Vector Calculus in Curvilinear Coordinates

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$*$ We assume that our readers have mastered some knowledge on *tensor algebra* and *differential geometry*.

Contents

1 Basis vectors and metric tensor **1**
- 1.1 Basis vectors in curvilinear coordinates . . . 1
- 1.2 Metric tensor . . . 1
- 1.3 Scalar product and cross product . . . 1
  - 1.3.1 Scalar product . . . 1
  - 1.3.2 Cross product . . . 2

2 Vector calculus in curvilinear coordinates **3**
- 2.1 Christoffel symbols . . . 3
- 2.2 Vector calculus in curvilinear coordinates . . . 3
  - 2.2.1 Gradient . . . 3
  - 2.2.2 Divergence . . . 3
  - 2.2.3 Laplacian . . . 4
  - 2.2.4 Curl . . . 4

3 Calculus properties **6**
- 3.1 Vector derivatives . . . 6
  - 3.1.1 Properties . . . 6
  - 3.1.2 Applications . . . 6
- 3.2 Vector integrals . . . 6
  - 3.2.1 Stokes-Cartan theorem . . . 6
  - 3.2.2 Gauss' divergence theorem . . . 7
  - 3.2.3 Stokes' curl theorem . . . 7
  - 3.2.4 Properties . . . 8
  - 3.2.5 Curvilinear coordinates case . . . 8

Appendix A Conclusions **9**
- A.1 Conclusions in arbitrary coordinates . . . 9
- A.2 Conclusions in spherical coordinates . . . 9
- A.3 Corresponding conclusions in cylindrical coordinates . . . 10

Appendix B Levi-civita tensor **11**

Appendix C Some supplementary proofs **13**
- C.1 Proof for Eq.(3.1) . . . 13
- C.2 Proof for Eq.(3.6) . . . 13
- C.3 Proof for Eq.(2.11) . . . 13

Appendix D Unit bases, a set of non-coordinate bases **15**

1 Basis vectors and metric tensor

1.1 Basis vectors in curvilinear coordinates

In some particular coordinate transformation, the basis vectors follow

$$\vec{e}_\beta = \Lambda^{\bar{\alpha}}{}_\beta \vec{e}_{\bar{\alpha}}, \tag{1.1}$$

where the barred indices represents Cartesian coordinates, and where

$$\Lambda^{\bar{\alpha}}{}_\beta = \frac{\partial x^{\bar{\alpha}}}{\partial x^\beta} \equiv x^{\bar{\alpha}}{}_{,\beta}. \tag{1.2}$$

Let us take the spherical coordinates for example. We know the following relations

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}\right\} \tag{1.3}$$

Therefore,

$$\left.\begin{aligned} \vec{e}_r &= x^{\bar{\alpha}}{}_{,r}\vec{e}_{\bar{\alpha}} = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z, \\ \vec{e}_\theta &= x^{\bar{\alpha}}{}_{,\theta}\vec{e}_{\bar{\alpha}} = r\cos\theta\cos\phi\,\vec{e}_x + r\cos\theta\sin\phi\,\vec{e}_y - r\sin\theta\,\vec{e}_z, \\ \vec{e}_\phi &= x^{\bar{\alpha}}{}_{,\phi}\vec{e}_{\bar{\alpha}} = -r\sin\theta\sin\phi\,\vec{e}_x + r\sin\theta\cos\phi\,\vec{e}_y. \end{aligned}\right\} \tag{1.4}$$

These are called *coordinate bases*, and somewhat different from *unit bases*.

1.2 Metric tensor

Metric tensor is a $\binom{0}{2}$ tensor that maps two vectors into a real number, which is their scalar product:

$$\mathbf{g}(\vec{A}, \vec{B}) = \vec{A}\cdot\vec{B}. \tag{1.5}$$

Its components are given by

$$g_{\alpha\beta} = \mathbf{g}(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha\cdot\vec{e}_\beta. \tag{1.6}$$

A general way to express metric tensor is to write out the spatial interval

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta. \tag{1.7}$$

In Cartesian coordinates, certainly

$$g_{\bar{\alpha}\bar{\beta}} \equiv \eta_{\bar{\alpha}\bar{\beta}} = \delta_{\bar{\alpha}\bar{\beta}}, \quad \text{and} \quad \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{1.8}$$

while in curvilinear coordinates things are not so trivial. Using Eqs.(1.4) and (1.6) yields

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{1.9}$$

indicating $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$, leaving all the others zero.

Metric tensor can also be obtained from coordinate transformations, noting that it is a $\binom{0}{2}$ tensor,

$$g_{\alpha\beta} = \Lambda^{\bar{\mu}}{}_\alpha\Lambda^{\bar{\nu}}{}_\beta g_{\bar{\mu}\bar{\nu}}. \tag{1.10}$$

One more important notion that would be used later is the inverse of the metric, of which the components are $g^{\alpha\beta}$. In *Euclidean* space, the metric is diagonal thus $g^{\alpha\beta}$ are just the reciprocals of $g_{\alpha\beta}$, which means, specifically in spherical coordinates, $g^{rr} = 1$, $g^{\theta\theta} = r^{-2}$, $g^{\phi\phi} = r^{-2}\sin^{-2}\theta$, with the others zero.

1.3 Scalar product and cross product

1.3.1 Scalar product

In arbitrary coordinate system, scalar product is often calculated by the metric tensor,

$$\vec{A}\cdot\vec{B} = \mathbf{g}(\vec{A}, \vec{B}) = \mathbf{g}(A^\alpha\vec{e}_\alpha, B^\beta\vec{e}_\beta) = A^\alpha B^\beta\mathbf{g}(\vec{e}_\alpha, \vec{e}_\beta),$$

so with Eq.(1.6) it is always expressed as

$$\vec{A} \cdot \vec{B} = g_{\alpha\beta} A^\alpha B^\beta. \tag{1.11}$$

Eq.(1.8) implies that in Cartesian coordinates

$$\vec{A} \cdot \vec{B} = A^x B^x + A^y B^y + A^z B^z,$$

while in spherical coordinates, with Eq.(1.9),

$$\vec{A} \cdot \vec{B} = A^r B^r + r^2 A^\theta B^\theta + r^2 \sin^2\theta A^\phi B^\phi. \tag{1.12}$$

1.3.2 Cross product

The cross product is somewhat tricky. In Cartesian coordinates, it is familiar to us,

$$\vec{A} \times \vec{B} = (A^y B^z - A^z B^y)\vec{e}_x + (A^z B^x - A^x B^z)\vec{e}_y + (A^x B^y - A^y B^x)\vec{e}_z.$$

We usually introduce a shorthand notation, or the so-called Levi-civita *symbol* ($\varepsilon_{\alpha\beta\gamma}$), to express the equation above as

$$\vec{A} \times \vec{B} = \eta^{\bar{\mu}\bar{\nu}} \varepsilon_{\bar{\nu}\bar{\alpha}\bar{\beta}} A^{\bar{\alpha}} B^{\bar{\beta}} \vec{e}_{\bar{\mu}}, \tag{1.13}$$

since the basis vectors follow

$$\vec{e}_{\bar{\alpha}} \times \vec{e}_{\bar{\beta}} = \eta^{\bar{\mu}\bar{\nu}} \varepsilon_{\bar{\nu}\bar{\alpha}\bar{\beta}} \vec{e}_{\bar{\mu}}. \tag{1.14}$$

However, in other coordinates, Eq.(1.13) needs to be slightly modified into

$$\vec{A} \times \vec{B} = \sqrt{g} g^{\mu\nu} \varepsilon_{\nu\alpha\beta} A^\alpha B^\beta \vec{e}_\mu, \tag{1.15}$$

where $g \equiv |\det(\mathbf{g})|$, and in spherical coordinates $\sqrt{g} = r^2 \sin\theta$. One can prove the correctness of Eq.(1.15) by showing that

$$\vec{e}_\alpha \times \vec{e}_\beta = g^{\mu\nu} E_{\nu\alpha\beta} \vec{e}_\mu \tag{1.16}$$

is a frame-invariant equation, and where $E_{\nu\alpha\beta} = \sqrt{g}\,\varepsilon_{\nu\alpha\beta}$ is the Levi-civita *tensor*, (see Appendix B to get rid of confusion,) in curvilinear coordinates.

