浙江大学 2010－2011 学年秋、冬学期

《线性代数（A 卷）》课程期末考试试卷答案

一. 解答题.

1. 解. $D=\begin{vmatrix} a & b & b & b \\ a & b & a & b \\ b & a & b & a \\ b & b & b & a \end{vmatrix}=\begin{vmatrix} a & b & b & b \\ 0 & 0 & a-b & 0 \\ b-a & a-b & 0 & a-b \\ b-a & 0 & 0 & a-b \end{vmatrix}=-(a-b)^3\begin{vmatrix} a & b & b \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix}$

$=(b-a)^3\begin{vmatrix} a & b & a+b \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix}=(b-a)^3(a+b).$

2. 解. (1). $\mathrm{A}^2=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$,

(2).因为 $\mathrm{A}^2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}=\mathrm{E+B}$，其中 $\mathrm{B}=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$，且 $\mathrm{B}^2=0$，

$\mathrm{A}^{2011}=\mathrm{A}^{2010+1}=\left(\mathrm{A}^2\right)^{1005}\cdot\mathrm{A}=(\mathrm{E+B})^{1005}\cdot\mathrm{A}=\left(E+C_{1005}^{1}B\right)\cdot A=(\mathrm{E}+1005\mathrm{B})\mathrm{A}=$

$=\begin{pmatrix} 1 & 0 & 0 \\ 1005 & 1 & 0 \\ 1005 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 1006 & 0 & 1 \\ 1005 & 1 & 0 \end{pmatrix}.$

3.解. $\begin{pmatrix} 1 & -2 & 1 & 2 & 2 \\ 2 & 1 & -3 & 9 & -1 \\ 1 & -1 & 2 & 3 & 3 \end{pmatrix}\to\begin{pmatrix} 1 & -2 & 1 & 2 & 2 \\ 0 & 5 & -5 & 5 & -5 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}\to\begin{pmatrix} 1 & 0 & 3 & 4 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -10 & 0 & -10 \end{pmatrix}$

$\to\begin{pmatrix} 1 & 0 & 0 & 4 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$,所以 $\begin{cases} x_1=1-4x_4 \\ x_2=\ \ -x_4 \\ x_3=1 \end{cases}$,(其中 x_4 是自由未知量),

特解是 $\xi_0=\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T$,基础解系是 $\eta=\begin{pmatrix}-4 & -1 & 0 & 1\end{pmatrix}^T$,

通解是 $\xi=\xi_0+k\eta=\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}^T+k\begin{pmatrix}-4 & -1 & 0 & 1\end{pmatrix}^T$ (其中 k 是任意常数),

4.解.(1).令 $e_{11}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},e_{12}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},e_{21}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},e_{22}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

因为$(e_1,e_2,e_3,e_4)=(\mathrm{e}_{11},\mathrm{e}_{12},\mathrm{e}_{21},\mathrm{e}_{22})\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}=(\mathrm{e}_{11},\mathrm{e}_{12},\mathrm{e}_{21},\mathrm{e}_{22})A\cdots\cdots(1)$

又因为$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(\mathrm{e}_{11},\mathrm{e}_{12},\mathrm{e}_{21},\mathrm{e}_{22})\begin{pmatrix}1&1&1&1\\0&2&2&2\\0&0&3&3\\0&0&0&4\end{pmatrix}=(\mathrm{e}_{11},\mathrm{e}_{12},\mathrm{e}_{21},\mathrm{e}_{22})B\cdots\cdots(2)$

而矩阵$A=\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}$和矩阵$B=\begin{pmatrix}1&1&1&1\\0&2&2&2\\0&0&3&3\\0&0&0&4\end{pmatrix}$是可逆的,

所以向量组(Ⅰ) e_1,e_2,e_3,e_4和向量组(Ⅱ)$\alpha_1,\alpha_2,\alpha_3,\alpha_4$都是基.

