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| | Undergraduate Course |
| **DISCRETE MATHEMATICS** | College of Computer Science |
| HOMEWORK 7 SOL | **Zhejiang University** |
| | Fall-Winter 2014 |

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## HOMEWORK 7

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### P458-459

**25.** a) Find a recurrence relation for the number of bit strings of length $n$ that contain three consecutive 0s.
b) What are the initial conditions?
c) How many bit strings of length seven contain three consecutive 0s?

*Solution* : a) Let $a_n$ be the number of bit strings of length $n$ contain three consecutive 0s.

| | | |
|---|---|---|
| 1 | ****** | —— $a_{n-1}$ |
| 0 | 1***** | —— $a_{n-2}$ |
| 0 | 01**** | —— $a_{n-3}$ |
| 0 | 00**** | —— $2^{n-3}$ |

We can immediately write down the recurrence relation, valid for all $n \geq 3$:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}.$$

b) The initial conditions are $a_0 = a_1 = a_2 = 0$.
c) We can compute $a_3$ through $a_7$ using the recurrence relation:

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 + 2^0 = 0 + 0 + 0 + 1 = 1 \\
a_4 &= a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3 \\
a_5 &= a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8 \\
a_6 &= a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20 \\
a_7 &= a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47.
\end{aligned}$$

**26.** a) Find a recurrence relation for the number of bit strings that contain the string 01.
b) what are the initial conditions?
c) How many bit strings of length seven contain the 01?

*Solution* : a) Let $a_n$ be the number of bit strings of length $n$ that contain the string 01.

| | | |
|---|---|---|
| 1 | ********* | —— $a_{n-1}$ |
| 0 | 1******** | —— $2^{n-2}$ |
| 0 | 01******* | —— $2^{n-3}$ |
| 0 | 001****** | —— $2^{n-4}$ |
| | ... | |

Thus, the recurrence relation for all $n \geq 2$ is:

$$a_n = a_{n-1} + 2^{n-1} - 1.$$

b) The initial conditions are $a_0 = a_1 = 0$.
c) We can compute $a_2$ through $a_7$ using the recurrence relation:

$$\begin{aligned}
a_2 &= a_1 + 2^1 - 1 = 0 + 2 - 1 = 1\\
a_3 &= a_2 + 2^2 - 1 = 1 + 4 - 1 = 4\\
a_4 &= a_3 + 2^3 - 1 = 4 + 8 - 1 = 11\\
a_5 &= a_4 + 2^4 - 1 = 11 + 16 - 1 = 26\\
a_6 &= a_5 + 2^5 - 1 = 26 + 32 - 1 = 57\\
a_7 &= a_6 + 2^6 - 1 = 57 + 64 - 1 = 120.
\end{aligned}$$

**28.** a) Find a recurrence relation for the number of ways to climb $n$ stairs if the person climbing the stairs can take one, two, or three stairs at a time.
b) what are the initial conditions?
c) How many ways can this person climb a flight of eight stairs?

*Solution* : a) Let $a_n$ be the number of ways to climb $n$ stairs. In order to climb $n$ stairs, a person must either

(1) start with a step of one stair and then climb $n-1$ stairs

or

(2) start with a step of two stairs and then climb $n-2$ stairs

or else

(3) start with a step of three stairs and then climb $n-3$ stairs

Thus, the recurrence relation for all $n \geq 3$ is:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

b) The initial conditions are $a_0 = a_1 = 0$ and $a_2 = 2$.
c) Each term in our sequence $\{a_n\}$ is the sum of the previous three terms, so the sequence begins $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 7, a_5 = 13, a_6 = 24, a_7 = 44, a_8 = 81$.

**42.** a) Find a recurrence relation for the number of ways to completely cover a $2 \times n$ chessboard with $1 \times 2$ dominos.
b) what are the initial conditions in part(a)?
c) How many ways completely cover a $2 \times 17$ chessboard with $1 \times 2$ dominos.

*Solution* : Let $a_n$ be the number of coverings.

a) If the right-most domino is positioned vertically, then we have a covering of the left-most $n-1$ columns, and this can be down in $a_{n-1}$ ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominos, and this can be done in $a_{n-2}$ ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.
b) Clearly $a_1 = 1$ and $a_2 = 2$.
c) The sequence we obtain is just the Fibonacci sequence, shift by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, $\cdots$, so the answer to this part is 2584.

**48.** In the Tower of Hanoi puzzle, suppose our goal is to transfer all $n$ disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.
a) Find a recurrence relation for the number of moves required to solve the puzzle for $n$ disks with this added restriction.
b) Solve this recurrence relation to find a formula for the number of moves required to solve the puzzle for $n$ disks.
c) How many different arrangements are there of $n$ disks on three pegs so that no disk is on top of a smaller disk?
d) Show that every allowable arrangement of the $n$ disks occurs in the solution of this variation of the puzzle.

