Fixed Income Securities

Topic 1

Definition

- Fixed Income Securities = An investment provides you with fixed amount on a fixed schedule
- The income stream is almost fixed, in comparison to other assets, such as equity.
- Most of Fixed Income Securities nowadays **do not** have ”Fixed Income”.

  e.g. US Treasury Inflation Protected Securities (TIPS) （通货膨胀保值债券）has a coupon payment fluctuating with inflation. Swaps(掉期), mortgage backed securities(抵押支持债券), and options(期权).

Why important

- One of the two most important asset classes along with **equity**.
- Fixed income markets move together, a good chance to better understand the most fundamental principle in finance: **no-arbitrage**.

Example

- Treasury Securities 国库券
- Agency Securities 代理证券(政府机构发行的)
- Mortgage-Backed Securities 抵押支持债券
- Corporate Securities 公司证券
- Asset-Backed Securities 资产支持证券：由资产组合支持的证券，例如信用卡应收账款
- Municipal issues 州政府&市政发行的

Government Debt Market

- Zero Coupon Bonds 零息债券
- Fixed Rate Coupon Bonds 固定利率债券 （半年期）
- Floating Rate Coupon Bonds 浮动利率债券
- Municipal Debt Market
  - Separate Trading of Registered Interest and Principal of Securities (STRIPS): Artificial zero coupon bonds constructed by stripping off separate interest and principal payments from a coupon bond

> 单独交易注册利息和证券本金（STRIPS）：通过剥离息票债券的单独利息和本金支付构建的人工零息债券

  - Treasury Inflation Protected Securities (TIPS) 通货膨胀保值债券

Money Market

The Money Market refers to the market for short term borrowing and lending, usually undertaken by banks.

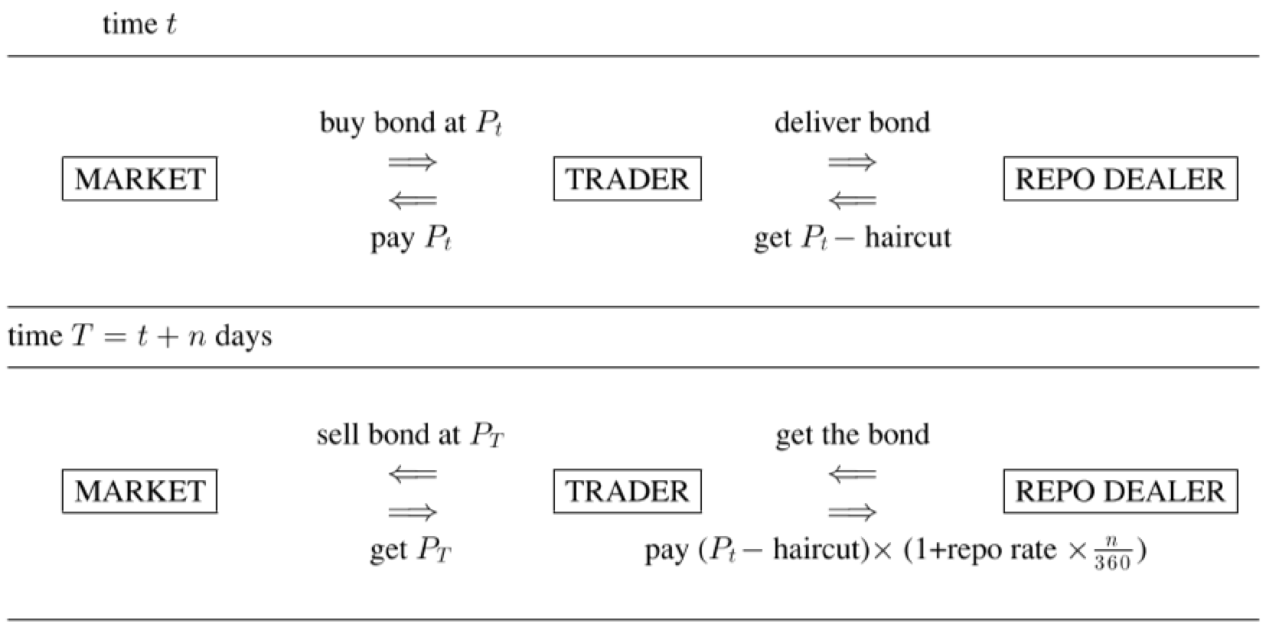
- Federal Funds Rate 美国联邦基金利率(美国同业拆借市场的利率)

- Eurodollar Rate 欧洲美元利率
- LIBOR 伦敦银行行间同业拆借利率
- Repo Rate 再回购利率

> Rank
>
> LIBOR > repo > treasury
>
> 因为repo有抵押（haircut）所以相对低

Repo

Figure 1.4 Schematic Repo Transaction



P_t 是市场上证券的价格

haircut 是 repo dealer 赚的一部分钱

过程

1. trader 花 P_t 在市场上买了 bond
2. trader 把证券卖给 repo dealer ，约定以后赎回。换回来 $P_t - haircut$ 的钱。
3. 到期后，trader 从 repo dealer 拿回债券，并支付 $(P_t - haircut) \times (1 + repo\ rate \times \frac{n}{360})$
4. trader 再把证券放到市场上卖掉，换回 P_T

所以

$$Profit = P_T - P_t - \frac{n}{360} \times Repo\ rate \times (P_t - haircut)$$

定义

$$Repo\ Interest = (P_t - haircut) \times Repo\ rate \times \frac{n}{360}$$

即回购利息。所以也可以说

$$Profit = P_T - P_t - Repo\ Interest$$

Capital at risk = haircut (very small)

风险收益(return)即

$$return = \frac{P_T - P_t - Repo\ interest}{Haircut}$$

杠杆非常高.

