right]_0^a$$

$$= \left[\frac{1}{\sqrt{3a}}\left[2a^{3/2}+2a^{3/2}\right]\right]$$

$$= \left[\frac{4a^{3/2}}{\sqrt{3a}}\right]$$

$$= \left[\frac{4a^{3/2}}{\sqrt{3}a^{1/2}}\right] = \frac{4a^{3/2}\cdot a^{-1/2}}{\sqrt{3}}$$

$$s = \frac{4a}{\sqrt{3}} \text{ or } \frac{4a}{\sqrt{3}}$$

Ex. 2. Find the length of the loop of the curve $9y^2 = (x+7)(x+4)^2$

$\Rightarrow$ i) Curve is symmetrical about x-axis

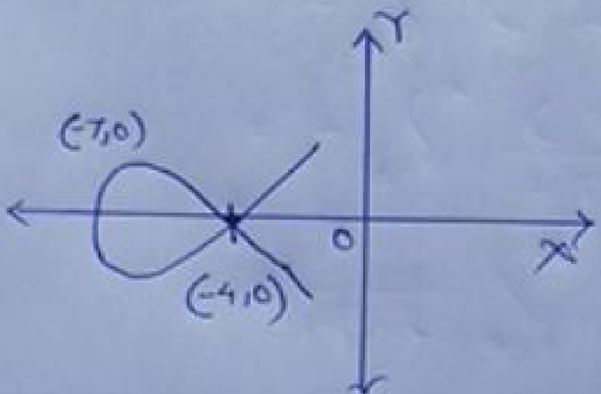
ii) If $x = -7$ then $y = 0$

$\Rightarrow$ Curve intersects x-axis at $(-7, 0)$

Also $x = 4$ then also $y = 0$

$\Rightarrow$ Curve intersect x-axis at $(-4, 0)$

iii) Curve does not exist for $x < -7$ and has loop betn the points $(-7, 0)$ to $(-4, 0)$



Eqn of curve,

$$9y^2 = (x+7)(x+4)^2$$

$\therefore$ Diff. w.r.t. x

$$18y\frac{dy}{dx} = (x+4)^2 + 2(x+7)(x+4)$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \left[\frac{(x+4)(3x+18)}{18y}\right]$$

$$\therefore \frac{dy}{dx} = \left[\frac{(x+4)(x+6)}{6y}\right]$$

$$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left[\frac{(x^2+16x+64)}{4(x+7)}\right]$$

$$= \frac{(x+8)^2}{4(x+7}$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \left[\frac{(x+8)^2}{4(x+7)}\right]$$

$\therefore$ length of loop of curve betn the points $(-7, 0)$ to $(-4, 0)$ is

$$S = 2\int_{-7}^{-4}\left[\sqrt{\frac{(x+8)^2}{4(x+7)}}\right]dx$$

$$= 2\int_{-7}^{-4}\left[\frac{(x+8)}{2\sqrt{x+7}}\right]dx$$

$$= \int_{-7}^{-4}\left(\frac{(x+8)}{\sqrt{x+7}}\right)dx$$

$$= \int_{-7}^{-4}\left[\frac{(x+7)+1}{\sqrt{x+7}}\right]dx$$

$$= \int_{-7}^{-4}\left[\sqrt{x+7} + \frac{1}{\sqrt{x+7}}\right]dx$$

$$S = \left[\frac{(x+7)^{3/2}}{3/2} + \frac{(x+7)^{1/2}}{1/2}\right]_{-7}^{-4}$$

$$= \left[\frac{2}{3}(3)^{3/2} + 2(3)^{1/2}\right] - [0]$$

$$= \frac{2}{3}(3)^{3/2} + \frac{6(3)^{1/2}}{3}$$

$$= \frac{2}{3}\sqrt[3]{3} + \frac{6}{3}\sqrt[2]{3}$$

$$= \sqrt{3}\cancel{\sqrt{3}}\cancel{\sqrt{3}}\frac{2}{\cancel{\sqrt{3}}\cancel{\sqrt{3}}} + \frac{6\cancel{\sqrt{3}}\sqrt{3}}{\cancel{\sqrt{3}}\sqrt{3}}$$

