MATH5000M: Dissertation in Mathematics

Banach and C*-algebras

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Summary

Whilst the road to the Gelfand-Naimark Theorems may not be paved with gold, it is certainly paved with beautiful mathematics, and this mathematics is what we shall explore in the first 5 chapters of this thesis.

We begin with a brief look at continuous functions, the main intention being to properly introduce the space $C_0(X)$ for a locally compact Hausdorff space X. For the sake of brevity, we focus on the definitions and basic properties rather than proving big theorems, for example we show that $C_c(X)$ is not a Banach space under the uniform norm, but we will not prove the Stone-Weierstrass Theorem or Urysohn's Lemma.

We move on to look at Banach algebras, quickly specialising to those defined over the complex numbers. In Chapter 2 we introduce many important definitions and so we spend time looking at a good number of examples. One definition in particular is that of the spectrum, this set of complex numbers is of crucial importance and we here we shall prove that the spectrum of an element in a complex Banach algebra is never empty, and the spectral radius formula.

Moving from the basics, we look at the theory of commutative Banach algebras developed by Gelfand. We will see how characters can be used to conjure up a locally compact Hausdorff space for every commutative Banach algebra. This topological space allows for the definition of the Gelfand representation, an algebra homomorphism which takes values in $C_0(X)$ and we use this in later chapters to prove that $C_0(X)$ is essentially the only commutative C*-algebra.

From Banach algebras we move to involutive algebras. Of this class of algebras we are primarily interested in C*-algebras so only spend a couple of pages talking about *-algebras and Banach *-algebras. The main goal of Chapters 4 and 5 is to present full proofs of both Gelfand-Naimark Theorems, so we cover everything necessary in order to achieve this, starting with elementary consequences of the innocent looking C*-identity. Thanks to the work we do in Chapter 3, proving the commutative Gelfand-Naimark Theorem takes surprisingly little effort, and is essentially just an application of the Stone-Weierstrass Theorem.

More theory is required for the main Gelfand-Naimark Theorem, and we introduce the continuous functional calculus, with which we can define a partial order on the self adjoint elements of a C*-algebra. It is at this point most textbooks look at approximate units for C*-algebras, a topic which we avoid going into much detail on. To conclude Chapter 5 we look at representations and states of a C*-algebras, covering in detail the GNS construction for unital C*-algebras with which we can prove the Gelfand-Naimark Theorem.

In Chapter 6 we look at a certain C*-algebra which is used by physicists. We prove that the canonical commutation relations of quantum mechanics cannot be satisfied by bounded linear operators on a Hilbert space, and finish the thesis with a proof of Slawny's Theorem.

Sources

The main sources used are a combination of textbooks, online lecture notes and journal articles. Specifically, the main sources are the textbooks

(i) *C*-algebras and Operator Theory* by G. Murphy,
(ii) *A Short Course on Spectral Theory by* W. Arveson,
(iii) *An Introduction to Banach Spaces and Algebras* by G.R Allen and H.G Dales,
(iv) *Real Analysis: Modern Techniques and Their Applications* by G. Folland,
(v) *An Introduction to the Algebra of Canonical Commutation Relations* by D. Petz,

the online lecture notes

(vi) *A (Very) Short Course on C*-algebras* by D.P Williams,
(vii) *C*-algebras* by C. Bär and C. Becker,

and the journal articles

(viii) *On Factor Representations and the C*-algebra of Canonical Commutation Relations* by J. Slawny,
(ix) *The Smallest C*-algebra for Canonical Commutations Relations* by J. Manuceau et.al.

Full references to all of these sources may be found in the bibliography.

Contents

Note 0.1. Throughout we shall adopt the following conventions,

(i) $\mathbb{K}$ shall denote either $\mathbb{R}$ or $\mathbb{C}$,
(ii) 0 belongs to $\mathbb{N}$, and we shall write $\mathbb{N}^{>0}$ for the positive integers,
(iii) Inner products are linear in their second argument.

1. **Spaces of Continuous Functions**

Seeing as we shall be talking about analysis in this thesis, it is inevitable that we will encounter continuous functions at some point. In this chapter we shall introduce certain spaces of continuous functions which will appear throughout our later study of Banach and C*-algebras. For the sake of space we will not give a full account of the theory, but we shall present the basic definitions, along with a couple of interesting results and properties. For any results we use later but do not cover here, we shall give a reference to to where one may find them in the literature.

1.1. **The Spaces $C(X)$ and $C_b(X)$.**

Definition 1.1. If X is any topological space then we write $C(X)$ for the collection of all continuous functions from X to $\mathbb{K}$. It is not difficult to show that $C(X)$ is a vector space under the operations of pointwise addition and scalar multiplication which are defined by

$$(f+g)(x) = f(x) + g(x),$$
$$(\lambda f)(x) = \lambda f(x),$$

where $f, g \in C(X)$, $\lambda \in \mathbb{K}$ and $x \in X$.

Remark 1.2. It is similarly straightforward to show that $C(X)$ is also closed under pointwise multiplication of functions. Moreover we can show that this multiplication is associative and satisfies,

$$f(g + \lambda h) = fg + \lambda fh,$$
$$(f + \lambda g)h = fh + \lambda gh,$$

for any $f, g, h \in C(X)$ and any $\lambda \in \mathbb{K}$. These properties mean that pointwise multiplication gives $C(X)$ the structure of an *algebra* and we shall say more about algebras in Chapter 2.

Definition 1.3. For any topological space X, we write $C_b(X)$ for the space of *bounded* continuous functions from X to $\mathbb{K}$. This is a vector subspace of $C(X)$ and we define the *uniform norm*[1] on $C_b(X)$ by

$$\|f\|_\infty = \sup\{|f(x)|;\, x \in X\},$$

for $f \in C_b(X)$. One can show that $C_b(X)$ is a Banach space under this norm, see [6, Chapter 1, Example 1.6].

Remark 1.4. If X is a compact space, then every continuous function $f : X \to \mathbb{K}$ is bounded, so $C(X) = C_b(X)$ in this case.

So far we have not imposed any restrictions on our topological spaces, which means that $C(X)$ and $C_b(X)$ may be un-interesting. For example if we equip an infinite set X with the cofinite topology, then the only continuous functions from X to $\mathbb{K}$ are constant functions. To avoid such problems we shall restrict ourselves to *locally compact Hausdorff spaces*.

Definition 1.5. A Hausdorff topological space X is *locally compact* if for every point $x \in X$ there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ with $x \in U \subseteq K$.

Example 1.6. $\mathbb{R}$ with its standard topology is Hausdorff and if x is any real number, then $(x-1, x+1)$ is open, $[x-1, x+1]$ is compact and $x \in (x-1, x+1) \subseteq [x-1, x+1]$ so that $\mathbb{R}$ is locally compact. Similarly we can show that $\mathbb{R}^n$ with its standard topology is locally compact.

Example 1.7. Let X be any set equipped with the discrete topology. Then X is Hausdorff and for any point $x \in X$ the singleton $\{x\}$ is open and compact (because it is finite!), thus X is locally compact.

[1]The name *uniform norm* is used because a sequence of functions (f_n) in $C_b(X)$ converges in the uniform norm to $f \in C_b(X)$ if and only if (f_n) converges to f uniformly.

Assumption 1.8. Unless otherwise stated, for the rest of this chapter X will be a locally compact Hausdorff space.

Because locally compact Hausdorff spaces possess a version of Urysohn's Lemma [8, Theorem 4.32], by restricting our attention to such topological spaces we avoid the scenario where $C(X)$ or $C_b(X)$ is just the constant functions.

1.2. **Continuous Functions With Compact Support.** Rather than working with the space $C_b(X)$, it is often convenient to work with certain subspaces. The first subspace which we shall look at is the one containing the functions which have *compact support*.

Definition 1.9. For a continuous function $f : X \to \mathbb{K}$, the *support* of f is the set

$$\operatorname{Supp}(f) = \overline{\{x \in X;\ f(x) \neq 0\}}.$$

We write $C_c(X)$ for the collection of all continuous functions from X to $\mathbb{K}$ which have *compact support*.

Remark 1.10. If X is a compact space, then for any $f \in C(X)$ the support of f is (by definition) a closed subset of a compact space, so is compact. Hence $C(X) = C_c(X)$ in this case.

Example 1.11. Define a continuous function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq -\pi, \\ \sin(x) & \text{if } -\pi \leq x \leq \pi, \\ 0 & \text{if } \pi \leq x. \end{cases}$$

Then the support of f is $[-\pi, \pi]$ which is a compact subset of $\mathbb{R}$, so $f \in C_c(\mathbb{R})$.

Proposition 1.12. *$C_c(X)$ is a vector subspace of $C_b(X)$.*

Proof. First we observe that $C_c(X)$ is nonempty, because the support of the zero function is the empty set, which is compact. For a function $f \in C_c(X)$, we have

$$f(X) = f(X \setminus \operatorname{Supp}(F)) \cup f(\operatorname{Supp}(f)) = \{0\} \cup f(\operatorname{Supp}(f)),$$

therefore $f(X)$ is a compact subset of $\mathbb{K}$ and in particular it must be bounded, so $C_c(X)$ is a non-empty subset of $C_b(X)$. Now take any two functions $f, g \in C_c(X)$ and any scalar $\lambda \in \mathbb{K}$. First observe that λf and $f + g$ are both continuous functions and second observe that if $\lambda = 0$ then λf is the zero function which we noted previously has compact support. If $\lambda \neq 0$ then

$$\operatorname{Supp}(\lambda f) = \overline{\{x \in X;\ \lambda f(x) \neq 0\}} = \overline{\{x \in X;\ f(x) \neq 0\}} = \operatorname{Supp}(f),$$

so the support of λf is compact and $\lambda f \in C_c(X)$. Now let $h = f + g$, then for $x \in X \setminus \operatorname{Supp}(f) \cap X \setminus \operatorname{Supp}(g)$ we have $f(x) = g(x) = 0$ and so $h(x) = 0$, therefore

$$X \setminus \operatorname{Supp}(f) \cap X \setminus \operatorname{Supp}(g) = X \setminus (\operatorname{Supp}(f) \cup \operatorname{Supp}(g)) \subseteq X \setminus \operatorname{Supp}(h).$$

This shows that $\operatorname{Supp}(h) \subseteq \operatorname{Supp}(f) \cup \operatorname{Supp}(g)$ and thus the support of h is compact for it is a closed subset of a compact set, hence $h = f + g \in C_c(X)$. □

An obvious choice of norm for us to place on $C_c(X)$ is the restriction of the uniform norm on $C_b(X)$, but as the next example will show, in general $C_c(X)$ fails to be a Banach space under this norm.

Example 1.13. For each $n \in \mathbb{N}^{>0}$, we define a continuous function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq -n, \\ \frac{1 - \left|\frac{x}{n}\right|}{1 + |x|} & \text{if } -n \leq x \leq n, \\ 0 & \text{if } n \leq x. \end{cases}$$

Note that for each $n \in \mathbb{N}^{>0}$, the support of f_n is $[-n, n]$, which is compact and so (f_n) is a sequence in $C_c(\mathbb{R})$. We now show that (f_n) is a Cauchy sequence with respect to the uniform norm. For positive integers $n > m$ we have

$$f_n(x) - f_m(x) = \begin{cases} 0 & \text{if } x \leq -n \\ \frac{1-\left|\frac{x}{n}\right|}{1+|x|} & \text{if } -n \leq x \leq -m \\ \frac{\left|\frac{x}{m}\right|-\left|\frac{x}{n}\right|}{1+|x|} & \text{if } -m \leq x \leq m \\ \frac{1-\left|\frac{x}{n}\right|}{1+|x|} & \text{if } m \leq x \leq n \\ 0 & \text{if } n \leq x. \end{cases}$$

We want to bound the quantity $|f_n(x) - f_m(x)|$, so we consider the different cases. If $-n \leq x \leq -m$, then $m \leq |x| \leq 1 + |x|$, so

$$\frac{1 - \left|\frac{x}{n}\right|}{1 + |x|} \leq \frac{1}{1 + |x|} \leq \frac{1}{m}.$$

Similarly if $m \leq x \leq n$, then

$$\frac{1}{1 + |x|} \leq \frac{1}{m},$$

and if $-m \leq x \leq m$, then

$$\frac{\left|\frac{x}{m}\right|}{1 + |x|} - \frac{\left|\frac{x}{n}\right|}{1 + |x|} \leq \frac{\frac{|x|}{m}}{1 + |x|} \leq \frac{|x|}{m|x|} = \frac{1}{m}.$$

We combine the above to see that for $n > m$ we have

$$\|f_n - f_m\|_\infty = \sup\{|f_n(x) - f_m(x)|;\, x \in \mathbb{R}\} \leq \frac{1}{m},$$

which shows that (f_n) is a Cauchy sequence. Now we show that (f_n) converges to $\frac{1}{1+|x|}$ in the uniform norm. Fix $n \in \mathbb{N}^{>0}$, then

$$\frac{1}{1 + |x|} - f_n(x) = \begin{cases} \frac{1}{1+|x|} & \text{if } x \leq -n, \\ \frac{\left|\frac{x}{n}\right|}{1+|x|} & \text{if } -n \leq x \leq n, \\ \frac{1}{1+|x|} & \text{if } n \leq x. \end{cases}$$

As before, we consider the different cases in order to bound $\left|\frac{1}{1+|x|} - f_n(x)\right|$. If $x \leq -n$ or $n \leq x$ then $n \leq |x|$, so

$$\frac{1}{1 + |x|} \leq \frac{1}{n}.$$

If $-n \leq x \leq n$ then note that $|x| \leq |x| + 1$, so

$$\frac{\left|\frac{x}{n}\right|}{1 + |x|} \leq \frac{\left|\frac{x}{n}\right|}{|x|} = \frac{1}{n}.$$

Combining, we see that for any $n \in \mathbb{N}^{>0}$ we have

$$\left\|\frac{1}{1 + |x|}\right\|_\infty = \sup\left\{\left|\frac{1}{1 + |x|} - f_n(x)\right|;\, x \in \mathbb{R}\right\} \leq \frac{1}{n},$$

so (f_n) converges uniformly to $\frac{1}{1+|x|}$. But $\frac{1}{1+|x|}$ never vanishes on $\mathbb{R}$, so does not have compact support and hence we have shown that $C_c(\mathbb{R})$ is not complete.

In the next section we will answer the question of what the completion of $C_c(X)$ is under the uniform norm, but for now we shall define another norm, this time on the space $C_c(\mathbb{R})$.

Definition 1.14. We define the L^2 norm on $C_c(\mathbb{R})$ by

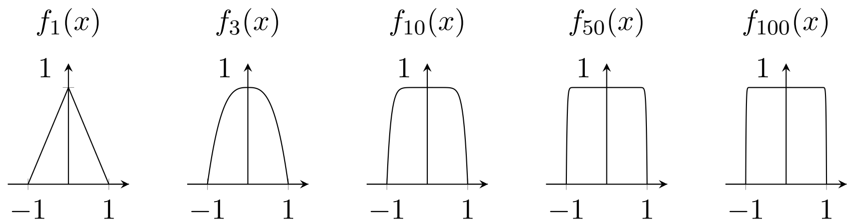
$$\|f\|_2 = \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{\frac{1}{2}},$$

for $f \in C_c(\mathbb{R})$. Note that the integral is well defined because the function has compact support. The next example will show that $C_c(\mathbb{R})$ is not a Banach space under this norm.

Example 1.15. Define a sequence of continuous functions (f_n) by

$$f_n = \begin{cases} 0 & \text{if } x \leq -1, \\ 1 - |x|^n & \text{if } -1 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Note that for each $n \in \mathbb{N}$ the support of f_n is $[-1, 1]$, so $f_n \in C_c(\mathbb{R})$. To get an idea of what is going on, we plot some of the f_n below.



Now we show that (f_n) is a Cauchy sequence with respect to the L^2 norm. For natural numbers $n > m$ we have

$$f_n(x) - f_m(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ |x|^m - |x|^n & \text{if } -1 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Therefore,

$$\begin{aligned}
\|f_n - f_m\|_2^2 &= \int_{-1}^{1} (|x|^m - |x|^n)^2 dx \\
&= \int_0^1 x^{2m} - 2x^{m+n} + x^{2n} dx + \int_{-1}^{0} (-x)^{2m} - 2(-x)^{m+n} + (-x)^{2n} dx \\
&= \left[\frac{x^{2m+1}}{2m+1} - \frac{2x^{m+n+1}}{m+n+1} + \frac{x^{2n+1}}{2n+1} \right]_0^1 + \left[-\frac{(-x)^{2m+1}}{2m+1} + \frac{2(-x)^{m+n+1}}{m+n+1} - \frac{(-x)^{2n+1}}{2n+1} \right]_{-1}^{0} \\
&= \frac{1}{2m+1} - \frac{2}{m+n+1} + \frac{1}{2n+1} + \frac{1}{2m+1} - \frac{2}{m+n+1} + \frac{1}{2n+1} \\
&< \frac{2}{2m+1} + \frac{2}{2n+1} \\
&< \frac{2}{2m+1} + \frac{2}{2m+1} \\
&= \frac{4}{2m+1}.
\end{aligned}$$

We take square roots to see that $\|f_n - f_m\|_2 \leq \frac{2}{\sqrt{2m+1}}$, so that (f_n) is a Cauchy sequence. Now we show that (f_n) converges in the L^2 norm to $\chi_{[-1,1]}$, where $\chi_{[-1,1]}$ denotes the *indicator function* of the interval $[-1, 1]$. First note that for fixed $n \in \mathbb{N}$,

$$f_n(x) - \chi_{[-1,1]}(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ -|x|^n & \text{if } -1 \leq x \leq 1, \\ 0 & \text{if } 1 \leq x. \end{cases}$$

Then

$$\begin{aligned}
\|f_n - \chi_{[-1,1]}\|_2^2 &= \int_{-1}^{1} |x|^{2n} dx \\
&= \int_0^1 x^{2n} dx + \int_{-1}^{0} (-x)^{2n} dx \\
&= \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 + \left[-\frac{(-x)^{2n+1}}{2n+1}\right]_{-1}^{0} \\
&= \frac{1}{2n+1} + \frac{1}{2n+1} \\
&= \frac{2}{2n+1}.
\end{aligned}$$

It follows that $\|f_n - \chi_{[-1,1]}\|_2 = \frac{\sqrt{2}}{\sqrt{2n+1}} \to 0$ as $n \to \infty$ and so (f_n) converges in norm to $\chi_{[-1,1]}$. Since $\chi_{[-1,1]}$ is not a continuous function, this shows that $C_c(\mathbb{R})$ is not complete under this norm.

In view of the previous example, we may ask what is the completion of $C_c(\mathbb{R})$? Answering this question requires the theory of L^p function spaces and Radon measures[2], with which we have the following result.

Theorem 1.16. *[8, Proposition 7.9]*
If X is a locally compact Hausdorff space and μ is a Radon measure on X, then $C_c(X)$ is dense in $L^p(X,\mu)$.

1.3. **Continuous Functions Which Vanish at Infinity.** Now we shall look at a different subspace of $C_b(X)$, this one contains the functions which *vanish at infinity*.