- The repo rate for most Treasury securities is called the **General Collateral Rate**.(一般抵押率)
- **Special Repo Rate** 特殊回购利率

Dealers often short on-the-run Treasuries to hedge other securities.

通过short Treasuries 来为其他资产套期保值。

Purchased via reverse repo, causing repo rate to fall. Because

$$Profit = (P_t - P_T) + Repo\ interest$$

Mortgage Backed Securities

These securities allow local banks, which issue mortgages to individuals, to diversify their risk.

A bank issues mortgages to individuals living nearby.

These mortgages are susceptible(容易被影响的) to local events (e.g. a local company goes bankrupt leaving many mortgage holders without a job) I By pooling its mortgages and buying into a mortgage backed security, the bank reduces the effect of a local event affecting its balance sheet

Asset Backed Securities (ABS)

are similar to MBS, but instead they are collateralized by other types of loans (auto loans, credit cards, etc.)

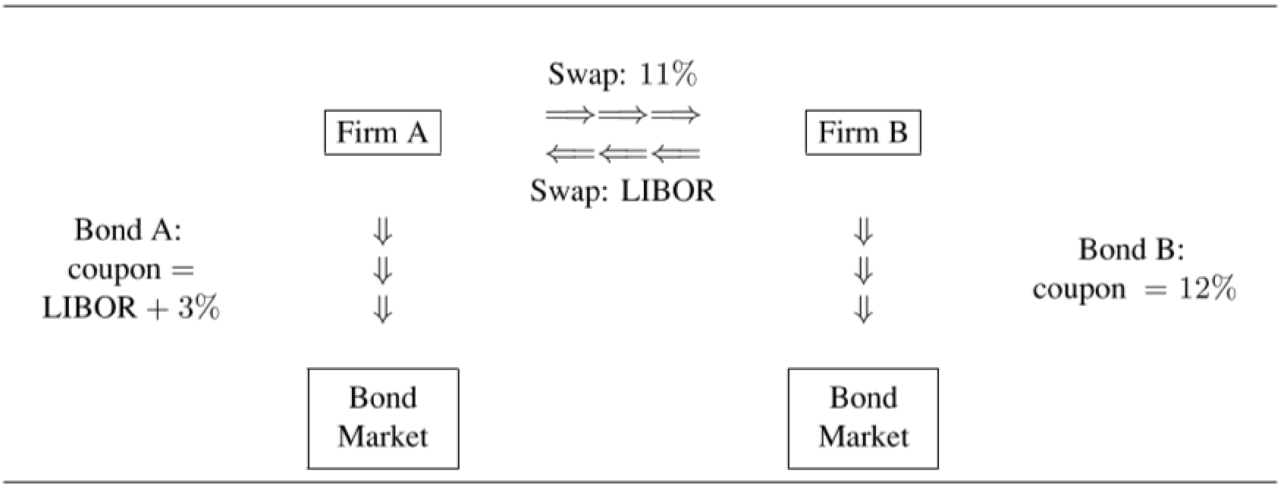
Swaps

A B 两公司想发行面值相等的债券，A公司想用固定利率，B公司想用浮动利率。

Firm	fixed rate	floating rate
A	15%	LIBOR +3 %
B	12%	LIBOR +2 %

B 公司借钱的浮动利率和固定利率都低于 A 公司，但仍然能通过比较优势进行掉期交易来使双方利益最大化。这就是 SWAP 的精髓。

Figure 1.8 A Swap Deal



若B公司使用固定利率发行，A 公司采用浮动利率发行，则 A 公司需要 LIBOR +3 % 的利率，B 公司需要使用 12% 的利率。

然后，A 公司付给 B 公司 11% 的利息，而 B 公司付给 A 公司 LIBOR 的利息。

则 A 公司支付 $(LIBOR + 3\%) + 11\% - LIBOR = 14\%$

而 B 公司支付 $12\% - 11\% + LIBOR = LIBOR + 1\%$

选择 `11%` 是为了让两家公司从掉期交易中获得同等的收益（1%）

Futures and Forwards 远期和期货

> 双方谈好价格约定在未来交易

期货通常有正规成体系的合同单

远期通常是场外交易 OTC (Over-the-counter) market

Option

> 购买 option 以获得在未来以合约价格购买或卖出的权力

Topic 2

Discount Factor

- 两个日期间的 Discount Factor = Z(t,T) 表示在T期的1单位价格在t期值多少钱。
- Z(t,T) 反映了t到T的时间价值。
- 给定 t 。T越大，Z(t,T)越小。