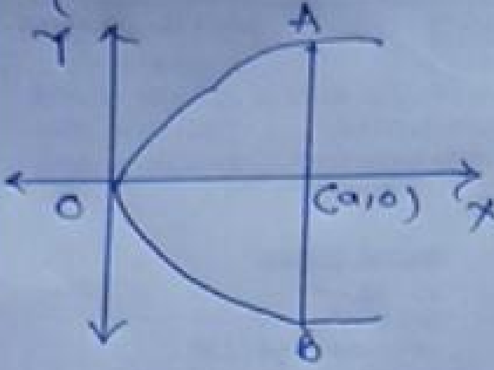
$$= 2\sqrt{3} + \left(\frac{2\times 3}{\sqrt{3}}\right)$$

$$= 2\sqrt{3} + \left(\frac{2\times\sqrt{3}\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}}\right) = 2\sqrt{3} + 2\sqrt{3}$$

$$S = 4\sqrt{3}$$

Ex. Find the length of arc of parabola $y^2 = 4ax$ cutt of the latus rectum.

$\Rightarrow$



In fig. AOB is the length of curve $y^2 = 4ax$ cut off by its latus rectum.

$\therefore s = 2$ (length OA)

$$= 2\int_0^a \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)dx$$

As, $y^2 = 4ax$

Diff. w.r.t. x

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore s = 2\int_0^a \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)dx = 2\int_0^a \left(\sqrt{1+\left(\frac{2a}{y}\right)^2}\right)dx$$

$$= 2\int_0^a \left(\sqrt{\frac{y^2+4a^2}{y^2}}\right)dx$$

$$= 2\int_0^a \left(\sqrt{\frac{4ax+4a^2}{4ax}}\right)dx$$

$$= 2\int_0^a \left(\sqrt{\frac{x+a}{x}}\right)dx$$

$\therefore$ length of parabola cuts of by latus rectum is, Put $x = t^2 \Rightarrow dx = 2t\,dt$

if $x = 0 \Rightarrow t = 0$

$x = a \Rightarrow t = \sqrt{a}$

$$\therefore S = 2\int_0^{\sqrt{a}} \left(\sqrt{\frac{t^2+a}{t^2}}\right) \cancel{dt}\; 2t\,dt$$

$$= 4\int_0^{\sqrt{a}} \left(\sqrt{t^2+a}\right) dt$$

$$= 4\left[\frac{t}{2}\sqrt{t^2+a} + \frac{a}{2}\log\left(t+\sqrt{t^2+a}\right)\right]_0^{\sqrt{a}}$$

$$= 4\left[\frac{\sqrt{a}}{2}\sqrt{a+a} + \frac{a}{2}\log\left(\sqrt{a}+\sqrt{a+a}\right]\right.$$

$$\left. - \left[0 + \frac{a}{2}\log\sqrt{a}\right]\right.$$

$$= 4\left[\frac{\sqrt{a}\cdot\sqrt{2a}}{2} + \frac{a}{2}\log\left(\sqrt{a}+\sqrt{2a}\right)\right]$$

$$- \left[\frac{a}{2}\log\sqrt{a}\right]$$

$$= 4\left[\frac{\sqrt{2}\,a}{2} + \frac{a}{2}\log\left(\sqrt{a}+\sqrt{2a}\right)\right.$$

$$\left. - \left[\frac{a}{2}\log\sqrt{a}\right]\right.$$

$$= 2a\left[\sqrt{2} + \log\left(\sqrt{a}+\sqrt{2a}\right)\right.$$

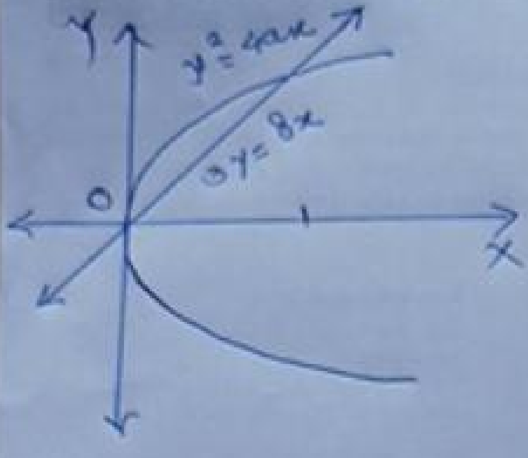
$$\left. - \log\sqrt{a}\right]$$

$$= 2a\left[\sqrt{2} + \log\left(\sqrt{a}(1+\sqrt{2})\right) - \log\sqrt{a}\right]$$

$$= 2a\left[\sqrt{2} + \log\sqrt{a} + \log(1+\sqrt{2}) - \log\sqrt{a}\right]$$

$$S = 2a\left[\sqrt{2} + \log(1+\sqrt{2})\right] \text{ units}$$

Ex. 4) Find the length of arc of parabola $y^2 = 4ax$ cuts of by the line $3y = 8x$