Definition 1.17. A continuous function $f : X \to \mathbb{K}$ is said to *vanish at infinity* if for every $\epsilon > 0$, there is a compact subset $K \subseteq X$ such that

$$|f(x)| < \epsilon,$$

for all $x \in X \setminus K$. We write $C_0(X)$ for the collection of all such functions from X to $\mathbb{K}$.

Remark 1.18. If X is a compact space, then for any $f \in C(X)$ and any $\epsilon > 0$ we have $|f(x)| < \epsilon$ for all $x \in X \setminus X = \emptyset$, so that $f \in C_0(X)$ and in this case $C(X) = C_0(X)$.

Example 1.19. Define a continuous function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{1}{1+|x|},$$

and for each $\epsilon > 0$ define a compact set $K_\epsilon \subseteq \mathbb{R}$ by

$$K_\epsilon = \left[-\frac{1}{\epsilon}, \frac{1}{\epsilon}\right].$$

Then if $x \in \mathbb{R} \setminus K_\epsilon$ we have

$$1 + |x| > |x| > \frac{1}{\epsilon} \iff \epsilon > \frac{1}{1+|x|},$$

so that $f \in C_0(\mathbb{R})$.

At first glance, it is perhaps not clear why the phrase vanishes at infinity is used. Certainly if $X = \mathbb{R}$ then one can show that $\lim_{|x|\to\infty} f(x) = 0$ for a function $f \in C_0(X)$, but for the general case, the next result should provide some insight.

[2] A very readable treatment of Radon measures is given by Bauer in [5, Chapter 4].

Proposition 1.20. [24, Exercise E 1.1.9.]
If X is a non-compact locally compact Hausdorff space, we let $\widehat{X}$ denote its one-point compactification[3]*. Then for each $f \in C(X)$, we define a function $\widehat{f} : \widehat{X} \to \mathbb{K}$ by*

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = \infty. \end{cases}$$

Then for any $f \in C(X)$, f belongs to $C_0(X)$ if and only if $\widehat{f}$ belongs to $C(\widehat{X})$.

Proof. First we take any function $f \in C(X)$ and suppose that $f \in C_0(X)$. So for every $\epsilon > 0$ there is a compact set $K \subseteq X$ with

$$|f(x)| \geq \epsilon,$$

for all $x \in K$. Now take an open set $U \subseteq \mathbb{K}$. If $0 \notin U$ then certainly $\infty \notin \widehat{f}^{-1}(U)$, so

$$\widehat{f}^{-1}(U) = f^{-1}(U),$$

and by continuity of f it follows that $\widehat{f}^{-1}(U)$ is open in X so is open in $\widehat{X}$. If $0 \in U$, then $\widehat{f}^{-1}(U) = \{\infty\} \cup f^{-1}(U)$ and we claim that $X \setminus f^{-1}(U)$ is compact, so that $\{\infty\} \cup f^{-1}(U)$ is open in $\widehat{X}$. Note that since U is open we can find $\delta > 0$ so that $B_\delta(0) \subseteq U$ and moreover $U = U \cup B_\delta(0)$, so

$$f^{-1}(U) = f^{-1}(U) \cup f^{-1}(B_\delta(0)),$$

and De Morgan's laws show that

$$\begin{aligned} \widehat{X} \setminus \widehat{f}^{-1}(U) &= X \setminus f^{-1}(U) \cap X \setminus f^{-1}(B_\delta(0)) \\ &= X \setminus f^{-1}(U) \cap \{x \in X;\, |f(x)| \geq \delta\}. \end{aligned}$$

Because $f \in C_0(X)$, the set $\{x \in X;\, |f(x)| \geq \delta\}$ must be compact and because $f \in C(X)$ the set $X \setminus f^{-1}(U)$ must be closed. It follows that $\widehat{X} \setminus \widehat{f}^{-1}(U)$ is the intersection of a closed set with a compact set, so is therefore closed. This means that $\widehat{f}^{-1}(U)$ is open in $\widehat{X}$ and so $\widehat{f} \in C(\widehat{X})$.

Conversely we take any function $f \in C(X)$ whose extension $\widehat{f}$ belongs to $C(\widehat{X})$. Given any $\epsilon > 0$, we consider the open set $(-\epsilon, \epsilon) \subseteq \mathbb{R}$ and the continuous map $|\widehat{f}| : \widehat{X} \to \mathbb{R}$ in order to define an open set

$$P_\epsilon = |\widehat{f}|^{-1}((-\epsilon, \epsilon)) = \{x \in \widehat{X};\, |\widehat{f}(x)| < \epsilon\}.$$

Since $\widehat{f}(\infty) = 0$ it follows that $\infty \in P_\epsilon$, so by definition of the topology on $\widehat{X}$ there must be a compact set $K_\epsilon \subseteq X$ with

$$P_\epsilon = (X \setminus K_\epsilon) \cup \{\infty\}.$$

Therefore $X \setminus K_\epsilon = \{x \in X;\, |f(x)| < \epsilon\}$ which shows that $f \in C_0(X)$. □

Proposition 1.21. *$C_0(X)$ is a vector subspace of $C_b(X)$.*

Proof. We first note that $C_0(X)$ is nonempty, for if f is the zero function then for any $\epsilon > 0$ we have $|f(x)| = 0 < \epsilon$, for all $x \in X = X \setminus \emptyset$ and so $f \in C_0(X)$ because the empty set is compact. Now take a function $f \in C_0(X)$, then for $\epsilon = 1$ we have a compact set $K \subseteq X$ such that $|f(x)| < 1$ whenever $x \in X \setminus K$. Therefore f is bounded on $X \setminus K$. However by compactness of K we also have that f is bounded on K, so it follows that f is bounded and $C_0(X)$ is a non-empty subset of $C_b(X)$.

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[3]If (X, τ) is a Hausdorff space, then $\widehat{X}$ is the set $X \cup \{\infty\}$, where ∞ is some abstract point outside of X, and we define a topology $\widehat{\tau}$ on $\widehat{X}$ as follows. If $A \subseteq \widehat{X}$, then $X \in \widehat{\tau}$ if either

(i) $A \in \tau$, or
(ii) $A = (X \setminus K) \cup \{\infty\}$ for a compact set $K \subseteq X$.

If X is a locally compact Hausdorff space, then $(\widehat{X}, \widehat{\tau})$ is a compact Hausdorff space and X is open in $\widehat{X}$. For more details see [12, Theorem 5.21].

To finish the proof, we take any two functions $f, g \in C_0(X)$ and any two scalars $\lambda, \mu \in \mathbb{K}\backslash\{0\}$. Then given any $\epsilon > 0$ we can find compact subsets $F_\epsilon, G_\epsilon \subseteq X$ such that

$$F_\epsilon = \left\{ x \in X;\ |f(x)| \geq \frac{\epsilon}{2|\lambda|} \right\} \text{ and } G_\epsilon = \left\{ x \in X;\ |g(x)| \geq \frac{\epsilon}{2|\mu|} \right\}.$$

Now we define $h = \lambda f + \mu g$ and $H_\epsilon = \{x \in X;\ |h(x)| \geq \epsilon\}$, noting that h is a linear combination of continuous functions so is continuous and H_ϵ is closed for it is the preimage of the closed set $[\epsilon, \infty)$ under the continuous map $|h|$. Moreover for $x \in (X \setminus F_\epsilon) \cap (X \setminus G_\epsilon) = X \setminus (F_\epsilon \cup G_\epsilon)$, we have

$$|h(x)| = |\lambda f(x) + \mu g(x)| \leq |\lambda||f(x)| + |\mu||g(x)| < |\lambda| \frac{\epsilon}{2|\lambda|} + |\mu| \frac{\epsilon}{2|\mu|} = \epsilon.$$

Therefore $x \in X \setminus H_\epsilon$ which shows that

$$H_\epsilon \subseteq F_\epsilon \cup G_\epsilon,$$

and so H_ϵ is compact as it is a closed subset of a compact set. Finally we note that if either of λ or μ is zero then the above argument still works with a minor adjustment to the sets F_ϵ and G_ϵ, and if $\lambda = \mu = 0$, then $\lambda h = 0$ and by the first sentence of the proof the zero function belongs to $C_0(X)$. □

Remark 1.22. In Example 1.19 we saw that $\frac{1}{1+|x|} \in C_0(\mathbb{R})$ and in Example 1.13 we constructed a sequence of continuous functions, each with compact support that converged uniformly to $\frac{1}{1+|x|}$. In fact the metric space completion of $C_c(X)$ with the uniform norm is $C_0(X)$, as is shown by Rudin in [21, Theorem 3.17].

We shall finish this section by combining previous remarks into the following.

Remark 1.23. If X is a compact topological space, then $C(X) = C_b(X) = C_c(X) = C_0(X)$.

2. **Banach Algebras**

In Remark 1.2 we mentioned that $C(X)$ possesses a multiplicative structure which makes it an algebra. In this chapter we shall begin by properly introducing algebras and see how we can place norms on them in order for us to do analysis. In particular we shall develop the basic theory of a particular class of algebras; Banach algebras.

2.1. **Definitions and Examples.**

Before we can talk about Banach algebras, we of course need to know what an algebra is.

Definition 2.1. An *algebra* consists of a $\mathbb{K}$-vector space together with a multiplication $A \times A \to A$ such that,

(i) $x(yz) = (xy)z$,
(ii) $x(y + \lambda z) = xy + \lambda xz$,
(iii) $(x + \lambda y)z = xz + \lambda yz$,

for all $x, y, z \in A$ and all $\lambda \in \mathbb{K}$. If we wish to make reference to the scalar field of an algebra then we use the phrase *algebra over* $\mathbb{K}$ or $\mathbb{K}$-*algebra*. If the multiplication is commutative, then we call A a *commutative algebra* and if there is an element $1_A \in A$ such which satisfies

$$1_A x = x = x 1_A$$

for all $x \in A$, then we call A a *unital algebra*. In this case we shall call the element 1_A the *unit* of A and will always denote the unit of a unital algebra A by 1_A, in order to avoid any confusion between the unit and the number 1.

Remark 2.2. Should a unit exist, then it is unique, for if $1_A, 1'_A \in A$ satisfy the condition above, then $1_A = 1_A 1'_A = 1'_A$.

Example 2.3. Perhaps the most basic example of an algebra is the field $\mathbb{K}$, where the multiplication is just multiplication of real or complex numbers and its unit is $1_{\mathbb{K}} = 1$.

Example 2.4. As we hinted earlier in Remark 1.2, $C(X)$ for any topological space X under pointwise multiplication is an algebra. All the necessary conditions follow immediately from the properties of $\mathbb{K}$, so we omit the calculations. The unit of $C(X)$ is the function which takes a constant value of 1.

Of course both of these examples are examples of commutative algebras because multiplication in $\mathbb{K}$ is commutative.

Definition 2.5. A *normed algebra* consists of an algebra A, together with a norm $\|\cdot\|$ which is submultiplicative. In other words, the norm must satisfy

$$\|xy\| \leq \|x\|\|y\|,$$

for all $x, y \in A$. A normed algebra A is called a *Banach algebra* if the pair $(A, \|\cdot\|)$ is a Banach space. If there is the potential for confusion then we shall denote the norm on an algebra A by $\|\cdot\|_A$.

Lemma 2.6. *Let A be a normed algebra, then the algebra multiplication is continuous.*

Proof. Take sequences $(x_n), (y_n)$ in A with norm limits $x, y \in A$ respectively. Then for each $n \in \mathbb{N}$, the triangle inequality and submultiplicativity allow us to make the estimate

$$\begin{aligned} \|x_n y_n - xy\| &= \|(x_n y_n - x_n y) + (x_n y - xy)\| \\ &\leq \|x_n y_n - x_n y\| + \|x_n y - xy\| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|. \end{aligned}$$

The convergence of (x_n) to x and of (y_n) to y allows us to make the right hand side arbitrarily small, showing that $\lim_{n\to\infty} \|x_n y_n - xy\| = 0$ so that the algebra multiplication is continuous. □

Example 2.7. If X is a topological space, then under the uniform norm $C_b(X)$ is a Banach space. We will prove submultiplicativity of this norm and hence that $C_b(X)$ is a Banach algebra. Take functions $f, g \in C_b(X)$, then for each $x \in X$ we have

$$|f(x)g(x)| = |f(x)||g(x)| \leq \|f\|_\infty |g(x)|,$$

which implies that

$$\|fg\|_\infty \leq \sup\{\|f\|_\infty |g(x)|;\, x \in X\} = \|f\|_\infty \sup\{|g(x)|;\, x \in X\} = \|f\|_\infty \|g\|_\infty.$$

Showing that $C_b(X)$ is an algebra is now easy. The pointwise product of functions $f, g \in C_b(X)$ is continuous and the above shows that it is also bounded. The properties of $\mathbb{K}$ then show that the additional properties for the multiplication are satisfied. Moreover, $C_b(X)$ is unital and commutative.

Remark 2.8. The last example of course shows that for a compact Hausdorff space X, $C(X)$ is a unital commutative Banach algebra.

Example 2.9. This example may be found in [3, Example 1.3.10], but it is stated without details, which we shall provide here.

Let $\mathbb{M}_n(\mathbb{K})$ denote the collection of $n \times n$ matrices with entries from the field $\mathbb{K}$. Under the usual operations of scalar multiplication, matrix addition and matrix multiplication, $\mathbb{M}_n(\mathbb{K})$ is a $\mathbb{K}$-algebra. We define $\|\cdot\| : \mathbb{M}_n(\mathbb{K}) \to \mathbb{R}^{\geq 0}$ by

$$\|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|,$$

for any $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{K})$ and claim that $\mathbb{M}_n(\mathbb{K})$ with this norm is a Banach algebra. First we check that the mapping is a norm, let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{M}_n(\mathbb{K})$ and $\lambda \in \mathbb{K}$, then $\|A\| \geq 0$ because $\|A\|$ is a sum of positive real numbers, moreover

$$\|A\| = 0 \iff a_{ij} = 0 \text{ for all } 1 \leq i,j \leq n \iff A = (0).$$

For the other properties, we have

$$\|\lambda A\| = \sum_{i=1}^{n}\sum_{j=1}^{n} |\lambda a_{ij}| = \sum_{i=1}^{n}\sum_{j=1}^{n} |\lambda||a_{ij}| = |\lambda|\|A\|,$$

and

$$\|A + B\| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij} + b_{ij}| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) = \|A\| + \|B\|.$$

We check submultiplicativity,

$$\|AB\| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}b_{ij}| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}||b_{ij}| \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}|\right) = \|A\|\|B\|,$$

which proves that $(\mathbb{M}_n(\mathbb{K}), \|\cdot\|)$ is a normed algebra. For completeness, we take a Cauchy sequence (A_n) of matrices in $\mathbb{M}_n(\mathbb{K})$ (where we denote the n^{th} matrix by $A_n = (a_{ij}^{(n)})$), then for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ so that

$$p, q > N \Rightarrow \epsilon > \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}^{(p)} - a_{ij}^{(q)}|.$$

Since $|a_{ij}^{(p)} - a_{ij}^{(q)}| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}^{(p)} - a_{ij}^{(q)}|$ for each pair $1 \leq i,j \leq n$ it follows that for each pair $1 \leq i,j \leq n$ we have a Cauchy sequence $(a_{ij}^{(n)})_{n\in\mathbb{N}}$ in $\mathbb{K}$, so by completeness we have a limit $a_{ij} \in \mathbb{K}$. We form a matrix $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{K})$ and claim that $A_n \to A$. Indeed the convergence is immediate once we take the limit $q \to \infty$ in the inequality

$$\epsilon > \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}^{(p)} - a_{ij}^{(q)}|.$$

Example 2.10. Let X be a Banach space and consider $B(X)$, the Banach space of bounded linear operators on X under the operator norm[4]. Define a multiplication on $B(X)$ by

$$AB = A \circ B,$$

for all operators $A, B \in B(X)$. This is certainly associative and for $A, B, C \in B(X)$ and $\lambda \in \mathbb{K}$ we have for each $x \in X$ that

$$A(B + \lambda C)(x) = A(B(x) + \lambda C(x)) = AB(x) + \lambda AC(x),$$

by linearity of the operators. Therefore $A(B+\lambda C) = AB+\lambda AC$ and by a similar calculation we see that $(A+\lambda B)C = AC+\lambda BC$, so that $B(X)$ is an algebra under this multiplication. Finally we check submultiplicativity of the operator norm. For any $x \in X$ we have the inequality

$$\|AB(x)\|_A \leq \|A\|_{\text{op}}\|B(x)\|_A \leq \|A\|_{\text{op}}\|B\|_{\text{op}}\|x\|_A,$$

because A and B are bounded. Taking the supremum over all $x \in X$ with $\|x\|_A = 1$ then gives $\|AB\|_{\text{op}} \leq \|A\|_{\text{op}}\|B\|_{\text{op}}$ which is what we want. $B(X)$ is therefore a Banach algebra, its unit is the operator which takes constant value 1 and the multiplication this time is non-commutative.

2.2. **Subalgebras, Ideals and Homomorphisms.** As well as giving the definitions of subalgebras, ideals and homomorphisms, in this section we shall present many examples, a good number of which we shall use frequently in the later chapters.

Definition 2.11. Let A be an algebra. A subset $B \subseteq A$ is a *subalgebra* of A if it satisfies the following,

(i) A is a vector subspace of A,
(ii) For every $x, y \in B$, we have $xy \in B$.

If in addition A is unital and B is a subalgebra which contains the unit of A, then we call B a *unital subalgebra*. If A is a Banach algebra, then a subalgebra $B \subseteq A$ is itself a Banach algebra if and only if it is closed, and we call a complete subalgebra of a Banach algebra a *Banach subalgebra*.

Example 2.12. Let X be a locally compact Hausdorff space. We mentioned in Chapter 1 that $C_0(X)$ is a Banach space under the uniform norm and we shall now prove that it is a subalgebra of $C_b(X)$. From Proposition 1.21 we know that $C_0(X)$ is a subspace of $C_b(X)$. We take two functions $f, g \in C_0(X)$ and for $\epsilon > 0$ we have compact sets $F_\epsilon, G_\epsilon \subseteq X$ such that

$$F_\epsilon = \{x \in X;\ |f(x)| \geq \sqrt{\epsilon}\} \text{ and } G_\epsilon = \{x \in X;\ |g(x)| \geq \sqrt{\epsilon}\}.$$

We take $x \in (X \setminus F_\epsilon) \cap (X \setminus G_\epsilon) = X \setminus (F_\epsilon \cup G_\epsilon)$, then

$$|f(x)g(x)| = |f(x)||g(x)| \leq \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

If we define $H_\epsilon = \{x \in X;\ |f(x)g(x)| \geq \epsilon\}$ then the above shows that our x belongs to $X \setminus H_\epsilon$ which of course shows that $H_\epsilon \subseteq F_\epsilon \cup G_\epsilon$. H_ϵ is closed because $H_\epsilon = |fg|^{-1}([\epsilon, \infty))$ and therefore H_ϵ is a closed subset of the compact set $F_\epsilon \cup G_\epsilon$, so is compact. This proves that $fg \in C_0(X)$, so $C_0(X)$ is a Banach subalgebra of $C_b(X)$. Whilst $C_0(X)$ is certainly commutative, checking whether it is unital or not takes a bit of care.

