Nonparametrics and Local Methods

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Microeconometrics

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Something Familiar

Let's start with something we all know, how to calculate:

$$E[Y_i|X_i] = \beta_0 + \beta_1 X_1 + \ldots + \varepsilon_i$$

- The Gauss-Markov theorem (remember that?) tells us that OLS is best among linear unbiased estimators.

Identification

What does it mean for a model to be identified?

$$\hat{\beta}_{OLS} = \arg\max_{\beta} (\mathbf{y} - \mathbf{x}\beta)'(\mathbf{y} - \mathbf{x}\beta)$$

- We need there to be a unique value of β that solves the above equation.
- For OLS this reduces to solving a linear system of $\dim(X) = k$ equations and unknowns.
- The OLS estimator produces $\hat{\beta}_{OLS} = \beta + (X'X)^{-1}X'\varepsilon$.
  - Identification: means that we require $(X'X)$ is invertible (no perfect multicollinearity).
  - When $E[X'\varepsilon] = 0$ then this converges in probability to true β.

A Little More Complicated

- For many cases that we care about, $E[X'\varepsilon] \neq 0$. (Endogeneity)
- We know this is not the end of the world if we have an instrument Z:
- Now we need that $Z'X$ is invertible.

$$\begin{aligned} E[\mathbf{x}'(\mathbf{y} - \mathbf{x}\beta)] &\neq 0 \\ E[\mathbf{z}'(\mathbf{y} - \mathbf{x}\beta)] &= 0 \\ \rightarrow \hat{\beta}_{TSLS} &= E(Z'X)^{-1}E[Z'Y] \end{aligned}$$

Some more information

This leads to the common terminology:

- Under-identified: if $\dim(X) > \dim(Z)$.
- Just-identified: if $\dim(X) = \dim(Z)$.
- Over-identified: if $\dim(X) < \dim(Z)$.

What does this mean exactly

To understand this we usually construct the GMM form of the moment conditions:

$$\begin{aligned} E[\mathbf{z}'(\mathbf{y} - \mathbf{x}\beta)] &= 0 \\ f(\beta) &= (\mathbf{y} - \mathbf{x}\beta)'\mathbf{z} \cdot W \cdot \mathbf{z}'(\mathbf{y} - \mathbf{x}\beta) \end{aligned}$$

- Because $f(\beta)$ is a quadratic form, and when W is PSD then $f(\beta) \geq 0$.
- Just-identified: There is exactly one solution to $f(\beta) = 0$ which does not depend on W.
- Under-identified: $f(\beta) = 0$ has many solutions because we have fewer instruments (or moment restrictions) than parameters.
- Over-identified: There are no solutions and $f(\beta) > 0$. Instead we choose β to minimize the distance from zero with distance metric W.

The Binary Case

Now let's think about a case where $Y \in \{0, 1\}$:

$$y_i = \mathbf{1}[F(X_i) - \varepsilon_i > 0]$$

- There are different choices of $F(X_i)$ and $f(\varepsilon)$
- Probit: $F(X_i) = \beta X_i$ where $\varepsilon_i \sim N(0, 1)$.
- Logit: same $F(\cdot)$ but different $f(\varepsilon)$ (Type I Extreme Value). $P(Y_i = 1|X_i) = 1/(1 + exp[-\beta X_i])$.
- These choices of F are strong and somewhat arbitrary, we choose them out of convenience because both transformations are monotonic in βX_i, and continuously map $[-\infty, \infty] \rightarrow [0, 1]$

The Binary Case

What is the least I can assume in the binary case and still learn something?

$$y_i = \mathbf{1}[F(X_i) - \varepsilon_i > 0]$$

- Suppose instead we observe $P(x) = Pr(y_i = 1|x_i)$ at many values of x_i. (Often refer to this as the CCP).
- Sometimes CCPs are all we care about. (if we are lucky).

Instead we can try and solve:

$$P(x) = \int \mathbf{1}(F(x) - \varepsilon > 0) dH(\varepsilon|x) = H(F(x)|x)$$

- Identification asks, what is the least we need to assume in order to recover F, H?
- Even with $\varepsilon \perp X$ we still have too many degrees of freedom.

Are we stuck?

- We know that Logit and Probit are identified.
- Start with $\varepsilon \perp X$.
- $F(X_i) = G(X_i\beta)$ is a very powerful assumption often called (single) index model
- Can use Marginal Rate of Substitution (MRS) of $P(x)$ to identify β_k/β_l.
  - How do conditional probabilities respond to changes in $X^{(k)}$ versus $X^{(l)}$?
  - Suggests the average derivative estimator which doesn't require normality assumptions of probit
  - More robust, but potentially less efficient if true distribution of $\varepsilon \sim N(0,1)$.
- Can we extend the general intuition to more cases?

A Fake Data Example

Following the THF textbook example, we can generate some fake data and let:

$$\begin{aligned} Y = & \quad ORANGE \text{ if } Y* \quad > 0.5 \\ Y = & \quad\quad BLUE \text{ if } Y* \quad\quad \leq 0.5 \end{aligned}$$

- Easiest way to recover $Y*$ is by running OLS on the linear probability model.
- Draws from bivariate normal distribution with uncorrelated components but different means (2 overlapping types)
- Mixture of 10 low variance (nearly point mass) normal distributions where the individual means were drawn from another normal distribution. (10 nearly distinct types).