Hint: Point of intersection of parabolla and line $y = \frac{8}{3}x$ are $(0,0)$ & $\left(\frac{39}{6}, \frac{729}{48}\right)$

$$\therefore s = \int_0^{39/16} \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

$$= \int_0^{39/16} \sqrt{\frac{x + a}{x}}\, dx$$

As ex. 3 Evaluate above integral and find req. length.

(Ans: $s = a\left[\log 2 + \frac{15}{16}\right]$ Units.

Ex. 5] find ~~pou~~ Perimeter of circle $x^2 + y^2 = a^2$

Ex. 6] find length of loop of curve

$$x^2(a^2 - x^2) = 8a^2y^2$$

Ans: $\left(\frac{\pi a}{\sqrt{2}}\right)$

Lecture: 7

* Rectification of Polar curves :

Result: i) If $r = f(\theta)$ be a polar curve then length of an arc of curve between points $\theta = \theta_1$ and $\theta = \theta_2$ is given by,

$$s = \int_{\theta_1}^{\theta_2} \left(\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right) d\theta$$

ii) If $\theta = f(r)$ be a polar curve then the length of an arc of curve bet^n the points $r = r_1$ & $r = r_2$ is given by,

$$s = \int_{r_1}^{r_2} \left(\sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} \right) d\theta$$

Imp formulae

1) If $r = f(\theta)$ be polar curve

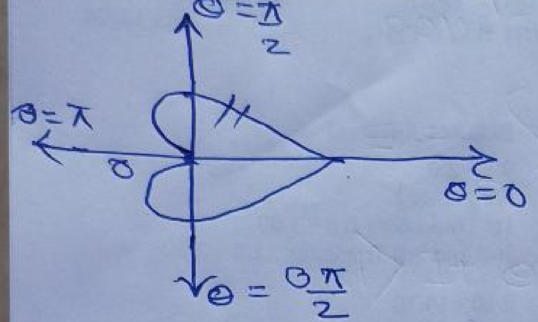
$$\therefore s = \int_{\theta_1}^{\theta_2} \left(\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \right) d\theta$$

2) If $\theta = f(r)$ be polar curve

$$\therefore s = \int_{r_1}^{r_2} \left(\sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} \right) d\theta$$

Ex 1) Find the perimeter of the cardioid $r = a(1+\cos\theta)$

$\Rightarrow$ The cardioid is symmetrical about the line $\theta = 0$ and its upper half lies betn $\theta = 0$ to $\theta = \pi$ as shown in figure,



$\therefore$ length of cardioid (Perimeter) is given by,

$$s = 2\int_0^{\pi}\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta \quad \ldots (1)$$

hence $r = a(1+\cos\theta) \Rightarrow \frac{dr}{d\theta} = (-a\sin\theta)$

$$\therefore \left[r^2+\left(\frac{dr}{d\theta}\right)^2\right] = a^2(1+\cos\theta)^2 + (-a\sin\theta)^2$$

$$= a^2(1+2\cos\theta+\cos^2\theta) + (a^2\sin^2\theta)$$

$$= a^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta)$$

$$= a^2(1+2\cos\theta+1)$$

$$= a^2(2+2\cos\theta)$$

$$= 2a^2(1+\cos\theta)$$

$$= 2a^2\left(2\cos^2\left(\frac{\theta}{2}\right)\right)$$

$$= 4a^2\cos^2\left(\frac{\theta}{2}\right) \quad \ldots (2)$$

from (1) & (2) we get,

$$s = 2\int_0^{\pi}\left(\sqrt{4a^2\cos^2\left(\frac{\theta}{2}\right)}\right)d\theta$$

$$= 4a\int_0^{\pi}\left(\cos\frac{\theta}{2}\right)d\theta$$

$$= 4a\left[\frac{\sin\left(\frac{\theta}{2}\right)}{1/2}\right]_0^{\pi} = 2\times 4a\left[\sin\left(\frac{\pi}{2}\right)-\sin(0)\right]$$