*Solution* : Let $a_n$ be the number of moves required for this puzzle.
a) In order to move the bottom disk off peg 1,
1) we must have transferred the other $n-1$ disks to peg 3(Since we must move the bottom disk to peg 2); this will require $a_{n-1}$ steps.
2) Then we can move bottom disk to peg 2.
3) Our goal, though, was to move it to peg 3, so now we must move the other $n-1$ disks from peg 3 to peg 1, leaving the bottom disk quietly resting on peg 2. By symmetric, this again take $a_{n-1}$ steps.
4) One more step lets us move the bottom disk from peg 2 to peg3.
5) Now it takes $a_{n-1}$ steps to move the remaining disks from peg 1 to peg 3.
So our recurrence relation is $a_n = 3a_{n-1} + 2$. The initial condition is of course that $a_0 = 0$.
b) Computing the first few values, we find that $a_1 = 2, a_2 = 8, a_3 = 26,$ and $a_4 = 80$. It appears that $a_n = 3^n - 1$. This is easily verified by induction.
c) The only choice in distributing the disks is which peg each disk goes on, since the order of the disk on a given peg is fixed. Since there are 3 choice for each disk, the answer is $3^n$.
d) The puzzle involves $1 + a_n = 3^n$ arrangements of disks during its solution——the initial arrangement and the arrangement after each move. None of these arrangements can be repeat a previous arrangement, since if it did so, there have been no point in making the moves between the two occurrences of the same arrangement. Therefore these $3^n$ arrangements are

all distinct. We saw in part(c) that there are exactly $3^n$ arrangements so every arrangement was used.

**P471**

**2. Determine which of the following are linear homogenous recurrence relations with constant coefficients. Also, find the degree of those that are.**

a) $a_n = 3a_{n-2}$ b) $a_n = 3$
c) $a_n = a_{n-1}^2$ d) $a_n = a_{n-1} + 2a_{n-3}$
e) $a_n = a_{n-1}/n$ f) $a_n = a_{n-1} + a_{n-3} + n + 3$
g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

**Solution** :

a) Linear, homogenous, with constant coefficients; degree 2.
b) Linear with constant coefficients but not homogenous.
c) Nonlinear.
d) Linear, homogenous, with constant coefficients; degree 3.
e) Linear and homogenous, but not with constant coefficients.
f) Linear, homogeneous, with constant coefficients, degree 7.

**4. Solve the following recurrence relations together with the initial conditions given.**

g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2, a_1 = 8$.

**Solution** : $r^2 + 4r - 5 = 0$ $r = -5, 1$

$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$

$2 = \alpha_1 + \alpha_2$

$8 = -5\alpha_1 + \alpha_2$

$\alpha_1 = -1,\ \alpha_2 = 3$

$a_n = -(-5)^n + 3$

**20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.**

**Solution** : This is a fourth degree recurrence relation. The characteristic polynomial is $r^4 - 8r^2 + 16$, which factors as $(r-2)^2(r+2)^2$. The roots are 2 and $-2$, each multiplicity 2. Thus we can write down the general solution as usual: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3(-2)^n + \alpha_4 n \cdot (-2)^n$.

**30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.**
**b) Find the solutions of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.**

**Solution** : a) The associated homogeneous recurrence relation is $a_n = -5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 + 5r + 6 = 0$, find that $r = -2$ and $r = -3$ are its solutions and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha(-2)^n + \beta(-3)^n$. Next we need a particular solution to the given recurrence relation. By theorem we want to look for a function of the form $a_n = c \cdot 4^n$. We plug this into the given recurrence relation and obtain $c \cdot 4^n = -5c \cdot 4^{n-1} - 6c \cdot 4^{n-2} + 42 \cdot 4^n$. We divide through by $4^{n-2}$, obtain that $c = 16$. Therefore the particular solution we seek is $a_n^{(p)} = 16 \cdot 4^n = 4^{n+2}$. So the general solution is the sum of the homogenous solution and this particular solution, namely $a_n^{(h)} = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$.

b) We plug the initial conditions into our solution from part (a) to obtain $56 = a_1 = -2\alpha - 3\beta + 64$ and $278 = a_2 = 4\alpha + 9\beta + 256$. A little algebra yields $\alpha = 1$ and $\beta = 2$. So the solution is $a_n = (-2)^n + 2(-3)^n + 4^{n+2}$.

**32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.**

**Solution** :The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By theorem we want to look for a function of the form $a_n = cn \cdot 2^n$. We plug this into the given recurrence relation and obtain $2c \cdot 2^n = 2c \cdot (n-1)2^{n-1} + 3 \cdot 2^n$. We divide through by $2^{n-1}$, obtain that $c = 3$. Therefore the particular solution we seek is $a_n^{(p)} = 3n \cdot 2^n$. So the general solution is the sum of the homogenous solution and this particular solution, namely $a_n^{(h)} = \alpha 2^n + 3n \cdot 2^n = (3n + \alpha)2^n$.

**40. Solve the simultaneous recurrence relations**

$$a_n = 3a_{n-1} + 2b_{n-1}$$

$$b_n = a_{n-1} + 2b_{n-1}$$

**where $a_0 = 1$ and $b_0 = 2$.**

**Solution** : First we reduce this system to a recurrence relation and initial conditions involving only $a_n$. If we subtract the two equations, we obtain $a_n - b_n = 2a_{n-1}$, which gives us $b_n = a_n - 2a_{n-1}$. We plug this back into the first equation to get $a_n = 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) = 5a_{n-1} - 4a_{n-2}$, our desired recurrence relation in one variable. Note also that the first of the original equation gives us the necessary second initial condition, namely $a_1 = 3a_0 + 2b_0 = 7$. We now solve this problem for $\{a_n\}$ in the usual way. The roots of the characteristic equation $r^2 - 5r + 4 = 0$ are 1 and 4, and the solution, after solving for the arbitrary constants ia $a_n = -1 + 2 \cdot 4^n$. Finally, we plug this back into the equation $b_n = a_n - 2a_{n-1}$ to find that $b_n = 1 + 4^n$.