Homework Assignment 2

i don't tell u who i am

2018.10.18

1 Chapter 3

1.1 5)

Suppose there exists 3 states of the world $s = 1, 2, 3$ and 2 assets $x^1 = (2, 1, 0)'$ and $x^2 = (0, 1, 0)'$.

1.1.1 1.

Suppose $p_1 = 1$ and $p_2 = 0.3$. What state prices are consistent with these prices?

Answer Since $p = X'q$, we have

$$\begin{pmatrix} 1 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{1}$$

So we have

$$1 = 2q_1 + q_2 \tag{2}$$

$$0.3 = q_2 \tag{3}$$

Thus we have

$$\begin{cases} q_1 = & 0.35 \\ q_2 = & 0.3 \\ q_3 \in & R \end{cases} \tag{4}$$

1.1.2 2.

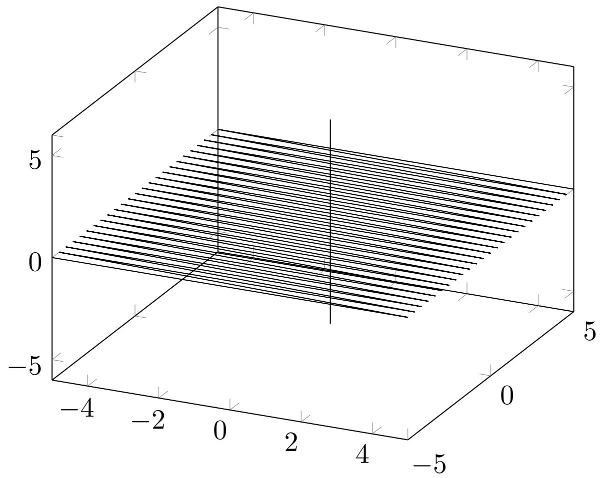
Solve for the unique pricing kernel q^*.

Answer The seccurity structure $\langle X \rangle$ is like a plane in a three dimension space. Since we have q_1 and q_2 and have no idea of q_3, the vector q is like a line in the three dimension space. So the pricing kernel has to be in the intersection of the plane and the line.

Let's focus on the plane. Since it has the vector $(2, 1, 0)$ and vector $(0, 1, 0)$. With these vectors we can determine the plane. Because they all have 0 for the z axis, we can easily determine the plane to be $z = 0$.

Then we have a line determined by q, with q_3 unknown. It's a line like $(0.35, 0.4)$.The kernel must be inside the $\langle X \rangle$, namely the intersection.

The three dimension space looks like:



$$(5)$$

So the kernel has to be

$$q* = \begin{pmatrix} 0.35 \\ 0.3 \\ 0 \end{pmatrix} \tag{6}$$

If we cannot use the data from question 1, then the kernel is

$$q* = \begin{pmatrix} \frac{p_1 - p_2}{2} \\ p_2 \\ 0 \end{pmatrix} \tag{7}$$

1.1.3 3.

Use the pricing kernel to value a third asset $x^3 = (0, 1, 1)'$. What other state prices (different from q^*) are consistent with no arbitrage?

Answer Since $p = X'q$, we have

$$p_3 = 0 \times q_1 + 1 \times q_2 + 1 \times q_3 \tag{8}$$

That is

$$p_3 = 0.3 \tag{9}$$

To make the state prices exclude arbitrage, we have $p >> 0$, so the state price will be

$$q = \begin{pmatrix} 0.35 \\ 0.3 \\ q_3 \end{pmatrix} \tag{10}$$

where $q_3 > 0$.

If we cannot use the data from question 1, we have $p_3 = q_2 + q_3$. Then $q_3 = p_3 - q_2$. Then $q_3 = p_3 - p_2$. So we have

$$\begin{cases} \dfrac{p_1 - p_2}{2} > 0 \\ p_2 > 0 \\ p_3 - p_2 > 0 \end{cases} \tag{11}$$

which means the state price should satisfy

$$\begin{cases} p_1 > & p_2 \\ p_2 > & 0 \\ p_3 > & p_2 \end{cases} \tag{12}$$

simutaneously to exclude arbitrage.

1.1.4 4.

Now suppose $p_3 = 0.6$. Solve for the state price vector. Does this market permit arbitrage?

Answer According to $p = X'q$, we have

$$\begin{pmatrix} 1 \\ 0.3 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{13}$$

So we have

$$\begin{cases} 1 = & 2q_1 + q_2 \\ 0.3 = & q_2 \\ 0.6 = & q_2 + q_3 \end{cases} \tag{14}$$

Hence the state price vector is $q = (0.35, 0.3, 0.3)'$. For $q >> 0$, it certainly exclude arbitrage.

1.1.5 5.

Solve for the stochastic discount factor assuming the physical probablity is such that $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$.

Answer we simply have

$m_1 = 0.35 \div \frac{1}{3} = 1.05$
$m_2 = 0.3 \div \frac{1}{3} = 0.9$
$m_3 = 0.3 \div \frac{1}{3} = 0.9$

1.1.6 6.