$$= 8a\left[\sin\left(\frac{\pi}{2}\right)-\sin(0)\right].$$

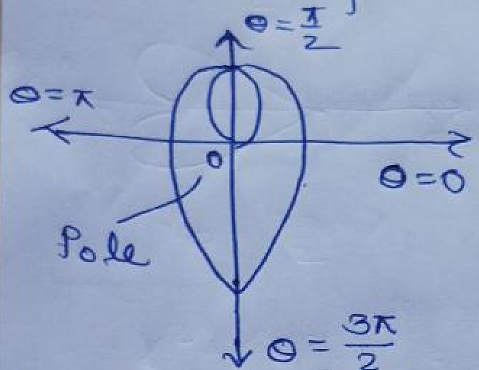
$$= 8a(1-0) = 8a \text{ unit}$$

$\therefore$ Perimeter or entire length of cardioid $r = a(1+\cos\theta)$ is $\underline{s = 8a \text{ units}}$

Ex. 2) Find the total length of the curve $r = a\sin^3\left(\frac{\theta}{3}\right)$ taking Pole as fixed point

$\Rightarrow$ The curve is symmetrical about the line $\theta = \frac{\pi}{2}$

and half of the curve lies betn $\theta = 0$ to $\theta = \frac{3\pi}{2}$ from pole to a point on the line $\theta = \frac{3\pi}{2}$



$\therefore$ Total length of the curve is given by,

$$S = 2\int_0^{3\pi/2}\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta \quad \text{---} (1)$$

Now $r = a\sin^3\left(\frac{\theta}{3}\right)$

after diffn w.r.t. θ

$$\left(\frac{dr}{d\theta}\right) = \left[3a\sin^2\left(\frac{\theta}{3}\right)\cdot\cos\left(\frac{\theta}{3}\right)\cdot\frac{1}{3}\right]$$

$$= \left[a\sin^2\left(\frac{\theta}{3}\right).\cos\left(\frac{\theta}{3}\right)\right]$$

$$\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right] = \left[a^2\sin^6\left(\frac{\theta}{3}\right) + a^2\sin^4\left(\frac{\theta}{3}\right).\cos^2\left(\frac{\theta}{3}\right)\right]$$

$$= a^2\sin^4\left(\frac{\theta}{3}\right)\left[\sin^2\left(\frac{\theta}{3}\right)+\cos^2\left(\frac{\theta}{3}\right)\right]$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2\sin^4\left(\frac{\theta}{3}\right) \quad \text{...} (2)$$

from (1) & (2)

$$S = 2\int_0^{\frac{3\pi}{2}}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$= 2\int_0^{\frac{3\pi}{2}}\left(\sqrt{a^2\sin^4\left(\frac{\theta}{3}\right)}\right)d\theta$$

$$= 2a\int_0^{3\pi/2}\left(\sin^2\left(\frac{\theta}{3}\right)\right)d\theta$$

$$= 2a\int_0^{3\pi/2}\left[\frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\right]d\theta$$

$$S = 2a\int_0^{3\pi/2}\left[\frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\right]d\theta$$

$$= a\left[\theta - \frac{\sin\left(\frac{2\theta}{3}\right)}{2/3}\right]_0^{3\pi/2}$$

$$= a\left[\frac{3\pi}{2} - \frac{3}{2}\sin(\pi)\right]$$

$$S = \left(\frac{3\pi}{2}\right)a \text{ units}$$

Ex. 3) Find the length of the equiangular spiral.

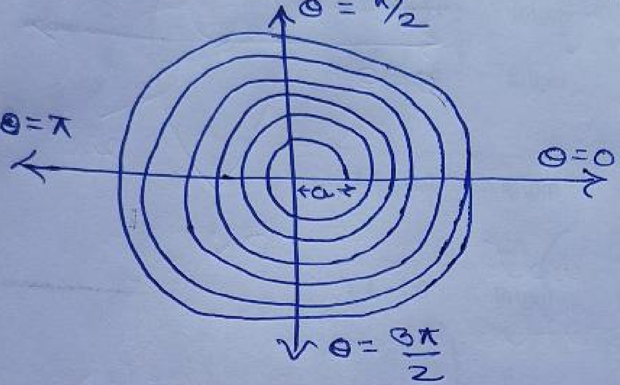
$\boxed{r = a \cdot e^{m\theta}}$ Type of $\boxed{r = f(\theta)}$