Claim 2.13. *$C_0(X)$ is unital if and only if X is compact.*

Proof. Let $1_X : X \to \mathbb{K}$ be the function which takes constant value one at all points of X. Clearly this function satisfies $1_X f = f = f 1_X$ for every $f \in C_0(X)$. If X is compact then as we remarked earlier, $C_0(X) = C(X)$ and 1_X is clearly the unit of $C(X)$. Conversely, if $1_X \in C_0(X)$ then for $\epsilon < 1$ we may find a compact set $K \subseteq X$ so that

$$|1_X(x)| < \epsilon < 1,$$

for every $x \in X \setminus K$. But since $1_X(x) = 1$ for every $x \in X$ it must be the case that $X \setminus K = \emptyset$, which implies that $K = X$ and so X is compact. $\square$

[4]We shall always denote the operator norm by $\|\cdot\|_{\text{op}}$, and will denote the identity operator either by I or id.

Example 2.14. Let $\mathbb{U}_n(\mathbb{K})$ denote the set of upper-triangular $n \times n$ matrices with entries from $\mathbb{K}$, that is

$$\mathbb{U}_n(\mathbb{K}) = \{(a_{ij}) \in \mathbb{M}_n(\mathbb{K});\ a_{ij} = 0 \text{ if } i > j\}.$$

We claim that this is a subalgebra of $\mathbb{M}_n(\mathbb{K})$. Take any two matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{U}_n(\mathbb{K})$ and any scalar $\lambda \in \mathbb{K}$, then

$$A + \lambda B = (a_{ij} + \lambda b_{ij}),$$

so $A + \lambda B$ is clearly an upper triangular matrix. Moreover the product of A and B is given by

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right),$$

so that if $i > j$ we have

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = 0,$$

since $a_{ik} = 0$ for $1 \leq k \leq i - 1$ and $b_{kj} = 0$ for $i \leq k \leq n$. Therefore $AB \in \mathbb{U}_n(\mathbb{K})$ as well which shows that $\mathbb{U}_n(\mathbb{K})$ is a (unital) subalgebra of $\mathbb{M}_n(\mathbb{K})$.

We could show that $\mathbb{U}_n(\mathbb{K})$ is complete directly, however we shall argue differently. As a vector space, $\mathbb{M}_n(\mathbb{K})$ has a basis $\{E_{ab};\ a, b \in \{1, \dots, n\}\}$ which consists of the matrices $E_{ab} = (e_{ij})$ whose entries are given by

$$e_{ij} = \begin{cases} 1 & \text{if } (i,j) = (a,b), \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that $\mathbb{U}_n(\mathbb{K})$ is a finite dimensional subspace of $\mathbb{M}_n(\mathbb{K})$ and so is closed, hence complete.

Example 2.15. Let $C^1([-1,1])$ be the collection of all functions $f : [-1,1] \to \mathbb{R}$ whose first derivative exists and is continuous. Thanks to the sum and product rules for differentiation it follows that $C^1([-1,1])$ is a unital subalgebra of $C([-1,1])$, however if we simply restrict the uniform norm on $C([-1,1])$ to $C^1([-1,1])$, this space fails to be complete. To prove this, we define for each $n \in \mathbb{N}^{>0}$ a function $f_n : [-1,1] \to \mathbb{R}$ by

$$f_n(x) = \sqrt{\frac{1}{n} + x^2},$$

for $x \in [-1,1]$. The chain rule shows that

$$f_n'(x) = \frac{x}{\sqrt{\frac{1}{n} + x^2}},$$

and we note that this is continuous, so f_n belongs to $C^1([-1,1])$. For fixed $x \in [-1,1]$ and $n \in \mathbb{N}^{>0}$ we have the following chain of inequalities

$$x^2 \leq \frac{1}{n} + x^2 \leq \left(\frac{1}{\sqrt{n}} + |x| \right)^2.$$

Taking square roots yields

$$|x| \leq f_n(x) \leq \frac{1}{\sqrt{n}} + |x|,$$

and therefore $|f_n(x) - |x|| \leq \frac{1}{\sqrt{n}}$, so that (f_n) converges uniformly to the absolute value function. We recall from real analysis that the absolute value function is not differentiable at the origin, so does not belong to $C^1([-1,1])$, proving that this space is not a Banach algebra under the uniform norm. Lets consider a different norm on $C^1([-1,1])$, define $\|\cdot\| : C^1([-1,1]) \to \mathbb{R}^{\geq 0}$ by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

Checking that this is a norm follows easily from the properties of $\|\cdot\|_\infty$. For submultiplicativity we take $f, g \in C^1([-1,1])$, then the product rule gives the inequality

$$\|fg\| \le \|f\|_\infty\|g\|_\infty + \|f'\|_\infty\|g\|_\infty + \|f\|_\infty\|g'\|_\infty,$$

which implies that

$$\|fg\| \le (\|f\|_\infty + \|f'\|_\infty)(\|g\|_\infty + \|g'\|_\infty) = \|f\|\|g\|.$$

To prove completeness, let (f_n) be a Cauchy sequence of functions in $C^1([-1,1])$, then for each $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$m, n > N \Rightarrow \epsilon > \|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty,$$

so (f_n) and (f_n') are Cauchy sequences in $(C([-1,1]), \|\cdot\|_\infty)$. By completeness these sequences have limits $f, g \in C([-1,1])$ respectively and we must show that f is differentiable, with derivative g. We fix $n \in \mathbb{N}$ and then for each $x \in [-1,1]$ the fundamental theorem of calculus gives us

$$f_n(x) - f_n(-1) = \int_{-1}^{x} f_n'(t)dt.$$

Now we use the uniform convergence of (f_n) to f and of (f_n') to g to conclude that

$$f(x) = f(-1) + \int_{-1}^{x} g(t)dt,$$

so f is a differentiable function and it now follows that $f' = g$. The last thing that we need to do is observe that for each $n \in \mathbb{N}$ we have

$$\|f_n - f\| = \|f_n - f\|_\infty + \|f_n' - f'\|_\infty,$$

which converges to zero as $n \to \infty$, therefore $(C^1([-1,1]), \|\cdot\|)$ is complete so is a Banach algebra.

The final example of a Banach subalgebra which we give is an important one.

Example 2.16. Let A be a Banach algebra over $\mathbb{K}$ and let $S \subseteq A$ be any subset. We define the *Banach subalgebra of A generated by S* to be the intersection of all closed subalgebras of A which contain S and we will denote this algebra by $\mathbb{B}(S)$. There is another way to define $\mathbb{B}(S)$, which is perhaps more useful in practise as it provides a very nice dense subset of $\mathbb{B}(S)$.

Claim 2.17. *If we let $P(S) \subseteq A$ denote the set of all polynomials in the elements of S, then $\mathbb{B}(S)$ is equal to the closure of $P(S)$ in A.*

Proof. Let f be any polynomial in $P(S)$, then f is a finite sum of terms of the form

$$\lambda s_1 \cdot \ldots \cdot s_k,$$

where $\lambda \in \mathbb{K}$, $s_1, \ldots, s_k \in S$ and $k \in \mathbb{N}$. It is clear that f also belongs to $\mathbb{B}(S)$, which gives the inclusion $P(S) \subseteq \mathbb{B}(S)$. Now we take the closure with respect to $\|\cdot\|_A$ to give $\overline{P(S)} \subseteq \mathbb{B}(S)$. Conversely, $\overline{P(S)}$ is a closed subalgebra of A which contains S, so it must be one of the algebras in the intersection which defines $\mathbb{B}(S)$, therefore $\mathbb{B}(S) \subseteq \overline{P(S)}$. □

One final observation for us to make is that if the set S has the property that $xy = yx$ for every pair of elements $x, y \in S$, then the Banach algebra $\mathbb{B}(S)$ is commutative. It is straightforward to see why this is true; it is certainly true for polynomials $p, q \in P(S)$ that $pq = qp$ and this extends to $\mathbb{B}(S)$ by continuity of multiplication and density of $P(S)$ in $\mathbb{B}(S)$.

Definition 2.18. Let A be an algebra. A *two-sided ideal* of A is a subset $I \subseteq A$ such that

(i) I is a vector subspace of A,
(ii) $IA \subseteq I$,
(iii) $AI \subseteq I$.

If we only have points (i) and (ii), then I is a *right ideal* and if we only have points (i) and (iii) then I is a *left ideal.* Henceforth we shall simply say *ideal* to mean two-sided ideal.

Remark 2.19. Suppose that I is an ideal of a unital algebra A. If $1_A \in I$, then for every $x \in A$ we have $x1_A \in I$ so that $I = A$. It follows that $I \subseteq A$ is a proper ideal if and only if $1_A \notin I$.

Example 2.20. If I is an ideal of an algebra A, then the quotient vector space A/I is an algebra under the multiplication

$$(x + I)(y + I) = xy + I.$$

If additionally A is a Banach algebra and I is closed, then the mapping $\|\cdot\|_{A/I} \to \mathbb{R}^{\geq 0}$ which for $x + I \in A/I$ is given by

$$\|x + I\|_{A/I} = \inf_{y \in I} \|x + y\|_A,$$

is a well defined norm on A/I. Under this norm, A/I is a Banach space and we may consult [17, Proposition 2.15] for details of this. We shall prove submultiplicativity of this norm, so take $x_1 + I, x_2 + I \in A/I$, then for any $\epsilon > 0$ there are $y_1, y_2 \in I$ such that

$$\begin{aligned}\|x_1 + I\|_{A/I}\|x_2 + I\|_{A/I} + \epsilon &\geq \|x_1 + y_1\|_A \|x_2 + y_2\|_A \\ &\geq \|(x_1 + y_1)(x_2 + y_2)\|_A \\ &= \|x_1x_2 + (x_1y_2 + y_1x_2 + y_1y_2)\|_A \\ &\geq \inf_{y \in I} \|x_1x_2 + y\|_A.\end{aligned}$$

Here we have made use of the facts that the norm on A is submultiplicative and that I is an ideal. It follows from this that A/I is also a Banach algebra.

Example 2.21. [24, Exercise E 1.1.8]
Let X be a compact Hausdorff space and let $E \subseteq X$ be closed. We define

$$I(E) = \{f \in C(X);\ f(x) = 0 \text{ for all } x \in E\}.$$

It's very easy to check that $I(E)$ is an ideal of $C(X)$, so we omit the calculations. Remarkably we can completely classify the closed ideals of $C(X)$.

Claim 2.22. *If $J \subseteq C(X)$ is a closed ideal, then $J = I(E)$ for some closed set $E \subseteq X$.*

Proof. Define $E_J = \{x \in X;\ f(x) = 0 \text{ for all } f \in J\}$. We note that

$$E_J = \bigcap_{f \in J} f^{-1}(\{0\}),$$

so by continuity of each $f \in J$ and the properties of compact Hausdorff spaces, it follows that E_J is closed. To show that $J = I(E_J)$, we first observe that if $f \in J$, then $f(x) = 0$ for every $x \in E_J$ so that $f \in I(E_J)$.

Conversely we take any function $f \in I(E_J)$ and shall show that f belongs to the closure of J. We shall do this by showing that $B_\epsilon(f) \cap J \neq \emptyset$ for every $\epsilon > 0$. Given $\epsilon > 0$ we define

$$U_\epsilon(f) = \{x \in X;\ |f(x)| < \epsilon\},$$

and note that $U_\epsilon(f) = f^{-1}((-\epsilon, \epsilon))$, so is open in X. Furthermore we note that

$$U_\epsilon(f) = \{x \in X;\ f(x) = 0\} \supseteq E_J,$$

so that $X \setminus U_\epsilon(f) \subseteq X \setminus E_J$. If we pick any point x in the compact set $X \setminus U_\epsilon(f)$, then $x \in X \setminus E_J$ and by definition of E_J we can find a function $g_x \in J$ with $g_x(x) \neq 0$. Because g_x is a continuous function, there is an open neighbourhood $V_x \subseteq X$ of x such that g_x does not vanish on V_x.

The collection $\{V_x\}_{x \in X \setminus U_\epsilon(f)}$ forms an open cover of $X \setminus U_\epsilon(f)$ so by compactness we can find a finite subcover $\{V_{x_i}\}_{i=1}^n$ and we define a function $g : X \to \mathbb{R}$ by

$$g(x) = \sum_{i=1}^{n} |g_{x_i}(x)|,$$

for $x \in X$. This g is obviously continuous and since for each $1 \leq i \leq n$ we have $g_{x_i} \in J$ and $\overline{g_{x_i}} \in C(X)$, it follows that the product $g_{x_i}\overline{g_{x_i}} = |g_{x_i}|$ lies in J, so in fact $g \in J$. For $k \in \mathbb{N}$ we define another function $h_k : X \to \mathbb{R}$ by

$$h_k(x) = 1 + kg(x),$$

for $x \in X$. Then h_k is continuous and does not vanish on X, so $\frac{1}{h_k} \in C(X)$ and the following defines a sequence of functions (f_k) in J,

$$f_k = \frac{1}{h_k} \cdot kg \cdot f = \frac{kg}{1+kg} f.$$

We observe that for any $x \in X$ we have $0 \leq kg(x) \leq 1 + kg(x)$, so that $0 \leq \frac{kg(x)}{1+kg(x)} \leq 1$ and

$$|f_k(x) - f(x)| = \left| f(x) \left(\frac{kg(x)}{1+kg(x)} - 1 \right) \right| = |f(x)| \left| \frac{kg(x)}{1+kg(x)} - 1 \right|.$$

It will be shown that $N \in \mathbb{N}$ can be found which satisfies

$$|f_{N+1}(x) - f(x)| < \epsilon,$$

for every $x \in X$ and to do so we first prove that $\frac{kg(x)}{1+kg(x)}$ converges uniformly to 1 on $X \setminus U_\epsilon(f)$. Indeed for $x \in X \setminus U_\epsilon(f)$ we have $g(x) \neq 0$ and

$$\left| \frac{kg(x)}{1+kg(x)} - 1 \right| = \frac{1}{1+kg(x)} < \frac{1}{kg(x)} = \frac{1}{k}\frac{1}{g(x)}.$$

We note that by continuity of g and compactness of $X \setminus U_\epsilon(f)$, we have real numbers $0 < a \leq b$ such that $g(X \setminus U_\epsilon(f)) \subseteq [a, b]$ and this fact implies that $\frac{1}{b} \leq \frac{1}{g(x)} \leq \frac{1}{a}$ for all $x \in X \setminus U_\epsilon(f)$ which proves the uniform convergence. Using this convergence, we may find $N \in \mathbb{N}$ so that

$$\left| \frac{(N+1)g(x)}{1+(N+1)g(x)} - 1 \right| < \frac{\epsilon}{\|f\|_\infty},$$

for all $x \in X \setminus U_\epsilon(f)$ and hence

$$|f_{N+1}(x) - f(x)| < \frac{\epsilon}{\|f\|_\infty} f(x) \leq \epsilon,$$

for all $x \in X \setminus U_\epsilon(f)$.

Now we shall show that $|f_k(x) - f(x)| < \epsilon$, for $x \in U_\epsilon(f)$, but thankfully this is straightforward. By definition, $|f(x)| < \epsilon$ for $x \in U_\epsilon(f)$ and since $0 \leq \frac{kg(x)}{1+kg(x)} \leq 1$ for every $x \in X$, it follows that

$$\left| \frac{kg(x)}{1+kg(x)} - 1 \right| \leq 1,$$

so that

$$|f_k(x) - f(x)| < \epsilon,$$

for all $k \in \mathbb{N}$ and all $x \in X \setminus U_\epsilon(f)$. Combining our work we see that we can find $N \in \mathbb{N}$ with

$$\|f_{N+1} - f\|_\infty < \epsilon,$$

which shows that $B_\epsilon(f) \cap J \neq \emptyset$. Of course our choice of ϵ was arbitrary so this proves that $f \in \overline{J} = J$. $\square$

Consider now the assignment

$$I : \{\text{Closed subsets } E \subseteq X\} \longrightarrow \{\text{Closed ideals } I \subseteq C(X)\}$$

sending $E \mapsto I(E)$. What we have just shown is that I is surjective. If we now consider disjoint closed subsets $E, F \subseteq X$, then by Urysohn's Lemma [8, Theorem 4.15] there is a a continuous function $f \in C(X)$ with $f|_E = 0$ and $f|_F = 1$. Therefore $f \in I(E)$ but $f \notin I(F)$ so $I(E) \neq I(F)$ and in fact I gives a bijection between the closed subsets of X and closed ideals of $C(X)$.

One particular type of ideal which we will encounter later on is the following.

Definition 2.23. Let A be an algebra. A *maximal* ideal is a proper ideal I such that for any other ideal $J \subseteq A$ with $I \subseteq J$, either $I = J$ or $J = A$.

Example 2.24. [24, Exercise 1.1.10]
For this example, we let X be a locally compact Hausdorff space. Observe that since $\widehat{X}$ is Hausdorff, the singleton $\{\infty\}$ is closed in $\widehat{X}$ and so by Example 2.21 $I(\{\infty\})$ is a closed ideal of $C(\widehat{X})$. Using Proposition 1.20 we can see that there is a bijective correspondence between $C_0(X)$ and $I(\{\infty\})$.

Claim 2.25. *$I(\{\infty\})$ is a maximal ideal of $C(\widehat{X})$.*

Proof. Lets suppose that $J \subseteq C(\widehat{X})$ is an ideal that satisfies

$$I(\{\infty\}) \subsetneq J \subseteq C(\widehat{X}).$$

Then there is a function $f \in J \setminus I(\{\infty\})$ so that $f(\infty) \neq 0$. We consider the expression

$$1 = \frac{f(\infty) - f}{f(\infty)} + \frac{f}{f(\infty)},$$

noting that

(i) $f \in J$ and $\frac{1}{f(\infty)} \in \mathbb{K}$, so $\frac{f}{f(\infty)} \in J$,
(ii) If $x = \infty$ then $\frac{f(\infty) - f(x)}{f(\infty)} = 0$, so $\frac{f(\infty) - f}{f(\infty)} \in I(\{\infty\}) \subsetneq J$.

So 1 belongs to J because we have written it as a linear combination of functions in J. This proves that $J = C(\widehat{X})$ and hence that $I(\{\infty\})$ is maximal. $\square$

Definition 2.26. Let A and B be algebras. A map $\varphi : A \to B$ is called an *algebra homomorphism* if

(i) φ is a linear map,
(ii) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$.

If additionally A and B are unital and $\varphi(1_A) = 1_B$, then we call φ a *unital homomorphism*, and if φ is a bijection then we say that it is an *isomorphism*. It is easily checked that the image of φ is a subalgebra of B and the kernel of φ is an ideal of A.

Remark 2.27. We have the familiar result that an algebra homomorphism is injective if and only if its kernel is the singleton $\{0\}$.