Again, we use Eq.(1.9) to obtain, in spherical coordinates,

$$\begin{aligned}
\vec{A} \times \vec{B} &= \sqrt{g}(g^{rr} \varepsilon_{r\alpha\beta} A^\alpha B^\beta \vec{e}_r + g^{\theta\theta} \varepsilon_{\theta\alpha\beta} A^\alpha B^\beta \vec{e}_\theta + g^{\phi\phi} \varepsilon_{\phi\alpha\beta} A^\alpha B^\beta \vec{e}_\phi) \\
&= r^2 \sin\theta \Bigg[(A^\theta B^\phi - A^\phi B^\theta)\vec{e}_r + \frac{1}{r^2}(A^\phi B^r - A^r B^\phi)\vec{e}_\theta \\
&\quad + \frac{1}{r^2 \sin^2\theta}(A^r B^\theta - A^\theta B^r)\vec{e}_\phi \Bigg] \\
&= r^2 \sin\theta (A^\theta B^\phi - A^\phi B^\theta)\vec{e}_r + \sin\theta (A^\phi B^r - A^r B^\phi)\vec{e}_\theta \\
&\quad + \frac{1}{\sin\theta}(A^r B^\theta - A^\theta B^r)\vec{e}_\phi.
\end{aligned} \tag{1.17}$$

2 Vector calculus in curvilinear coordinates

2.1 Christoffel symbols

In Cartesian coordinates, the derivatives of the bases with respect to some coordinate are trivially zero, but in curvilinear coordinates

$$\partial_\alpha \vec{e}_\beta = \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu, \tag{2.1}$$

where $\Gamma^\mu{}_{\alpha\beta}$ are called Christoffel symbols. It is not a tensor, but can be calculated by

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}). \tag{2.2}$$

In spherical coordinates, with Eqs.(1.9) and (2.2), we find

$$\left.\begin{array}{llll}
(1) & \Gamma^r{}_{rr} = 0, & \Gamma^\theta{}_{rr} = 0, & \Gamma^\phi{}_{rr} = 0; \\
(2) & \Gamma^r{}_{r\theta} = 0, & \Gamma^\theta{}_{r\theta} = 1/r, & \Gamma^\phi{}_{r\theta} = 0; \\
(3) & \Gamma^r{}_{r\phi} = 0, & \Gamma^\theta{}_{r\phi} = 0, & \Gamma^\phi{}_{r\phi} = 1/r; \\
(4) & \Gamma^r{}_{\theta r} = 0, & \Gamma^\theta{}_{\theta r} = 1/r, & \Gamma^\phi{}_{\theta r} = 0; \\
(5) & \Gamma^r{}_{\theta\theta} = -r, & \Gamma^\theta{}_{\theta\theta} = 0, & \Gamma^\phi{}_{\theta\theta} = 0; \\
(6) & \Gamma^r{}_{\theta\phi} = 0, & \Gamma^\theta{}_{\theta\phi} = 0, & \Gamma^\phi{}_{\theta\phi} = \cot\theta; \\
(7) & \Gamma^r{}_{\phi r} = 0, & \Gamma^\theta{}_{\phi r} = 0, & \Gamma^\phi{}_{\phi r} = 1/r; \\
(8) & \Gamma^r{}_{\phi\theta} = 0, & \Gamma^\theta{}_{\phi\theta} = 0, & \Gamma^\phi{}_{\phi\theta} = \cot\theta; \\
(9) & \Gamma^r{}_{\phi\phi} = -r\sin^2\theta, & \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, & \Gamma^\phi{}_{\phi\phi} = 0.
\end{array}\right\} \tag{2.3}$$

2.2 Vector calculus in curvilinear coordinates

First we introduce the notation ∇, which can be considered as a vector and an operator, generally

$$\hat{\nabla} = \vec{e}_\mu g^{\mu\nu}\partial_\nu. \tag{2.4}$$

Here we would not go deep on how it is derived.

2.2.1 Gradient

We act the $\hat{\nabla}$ directly on a scalar field φ to get its vector gradient

$$\nabla\varphi = (\vec{e}_\mu g^{\mu\nu}\partial_\nu)\varphi = g^{\mu\nu}\varphi_{,\nu}\vec{e}_\mu. \tag{2.5}$$

Specifically, in spherical coordinates

$$\begin{aligned}
\nabla\varphi &= g^{\mu\nu}\varphi_{,\nu}\vec{e}_\mu \\
&= g^{rr}\varphi_{,r}\vec{e}_r + g^{\theta\theta}\varphi_{,\theta}\vec{e}_\theta + g^{\phi\phi}\varphi_{,\phi}\vec{e}_\phi \\
&= \frac{\partial\varphi}{\partial r}\vec{e}_r + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\vec{e}_\theta + \frac{1}{r^2\sin^2\theta}\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi.
\end{aligned} \tag{2.6}$$

2.2.2 Divergence

Let us look at the dot product of $\hat{\nabla}$ with a vector field $\vec{A}$, noting that ∂_ν acts on both A^α and $\vec{e}_\alpha$, we obtain the divergence of the vector field, with the help of Eq.(1.6),

$$\begin{aligned}
\nabla\cdot\vec{A} &= (\vec{e}_\mu g^{\mu\nu}\partial_\nu)\cdot(A^\alpha\vec{e}_\alpha) \\
&= (\vec{e}_\mu\cdot\vec{e}_\alpha)g^{\mu\nu}\partial_\nu A^\alpha + (\vec{e}_\mu g^{\mu\nu}A^\alpha)\cdot(\partial_\nu\vec{e}_\alpha) \\
&= g_{\mu\alpha}g^{\mu\nu}A^\alpha{}_{,\nu} + (\vec{e}_\mu g^{\mu\nu}A^\alpha)\cdot(\Gamma^\lambda{}_{\alpha\nu}\vec{e}_\lambda) \\
&= \delta^\nu{}_\alpha A^\alpha{}_{,\nu} + g_{\mu\lambda}g^{\mu\nu}A^\alpha\Gamma^\lambda{}_{\alpha\nu} \\
&= A^\nu{}_{,\nu} + \delta^\nu{}_\lambda A^\alpha\Gamma^\lambda{}_{\alpha\nu} \\
&= A^\nu{}_{,\nu} + A^\alpha\Gamma^\nu{}_{\alpha\nu}.
\end{aligned} \tag{2.7}$$

The last equation in Eq.(2.7) is denoted by $A^\nu{}_{;\nu}$, where $A^\alpha{}_{;\beta}$ are so-called *covariant derivatives*, thus divergence is the self-contraction of covariant derivatives, i.e.