假如在日期 t ，一个T期到期面值为 100 的零息债券价格为 97，则

$Z(t,T) = \frac{97}{100}$

影响Z的一个因素是**通货膨胀**

> 高预期通胀会使 discount factor 减少

Interest Rate

半年期复利

> N 表示一年付几次息 T - t 以年计

容易理解 本金+利息=收益 即

$$Z(t,T) * (1+\frac{r_n(t,T)}{n})^{n(T-t)} = 1$$

从中可推导出

$$r_n(t,T) = n * (\frac{1}{Z(t,T)^{\frac{1}{n*(T-t)}}} - 1)$$

连续复利

$$Z(t,T) = e^{-r_{t,T}*(T-t)}$$

或者

$$r_{t,T} = \frac{-ln(Z(t,T))}{(T-t)}$$

可得

$$P_z(t,T) = e^{-r(t,T)*(T-t)} * 100(\text{本金})$$

Discount Factor 与 Interest Rate

算出来了利率，则 Discount Factor 迎刃而解

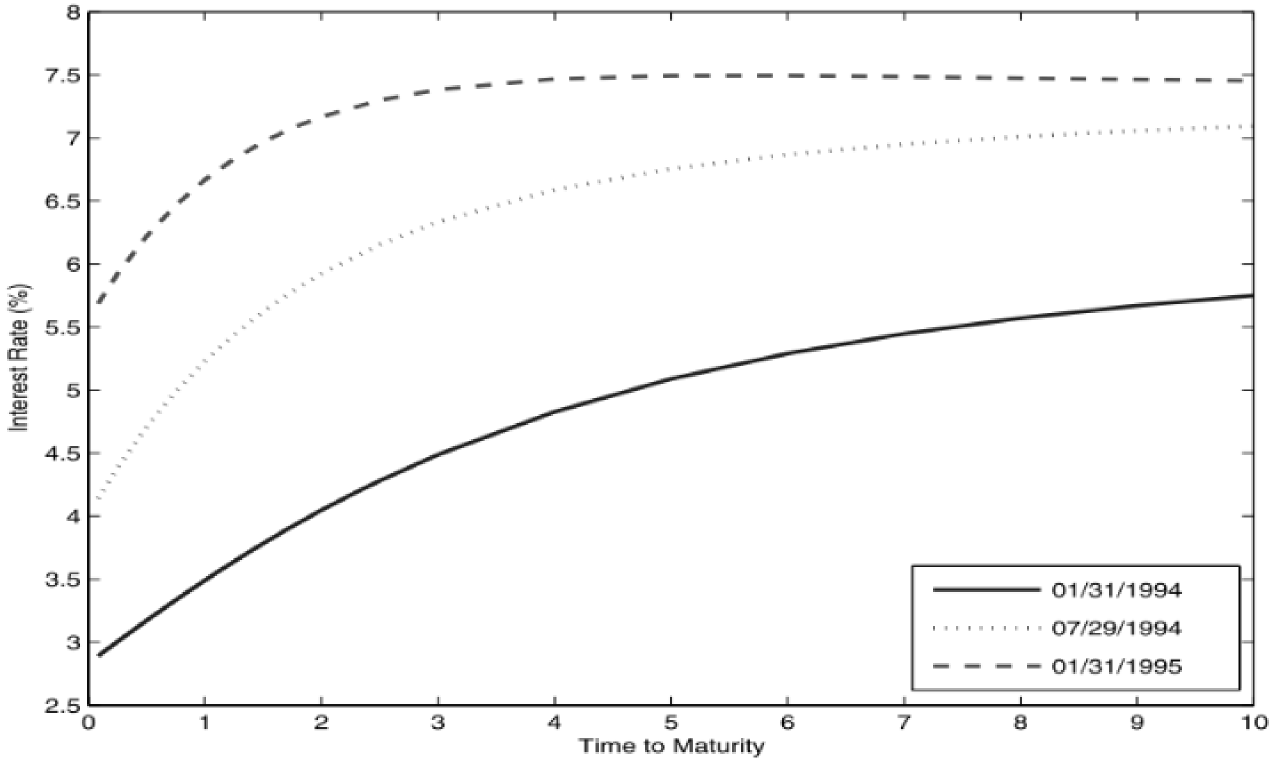
给出不同 frequency 的 Interest Rate 转换公式：

$$r_{t,T} = n * ln(1+\frac{r_n(t,T)}{n})$$

$$r_n(t,T) = n * (e^{\frac{r_{t,T}}{n}} - 1)$$

Term Structure

Figure 2.4 The Term Structure of Interest Rates on Three dates



Coupon Bonds

$$P_c(t, T_n) = \frac{c * 100}{2} * \sum_{i=1}^{n} Z(t, T_i) + 100 * Z(t, T_n)$$

Also,

$$P_c(t, T_n) = \frac{c}{2} * \sum_{i=1}^{n} P_z(t, T_i) + P_z(t, T_n)$$

P_c 表示 coupon bond P_z 表示zero-coupon bond

c 代表 coupon rate

Bootstrap

1. The first discount factor $Z(t, T1)$ is given by:

$$Z(t, T_1) = \frac{P_c(t, T_1)}{100 * (1 + c_1/2)}$$

2. Any other discount factor$Z(t, Ti)$ for $i = 2, , n$ is given by:

$$Z(t, T_i) = \frac{P_c(t, T_i) - c_i/2 * 100 * \sum_{j=1}^{i-1} Z(t, T_j)}{100 * (1 + c_i/2)}$$

Yield to Maturity 到期收益率

也叫 internal rate of return

Yield to maturity is a kind of weighted average of the spot rates corresponding to the different cash flows paid by a bond.

The yield to maturity y can be considered an average of the semi-annually compounded spot rates $r_2(0, T)$, which define the discount $Z(0, T)$.

IMPORTANT: Yield to maturity is often not a good way to compare investment decisions.

The bond with the higher coupon has lower yield to maturity y.

Floating Rate Bonds

债券利率浮动，利率与参考利率挂钩，通常再加一个 $spread$

这里是 coupon 不是 rate

$$c(T_i) = 100 * (r_2(T_{i-0.5}) + s)$$

$r_2(t)$ 是 6-month reference date at t.

1. spread $s == 0$,

   the ex-coupon price of a floating rate bond on any coupon date is equal to the bond par value.