Linear Probability Model

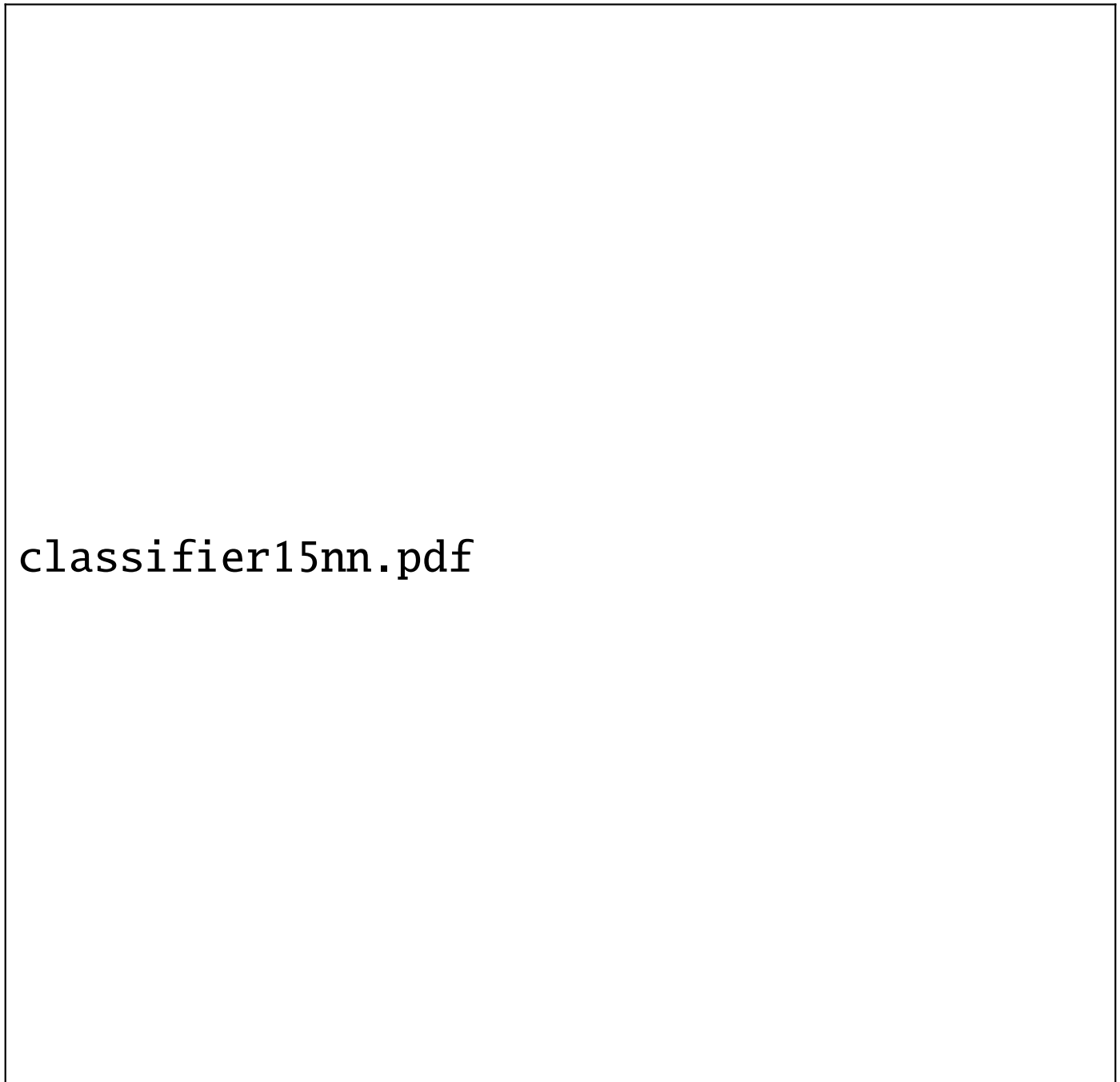
Alternative

- Lots of potential alternatives to our decision rule.
- A simple idea is to hold a majority vote of neighboring points

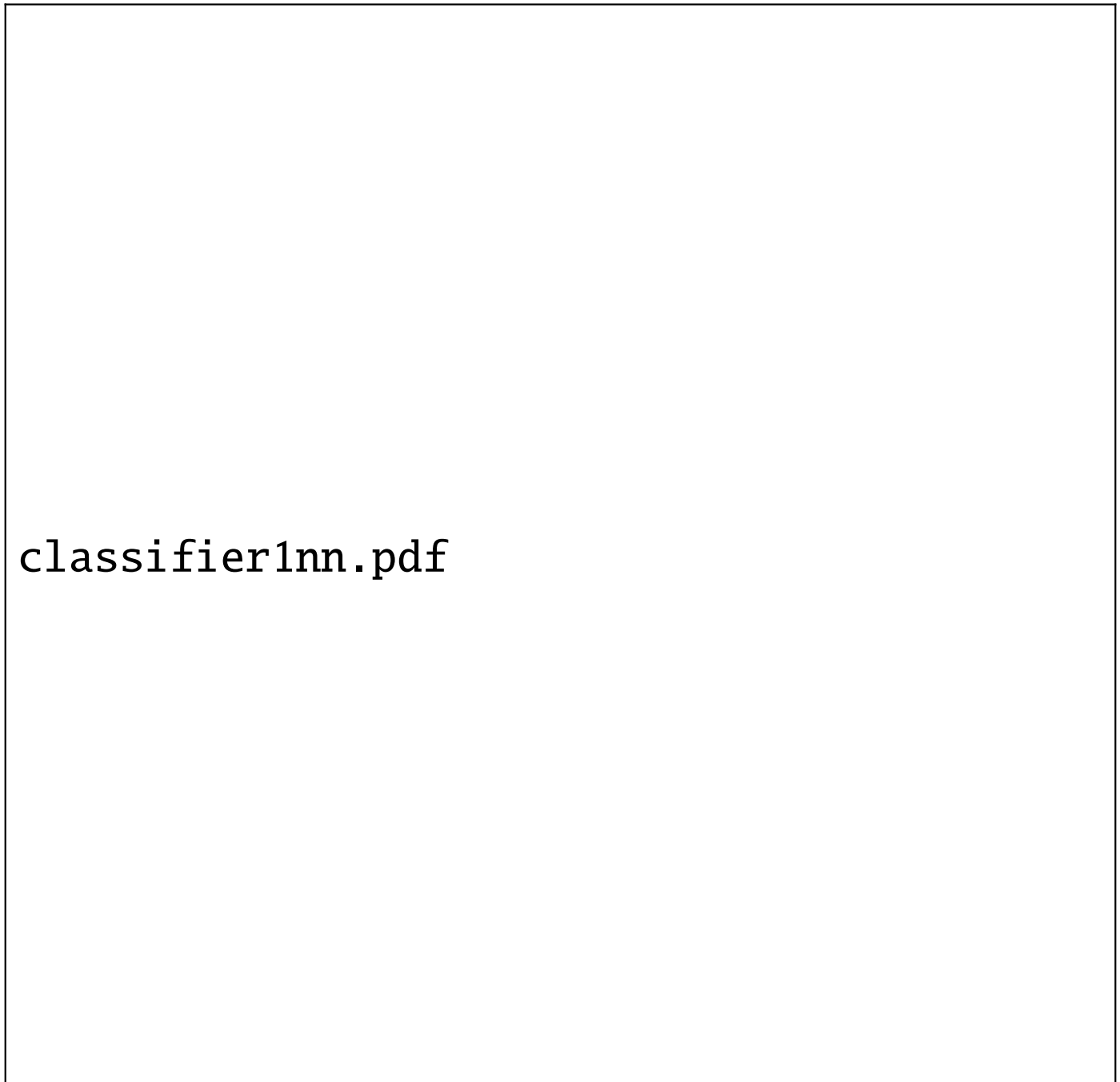
$$Y^* = \frac{1}{k} \sum_{x_i \in N_k(x)} y_i$$

- To avoid including "yourself" in your neighborhood, we often estimate on one sample and validate on another
- How many parameters does this model have: None? One? k?
- Technically it has something like N/k.
- As $N \to \infty$ this means we have an infinite number of parameters! (This is a defining characteristic of non-parametrics).