Example 2.28. Let A be an algebra and $I \subseteq A$ an ideal. We recall from Example 2.20 that A/I is again an algebra. There is a map $\mathcal{Q} : A \to A/I$ given by $\mathcal{Q}(x) = x + I$ for $x \in A$ which we call the *quotient map*, and this is a surjective, unital algebra homomorphism. Suppose further that A is a Banach algebra and I is a closed ideal, then notice that we have the bound

$$\|\mathcal{Q}(x)\|_{A/I} = \|x + I\|_{A/I} \leq \|x\|_A,$$

for every $x \in A$, so the quotient map is continuous and this example shows that every closed ideal of A is the kernel of some homomorphism. Conversely, we observe that if $\varphi : A \to B$ is a continuous homomorphism of Banach algebras, then its kernel $\ker(\varphi) = \varphi^{-1}(\{0\})$ is a closed ideal of A.

Example 2.29. Fix two numbers $m, n \in \mathbb{N}^{>0}$ such that $m > n$. Now define a map $\varphi : \mathbb{M}_n(\mathbb{R}) \to \mathbb{M}_m(\mathbb{R})$ by

$$\varphi(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

for each $A \in \mathbb{M}_n(\mathbb{R})$. The matrix $\varphi(A)$ is a *block matrix* and the 0's denote zero matrices of the appropriate dimensions. It is easy to check that φ is indeed an algebra homomorphism. We note further that φ is injective because its kernel only contains the zero matrix and also see that φ does not map the unit of $\mathbb{M}_n(\mathbb{R})$ to the unit of $\mathbb{M}_m(\mathbb{R})$.

Example 2.30. We define a map $\varphi : \mathbb{U}_2(\mathbb{C}) \to \mathbb{C}$ as follows; for any $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathbb{U}_2(\mathbb{C})$ we put

$$\varphi(A) = a.$$

It is again easy to check that this is a homomorphism, and that this time φ is not injective but is unital. Note that if we were to define the same map on $\mathbb{M}_2(\mathbb{C})$ then it would fail to be multiplicative thus not a homomorphism.

Example 2.31. Let A be a unital algebra. We write $\mathbb{C}[z]$ for the algebra of polynomials in one variable with complex coefficients, then for $x \in A$ and $p(z) = \sum_{k=0}^{n} \lambda_k z^k \in \mathbb{C}[z]$ we define

$$p(x) := \lambda_0 1_A + \sum_{k=1}^{n} \lambda_k x^k.$$

If we let $\varphi_x : \mathbb{C}[z] \to A$ be the map which sends $p(z) \mapsto p(a)$, then φ_x is a unital homomorphism.

2.3. **Re-norming a Banach Algebra.** Let A be a unital normed algebra, then we note that $\|1_A\| = \|1_A^2\| \leq \|1_A\|^2$ so that $1 \leq \|1_A\|$. It would perhaps be convenient if $\|1_A\| = 1$ but there is no way to guarantee this in an arbitrary normed algebra, for instance $\mathbb{M}_n(\mathbb{K})$ is unital, but if we equip $\mathbb{M}_n(\mathbb{K})$ with the norm introduced in Example 2.9 we have

$$\|I_n\| = n.$$

However the next result will show that we can always re-norm a unital Banach algebra so that the unit has norm one, without changing the norm topology.

Proposition 2.32. *Let A be a unital Banach algebra, then there is a norm $\|\cdot\|_{\#}$ on A which is equivalent to $\|\cdot\|_A$ and such that $(A, \|\cdot\|_{\#})$ is a Banach algebra with $\|1_A\|_{\#} = 1$.*

Proof. For $x \in A$, we consider the map $L_x : A \to A$ given by

$$L_x(y) := xy,$$

for all $y \in A$. L_x is clearly linear and for $y \in A$ the inequality $\|L_x y\|_A = \|xy\|_A \leq \|x\|_A \|y\|_A$ shows that L_x is bounded, hence $L_x \in B(A)$ for each $x \in A$. It is easily checked that $x \mapsto L_x$ gives an injective algebra homomorphism, so gives an embedding of A into $B(A)$ and the norm $\|\cdot\|_{\#}$ is defined to be the restriction of the operator norm on $B(A)$ to the image of A under this map. The inequality $\|L_x y\|_A \leq \|x\|_A \|y\|_A$ gives us $\|x\|_{\#} \leq \|x\|_A$ for each $x \in A$, and moreover if we let $y_0 = \frac{1_A}{\|1_A\|}$ then

$$\|x\|_{\#} = \|L_x\|_{\mathrm{op}} = \sup\{\|L_x y\|_A;\ \|y\|_A = 1\} \geq \|L_x y_0\|_A = \frac{\|x\|_A}{\|1_A\|_A}.$$

Therefore $\frac{\|x\|_A}{\|1_A\|_A} \leq \|x\|_{\#} \leq \|x\|_A$ and this proves that the two norms are equivalent.

Because A is closed with respect to $\|\cdot\|_A$ and $\|\cdot\|_{\#}$ is an equivalent norm, it follows that A is closed with respect to $\|\cdot\|_{\#}$. So A is a closed subalgebra of $B(A)$ so the pair $(A, \|\cdot\|_{\#})$ is a Banach algebra.

The fact that $\|1_A\|_{\#} = 1$ is an immediate consequence of how we have defined $\|\cdot\|_{\#}$. □

Proposition 2.32 is the reason that many authors include the condition $\|1_A\| = 1$ as part of their definition, see for example [15, Section 1.1] or [6, Chapter VII, Definition 1.1].

Assumption 2.33. From now on, if A is a unital Banach algebra then we assume without any loss of generality that $\|1_A\| = 1$.

2.4. **The Unitization of a Banach Algebra.** It is certainly not true that every algebra we encounter will have a unit. The purpose of this section is to give the details of a procedure known as unitization, which allows us to adjoin a unit to a non-unital algebra. We shall frequently use this process in proofs, and often we first prove a statement for unital Banach algebras, then use the unitization to extend to the general case.

Definition 2.34. Let A be a non-unital algebra. We define a new vector space by

$$\widetilde{A} = A \oplus \mathbb{K},$$

and we define a multiplication $\widetilde{A} \times \widetilde{A} \to \widetilde{A}$ by

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu),$$

for any $(x, \lambda), (y, \mu) \in \widetilde{A}$. Under this multiplication $\widetilde{A}$ is also an algebra and note that if A is commutative, then $\widetilde{A}$ is also commutative. Now observe that for any $(x, \lambda) \in \widetilde{A}$ we have

$$(x, \lambda)(0, 1) = (0 + \lambda 0 + 1x, \lambda) = (x, \lambda),$$
$$(0, 1)(x, \lambda) = (0 + 1x + \lambda 0, \lambda) = (x, \lambda),$$

so $\widetilde{A}$ is unital with unit $1_{\widetilde{A}} = (0, 1)$. We call our new algebra $\widetilde{A}$ the *unitization of* A.

If A is a non-unital algebra, then we define a map $\varphi : A \to \widetilde{A}$ by $\varphi(x) = (x, 0)$. It is easily checked that this is an injective algebra homomorphism with image

$$\varphi(A) = \{(x, 0);\, x \in A\}.$$

We shall refer to this map as the *canonical embedding of* A *into* $\widetilde{A}$ and can also check that $\varphi(A)$ is an ideal of $\widetilde{A}$.

Definition 2.35. Suppose that A is a non-unital normed algebra, then we define a norm $\|\cdot\|_{\widetilde{A}} : \widetilde{A} \to \mathbb{R}^{\geq 0}$ on $\widetilde{A}$ by

$$\|(x, \lambda)\|_{\widetilde{A}} = \|x\|_A + |\lambda|,$$

for any $(x, \lambda) \in \widetilde{A}$. We note that $\|\cdot\|_{\widetilde{A}}$ satisfies all the properties of a norm because $\|\cdot\|_A$ and $|\cdot|$ are norms on A and $\mathbb{K}$ respectively.

Lemma 2.36. *Let A be a non-unital normed algebra and let $\varphi : A \to \widetilde{A}$ be the canonical embedding. Then φ is an isometric isomorphism of A onto $\varphi(A)$ and $\varphi(A)$ is closed with respect to the norm $\|\cdot\|_{\widetilde{A}}$ defined above.*

Proof. By our previous discussion, $\varphi : A \to \varphi(A)$ is an isomorphism and for any $x \in A$ we have

$$\|\varphi(x)\|_{\widetilde{A}} = \|(x, 0)\|_{\widetilde{A}} = \|x\|_A + |0| = \|x\|_A,$$

so that φ is an isometric isomorphism. If $((x_n, 0))$ is a sequence in $\varphi(A)$ with limit $(x, \lambda) \in \widetilde{A}$, then for each $n \in \mathbb{N}$ we have

$$\|(x_n, 0) - (x, \lambda)\|_{\widetilde{A}} = \|x_n - x\|_A + |\lambda|.$$

We take the limit $n \to \infty$ to show that $|\lambda| = 0$, so $\lambda = 0$ and $(x, \lambda) \in \varphi(A)$. □

Lemma 2.37. *If A is a non-unital normed algebra then A is a Banach algebra if and only if $\widetilde{A}$ is a Banach algebra.*

Proof. It follows from the previous lemma that if $\widetilde{A}$ is a Banach algebra then A is a Banach algebra. Conversely if A is a Banach algebra and $((x_n, \lambda_n))$ is a Cauchy sequence in $\widetilde{A}$, then for any $\epsilon > 0$ we may find $N \in \mathbb{N}$ so that

$$n > N \Rightarrow \epsilon > \|(x_n, \lambda_n) - (x_m, \lambda_m)\|_{\widetilde{A}} = \|x_n - x_m\|_A + |\lambda_n - \lambda_m|,$$

which implies that (x_n) and (λ_n) are Cauchy sequences in A and $\mathbb{K}$ respectively. By completeness these sequences have limits $x \in A$ and $\lambda \in \mathbb{K}$ respectively, so $(x_n, \lambda_n) \to (x, \lambda) \in \widetilde{A}$ and this proves that $\widetilde{A}$ is a Banach algebra. □

2.5. **Invertible Elements.**

Definition 2.38. Let A be a unital algebra, then an element $x \in A$ is called,

(i) *Left invertible* if there is $l \in A$ with $lx = 1_A$,
(ii) *Right invertible* if there is $r \in A$ with $xr = 1_A$,
(iii) *Invertible* if there is $y \in A$ with $xy = yx = 1_A$.

If x is left invertible, then the element l in the above is called the *left inverse* of x, similarly one defines a *right inverse* of x. If x is invertible, then the element y in the above is called the *inverse* of x and is denoted x^{-1}. We write $\mathrm{Inv}(A)$ for the set of all invertible elements of A.

Remark 2.39. Suppose that x is an element of a unital algebra A and the elements $l, r \in A$ satisfy $lx = 1_A$ and $xr = 1_A$, then by associativity of the algebra multiplication we have

$$l = l(xr) = (lx)r = r.$$

In other words, if x is both left and right invertible, then x is invertible and its left and right inverses coincide.

Example 2.40. Let A be a unital normed algebra and suppose that $F = \{x_1, \ldots, x_n\} \subseteq A$ is a finite subset with the property that $x_j x_k = x_k x_j$ for every $1 \leq j, k \leq n$. If each $x_k \in F$ is invertible in A, then the product $x := \prod_{k=1}^{n} x_k$ is obviously invertible. On the other hand, if $x \in \mathrm{Inv}(A)$ then consider the following,

$$x_1 \left(\prod_{k=2}^{n} x_k \right) x^{-1} = x x^{-1} = 1_A = x^{-1} x = \left(x^{-1} \prod_{k=2}^{n} x_k \right) x_1.$$

When combined with Remark 2.39, this shows that x_1 is invertible. Note that we have used the commuting property of F to bring x_1 to the right hand side of the expression.

We may apply a similar argument to show that every element of F is invertible and deduce that

$$x \in \mathrm{Inv}(A) \iff x_k \in \mathrm{Inv} \text{ for each } 1 \leq k \leq n.$$

Proposition 2.41. *(*The Neumann Criterion*)*
Let A be a unital Banach algebra. If $x \in A$ satisfies $\|1_A - x\| < 1$, then $x \in Inv(A)$ and

$$x^{-1} = 1_A + \sum_{n=1}^{\infty} (1_A - x)^n.$$

Proof. For each $n \in \mathbb{N}$, submultiplicativity of the norm on A gives us $\|(1_A - x)^n\| \leq \|1_A - x\|^n < 1$, so that the series $\sum_{n=1}^{\infty} (1_A - x)^n$ converges absolutely. Since A is a Banach space, this implies that the series converges to some element of A. For fixed $k \in \mathbb{N}$ we have

$$\begin{aligned}
\left\| \left(1_A + \sum_{n=1}^{k} (1_A - x)^n \right) x - 1_A \right\| &= \left\| x + \left(\sum_{n=1}^{k} (1_A - x)^n \right) x - 1_A \right\| \\
&= \left\| \left(\sum_{n=1}^{k} (1_A - x)^n \right) (1_A - (1_A - x)) - (1_A - x) \right\| \\
&= \left\| \sum_{n=1}^{k} (1_A - x)^n - \sum_{n=1}^{k} (1_A - x)^{n+1} - (1_A - x) \right\| \\
&= \|(1_A - x)^{k+1}\| \\
&\leq \|1_A - x\|^{k+1}.
\end{aligned}$$

Since $\|1_A - x\|^{k+1} \to 0$ as $k \to \infty$, it follows that $(1_A + \sum_{n=1}^{\infty} (1_A - x)^n) x = 1_A$. Similarly we may show that $x(1_A + \sum_{n=1}^{\infty} (1_A - x)^n) = 1_A$, which proves that

$$x^{-1} = 1_A + \sum_{n=1}^{\infty} (1_A - x)^n.$$

$\square$

Remark 2.42. An immediate corollary is that if A is a unital Banach algebra and $y \in A$ satisfies $\|y\| < 1$ then $(1_A - y) \in \mathrm{Inv}(A)$ and

$$(1_A - y)^{-1} = 1_A + \sum_{n=1}^{\infty} y^n.$$

To see this, simply put $x = 1_A - y$ in the previous result.

Example 2.43. Let $A \in \mathbb{M}_2(\mathbb{R})$ be a matrix of the following form,

$$A = \begin{pmatrix} 1 & \epsilon \\ -\epsilon^2 & 1 \end{pmatrix},$$

where $\epsilon \in \mathbb{R}$ satisfies $|\epsilon| < 1$. We shall use the Neumann criterion to compute the inverse of A and for this we let $B = I_2 - A$ and let $x = (a, b)$ be a vector in $\mathbb{R}^2$ with $\|x\| = 1$. Then,

$$\|Bx\| = \|(-\epsilon b, \epsilon^2 a)\| = \sqrt{\epsilon^2 b^2 + \epsilon^4 a^2} = |\epsilon|\sqrt{b^2 + \epsilon^2 a^2} \le |\epsilon|\sqrt{a^2 + b^2} < 1.$$

The Neumann criterion shows that $I_2 - B = A$ is invertible and is given by

$$A^{-1} = (I_2 - B)^{-1} = \sum_{n=0}^{\infty} B^n.$$

Now we examine the powers of B, first we see that

$$B^2 = \begin{pmatrix} -\epsilon^3 & 0 \\ 0 & -\epsilon^3 \end{pmatrix} = -\epsilon^3 I,$$

which implies that for any $k \in \mathbb{N}$, $B^{2k} = (B^2)^k = (-1)^k \epsilon^{3k} I_2$ and $B^{2k+1} = (-1)^k \epsilon^{3k} B$. Therefore,

$$A^{-1} = (I_2 - B)^{-1} = \sum_{k=0}^{\infty} B^{2k} + \sum_{k=0}^{\infty} B^{2k+1} = \left(\sum_{k=0}^{\infty} (-1)^k \epsilon^{3k}\right)(I_2 + B) = \frac{1}{1+\epsilon^3}(I_2 + B),$$

or rather,

$$A^{-1} = \frac{1}{1+\epsilon^3}\begin{pmatrix} 1 & -\epsilon \\ \epsilon^2 & 1 \end{pmatrix},$$

which is the answer we would have gotten had we used the usual method for computing the inverse of A.

The following two corollaries of the Neumann criterion may be combined to see that the invertible elements of a unital Banach algebra form a topological group.

Corollary 2.44. *If A is a unital Banach algebra, then $Inv(A)$ is an open subset of A.*

Proof. Given $x \in \mathrm{Inv}(A)$, put $r = \frac{1}{\|x^{-1}\|}$. We shall show that $B_r(x) \subseteq \mathrm{Inv}(A)$. Take any $y \in B_r(x)$, then $\|x - y\| \le \frac{1}{\|x^{-1}\|}$ and by submultiplicativity we have

$$\|1_A - yx^{-1}\| = \|(x - y)x^{-1}\| \le \|x - y\|\|x^{-1}\| < \frac{1}{\|x^{-1}\|}\|x^{-1}\| = 1.$$

So by the Neumann criterion, $yx^{-1} \in \mathrm{Inv}(A)$ and hence $y = (yx^{-1})x \in \mathrm{Inv}(A)$, which proves that $B_r(x) \subseteq \mathrm{Inv}(A)$. $\square$

Corollary 2.45. *If A is a unital Banach algebra, then the inversion map which sends $x \in Inv(A)$ to $x^{-1} \in Inv(A)$ is a homeomorphism.*

Proof. The map is clearly bijective and self inverse, so all we check is continuity. First we prove continuity at 1_A. Suppose that $\delta \le 1$, then if $\|1_a - x\| < \delta$ the Neumann criterion tells us that $x \in \mathrm{Inv}(A)$ and $x^{-1} = 1_A + \sum_{n=1}^{\infty}(1_A - x)^n$, hence

$$\|x^{-1} - 1_A\| = \left\|\sum_{n=1}^{\infty}(1_A - x)^n\right\| \le \sum_{n=1}^{\infty}\|(1_A - x)\|^n < \sum_{n=1}^{\infty}\delta^n.$$

So given $\epsilon > 0$, we put $\delta = \frac{\epsilon}{1+\epsilon}$ (noting that $\delta < 1$) and we obtain

$$\|1_A - x\| < \delta \Rightarrow \|x^{-1} - 1_A\| < \sum_{n=1}^{\infty} \delta^n = \frac{\delta}{1-\delta} = \epsilon,$$

which proves continuity at 1_A. For a general element $x \in \mathrm{Inv}(A) \setminus \{1_A\}$ and any sequence (x_n) in $\mathrm{Inv}(A)$ with $x_n \to x$, continuity of multiplication shows that

$$\lim_{n\to\infty} x_n x^{-1} = xx^{-1} = 1_A.$$

Because the inversion map is continuous at 1_A, it follows that

$$\lim_{n\to\infty} (x_n x^{-1})^{-1} = (xx^{-1})^{-1} = 1_A.$$

Finally we see that

$$\lim_{n\to\infty} x_n^{-1} = \lim_{n\to\infty} (x^{-1}x)x_n^{-1} = \lim_{n\to\infty} x^{-1}(xx_n^{-1}) = \lim_{n\to\infty} x^{-1}(x_n x^{-1})^{-1} = x^{-1}.$$

□

2.6. **The Spectrum and Resolvent Set.** To conclude our introduction to Banach algebras we shall introduce two important sets of complex numbers. The definition of the spectrum is a very simple one, yet it provides to be one of the most useful tools we have. We begin with an assumption.