   因为 cash flow effect 和 discount factor 互相抵消了。

2. spread $s! = 0$

   $$Price\ with\ spread = Price\ of\ no\ spread\ bond + s * \sum_{t=0.5}^{n} Z(0, t)$$

   Price of no spread bond = 100

   $P_{F\ R}(t)$ 表示 Ex-Coupon Price (恒等于100)

   $P_{F\ R}^{C}(t)$ 表示 Cum-Coupon Price = $100 + c(t)$

在 $T_i < t < T_{i+1}$ 的情况下（no spread)

$$P_{FR}(t, T) = Z(t, T_{i+1}) * 100 * (1 + \frac{r_2(T_i)}{2})$$

Topic 3

The Federal Reserve

美联储

通过

1. 控制纸币发放
2. 货币乘数 （给银行钱，银行再借贷）

来控制货币供给。

货币政策工具

1. Open market operations 公开市场交易

   Federal Open Market Committee (FOMC), 买卖国库券，OMC是最主要调节货币市场工具
2. reserve requirements 准备金
3. Federal discount rate 联邦贴现率

Federal Funds Rate

美国联邦基金利率(*Federal funds rate*)是指美国同业拆借市场的利率。短期预测OK，长期较难。

预测它最好的工具就是前一天的利率。

$$r^{FF}(t+1)=\alpha+\beta_1*r^{FF}(t)+\epsilon(t+1)$$

它也对商业周期很敏感（就业，通胀）

$$r^{FF}(t+1)=\alpha+\beta_1*r^{FF}(t)+\beta_2*X^{Pay}(t)+\beta_3*X^{lnf}(t)+\epsilon(t+1)$$

其中，$X^{pay}(t)$ 是非农就业人口年增长率

X^{Inf} 是 CPI 指数的年增长率

预测多期未来时，解释变量这么处理

$$X^{Pay}(t+1)=\alpha+\beta_21*r^{FF}(t)+\beta_22*X^{Pay}(t)+\beta_23*X^{lnf}(t)+\epsilon^{pay}(t+1)$$

也可以利用 Fed Funds Futures 来预测，即

$$r^{FF}(t+h)=\alpha+\beta*f^{Fut}(t,t+h)+\epsilon(t+h)$$

Term Structure of Interest Rates

The long-term yield is a weighted average of the current short-term yield and the short-term yield next period.

e.g.

$$P_z(0,2)=e^{-r(0,1)}\times P_z(1,2)=e^{-r(0,1)}\times e^{-r(1,2)}\times 100=e^{-r(0,1)-r(1,2)}\times 100$$

$$P_z(0,2)=e^{-r(0,2)}\times 2\times 100$$

$$r(0,2)=\frac{r(0,1)+r(1,2)}{2}$$

对利率预期高，则 Term Structure 增加。

但是！高的 Term Structure 不一定表示投资者对于未来利率预期增加，也可能时他们对于风险溢价的要求高。

$$
\begin{aligned}
r(t,T) \quad = & \quad \left[\frac{1}{\tau} \times r(t,t+1) + \frac{(\tau-1)}{\tau} \times E_t\left(r(t+1,T)\right)\right] \quad \text{(Expected future yield)} \\
& + \frac{\lambda}{\tau} \qquad \text{(Risk premium)} \\
& - \frac{(\tau-1)^2}{2\tau} V_t\left(r(t+1,T)\right) \qquad \text{(Convexity)}
\end{aligned}
$$

The first term in brackets on the right-hand side is the weighted average between the current short- term rate, and the expected long-term yield next year.

The second term, λ, is a risk premium that market participants require to hold long-term zero coupon bonds with maturity T over safe short-term bonds with maturity $t+1$.

The last term is related to the variance of the long-term yield $r(t+1,T)$, and it is called convexity term. 后面章节会再提到。

Expectation Hypothesis

Definition

The positive relation between market participants' expectations about future rates and the current shape of the yield curve.

$$E_t[r(t+1,t+\tau) - r(t,t+\tau)] = \frac{[r(t,t+\tau) - r(t,t+1)]}{\tau - 1}$$

The truth is opposite. Actually it is a **negative** one.

这说明 LRP 依赖于 Term Structure 的斜率。

LRP(log-risk premium)

$$LRP_t^{(\tau)} = \lambda - \frac{(\tau-1)^2 * V_t(r(t+1,T))}{2}$$

所以方程应该写为

$$E_t[r(t+1,t+\tau) - r(t,t+\tau)] = \frac{[r(t,t+\tau) - r(t,t+1)]}{\tau - 1} - LRP_t(\tau)$$

A strongly sloped term structure predicts lower future yields, on average.

Predicting Excess Returns

$$LER_t(\tau) = [log(\frac{P_z(t+1,t+\tau)}{P_z(t,t+\tau)}) - log(\frac{100}{P_z(t,t+1)})]$$

where $LER_t()$ is the log-excess return from holding the long-term zero coupon bond with time to maturity τ over the short-term one year zero coupon bond. Note that $LRP_t(\tau) = E_t[LER_t(\tau)]$.

Fama and Bliss (1987) run:

$$LERt(\tau) = \alpha + \beta[f(t, t+\tau-1, t+\tau) - r(t, t+1)] + \epsilon(t)$$

根据 EP，α 和 β 应该为 0 ，然而 β 显著为正。

Excess log return is in fact predictable.

when the forward spread is strongly positive, that is, the term structure is positively sloped, on average investments in long-term bonds generate a higher return compared to short term bonds.

the forward spread the difference between the forward rate and the current short term spot rate predicts well monthly and annual returns on long term bonds

In short, returns on zero coupon bonds are predictable by using some predicting factors and this is due to a variation in risk premia, rather than variation in expectation of future yields.

Treasury Inflation Protected Securities

The principal changes over time in response to inflation.

The coupon rate of TIPS is a constant fraction of the principal

$$CouponPayment = 0.5 * couponrate * 100 * IndexRatio$$

Index Ratio = the change in the CPI index between the issuance of the TIPS and the reference CPI reading

The reference CPI reading = the average of the CPI value at the beginning of the month of the coupon payment and the CPI value at the beginning of the previous month.

Real Bonds and the Real Term Structure of Interest Rates

Real bonds are bonds that are denominated in units of a good, such as gold, instead of dollars.