$$\nabla\cdot\vec{A} \equiv A^\nu{}_{;\nu} = A^\nu{}_{,\nu} + A^\alpha\Gamma^\nu{}_{\alpha\nu}. \tag{2.8}$$

Using Eq.(2.2) gives

$$\begin{aligned}\Gamma^{\nu}{}_{\alpha\nu} &= \frac{1}{2}g^{\mu\nu}(g_{\mu\alpha,\nu} + g_{\mu\nu,\alpha} - g_{\alpha\nu,\mu}) \\ &= \frac{1}{2}g^{\mu\nu}(g_{\mu\alpha,\nu} - g_{\alpha\nu,\mu}) + \frac{1}{2}g^{\mu\nu}g_{\mu\nu,\alpha}. \end{aligned} \tag{2.9}$$

Notice that the term in the parentheses is antisymmetric in μ and ν, while contracted with $g^{\mu\nu}$ the first term vanishes. Therefore we find

$$\Gamma^{\nu}{}_{\alpha\nu} = \frac{1}{2}g^{\mu\nu}g_{\mu\nu,\alpha}. \tag{2.10}$$

Since $(g^{\mu\nu})$ is the inverse matrix of $(g_{\mu\nu})$, it can be shown that the derivative of the determinant g of the matrix $(g_{\mu\nu})$ is

$$g_{,\alpha} = gg^{\mu\nu}g_{\mu\nu,\alpha}. \tag{2.11}$$

Substituting this equation in Eq.(2.10), we find

$$\Gamma^{\nu}{}_{\alpha\nu} = (\sqrt{g})_{,\alpha}/\sqrt{g}. \tag{2.12}$$

Then we can write the divergence, Eq.(2.8), as

$$A^{\alpha}{}_{;\alpha} = A^{\alpha}{}_{,\alpha} + \frac{1}{\sqrt{g}}A^{\alpha}(\sqrt{g})_{,\alpha} = \frac{1}{\sqrt{g}}(\sqrt{g}A^{\alpha})_{,\alpha}. \tag{2.13}$$

In spherical coordinates, $\sqrt{g} = r^2\sin\theta$ and

$$\begin{aligned}\nabla\cdot\vec{A} &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^{\alpha}}(r^2\sin\theta A^{\alpha}) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A^r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A^{\theta}) + \frac{\partial A^{\phi}}{\partial\phi}. \end{aligned} \tag{2.14}$$

2.2.3 Laplacian

To calculate Laplacian is trivial, for a scalar field φ, it is just $\nabla^2\varphi = \nabla\cdot(\nabla\varphi)$. If we let $\vec{A} = \nabla\varphi$, and then

$$A^{\alpha} = g^{\alpha\beta}\varphi_{,\beta}. \tag{2.15}$$

Putting this into Eq.(2.7) yields

$$\nabla^2\varphi = \frac{1}{\sqrt{g}}(\sqrt{g}g^{\alpha\beta}\varphi_{,\beta})_{,\alpha}. \tag{2.16}$$

In spherical coordinates

$$\begin{aligned}\nabla^2\varphi &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial x^{\alpha}}\left(r^2\sin\theta\, g^{\alpha\beta}\frac{\partial\varphi}{\partial x^{\beta}}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}. \end{aligned} \tag{2.17}$$

2.2.4 Curl

In the same manner as we deal with divergence, and using Eq.(1.16), we can calculate the curl as follows

$$\begin{aligned}\nabla\times\vec{A} &= (\vec{e}_{\mu}g^{\mu\nu}\partial_{\nu})\times(A^{\alpha}\vec{e}_{\alpha}) \\ &= (\vec{e}_{\mu}\times\vec{e}_{\alpha})g^{\mu\nu}\partial_{\nu}A^{\alpha} + (\vec{e}_{\mu}g^{\mu\nu}A^{\alpha})\times(\partial_{\nu}\vec{e}_{\alpha}) \\ &= (\sqrt{g}g^{\gamma\lambda}\varepsilon_{\lambda\mu\alpha}\vec{e}_{\gamma})g^{\mu\nu}A^{\alpha}{}_{,\nu} + (\vec{e}_{\mu}g^{\mu\nu}A^{\alpha})\times(\Gamma^{\beta}{}_{\alpha\nu}\vec{e}_{\beta}) \\ &= \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\alpha}A^{\alpha}{}_{,\nu}\vec{e}_{\gamma} + (\sqrt{g}g^{\gamma\lambda}\varepsilon_{\lambda\mu\beta}\vec{e}_{\gamma})g^{\mu\nu}A^{\alpha}\Gamma^{\beta}{}_{\alpha\nu} \\ &= \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\alpha}A^{\alpha}{}_{,\nu}\vec{e}_{\gamma} + \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\beta}A^{\alpha}\Gamma^{\beta}{}_{\alpha\nu}\vec{e}_{\gamma} \\ &= \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}(\varepsilon_{\lambda\mu\alpha}A^{\alpha}{}_{,\nu} + \varepsilon_{\lambda\mu\beta}A^{\alpha}\Gamma^{\beta}{}_{\alpha\nu})\vec{e}_{\gamma} \\ &= \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\alpha}(A^{\alpha}{}_{,\nu} + A^{\beta}\Gamma^{\alpha}{}_{\beta\nu})\vec{e}_{\gamma} \\ &= \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\alpha}A^{\alpha}{}_{;\nu}\vec{e}_{\gamma}. \end{aligned} \tag{2.18}$$

Now in spherical coordinates, we work out its components one-by-one

$$\begin{aligned}
(\nabla\times\vec{A})^r &= \sqrt{g}g^{rr}g^{\mu\nu}\varepsilon_{r\mu\alpha}A^\alpha{}_{;\nu}\\
&= \sqrt{g}g^{rr}(g^{\theta\theta}\varepsilon_{r\theta\phi}A^\phi{}_{;\theta}+g^{\phi\phi}\varepsilon_{r\phi\theta}A^\theta{}_{;\phi})\\
&= r^2\sin\theta\left(\frac{1}{r^2}A^\phi{}_{;\theta}-\frac{1}{r^2\sin\theta}A^\theta{}_{;\phi}\right), \qquad (2.19)\\
(\nabla\times\vec{A})^\theta &= \sqrt{g}g^{\theta\theta}g^{\mu\nu}\varepsilon_{\theta\mu\alpha}A^\alpha{}_{;\nu}\\
&= \sqrt{g}g^{\theta\theta}(g^{\phi\phi}\varepsilon_{\theta\phi r}A^r{}_{;\phi}+g^{rr}\varepsilon_{\theta r\phi}A^\phi{}_{;r})\\
&= \sin\theta\left(\frac{1}{r^2\sin^2\theta}A^r{}_{;\phi}-A^\phi{}_{;r}\right), \qquad (2.20)\\
(\nabla\times\vec{A})^\phi &= \sqrt{g}g^{\phi\phi}g^{\mu\nu}\varepsilon_{\phi\mu\alpha}A^\alpha{}_{;\nu}\\
&= \sqrt{g}g^{\phi\phi}(g^{rr}\varepsilon_{\phi r\theta}A^\theta{}_{;r}+g^{\theta\theta}\varepsilon_{\phi\theta r}A^r{}_{;\theta})\\
&= \frac{1}{\sin\theta}\left(A^\theta{}_{;r}-\frac{1}{r^2}A^r{}_{;\theta}\right). \qquad (2.21)
\end{aligned}$$