Solve for the distribution under the equivalent martingale measure.

Answer To make a risk-free seccurity as a equivalent martingale measure, we buy $1x_1$ and $2x_3$ and sell $1x_2$, thus we have the payoff $(2, 2, 2)'$, and the price of which is $1+2\times 0.6-0.3 = 1.9$. In other words, $(1, 1, 1)$ costs 0.95. We can calculate r^f by

$$p_b = \sum_{S=1}^{S} q_s = \frac{1}{1+r^f} \tag{15}$$

$$0.95 = \frac{1}{1+r^f} \tag{16}$$

$$r^f = \frac{1}{19} \tag{17}$$

$\hat{\pi}_1 = \frac{q_1}{q_1+q_2+q_3} = 0.368$
$\hat{\pi}_2 = \frac{q_2}{q_1+q_2+q_3} = 0.316$
$\hat{\pi}_3 = \frac{q_3}{q_1+q_2+q_3} = 0.316$

1.2 6)

Suppose a stock index is currently trading at \$25, and there are 5 possible states of the world in $t = T$ such that $S_T \in \{15, 20, 25, 30, 35\}$.

1.2.1 1.

Given a zero risk-free interest rate, describe a valid equivalent martingale measure.

Answer A valid equivalent martingale measure may be

$$\hat{\pi}_1 = \hat{\pi}_2 = \hat{\pi}_3 = \hat{\pi}_4 = \hat{\pi}_5 = 0.2 \tag{18}$$

1.2.2 2.

Under this measure, price call options at $K = 15, 20, 25, 30, 35$

Answer When $K = 15$, we have $p = 0.2 * 5 + 0.2 * 10 + 0.2 * 15 + 0.2 * 20 = 10$
When $K = 20$, we have $p = 0.2 * 5 + 0.2 * 10 + 0.2 * 15 = 6$
When $K = 25$, we have $p = 0.2 * 5 + 0.2 * 10 = 3$
When $K = 30$, we have $p = 0.2 * 5 = 1$
When $K = 35$, we have $p = 0$.

1.2.3 3.

Use this information to recover state prices.

Answer $q_2 = 10 - 2 * 6 + 3 = 1$
$q_3 = 6 - 2 * 3 + 1 = 1$
$q_4 = 3 - 2 * 1 + 0 = 1$
As for p_1 and p_5, we have to approximate the prices of the call options. And we have $q_1 = 15 - 2 * 10 + 6 = 1$ $q_5 = 1 - 0 + 0 = 1$

1.3 7.

Suppose there are S possible states of the world of the world in $t = T$ and each has a (physical) probablity of occurence $\eta_s > 0$ with $\sum\limits_{s=1}^{S} \eta_s = 1$. Consider the vector $\mu \in R^S$ with for $s = 1, \cdots, S \mu_s = \frac{q_s}{\eta_s}$, where q is a state-price vector. Write $R(y) = \sum\limits_{S=1}^{S} \eta_s y_s$ for any $y \in R^S$.

1.3.1 1.

Consider an asset with payoff $x = (x_1, \cdots, x_s)$. Show that the price of this asset must be $E(z)$, where $z_s = \eta_s x_s$. Interpret this result.

Answer We have $p = X'q$. That is $p = \sum\limits_{S=1}^{S} q_s x_s$. Since $q_s = \mu_s \eta_s$, we can write

$$p = \sum_{S=1}^{S} \eta_s \mu_s x_s \tag{19}$$

Thus, we have $p = E(z)$, where $z_s = \mu_s x_s$.
This result indicates that the price of an asset equals to the sum of the state price times the probablity of the state occurred times a dicscount factor.

1.3.2 2.

Let the rate of return for an asset with price $p > 0$ in state s be $r_s = \frac{x_s}{p}$ and let $w_s = r_s \mu_s$. Show that $E(w) = 1$. Is there some function of the excess return $r - r^f$ of the asset such that $E[f(r - r^f)] = 0$?

Answer We have

$$E(w) = \sum_{s=1}^{S} \eta_s w_s \tag{20}$$

$$E(w) = \sum_{s=1}^{S} \eta_s r_s \mu_s \tag{21}$$

$$E(w) = \sum_{s=1}^{S} \eta_s \frac{x_s}{p} \frac{q_s}{\eta_s} \tag{22}$$

$$E(w) = \sum_{s=1}^{S} \frac{x_s}{p} q_s \tag{23}$$

$$E(w) = \frac{\sum\limits_{s=1}^{S} x_s q_s}{p} = 1 \tag{24}$$

In order to creat a function to make $E[m(R - R^f)] = 0$, we need to use the conclusion of $E(w) = 1$. For a risk-free bond, $E(w^f) = 1$ also holds. So the function can be $w - w^f$. To use $r - r^f$ as the independent variable, we have

$$f(r - r^f) = \mu(r - r^f) \tag{25}$$

1.4 8)

Show in detail how to retrieve state prices using put options both in a continuous and discrete states setup.