The real discount factor $Z_{real}(t, T)$ defines the exchange rate between consumption goods at t versus consumption goods at a later date T

The continuously compounded real interest rate can be obtained from the real discount factor as the solution to the equation.

$$Z_{real}(t;T) = e^{-r_{real}(t;T)(T-t)} * 1$$

$$r_{real} = -\frac{ln(Z_{real}(t;T))}{T-t}$$

So the value of a real coupon bond with maturity T and coupon c is:

$$P_c^{real} = \frac{c*100}{2}\sum_{i=1}^{n} Z^{real}(t;T_i) + 100 * Z^{real}(t;T)$$

这里公式都和以前的一样。

Real Bonds and TIPS

$Idx(T)$ refers to the CPI index for maturity T

Strip the coupon payments on TIPS generating a series of zero coupons

$Zero\ coupon\ TIPS\ payoff\ at\ T = 100 * Idx(T)/Idx(0)$

Given the real discount factor, it follows that the value at t of the payoff is:

$$PV\ of\ 100 * Idx(T) = Z_{real}(t;T) * 100$$

To obtain the dollar present value we multiply by $Idx(t)$

$$Dollar\ PV\ of\ 100 * Idx(T) = Z_{real}(t;T) * Idx(t) * 100$$

The left hand side is not exactly the payoff, we need to divide it by $Idx(0)$

$$P_z^{TIPS}(t;T) = Z_{real}(t;T) * Idx(t)/Idx(0) * 100$$

So the value of a coupon bearing TIPS is:

$$P_c^{TIPS} = \frac{Idx(t)}{Idx(0)} * [\frac{c * 100}{2} * \sum_{i=1}^{n} Z^{real}(t;T_i) + Z^{real}(t;T)]$$

Topic 4

The variation in interest rates

1. The Savings and Loan Debacle

> 存贷崩溃：在1980年代，美国利率上升，银行需要付给存款人的利息增加，而收到的贷款利息还是按照以往的利率。

2. The Bankruptcy of Orange County

> portfolio 对利率太敏感

Duration

久期就是债券价格相对于利率的敏感性。

Duration of a security is the (negative of the) percent sensitivity of its price to a small parallel shift in the level of interest rates

$$D_T = -\frac{1}{P(r,t,T)}[\frac{dP(r,t,T)}{d_r}]$$

对于零息债券，易得 $D_{t,T} = T - t$

资产组合的 Duration 就是所有资产的 duration 的加权平均。

$$D_w = \sum_{i=1}^{n} w_i D_i$$

Duration of a coupon bond

$$D_c = \sum_{i=1}^{n} w_i D_i = \sum_{i=1}^{n} w_i T_i$$

也就是说把它想象成一个资产组合。而 T_i 指payment time。

由于

$$P_c(0,T_n)=\sum_{i=1}^{n-1}\frac{c}{2}\times P_z(0,T_i)+(1+\frac{c}{2})P_z(0,T_n)$$

所以对w_i可以这么算，其中对$i<n$

$$w_i=\frac{c/2*P_z(0,T_i)}{P_c(0,T_n)}$$

而

$$w_n=\frac{(1+c/2)*P_z(0,T_n)}{P_c(0,T_n)}$$

对于浮动利率证券来说，Duration != average time of payments

事实上

$$D_{F\ R}=T_i-t$$

即下一发放coupon的日期和今天的差就是浮动利率证券 Duration

coupon rate increases, duration falls

传统 Duration

$$D=-\frac{1}{P}\frac{dP}{dr}$$

Traditionally duration is defined against the semi-annually compounded yield to maturity

还有麦考利久期和 Modified Duration 不考公式

- 麦考利久期是使用加权平均数的形式计算债券的平均到期时间。
- 修正久期是对于给定的到期收益率的微小变动，债券价格的相对变动与其麦考利久期的比例。

Dollar Duration

$$D_P^{\$}=-\frac{dP}{dr}$$

也就是

$$D_P^{\$}=P*D_p$$

Value-at-Risk

Value-at-Risk (VaR) is a risk measure that attempts to quantify the amount of risk in a portfolio.

In brief, VaR answers the following question: With 95% probability, what is the maximum portfolio loss that we can expect within a given horizon, such as a day, a week or a month?

Normal Distribution Approach

令

$$\mu_p = -D_p \times P \times \mu \quad and \quad \sigma_P = D_P \times P \times \sigma$$

则在 dr 服从正太分布的情况下，dP 满足

$$dr \sim N(\mu, \sigma^2) \quad \Longrightarrow \quad dP \sim N(\mu_P, \sigma_P^2)$$

95% VaR 则是

$$95\% VaR = -(\mu_P - 1.645 \times \sigma_P)$$

Historical Distribution Approach

用历史的分布来预测 95%VaR

Warnings

VaR 仅对小波动和短期才可靠。

更常用的是计算 unexpected VaR by setting $\mu_p = 0$

Duration and Expected Shortfall

Expected Shortfall is the expected loss on a portfolio P over the horizon T conditional on the loss being larger than the (100−α)% T VaR:

$Expected\ shortfall = E[L_T | L_T > VaR]$

Under these assumptions:

95% Expected Shortfall $= -(\mu - \sigma_p * \frac{f(-1.645)}{N(-1.645)})$

where $f(x)$ denotes the standard normal density and

$N(x)$ is the standard normal cumulative density.

Interest Rate Risk Management

An institution may consider two alternatives to hedge interest rate risk:

1. **Cash Flow Matching**: buy a set of securities that payoff the exact required cash flow over the period
2. **Immunization**: choose a portfolio of securities with the same present value and duration of the cash flow commitment to pay

Asset Liability Management

资产负债管理

资产 = 所有者权益 + 负债

Equity = Asset - Liability

$D_E = \frac{A}{A-L} * D_A - \frac{L}{A-L} * D_L$

This is a strategy of choosing the (dollar) duration of liabilities to match the (dollar) duration of assets.