Using the Christoffel symbols in the table of Eq.(2.3) gives

$$\begin{aligned}
&A^\phi{}_{;\theta}=A^\phi{}_{,\theta}+\cot\theta A^\phi, && A^\theta{}_{;\phi}=A^\theta{}_{,\phi}-\sin\theta\cos\theta A^\phi;\\
&A^r{}_{;\phi}=A^r{}_{,\phi}-r\sin^2\theta A^\phi, && A^\phi{}_{;r}=A^\phi{}_{,r}+\frac{1}{r}A^\phi;\\
&A^\theta{}_{;r}=A^\rho{}_{,r}+\frac{1}{r}A^\theta, && A^r{}_{;\theta}=A^r{}_{,\theta}-rA^\theta.
\end{aligned}$$

Substituting these into the components respectively, we get

$$\begin{aligned}
(\nabla\times\vec{A})^r &= r^2\sin\theta\left(\frac{1}{r^2}A^\phi{}_{;\theta}-\frac{1}{r^2\sin\theta}A^\theta{}_{;\phi}\right)\\
&= r^2\sin\theta\left(\frac{1}{r^2}A^\phi{}_{,\theta}+\frac{2}{r^2}\cot\theta A^\phi-\frac{1}{r^2\sin\theta}A^\theta{}_{,\phi}\right)\\
&= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin^2\theta A^\phi)-\frac{\partial A^\theta}{\partial\phi}\right], \qquad (2.22)\\
(\nabla\times\vec{A})^\theta &= \sin\theta\left(\frac{1}{r^2\sin^2\theta}A^r{}_{;\phi}-A^\phi{}_{;r}\right)\\
&= \sin\theta\left(\frac{1}{r^2\sin^2\theta}A^r{}_{,\phi}-A^\phi{}_{,r}-\frac{2}{r}A^\phi\right)\\
&= \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial A^r}{\partial\phi}-\sin\theta\frac{\partial}{\partial r}(r^2A^\phi)\right], \qquad (2.23)\\
(\nabla\times\vec{A})^\phi &= \frac{1}{\sin\theta}\left(A^\theta{}_{;r}-\frac{1}{r^2}A^r{}_{;\theta}\right)\\
&= \frac{1}{\sin\theta}\left(A^\rho{}_{,r}+\frac{2}{r}A^\theta-\frac{1}{r^2}A^r{}_{,\theta}\right)\\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2A^\theta)-\frac{\partial A^r}{\partial\theta}\right]. \qquad (2.24)
\end{aligned}$$

Thus altogether, the curl in spherical coordinates is

$$\begin{aligned}
\nabla\times\vec{A} &= \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin^2\theta A^\phi)-\frac{\partial A^\theta}{\partial\phi}\right]\vec{e}_r\\
&\quad+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial A^r}{\partial\phi}-\sin\theta\frac{\partial}{\partial r}(r^2A^\phi)\right]\vec{e}_\theta\\
&\quad+\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2A^\theta)-\frac{\partial A^r}{\partial\theta}\right]\vec{e}_\phi. \qquad (2.25)
\end{aligned}$$

3 Calculus properties

3.1 Vector derivatives

3.1.1 Properties

One can derive the following identities with proper choice of equations above.

$$\nabla \cdot (\varphi \vec{A}) = (\nabla \varphi) \cdot \vec{A} + \varphi(\nabla \cdot \vec{A}), \tag{3.1}$$
$$\nabla \times (\varphi \vec{A}) = (\nabla \varphi) \times \vec{A} + \varphi(\nabla \times \vec{A}), \tag{3.2}$$
$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - A \cdot (\nabla \times \vec{B}), \tag{3.3}$$
$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}, \tag{3.4}$$
$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}), \tag{3.5}$$
$$\nabla \times (\nabla \varphi) = 0, \tag{3.6}$$
$$\nabla \cdot (\nabla \times \vec{A}) = 0, \tag{3.7}$$
$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}. \tag{3.8}$$

More properties in vector calculus could be found than the eight above. We here just provide the proofs for identities Eqs.(3.1) and (3.6) in Appendix C.

3.1.2 Applications

One important application is *integration by parts*, we are familiar with the scalar case

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u. \tag{3.9}$$

This is also available for vectors. Take Eq.(3.1) for example, one hope to find the integral of $(\nabla\varphi) \cdot \vec{A}$ when $\varphi\vec{A}$ and $\varphi(\nabla \cdot \vec{A})$ are given (rather than given φ and $\vec{A}$ directly), then

$$\begin{aligned}\int (\nabla\varphi) \cdot \vec{A} \mathrm{d}^3 x &= \int \nabla \cdot (\varphi\vec{A}) \mathrm{d}^3 x - \int \varphi(\nabla \cdot \vec{A}) \mathrm{d}^3 x \\ &= \oint (\varphi\vec{A}) \mathrm{d}^2 \vec{S} - \int \varphi(\nabla \cdot \vec{A}) \mathrm{d}^3 x.\end{aligned}$$

This is a "3-integral to 2-integral" law, or integration by parts in vector case.

To carry it out we need integral properties, which is introduced below.

3.2 Vector integrals

3.2.1 Stokes-Cartan theorem

All integral properties are concluded as *Stokes-Cartan theorem*:

If ω is a smooth $(n-1)$-form with compact support on smooth n-dimensional manifold-with-boundary Ω, $\partial\Omega$ denotes the boundary of Ω given the induced orientation, then

$$\int_\Omega \mathrm{d}\omega = \oint_{\partial\Omega} \omega, \tag{3.10}$$

where $\mathrm{d}\omega$ is its exterior derivative. In *our* language, if ω is a (general) k-form

$$\omega = f_{i_1 i_2 \cdots i_k} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} \equiv f_I \mathrm{d}x^I, \tag{3.11}$$

(where $a \wedge b$ is *exterior product*, satisfying $a \wedge b = -b \wedge a$) then

$$\mathrm{d}\omega = \frac{\partial f_I}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^I \tag{3.12}$$

is a $(k+1)$-form, the exterior derivative of ω. Here I is a multi-index, and f_I need not to be a $\binom{0}{k}$ tensor. A simple k-form could be

$$\omega = g \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} \equiv g \mathrm{d}x^I, \tag{3.13}$$

where g is some scalar function.

3.2.2 Gauss' divergence theorem

Specifically for instance, Gauss' divergence theorem in three-dimensional Euclidean space (expressed in Cartesian coordinates) is

$$\int_\Omega (\nabla\cdot\vec{A})\mathrm{d}^3x = \oint_{\partial\Omega} \vec{A}\cdot\mathrm{d}^2\vec{S}, \tag{3.14}$$

or expressed with index

$$\int_\Omega A^\alpha{}_{,\alpha}\mathrm{d}^3x = \oint_{\partial\Omega} A^\alpha n_\alpha \mathrm{d}^2S. \tag{3.15}$$

Here $\tilde{n}$ is the unit-normal *covector* of the boundary.