Assumption 2.46. Unless otherwise stated, from now on we shall assume that all our algebras are over $\mathbb{C}$.

Definition 2.47. Let A be a unital algebra. The *spectrum* of an element $x \in A$ is the set

$$\sigma_A(x) = \{\lambda \in \mathbb{C};\ \lambda 1_A - x \notin \mathrm{Inv}(A)\},$$

and the *resolvent set* of x is the set

$$\mathrm{Res}_A(x) = \mathbb{C} \setminus \sigma_A(x).$$

We will usually omit the subscripts when writing these sets and often write $\lambda 1_A - x$ simply as $\lambda - x$, but it is very important to keep in mind λ means $\lambda 1_A$ in this case.

Remark 2.48. Observe that it doesn't matter if we write $\lambda 1_A - x \in \mathrm{Inv}(A)$ or $x - \lambda 1_A \in \mathrm{Inv}(A)$ in the definition of the spectrum. Indeed $\lambda 1_A - x$ is invertible if and only if $x - \lambda 1_A$ is invertible.

Example 2.49. Let A be any matrix in $\mathbb{M}_n(\mathbb{C})$. Then the spectrum of A is precisely the set of eigenvalues of A.

Example 2.50. Let X be a compact Hausdorff space and take any function $f \in C(X)$. If $\lambda \notin f(X)$ then the function $x \mapsto \lambda - f(x)$ never vanishes on X, so the function

$$g(x) = \frac{1}{\lambda - f(x)},$$

belongs to $C(X)$. Since we clearly have $(\lambda - f)g = g(\lambda - f) = 1$ it follows that $\lambda \notin \sigma(f)$. On the other hand, if $\lambda \notin \sigma(f)$ then $\lambda - f$ is invertible, so we can find $g \in C(X)$ such that $(\lambda - f)g = 1$. g cannot vanish on X, so we rearrange to give

$$f = \frac{1}{g} - \lambda.$$

It follows that there is no $x \in X$ for which $f(x) = \lambda$ and so $\lambda \notin f(X)$. Our conclusion is that $\sigma(x) = f(X)$.

Proposition 2.51. *Let A be a unital Banach algebra. Then for every $x \in A$, we have $\sigma(x) \subseteq \overline{B}_{\|x\|}(0)$ and moreover $\sigma(x)$ is a compact subset of $\mathbb{C}$.*

Proof. First we shall show that $\sigma(x) \cap \mathbb{C} \setminus \overline{B}_{\|x\|}(0) = \emptyset$. Take any complex number $\lambda \in \mathbb{C} \setminus \overline{B}_{\|x\|}(0)$, then $\|x\| < |\lambda|$, so $\frac{\|x\|}{|\lambda|} < 1$ and hence $1_A - \frac{x}{\lambda} \in \mathrm{Inv}(A)$ by the Neumann criterion. Moreover, $\lambda - x = \lambda(1_A - \frac{x}{\lambda}) \in \mathrm{Inv}(A)$ which shows that $\lambda \notin \sigma(x)$. Therefore $\sigma(x) \cap \mathbb{C} \setminus \overline{B}_{\|x\|}(0) = \emptyset$ so that $\sigma(x) \subseteq \overline{B}_{\|x\|}(0)$.

Note that in particular, the above shows that $\sigma(x)$ is bounded, so by the Heine-Borel theorem all that remains for us to prove is that $\sigma(x)$ is closed. For this, we recall that by Corollary 2.44 $\mathrm{Inv}(A)$ is open in A and consider the continuous map $\varphi_x : \mathbb{C} \to A$ given by $\varphi_x(\lambda) = \lambda 1_A - x$. Then

$$\varphi_x^{-1}(A \setminus \mathrm{Inv}(A)) = \varphi_x^{-1}(A) \setminus \varphi_x^{-1}(\mathrm{Inv}(A)) = \mathbb{C} \setminus \mathrm{Res}(x) = \sigma(x).$$

So $\sigma(x)$ is closed for it is the preimage of a closed set under a continuous map. □

Definition 2.52. Let A be a unital Banach algebra. Then for $x \in A$ and $\lambda \in \mathrm{Res}(x)$ we define the *resolvent of x and λ* by

$$R_x(\lambda) = (\lambda - x)^{-1}.$$

If A is a unital Banach algebra, then the above definition defines a map $R_x : \mathrm{Res}(x) \to \mathrm{Inv}(A)$, which is continuous for it is the composition of the continuous maps

$$\mathrm{Res}(x) \to \mathrm{Inv}(A),\ \lambda \mapsto \lambda - x,$$

and

$$\mathrm{Inv}(A) \to \mathrm{Inv}(A),\ x \mapsto x^{-1}.$$

If $\lambda, \mu \in \mathrm{Res}(x)$, then a simple calculation shows that

$$\begin{aligned} R_x(\lambda) - R_x(\mu) &= R_x(\lambda)(R_x(\mu)^{-1} - R_x(\lambda)^{-1})R_x(\mu) \\ &= R_x(\lambda)((\mu - x) - (\lambda - x))R_x(\mu) \\ &= (\mu - \lambda)R_x(\lambda)R_x(\mu). \end{aligned}$$

An expression which is often referred to as the *resolvent identity.* Before proceeding, we should recall the following definition from complex analysis.

Definition 2.53. A function $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$ is an open set, is called *holomorphic* on U if for every $z_0 \in U$ the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

Lemma 2.54. *If A is a unital Banach algebra, then for every $x \in A$ and every $\varphi \in A^*$, the composition*

$$\varphi \circ R_x : Res(x) \to \mathbb{C}$$

is holomorphic on $Res(x)$.

Proof. For distinct complex numbers $\lambda, \mu \in \mathrm{Res}(x)$ we have

$$\frac{R_x(\lambda) - R_x(\mu)}{\lambda - \mu} + R_x(\mu)^2 = \frac{(\mu - \lambda)R_x(\lambda)R_x(\mu)}{-(\mu - \lambda)} + R_x(\mu)^2 = -(R_x(\lambda) - R_x(\mu))R_x(\mu).$$

Therefore, for an arbitrary linear functional $\varphi \in A^*$,

$$\begin{aligned} \left| \frac{\varphi(R_x(\lambda)) - \varphi(R_x(\mu))}{\lambda - \mu} - \varphi(-R_x(\mu)^2) \right| &= \left| \varphi \left(\frac{R_x(\lambda) - R_x(\mu)}{\lambda - \mu} + R_x(\mu)^2 \right) \right| \\ &= |\varphi((R_x(\lambda) - R_x(\mu))R_x(\mu))| \\ &\leq \|\varphi\|_{\mathrm{op}} \|R_x(\mu)\|_A \|R_x(\lambda) - R_x(\mu)\|_A. \end{aligned}$$

By continuity of R_x, we can make the quantity $\|\varphi\|_{\mathrm{op}} \|R_x(\mu)\|_A \|R_x(\lambda) - R_x(\mu)\|_A$ arbitrarily small and the result follows. □

This brings us to the first of two fundamental results about the spectrum. The proofs of Theorems 2.55 and 2.60 bring together a number of classic results from analysis. Of course proofs of these results may be found in practically every textbook on the subject and we shall follow the ideas presented by Murphy [15, Theorems 1.2.5 and 1.2.7], but provide more details in each proof.

Theorem 2.55. *Let A be a unital Banach algebra. Then for any $x \in A$ the spectrum $\sigma(x)$ is non-empty.*

Proof. Toward a contradiction, we assume that $\sigma(x) = \emptyset$ for some $x \in A$, then the map $R_x : \mathrm{Res}(x) \to \mathrm{Inv}(A)$ is a map from $\mathbb{C}$ to $\mathrm{Inv}(A)$. Given any $\lambda \in \mathbb{C}$, if $|\lambda| > 2\|x\|$ then $\frac{1}{2} > \|\lambda^{-1}x\|$, so by the Neumann criterion we have the expression

$$(1_A - \lambda^{-1}x)^{-1} = 1_A + \sum_{n=1}^{\infty}(\lambda^{-1}x)^n.$$

Taking norms we get,

$$\|(1_A - \lambda^{-1}x)^{-1} - 1_A\| = \left\|\sum_{n=1}^{\infty}(\lambda^{-1}x)^n\right\| \leq \sum_{n=1}^{\infty}\|\lambda^{-1}x\|^n = \frac{\|\lambda^{-1}x\|}{1 - \|\lambda^{-1}x\|} < 2\|\lambda^{-1}x\| < 1.$$

By the triangle inequality, we have

$$\|(1_A - \lambda^{-1}x)^{-1}\| = \|((1_A - \lambda^{-1}x)^{-1} - 1_A) + 1_A\| < 1 + \|1_A\| = 2,$$

so it follows that

$$\|R_x(\lambda)\| = \|(\lambda - x)^{-1}\| = \|\lambda^{-1}(1_A - \lambda^{-1}x)^{-1}\| = |\lambda^{-1}|\|(1_A - \lambda^{-1}x)^{-1}\| < 2|\lambda^{-1}| < \frac{1}{\|x\|}.$$

This shows that the function R_x is bounded outside of the closed ball $\overline{B}_{2\|x\|}(0)$. On the ball $\overline{B}_{2\|x\|}(0)$, R_x is a continuous function on a compact set so is bounded there and therefore R_x is bounded on the whole complex plane.

Now take any bounded linear functional $\varphi \in A^*$, Lemma 2.54 tells us that the composition $\varphi \circ R_x : \mathbb{C} \to \mathbb{C}$ is holomorphic on $\mathbb{C}$ and moreover the above shows that $\varphi \circ R_x$ is bounded on $\mathbb{C}$. By Liouville's Theorem [21, Theorem 10.23], $\varphi \circ R_x$ is a constant function, so in particular $(\varphi \circ R_x)(0) = (\varphi \circ R_x)(1)$ and we have

$$\varphi((-x)^{-1} - (1 - x)^{-1}) = 0,$$

for every $\varphi \in A^*$. By a corollary to the Hahn-Banach Theorem [7, Corollary 1.28], it follows that $(-x)^{-1} - (1 - x)^{-1} = 0$ and hence $-x = 1 - x$ which is a contradiction. □

If A is a non-unital algebra, then we may still define the spectrum of an element, but we do so via the unitisation of A. If $x \in A$, then the spectrum of x is the set

$$\sigma_A(x) = \sigma_{\widetilde{A}}(x).$$

In this case, we always have $0 \in \sigma_A(x)$. If in addition A is a Banach algebra, then since $\widetilde{A}$ is a unital algebra it follows from Theorem 2.55 that $\sigma_A(x)$ is non-empty.

Corollary 2.56. (The Gelfand-Mazur Theorem)
If A is a unital Banach algebra in which every non-zero element is invertible, then there is an isometric isomorphism from A to $\mathbb{C}$.

Proof. Take any element $0 \neq x \in A$, then $\sigma(x) \neq \emptyset$ so there is $\lambda_x \in \sigma(x)$ which satisfies $\lambda_x - x \notin \mathrm{Inv}(A)$. But the assumption that $x \in \mathrm{Inv}(A)$ implies that $\lambda_x - x = 0$. Now consider the map $\varphi : \mathbb{C} \to A$ given by $\varphi(\mu) = \mu 1_A$. This is clearly an injective homomorphism and it is even surjective; the above shows that any $0 \neq x \in A$ may be written as $x = \lambda_x 1_A$ with $\lambda_x \in \sigma(x)$ so that $\varphi(\lambda_x) = x$, and $\varphi(0) = 0$ because φ is a linear map. To see that φ is isometric, we take $0 \neq x \in A$ and $\lambda_x \in \sigma(x)$ such that $\varphi(\lambda_x) = x$, then

$$\|\varphi(\lambda_x)\| = \|\lambda_x 1_A\| = |\lambda_x|.$$

□

Lemma 2.57. (The Spectral Mapping Property for Polynomials)
Let A be a unital algebra. Then for any element $x \in A$ with $\sigma(x) \neq \emptyset$ and any polynomial $p \in \mathbb{C}[z]$, we have

$$\sigma(p(x)) = \{p(\lambda);\, \lambda \in \sigma(x)\}.$$

Proof. If p is a constant polynomial, say $p(z) = \alpha \in \mathbb{C}$ then $p(x) = \alpha 1_A$ so that $\sigma(p(x)) = \{\alpha\}$ and conversely $\{p(\lambda);\, \lambda \in \sigma(x)\} = \{\alpha\}$. Now we assume that $\deg(p) \geq 1$, then for $\nu \in \mathbb{C}$ we decompose the polynomial $p(z) - \nu$ into factors,

$$p(z) - \nu = \mu_0 \prod_{k=1}^{n} (z - \mu_k),$$

for some $\mu_0, \ldots, \mu_n \in \mathbb{C}$. Evaluating this polynomial at x gives us

$$p(x) - \nu 1_A = \mu_0 \prod_{k=1}^{n} (x - \mu_k 1_A),$$

and because all of the terms in the product commute with one another, it follows from Example 2.40 that invertability of $p(x) - \nu 1_A$ is equivalent to invertability of $x - \mu_k 1_A$, for each $1 \leq k \leq n$. The consequence of this is that ν belongs to $\sigma(p(x))$ if and only if at least one of the factors $x - \mu_k 1_A$ is non-invertible, which happens if and only if $\mu_k \in \sigma(x)$ for some $1 \leq k \leq n$, which in turn happens if and only if $\nu = p(\mu_k) \in \{p(\lambda);\, \lambda \in \sigma(x)\}$. This completes our proof. $\square$

Definition 2.58. Let A be a unital Banach algebra. Then for $x \in A$ the *spectral radius* of x is

$$r(x) = \sup\{|\lambda|;\, \lambda \in \sigma(x)\}.$$

Note that $r(x) \leq \|x\|$ because $\sigma(x) \subseteq \overline{B}_{\|x\|}(0)$.

Example 2.59. If x is a matrix in $\mathbb{M}_n(\mathbb{C})$, then $r(x)$ is simply the largest absolute value of its eigenvalues. If X is a compact Hausdorff space and $f : X \to \mathbb{C}$ is a continuous function, then because $\sigma(f) = f(X)$ we have $r(f) = \|f\|_\infty$.

Theorem 2.60. *Let A be a unital Banach algebra. Then for any $x \in A$ the spectral radius of x is given by*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

Proof. First observe that if $x = 0$, then $\sigma(x) = \{0\}$ and in this case we have $r(x) = 0 = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$, so henceforth we assume that $x \neq 0$. By the spectral mapping property for polynomials, if $\lambda \in \sigma(x)$ then $\lambda^n \in \sigma(x^n)$ for every $n \in \mathbb{N}$, so $\|x^n\| \geq |\lambda^n| = |\lambda|^n$ which implies that $\|x^n\|^{\frac{1}{n}} \geq |\lambda|$. Because this holds for every λ in the spectrum of x, it follows that $r(x) \leq \|x^n\|^{\frac{1}{n}}$ which implies the inequality

$$r(x) \leq \inf_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty} \|x^n\|^{\frac{1}{n}}.$$

We define $B := B_{\frac{1}{r(x)}}(0)$ and then if $\lambda \in B$, precisely one of the following holds,

(i) $\lambda = 0$, so $1_A - \lambda x = 1_A \in \mathrm{Inv}(A)$, or
(ii) $0 < r(x) < \frac{1}{\lambda}$, so $\frac{1}{\lambda} \notin \sigma(x)$ and therefore $1_A - \lambda x = \lambda\left(\frac{1}{\lambda} - x\right) \in \mathrm{Inv}(A)$.

For each bounded linear functional $\varphi \in A^*$, we define a function $f_\varphi : B \to \mathbb{C}$ by

$$f_\varphi(\lambda) = \varphi((1_A - \lambda x)^{-1}),$$

noting that by our previous discussion f_φ is properly defined on B. It turns out that f_φ is actually holomorphic on B, so by [21, Theorem 10.16] f_φ is representable by power series in B, hence there is a (unique) sequence of complex numbers (α_n) such that

$$f_\varphi(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n,$$

for any $\lambda \in B$. If $\lambda \in B$ satisfies $|\lambda| < \frac{1}{\|x\|} < \frac{1}{r(x)}$, then $\|\lambda x\| < 1$ and by the Neumann criterion we have

$$(1_A - \lambda x)^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n,$$

which combined with linearity of φ[5] implies that

$$f_\varphi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \varphi(x^n).$$

In other words, for each $n \in \mathbb{N}$ the number $\alpha_n \in \mathbb{C}$ is given by the relation

$$\alpha_n = \varphi(x^n).$$

If we now fix $\lambda \in B$ then we get a sequence $(\lambda^n \varphi(x^n))$ which must converge to zero and is therefore a bounded sequence. Our goal now is to prove that in fact $(\lambda^n \varphi(x^n))$ is a bounded sequence, and for this we shall use the Principle of Uniform Boundedness [8, Theorem 5.12]. Recall that because A is a Banach space, we have an isometric embedding $\psi : A \to A^{**}$ where for $x \in A$ the functional $\psi(x) : A^* \to \mathbb{C}$ is defined by

$$(\psi(x))\, \pi = \pi(x),$$

for every $\pi \in A^*$. If we take $\varphi \in A^*$, then because $(\lambda^n \varphi(x^n))$ is a bounded sequence we have a constant $M_\varphi \in \mathbb{R}^{>0}$ which satisfies

$$|\left(\psi(\lambda^n x^n)\right)(\varphi)| = |\varphi(\lambda^n x^n)| \leq M_\varphi,$$

for every $n \in \mathbb{N}$. The set $\{\psi(\lambda^n x^n)\}_{n \in \mathbb{N}} \subseteq A^{**}$ is a collection of bounded linear operators and the inequality stated above shows that for any $\varphi \in A^*$ we have

$$\sup_{n \in \mathbb{N}} |\left(\psi(\lambda^n x^n)\right)(\varphi)| < \infty.$$

The Principle of Uniform Boundedness now asserts that

$$\sup_{n \in \mathbb{N}} \|\psi(\lambda^n x^n)\|_{\mathrm{op}} < \infty,$$

but of course ψ is an isometry so it follows that $(\lambda^n x^n)$ is a bounded sequence in A, thus we have $M_\lambda \in \mathbb{R}^{>0}$ which satisfies $\|\lambda^n x^n\| \leq M_\lambda$ for all $n \in \mathbb{N}$. So if $\lambda \neq 0$ then

$$\|x^n\| \leq \frac{M_\lambda}{|\lambda|^n}.$$

We take n^{th} roots to give

$$\|x^n\|^{\frac{1}{n}} \leq \frac{(M_\lambda)^{\frac{1}{n}}}{|\lambda|},$$

and this yields the inequality

$$\limsup_{n \to \infty} \|x^n\|^{\frac{1}{n}} \leq \limsup_{n \to \infty} \frac{(M_\lambda)^{\frac{1}{n}}}{|\lambda|} = \frac{1}{|\lambda|}.$$

Now we note that $r(x) = \inf \left\{ \frac{1}{|\lambda|};\, \lambda \in \mathbb{C} \text{ and } \frac{1}{|\lambda|} > r(x) \right\}$[6] and combine our efforts to give

$$\limsup_{n \to \infty} \|x^n\|^{\frac{1}{n}} \leq r(x) \leq \liminf_{n \to \infty} \|x^n\|^{\frac{1}{n}},$$

so of course $r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$ as required. □

[5] Of course we also use continuity of φ to deduce that φ is linear on infinite sums.