(2)$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(e_1,e_2,e_3,e_4)M=(\mathrm{e}_{11},\mathrm{e}_{12},\mathrm{e}_{21},\mathrm{e}_{22})AM$

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由(1)和(2)得到$B=AM\Rightarrow M=\boxed{A^{-1}B}=\begin{pmatrix}1&-1&-1&-1\\0&2&-1&-1\\0&0&3&-1\\0&0&0&4\end{pmatrix}$

由基(Ⅰ)到基(Ⅱ)的过渡矩阵是$M=\begin{pmatrix}1&-1&-1&-1\\0&2&-1&-1\\0&0&3&-1\\0&0&0&4\end{pmatrix}$, 5

(3).向量α在基Ⅰ下的坐标是$(1\quad 1\quad 1\quad 1)^T$,

3'

在基Ⅱ下的坐标是$M^{-1}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&\frac{1}{6}&\frac{1}{6}\\0&0&\frac{1}{3}&\frac{5}{6}\\0&0&0&\frac{1}{4}\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}\frac{5}{2}\\\frac{5}{6}\\\frac{5}{12}\\\frac{1}{4}\end{pmatrix}$ 5'

5.解.因为$A-E,A+2E,5A-3E$都不可逆,所以A的特征值是$1,-2,\frac{3}{5}$, $|A|=-\frac{6}{5}$,

A^{-1}的特征值是$1,-\frac{1}{2},\frac{5}{3}$;

(1).因为$A^*=|A|A^{-1}$,所以

$$\left|A^*-2A^{-1}\right|=\left||A|A^{-1}-2A^{-1}\right|=\left|-\frac{4}{5}A^{-1}\right|=\left(-\frac{4}{5}\right)^3\times\left|A^{-1}\right|=-\frac{64}{125}\times\left(-\frac{5}{6}\right)=\frac{32}{75},$$

(2).$\varphi(A)=A^3+2A^2+A-E$的特征值是

$$1^3+2\times1^2+1-1=3$$
$$(-2)^3+2(-2)^2+(-2)-1=-3$$
$$\left(\frac{3}{5}\right)^3+2\left(\frac{3}{5}\right)^2+\left(\frac{3}{5}\right)-1=\frac{67}{125},$$

(3).$|\varphi(A)|=3\times(-3)\times\left(\frac{67}{125}\right)=-\frac{603}{125}.$

6.解.(1).二次型的矩阵是$A=\begin{pmatrix}3&2&2\\2&3&2\\2&2&3\end{pmatrix}$,

(2). ①特征多项式是$|\lambda E-A|=\begin{vmatrix}\lambda-3&-2&-2\\-2&\lambda-3&-2\\-2&-2&\lambda-3\end{vmatrix}=(\lambda-1)^2(\lambda-7)$,

特征值是 1,1,7.

②对应于特征值 1 的线性无关的特征向量是

$$\alpha_1=\begin{pmatrix}1&-1&0\end{pmatrix}^T,\alpha_2=\begin{pmatrix}1&0&-1\end{pmatrix}^T,$$

对应于特征值 7 的线性无关的特征向量是

$$\alpha_3=\begin{pmatrix}1&1&1\end{pmatrix}^T,$$

③用施密特方法把特征向量$\alpha_1,\alpha_2,\alpha_3$标准正交化,得

$$\eta_1=\begin{pmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}&0\end{pmatrix}^T,\eta_2=\begin{pmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt6}&-\frac{2}{\sqrt6}\end{pmatrix}^T,\begin{pmatrix}\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\end{pmatrix}^T,$$

④.令正交矩阵$U=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}$

令正交线性变换$X=UY$,

$$f\left(x_1,x_2,x_3\right)=X^TAX=\left(UY\right)^TA\left(UY\right)=Y^T\left(U^TAU\right)Y=Y^T\begin{pmatrix}1 & & \\ & 1 & \\ & & 7\end{pmatrix}Y$$

$$=y_1^2+y_2^2+7y_3^2.$$

二.证明题:

7.证明.因为A与B相似,则存在可逆矩阵P,使得

$$P^{-1}AP=B,$$

所以$B^2=\left(P^{-1}AP\right)^2=\left(P^{-1}AP\right)\left(P^{-1}AP\right)=P^{-1}A^2P=P^{-1}AP=B$,

故B是幂等矩阵.

8.证明.因为方程组(1)的解是方程(2)的解,所以方程组(1)

$$\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=0\\ \cdots\cdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=0\end{cases}\cdots(1)$$

和
$$\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=0\\ \cdots\cdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=0\\ b_1x_1+b_2x_2+\cdots+b_nx_n=0\end{cases}\cdots(3)$$

同解,

所以
$$r\left(\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\right)=r\left(\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\\ b_1 & b_2 & \cdots & b_n\end{pmatrix}\right),$$

故 $r\left(\alpha_1,\alpha_2,\cdots,\alpha_m\right)=r\left(\alpha_1,\alpha_2,\cdots,\alpha_m,\beta\right)$,

所以β可以由向量组$\alpha_1,\alpha_2,\cdots\alpha_m$,线性表示.