$\Rightarrow$ If $\theta = 0 \Rightarrow r = a$

As θ increases "r" goes on increases

But for $\theta = \infty \Rightarrow r = \infty$ & if $\theta = -\infty \Rightarrow r = 0$

$\therefore$ We can find length of such curve betn $r = r_1$ & $r = r_2$ points as follows,



$$s = \int_{r_1}^{r_2} \left(\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\right) dr \quad \cdots (1)$$

$\therefore r = ae^{m\theta}$

diff. w.r.t. r we get

$$1 = mae^{m\theta}\left(\frac{d\theta}{dr}\right)$$

$$\left(\frac{d\theta}{dr}\right) = \frac{1}{ae^{m\theta} \cdot m} = \left(\frac{1}{mr}\right)$$

$$\therefore \left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right] = 1 + r^2\left[\frac{1}{m^2r^2}\right]$$

$$= \left(1 + \frac{1}{m^2}\right)$$

$$\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right] = \left(\frac{m^2+1}{m^2}\right)$$

$$\therefore \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2} = \left(\frac{\sqrt{m^2+1}}{m}\right) \quad \cdots (2)$$

from eqn (1) & (2)

length of spiral

$$s = \int_{r_1}^{r_2} \left(\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\right) dr = \int_{r_1}^{r_2} \left(\sqrt{\frac{m^2+1}{m^2}}\right) dr$$

$$= \left(\frac{\sqrt{m^2+1}}{m}\right)(r)_{r_1}^{r_2} = \left(\frac{\sqrt{m^2+1}}{m}\right)(r_2 - r_1)$$

$$s = \left[\frac{\sqrt{m^2+1}}{m}(r_2 - r_1)\right] \text{units}$$

Ex. 4) find length of spiral $r = a e^{\theta \cot\alpha}$ for $r = r_1$ to $r = r_2$

$\Rightarrow$ $r = a \cdot e^{\theta \cot\alpha}$

diff. w.r.t. r

$1 = a \cot\alpha \cdot e^{\theta\cot\alpha}\left(\frac{d\theta}{dr}\right)$

$\Rightarrow \left(\frac{d\theta}{dr}\right) = \left(\frac{1}{r\cot\alpha}\right)$

$\therefore \left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right] = 1 + r^2\left[\frac{1}{r^2\cot^2\alpha}\right] = 1 + \frac{1}{\cot^2\alpha}$

$= \frac{\cot^2\alpha + 1}{\cot^2\alpha}$

$= \left(\frac{\operatorname{cosec}^2\alpha}{\cot^2\alpha}\right)$

$\therefore \left(\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\right) = \sqrt{\left(\frac{\operatorname{cosec}^2\alpha}{\cot^2\alpha}\right)} = \frac{\operatorname{cosec}\alpha}{\cot\alpha}$

$= \frac{1/\sin\alpha}{\frac{\cos\alpha}{\sin\alpha}} = \frac{1}{\cos\alpha} = \sec\alpha$

$\therefore$ length of spiral is given by,

$s = \int_{r_1}^{r_2}\left(\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\right)dr$

$= \int_{r_1}^{r_2}(\sec\alpha)\,dr$

$= \sec\alpha\,[r]_{r_1}^{r_2}$

$s = \sec\alpha\,[r_2 - r_1]$

Req. length of spiral.

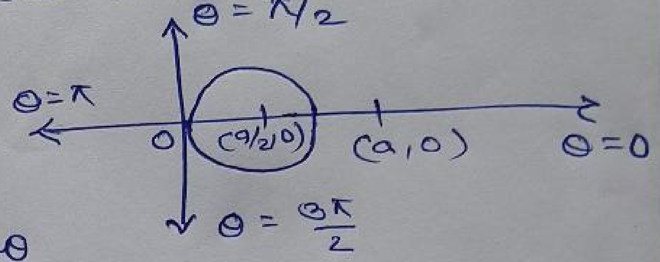
Ex. 5) find the perimeter of the curve $\boxed{r = a\cos\theta}$ by rectification method

