MATH 119 - Calculus 2 for Engineering

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Appoximation Methods

Some methods (ie unintegratable ones) must be approximated. There are two such methods for approximation, analytic and numerical.

For **analytic approximation** we make a simplification, ie

$$\sin x^2 = x^2$$

for any small x.

Numerical approximation is the brute force approach. Using the midpoint rule we have

$$\int_0^4 \sin x^2 \cos(\sin x)\ \mathrm{dx} \approx 0.52725$$

Linear Approximation

Linear approximation is also known as Tangent Line Approsimation or Linearization. The definition of a derivative is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The linear approximation near $x = a$ is

$$L(x) = f(a) + f'(a)(x - a)$$

Note that differentials are approximations of this.

For a pendulum we have

$$\begin{aligned}
m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} &= -mg\sin\theta \\
mL\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} &= -mg\sin\theta \\
\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} &= -\frac{g}{L}\sin\theta \\
\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} &= -\frac{g}{L}\theta \\
\theta(t) &= \theta_0 \cos\left(t\sqrt{\frac{g}{L}}\right)
\end{aligned}$$

In general form, for

$$f(x) = ae^{b(x+c)}$$

we have

$$L(x) = a + ab(x+c)$$

Newton's Method

Find a root of $x^3 - 2x - 5 = 0$. With linear approximatinon we have

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

thus for

$$\begin{aligned}
f(0) &= -5 \\
f(1) &= -6 \\
f(2) &= -1 \\
f(3) &= 16
\end{aligned}$$

and

$$f'(x) = 3x^2 - 2$$

so

$$L(x) = -1 + 10(x-2) = 10x - 21$$

gives us

$$x = 2.1$$

We then take

$$f(2.1) + f'(2.1)(x - 2.1)$$

leads to

$$x = 2.09457$$

through repetition.

Formally, **Newton's Method** is defined as

1. Pick x_0
2. Do linear approximation at x_0
3. Set approximation to zero, solve for x
4. $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$

Note: If Newton's Method fails, use bisection to improve your initial guess.

Fixed Point Iteration

A simpler alternative to Newton's Method is to rewrite $F(x) = 0$ as $x = g(x)$. Thus we find an approximate solution via

$$x_{n+1} = g(x_n)$$

This converges slower than Newton's Method, but is simpler to calculate.

Conditions for Convergence

A fixed point iteration scheme $x_{n+1} = g(x_n)$ will converge if $\left|g'(x)\right| < 1$ in an interval about the fixed point.

If the iteration scheme diverges, we can solve for x in a different way.

Polynomial Interpolation

Consider that we are given $n+1$ points (x, y) and we want to find a polynomial of degree n passing through them. We could either solve this with matrices, or we can use **Newton's Forward Difference Formula**

$$\Delta^m y_n = \Delta^{m-1} y_{n+1} - \Delta^{m-1} y_n$$

This will reduce the system to one of size $n-1$. By iterating through this method until we have an $n = 1$ system, we can solve for each of the coefficient by substituting them into the general polynomial. This will give us a general solution which can then be used for any dataset. This solution is of the form

$$y = y_0 + x\Delta y_0 + ... + x(x-1)...(x-n+1)\frac{\Delta^n y_0}{n!}$$

If we have non-unit spacing, this formula becomes

$$y = y_0 + \frac{x - x_0}{h}\Delta y_0 + ... + \frac{(x - x_0)...(x - x_{n-1})}{n!h^n}\Delta^n y_0$$

Note that this is mostly a generalized version, and you may assume equal unit spacing by $x_z = z$ and $h = 1$. Also note theat $x_n = x_0 + nh$ where $h = \Delta x$.

If we have both non-unit and non-equal spacing, we use **Newton's Divided Differences**, which is generalized from

$$\Delta^m f(x)_n = \frac{\Delta^{m-1} f(x)_{n+1} - \Delta^{m-1} f(x)_n}{x_{n+1} - x_n}$$

Note that high-order polynomials are known to be innaccurate, and oscillate wildly at each end.

The Lagrange Linear Interpolation Formula is

$$f(x) \approx \left(\frac{x - x_1}{x_0 - x_1} \right) f(x_0) + \left(\frac{x - x_0}{x_1 - x_0} \right) f(x_1)$$

Taylor Polynomials

The nth order Taylor Polynomial is

$$P_{n,x_0}(x) = f(x_0) + (x - x_0) f'(x_1) + \frac{(x - x_0)^2}{2!} f''(x_2) + ... + \frac{(x - x_0)^n}{n!} f^{n\prime}(x_n)$$

Note that high order Taylor Polynomials completely break down.

More generally, we have

$$P_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!} (x - x_0)^k$$

Taylor's Theorem with Integral Remainders

If $f(x)$ ihas $n+1$ derivatives at x_0, then

$$f(x) = \sum_{k=0}^{n} f^k(x_0)(x - x_0)^k + R_{n,x_0}(x)$$

where

$$R_{n,x_0}(x) = \int_{x_0}^{x} \frac{(x - t)^n}{n!} f^{n+1}(t) \text{ dt}$$

If we can bound

$$\left| f^{n+1}(t) \right| \leq k$$

for all t between x and x_0 then

$$\begin{aligned}
E &= \left|f(t) - P_{n,x_0}(x)\right| \\
&= \left|R_{n,x_0}(x)\right| \\
&= \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)\ \mathrm{dt} \\
&\leq \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)\ \mathrm{dt} \\
&\leq \int_{x_0}^{x} \frac{\left|x-t\right|^n}{n!} \left|f^{n+1}(t)\right|\ \mathrm{dt} \\
&\leq k \int_{x_0}^{x} \frac{\left|x-t\right|^n}{n!}\ \mathrm{dt} \\
&\leq k \frac{\left|x-t\right|^{n+1}}{(n+1)!}\ \Bigg|_{x_0}^{x} \\
&= k \frac{\left|x-x_0\right|}{(n+1)!} \\
&\geq \left|R_{n,x_0}(x)\right| \\
E &= \left|R_{n,x_0}(x)\right| \leq k \frac{\left|x-x_0\right|^{n+1}}{(n+1)!}
\end{aligned}$$

This is called **Taylor's Inequality**.