We define a correspondence relation

$$n_\alpha \mathrm{d}^2S \sim \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma, \tag{3.16}$$

$$\mathrm{d}^3x \sim \varepsilon_{\alpha\beta\gamma}\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma, \tag{3.17}$$

and $A^\alpha n_\alpha \mathrm{d}^2S = \vec{A}\cdot\mathrm{d}^2\vec{S}$ corresponds (*not* equals) to a 2-form

$$\omega = \frac{1}{2}\varepsilon_{\alpha\beta\gamma}A^\alpha\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma \sim A^\alpha n_\alpha\mathrm{d}^2S.$$

Then

$$\begin{aligned}
\mathrm{d}\omega &= \frac{1}{2}\varepsilon_{\alpha\beta\gamma}\frac{\partial A^\alpha}{\partial x^\lambda}\mathrm{d}x^\lambda\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma \\
&\sim \frac{1}{2}\varepsilon_{\alpha\beta\gamma}A^\alpha{}_{,\lambda}\varepsilon^{\lambda\beta\gamma}\mathrm{d}^3x \\
&= \frac{1}{2}A^\alpha{}_{,\lambda}(2\delta^\lambda{}_\alpha)\mathrm{d}^3x = A^\alpha{}_{,\alpha}\mathrm{d}^3x.
\end{aligned}$$

We find that Gauss' theorem is one special case of Eq.(3.10).

Note that here A^α is not necessary a vector's, but also the α-component of something (perhaps a tensor) else.

3.2.3 Stokes' curl theorem

More particularly, the Stokes' curl theorem on a two-dimensional manifold is

$$\int_\Omega (\nabla\times\vec{A})\cdot\mathrm{d}^2\vec{S} = \oint_{\partial\Omega}\vec{A}\cdot\mathrm{d}\vec{l}. \tag{3.18}$$

Here

$$\omega = A_\alpha\mathrm{d}x^\alpha = A_1\mathrm{d}x^1 + A_2\mathrm{d}x^2 + A_3\mathrm{d}x^3 \sim \vec{A}\cdot\mathrm{d}\vec{l}$$

is a 1-form (in dual space), and

$$\mathrm{d}\omega = \frac{\partial A_\alpha}{\partial x^\beta}\mathrm{d}x^\beta\wedge\mathrm{d}x^\alpha = A_{\alpha,\beta}\mathrm{d}x^\beta\wedge\mathrm{d}x^\alpha.$$

According to Eq.(B.17), we find that in Cartesian coordinates

$$\begin{aligned}
(\nabla\times\vec{A})\cdot\mathrm{d}^2\vec{S} &= (\varepsilon^{\lambda\beta}{}_\alpha A^\alpha{}_{,\beta}\vec{e}_\lambda)\cdot\mathrm{d}^2\vec{S} \\
&= \varepsilon^{\lambda\beta\alpha}A_{\alpha,\beta}n_\lambda\mathrm{d}^2S \\
&\sim \varepsilon^{\lambda\beta\alpha}A_{\alpha,\beta}\left(\frac{1}{2}\varepsilon_{\lambda\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\right) \\
&= \frac{1}{2}(\delta^\beta{}_\mu\delta^\alpha{}_\nu - \delta^\beta{}_\nu\delta^\alpha{}_\mu)A_{\alpha,\beta}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \\
&= \frac{1}{2}(A_{\alpha,\beta}\mathrm{d}x^\beta\wedge\mathrm{d}x^\alpha - A_{\alpha,\beta}\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta) \\
&= A_{\alpha,\beta}\mathrm{d}x^\beta\wedge\mathrm{d}x^\alpha = \mathrm{d}\omega.
\end{aligned}$$

This is another special case of Eq.(3.10).

3.2.4 Properties

Including Gauss' and Stokes' theorem, in *your familiar* notations, we have the following properties

$$\int (\nabla \cdot \vec{A}) \mathrm{d}^3 x = \oint \vec{A} \cdot \mathrm{d}^2 \vec{S}, \tag{3.19}$$

$$\int (\nabla \times \vec{A}) \mathrm{d}^3 x = -\oint \vec{A} \times \mathrm{d}^2 \vec{S}, \tag{3.20}$$

$$\int (\nabla \varphi) \mathrm{d}^3 x = \oint \varphi \, \mathrm{d}^2 \vec{S}; \tag{3.21}$$

$$\int (\nabla \times \vec{A}) \cdot \mathrm{d}^2 \vec{S} = \oint \vec{A} \cdot \mathrm{d}\vec{l}, \tag{3.22}$$

$$\int (\nabla \varphi) \times \mathrm{d}^2 \vec{S} = -\oint \varphi \, \mathrm{d}\vec{l}. \tag{3.23}$$

These identities are bound to benefit you a lot, and you are welcome to prove them with Stokes-Cartan theorem Eq.(3.10).

3.2.5 Curvilinear coordinates case

In curvilinear coordinates, $\partial\Omega$ is usually a curved manifold, this time proper volume element should be $\sqrt{g}\mathrm{d}^3 x$ rather than simply $\mathrm{d}^3 x$. We hope to find the corresponding Eq.(3.15) now, with the help of Eq.(2.13)

$$A^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{g}} (\sqrt{g} A^{\alpha})_{,\alpha},$$

we find

$$\int A^{\alpha}{}_{;\alpha} \sqrt{g} \mathrm{d}^3 x = \int (\sqrt{g} A^{\alpha})_{,\alpha} \mathrm{d}^3 x. \tag{3.24}$$

Since the final term involves simple partial derivatives, the mathematics of Gauss's law Eq.(3.15) applies to it:

$$\int (\sqrt{g} V^{\alpha})_{,\alpha} \mathrm{d}^3 x = \oint_{\partial\Omega} (\sqrt{g} V^{\alpha}) n_{\alpha} \mathrm{d}^2 S. \tag{3.25}$$

This means (explaining why above I said A^{α} need not to be a vector's α-component, here $A^{\alpha} \to \sqrt{g} A^{\alpha}$.)

$$\int A^{\alpha}{}_{;\alpha} \sqrt{g} \mathrm{d}^3 x = \oint_{\partial\Omega} V^{\alpha} n_{\alpha} \sqrt{g} \mathrm{d}^2 S. \tag{3.26}$$

So Gauss's law does apply on a curved manifold, we need to integrate the divergence ($\nabla \cdot \vec{A} = A^{\alpha}{}_{;\alpha}$) over proper volume $\sqrt{g}\mathrm{d}^3 x$ and to use the proper surface element, $n_{\alpha}\sqrt{g}\mathrm{d}^2 S$, in the surface integral.

The other identities slightly modifies in the same manner, in curvilinear coordinates.

Appendices

A Conclusions

A.1 Conclusions in arbitrary coordinates

Spatial interval

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta. \tag{A.1}$$

Scalar product

$$\vec{A}\cdot\vec{B} = g_{\alpha\beta}A^\alpha B^\beta. \tag{A.2}$$

Cross product

$$\vec{A}\times\vec{B} = \sqrt{g}g^{\mu\nu}\varepsilon_{\nu\alpha\beta}A^\alpha B^\beta\vec{e}_\mu. \tag{A.3}$$

Gradient

$$\nabla\varphi = g^{\mu\nu}\varphi_{,\nu}\vec{e}_\mu. \tag{A.4}$$

Divergence

$$\nabla\cdot\vec{A} = A^\alpha{}_{;\alpha} = \frac{1}{\sqrt{g}}(\sqrt{g}A^\alpha)_{,\alpha}. \tag{A.5}$$

Laplacian

$$\nabla^2\varphi = \frac{1}{\sqrt{g}}(\sqrt{g}g^{\alpha\beta}\varphi_{,\beta})_{,\alpha}. \tag{A.6}$$

Curl

$$\nabla\times\vec{A} = \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\alpha}A^\alpha{}_{;\nu}\vec{e}_\gamma. \tag{A.7}$$