$\Rightarrow$ i) The curve $r = a\cos\theta$ is symmetrical about initial line $\theta = 0$ (As $\theta = -\theta$ curve remains same)

ii) If $\theta = 0 \Rightarrow r = a$ and if $\theta = \pi/2 \Rightarrow r = 0$
Hence curve passes through the pole and line $\theta = \frac{\pi}{2}$ is a tangent at pole.

iii) The max value of $r = a$ & min value of $r = 0$

$\therefore$ The curve is a circle with radius 'a' as shown in figure



$\therefore$ Perimeter of circle is given by,

$$s = 2\int_0^{\pi/2}\left[\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right]d\theta \quad \ldots (1)$$

where, $r = a\cos\theta \Rightarrow \left(\frac{dr}{d\theta}\right) = (-a\sin\theta)$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = (a^2\cos^2\theta + a^2\sin^2\theta)$$
$$= a^2[\cos^2\theta + \sin^2\theta]$$

$$\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = a^2 \quad \ldots (2)$$

from (1) & (2)

$$s = 2\int_0^{\pi/2}\left(\sqrt{a^2}\right)d\theta = 2a\int_0^{\pi/2} d\theta$$

$$= 2a[\theta]_0^{\pi/2}$$

$$= 2a\left[\frac{\pi}{2} - 0\right]$$

$$= a\pi$$

$\therefore$ Perimeter of a curve $r = a\cos\theta$ is $\boxed{s = \pi a}$ units

Lecture - 8

Ex. find length of the curve $\boxed{r = a(1+\cos\theta)}$ which is cutt off by line $\boxed{4r = 3a\sec\theta}$ remote from the pole.

$\Rightarrow$ As $4r = 3a\sec\theta$ is a line given hence for $\theta = 0 \Rightarrow r = \frac{3a}{4}$

$\Rightarrow$ The line intersects to initial line at $r = \frac{3a}{4}$ on it.

Hence its parallel to line $\theta = \frac{\pi}{2}$ line.

Also, $r = a(1+\cos\theta)$ & $4r = 3a\sec\theta$ intersects at the points which can be obtain by solving eq^n,

$\Rightarrow 4\cancel{a}(1+\cos\theta) = 3\cancel{a}\sec\theta$

$4 + 4\cos\theta = 3\sec\theta$

$\Rightarrow \frac{4+4\cos\theta}{\sec\theta} = 3$

$\Rightarrow 4\cos\theta + 4\cos^2\theta - 3 = 0$ ... Quadratic eq^n in $\cos\theta$

$\therefore 4\cos^2\theta + 6\cos\theta - 2\cos\theta - 3 = 0$

$\Rightarrow 2\cos\theta(2\cos\theta + 3) - 1(2\cos\theta + 3) = 0$

$\Rightarrow (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{3}{2}$

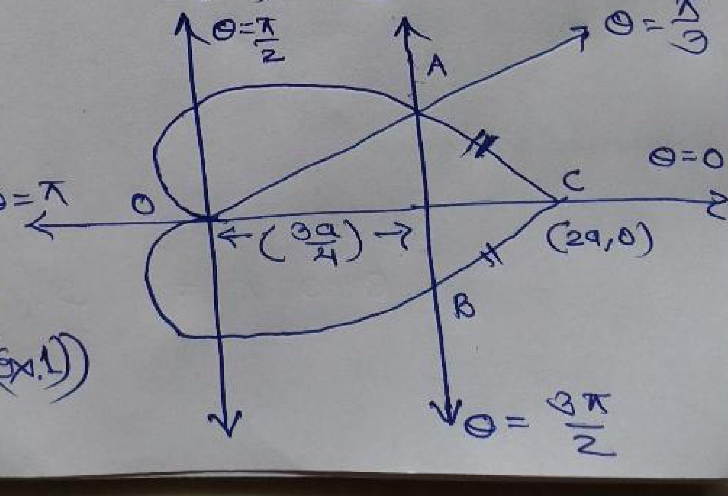
$\therefore \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Now, $r = a(1+\cos\theta)$

$\frac{dr}{d\theta} = (-a\sin\theta)$

$\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = 4a^2\cos^2\left(\frac{\theta}{2}\right)$

(seen in lecture 6 (Ex.1))



$\therefore$ length of arc (ACB) = 2 [length of arc AC]

$$\therefore S = 2\int_0^{\pi/3}\left(\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right) d\theta$$

$$= 2\int_0^{\pi/3}\left[2a\cos(\theta/2)\right] d\theta$$

$$= 4a\left[\frac{\sin(\theta/2)}{1/2}\right]_0^{\pi/3}$$

$$= 8a\left[\sin(\pi/6) - \sin(0)\right]$$

$$S = 8a\left[\frac{1}{2} - 0\right] = 4a \text{ units}$$

$\therefore$ Req. length of (ACB)

$\Rightarrow S = (4a)$ units.

