Passing Stones Problem

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1 Definitions

s- number of initial stones
n- number of vertices on graph
v_i- vertex that is given the initial s stones
step - an iteration from one state of stones to another
saturation- when the number of stones on each vertex is equal to the degree of the vertex
oversaturated- when a vertex contains more stones than the degree of said vertex
n-state equilibrium- the number of different steps required to reach the last step of the first equilibrium
endpoint of path- one of the two vertices with degree of one

2 General

The mechanism by which the stones are passed are as follows:

- All stones will start on one initial vertex.
- On any step, if the number of stones on a vertex is greater than or equal to the degree of the vertex, then the vertex will pass along one stone to each other adjacent vertex for the next step.

2.1 Theorem 2.1

Theorem: Every graph with any initial concentration of stones requires a finite number of steps to reach any equilibrium.

Proof: Consider any graph with n vertices and s stones. Since we know that the graph is finite, there is only a finite number of possible distribution of stones. During the iterations, if any iteration repeats, then we know that the graph has reached an equilibrium because any iterations afterwards will be the same as the mechanism by which the stones distribute is always the same. This means that the graph has reached an equilbrium of some state.

2.2 Theorem 2.2

Theorem: In a disconnected graph, the sub-graph that does not contain v_i will have zero stones in the equilibrium.

Proof: There are no connections between the sub-graph that contains v_i and the sub-graph that does not. So, stones have no edges by which to transfer to the second sub-graph. This means that while the sub-graph with v_i will have stones in its vertices, the sub-graph without v_i will have zero stones indefinitely, including after equilibrium is reached.

2.3 Theorem 2.3

Theorem: On all vertices other than v_i, the number of stones on a vertex cannot ever be greater than $2 * deg(v) - 1$.

Proof: Obviously, the number of stones on v_i can be greater than $2 * deg(v) - 1$ as the number of stones on v_i is manipulated. On any other vertex, assume that all vertices other than any vertex v are either saturated or oversaturated. Since v is not saturated, it will be passed 1 stone from each vertex that it is connected to. This number is equal to $deg(v)$. If v has $deg(v)$ stones when the iteration starts, then it will pass along stones along with all of the other vertices. The maximum number of stones that v can have without passing them to other vertices is $deg(v)$ is $deg(v) - 1$. So, if v is passed $deg(v)$ stones, it will contain at most $2 * deg(v) - 1$.

3 Path Graphs

3.1 Conjecture 3.1

Conjecture: If $s < 2n - 2$, then the path graph will always have a two-step equilibrium state.

Proof: For a path graph, the degree of each vertex that is not an endpoint will be 2. The two outer vertices have degree 1. This means that the sum of the degrees of the vertices will be 2(n-2)+2 = 2n-2. If s in less than this number, then this means that on any step of the distribution of stones, there will be at least one vertex that is not saturated. If a vertex is not saturated, it will not pass stones along in the next step, by definition. This means that the equilibrium cannot be one step.
Consider any three vertices in order: P_n,P_{n+1},P_{n+2}. Without loss of generality, assume that these three vertices are not all saturated at the same time but the graph is approaching an equilibrium state without a net influence from other vertices. Assume P_n passes a stone to P_{n+1}, and one is not passed back. In the next step, assume a net of one stone is passed from P_{n+1} to P_{n+2}. After this, the stone can only have one place to be passed to, and that is back to P_{n+1}. Since we know that nothing else has been affecting the steps thus far, this means that the stone will keep being passed between the two vertices. Since we are only looking at path graphs, there is no edge that P_{n+2} can use to pass the stone back to P_n. This means that the state of the equilibrium can only be

two. The same argument applies for longer chains, namely that any stones that are passed will eventually be passed only between two vertices, causing what can only be a 2-state equilibrium.

3.2 Conjecture 3.2

Conjecture: If $s \geq 2n - 2$, then the path graph will always have a one-step equilibrium state that is saturated, so every non-endpoint vertex other than v_i will be a 2 stone vertex and every endpoint vertex other than v_i will be a 1 stone vertex. If v_i is an endpoint, it will have $s - (n-2) * 2 - 1$ stones, and if it is not an endpoint, it will have $s - (n-2) * 2 - 2$ stones.

3.3 Conjecture 3.3

Conjecture: Let us say that the path graph has vertices $P_1, P_2, P_3, ..., P_{n-1}, P_n$ and that P_k is a specific vertex on that graph. If $s \geq 2n$, then the path graph will reach its equilibrium state after at most $S(\lfloor \frac{n}{2} \rfloor + dist(P_{\lceil \frac{n}{2} \rceil}, P_k))$ steps, where $S(n) = \frac{1}{2}n(3n-1)$ and $dist(P_a, P_b)$ is the number of edges between the vertices P_a and P_b.

4 Cycle Graphs

4.1 Conjecture 4.1

Conjecture: If $s < 2n$ and $s > n$, then the cycle graph will always have a two-step equilibrium state.