A.2 Conclusions in spherical coordinates

Spatial interval

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2. \tag{A.8}$$

Scalar product

$$\vec{A}\cdot\vec{B} = A^rB^r + r^2A^\theta B^\theta + r^2\sin^2\theta A^\phi B^\phi. \tag{A.9}$$

Cross product

$$\vec{A}\times\vec{B} = r^2\sin\theta(A^\theta B^\phi - A^\phi B^\theta)\vec{e}_r + \sin\theta(A^\phi B^r - A^rB^\phi)\vec{e}_\theta + \frac{1}{\sin\theta}(A^rB^\theta - A^\theta B^r)\vec{e}_\phi.$$

Gradient

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\vec{e}_r + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\vec{e}_\theta + \frac{1}{r^2\sin^2\theta}\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi. \tag{A.10}$$

Divergence

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A^r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A^\theta) + \frac{\partial A^\phi}{\partial\phi}. \tag{A.11}$$

Laplacian

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}. \tag{A.12}$$

Curl

$$\begin{aligned}\nabla\times\vec{A} = &\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin^2\theta A^\phi) - \frac{\partial A^\theta}{\partial\phi}\right]\vec{e}_r\\ &+ \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial A^r}{\partial\phi} - \sin\theta\frac{\partial}{\partial r}(r^2A^\phi)\right]\vec{e}_\theta\\ &+ \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2A^\theta) - \frac{\partial A^r}{\partial\theta}\right]\vec{e}_\phi.\end{aligned} \tag{A.13}$$

A.3 Corresponding conclusions in cylindrical coordinates

Spatial interval

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2 + \mathrm{d}z^2. \tag{A.14}$$

Scalar product

$$\vec{A} \cdot \vec{B} = A^\rho B^\rho + \rho^2 A^\phi B^\phi + A^z B^z. \tag{A.15}$$

Cross product

$$\vec{A} \times \vec{B} = \rho(A^\phi B^z - A^z B^\phi)\vec{e}_r + \frac{1}{\rho}(A^z B^\rho - A^\rho B^z)\vec{e}_\phi + \rho(A^\rho B^\phi - A^\phi B^\rho)\vec{e}_z. \tag{A.16}$$

Gradient

$$\nabla\varphi = \frac{\partial\varphi}{\partial\rho}\vec{e}_\rho + \frac{1}{\rho^2}\frac{\partial\varphi}{\partial\phi}\vec{e}_\phi + \frac{\partial\varphi}{\partial z}\vec{e}_z. \tag{A.17}$$

Divergence

$$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A^\rho) + \frac{\partial A^\phi}{\partial\phi} + \frac{\partial A^z}{\partial z}. \tag{A.18}$$

Laplacian

$$\nabla^2\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2}. \tag{A.19}$$

Curl

$$\nabla \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A^z}{\partial\phi} - \rho\frac{\partial A^\phi}{\partial z}\right)\vec{e}_\rho + \frac{1}{\rho}\left(\frac{\partial A^\rho}{\partial z} - \frac{\partial A^z}{\partial\rho}\right)\vec{e}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho^2 A^\phi) - \frac{\partial A^\rho}{\partial\phi}\right]\vec{e}_z. \tag{A.20}$$

B Levi-civita tensor

Like metric tensor, Levi-civita tensor (in three dimensions) is a $\binom{0}{3}$ tensor that maps *three* vectors into a real number, which is their *triple product*, denoted by

$$\mathbf{E}(\vec{A},\vec{B},\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C}) = E_{\alpha\beta\gamma}A^\alpha B^\beta C^\gamma. \tag{B.1}$$

In Cartesian coordinates, the components of Levi-civita tensor $E_{\bar\alpha\bar\beta\bar\gamma}$ are simply Levi-civita symbols $\varepsilon_{\bar\alpha\bar\beta\bar\gamma}$, which is familiar to you. This is a *frame-invariant* symbol, so that in *any* coordinates, $\varepsilon_{\alpha\beta\gamma}$ have the same values

$$\varepsilon_{\alpha\beta\gamma} = \begin{cases} +1, & \text{if } (\alpha,\beta,\gamma) \text{ is an even permutation of } (1,2,3); \\ -1, & \text{if } (\alpha,\beta,\gamma) \text{ is an odd permutation of } (1,2,3); \\ 0, & \text{otherwise.} \end{cases} \tag{B.2}$$

We can use this symbol to describe the determinant of a 3×3 matrix

$$\det(A) = \frac{1}{3!}\sum_{i,j,k}\sum_{l,m,n}\varepsilon_{ijk}\varepsilon_{lmn}A_{il}A_{jm}A_{kn}. \tag{B.3}$$

We also have the property which is quite easy to prove,

$$\varepsilon_{\alpha\beta\gamma}\varepsilon^{\alpha\beta\gamma} = 3! = 6. \tag{B.4}$$

However, in curvilinear coordinates $E_{\alpha\beta\gamma}\neq\varepsilon_{\alpha\beta\gamma}$, but they have the same index-dependent manner, thus we set $E_{\alpha\beta\gamma}=\kappa\varepsilon_{\alpha\beta\gamma}$, where κ is a constant.

Notice that Eq.(B.1) is a frame-invariant equation since $\mathbf{E}$ is a well-defined tensor, therefore

$$\mathbf{E}(\vec{A},\vec{B},\vec{C}) = E_{\alpha\beta\gamma}A^\alpha B^\beta C^\gamma = E_{\bar\alpha\bar\beta\bar\gamma}A^{\bar\alpha}B^{\bar\beta}C^{\bar\gamma}, \tag{B.5}$$

where barred indices refer to Cartesian coordinates. While

$$E_{\alpha\beta\gamma}A^\alpha B^\beta C^\gamma = E_{\alpha\beta\gamma}\Lambda^\alpha{}_{\bar\alpha}\Lambda^\beta{}_{\bar\beta}\Lambda^\gamma{}_{\bar\gamma}A^{\bar\alpha}B^{\bar\beta}C^{\bar\gamma}, \tag{B.6}$$

and then we get

$$E_{\bar\alpha\bar\beta\bar\gamma} = E_{\alpha\beta\gamma}\Lambda^\alpha{}_{\bar\alpha}\Lambda^\beta{}_{\bar\beta}\Lambda^\gamma{}_{\bar\gamma}, \tag{B.7}$$

or, carrying out what we've discussed above,

$$\varepsilon_{\bar\alpha\bar\beta\bar\gamma} = \kappa\varepsilon_{\alpha\beta\gamma}\Lambda^\alpha{}_{\bar\alpha}\Lambda^\beta{}_{\bar\beta}\Lambda^\gamma{}_{\bar\gamma}. \tag{B.8}$$

Using Eq.(B.4) to contract indices $\bar\alpha$, $\bar\beta$, and $\bar\gamma$ on both sides of Eq.(B.8) indicates

$$3! = \varepsilon^{\bar\alpha\bar\beta\bar\gamma}\varepsilon_{\bar\alpha\bar\beta\bar\gamma} = \kappa\varepsilon^{\bar\alpha\bar\beta\bar\gamma}\varepsilon_{\alpha\beta\gamma}\Lambda^\alpha{}_{\bar\alpha}\Lambda^\beta{}_{\bar\beta}\Lambda^\gamma{}_{\bar\gamma}, \tag{B.9}$$

or, with the help of Eq.(C.8),

$$1 = \kappa\left(\frac{1}{3!}\varepsilon^{\bar\alpha\bar\beta\bar\gamma}\varepsilon_{\alpha\beta\gamma}\Lambda^\alpha{}_{\bar\alpha}\Lambda^\beta{}_{\bar\beta}\Lambda^\gamma{}_{\bar\gamma}\right) = \kappa\det(\Lambda^\alpha{}_{\bar\alpha}). \tag{B.10}$$