Ex. Find the length of the cardioid $\boxed{r = a(1-\cos\theta)}$ which lies inside the circle $\boxed{r = a\cos\theta}$

$\Rightarrow$ Solving Eqn for their points of intersections

$r = a(1-\cos\theta)$ & $r = a\cos\theta$

$\Rightarrow a(1-\cos\theta) = a\cos\theta$

$\Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \frac{\pi}{3}$

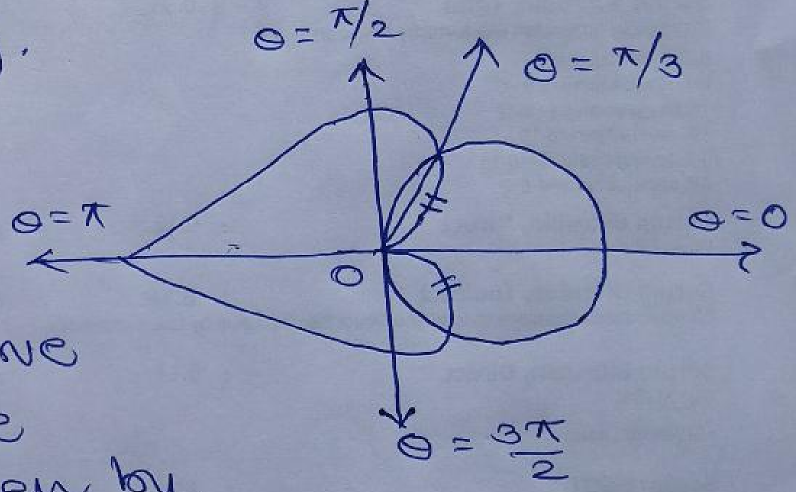
Curve and circle intersect at line $(\pi/3)$

As well as both curve ~~parallel~~ passes through pole.

Hence limit of intersection are

$\theta = \pi/3$ to $\theta = \pi$

As shown in fig.



Req. length of curve inside the circle $r = a\cos\theta$ is given by,

$$S = 2\int_{\pi/3}^{\pi}\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta \quad \ldots (1)$$

Now, $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = +(a\sin\theta)$$

$$\therefore \left[r^2+\left(\frac{dr}{d\theta}\right)^2\right] = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[2-2\cos\theta]$$

$$= 2a^2(1-\cos\theta)$$

$$\therefore \left[r^2 + \left(\frac{dr}{d\theta}\right)^2 \right] = 4a^2 \sin^2\left(\frac{\theta}{2}\right) \quad \cdots (2)$$

from (1) & (2) req. length is,

$$\therefore s = 2 \int_{\pi/3}^{\pi} \left(\sqrt{4a^2 \sin^2(\theta/2)}\right) d\theta$$

$$= 4a \int_{\pi/3}^{\pi} \left(\sin(\theta/2)\right) d\theta$$

$$= 4a \left[- \frac{\cos(\theta/2)}{1/2} \right]_{\pi/3}^{\pi}$$

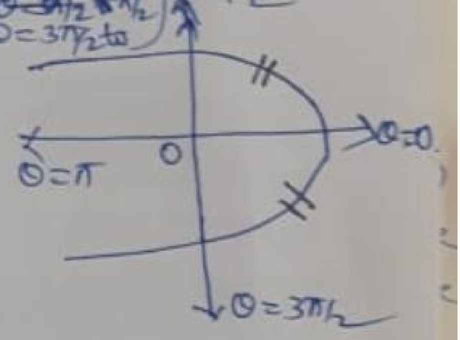
$$= 8a \left[0 - \left(- \frac{\sqrt{3}}{2} \right) \right]$$

$$s = (4a\sqrt{3}) \text{ units}$$

Unit (01) Curve Tracings And Rectifications

Ex:-(08) find the length of the parabola $\boxed{r=\left(\frac{2a}{1+\cos\theta}\right)}$ from $\theta=0$ to $\theta=\pi/2$ (Also for $\theta=3\pi/2$ to $\theta=\pi/2$)

