ECE 106 - Electricity and Magnetism

Kevin Carruthers

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Structure of an Atom

An atom has a nucleus comprised of protons (p+) and neutrons (n). It is surrounded by an electron (e-) cloud. Each has a charge of $\pm 1.6 * 10^{-19} C$ (note that a single Coloumb would be fucking huge.). Any atom with an equal amount of electrons and protons is electrically neutral. If one or more electrons is removed, the atom is positively charged. Extra electron(s) make the atom negatively charged.

In a **conductor**, the electrons' bond to the nucleus is very weak. As such, they are free to move. By bringing a charged object nearby, we can induce a charge into such a medium.

In an **insulator**, the electrons are not free to move; however, the atoms can be slightly "re-shaped" internally. We can also induce charge in inductors. This is known as polarizability.

Electromagnetic Force

Electromagnetic force is the second strongest of the four fundamental forces, weaker only than the strong force, which has extremely short range. The electromagnetic force has infinite range, as it is inversely proportional to the distance between interacting charges; it may grow extremely weak, but will never lose all effect.

It can also both attract and repel, depending on the relative differences (from zero) of interacting charges.

Electric Charge

Electric charge is the fundamental unit upon which electromagnetism acts, much like mass/energy is for gravity. Charge is discrete in nature, which means all values of charge are some integer multiple of the charge of an electron: electromagnetism's fundamental unit. Note that charge can thus be negative or positive.

Charge is conserved - it can neither be created nor destroyed, and thus in any isolated (closed) system, the total charge can never change.

Electric Field

The electric field E is defined as the force created by a charge, divided by the charge of the interacting particle

$$E = \frac{kq}{d^2}\vec{d}$$

(I'll define k soon). Note that positive charges will create a field pointing away from them, and vice versa, as evidenced by the $\vec{d}$. In the future, I will not show this **direction vector** unless necessary for clarity, but it is present in virtually all calculations.

For any charge q in an electric field, we can calculate the **electric force** acting on it by

$$F = qE$$

This gives us the general equation for electric force

$$F = \frac{kq_0q_1}{d^2}$$

Note the similarity to the equation for gravitational force.

Vector Diagrams of the Electric Field

When sketching fields, the direction of arrows shows the direction of the field, and the density of arrows shows the relative strength of the field (qualitatively). Since fields are vectors, we can then add multiple fields through vectorial addition (and/or superposition).

Coloumb's Law

The force acting between two charges is equal to

$$k\frac{q_0q_1}{d^2}$$

where k is Coloumb's constant and $k = \frac{1}{4\pi\varepsilon_0} = 9 * 10^{-9}$. $\varepsilon_0 = 8.854 * 10^{-12}$ is the permitivity of free space, which does indicate that the value of ε_0 (and thus k) will change depending on the medium through which the field travels.

Dipoles

A dipole is a area where two equal but opposite charge are in close proximity (separated by d, where d is very small). When a dipole exists, the approximate electric field is

$$E \propto \frac{1}{r^n}$$

where $n < 2$. We can also do vector addition with the following

$$r_1 + \frac{d}{2} = r_2 + \frac{d}{2} = r$$

thus giving us the simplification for the electric field of a dipole system

$$E = \frac{kqd}{r^2}$$

Dipole Moment

The dipole moment is written as

$$p = qd$$

which is the charge multiplied by its displacement from the other dipole. Note that this simplifies our earlier dipole equation to

$$E = \frac{kp}{r^2}$$

Charge Distribution

Charge Density

The charge density rho is equal to

$$\rho = \frac{Q}{V}$$

This gives

$$\mathrm{d}E = \frac{k\rho}{r^2} \;\; \mathrm{d}V$$

and

$$E = \int \frac{k\rho}{r^2} \, \mathrm{dV}$$

Note that this is a vectorial integration, as r is changing in direction and size. This can only be easily done in symmetrical cases.