15 Nearest Neighbor



Extreme: 1 Nearest Neighbor



Comparisons

- What would happen if $K \to N$?
- The k-NN model is locally constant.
- The k-NN approach tends to be really bumpy which can be undesirable.
- The OLS model is globally linear (is this always true?)

What about?

- If we fixed the fact that there are discrete jumps in who is in the neighborhood by smoothly weighting observations and varying those weights instead (Kernels).
- Another drawback of $k - NN$ is that we consider distance in each X dimension on the same scale, perhaps we could rescale the data to improve our "closeness" measure.
- Instead of fitting a constant locally, we fit a linear function locally (Lowess).
- Instead of using a global linear approximation in OLS use a more flexible nonlinear one.
- There is a bias/variance tradeoff. explain.

Bias Variance Decomposition

We can decompose any estimator into two components

$$\underbrace{E[(y-\hat{f}(x))^2]}_{MSE} = \underbrace{\left(E[\hat{f}(x)-f(x)]\right)^2}_{Bias^2} + \underbrace{E\left[\left(\hat{f}(x)-E[\hat{f}(x)\right]\right)^2}_{Variance}$$

- In general we face a tradeoff between bias and variance.
- In k-NN as k gets large we reduce the variance (each point has less influence) but we increase the bias since we start incorporating far away and potentially irrelevant information.
- In OLS we minimize the variance among unbiased estimators assuming that the true f is linear.

Bias Variance Decomposition

What minimizes MSE?

$$f(x_i) = E[Y_i|X_i]$$

- Seems simple enough (but we are back where we started).
- How do we compute the expectation ?
- k-NN tries to use local information to estimate conditional mean
- OLS uses entire dataset and adds structure $y = x\beta$ to the problem.

Big Data

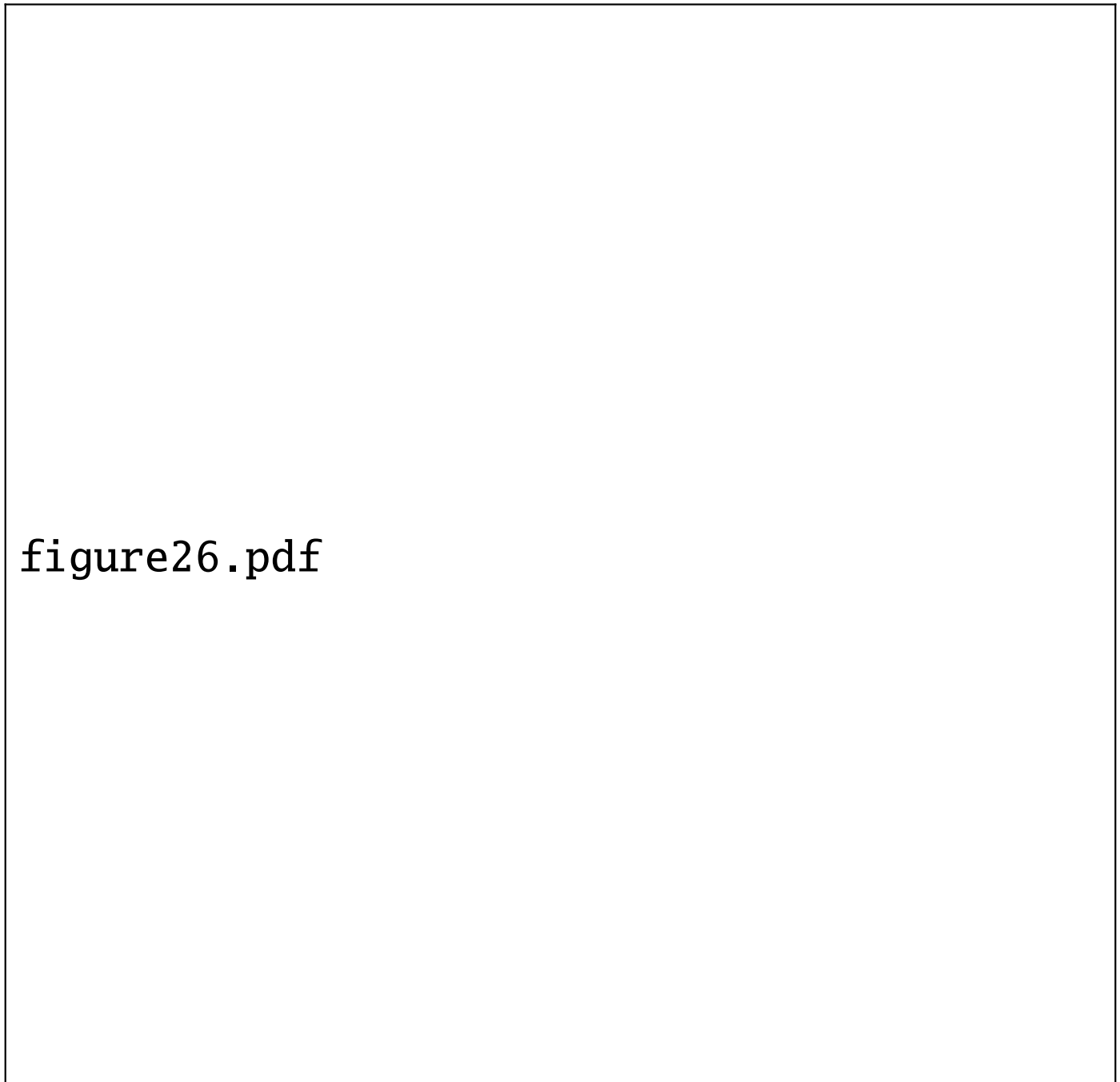
- It used to be that if you had $N = 50$ observations then you had a lot of data.
- Those were the days of finite-sample adjusted t-statistics.
- Now we frequently have 1 million observations or more, why can't we use k-NN type methods everywhere?

Curse of Dimensionality

Take a unit hypercube in dimension p and we put another hypercube within it that captures a fraction of the observations r within the cube

- Since it corresponds to a fraction of the unit volume, r each edge will be $e_p(r) = r^{1/p}$.
- $e_{10}(0.01) = 0.63$ and $e_{10}(0.1) = 0.80$, so we need almost 80% of the data to cover 10% of the sample!
- If we choose a smaller r (include less in our average) we increase variance quite a bit without really reducing the required interval length substantially.

