DISCRETE MATHEMATICS HOMEWORK 8 SOL	Undergraduate Course Chukechen Honors College **Zhejiang University** Fall-Winter 2014

HOMEWORK 8

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6. Find a closed form for the generating function for the sequence $\{a_n\}$ where

a) $a_n = -1$ for all $n = 0, 1, 2 \cdots$
b) $a_n = 2^n$ for $n = 1, 2, 3, \cdots$ and $a_0 = 0$.
c) $a_n = n - 1$ for $n = 0, 1, 2 \cdots$
d) $a_n = 1/(n + 1)!$ for $n = 0, 1, 2 \cdots$
e) $a_n = \binom{n}{2}$ for $n = 0, 1, 2 \cdots$
f) $a_n = \binom{10}{n+1}$ for $n = 0, 1, 2 \cdots$

Solution : a) Since the sequence with $a_n = 1$ for all n has generating function $1/(1 - x)$, this sequence has generating function $-1/(1 - x)$.
b) $2x/(1 - 2x)$.
c) We need to split this into two parts. Since we know that the generating function for the sequence $\{n + 1\}$ is $1/(1 - x)^2$, we write $n - 1 = (n + 1) - 2$. Therefore the generating function is $1/(1 - x)^2 - (2/(1 - x)) = (2x - 1)/(1 - x)^2$.
d) The power series for the function e^x is $\sum_{n=0}^{\infty} x^n/n!$. That is almost what we have here; the difference is that the denominator is $(n + 1)!$ in stead of $n!$. So we have

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

by change of variable. The answer is $\frac{1}{x}(e^x - 1)$.
e) Let $f(x)$ be the generating function we seek. We note that $1/(1 - x)^3 = \sum_{n=0}^{\infty} C(n + 2, 2)x^n$, and that is almost what we have here. To transform this to $f(x)$ need to factor out x^2 and change the variable of summation:

$$\begin{aligned} \frac{1}{(1-x)^3} &= \sum_{n=0}^{\infty} C(n+2,2)x^n = \frac{1}{x^2} \sum_{n=2}^{\infty} C(n,2)x^n \\ &= \frac{1}{x^2} \cdot (f(x) - f(0) - f(1)). \end{aligned} \tag{1}$$

Noting that $f(0) = f(1) = 0$ by definition, we have $f(x) = x^2/(1 - x)^3$.
f) $\sum_{n=0}^{\infty} C(10, n + 1)x^n = \sum_{n=1}^{\infty} C(10, n)x^{n-1} = \frac{1}{x} \cdot \sum_{n=1}^{\infty} C(10, n)x^n = \frac{1}{x}((1 + x)^{10} - 1)$.

10. Find the coefficient of x^9 in the power series of each of the following functions.

c) $(x^3 + x^5 + x^6)(x^3 + x^4)(x + x^2 + x^3 + x^4 + \cdots)$

d) $(x + x^4 + x^7 + x^{10} + \cdots)(x^2 + x^4 + x^6 + x^8 + \cdots)$

e) $(1 + x + x^2)^3$

Solution: c) If we factor out as high power of x from each factor as we can, then we can write this as

$$x^7(1 + x^2 + x^3)(1 + x)(1 + x + x^2 + x^3 + \cdots)$$

and so we seek the coefficient of x^2 in $(1+x^2+x^3)(1+x)(1+x+x^2+x^3+\cdots)$. We could do this by brute force, but let's try it more analytically. We write our expression in closed form as

$$\begin{aligned} \frac{(1+x^2+x^3)(1+x)}{(1-x)} &= \frac{(1+x+x^2+\text{higher order terms })}{1-x} \\ &= \frac{1}{1-x} + x \cdot \frac{1}{1-x} + x^2 \cdot \frac{1}{1-x} + \text{ irrelevant terms.} \end{aligned} \tag{2}$$

The coefficient of x^2 in this power series comes either from the coefficient of x^2 in the first term in the final expression displayed above, or from the coefficient of x^1 in the second factor of the second term of the expression, or from the coefficient of x^0 in the second factor of the third term. Each of this coefficients is 1, so our answer is 3.

d) The easier approach here is simple to note that there are only two combinations of terms that will give us an x^9 term in the product: $x \cdot x^8$ and $x^7 \cdot x^2$. So our answer is 2.

$$\begin{aligned} &(x + x^4 + x^7 + x^{10} + \cdots)(x^2 + x^4 + x^6 + x^8 + \cdots) \\ &\quad = x^3(1 + x^3 + x^6 + \cdots)(1 + x^2 + x^4 + \cdots) \\ &\quad = x^3 \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^2} \\ &\quad = x^3 \cdot \frac{1}{1-(x^2+x^3-x^5)} \\ &\quad = x^3[1 + (x^2+x^3-x^5) + (x^2+x^3-x^5)^2 + (x^2+x^3-x^5)^3 + \text{ irrelevant terms.}] \end{aligned} \tag{3}$$

e) The higher power of x appearing in this expression when multiplied out is x^6. Therefore the answer is 0.

24. a) What is the generating function for $\{a_k\}$, where a_k is the number of solution of $x_1 + x_2 + x_3 + x_4 = k$ when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3, 1 \leq x_2 \leq 5, 0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

b) Use your answer to part (a) to find a_7.