[6] If we let $\Lambda := \left\{ \frac{1}{|\lambda|};\, \lambda \in \mathbb{C} \text{ and } \frac{1}{|\lambda|} > r(x) \right\}$, then for any $\epsilon > 0$ we put $\frac{1}{\lambda_\epsilon} = r(x) + \frac{\epsilon}{2}$. Then because both $r(x)$ and $\frac{\epsilon}{2}$ are positive and real, we have $\frac{1}{|\lambda_\epsilon|} = r(x) + \frac{\epsilon}{2} > r(x)$, so that $\frac{1}{|\lambda_\epsilon|} \in \Lambda$. Moreover we have $\frac{1}{|\lambda_\epsilon|} < r(x) + \epsilon$, which implies that $\inf \Lambda = r(x)$.

Remark 2.61. In this proof we stated that the function $f_\varphi : B \to \mathbb{C}$ is holomorphic on the open disk $B := B_{\frac{1}{r(x)}}(0)$. We can prove this using similar ideas to those we used in Lemma 2.54. For the sake of neatness, we write $\phi_x(\lambda) := (1_A - \lambda x)^{-1}$, then for $\lambda, \mu \in B$ we compute that

$$\phi_x(\lambda) - \phi_x(\mu) = (\lambda - \mu)\phi_x(\lambda)x\phi_x(\mu).$$

Then for distinct complex numbers $\lambda, \mu \in B$ we observe that

$$\begin{aligned}\left|\frac{\varphi(\phi_x(\lambda) - \varphi(\phi_x(\mu))}{\lambda - \mu} - \varphi(\phi_x(\mu)x\phi_x(\mu))\right| &= |\varphi((\phi_x(\lambda) - \phi_x(\mu))x\phi_x(\mu)| \\ &\leq \|\varphi\|_{\mathrm{op}}\|\phi_x(\lambda) - \phi_x(\mu)\|\|x\phi_x(\mu)\|,\end{aligned}$$

where the inequality is due to continuity of φ and submultiplicativity of the norm on A. Holomorphicity of $f_\varphi = \varphi \circ \phi_x$ follows once we observe that ϕ_x is continuous, so the right hand side of the inequality may be made arbitrarily small.

3. **Commutative Banach Algebras**

The goal of this chapter is to construct a map known as the *Gelfand representation*, this will be defined on any commutative Banach algebra and takes values $C_0(X)$, for some topological space X which we will generate from the algebra. Our main use of this map will be in the later chapters on C*-algebras, to prove the commutative Gelfand-Naimark Theorem. First though we shall look at maximal ideals of Banach algebras.

3.1. **Maximal Ideals and Characters.**

Lemma 3.1. *Let A be a unital Banach algebra, then every maximal ideal of A is closed and every proper ideal of A is contained in a maximal ideal.*

Proof. Suppose that $I \subsetneq A$ is a proper ideal. If $x \in I \cap \mathrm{Inv}(A)$, then x is invertible in A and of course this would mean that $1_A = xx^{-1} \in I$, hence $I \cap \mathrm{Inv}(A) = \emptyset$. Because $\mathrm{Inv}(A)$ is open, we may find an $r > 0$ such that $B_r(1_A) \subseteq \mathrm{Inv}(A)$ and it follows that $I \cap B_r(1_A) = \emptyset$, so there cannot be a sequence in I which converges to 1_A. This implies that $1_A \notin \overline{I}$ and because $\overline{I}$ is also an ideal of A, this means that $\overline{I}$ is a proper ideal of A.

Now suppose that $I \subsetneq A$ is a maximal ideal. The previous paragraph shows that $I \subseteq \bar{I} \subsetneq A$ and maximality of I then shows that $I = \bar{I}$, so I is closed.

For the second statement, we again let $I \subsetneq A$ be a proper ideal and consider the set

$$\Sigma := \{J \subseteq A;\ J \text{ is a proper ideal and } I \subseteq J\}.$$

We note that Σ is non-empty, because $I \in \Sigma$ and we order Σ by inclusion. Then if $\{J_\lambda\}_{\lambda\in\Lambda}$ is a totally ordered subset of Σ, the union

$$J := \bigcup_{\lambda\in\Lambda} J_\lambda,$$

is disjoint from $\mathrm{Inv}(A)$ (because each J_λ is) and moreover $J_\lambda \subseteq J$ for each $\lambda \in \Lambda$. Therefore J is a proper ideal of A which is an upper bound for $\{J_\lambda\}_{\lambda\in\Lambda}$, so by Zorn's Lemma Σ has a maximal element. This maximal element is a proper ideal of A which contains I and is not contained in any proper ideal of A, thus is a maximal ideal of A. $\square$

Assumption 3.2. For the remainder of this chapter, we will assume that all algebras are commutative.

Now we shall move on and look at a special type of algebra homomorphism, called characters. These are defined on commutative Banach algebras and will be the key ingredient for us in order to construct a locally compact Hausdorff space from any given Banach algebra.

Definition 3.3. Let A be a Banach algebra. A *character* on A is a non-zero algebra homomorphism $\varphi : A \to \mathbb{C}$. We denote by $\Omega(A)$ the set of all characters on A.

Whilst it is elementary, it is important that we note the following.

Lemma 3.4. *If A is a unital Banach algebra, then any character φ on A is a surjective, unital homomorphism.*

Proof. Take any $\varphi \in \Omega(A)$, then

$$\varphi(1_A) = \varphi(1_A^2) = \varphi(1_A)^2.$$

We divide by $\varphi(1_A)$ to give $\varphi(1_A) = 1$ and of course we should note that in order for φ to be a character we must have $\varphi(1_A) \neq 0$. For any complex number $\lambda \in \mathbb{C}$, we have

$$\varphi(\lambda 1_A) = \lambda\varphi(1_A) = \lambda,$$

and this shows that φ is surjective. $\square$

Example 3.5. Let X be a locally compact Hausdorff space. For $x \in X$ we define a function $\phi_x : C_0(X) \to \mathbb{C}$ by

$$\phi_x(f) = f(x),$$

for $f \in C_0(X)$ and can easily check that this is a character. For example multiplicativity holds because

$$\phi_x(fg) = (fg)(x) = f(x)g(x),$$

for every $f, g \in C_0(X)$.

Example 3.6. If A is a non unital Banach algebra and $\varphi \in \Omega(A)$ is any character, then we may extend φ to a character $\widetilde{\varphi}$ on $\widetilde{A}$ by putting

$$\widetilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda.$$

Example 3.7. Lets define

$$A = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} ; \lambda, \mu \in \mathbb{C} \right\},$$

which we can easily check to be a commutative unital subalgebra of $\mathbb{U}_2(\mathbb{C})$. Because A is two dimensional, it follows that A is closed in $\mathbb{U}_2(\mathbb{C})$ and so is a Banach algebra. Suppose that $\varphi : A \to \mathbb{C}$ is a character, then because φ must be a linear map we have for $x = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} \in A$ that

$$\varphi(x) = \lambda\varphi(I_2) + \mu\varphi(y),$$

where $y := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since A is unital, we must have $\varphi(I_2) = 1$ and it follows that φ is completely determined by its value on y. Notice however that $y^2 = 0$, so because φ is a character we must have

$$\varphi(y)^2 = \varphi(y^2) = \varphi(0) = 0,$$

implying that $\varphi(y) = 0$. We now see that $\varphi(x) = \lambda$, and because all characters must take the same value when we plug in the identity matrix we conclude that this is the only character on A.

Proposition 3.8. *Let A be a unital Banach algebra and let $\varphi \in \Omega(A)$. Then φ is continuous and $\|\varphi\|_{op} = 1$.*

Proof. Given a character $\varphi \in \Omega(A)$, pick any element $x \in A$ and then if $\varphi(x)1_A - x$ were invertible in A, there would be a $y \in A$ with $(\varphi(x)1_A - x)y = 1_A$. One applies φ to this and obtains

$$\varphi(x)\varphi(y) - \varphi(x)\varphi(y) = 1,$$

which is clearly a contradiction. It follows that $\varphi(x) \in \sigma(x)$ and therefore $|\varphi(x)| \le r(x) \le \|x\|$, so $\|\varphi\|_{\mathrm{op}} \le 1$. Since $\varphi(1_A) = 1$, we have $\|\varphi\|_{\mathrm{op}} = 1$. $\square$

Corollary 3.9. *Let A be a non-unital Banach algebra and let $\varphi \in \Omega(A)$. Then φ is continuous and $\|\varphi\|_{op} \le 1$.*

Proof. We consider the extension of φ to $\widetilde{A}$ as described in Example 3.6. By the preceding proposition this extension satisfies $\|\widetilde{\varphi}\|_{\mathrm{op}} = 1$ and for $(x, \lambda) \in \widetilde{A}$ we have

$$|\widetilde{\varphi}((x, \lambda))| \le \|(x, \lambda)\|_{\widetilde{A}} = \|x\|_A + |\lambda|.$$

This then implies that

$$|\varphi(x)| = |\widetilde{\varphi}((x, 0))| \le \|(x, 0)\|_{\widetilde{A}} = \|x\|_A,$$

for every $x \in A$, which shows that $\|\widetilde{\varphi}\|_{\mathrm{op}} \le 1$. $\square$

Theorem 3.10. *Let A be a unital Banach algebra, then*

(i) $\Omega(A) \neq \emptyset$,
(ii) *I is a maximal ideal of A if and only if $I = \ker(\varphi)$ for some $\varphi \in \Omega(A)$.*

Proof. (i) By Lemma 3.1, A has a least one maximal ideal I and this maximal ideal is closed. Suppose that $x \in A$ satisfies $\mathcal{Q}(x) \neq 0 + I$, where $\mathcal{Q} : A \to A/I$ denotes the quotient map from Example 2.28, and define

$$I_x := \{ax + j;\, a \in A,\, j \in I\}.$$

Since A is commutative, I_x is an ideal of A and we have $I \subseteq I_x$. We now use maximality of I to conclude that $I_x = A$, so we can find $a \in A$ and $j \in I$ such that $1_A = ax + j$. Applying the quotient map gives us

$$1_{A/I} = \mathcal{Q}(1_A) = \mathcal{Q}(ax + j) = \mathcal{Q}(a)\mathcal{Q}(x) + \mathcal{Q}(j) = \mathcal{Q}(a)\mathcal{Q}(x),$$

so that $\mathcal{Q}(x)$ is invertible in A/I. Because our choice of $x + I$ was arbitrary, we can apply the Gelfand-Mazur Theorem and deduce that there is an isometric isomorphism $\tau : A/I \to \mathbb{C}$. The composition $\tau \circ \mathcal{Q} : A \to \mathbb{C}$ is a character on A, so $\Omega(A) \neq \emptyset$ and moreover $\ker(\tau \circ \mathcal{Q}) = I$.

(ii) By part (i), if I is a maximal ideal of A then $I = \ker(\varphi)$ for some character $\varphi \in \Omega(A)$, so we only need to prove that the kernel of every character on A is a maximal ideal of A. Let $\varphi \in \Omega(A)$ be any character and suppose that $\ker(\varphi) \subsetneq I$ for some ideal $I \subseteq A$. Then there is $j \in I$ with $\varphi(j) \neq 0$, so for any $x \in A$ it follows that

$$x - \varphi(x)\varphi(j)^{-1}j \in \ker(\varphi) \subsetneq I,$$

which implies that

$$x = (x - \varphi(x)\varphi(j)^{-1}j) + \varphi(x)\varphi(j)^{-1}j \in I.$$

We conclude that $A = I$ and this proves maximality of $\ker(\varphi)$.

□

In the previous theorem, the assumption that A is unital is essential, as we will now discuss. Let A be any $\mathbb{K}$-algebra, then an element $x \in A$ is said to be *nilpotent* if there exists $n \in \mathbb{N}$ which satisfies $x^n = 0$. Now suppose that A is a non-unital Banach algebra (over $\mathbb{C}$ of course) in which every element is nilpotent. If $\varphi : A \to \mathbb{C}$ is an algebra homomorphism, then for every $x \in A$ we have some $n_x \in \mathbb{N}$ with the property that $x^{n_x} = 0$. Hence

$$0 = \varphi(x^{n_x}) = \varphi(x)^{n_x},$$

by multiplicativity of φ, which implies that $\varphi(x) = 0$ and so φ is the zero homomorphism. It now follows that $\Omega(A) = \emptyset$. Of course it might be the case that this is no issue, so far as there are no Banach algebras in which every element is nilpotent, but alas, we can construct such an algebra. Let A be any non-zero Banach space over $\mathbb{C}$ and define a multiplication on A by

$$xy = 0,$$

for every $x, y \in A$. This makes A into an algebra and because

$$0 = \|xy\| \leq \|x\|\|y\|,$$

holds for every $x, y \in A$, A is in fact a Banach algebra. A cannot have a unit, since for each possible candidate $1_A \in A$ we pick an element $0 \neq x \in A$ and then $1_A x = 0 = x 1_A$, so 1_A cannot possibly be a unit. With this multiplication, $x^2 = 0$ holds for every $x \in A$, so that each element of A is nilpotent and therefore $\Omega(A)$ is empty.

Theorem 3.11. *Let A be a Banach algebra, then for any $x \in A$ we have the following,*

(i) If A is unital then

$$\sigma(x) = \{\varphi(x);\, \varphi \in \Omega(A)\},$$

(ii) If A is non-unital then

$$\sigma(x) = \{\varphi(x);\, \varphi \in \Omega(A)\} \cup \{0\}.$$

Proof. (i) Let $x \in A$ and take any complex number $\lambda \in \sigma(x)$. The ideal $I_\lambda := (x-\lambda)A$ is a proper ideal of A, since if $1_A \in I_\lambda$ then we could find $y \in A$ such that $(x-\lambda)y = 1_A$ which would contradict $\lambda \in \sigma(x)$. It follows from Lemma 3.1 that I_λ is contained in a maximal ideal of A, which by Theorem 3.10(ii) has the form $\ker(\varphi)$ for some $\varphi \in \Omega(A)$. Then we have

$$x - \lambda = (x-\lambda)1_A \in I_\lambda \subseteq \ker(\varphi),$$

so $0 = \varphi(x-\lambda) = \varphi(x) - \varphi(\lambda)$, or equivalently $\varphi(x) = \varphi(\lambda)$. Hence $\varphi(x) = \lambda$ which gives us the inclusion $\sigma(x) \subseteq \{\varphi(x);\, \varphi \in \Omega(A)\}$. Conversely we take $\varphi \in \Omega(A)$, then if $\varphi(x) \notin \sigma(x)$ we must have $x - \varphi(x) \in \mathrm{Inv}(A)$, so

$$\varphi(x - \varphi(x))\varphi((x-\varphi(x))^{-1}) = 1_A.$$

But $\varphi(x - \varphi(x)) = \varphi(x) - \varphi(x) = 0$ which is a contradiction. Therefore we must have $\varphi(x) \in \sigma(x)$ which finishes the proof.

(ii) Let $\tau : \widetilde{A} \to \mathbb{C}$ be the homomorphism given by $\tau((x,\lambda)) = \lambda$, for $(x,\lambda) \in \widetilde{A}$. τ is a character on $\widetilde{A}$ and so it follows that

$$\Omega(\widetilde{A}) = \{\widetilde{\varphi};\, \varphi \in \Omega(A)\} \cup \{\tau\},$$

where $\widetilde{\varphi}$ is the unique character which extends $\varphi \in \Omega(A)$ to $\widetilde{A}$. By part (i) we have that

$$\sigma(x) = \sigma_{\widetilde{A}}(x) = \{\varphi(x);\, \varphi \in \Omega(\widetilde{A})\} = \{\varphi(x);\, \varphi \in \Omega(A)\} \cup \{0\},$$

where the final equality follows from our description of $\Omega(\widetilde{A})$ and the definition of τ.

□

Since characters are bounded linear maps whose norm is at most one, we may view $\Omega(A)$ as a subset of the closed unit ball in A^*. The Banach-Alaoglu Theorem [8, Theorem 5.18] asserts that if A is any Banach space, then the closed unit ball of A^* is compact in the weak-* topology, and we will make full use of this fact in the following results.

Definition 3.12. Let A be a Banach algebra. We equip the set $\Omega(A)$ with the subspace topology, inherited from the weak-* topology on A^* and call the corresponding topological space the *character space* of A.

Theorem 3.13. *If A is a Banach algebra, then $\Omega(A)$ is a locally compact Hausdorff space. If in addition A is unital, then $\Omega(A)$ is compact.*

Proof. By our previous discussion, $\Omega(A) \subseteq \overline{B_1(0)} \subseteq A^*$ and the Banach-Alaoglu Theorem tells us that $\overline{B_1(0)}$ is compact in the weak-* topology. Consider the closure of the character space, this is a closed subset of $\overline{B_1(0)}$ and is therefore compact, moreover it is Hausdorff simply because the weak-* topology is.

For any $\varphi \in \overline{\Omega(A)}$ there is a net (φ_λ) which converges to φ in the weak-* topology, so $\varphi_\lambda(x) \to \varphi(x)$ for every $x \in A$. Because each φ_λ is a character and multiplication of complex numbers is continuous, we have for any $x, y \in A$ that

$$\varphi(xy) = \lim_\lambda \varphi_\lambda(xy) = \lim_\lambda \varphi_\lambda(x)\varphi_\lambda(y) = \varphi(x)\varphi(y).$$

Therefore either $\varphi \in \Omega(A)$ or $\varphi = 0$, so that $\overline{\Omega(A)} = \Omega(A) \cup \{0\}$. Now we observe that

$$\Omega(A) = (\Omega(A) \cup \{0\}) \setminus \{0\},$$

is an open subset of the compact space $\overline{\Omega(A)}$, so is locally compact. Finally, if A is unital then $\varphi(1_a) = 1$ for every character $\varphi \in \Omega(A)$ which implies that $0 \notin \overline{\Omega(A)}$, so in this case $\overline{\Omega(A)} = \Omega(A)$ and $\Omega(A)$ is compact. □

We shall now take a short break to present a couple of examples in which we will make use of the character space.