Curse of Dimensionality



Curse of Dimensionality

Don't worry, it only gets worse:

$$d(p,N) = \left(1 - \left(\frac{1}{2}\right)^{1/N}\right)^{1/p}$$

- $d(p,N)$ is the distance from the origin to the closest point.
- $N = 500$ and $p = 10$ means $d = 0.52$ or that the closest point is closer to the boundary than the origin!
- Why is this a problem?
- In some dimension nearly every point is the closest point to the boundary – when we average over nearest neighbors we are extrapolating not interpolating.

Density/Distribution Estimation

One of the more successful and popular uses of nonparametric methods is estimating the density or distribution function $f(x)$ or $F(x)$.

- Estimating the CDF is easy and something you have already done
- Q-Q plots, etc.

$$\hat{F}_{ECDF}(x_0) = \frac{1}{N}\sum_{i=1}^{N}(x_i \leq x_0)$$

- Differentiating to get density is unhelpful : $F'_{ECDF}(x) = 0$ in most places.

Density/Distribution Estimation

One of the more successful and popular uses of nonparametric methods is estimating the density or distribution function $f(x)$ or $F(x)$.

- Think about the histogram (definition of derivative):

$$\hat{f}_{HIST}(x_0) = \frac{1}{N} \sum_{i=1}^{N} \frac{\mathbf{1}(x_0 - h < x_i < x_0 + h)}{2h}$$

- Divide the dataset into bins, count up fraction of observations in each bins
- Similar to k-NN except instead of windows that vary with x_i we have fixed width bins
- Larger bin width $\rightarrow$ More Bias, Less Variance.
- Histogram will never be smooth! (Just like k-NN).

$$\hat{f}_{HIST}(x_0) = \frac{1}{Nh} \sum_{i=1}^{N} \frac{1}{2} \cdot \mathbf{1}\left(\left|\frac{x_i - x_0}{h}\right| < 1\right)$$

Smooth Kernels

We can take our histogram and smooth it out:

$$\hat{f}(x_0) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{x_i - x_0}{h}\right) \frac{1}{n} \sum_{i=1}^{n} K_h\left(y - y_i\right).$$

We call $K(\cdot)$ a Kernel function and h the bandwidth. We usually assume

(i) $K(z)$ is symmetric about 0 and continuous.

(ii) $\int K(z)dz = 1$, $\int zK(z)dz = 0$, $\int |K(z)|dz < \infty$.

(iii) Either (a) $K(z) = 0$ if $|z| \geq z_0$ for some z_0 or (b) $|z|K(z) \to 0$ as $|z| \to \infty$.

(iv) $\int z^K(z)dz = \kappa$ where κ is a constant.

Kernel Smoothers

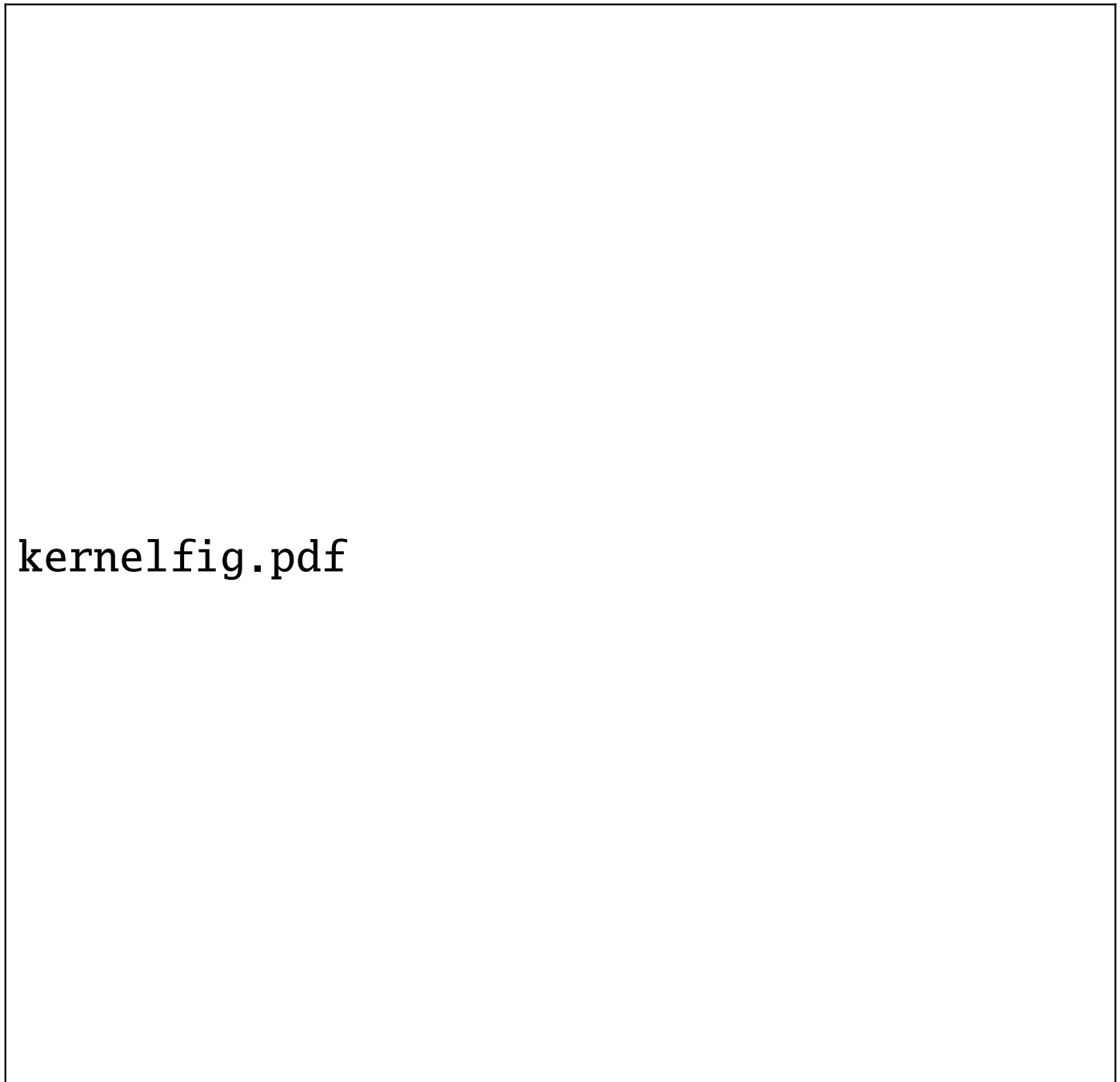
If K is C^k, then so is $\hat{f}_n$, so we can plot it nicely.
Usually we choose a smooth, symmetric K:

- $K = \phi$, density of $N(0,1)$ (or some other symmetric density);
- K with compact support: Epanechnikov (mildly) optimal

$$K(x) = \frac{3}{4}\max(1 - x^2, 0).$$

A common nonsmooth choice: $K(x) = (|x| < 1/2)$ gives the *histogram* estimate.

