DISCRETE MATHEMATICS HOMEWORK 5 SOL	Undergraduate Course College of Computer Science **Zhejiang University** Fall-Winter 2014

HOMEWORK 5

P344

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) How many ways are there to pick two representatives, so that one is a mathematics majors and the other is a computer science major?

b) How many ways are there to pick one representative who is either a mathematics majors or a computer science major?

Solution : a) $C(18,1)C(325,1) = 5850$ b) $C(18,1) + C(325,1) = 343$

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) How many ways can a student answer the questions on the test if every question is answered?

b) How many ways can a student answer the questions on the test if the student can leave answers blank?

Solution : a) 4^{10} b) 5^{10}

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

Solution : By product rules there are $12 \cdot 2 \cdot 3 = 72$ different types of shirt.

8. How many different three-letter initials which none of the letters repeated can people have?

Solution : $26 \cdot 25 \cdot 24 = 15,600$.

12. How many bit strings are there of length six less?

Solution : $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^7 - 1 = 127$.

14. How many bit strings are there of length n, where n is a positive integer,

start and end with 1s?

Solution : 2^{n-2}.

20. How many positive integers less than 1000 are
a) divisible by 7?
b) divisible by 7 but not by 11?
c) divisible by both 7 and 11?
d) divisible by either 7 or 11?
e) divisible by exactly one of 7 and 11?
f) divisible by neither 7 or 11?
g) have distinct digits?
h) have distinct digits and are even?

Solution : a) There are $\lfloor 999/7 \rfloor = 142$ numbers in our range divisible by 7.
b) There are $\lfloor 999/11 \rfloor = 90$ in our range divisible by 11, and there are $\lfloor 999/77 \rfloor = 12$ in our range divisible by both 7 and 11.
Hence, we see that there are $142 - 12 = 130$ numbers divisible 7 but not by 11.
c) $\lfloor 999/77 \rfloor = 12$
d) By the principle of inclusion-exclusion, there are $\lfloor 999/7 \rfloor + \lfloor 999/11 \rfloor - \lfloor 999/77 \rfloor = 142 + 90 - 12 = 220$ numbers divisible by either 7 or 11.
e) $220 - 12 = 208$
f) $999 - 220 = 779$
g) one-digit numbers: 9;
two-digit numbers: $9 \cdot 9 = 81$;
three-digit numbers: $9 \cdot 9 \cdot 8 = 648$.
The final answer is $9 + 81 + 648 = 738$.

h) one-digit numbers and odd: 5;
two-digit numbers and odd: $8 \cdot 5 = 40$;
three-digit numbers and odd: $8 \cdot 8 \cdot 5 = 320$.
So there are $5 + 40 + 320 = 365$ odd numbers with distinct digits. Thus the final answer is $738 - 365 = 373$.

28. How many license plates can be made using either three letters followed by three digits or four letters followed by two digits?

Solution : $26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 63,273,600$.

36. How many partial functions are there from a set with five elements to sets with the following number of elements?
a) 1 b) 2 c) 5 d) 9

Definition: (P69) A **partial function** f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b

in B.

Solution : The answer is $(n+1)^5$ in each case, where n is the number of elements in the codomain.

a) 2^5 b) 3^5 c) 6^5 d) 10^5

44. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Solution : First we count the number of bit strings of length 10 that contain five consecutive 0's. We will base the count on where the string of five or more consecutive 0s starts.

i) If it starts in the first bit, then the first five bits are all 0s, but there is free choice for the last five bits; therefore there are $2^5 = 32$ such strings.

ii) If start in the second bit, then the first bit must be a 1, then the next five bits are all 0s; but there is free choice for the last four bits; therefore there are $2^4 = 16$ such strings.

Similarly, there are 2^4 such strings that have five consecutive 0's starting in each position third, four, five, six. This gives us a total of $32+5 \cdot 16 = 112$ strings that have five consecutive 0's.

Symmetrically, there 112 strings that have five consecutive 1's. Clearly there are exactly two strings that contain both (1111100000 and 0000011111). Therefore by the inclusion-exclusion principle, the answer is $112 + 112 - 2 = 222$.

P353-354

6. Let d be a positive integer. Show that in any set of $d+1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

Solution : There are only d possible remainders when an integer is divided by d, namely $0, 1, \cdots, d-1$. By the pigeonhole principle, if we have $d+1$ remainders, then at least two must be the same.

10. Let $(x_i, y_i), i = 1, 2, 3, 4, 5$ be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

Solution : The midpoint of the segment whose endpoints are (a, b) and (c, d) is $((a+c)/2, (b+d)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and c have the same parity (both odd or both even). Thus what matters in the problem is the parity of the coordinates. There are four possible pairs of parities:

$$(odd, odd), (odd, even), (even, odd), (even, even)$$

Since we are given five points, the pigeonhole principle of guarantees that at least two of them will have the same parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

36. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

Solution : Let $K(x)$ be the number of other people at the party that person x knows. The possible values of $K(x)$ are $0, 1, \cdots, n-1$ where $n \geq 2$ is the number of people at the party. We can not apply the pigeonhole principle directly, since there are n pigeons and n pigeonholes. However, it is impossible for both 0 and $n-1$ in the range of K,*since if one person knows everybody else, then no body can know no one else.* (We assume that "knowing" is symmetric). Therefore the range of K has at most $n-1$ elements, whereas the domain has n elements, so K is not one-to-one, precisely what we wanted to prove.

37-38. An arm wrestler is the champion for a period of 75 hours. The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches. If 24 is replaced by

a) 2? b) 23? c) 25? d) 30?

Solution : a) Let a_i be the number of matches played up through and including hour i. Then $1 \leq a_1 < \cdots < a_{75} \leq 125$ and $3 \leq a_1 + 2 < \cdots < a_{75} + 2 \leq 127$.

By the pigeonhole principle, we get $a_i = a_j + 2$ for some i and j. This means that exactly 2 matches were played from hour $j+1$ to i.

b) The solution of a), with 2 replaced by 23 and 127 replaced by 148, tell us that the statement is true.

c) Let a_i be the number of matches played up through and including hour i. Then $1 \leq a_1 < \cdots < a_{75} \leq 125$ and $26 \leq a_1 + 25 < \cdots < a_{75} + 25 \leq 150$.

Now either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get, $a_i = a_j + 25$, some i and j, and we are done.

In the former case, however, since each numbers $a_i + 25$ is greater than or equal to 26, the numbers $1, 2, \cdots, 25$ must all appear among a_i's. But the a_i's are increasing the only way this can happen is if $a_1 = 1, a_2 = 2, \cdots, a_{25} = 25$. Thus there were 25 matches in the first 25hours.

d) Let a_i be the number of matches played up through and including hour i and note that $1 \leq a_1 < \cdots < a_{75} \leq 125$.

By the pigeonhole principle two of the numbers among $a_1, a_2, \cdots, a_{31}$ are congruent modulo 30. If the differ by 30, then we have our solution. Otherwise, they differ by 60 or more, so $a_{31} \geq 61$.

Similarly, among a_{31} through a_{61}, either we find the solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{61} \geq 121$. But this means that $a_{66} \geq 126$, a contradiction.

42. Let $n_1, n_2, \cdots, n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, \cdots, t$ the ith box contains at least n_i objects.

Solution: Suppose this statement are not true. Then for each i, the i^{th} box contains at most $n_i - 1$ objects. Adding, we have at most $(n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1) = n_1 + n_2 + \cdots + n_t - t$ objects in all, contradicting the fact that there were $n_1 + n_2 + \cdots + n_t - t + 1$ objects in all. Therefore the statement must be true.