Solution : a) The restriction on x_1 gives us the factor $x^3 + x^4 + x^5 + \cdots$. The restriction on x_2 gives us the factor $x + x^2 + x^3 + x^4 + x^5$. The restriction on x_3 gives us the factor

$1+x+x^2+x^3+x^4$.The restriction on x_4 gives us the factor $x+x^2+x^3+\cdots$. Thus the answer is the product of these

$$(x^3+x^4+\cdots)(x+x^2+x^3+x^4+x^5)(1+x+x^2+x^3+x^4)(x+x^2+x^3+\cdots).$$

We can use algebra to rewrite this in closed form as $x^5(1+x+x^2+x^3+x^4)^2/(1-x)^2$.
b) We want the coefficient of x^7 in this series, which is the same as the coefficient of x^2 in the series for

$$\frac{(1+x+x^2+x^3+x^4)^2}{(1-x)^2}=\frac{(1+2x+3x^2+\text{higher order terms })}{(1-x)^2}.$$

Since the coefficient of x^n in $1/(1-x)^2$ is $n+1$, our answer is $1\cdot 3+2\cdot 2+3\cdot 1=10$.

30. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of the following sequence?

a) $2a_0, 2a_1, 2a_2, 2a_3, \cdots$
b) $0, a_0, a_1, a_2, a_3, \cdots$
c) $0, 0, 0, 0, a_2, a_3, \cdots$
d) $a_2, a_3, a_4, \cdots$
e) $a_1, 2a_2, 3a_3, 4a_4, \cdots$
f) $a_0^2, 2a_0a_1, a_1^2+2a_0a_2, 2a_0a_3+2a_1a_2, 2a_0a_4+2a_1a_3+a_2^2, \cdots$

Solution : a) Multiplication distributes over addition, even when we are taking about infinite sums, so the generating function is just $2G(x)$.
b) $a_0x+a_1x^2+a_2x^3+\cdots=x(a_0+a_1x+a_2x^2+\cdots)=xG(x)$.
c) The term involving a_0 and a_1 are missing; $G(x)-(a_0+a_1x)=a_2x^2+a_3x^3+\cdots$. Here, however, we want a_2 to be the coefficient of x^4, not x^2 (and similarly for the other power), so we must throw in an extra factor. Thus the answer is $x^2(G(x)-a_0-a_1x)$.
d) This is just like part (c). The answer is $(G(x)-a_0-a_1x)/x^2$.
e) Following the hint, we differentiate $G(x)=\sum_{n=0}^{\infty}a_nx^n$ to obtain $G'(x)=\sum_{n=0}^{\infty}na_nx^{n-1}$. By a change of variable this becomes $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n=a_1+2a_2x+3a_3x^2+\cdots$, which is the generating function for precisely the sequence we are given. Thus $G'(x)$ is the generating function for this sequence.
f) If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x)\cdot G(x)$.

34. Use generating function to solve the recurrence relation $a_k=3a_{k-1}+4^{k-1}$ with the initial condition $a_0=1$.

Solution : Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$. Thus

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1} x^k = a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k \\ &= 1 + \sum_{k=1}^{\infty} 4^{k-1} x^k = 1 + x \sum_{k=0}^{\infty} 4^k x^k \\ &= 1 + x \cdot \frac{1}{1-4x} = \frac{1-3x}{1-4x}. \end{aligned} \tag{4}$$

Thus $G(x)(1-3x) = (1-3x)/(1-4x)$, so $G(x) = 1/(1-4x)$. Therefore $a_k = 4^k$.

42. Use generating function to prove Paccal's Identity: $C(n,r) = C(n-1,r) + C(n-1,r-1)$ **when** n **and** r **are positive integers with** $r < n$.

Solution : First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$.
Expanding both sides of this equality using the Binomial Theorem, we have

$$\begin{aligned} \sum_{r=0}^{n} C(n,r) x^r &= \sum_{r=0}^{n-1} C(n-1,r) x^r + \sum_{r=0}^{n-1} C(n-1,r) x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r) x^r + \sum_{r=1}^{n} C(n-1,r-1) x^r \end{aligned} \tag{5}$$

Thus

$$1 + \sum_{r=1}^{n-1} C(n,r) x^r + x^n = 1 + \sum_{r=1}^{n-1} (C(n-1,r) + C(n-1,r-1)) x^r + x^n$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r) + C(n-1,r-1)$ for $1 \le r \le n-1$, as desired.

46. Find a closed form for the exponential generating function for the sequence $\{a_n\}$**, where**
a) $a_n = (-2)^n$

Solution : We will make use of the identity $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.
a)

$$\sum_{n=0}^{\infty} \frac{1}{n!} (-2)^n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-2x)^n = e^{-2x}.$$

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4. Find the number of solutions of equation $x_1 + x_2 + x_3 + x_4 = 17$**, where** x_i, $i = 1,2,3,4$ **are nonnegative integers such that** $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 5$ **and** $x_4 \le 8$.

Solution: $C(4+17-1,17)-C(4+13-1,13)-C(4+12-1,12)-C(4+11-1,11)-C(4+8-1,8)+C(4+8-1,8)+C(4+7-1,7)+C(4+4-1,4)+C(4+6-1,6)+C(4+3-1,3)+C(4+2-1,2)-C(4+2-1,2)=20.$

5. Find the number of primes less than 200 using the principle of inclusion-exclusion.
Solution: (omitted)

10. How many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?
Solution: This problem is asking for the number of onto functions from a set with 8 elements (the balls) to a set with 3 elements (the urns). Therefore the answer is $3^8 - C(3,1)2^8 + C(3,2)1^8 = 5796.$