Friday, October 8, 2021
8:53 AM

Definition: Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, 'a' is the greatest element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, 'a' is the least element of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists.

Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

**a)** Find the maximal elements.
**b)** Find the minimal elements.
**c)** Is there a greatest element?
**d)** Is there a least element?
**e)** Find all upper bounds of $\{2, 9\}$.
**f)** Find the least upper bound of $\{2, 9\}$, if it exists.
**g)** Find all lower bounds of $\{60, 72\}$.
**h)** Find the greatest lower bound of $\{60, 72\}$, if it exists.

if $\forall b \in S,\ b \preccurlyeq a$.

$\forall b \in S,\ b | a$

$a \preccurlyeq b\ \forall b \in S$

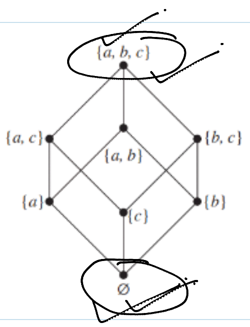
$a | b\ \forall b \in S$.

$\boxed{27, 48, 60, 72.}$

$2, 9$

$[\{2, 4, 8, 16, 32\}, 1]$

Let $S$ be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.




$\emptyset \preccurlyeq A \quad \forall A \in P(S)$

$\subseteq$

$A \subseteq \{a, b, c\}$

$\preccurlyeq$

Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

Definition:

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset $A$ of a poset $(S, \preccurlyeq)$. If $u$ is an element of $S$ such that $a \preccurlyeq u$ for all elements $a \in A$, then $u$ is called an **upper bound** of $A$. Likewise, there may be an element less than or equal to all the elements in $A$. If $l$ is an element of $S$ such that $l \preccurlyeq a$ for all elements $a \in A$, then $l$ is called a **lower bound** of $A$.

Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

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$(S, \preccurlyeq)$ $\boxed{A \subseteq S.}$

$u \in S.$ if $\forall a \in A,\ a \preccurlyeq u.$

$A = \{2, 9\} \subseteq S.$

$2 \preccurlyeq 4$

$9 \preccurlyeq 4$

$\boxed{18, 36, 72}$

$u_1, u_2, u_3.$

Definition:

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset $A$ of a poset $(S, \preccurlyeq)$. If $u$ is an element of $S$ such that $a \preccurlyeq u$ for all elements $a \in A$, then $u$ is called an **upper bound** of $A$. Likewise, there may be an element less than or equal to all the elements in $A$. If $l$ is an element of $S$ such that $l \preccurlyeq a$ for all elements $a \in A$, then $l$ is called a **lower bound** of $A$.

Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

**a)** Find the maximal elements.
**b)** Find the minimal elements.
**c)** Is there a greatest element?
**d)** Is there a least element?

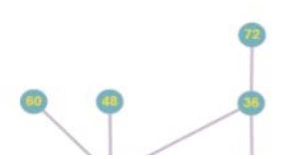
$l$ if $l \preccurlyeq a\ \forall a \in A.$

$l_1 = 2$

$2 \preccurlyeq a\ \forall a \in A.$

$\{60, 72\}$

$l = 4$



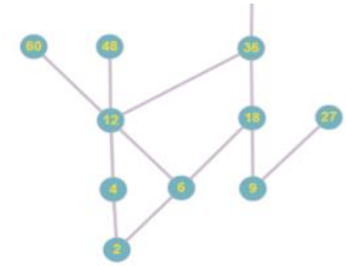
**b)** Find the minimal elements.
**c)** Is there a greatest element?
**d)** Is there a least element?
**e)** Find all upper bounds of {2, 9}.
**f)** Find the least upper bound of {2, 9}, if it exists.
**g)** Find all lower bounds of {60, 72}.
**h)** Find the greatest lower bound of {60, 72}, if it exists.

$l_1 = 2$
$l_2 = 4$
$l_3 = 6$
$l_4 = 12.$

$2 \preccurlyeq a \ \forall a \in A$
$\{60, 72\}$

$B = \{2, 9, 18\} \subseteq S$

18, 36, 72.




Definition: The element '$x$' is called the least upper bound of the subset 'A' if '$x$' is an upper bound that is less than every other upper bound of A. Because there is only one such element, if it exists, it makes sense to call this element the least upper bound. That is, '$x$' is the least upper bound of A if $a \preccurlyeq x$ whenever a ∈ A, and $x \preccurlyeq z$ whenever $z$ is an upper bound of A. Similarly, the element '$y$' is called the greatest lower bound of A if '$y$' is a lower bound of A and $z \preccurlyeq y$ whenever $z$ is a lower bound of A. The greatest lower bound of A is unique if it exists. The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively.

Answer these questions for the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |).
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18, 36, 72.

$x$ lub(A)
$\forall a \in A \quad a \preccurlyeq x$
& if $y$ is another upp. bound. of A.
then $x \preccurlyeq y$.

2, 4, 6, 12

$x$ glb(A) if $\forall a \in A$, $x \preccurlyeq a$
if $y$ is another lower bound of A, then $y \preccurlyeq x$

If A = S. upp bound of A.
{{2, 4, 8, 16, 32}, |} A = S.

Find the greatest lower bound and the least upper bound of the sets {3, 9, 12} and {1, 2, 4, 5, 10}, if they exist, in the poset ($\mathbf{Z}^+$, |).

Find the lower and upper bounds of the subsets {$a, b, c$}, {$j, h$}, and {$a, c, d, f$} in the poset with the Hasse diagram shown in Figure 7.

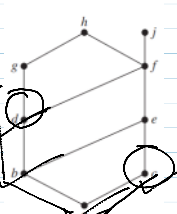


**FIGURE 7** The Hasse diagram of a poset.

(a) d (b) g
(c) e. (d) All of these

1, 3

36, 72,

e, f, j, h.

glb = hcf gcd.
lub = LCM.

*Solution:* The upper bounds of {$a, b, c$} are $e, f, j$, and $h$, and its only lower bound is $a$. There are no upper bounds of {$j, h$}, and its lower bounds are $a, b, c, d, e$, and $f$. The upper bounds of {$a, c, d, f$} are $f, h$, and $j$, and its lower bound is $a$.

Find the greatest lower bound and the least upper bound of {$b, d, g$}, if they exist, in the poset shown in Figure 7.