For a charge distributed evenly over a disk of radius r

$$\rho = \frac{Q}{\pi r^2}$$

and

$$\mathrm{d}Q = \rho 2\pi r \ \mathrm{d}r$$

The disk would create an electric field such that

$$\mathrm{d}E = \frac{k \ \mathrm{d}Q}{4\pi\varepsilon(r^2 + z^2)^{\frac{3}{2}}}$$

and

$$E = \int_0^r \frac{\mathrm{d}\rho \ 2\pi r \ z}{4\pi\varepsilon(r^2 + z^2)^{\frac{3}{2}}} \ \mathrm{dr}$$

which is

$$E = \frac{2\pi\rho z}{4\pi\varepsilon}\left[\frac{1}{2} * \frac{1}{\sqrt{r^2 + z^2}}\right] = \frac{\rho z}{4\varepsilon\sqrt{r^2 + z^2}}$$

Superposition

Superposition is basically a simple concept: in electromagnetism, forces add. In a given system with N charges, the force extered on one of them is

$$F = \sum_{i=1}^{N} \frac{kq_0q_1}{r^2}$$

Given charge density, we can integrate this sum to find

$$F = \sum_{i=1}^{N} \frac{kq\rho\Delta V}{r^2}$$

given N chunks of volume ΔV and charge $\rho\Delta V$. Technically, this formula is an approximation, but we can use integrals to determine the exact answer with

$$F = \int_V \frac{kq\rho}{r^2} \ \mathrm{dV}$$

Uniformly Distributed Charge

For any point on an object with uniformly distributed charge, we can give it a width of $\mathrm{d}y$ thus giving us

$$\begin{aligned}\mathrm{d}E &= \frac{k \ \mathrm{d}Q}{r_1^2} \\ &= \frac{k\rho \ \ \mathrm{d}y}{r^2 + y^2}\end{aligned}$$

So we have a general equation of the form

$$E = \int_0^L \mathrm{dE}$$

where L is the length of the object. Note that this is a vectorial integral: as such, the direction of the field will always be perpendicular to the object (ie symmetrical along the horizontal axis).

Because this integral is symmetrical, we can integrate only along the axis, thus giving us

$$\begin{aligned} E &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{dE} \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k\rho r}{r_1^3} \mathrm{dy} \\ &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{k\rho r}{(r^2+y^2)^{\frac{3}{2}}} \mathrm{dy} \\ &= \frac{k\rho L}{r\sqrt{r^2+\frac{L^2}{4}}} \end{aligned}$$

Thus as $L \to \infty$, $E \propto \frac{1}{r}$

For charges uniformly distriuted on a ring, we have

$$\mathrm{d}E = \frac{k\rho \, \mathrm{d}L}{(r')^2}$$

or for any one direction

$$\mathrm{d}E = \frac{k\rho \, \mathrm{d}L}{(r')^2} \frac{z}{(r')^2}$$

where z is the direction along it's z-axis, so

$$E = \frac{k\rho z}{(r')^2} \int \mathrm{dL}$$

where $\int$ dL is the circumferance.

Thus gives us the general form of this equation

$$E_z = \frac{\rho r z}{2\varepsilon(r^2+z^2)^{\frac{3}{2}}}$$

Electric Fields for Infinite Plains

A plane can be written as a disk with an infinite radius. Then

$$\frac{1}{\sqrt{r^2 + z^2}} \to 0$$

so

$$E = \frac{\rho}{2\varepsilon}$$

Note that this has no relation to z, and is constant with distance.

Electric Flux

Electric flux is the *flow* of some vectorial quantity (ie charge) through a given area. It is given as the overlap between the *amount of flow* and the *given area*, thus giving us

$$\mathrm{d}\Phi = E \ \mathrm{d}A$$

where $\mathrm{d}A$ is the area of an open object. This gives us

$$\Phi = \int E \ \mathrm{dA}$$

or as a non-vectorial implementation

$$\Phi = EA\cos\theta$$

where θ is the angle between the perpendicular vector to the object and the direction of the field lines.

Gauss' Law

An important thing to note is that for *any* closed surface, we have $\Phi = 0$, since as much flux leaves the object as enters it.

If we have a closed surface with an electric field pointing in only one direction (example: a sphere of radius r with a point charge q enclosed in it's center), this gives us

$$\Phi = \int E \ \mathrm{dA} = \int \frac{kq}{r^2} \ \mathrm{d\pi r^2} = \frac{kq}{r^2}\frac{r^2}{\varepsilon_0} = \frac{1}{\varepsilon_0}q$$

If the surface is not spherical, we have the same equation: we can create a spherical region around the charge, segregate this area, and perform superposition to find our answer ($k4\pi q + 0 = k4\pi q$). Note that for more than one charge this gives us

$$\Phi = \frac{1}{\varepsilon_0}\sum_{i=1}^{N} q_i$$

This is **Gauss' Law**: that *the electrix flux through a closed surface S is equal to the total charge contained inside S.* Mathematically, we have

$$\Phi = \frac{1}{\varepsilon_0} q_{within}$$

Using Gauss' Law

Gauss' Law is indescribable useful in situations involving symmetry. The simplest case is in that of spherical symmetry.