It helps reduce the sensitivity of equity to changes in interest rates, and ensures that cash flows received from assets are sufficient to pay the cash flows from liabilities.

Topic 5

Convexity

The Convexity of a security measures the percentage change in the price of a security due to the curvature of the price with respect to the interest rate

$$C = \frac{1}{P}\frac{d^2P}{dr^2}$$

一个比duration更为精准的近似

$$\frac{dP}{P} = -D * dr + \frac{1}{2} * C * dr^2$$

1. 零息债券的Convexity

$$C_z = (T-t)^2$$

2. 资产组合的Convexity

$$C_w = \sum_{i=1}^{n} w_i * C_i$$

$$w_i = \frac{N_i * P_i}{W}$$

就是convexity的加权和，肥肠简单。

3. coupon bond 的 convexity

   和资产组合的逻辑一样，也和 Duration 推导一样，不赘述了。

$$C = \frac{1}{P_c(t,T)} * [\sum_{i=1}^{n} \frac{c}{2} * P_z(t,T_i) * (T_i - t)^2 + (1 + \frac{c}{2}) * P_z(t,T_n) * (T_n - t)^2]$$

Convexity 为 **正** 是好事儿，从公式上来看，它意味着

1. 当利率上升，债券价格下跌没这么狠。
2. 当利率下降，债券价格上涨得更凶了。

假如波动的期望为0，即 $E(d_r) == 0$ 则 $-D * d_r == 0$，但是波动的平方项 !=0，即$E(dr^2)! = 0$，我们可以得到

$$E(\frac{dP}{P}) = -D * E(d_r) + 0.5 * C * E(dr^2)$$

$$E(\frac{dP}{P}) = 0.5 * C * E(dr^2)$$

即在长期中，即便波动的数学期望为0，价格变动的数学期望依然 > 0.

容易产生的误解

1. convexity 本身并不是 duration 的变动
2. 所以不要认为当 duration 为常数时，$convexity == 0$
3. 对 Dollar Convexity = $\frac{d^2P}{dr^2}$, 可以说当 dollar duration 为常数时，dollar convexity 为 0。
4. dollar convexity 对于 零息债券 并不是0

Risk Management

假如P是已经有的，你要你来挑选合适的k1,k2（比例），使得 V 受利率波动的冲击最小。

$$V = P + k_1 * P_1 + k_2 * P_2$$

也就是（dV=0）

$$dV = -(D * P + k_1 * D_1 * P_1 + k_2 * D_2 * P_2) * d_r + \frac{1}{2} * (C * P + k_1 * C_1 * P_1 + k_2 * C_2 * P_2) * dr^2$$

你可以得到

$$k_1 * D_1 * P_1 + k_2 * D_2 * P_2 = -D * P$$
$$k_1 * C_1 * P_1 + k_2 * C_2 * P_2 = -C * P$$

你可以解得

$$k_1 = -\frac{P}{P_1}\frac{D * C_2 - D_2 * C}{D_1 * C_2 - D_2 * C_1}$$
$$k_2 = -\frac{P}{P_1}\frac{D * C_1 - D_1 * C}{D_2 * C_1 - D_1 * C_2}$$

其实很简单，其实很自然。

Convexity Trading and the Passage of Time

Barbell-bullet portfolio Strategy

barbell 哑铃

就是 long 一个久期长的和一个久期短的来文体两开花实现hedge

bullet 子弹

就是 short 一个久期中等的，实现和barbell一样的久期。

1. Long a barbell bond position: a portfolio that is long both high duration and low duration assets.
2. hedged with a bullet bond position: a position long a medium duration asset with same duration as barbell

这样的策略使得对dV来说，Duration带来的影响恒为0，而 convexity 为正，也就是说不管利率上升还是下降，dV始终为正，始终有钱赚。

这个策略**不是** arbitrage 因为随时间得来的收益被convexity的收益抵消了。即

The gain in value from higher convexity is offset by a lower gain to the passage of time (Theta-Gamma relation).

Slope and Curvature

随着利率变动，长期收益和短期收益并不是同步跟随利率变动相同的幅度。

引入

slope Term Spread = Long Term Yield - Short term Yield

curvature Butterfly Spread = -Short Term Yield + 2*Medium Term Yield - Long Term Yield

来分别衡量坡度和曲率的变动。

所以对于 risk management 来说，要考虑 slope and curvature 的影响。

Factor Duration

是衡量 P 对一个因素变动而变动的敏感性。

$$D_j = -\frac{1}{P}\frac{dP}{d\phi_j}$$

portfolio 的 factor duration 依旧是加权平均。

假设收益的变化由这些factor影响(不同时期受这些因素影响的系数不同)，即

$$dr_1 = \beta_{11}d\phi_1 + \beta_{12}d\phi_2 + \cdots + \beta_{1m}d\phi_m$$
$$dr_2 = \beta_{21}d\phi_1 + \beta_{22}d\phi_2 + \cdots + \beta_{2m}d\phi_m$$

则 $\frac{dP}{d\phi_j}$ 可以表示成

$$\frac{P_z(t,T_i)}{d\phi_j} = \frac{dP_z(t,T_i)}{dr_i} \times \frac{dr_i}{d\phi_j} = \frac{dP_z(t,T_i)}{dr_i} \times \beta_{ij}$$

i是时间，j是 factor 的下标。

而

$$\frac{P_z(t,T_i)}{dr_i} = -(T_i - t) \times P_z(t,T_i)$$

所以

$$D_{ij} = (T_i - t) \times \beta_{ij}$$

则对于 coupon bond $P_c(t,T)$而言，它的 factor duration 为

$$D_j = \sum_{i=1}^{n} w_i \times (T_i - t) \times \beta_{ij}$$

其中

$$w_i = \frac{c}{2} \times \frac{P_z(t,T_i)}{P_c(t,T)}$$

对于 $i == n$，自然前面是$(1+c/2)$ 老套路了

所以我们现在衡量价格的变动率可以这么写

$$\frac{dP}{P} = -D_1 \times d\phi_1 - D_2 \times d\phi_2 - D_3 * d\phi_3$$

Factor Neutrality

和曲率差不多，就是选取用长期零息债券和短期零息债券来套期保值，选择它们俩合适的比例。只是衡量 dP 的标准不同罢了。

Estimation of the Factor Model

采取历史数据估测 β

3 components

即之前提到的 level slope 和 curvature .