Kernel Comparison



How to Choose h

- We want both bias and variance to be as small as possible, as usual.
- In parametric estimation, it is not a problem: they both go to zero as sample size increases.

Problem with nonparametrics:

$$E\hat{f}_n(y) = \int K((y-t)/h)f(t)dt/h = \int K(-u)f(y+uh)du = f(y)+O(h^2)$$

$\rightarrow$ bias can be made tiny by having a very concentrated kernel ($h \simeq 0$); but

$$V\hat{f}_n(y) = \frac{1}{nh^2}VK((y-Y)/h) = O\left(\frac{1}{nh}\right)$$

$\rightarrow$ a small h gives a high variance!
Reducing h reduces bias, but increases variance; how are we to trade off?

The AMISE

- Asymptotic Mean Integrated Square Error =asymptotic approximation of a quadratic loss function

$$E\left(\hat{f}_n(y) - f(y)\right)^2 dy$$

- Simple approximate expression (symmetric kernels of order 2):

$$\text{(bias)}^2 + \text{variance} = Ah^4 + B/nh$$

- **Why?**
  Bias in y is

$$\int K(-u)\left(f(y+uh) - f(y)\right) du \simeq h^2 \frac{f''(y)}{2} \int K(u)u^2 du.$$

  Intuition: if f is close to linear around y, then averaging does not hurt us: $f''(y) \simeq 0$ and the bias is small. The bias is larger (and negative) at the mode of f.

The Variance

$$V\hat{f}_n(y) = \frac{1}{nh^2} VK((y - Y)/h)$$

The important term in

$$VK((y - Y)/h)$$

is

$$h \int K(u)^2 f(y + uh) du \simeq hf(y) \int K(u)^2 du.$$

And we end up with

$$V\hat{f}_n(y) \simeq \frac{f(y)}{nh} \int K(u)^2 du.$$

Intuition: we are really taking an average over $nhf(y)$ points. In low-density region, this induces a high *relative* imprecision:

$$\frac{\sigma(\hat{f}_n(y))}{f(y)} = \frac{1}{nhf(y)} \int K(u)^2 du.$$

Optimal bandwidth

- The AMISE is

$$Ah^4 + B/nh$$

  with $A = \int (f''(y))^2 \left(\int u^2 K \right)^2 /4$ and $B = f(y) \int K^2$

- AMISE is smallest in $h_n^* = \left(\frac{B}{4An} \right)^{1/5}$. Then,
  - bias and standard error are *both* in $n^{-2/5}$
  - and the AMISE is $n^{-4/5}$—**not** $1/n$ as it is in parametric models.
- But: A and B both depend on K (known) and $f(y)$ (unknown), and especially "wiggliness" $\int (f'')^2$ (unknown, not easily estimated). Where do we go from here?

Silverman's Rule of Thumb

- If f is normal with variance σ^2 (may not be a very appropriate benchmark!), the optimal bandwidth is

$$h_n^* = 1.06\sigma n^{-1/5}$$

- Just do it with $\sigma = s$ empirical dispersion of the y_i's , or something more robust/slightly less smooth:

$h_n^* = 0.9 * \min(s, IQ/1.34) * n^{-1/5}$, IQ=interquartile distance

- Investigate changing it by a reasonable multiple.

Cross-validation

- General concept in the whole of nonparametrics: choose h to minimize a criterion $CV(h)$ that approximates

$$AMISE(h) = \int E(\hat{f}_n(x) - f(x))^2 dx.$$

- Usually programmed in metrics software. *If you can do it, do it on a subsample, and rescale.*
- Two problems:
  - it is costly; often it involves computing "leave-one-out" estimators

$$\hat{f}_{(-i)}(x_i) = \frac{1}{nh} \sum_{j \neq i} K\left(\frac{x_i - x_j}{h}\right),$$

    for *every* observation i.
  - the resulting h converges super-slowly ($n^{-1/10}$!) to the optimal one.

Alternatives to Cross-Validation

- LOOCV
- k-fold CV
- Sample Splitting
  - Training Set
  - Test Set
  - Validation Set

Local Bandwidths

If you only care about $f(y)$ at some given point, then

$$A = f''(y)^2 \left(\int u^2 K \right)^2 /4 \text{ and } B = f(y) \int K^2.$$

So in a low-density region, worry about variance and take h larger. In a curvy region, worry about bias and take h small.

Higher-Order Kernels

- K of order r iff $\int x^j K(x)dx = 0$ for $j < r$ and $\int x^r K(x)dx \neq 0$. Try $r > 2$?
- The beauty of it: bias in h^r if f is at least C^r . . . so AMISE can be reduced to $n^{-r/(2r+1)}$, almost $\sqrt{n}$-consistent if r is large.
- But gives wiggly (and sometimes negative) estimates $\rightarrow$ leave them to theorists.

Back to the CDF

Since now we have estimated the density with

$$\hat{f}_n(y) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{y - y_i}{h}\right),$$

a natural idea is to integrate; let $\mathcal{K}(y) = \int_{-\infty}^{y} K(t)dt$, try

$$\hat{F}_n(y) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}\left(\frac{y - y_i}{h}\right)$$

as a reasonable estimator of the cdf in y.
Very reasonable indeed:

- when $n \longrightarrow \infty$ and h goes to zero (at rate $n^{-1/3}\ldots$)
  it is consistent at rate $\sqrt{n}$
- it is nicely smooth
- by construction it accords well with the density estimator
- $\ldots$ it is a much better choice than the empirical cdf.

What if y is of dimension $p_y > 1$?

"Easy": use p_y-dimensional K (often a p_y-product of 1-dim kernels) and bandwidth h, and do

$$\hat{f}_n(y) = \frac{1}{nh_y^p} \sum_{i=1}^{n} K\left(\frac{y - y_i}{h}\right).$$

- **1st minor pitfall:** the various dimensions may have very different variances, so use $(h_1, \ldots, h_{p_y})$.
- **2nd minor pitfall:** they may be strongly correlated; then sphericize first.
- **Major problem:** next slide...