Remembering the transformation of the metric components

$$g_{\alpha\beta} = \Lambda^{\bar\mu}{}_\alpha\Lambda^{\bar\nu}{}_\beta\eta_{\bar\mu\bar\nu}, \tag{B.11}$$

in matrix terminology

$$(g) = (\Lambda)(\eta)(\Lambda^{\mathrm{T}}).$$

It follows that the determinants satisfy

$$\det(g) = \det(\Lambda)\det(\eta)\det(\Lambda^{\mathrm{T}}),$$

but for any matrix

$$\det(\Lambda) = \det(\Lambda^{\mathrm{T}}),$$

and we can easily see from Eq.(1.8) that

$$\det(\eta) = 1.$$

Therefore, we get

$$g = |\det(g)| = [\det(\Lambda)]^2, \tag{B.12}$$

which enables us to conclude that

$$\det(\Lambda^{\bar{\alpha}}{}_{\beta}) = g^{1/2}, \tag{B.13}$$

and the determinant of its inverse

$$\det(\Lambda^{\alpha}{}_{\bar{\beta}}) = g^{-1/2}. \tag{B.14}$$

Finally, from Eq.(B.10) we can tell that $\kappa = \sqrt{g}$, and hence

$$E_{\alpha\beta\gamma} = \sqrt{g}\varepsilon_{\alpha\beta\gamma}. \tag{B.15}$$

This accomplishes part of the proof for Eq.(1.15).

With the definition of Levi-civita tensor, we could rewrite cross product and curl in a more compact form with index raising.

$$\begin{aligned}
\vec{A} \times \vec{B} &= \sqrt{g} g^{\mu\nu} \varepsilon_{\nu\alpha\beta} A^{\alpha} B^{\beta} \vec{e}_{\mu} \\
&= g^{\mu\nu} E_{\nu\alpha\beta} A^{\alpha} B^{\beta} \vec{e}_{\mu} \\
&= E^{\mu}{}_{\alpha\beta} A^{\alpha} B^{\beta} \vec{e}_{\mu}, \qquad \text{(B.16)} \\
\nabla \times \vec{A} &= \sqrt{g} g^{\gamma\lambda} g^{\mu\nu} \varepsilon_{\lambda\mu\alpha} A^{\alpha}{}_{;\nu} \vec{e}_{\gamma} \\
&= g^{\gamma\lambda} g^{\mu\nu} E_{\lambda\mu\alpha} A^{\alpha}{}_{;\nu} \vec{e}_{\gamma} \\
&= E^{\gamma\nu}{}_{\alpha} A^{\alpha}{}_{;\nu} \vec{e}_{\gamma}. \qquad \text{(B.17)}
\end{aligned}$$

Sometimes, it is more convenient to express cross product or curl as above in actual problems.

C Some supplementary proofs

C.1 Proof for Eq.(3.1)

Applying Eq.(2.13), we have

$$\begin{aligned}\nabla\cdot(\varphi\vec{A}) &= \frac{1}{\sqrt{g}}(\sqrt{g}\,\varphi A^\alpha)_{,\alpha}\\ &= \frac{1}{\sqrt{g}}\varphi_{,\alpha}(\sqrt{g}A^\alpha)+\varphi\frac{1}{\sqrt{g}}(\sqrt{g}A^\alpha)_{,\alpha}\\ &= \varphi_{,\alpha}A^\alpha+\varphi(\nabla\cdot\vec{A}).\end{aligned} \tag{C.1}$$

Note that

$$\begin{aligned}\varphi_{,\alpha}A^\alpha &= \delta^\alpha{}_\beta\varphi_{,\alpha}A^\beta = g^{\alpha\mu}g_{\mu\beta}\varphi_{,\alpha}A^\beta\\ &= g_{\mu\beta}(g^{\alpha\mu}\varphi_{,\alpha})A^\beta = (\nabla\varphi)\cdot\vec{A}.\end{aligned} \tag{C.2}$$

Then we take Eq.(C.2) into Eq.(C.1) to get

$$\nabla\cdot(\varphi\vec{A}) = (\nabla\varphi)\cdot\vec{A}+\varphi(\nabla\cdot\vec{A}). \tag{C.3}$$

C.2 Proof for Eq.(3.6)

Substituting Eq.(2.5) into (2.18) illustrates, remembering $g^{\alpha\beta}{}_{;\mu}=0$,

$$\begin{aligned}\nabla\times(\nabla\varphi) &= \sqrt{g}g^{\gamma\lambda}g^{\mu\nu}\varepsilon_{\lambda\mu\alpha}(g^{\alpha\beta}\varphi_{,\beta})_{;\nu}\vec{e}_\gamma\\ &= \sqrt{g}g^{\gamma\lambda}\varepsilon_{\lambda\mu\alpha}g^{\mu\nu}g^{\alpha\beta}\varphi_{,\beta;\nu}\vec{e}_\gamma\\ &= \sqrt{g}\sum_{\mu=1}^{3}\sum_{\alpha=1}^{3}g^{\gamma\lambda}(\varepsilon_{\lambda\mu\alpha}g^{\mu\nu}g^{\alpha\beta}\varphi_{,\beta;\nu}+\varepsilon_{\lambda\alpha\mu}g^{\alpha\nu}g^{\mu\beta}\varphi_{,\beta;\nu})\vec{e}_\gamma\\ &= \sqrt{g}\sum_{\mu=1}^{3}\sum_{\alpha=1}^{3}\varepsilon_{\lambda\mu\alpha}g^{\gamma\lambda}(g^{\mu\nu}g^{\alpha\beta}\varphi_{,\beta;\nu}-g^{\alpha\nu}g^{\mu\beta}\varphi_{,\beta;\nu})\vec{e}_\gamma,\end{aligned}$$

In Euclidean space (or flat space), $g_{\alpha\beta}=g^{\alpha\beta}=0$ if $\alpha\neq\beta$, therefore

$$\begin{aligned}\nabla\times(\nabla\varphi) &= \sqrt{g}\sum_{\gamma=1}^{3}\sum_{\mu=1}^{3}\sum_{\alpha=1}^{3}\varepsilon_{\gamma\mu\alpha}g^{\gamma\gamma}(g^{\mu\mu}g^{\alpha\alpha}\varphi_{,\alpha;\mu}-g^{\alpha\alpha}g^{\mu\mu}\varphi_{,\mu;\alpha})\vec{e}_\gamma\\ &= \sqrt{g}\sum_{\gamma=1}^{3}\sum_{\mu=1}^{3}\sum_{\alpha=1}^{3}\varepsilon_{\gamma\mu\alpha}g^{\gamma\gamma}g^{\mu\mu}g^{\alpha\alpha}(\varphi_{,\alpha;\mu}-\varphi_{,\mu;\alpha})\vec{e}_\gamma.\end{aligned} \tag{C.4}$$

Notice that the second covariant derivative of a scalar field $\nabla\nabla\varphi$ has components $\varphi_{,\alpha;\beta}$ and is a $\binom{0}{2}$ tensor. In Cartesian coordinates

$$\varphi_{,\bar\alpha;\bar\beta}=\varphi_{,\bar\alpha,\bar\beta}=\varphi_{,\bar\beta,\bar\alpha}=\varphi_{,\bar\beta;\bar\alpha}, \tag{C.5}$$

and if a tensor is symmetric in one basis it is symmetric in all bases. Therefore

$$\varphi_{,\alpha;\beta}=\varphi_{,\beta;\alpha} \tag{C.6}$$

is valid in *any* basis, and putting this into Eq.(C.4) implies

$$\nabla\times(\nabla\varphi)=0. \tag{C.7}$$

C.3 Proof for Eq.(2.11)

Say A is an arbitrary $n\times n$ matrix with a_{ij} the element at row i and column j. Then Laplace's formula for the determinant is:

$$\det(A)=\sum_{j=1}^{n}a_{ij}C_{ij}=\sum_{j=1}^{n}a_{ij}(-1)^{i+j}M_{ij}, \tag{C.8}$$

where C_{ij} is the cofactor, i is any row, and M_{ij} is the "minor", that is, the determinant of the matrix formed by removing the ith row and jth column from A.