Example 3.14. [15, p31, Exercise 5(c)]
Let A be a unital Banach algebra and assume that the elements $x_1, \dots, x_n \in A$ are such that $A = \mathbb{B}(\{x_1, \dots, x_n\})$. We shall define

$$\sigma(x_1, \dots, x_n) := \{(\varphi(x_1), \dots, \varphi(x_n));\ \varphi \in \Omega(A)\} \subseteq \mathbb{C}^n,$$

which in the literature is known as the *joint spectrum* of $x_1, \dots, x_n$. We claim that $\Omega(A)$ is homeomorphic to $\sigma(x_1, \dots, x_n)$. Consider the map

$$\begin{aligned} \phi : \Omega(A) &\to \mathbb{C}^n \\ \varphi &\mapsto (\varphi(x_1), \dots, \varphi(x_n)) \end{aligned}$$

This is certainly surjective onto $\sigma(x_1, \dots, x_n)$. For $1 \le k \le n$ we let p_k denote the projection $(x_1, \dots, x_k, \dots, x_n) \mapsto x_k$ and observe that for each k the composition $p_k \circ \phi$ is clearly continuous, so ϕ is continuous. Finally, lets assume that $\tau, \varphi \in \Omega(A)$ satisfy $\phi(\tau) = \phi(\varphi)$, then $\tau(x_k) = \varphi(x_k)$ for each $k = 1, \dots, n$. Because $A = \mathbb{B}(\{x_1, \dots, x_n\})$, the collection of polynomials in $\{x_1, \dots, x_n\}$ is dense in A and because τ and φ are algebra homomorphisms they must agree on every such polynomial. The continuity of characters now implies that τ and φ agree on the whole of A, so $\varphi = \tau$ which proves that ϕ is injective. Using the facts that $\Omega(A)$ is compact, $\mathbb{C}^n$ is Hausdorff and ϕ is a continuous bijection, we conclude that ϕ is a homeomorphism.

Example 3.15. Let A and B be Banach algebras and let $\varphi : A \to B$ be an isomorphism. Consider the map

$$\begin{aligned} \pi : \Omega(A) &\to \Omega(B) \\ \tau &\mapsto \tau \circ \varphi^{-1} \end{aligned}$$

observing that for each $\tau \in \Omega(A)$, $\pi(\tau)$ is indeed a character on B. The map π is injective, since if $\pi(\tau) = \pi(\omega)$ for $\tau, \omega \in \Omega(A)$, then

$$\tau(\varphi^{-1}(y)) = \omega(\varphi^{-1}(y)),$$

for every $y \in B$ and because φ^{-1} is an isomorphism, this is equivalent to

$$\tau(x) = \omega(x),$$

for every $x \in A$, hence $\tau = \omega$. Furthermore, for any $\tau \in \Omega(B)$, the composition $\tau \circ \varphi$ is a character on A and we have

$$\pi(\tau \circ \varphi) = \tau,$$

so π is surjective. If (τ_λ) is a net in $\Omega(A)$ which converges to $\tau \in \Omega(A)$, then by definition of the weak-* topology we have for every $x \in A$ that $\tau_\lambda(x) \to \tau(x)$, hence for $y \in B$ we have

$$\lim_\lambda \pi(\tau_\lambda)(y) = \lim_\lambda \tau_\lambda(\varphi^{-1}(y)) = \tau(\varphi^{-1}(y)) = \pi(\tau)(y),$$

so that π is continuous.

We may employ a completely analogous argument to show that the inverse of π is also continuous, thus concluding that π is a homeomorphism.

3.2. **The Gelfand Representation.** We have now arrived at the main part of this chapter. We are going to introduce the so called Gelfand representation which will play a major role in proving the commutative Gelfand-Naimark Theorem in a few chapters time. Thanks to the work we have already done on characters and the character space, there isn't that much left for us to prove.

Definition 3.16. Let A be a Banach algebra with $\Omega(A) \neq \emptyset$, then for $x \in A$ the *Gelfand transform* of x is the map

$$\begin{aligned} \widehat{x} : \Omega(A) &\to \mathbb{C} \\ \varphi &\mapsto \varphi(x). \end{aligned}$$

Remark 3.17. The topology on $\Omega(A)$ is the restriction of the weak-* topology on A^* to the subset $\Omega(A) \subset A^*$, so it follows that this topology is the coarsest topology for which $\widehat{x}$ is continuous for every $x \in A$.

Theorem 3.18. (The Gelfand Representation)
Let A be a commutative Banach algebra such that $\Omega(A) \neq \emptyset$. The Gelfand Representation is the map

$$\begin{aligned} \Gamma : A &\to C_0(\Omega(A)) \\ x &\mapsto \widehat{x}. \end{aligned}$$

The Gelfand Representation enjoys the following properties,

(i) Γ is an algebra homomorphism, and $\|\Gamma(x)\|_\infty \leq \|x\|_A$ for every $x \in A$,
(ii) $r(x) = \|\widehat{x}\|_\infty$ for every $x \in A$,
(iii) If A is unital then for $x \in A$, $\sigma(x) = \widehat{x}(\Omega(A))$,
(iv) If A is non-unital then $\sigma(x) = \widehat{x}(\Omega(A)) \cup \{0\}$.

Proof. We shall begin by noting that if $x \in A$ and $\epsilon > 0$, the set

$$\{\varphi \in \Omega(A);\ |\widehat{x}(\varphi)| \geq \epsilon\} = \{\varphi \in \Omega(A);\ |\varphi(x)| \geq \epsilon\}$$

is a weak-* closed subset of $\overline{B_1(0)}$, so by the Banach-Alaoglu Theorem is weak-* compact and hence $\widehat{x} \in C_0(\Omega(A))$ for all $x \in A$.

For an arbitrary $x \in A$, Theorem 3.11 tells us that

$$\sigma(x) = \{\varphi(x);\ \varphi \in \Omega(A)\} = \widehat{x}(\Omega(A)),$$

if A is unital, and

$$\sigma(x) = \{\varphi(x);\ \varphi \in \Omega(A)\} \cup \{0\} = \widehat{x}(\Omega(A)) \cup \{0\},$$

if A is non-unital. It then follows that

$$r(x) = \sup\{|\varphi(x)|;\ \varphi \in \Omega(A)\} = \sup\{|\widehat{x}(\varphi)|;\ \varphi \in \Omega(A)\} = \|\widehat{x}\|_\infty.$$

Since we have the bound $r(x) \leq \|x\|$ if A is unital and $r(x) \leq \|x\|_{\widetilde{A}} = \|x\|_A$ if A is non-unital, it follows that $\|\widehat{x}\|_\infty \leq \|x\|$ so that Γ is a norm-decreasing map. Finally, the fact that Γ is an algebra homomorphism follows easily from the linear and multiplicative properties of characters. □

Remark 3.19. It is important to note that there are no claims about injectivity or surjectivity of the Gelfand representation. Indeed to see that it is not injective in the general case, we let A be the Banach algebra defined in Example 3.7, recalling that $\Omega(A)$ contains only one character which we denote here by τ. Considering the Gelfand transform of the (non-zero) matrix $x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we see that

$$\widehat{x}(\tau) = \tau(x) = 0,$$

so $\widehat{x} = 0$. Hence $\Gamma(x) = 0$ and so Γ is not injective.

Some authors call the character space of a Banach algebra A the *spectrum* of A, a convention which we have avoided in order to distinguish between the spectrum of an element and the character space of an algebra. We have already seen in Theorem 3.11 that there are links between the spectrum of an element and the characters on a Banach algebra, and the final result of this chapter provides another example of such links.

Theorem 3.20. *Let A be a unital Banach algebra and let $B \subseteq A$ be the Banach subalgebra generated by 1_A and an element x. Then the map*

$$\begin{aligned} \widehat{x} : \Omega(B) &\to \sigma(x) \\ \tau &\mapsto \tau(x), \end{aligned}$$

is a homeomorphism.

Proof. We note that B is commutative simply by its construction, we recall that continuity of $\widehat{x}$ was established in Remark 3.17 and that in Theorem 3.11 we proved that

$$\sigma(x) = \{\varphi(x);\ \varphi \in \Omega(B)\},$$

so $\widehat{x}$ is surjective. For injectivity, suppose that characters $\tau, \varphi \in \Omega(B)$ satisfy

$$\tau(x) = \widehat{x}(\tau) = \widehat{\varphi} = \varphi(x).$$

Now note that if $p(z)$ is any polynomial with complex coefficients, the assumption $\tau(x) = \varphi(x)$ and the fact that τ, φ are algebra homomorphisms implies that

$$\tau(p(x)) = \varphi(p(x)).$$

The set of polynomials in $\{1_A, x\}$ is dense in B, so it follows immediately that $\tau(y) = \varphi(y)$ for any $y \in B$ and therefore $\widehat{x}$ is injective. Once we remember that $\Omega(B)$ and $\sigma(x)$ are compact Hausdorff spaces, we see that $\widehat{x}$ is indeed a homeomorphism. $\square$

We have not presented many examples in this chapter, primarily because we are only going to start making use of this theory once we are talking about commutative C*-algebras. However this is not to say that this material doesn't have immediate applications, indeed it is used in commutative harmonic analysis, and we recommend Williams' notes [24, Chapter 1.4] for a brief introduction. Gelfand himself used this theory to give a slick proof of Wiener's Theorem about functions with absolutely convergent Fourier series, and Murphy provides a discussion of this [15, Example 1.3.1].

4. **Algebras With Involution**

We are now going to start to look at algebras which come equipped with involutions. We will be focussing our attention on a particular type of such algebras; C*-algebras and we shall gather all of the theory necessary in order to prove two major theorems due to Gelfand and Naimark. C*-algebras are a special type of Banach algebra, so the work we have done in previous chapters will carry over, for example we shall continue to make frequent use of the spectrum, and we shall revisit the Gelfand Representation when we discuss commutative C*-algebras.

4.1. *-algebras and Banach *-algebras.

In this section we present definitions which we need in order to introduce C*-algebras. In some textbooks these are just absorbed into the definition of a C*-algebra, however they are objects which are of interest on their own. For an encyclopaedic reference for these types of algebras, see the two volume work of Palmer [16].

Definition 4.1. Let A be an algebra. An *involution* on A is a map

$$\begin{aligned} {}^* : A &\to A \\ x &\mapsto x^* \end{aligned}$$

which satisfies the following, for all $x, y \in A$ and all $\lambda, \mu \in \mathbb{C}$,

(i) $(x^*)^* = x$,
(ii) $(\lambda x + \mu y)^* = \overline{\lambda} x^* + \overline{\mu} y^*$,
(iii) $(xy)^* = y^* x^*$.

An algebra which is equipped with an involution is called a *-algebra.

Remark 4.2. If A is a unital *-algebra, then $1_A = (1_A^*)^* = (1_A^* 1_A)^* = 1_A^* 1_A = 1_A^*$.

Example 4.3. The following are two easy examples of *-algebras.

(i) The complex numbers under complex conjugation,
(ii) $\mathbb{M}_n(\mathbb{C})$, where the involution is taking the conjugate-transpose of a matrix.

Definition 4.4. If A is a *-algebra and $S \subseteq A$ is a subset, then we define

$$S^* = \{x^*; \, x \in S\}.$$

If $S = S^*$ then we say that S is *self adjoint*. Observe that in order to prove that a subset $S \subseteq A$ is self adjoint, we only need to show that $x^* \in S$ for every $x \in S$.

Example 4.5. Let A be a *-algebra, then we make the following basic constructions.

(i) If $B \subseteq A$ is a self adjoint subalgebra, then B is called a *-subalgebra of A and is itself a *-algebra, where the involution is simply the involution on A restricted to B.
(ii) If $I \subseteq A$ is a self adjoint ideal, then A/I is a *-algebra, with involution given by $(x + I)^* = x^* + I$.
(iii) If A is a non-unital algebra, then we may define an involution on $\widetilde{A}$ by $(x, \lambda)^* = (x^*, \overline{\lambda})$. With this involution $\widetilde{A}$ is a *-algebra and A is a self adjoint ideal of $\widetilde{A}$.

Example 4.6. For a more involved construction, lets consider the following; suppose that A is a *-algebra and let $S \subseteq A$ be a self adjoint subset. We define the *commutant* of S by

$$S' = \{x \in A; \, xy = yx \text{ for all } y \in S\}.$$

We claim that S' is a *-subalgebra of A. First note that S' is certainly non-empty, then for $x_1, x_2 \in S'$ and $\lambda, \mu \in \mathbb{C}$ we have

$$(\lambda x_1 + \mu x_2)y = \lambda x_1 y + \mu x_2 y = \lambda y x_1 + \mu y x_2 = y(\lambda x_1 + \mu x_2),$$

for all $y \in S$, so S' is a vector subspace of A. Moreover we have

$$(x_1 x_2)y = x_1(y x_2) = (y x_1)x_2,$$

and

$$x_1^* y = (y^* x_1)^* = (x_1 y^*)^* = y x_1^*,$$

for all $y \in S$, where we have used the fact that S is self adjoint in the second equality. Therefore S' is indeed a *-subalgebra of A.

Now assume that A is equipped with a submultiplicative norm and consider a convergent sequence (x_n) in S' with limit $x \in A$. We fix a choice of $y \in S$ and for $n \in \mathbb{N}$ we make the estimate

$$\begin{aligned}\|xy - yx\| &= \|xy - x_n y + x_n y - yx\| \\ &\leq \|xy - x_n y\| + \|x_n y - yx\| \\ &\leq \|xy - x_n y\| + \|y x_n - yx\| \\ &\leq 2\|y\|\|x - x_n\|.\end{aligned}$$

The convergence of (x_n) to x means that we can make the quantity $\|x - x_n\|$ arbitrarily small, showing that $\|xy - yx\| = 0$ or rather $xy = yx$. Hence $x \in S'$ and S' is closed.

Definition 4.7. If A and B are *-algebras, then an algebra homomorphism $\varphi : A \to B$ is called a *-homomorphism if for all $x \in A$,

$$\varphi(x^*) = \varphi(x)^*.$$

If φ is also a bijection then we call it a *-isomorphism. We observe that if $\varphi : A \to B$ is a *-homomorphism, then $\varphi(A)$ is a self adjoint subalgebra of B and $\ker(\varphi)$ is a self adjoint ideal of A.

Example 4.8. If A is any *-algebra and $I \subseteq A$ is a self adjoint ideal, then consider the quotient map $\mathcal{Q} : A \to A/I$ defined in Example 2.28. For any $x \in A$ we have

$$\mathcal{Q}(x^*) = x^* + I = (x + I)^* = \mathcal{Q}(x)^*,$$

so that $\mathcal{Q}$ is a *-homomorphism.

Definition 4.9. If A is a *-algebra, then we define the following special types of element.

(i) $x \in A$ is called *self adjoint* if $x^* = x$,
(ii) $x \in A$ is called *normal* if $x^*x = xx^*$,
(iii) $x \in A$ is called *a projection* if $x^2 = x^* = x$.

If additionally A is unital, then we make a further definition,

(iv) $x \in A$ is *unitary* if $x^*x = xx^* = 1_A$.

As an exercise in playing around with our new definitions, we present the following examples.

Example 4.10. Let A be a (non-zero) *-algebra, then A always contains self adjoint elements. For example if $x \in A$ is an arbitrary non-zero element, then $x^*x \in A$ is self adjoint. Also $a := \frac{1}{2}(x + x^*)$ and $b := \frac{1}{2i}(x - x^*)$ are self adjoint elements of A with the property that $x = a + ib$.

Example 4.11. Let $\varphi : A \to B$ be a surjective *-homomorphism between *-algebras. If $y \in B$ is self adjoint, then by surjectivity the preimage $\varphi^{-1}(y)$ is non-empty, say with $x \in \varphi^{-1}(y)$. Because φ is a *-homomorphism, we have

$$\varphi(x^*) = \varphi(x)^* = y^* = y,$$

so that $x^* \in \varphi^{-1}(y)$, proving that the preimage $\varphi^{-1}(y)$ is self adjoint.

Example 4.12. Let A be a unital *-algebra and $x \in A$ any element, in this example we consider the spectrum and resolvent set of x^*. Take $\mu \in \mathrm{Res}(x^*)$, then $\mu - x^* \in \mathrm{Inv}(A)$ and we shall denote its inverse by y. This element $y \in A$ is a right inverse for $\mu - x^*$, so satisfies $(\mu - x^*)y = 1_A$ and we expand this to give

$$\mu y - x^* y = 1_A.$$

Applying the involution to both sides yields

$$\overline{\mu} y^* - y^* x = 1_A.$$

and once we bracket the left hand side as $y^*(\overline{\mu} - x)$, we see that y^* is a left inverse for $\overline{\mu} - x$. An analogous argument shows that y^* is also a right inverse for $\overline{\mu} - x$, meaning that $\overline{\mu} \in \mathrm{Res}(x)$ and we conclude that $\mathrm{Res}(x^*) \subseteq \{\overline{\lambda};\ \lambda \in \mathrm{Res}(x)\}$.

On the other hand, take $\overline{\mu} \in \{\overline{\lambda};\ \lambda \in \mathrm{Res}(x)\}$, then $\overline{\mu} - x \in \mathrm{Inv}(A)$ with inverse $y \in A$. Similarly to the above we can show that y^* is the inverse of $\mu - x$. This proves that we have the relations

$$\mathrm{Res}(x^*) = \{\overline{\lambda};\ \lambda \in \mathrm{Res}(x)\},$$
$$\sigma(x^*) = \{\overline{\lambda};\ \lambda \in \sigma(x)\}.$$

Definition 4.13. A *-algebra A is called a *Banach *-algebra* if it is equipped with a norm which makes it a Banach algebra and additionally satisfies

$$\|x^*\| = \|x\|,$$

for all $x \in A$. If A has a unit 1_A with $\|1_A\| = 1$, then A is a unital Banach *-algebra.

Remark 4.14. Requiring that the involution on a Banach *-algebra is isometric (and hence continuous) is by no means standard, for example Arveson [3, Definition 2.5.1] and Murphy [15, Chapter 2, p36] include this in their definitions but Palmer [16, Chapter 11.1] does not.

4.2. **C*-algebras.** For the remainder of this thesis, we shall only be interested in C*-algebras. These algebras arose from work in the early 20$^{\text{th}}$ century to provide a precise mathematical formulation for quantum mechanics and as such there is an overlap with some of the terminology. For a detailed historical background we recommend [13, Chapter 1.1].

The actual definition of a C*-algebra is fairly innocuous and from practically the moment we introduce them they prove to be quite exciting objects. We shall first introduce the definition and then proceed to spend the rest of this chapter exploring some elementary results and constructions. This is all preparation for the following chapter, where we shall start to look at the Gelfand-Naimark Theorems.

Definition 4.15. A non-empty Banach algebra A is called a C*-algebra if it is equipped with an involution which satisfies

$$\|x^*x\| = \|x\|^2,$$

for every $x \in A$. This equality is referred to as the *C*-identity.* If A is a C*-algebra and $B \subseteq A$ is a closed self adjoint subalgebra, then B is also a C*-algebra and we shall call such subalgebras *C*-subalgebras.*

Remark 4.16. If A is a unital C*-algebra, then the C*-identity shows that $\|1_A\|^2 = \|1_A^* 1_A\| = \|1_A\|^2$, so $\|1_A\| = 1$.

Before we look at examples of C*-algebras, we present a couple of lemmas. These are easy to deduce but are quite useful nevertheless.

Lemma 4.17. *If A is a C*-algebra, then A is a Banach *-algebra.*

Proof. Since A is a C^*-algebra, its norm is submultiplicative, so for $x \in A$ we have $\|x\|^2 = \|x^*x\| \leq \|x^*\|\|x\|$ and hence $\|x\| \leq \|x^*\|$. Because $x^{**} = x$, we also have $\|x^*\|^2 = \|(x^*)^*x^*\| = \|xx^*\| \leq \|x\|\|x^*\|$ and therefore $\|x^*\| = \|x\|$, so A is also a Banach *-algebra. ☐

Remark 4.18. It is important to realise that the converse to Lemma 4.17 is false; not every Banach *-algebra is a C*-algebra. We shall give a counter example but this is postponed until after the commutative Gelfand-Naimark Theorem.