The Curse of Dimensionality

- Computational cost increases exponentially.
- *Much worse:* to achieve precision ϵ in dimension p_y, the number of observations you need increases as

$$n \simeq \epsilon^{-(2+p_y/2)}.$$

- The *empty space* phenomenon: if $(y_1, \ldots, y_{p_y})$ all are iid uniform on $[-1, 1]$, then only $n/(10^{p_y})$ observations on average have all components in $[-0.1, 0.1]$. Bias still in h^2, but variance in $1/nh^{p_y}$ now.

Silverman's Table

Silverman (1986 book) provides a table illustrating the difficulty of kernel estimation in high dimensions. To estimate the density at 0 of a $N(0,1)$ with a given accuracy, he reports:

Dimensionality Required	Sample Size
1	4
2	19
5	786
7	10,700
10	842,000

Not to be taken lightly... in any case convergence with the optimal bandwidth is in $n^{-2/(4+p_y)}$ now—and Silverman's rule of thumb for choosing h_n^* must be adapted too.

Usually we care about conditional densities

That is: we have covariates x, we want the density $f(y|x)$. Again, "easy":

1. get a kernel estimator of the joint density $f(y, x)$;
2. and one of of the marginal density $f(x)$;
3. then define

$$\hat{f}_n(y|x) = \frac{\hat{f}_n(y,x)}{\hat{f}_n(x)} = \frac{\frac{1}{nh_y^{p_y}h_x^{p_x}}\sum_{i=1}^n K\left(\frac{y-y_i}{h_y}\right)K\left(\frac{x-x_i}{h_x}\right)}{\frac{1}{h_y^{p_y}}\sum_{i=1}^n K\left(\frac{x-x_i}{h_x}\right)}.$$

But the joint density is $(p_x + p_y)$ dimensional... and the curse strikes big time.

What if my distribution is discrete-continuous?

Very often in microeconometrics some covariates only take discrete values (e.g. gender, race, income bracket...). Say the only discrete variable is gender, we care about the density of income of men.

- The kernel approach adapts directly: we separately estimate a density for men (on the corresponding subsample).
- *Better:* mix the two subsamples! Add women, but **with a small weight** w.
- Intuition: by doing so we increase the bias (the density for women is probably different than for men) $\rightarrow$ bad, in w^2 but we reduce the variance, by $O(w)$; and this dominates for small w. (cf Li-Racine).

The Seminonparametric Approach

- If we are "pretty sure" that f is almost $f_{m,\sigma}$ for some family of densities indexed by (m, σ), then we can choose a family of positive functions of increasing complexity $P^1_\theta, P^2_\theta, \ldots$
- Choose some M that goes to infinity as n does (more slowly), and maximize over (m, σ, θ) the loglikelihood

$$\sum_{i=1}^{n} \log f_{m,\sigma}(y_i) P^M_\theta(y_i).$$

It works. . . but it is hard to constrain it to be a density for large M.

Mixtures of Normals

A special case of seminonparametrics, and usually a very good approach: Let $y|x$ be drawn from

$$N(m_1(x,\theta),\sigma_1^2(x,\theta)) \text{ with probability } q_1(x,\theta);$$

$$\ldots$$

$$N(m_K(x,\theta),\sigma_K^2(x,\theta)) \text{ with probability } q_K(x,\theta).$$

where you choose some parameterizations, and the q_k's are positive and sum to 1.
Can be estimated by maximum-likelihood:

$$\max_\theta \sum_{i=1}^n \log\left(\sum_{k=1}^K \frac{q_k(x_i,\theta)}{\sigma_k(x,\theta)}\phi\left(\frac{y_i - m_k(x_i,\theta)}{\sigma_k(x_i,\theta)}\right)\right).$$

Usually works very well with $K \leq 3$ (perhaps after transforming y to $\log y$, e.g).

Nonparametric Regression

Data $(y_i, x_i)_{i=1}^n$ now, we are after $E(g(y,x)|x) = m(x)$ for some function g.

- Best-fit approach, quite unbiased:
  - if $x = x_i$ then $\hat{m}_n(x) = g(y_i, x_i)$; otherwise . . . whatever.

  But: very jagged estimate; variance independent of n, so not consistent.
- Better and most usual: Nadaraya-Watson, inspired from kernel idea:

$$\hat{m}_n(x) = \frac{\sum_{i=1}^n g(y_i, x_i) K\left(\frac{x-x_i}{h}\right)}{\sum_{i=1}^n K\left(\frac{x-x_i}{h}\right)}.$$

  again, bias in h^2 and variance in $1/nh$ if $p_x = 1$.
  Pitfall 1: very unreliable where $f(x)$ is small.
  Pitfall 2: the formal for the optimal bandwidth h is very ugly.