Answer In a continuous setup, we have to

- Buy x call options with Strike $K - \epsilon$
- Sell $2x$ call options with strike K
- Buy x call options with $K + \epsilon$

In this case, $p = Xq$. And we can use

$$p = e^{-rT} E^Q[h(S_T)] = e^{-rT} \int_R f_{S_T}^Q(x)h(x)dx = \int_R \frac{\partial C(K,T)}{\partial K^2}(x)h(x)dx = \int_R q(x)h(x)dx \tag{26}$$

In a discrete setup, if all the option prices $C(K,T)$ for the strikes $K \in \{0, \Delta, 2\Delta, \cdots, N\Delta\}$ are observable (for N large and Δ ¿0 small enough) we can approximate $\frac{\partial^2 C(K,T)}{\partial K^2}$ in the following way

$$\frac{\partial^2 C(K,T)}{\partial K^2} \approx \frac{C(K+\Delta,T) - 2C(K,T) + C(K-\Delta,T)}{\Delta^2} \tag{27}$$

and therefore we can also calculate the empirical market-implied probablity distribution of S_T, $f_{S_T}^Q$ and the empirical state price density q.

Going back to our finite-state one-period model, if all the option prices $C(s,T)$ for the strikes $s \in \{0, 1, 2, \cdots, S\}$ are observable, we have

$$\Delta^2 C(s,T) \equiv C(s+1,T) - 2C(s,T) + C(s-1,T) = q_s = \frac{\hat{\pi}_s}{(1+r^f)} \tag{28}$$

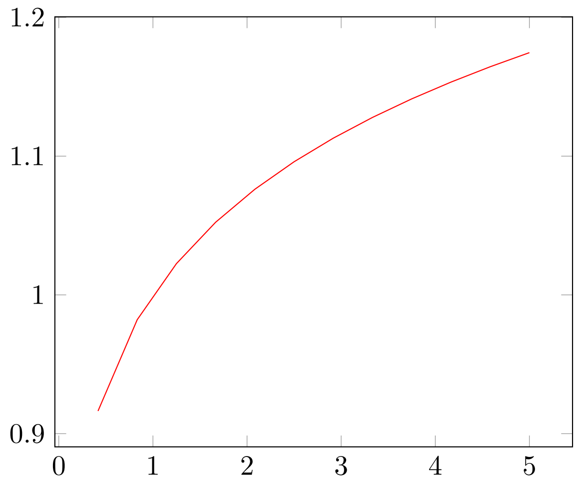
2 Addition Question

Describe yield curve.

Answer The yield curve is a curve showing several yields or interest rates across different contract lengths (2 month, 2 year, 20 year, etc. ...) for a similar debt contract. The curve shows the relation between the (level of the) interest rate (or cost of borrowing) and the time to maturity, known as the "term", of the debt for a given borrower in a given currency. More formal mathematical descriptions of this relation are often called the term structure of interest rates.

The shape of the yield curve indicates the cumulative priorities of all lenders relative to a particular borrower, or the priorities of a single lender relative to all possible borrowers. With other factors held equal, lenders will prefer to have funds at their disposal, rather than at the disposal of a third party. The interest rate is the "price" paid to convince them to lend. As the term of the loan increases, lenders demand an increase in the interest received. In addition, lenders may be concerned about future circumstances, e.g. a potential default (or rising rates of inflation), so they demand higher interest rates on long-term loans than they demand on shorter-term loans to compensate for the increased risk. Occasionally, when lenders are seeking long-term debt contracts more aggressively than short-term debt contracts, the yield curve ”inverts”, with interest rates (yields) being lower for the longer periods of repayment, because borrowers find it easier to attract long-term lending.

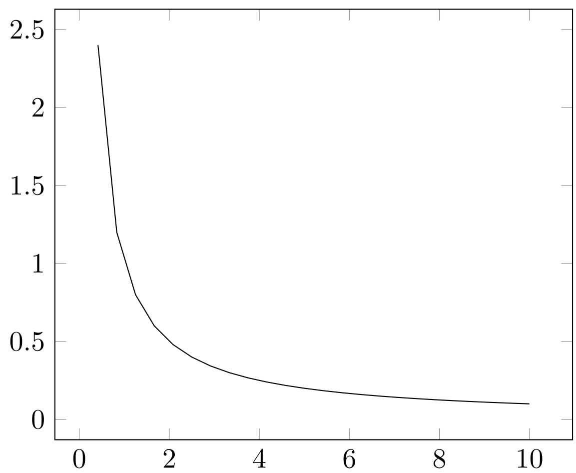
For instance, this curve



(29)

shows a good shape, which indicates the interest rate increase as the term increases.

While this curve



(30)

is a invert one which indicates that the economy is going down. As the term increases, the interest rate goes down.