4.2 Conjecture 4.2

Conjecture: If $s \geq 2n$, then the cycle graph will always have a one-step equilibrium state that is saturated, so every vertex other than v_i will be a 2 stone vertex and v_i will have $s - (n-1) * 2$ stones.

In this conjecture we noticed that vertex v_i distributes stones from high to low concentration throughout the cycle. $v_i$decreases by increments of 2, because every vertex on a cycle graph has degree 2. By Theorem 3, the maximum number of stones on any vertex except v_i must be less than or equal to 3. We noticed that the stones distribute outwards from v_i until 2 or more stones exist on the outmost vertices with stones. This causes a distribution in both directions if the vertices before the outer most vertices have 1 or less stone, which causes a rebounding process back towards v_i. This rebounding process results in the 2 immediate vertices next to v_i to have 1 stone, resulting in further stone passing from v_i in a high to low concentration mechanism. We analyzed and found that when the number of stones on v_i is $2n$, the stones eventually distribute evenly with 2 stones on each vertex. This results in all vertices passing stones back and forth with incident vertices, making no change in stone distribution. It is clear that as the number of stones on v_i increases, there will be one oversatured point on the cycle that will increase corresponding to the increase of S on v_i.

Namely, $n-1$ vertices will have 2 stones, based on our previous observation at $2n$ stones, and the oversaturated vertex has the remaining stones: $s-2*(n-1)$.

4.3 Conjecture 4.3

Conjecture: If $s \geq 2n$, then the cycle graph will reach its equilibrium state after at most $S(-1^n * \lfloor \frac{n}{2} \rfloor)$ where n is the number of vertices and $S(n) = \frac{1}{2}n(3n-1)$.

For this conjecture, we used a computer program to model the distribution of stones and the number of steps necessary to reach equilibrium. We noticed that starting with C_3, the numbers seemed to follow the pattern: $2, 5, 7, 12, 15, 22, 26, 35, 40, ...$ We discovered that these numbers are precisely what are known as generalized pentagonal numbers, which is defined explicitly by $S(n) = \frac{1}{2}n(3n-1)$ for the sequence $n = 0, 1, -1, 2, -2, 3, -3, 4....$ As the numbers repeat and alternte signs, we simply took the floor function of $\frac{n}{2}$ and multiplied by -1^n to alternate the sign. However, we noticed that after 40, the next value of the number of steps for the equilibrium is 50, while the pentagonal number counterpart is 51. We noticed that after this number, the numbers for the actual number of steps start decreasing at a rate that followed no observable pattern for us. So, we determined that the explicit function given is an upper bound for the number of steps required to reach equilibrium.

5 Complete Graphs

5.1 Conjecture 5.1

Conjecture: If $s \geq n(n-1)$, then the complete graph will always have a one-step equilibrium state that is saturated, and so every vertex other than v_i will have $n-1$ stones and v_i will have $s-(n-1)^2$ stones.

Proof: We know that all stones will start on any vertex v_i. We can assume this without loss of generality because every vertex is connected to every other vertex, so the mechanism by which the stones distribute will be the same. On every step, one vertex will be passed to all other vertices from v_i. However, when the degree of each of the other vertices are reached, then they will start passing back stones to v_i. If $s \geq n(n-1)$, then this means that each stone will have at least n-1 stones in equilibrium. This allows each vertex to continually pass along stones in each step, which is a one-step equilibrium. There will be n-1 vertices with n-1 stones as a result. Therefore, v_i will have $s-(n-1)^2$ stones left.

5.2 Conjecture 5.2

Conjecture: If $s \geq n-1$ and $s < n(n-1)$, then the complete graph will always have a two-state equilibrium that will start after n-1 steps.

Proof: We know that if $s < n-1$, then stones will not pass at all. We know that since our graph is complete, the degree of every vertex is n-1, and each vertex will be saturated when it contains n-1 stones. On the step where each vertex other than v_i is saturated, we know that v_i is unsaturated because there is not enough stones left over. This is because $s < n(n-1)$. This means that from this step onwards, a net of one stone will pass from all other vertices to v_i, and then v_i will pass stones to each other vertex. This process will indefinitely continue and is a two-state equilibrium.

5.3 Conjecture 5.3

Conjecture: If $s \geq n(n-1)$, then the complete graph will reach a one-state equilibrium in $n-1$ steps.

Proof: We will proceed by direct proof. Assume $s \geq n(n-1)$. Because we have a complete graph, then the degree of every vertex is n-1, and each vertex will be saturated when it contains n-1 stones. If there are s stones on vertex v_i, then on every step every other vertex on the graph will gain one more vertex as long as $s > n-1$ and the other vertices are not already saturated. It will take n-1 steps to saturate the other vertices, and so after n-1 steps, no stones will be passed without any being passed back, which is a one-state equilibrium.