Topic 6

Forward discount factor

Forward discount factor由两个贴现因子给出

$$F(t,T_1,T_2) = \frac{Z(t,T_2)}{Z(t,T_1)}$$

当 $T_1 == T_2$ 时，$F(t,T_1,T_2) == 1$

Forward Rate

给定付息频率n 则利率

$$f_n(t,T_1,T_2) = n * (\frac{1}{F(t,T_1,T_2)^{\frac{1}{n*(T_2-T_1)}}} - 1)$$

对连续复利

$$f_n(t,T_1,T_2) = -\frac{ln(F(t,T_1,T_2))}{T_2 - T_1}$$

和之前 interest rate 与 discount factor 的关系一样

forward rate 通过 No-arbitrage 原则实现

在 t 时期购买 T_2 的收益 应和

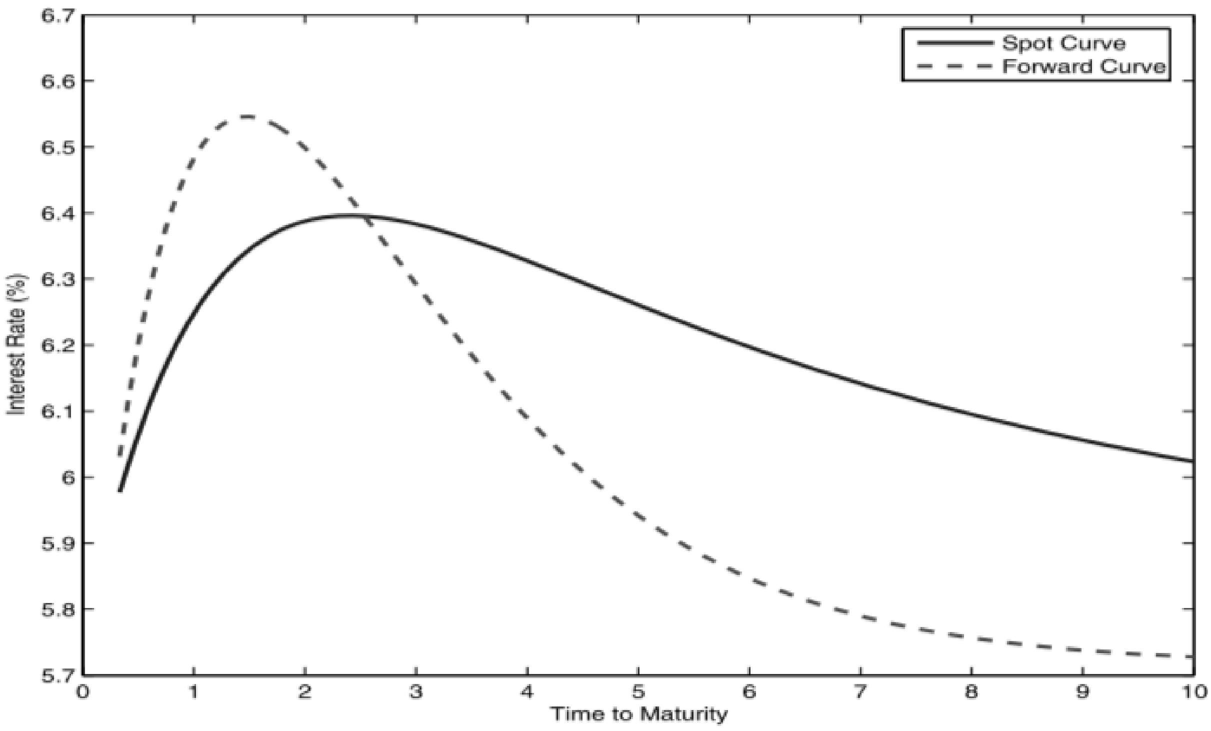
在 t 时期购买 T_1 在 T_1 时期用拿到的收益去购买 T_1 到 T_2 的收益

是一样的。

Forward Curve

The forward curve gives the relation between the forward rate $f(0, T, T + \Delta)$ and the time of the investment T.

Figure 5.2 The Spot Curve and the Forward Curve



Forward rate agreements

远期利率协议是一种远期合约，买卖双方(客户与银行或两个银行同业之间)商定将来一定时间点(指利息起算日)开始的一定期限的协议利率，并规定以何种利率为参照利率，在将来利息起算日，按规定的协议利率、期限和本金额，由当事人一方向另一方支付协议利率与参照利率利息差的贴现额。

在这种协议下，交易双方约定从将来某一确定的日期开始在某一特定的时期内借贷一笔利率固定、数额确定，以具体货币表示的名义本金。远期利率协议的买方就是名义借款人，如果市场利率上升的话，他按协议上确定的利率支付利息，就避免了利率风险；但若市场利率下跌的话，他仍然必须按协议利率支付利息，就会受到损失。远期利率协议的卖方就是名义贷款人，他按照协议确定的利率收取利息，显然，若市场利率下跌，他将受益；若市场利率上升，他则受损。

$$Net\ payment\ at\ T_2 = N \times \Delta \times [r_n(T_1, T_2) - f_n(0, T_1, T_2)]$$