Choosing h

- Plug-in estimates work badly.
- Fortunately, cross-validation amounts to

$$\min_h \sum_{i=1}^{n} \frac{(g(y_i, x_i) - \hat{m}_n(x_i; h))^2}{1 - k_i(h)}$$

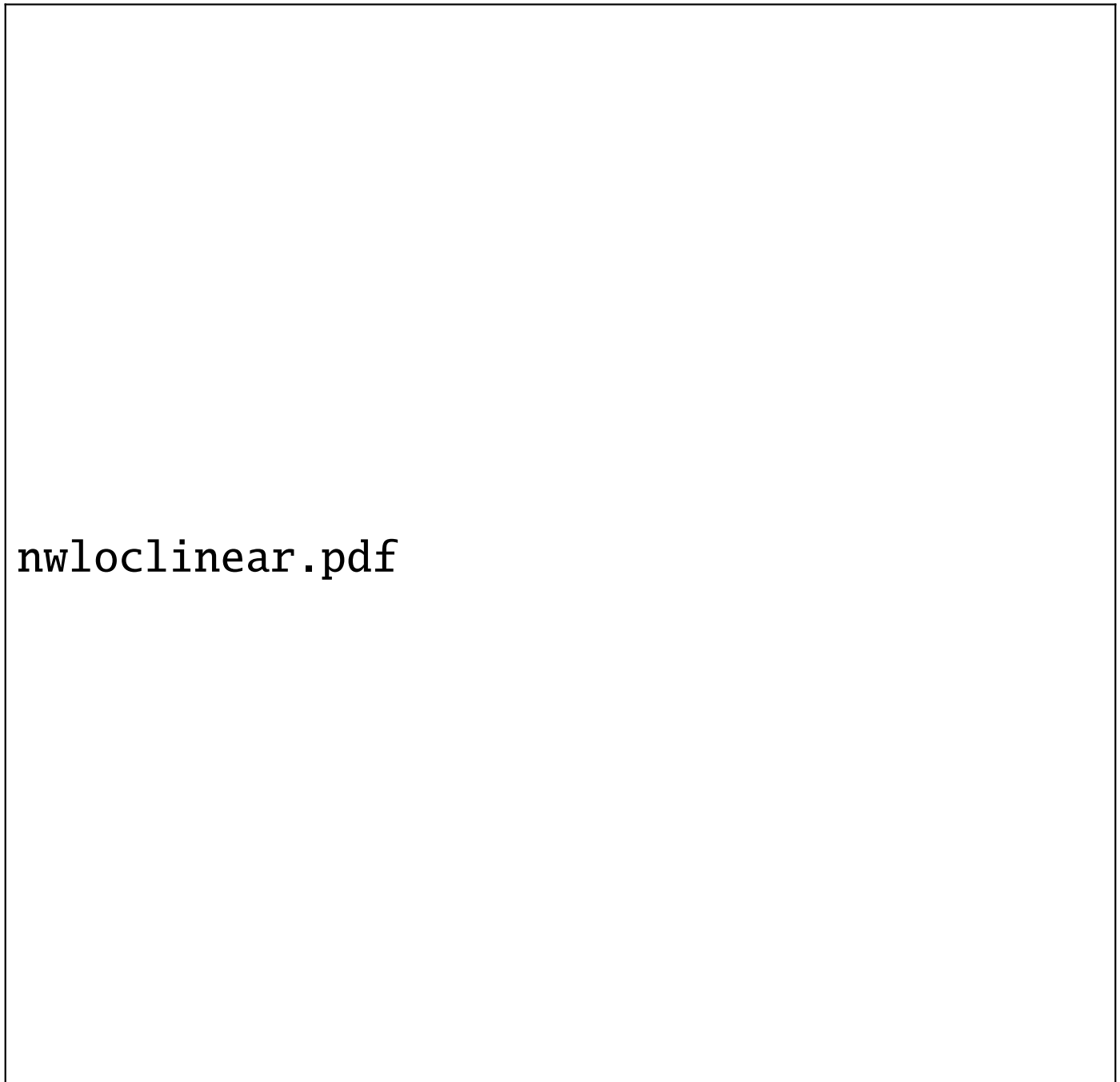
  where $k_i(h) = K_h(0) / \sum_{j=1}^{n} K_h(x_i - x_j)$.
- So not that hard, and can be done on a subsample and rescaled.

Local Linear Regression

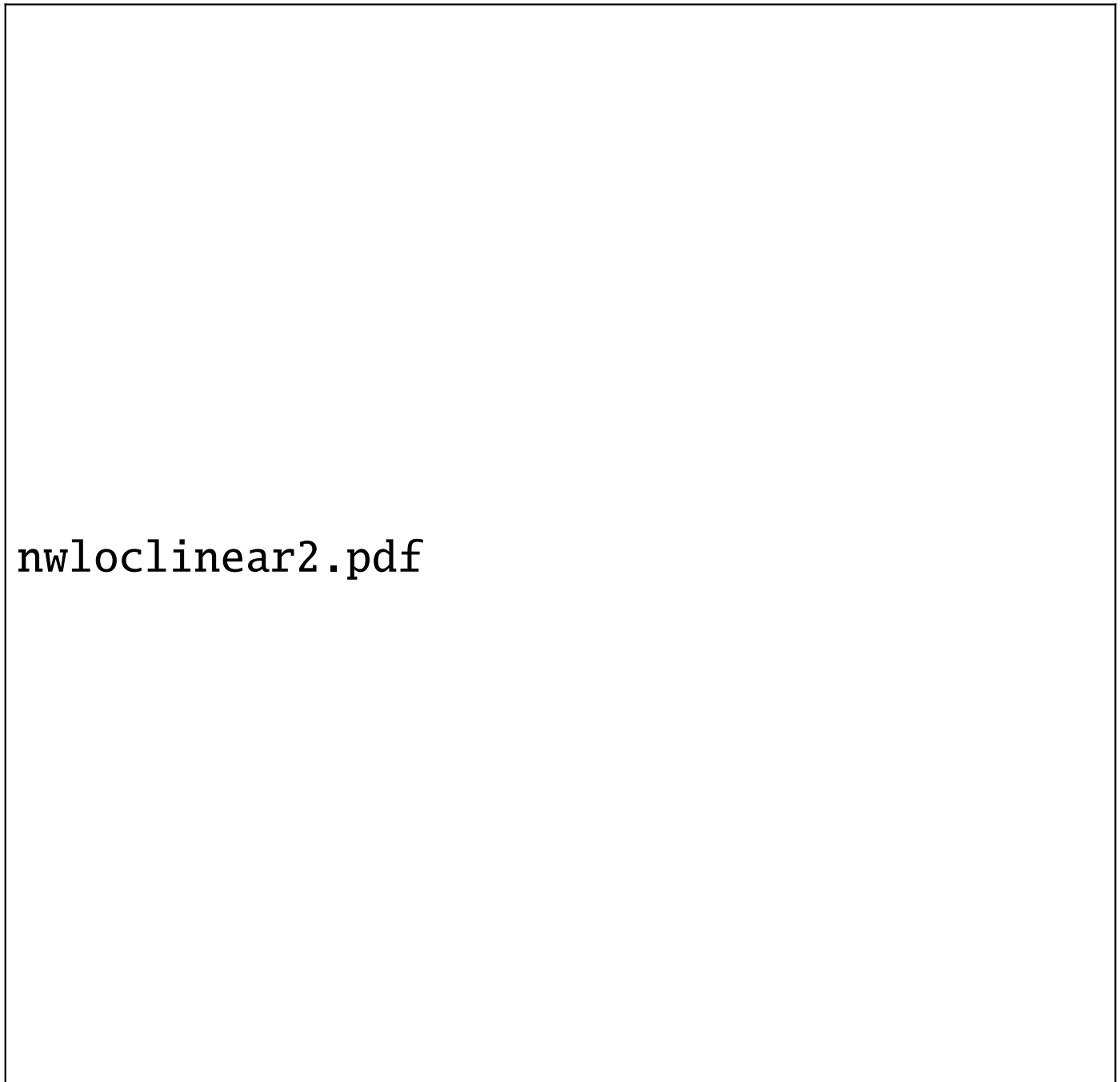
- The Nadaraya-Watson estimator in x can be obtained very simply by regressing $g(y_i, x_i)$ on 1, weighting each observation by $K((x - x_i)/h)$.
- We could also regress on 1 and $(x - x_i)$ (going to higher terms has problems) instead;
  Advantages:
  - the bias becomes 0 if the true $m(x)$ is linear.
  - the coefficient of $(x - x_i)$ estimates $m'(x)$.
  - behaves better in "almost empty" regions.