Example: suppose we have a spherical distribution of charge with density $\rho(r)$ (ie the density is a function of the radius).

Enclose this distribution within a spherical surface S of radius r. Gauss' Law gives

$$\Phi = \frac{1}{\varepsilon_0} q_{within}$$

Since the field is spherically symmetric, E must point radially and be proportional to r. It can only depend on r, which means it is constant over any spherical surface. Thus we have the integral

$$\Phi = \frac{1}{\varepsilon_0} r^2 E(r)$$

The charge within S is given by integrating the charge density over the volume of S

$$q = \int_V \rho(r) \text{ dV}$$

so

$$E(r) = \frac{\int_V \rho(r) \text{ dV}}{\varepsilon_0 r^2} = \frac{q(r)}{\varepsilon_0 r^2}$$

Note that outside of the sphere there is no charge (ρ is constant), so we have

$$\begin{aligned} E(d) &= \frac{\rho d}{3\varepsilon_0} \\ &= \frac{\rho r^3}{3\varepsilon_0 d^2} \end{aligned}$$

Or in other words: the electric field grows linearly with d inside the sphere, but falls off inversely proportional to the square distance outside of the sphere (ie when $d > r$). Thus the external field is exactly that of a point charge

$$q = \frac{\rho r^3}{3\varepsilon_0}$$

Energy

Work

Work is a measure of the force exerted to move a charge from one location to another. It is given by

$$W = \int F \text{ dr}$$

where F is the opposite of the force exerted upon that charge. This gives

$$\begin{aligned} W(r_1 \to r_2) &= \int F \text{ dr} \\ &= -\int_{r_1}^{r_0} \frac{kq_0q_1}{r^2} \text{ dr} \\ &= \frac{kq_0q_1}{r_0} - \frac{kq_0q_1}{r_1} \end{aligned}$$

An important fact to note is that the path travelled does not make a difference to the amount of work done; the only things which matter are the start and end positions.

Field Energy

Field energy can be equated with pressure: it is the measure of force caused by an object's electric field per it's area. It is calculated by multiplying the electric field of a "hole" within the object by the charge density of the disk formed by creating this hole. This gives us

$$P = (E_{obj} - E_{disk})\rho$$

We can use this quantity to help us calculate work done with

$$\mathrm{d}W = F \text{ d}r = PA \text{ d}r$$

In essence, this gives us "We had to put $\mathrm{d}W$ = (energy density) $\mathrm{d}V$ amount of work into the system *to create* that field."

Potential Difference

Potential difference is the measure of work which would have to be done to move a unit of charge from one location to another *per unit of charge.* It is given by

$$\phi_{ba} = -\int_a^b E \text{ ds}$$

which can be derived by solving for the amount of work, then dividng out the charge (since work done is proportional to the charge).

Potential

The electric potential of an object is its potential difference with respect to some fixed point. Though this point is generally infinity, that is not always the case. When it is we have

$$\phi(r) = -\int_{\infty}^{r} E \text{ ds}$$

We can also redefine potential difference in terms of potential (yes, this will be as painfully obvious as it sounds). Given a and b, the potential difference between them is equal to

$$\phi_{ba} = \phi(b) - \phi(a)$$

If we suppose our field is created from a point charge q at the origin we have

$$\phi(r) = -\int_{\infty}^{r} \frac{kq}{r^2} \text{ dr} = \frac{q}{r}$$

which gives us a formula we can use to generalize this. Given superposition, we have the potential at some point P from N contributing charges as

$$\phi(P) = \sum_{i=1}^{N} \frac{kq_i}{r_i}$$

this can then be further generalized to account for continuous distributions with

$$\phi(P) = \int_V \frac{k\rho}{r} \text{ dV}$$

Obviously, these equations are only useful if we *can* set the reference point to infinity. This, in turn, is only possible if the charge distribution is of a finite size. If the distribution is infinitely large, we will find the equation to diverge as the reference point approaches infinity. In such a case, simply pick a different reference point.