The value of a forward rate agreement

初始价值为0，但是随着时间变化和利率变动,FRA会随之发生改变。

对公司而言，在 t 时期的价值等于

$$V^{FRA}(t) = M \times P_{bill}(t, T_2) - P_{bill}(t, T_1)$$

而 $M = \frac{P_{bill}(0,T_1)}{P_{bill}(0,T_2)}$

（公司操作 相当于在0期short了一个T_1的bill，并在0期拿卖bill的钱买T_2，这样在T1收到钱偿还bill，在最终拿到T_2的收益）

而对 $Net\ Payment$ 而言

$$V^{FRA}(t) = N * [M * Z(t, T_2) - Z(t, T_1)]$$

其中

$$M = 1 + f_n(0, T_1, T_2)\Delta = \frac{Z(0, T_1)}{Z(0, T_2)}$$

也可以写成

$$V^{FRA}(t) = N * Z(t, T_2) * \Delta * [f_n(0, T_1, T_2) - f_n(t, T_1, T_2)]$$

这个推导有点复杂，看书把。

Forward Contract

远期合约

令 $P^{fwd}(0, T, T^*)$表示在时间0的时候，对于在T期交货一个在T*到期的证券的价格。

在时间T的时候，这个证券的价格记作$P(T, T^*)$

Long 这个证券的Payoff为

$$Payoff = P(T, T^*) - P^{fwd}(0, T, T^*)$$

short 这个证券的Payoff正好反一反。

$$P_z^{fwd}(0, T, T^*) = F(0, T, T^*) * 100$$

如果不等于这个，就有套利空间。

> 比如在时间0的时候
>
> 1. 约好用 $P_z^{fwd}(0, T, T^*)$ 的价格卖掉 $P_z(T, T^*)$
> 2. short $F(0, T, T^*)$ 的零息债券
> 3. 购买 T^* 的零息债券
>
> 在时间 T
>
> 1. 卖掉债券，拿到 $P_z^{fwd}(0, T, T^*)$ 的钱
> 2. 付出 $F(0, T, T^*) * 100$ 结束 short
>
> 那么最后的收获就是 $P_z^{fwd}(0, T, T^*) - F(0, T, T^*) * 100$

Forward contracts on Treasury Bonds

$$P_c^{fwd} = \frac{c}{2} * \sum_{i=1}^{n} P_z^{fwd}(0, T, T_i) + P_z^{fwd}(0, T, T_n)$$

Value of Forward Contract

$$V^{fwd}(t) = Z(t, T) * [P_c^{fwd}(t, T, T^*) - K]$$

其中 $K = P_c^{fwd}(0, T, T^*)$

interest rate swaps

$$net\ payment\ at\ T_i = N * \Delta * [r_n(T_i - 1) - c]$$

c is called swap rate

银行按浮动利率给钱（滞后一期） 公司按固定利率给钱。

这样的目的是为了公司用未来的收入填以前的用浮动利率支付的欠债坑。

Value of Swap

$$V_{swap}(t; c, T) = P_{FR}(t, T) - P_c(t, T)$$

相当于 long a floating bond 然后 short a fixed rate bond with coupon c

The Swap Rate

The Swap rate c is given by that number that makes $V^{swap}(0; c, T)$ equal to 0.

通过 swap curve 可以 bootstrap 出 discount factor

对$i = 1$

$$Z(t, T_1) = \frac{1}{1 + \frac{c(t, T_i)}{n}}$$

对 $i > 1$

$$Z(t, T_i) = \frac{1 - c(t, T_1) * \sum_{j=1}^{i-1} Z(t, T_i)}{1 + \frac{c(t, T_i)}{n}}$$

Forward swap contract

forward swap rate of a forward swap contract

$$f_n^s(0, T, T^*) = n * \frac{1 - F(0, T, T^*)}{\sum_{j=1}^{m} F(0, T, T_j)}$$

Interest Rate Risk Management Using Derivatives

dollar duration of a swap

$$D^{\$}_{swap} = D^{\$}_{floationg} - D^{\$}_{fixed}$$

Topic 7

Interest Rate Futures

从期货合约的 P&L (Profit and Loss) 是以日频每天积累的。

Convergence: futures price $P^{fut}(t,T)$ on a security with value P_t at time t has the property that at maturity

$$P^{fut}(T,T) = P(T)$$

这使得用远期合约和用期货合约来套期保值几乎是等价的。

例如：

在 0 期， $P^{Fut} = 100 - f_4^{Fut}(t,T)$

在 maturity, $P^{Fut}(T,T) = 100 - r_4^{LIBOR}(T)$

$Daily\ P\&L = N * \Delta * (P^{Fut}(t+dt,T) - P^{Fut}(t,T))/100$

Futures VS Forwards

当且仅当这些条件成立的情况下, $P_z^{Fut}(0,T_1,T_2) = P_z^{Fwd}(0,T_1,T_2)$

1. 忽略 Timing 的不同

   这个假设做不到

2. futures 和 forward 合同的 payoff 都在 T1 期实现

   经常是错的，尤其是远期经常是为了特殊需求，而 futures 是标准化的。

然而 forwards 还是 future 价格的很好的估计，尤其是短期和利率波动小的时候。

期货的缺陷

1. Basic Risk：标准化 无法满足特定需求
2. Tailing of the hedge: 企业需要考虑到从实现 cash flow 到 hedge maturity 的时间价值

期货的优点

1. Liquidity: 流动性大
2. credit risk: 信用风险小

Options

call option payoff

$$Payoff = max(F(t) - K, 0)$$

put option payoff

$$Payoff = max(K - F(t), 0)$$

K = strike price

$F(t) = P_c(t, T_B)$

欧式期权：严格 在 T 期交割

美式期权：随意 可以在 maturity 之前的任意一天交割