  Disadvantages: hardly any, just do it!

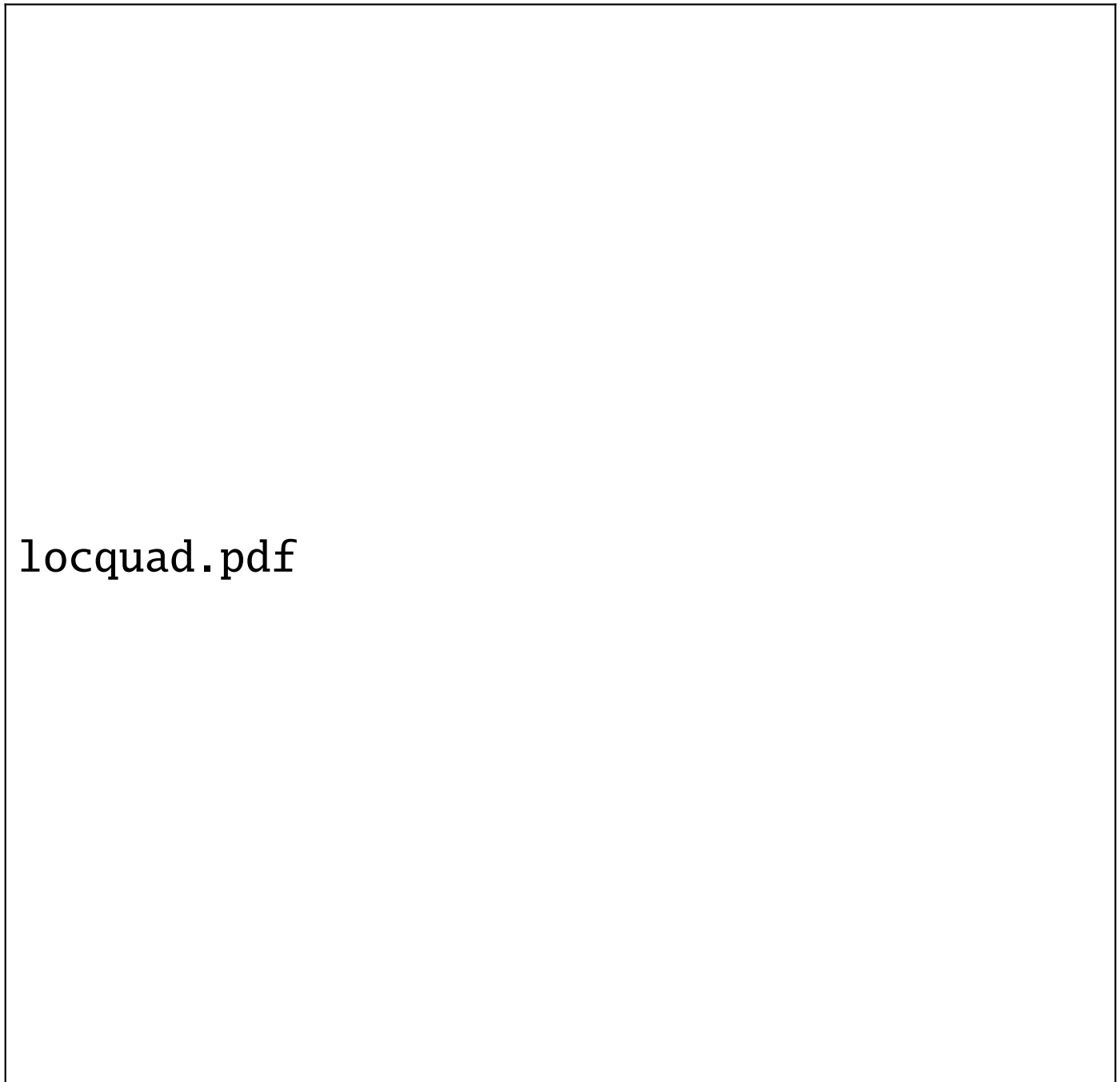
Local Linear



Local Linear



Local Quadratic



Nonparametric Regression, summary, 1

Nadaraya–Watson for $E(y|x) = m(x)$

$$\hat{m}(x) = \frac{\sum_i y_i K_h(x - x_i)}{\sum_i K_h(x - x_i)}$$

- bias in $O(h^2)$, variance in $1/(nh^{p_x})$
- optimal h in $n^{-1/(p+4)}$: then bias, standard error and RMSE all converge at rate $n^{-2/(p+4)}$
- to select h, no rule of thumb: cross-validate on a subsample and scale up.

Nonparametric Regression, summary, 2

Nadaraya–Watson=**local constant regression**: to get $\hat{m}(x)$,

1. regress y_i on 1 with weight $K_h(x - x_i)$
2. take the estimated coeff as your $\hat{m}(x)$.

Better: **local linear regression**

1. regress y_i on 1 and $(x_i - x)$ with weight $K_h(x - x_i)$
2. take the estimated coeffs as your $\hat{m}(x)$ and $\hat{m}'(x)$.

To estimate the standard errors: bootstrap on an *undersmoothed* estimate (so that bias is negligible.)

Seminonparametric (=Flexible) Regression

Idea: we add regressors when we have more data
$\rightarrow$ **series or sieve estimators**: choose a basis of functions $P_k(x_i)$ (x_i^k, or orthogonal polynomials, or sines...)
$\rightarrow$ run *linear regression* $y_i = \sum_{k=1}^{M} P_k(x_i)\theta_k + \epsilon_i$
a reasonable compromise (again, M must go to infinity, more slowly than n).
Still curse of dimensionality, and nonparametric asymptotics.

Splines: trading off fit and smoothness

Choose some $0 < \lambda < \infty$ and

$$\min_{m(.)} \sum_i (y_i - m(x_i))^2 + \lambda J(m),$$

Then we "obtain" the natural cubic spline with knots=$(x_1, \ldots, x_n)$:

- m is a cubic polynomial between consecutive x_i's
- it is linear out-of-sample
- it is C^2 everywhere.

"Consecutive" implies one-dimensional. . . harder to generalize to $p_x > 1$.
Orthogonal polynomials: check out Chebyshev,
$1, x, 2x^2 - 1, 4x^3 - 3x \ldots$ (on $[-1, 1]$ here.)

Review: What was the point?

- OLS is lowest variance among linear unbiased estimators.
- But there are nonlinear estimators and potentially biased estimators.
  - Everything faces a bias-variance tradeoff.
  - Nearly anything can be written as Kernel.