HOMEWORK 10

P563

9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) where $f(x) = f(y)$
a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

Solution : a)
i) Reflexive: $\forall x \in A$, $f(x) = f(x)$ i.e. $(x, x) \in R$.
ii) Symmetric: $\forall x, y \in A$, if $(x, y) \in R \Rightarrow f(x) = f(y) \Rightarrow f(y) = f(x) \Rightarrow (y, x) \in R$.
iii) Transitive: $\forall x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, i.e. $f(x) = f(y)$ and $f(y) = f(z)$. therefore, $f(x) = f(z)$, i.e. $(x, z) \in R$.
Hence, R is an equivalence relation on A.
b) $\forall x \in A$, then

$$[x]_R = \{a \mid x \in A \text{ and } f(x) = f(a)\}$$

16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b),(c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

Solution :
i) Reflexive: $\forall (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, since $ab = ba$ then $((a, b),(a, b)) \in R$.
ii) Symmetric: $\forall (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, if $((a, b),(c, d)) \in R \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow ((c, d),(a, b)) \in R$.
iii) Transitive: $\forall (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, $(c, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and $(e, f) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, if $((a, b),(c, d)) \in R$ and $((c, d),(e, f)) \in R \Rightarrow ad = bc$ and $cf = de \Rightarrow ad \cdot cf = bc \cdot de \Rightarrow af = be \Rightarrow ((a, b),(e, f)) \in R$.
Hence, R is an equivalence relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$.

21-23. Determine whether the relation with the directed graphs shown is an equivalence relation.
21. It is not an equivalence relation.
22. It is an equivalence relation

23. It is not an equivalence relation.

40 a) What is the equivalence class of $(1,2)$ with respect to the equivalence relation in Exercise 16?
b) Given an interpretation of the equivalence classes for the equivalence relation in Exercise 10.
Solution :
a) $[(1,2)]_R = \{(a,b) \mid b = 2a\}$
b) Again by our observation, the equivalence classes are the positive rational numbers.

41. Which of the following collections of subsets are partitions of $\{1,2,3,4,5,6\}$?
a) $\{1,2\},\{2,3,4\},\{4,5,6\}$ **b)** $\{1\},\{2,3,6\},\{4\},\{5\}$
c) $\{2,4,6\}$, $\{1,3,5\}$ **d)** $\{1,4,5\}$,$\{2,6\}$

Solution : b) and c) are partition of $\{1,2,3,4,5,6\}$.

54. Suppose that R_1 and R_2 are equivalence relations on set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

Definition A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

Solution : First, suppose that $R_1 \subseteq R_2$. We must show that P_1 is a refinement of P_2. Let $[a]_{R_1}$ be an equivalence class in P_1. We must show that $[a]_{R_1}$ is contained in an equivalence class in P_2. In fact, we will show that $[a]_{R_1} \subseteq [a]_{R_2}$. To this end, let $b \in [a]_{R_1}$. Then $(a,b) \in R_1 \subseteq R_2$. Therefore, $b \in [a]_{R_2}$, as desired.

Conversely, Suppose that P_1 is a refinement of P_2. Since $a \in [a]_{R_2}$, the definition of refinement force that $[a]_{R_1} \subseteq [a]_{R_2}$, for all $a \in A$. This means that for all $b \in A$ we have $(a,b) \in R_1 \to (a,b) \in R_2$; in other words, $R_1 \subseteq R_2$.

P578
7. Determine whether the relations represented by the following zero-one matrices are partial orders?

$$\text{a)} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{b)} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{c)} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Solution :
a) reflexive, antisymmetric, not transitive. The relation is not a partial order.
b) The relation is a partial order.

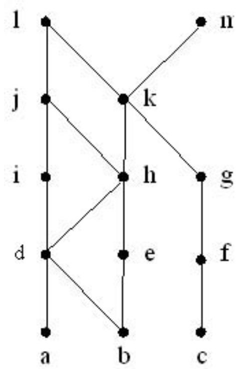
c) The relation is a partial order. Since the pair $(1,3)$ and $(3,4)$ are present, but not $(1,4)$.

14. omitted

19 Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$**.**

Solution : $0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$

32. Answer the following questions for the partial order represented by the following Hasse diagram.



a) Find the maximal elements.
b) Find the minimal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of $\{a, b, c\}$.
f) Find the least upper bound of $\{a, b, c\}$, if it exists.
g) Find all upper bounds of $\{f, g, h\}$.
h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

a)	m, l	b)	a, b, c
c)	no greatest element	d)	no least element
e)	k, l, m	f)	k
g)	no lower bound	h)	no greatest lower bound

44. Determine whether the following posets are lattices.
a) $(\{1, 3, 6, 9, 12\}, |)$
b) $(\{1, 5, 25, 125\}, |)$

Solution :

a) This is not a lattice, since the elements 6 and 9 have no upper bound.

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.