The most instructive derivation results from simply differentiating the expression for the determinant given in Eq.(C.8) above:

$$\begin{aligned}\frac{\partial det(A)}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu}\sum_{j=1}^{n} a_{ij}C_{ij} \\ &= \sum_{k,l}\frac{\partial}{\partial a_{kl}}\left(\sum_{j=1}^{n} a_{ij}C_{ij}\right)\frac{\partial a_{kl}}{\partial x^\mu} \\ &= \sum_{k,l}\frac{\partial}{\partial a_{kl}}\left(\sum_{j=1}^{n} a_{kj}C_{kj}\right)\frac{\partial a_{kl}}{\partial x^\mu} \\ &= \sum_{k,l,j}\left(\frac{\partial a_{kj}}{\partial a_{kl}}C_{kj} + a_{kj}\frac{\partial C_{kj}}{\partial a_{kl}}\right)\frac{\partial a_{kl}}{\partial x^\mu}. \end{aligned} \tag{C.9}$$

Recall that C_{kj} is proportional to the determinant of matrix formed from A with row k and column j removed, so it is independent of a_{kl}, giving $\frac{\partial C_{kj}}{\partial a_{kl}} = 0$. Hence the judicious choice of row $i = k$. So Eq.(C.9) simplies to

$$\begin{aligned}\frac{\partial det(A)}{\partial x^\mu} &= \sum_{k,l,j}\left(\frac{\partial a_{kj}}{\partial a_{kl}}C_{kj}\right)\frac{\partial a_{kl}}{\partial x^\mu} \ = \sum_{k,l,j}\delta_{jl}C_{kl}\frac{\partial a_{kl}}{\partial x^\mu} \\ &= \sum_{k,j}C_{kj}\frac{\partial a_{kj}}{\partial x^\mu} = \mathrm{tr}\left(\mathrm{adj}(A)\frac{\partial A}{\partial x^\mu}\right). \end{aligned} \tag{C.10}$$

The above result is known as Jacobi's formula. Here adj(A) is the adjugate of A, defined as the transpose of the matrix of cofactors of A. Now we can use,

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A), \tag{C.11}$$

to substitute for adj(A) in Eq.(C.10) to bring it closer to the form of Eq.(2.11):

$$\begin{aligned}\frac{\partial det(A)}{\partial x^\mu} &= \det(A)\mathrm{tr}\left(A^{-1}\frac{\partial A}{\partial x^\mu}\right), \\ \frac{\partial g}{\partial x^\mu} &= g g^{\alpha\beta}\frac{\partial g_{\beta\alpha}}{\partial x^\mu}. \end{aligned} \tag{C.12}$$

D Unit bases, a set of non-coordinate bases

Above, we discuss all the problems in so-called *coordinate bases*, but many physicists enjoy using *unit bases*, which all have magnitude 1. Conventionally, we put a 'hat' over the index of a unit basis, so that

$$|\vec{e}_{\hat{\alpha}}|^2 = 1, \quad \text{and} \quad \vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}} = g_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}}. \tag{D.1}$$

We won't go further on unit bases, which is a set of non-coordinate bases. We just tell you how to convert coordinate bases into unit bases. The rule is nothing but $\vec{e}_{\hat{\alpha}} = \vec{e}_\alpha / |\vec{e}_\alpha|$, say, in spherical coordinates

$$\vec{e}_{\hat{r}} = \vec{e}_r, \quad \vec{e}_{\hat{\theta}} = \frac{1}{r}\vec{e}_\theta, \quad \vec{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\vec{e}_\phi, \tag{D.2}$$

so that, when representing the same vector $\vec{A} = A^\alpha \vec{e}_\alpha = A^{\hat{\alpha}}\vec{e}_{\hat{\alpha}}$, its components in unit bases are

$$A^{\hat{r}} = A^r, \quad A^{\hat{\theta}} = rA^\theta, \quad A^{\hat{\phi}} = r\sin\theta A^\phi. \tag{D.3}$$

In the same manner, for cylindrical coordinates

$$\vec{e}_{\hat{\rho}} = \vec{e}_\rho, \quad \vec{e}_{\hat{\phi}} = \frac{1}{\rho}\vec{e}_\phi, \quad \vec{e}_{\hat{z}} = \vec{e}_z; \tag{D.4}$$

$$A^{\hat{\rho}} = A^\rho, \quad A^{\hat{\phi}} = \rho A^\phi, \quad A^{\hat{z}} = A^z. \tag{D.5}$$

With these rules, one can convert Eqs.(A.8)∼(A.20) from our coordinate bases into his unit bases. Simply for example, the curl in spherical coordinates with unit bases is

$$\begin{aligned}\nabla \times \vec{A} = &\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A^{\hat{\phi}}) - \frac{\partial A^{\hat{\theta}}}{\partial\phi}\right]\vec{e}_{\hat{r}} \\ &+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A^{\hat{r}}}{\partial\phi} - \frac{\partial}{\partial r}(rA^{\hat{\phi}})\right]\vec{e}_{\hat{\theta}} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA^{\hat{\theta}}) - \frac{\partial A^{\hat{r}}}{\partial\theta}\right]\vec{e}_{\hat{\phi}}.\end{aligned}$$

In fact, since the Christoffel symbols are well-defined also in non-coordinate basis, except for the symmetry $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$, we could find a way to calculate them too. Now if we denote

$$\partial_{\hat{\mu}}\vec{e}_{\hat{\nu}} - \partial_{\hat{\nu}}\vec{e}_{\hat{\mu}} \equiv c^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}}\vec{e}_{\hat{\alpha}}, \tag{D.6}$$

the Christoffel symbols in non-coordinate basis are hence derived

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} = \frac{1}{2}g^{\hat{\mu}\hat{\nu}}(g_{\hat{\nu}\hat{\alpha},\hat{\beta}} + g_{\hat{\nu}\hat{\beta},\hat{\alpha}} - g_{\hat{\alpha}\hat{\beta},\hat{\nu}} + c_{\hat{\nu}\hat{\alpha}\hat{\beta}} + c_{\hat{\nu}\hat{\beta}\hat{\alpha}} - c_{\hat{\alpha}\hat{\beta}\hat{\nu}}), \tag{D.7}$$

where $c_{\hat{\alpha}\hat{\beta}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}}c^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}$. Therefore, one may find all the corresponding equations directly using the Christoffel symbols in unit bases.

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