Soln:- $r=\left(\frac{2a}{1+\cos\theta}\right)$ is a Parabola in Polar form as shown in figure



$$\therefore r=\left(\frac{2a}{1+\cos\theta}\right).$$

$$\Rightarrow \log r = \log(2a) - \log(1+\cos\theta)$$

Diff. w.r.t. θ

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = 0 - \left[\frac{-\sin\theta}{(1+\cos\theta)}\right]$$

$$\therefore \left(\frac{dr}{d\theta}\right) = r\left[\frac{\sin\theta}{1+\cos\theta}\right] = \left[\frac{2a}{1+\cos\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right)\right]$$

$$= \frac{2a\sin\theta}{[1+\cos\theta]^2} = \frac{\cancel{2}a\cdot\cancel{2}\sin(\theta/2)\cos(\theta/2)}{[\cancel{2}\cos^2(\theta/2)]^2}$$

$$\left(\frac{dr}{d\theta}\right) = \frac{a\sin(\theta/2)}{\cos^3(\theta/2)} = a\tan(\theta/2)\cdot\sec^2(\theta/2)$$

Now, $$\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right] = \left[\frac{4a^2}{\{2\cos^2(\theta/2)\}^2} + a^2\tan^2(\theta/2)\sec^4(\theta/2)\right]$$

$$= a^2\left[\sec^4(\theta/2) + \tan^2(\theta/2)\sec^4(\theta/2)\right]$$

$$= a^2\sec^4(\theta/2)\left[1+\tan^2(\theta/2)\right]$$

$$\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right] = a^2\sec^4(\theta/2)\sec^2(\theta/2)$$

$$\therefore \left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right) = \left[a\sec^2(\theta/2)\sec(\theta/2)\right] \rightarrow (i)$$

Now, Reqd. length $= \boxed{S = 2\int_0^{\pi/2}\left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right)d\theta} \rightarrow (ii)$

→ (Contd. on Page (07))

(Ex: ③ contd.- - - -)

From Results (i) and (ii)

$$S = 2\int_0^{\pi/2} [a \sec^2(\theta/2) \cdot \sec(\theta/2)] d\theta$$

$$= 2a \int_0^{\pi/2} [1+\tan^2(\theta/2)] \sec(\theta/2) d\theta$$

$$= 2a \left\{ \int_0^{\pi/2} \sec(\theta/2) d\theta + \int_0^{\pi/2} \tan^2(\theta/2) \cdot \sec(\theta/2) d\theta \right\}$$

$$= 2a \left\{ \left[\frac{\log[\sec(\theta/2)+\tan(\theta/2)]}{(1/2)} \right]_0^{\pi/2} + \int_1^{\sqrt{2}} t \cdot 2dt \right\}$$

$$= 2a \left\{ \left[\frac{\log(\sqrt{2}+1) - \log(1+0)}{(1/2)} \right] + 2\left[\frac{t^2}{2}\right]_1^{\sqrt{2}} \right\}$$

$$= \left[4a \log(1+\sqrt{2}) + 4a\left(1-\frac{1}{2}\right) \right]$$

$$\boxed{S = 4a\left[\frac{1}{2} + \log(1+\sqrt{2})\right]}$$ → Reqd. length of Parabola

Put $\sec(\theta/2) = t$

$\frac{1}{2}\sec(\theta/2)\tan(\theta/2) = dt$

$\therefore \sec(\theta/2) \cdot \tan(\theta/2) = 2dt$)

For $\theta = 0 \Rightarrow t = 1$

$\theta = \pi/2 \Rightarrow t = \sqrt{2}$

H.W. (i) Explain the term "Rectification". Mention the formulae to find length of ~~arc~~ an arc of curve for cartesian and Polar curves. Also find Perimeters of (i) $x^2+y^2=a^2$ (ii) $r = a\sin\theta$

(ii) Find ~~total~~ length of curve in first quadrant for $x=0$ to $x=a$ if curve is $x^{2/3}+y^{2/3}=a^{2/3}$

(iii) Find length of the cardioid $r = a(1+\cos\theta)$ which lies outside the circle $r = a\cos\theta$.

(iv) Find Perimeter of $r = a(1-\cos\theta)$