Lemma 4.19. *If A is a Banach algebra equipped with an involution which satisfies $\|x\|^2 \leq \|x^*x\|$ for all $x \in A$, then A is a C*-algebra.*

Proof. Let $x \in A$ be arbitrary, we must show that the inequality $\|x^*x\| \leq \|x\|^2$ is true. Since A is a Banach algebra, submultiplicativity gives us

$$\|x^*x\| \leq \|x^*\|\|x\|.$$

Now we replace x by x^* to see that

$$\|x^*\|^2 \leq \|(x^*)^*x^*\| = \|xx^*\| \leq \|x\|\|x^*\|.$$

Notice that if $x = 0$ then the inequality $\|x^*x\| \leq \|x\|^2$ is trivially satisfied, so we may assume that $x \neq 0$ and divide the inequality $\|x^*\|^2 \leq \|x\|\|x^*\|$ by $\|x^*\|$ to obtain $\|x^*\| \leq \|x\|$. This means that $\|x^*x\| \leq \|x\|^2$ and once combined with the assumption that $\|x\|^2 \leq \|x^*x\|$, we get the C*-identity. □

We shall now give the archetypical examples of C*-algebras.

Example 4.20. Let $\mathcal{H}$ be a Hilbert space, then it follows from Example 2.10 that $B(\mathcal{H})$ is a Banach algebra under the operator norm. Recall that if $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, there exists a unique map $T^* \in B(\mathcal{H})$ called the *adjoint* of T which satisfies

$$\langle \nu, T\eta \rangle = \langle T^*\nu, \eta \rangle,$$

for all $\nu, \eta \in \mathcal{H}$. It may be checked that the correspondence $T \mapsto T^*$ is an involution on $B(\mathcal{H})$ and we shall show that the C*-identity is satisfied. Taking any operator $T \in B(\mathcal{H})$, if $\nu \in \mathcal{H}$ satisfies $\|\nu\| = 1$ then by the Cauchy-Schwarz inequality

$$\|T\nu\|^2 = \langle T\nu, T\nu \rangle = \langle T^*T\nu, \nu \rangle \leq \|T^*T\nu\|\|\nu\| \leq \|T^*T\|_{\text{op}}.$$

From this it follows that $\|T\|_{\text{op}}^2 \leq \|T^*T\|_{\text{op}}$ and we apply Lemma 4.19 to conclude that $B(\mathcal{H})$ is a C*-algebra.

Example 4.21. For $n \in \mathbb{N}$, we may view $\mathbb{M}_n(\mathbb{C})$ as $B(\mathbb{C}^n)$, so in light of the previous example $\mathbb{M}_n(\mathbb{C})$ is a C*-algebra.

Example 4.22. We let X be a locally compact Hausdorff space and consider the Banach algebra of continuous functions which vanish at infinity. An involution may be defined on this algebra via $f^* = \overline{f}$ for $f \in C_0(X)$, where $\overline{f}(x) = \overline{f(x)}$ for all $x \in X$. The C*-norm is satisfied because for any $f \in C_0(X)$, we have that

$$\|f^*f\|_\infty = \|\overline{f}f\|_\infty = \|f^2\|_\infty = \sup_{x \in X} |f^2(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|_\infty^2.$$

Our next two examples provide ways in which we can construct new C*-algebras from old ones we have lying around.

Example 4.23. If A and B are C*-algebras then we can make their vector space direct sum $A \oplus B$ into a C*-algebra. Define multiplication and involution by

$$(a, b)(c, d) = (ac, bd),$$
$$(a, b)^* = (a^*, b^*),$$

for $(a, b), (c, d) \in A \oplus B$. It's easy to check that these operations satisfy the required properties and it is similarly easy to check that the following defines a submultiplicative norm on $A \oplus B$,

$$\|(a, b)\| = \max\{\|a\|_A, \|b\|_B\}.$$

To show that this norm satisfies the C*-identity we take $(a, b) \in A \oplus B$ and a direct calculation yields

$$\|(a, b)^*(a, b)\| = \|(a^*a, b^*b)\| = \max\{\|a^*a\|_A, \|b^*b\|_B\} = \max\{\|a\|_A^2, \|b\|_B^2\} = \|(a, b)\|^2,$$

where $(a, b) \in A \oplus B$. To prove completeness requires a bit more effort and we shall give all the details. If $((a_n, b_n))$ is a Cauchy sequence in $A \oplus B$, then for any $\epsilon > 0$ there is $N \in \mathbb{N}$ so that

$$\epsilon > \|(a_n, b_n) - (a_m, b_m)\| = \max\{\|a_n - a_m\|_A, \|b_n - b_m\|_B\},$$

whenever $m, n > N$. This implies that (a_n) and (b_n) are Cauchy sequences in A and B respectively. Completeness of A and B means there is $a \in A$ and $b \in B$ so that $a_n \to a$, $b_n \to b$ and we claim that $(a_n, b_n) \to (a, b)$. Taking $\epsilon > 0$ arbitrary, we can choose $N \in \mathbb{N}$ so that

$$m, n > N \Rightarrow \|(a_n, b_n) - (a_m, b_m)\| < \frac{\epsilon}{2}.$$

We use the fact that $a_n \to a$ to choose $N_a \in \mathbb{N}$ such that

$$p > N_a \Rightarrow \|a_p - a\|_A < \frac{\epsilon}{2}.$$

Then for $n > \max\{N, N_a\}$ we have

$$\|a_n - a\|_A \leq \|a_n - a_{n+1}\|_A + \|a_{n+1} - a\|_A < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We carry out an identical procedure for the sequence (b_n) and conclude that for any $\epsilon > 0$ we can find $N \in \mathbb{N}$ so that

$$n > N \Rightarrow \epsilon > \max\{\|a_n - a\|_A, \|b_n - b\|_B\} = \|(a_n, b_n) - (a_m, b_m)\|.$$

Example 4.24. Recall from Example 2.16 how we constructed the Banach subalgebra generated by a set. If A is a C*-algebra and $S \subseteq A$, then we can perform an analogous construction to give the *C*-subalgebra of A generated by S*. Specifically, we let $\mathrm{C}^*(S)$ denote the intersection of all C*-subalgebras of A, or equivalently the norm-closure of the set of polynomials in $S \cup S^*$. Again, if S has the property that $xy = yx$ for every pair $x, y \in S$, then $\mathrm{C}^*(S)$ is commutative.

One particular example of this construction which we shall use frequently is to take the C*-subalgebra of a unital C*-algebra which is generated by the identity and a self adjoint element. If A was non-commutative then this gives us a commutative C*-subalgebra.

4.3. **Revisiting the Spectrum and Spectral Radius.** The spectrum continues to be a valuable tool for us and we shall begin this section by proving a few results regarding the spectra of projection, unitary and self adjoint elements in a C*-algebra. Afterwards we shall use the spectral radius formula derived in Chapter 2 to prove a remarkable result; the norm on a C∗-algebra is unique.

Lemma 4.25. *Let A be a C*-algebra.*

(i) If $x \in A$ is a projection, then $\sigma(x) \subseteq \{0, 1\}$,
(ii) If additionally A is unital and $x \in A$ is unitary, then $\sigma(u) \subseteq \mathbb{T}$.

Proof. (i) By definition, x satisfies $x^2 - x = 0$ and we use the spectral mapping property for polynomials to see that

$$0 = \sigma(x^2 - x) = \{\lambda^2 - \lambda;\, \lambda \in \sigma(p)\}.$$

Now observe that the solutions of the quadratic $\lambda^2 - \lambda = 0$ are precisely $\lambda = 0, 1$.

(ii) The C*-identity gives us $1 = \|1_A\| = \|x^*x\| = \|x\|^2$ and hence $\|x\| = 1$. Take any complex number $\lambda \in \sigma(x)$. Then if $\lambda^{-1} - x^*$ has inverse $y \in A$, we have

$$(\lambda - x)(-\lambda^{-1}x^*y) = -x^*y + \lambda^{-1}y = (\lambda^{-1} - x^*)y = 1_A,$$

so $\lambda - x$ is right invertible, and

$$(-\lambda^{-1}yx^*)(\lambda - x) = -yx^* + \lambda^{-1}y = y(\lambda^{-1} - x^*) = 1_A,$$

so $\lambda - x$ is left invertible, hence invertible.

Because $\lambda \in \sigma(x)$, we have the inequality $|\lambda| \leq \|x\| = 1$ and he above means that we have $\lambda^{-1} \in \sigma(x^*)$, so the inequality $|\lambda^{-1}| \leq \|x^*\| = \|x\| = 1$ holds true. This means that $|\lambda| = 1$ and therefore $\lambda \in \mathbb{T}$.

□

Now we want to prove that the spectrum of a self adjoint element in a C*-algebra is real, this will be more involved than the previous lemma and we must first prove a supplementary result.

Definition 4.26. Let A be a unital Banach algebra. We define a map exp: $A \to A$ by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Remark 4.27. With A and x as above, for fixed $k \in \mathbb{N}$ we have

$$\sum_{n=0}^{k} \left\| \frac{x^n}{n!} \right\| \leq \sum_{n=0}^{k} \frac{\|x\|^n}{n!} \leq e^{\|x\|} < \infty,$$

where e denotes the usual exponential function on $\mathbb{R}$. Therefore the series defining $\exp(x)$ converges absolutely, hence converges by completeness of A.

If A is a unital Banach algebra and $x, y \in A$ satisfy $xy = yx$, then one can show via manipulating the series that $\exp(x+y) = \exp(x)\exp(y)$. We may find details of this argument in [1, Proposition 4.98(i)].

Lemma 4.28. *Let A be a unital C*-algebra and let $x \in A$ be self adjoint. Then $\exp(ix)$ is unitary.*

Proof. First we shall show that $\exp(ix)^*$=$\exp(-ix)$. Fixing $k \in \mathbb{N}$, we have

$$\left(\sum_{n=0}^{k} \frac{(ix)^n}{n!} \right)^* = \sum_{n=0}^{k} \frac{(-i)^n (x^n)^*}{n!} = \sum_{n=0}^{k} \frac{(-ix)^n}{n!}.$$

The involution on A is continuous, so taking the limit $k \to \infty$ yields

$$\left(\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \right)^* = \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!},$$

so that $\exp(ix)^*$=$\exp(-ix)$. Since ix and $-ix$ commute, it now follows that

$$\exp(ix)^*\exp(ix) = \exp(-ix)\exp(ix) = \exp(0) = 1_A.$$

□

Theorem 4.29. *Let A be a C*-algebra and $x \in A$ a self adjoint element. Then $\sigma(x) \subseteq \mathbb{R}$.*

Proof. Recall that if A is non-unital, then for $x \in A$ we define $\sigma_A(x) = \sigma_{\widetilde{A}}(x)$, so we may safely assume that A is unital. We take any self adjoint $x \in A$, then because $\exp(ix)$ is unitary it follows that $\sigma(\exp(ix)) \subseteq \mathbb{T}$. Now take $\lambda \in \sigma(x)$ and put $y = \sum_{n=1}^{\infty} \frac{i^n (x - \lambda 1_A)^{n-1}}{n!}$. We shall denote by e the usual exponential function on $\mathbb{C}$, noting that $e^{i\lambda}1_A = \exp(i\lambda 1_A)$. Then

$$\begin{aligned}
\exp(ix) - e^{i\lambda}1_A &= \exp(i(x - \lambda 1_A))e^{i\lambda}1_A - e^{i\lambda}1_A \\
&= (\exp(i(x - \lambda 1_A)) - 1_A)e^{i\lambda}1_A \\
&= e^{i\lambda}1_A \left(\sum_{n=0}^{\infty} \frac{(i(x - \lambda 1_A))^n}{n!} - 1_A \right) \\
&= e^{i\lambda}1_A \left(\sum_{n=1}^{\infty} \frac{(i(x - \lambda 1_A))^n}{n!} \right) \\
&= e^{i\lambda}1_A(x - \lambda 1_A)y.
\end{aligned}$$

Since $x - \lambda 1_A$ is not invertible, it follows that $\exp(ix) - e^{i\lambda}1_A$ is not invertible, so $e^{i\lambda} \in \sigma(\exp(ix))$. But because $\exp(ix)$ is unitary, we must have $e^{i\lambda} \in \mathbb{T}$ and this implies that $\lambda \in \mathbb{R}$. □

Now we shall prove a couple of useful relations between the spectral radius and the norm of particular elements in a C*-algebra.

Lemma 4.30. *Let A be a C*-algebra and $x \in A$ a self adjoint element. Then $r(x) = \|x\|$.*

Proof. Clearly $\|x^2\| = \|x^*x\| = \|x\|^2$ and if $\|x^{2^{n-1}}\| = \|x\|^{2^{n-1}}$ for all $n \in \mathbb{N}$ then we have

$$\|x^{2^n}\| = \|(x^{2^{n-1}})^2\| = \|(x^{2^{n-1}})^* x^{2^{n-1}}\| = \|x^{2^{n-1}}\|^2 = \|x\|^{2^n},$$

so by induction $\|x^{2^n}\| = \|x\|^{2^n}$ for all $n \in \mathbb{N}$. Then it follows that

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|x\|^{\frac{2^n}{2^n}} = \|x\|.$$

□

Example 4.31. Let $x = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$, then x is self adjoint and because

$$\begin{vmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda(\lambda - 2),$$

it follows that the eigenvalues of x are 0 and 2, so $\sigma(x) = \{0, 2\}$. Using the lemma which we just proved, we see that

$$\|x\| = r(x) = \sup\{|\lambda|;\ \lambda \in \sigma(x)\} = 2.$$

Corollary 4.32. *Let A be a C*-algebra and $x \in A$ a normal element. Then $r(x) = \|x\|$.*

Proof. For $n \in \mathbb{N}$, we have by the C*-identity that

$$\|x\|^{2^{n+1}} = \|x^*x\|^{2^n}.$$

Because x^*x is self adjoint, the previous proof tells us that

$$\|x^*x\|^{2^n} = \|(x^*x)^{2^n}\|,$$

from which it follows that

$$\|x\|^{2^{n+1}} = \|(x^*x)^{2^n}\| = \|(x^{2^n})^* x^{2^n}\| = \|x^{2^n}\|^2.$$

We now conclude that $r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{\frac{1}{2^n}} = \|x\|$. □

Corollary 4.33. *If A is a *-algebra then there is at most one norm on A which makes it a C*-algebra.*

Proof. Take any *-algebra A and suppose that we have two submultiplicative norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on A, which satisfy the C*-identity and under which A is complete. Take any element $x \in A$, then because x^*x is self adjoint, it follows from Lemma 4.30 that

$$\|x\|_1^2 = \|x^*x\|_1 = r(x^*x) = \sup\{|\lambda|;\ \lambda \in \sigma(x^*x)\},$$
$$\|x\|_2^2 = \|x^*x\|_2 = r(x^*x) = \sup\{|\lambda|;\ \lambda \in \sigma(x^*x)\}.$$

Therefore $\|x\|_1 = \|x\|_2$ for all $x \in A$. □

Remark 4.34. Not only is the statement of Corollary 4.33 surprising, but so too is the way in which we prove it. Using the C*-identity we show that the norm of any element x is a C*-algebra depends only on the spectral radius of x^*x, which is something we define independently of the norm!

4.4. **The Unitisation of a C*-algebra.** Suppose that A is a non-unital C*-algebra. For the time being, just consider A as a Banach algebra and recall that the vector space $\widetilde{A} = A \oplus \mathbb{C}$ is given the structure of a Banach algebra via the multiplication and norm

$$(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu),$$
$$\|(x, \lambda)\| = \|x\|_A + |\lambda|,$$

where $(x, \lambda), (y, \mu) \in \widetilde{A}$. We may define an involution on $\widetilde{A}$ by

$$(x, \lambda)^* = (x^*, \overline{\lambda}),$$

and this satisfies

$$\|(x, \lambda)^*\| = \|x^*\|_A + |\overline{\lambda}| = \|x\|_A + |\lambda| = \|(x, \lambda)\|,$$

so that $\widetilde{A}$ is a Banach *-algebra. Unfortunately though, $\widetilde{A}$ is not a C*-algebra under this norm as we will now show. If $0 \neq x \in A$, then

$$\|(x^*x, i)^*(x^*x, i)\| = \|(x^*xx^*x + ix^*x + \overline{i}x^*x, i\overline{i})\| = \|x^*xx^*x\|_A + 1 = \|x^*x\|_A^2 + 1,$$

but on the other hand

$$\|(x^*x, i)\|^2 = (\|x^*x\|_A + |i|)^2 = (\|x^*x\|_A + 1)^2 = \|x^*x\|^2 + 2\|x^*x\| + 1.$$

Because $x \neq 0$ these quantities are not equal. It is possible for us to put a norm on $\widetilde{A}$ which makes it a C*-algebra, but the definition is a bit more involved.

First we recall that for any $x \in A$ we may define a map $L_x \in B(A)$ by $L_x(y) = xy$, for each $y \in A$ and the correspondence $x \mapsto L_x$ is an injective homomorphism between A and $B(A)$. For $x \in A$, submultiplicativity of the norm gives us the inequality $\|L_x\|_{\text{op}} \leq \|x\|_A$ and for $y = \frac{x^*}{x}$, we have $\|y\|_A = 1$ because the involution on a C*-algebra is isometric. Therefore,

$$\|L_x(y)\|_A = \frac{\|xx^*\|_A}{\|x\|_A} = \frac{\|(x^*x)^*\|_A}{\|x\|_A} = \frac{\|x\|_A^2}{\|x\|_A} = \|x\|_A,$$

which shows that $\|L_x\|_{\text{op}} = \|x\|_A$.

Theorem 4.35. *If A is a non-unital C*-algebra, then we define*

$$A^L = \{L_x + \lambda I;\ x \in A,\ \lambda \in \mathbb{C}\} \subseteq B(A),$$

where I denotes the identity operator. With involution defined for $L_x + \lambda I \in A^L$ by

$$(L_x + \lambda I)^* = L_{x^*} + \overline{\lambda} I,$$

the set A^L is a unital C-algebra under the operator norm. Moreover the map $\widetilde{A} \to A^L$ sending $(x, \lambda) \mapsto L_x + \lambda I$ is a *-isomorphism.*

Proof. Because we have already shown that $x \mapsto L_x$ is a homomorphism between A and $B(A)$, checking that A^L is a subalgebra of $B(A)$ is easy. It certainly contains the zero operator so is non empty. Now we take operators $L_x + \lambda I, L_y + \mu I \in A^L$, scalars $\eta, \xi \in \mathbb{C}$ and compute,

$$\begin{aligned} \eta(L_x + \lambda I) + \xi(L_y + \mu I) &= \eta L_x + \xi L_y + (\eta\lambda + \xi\mu)I \\ &= L_{\eta x + \xi y} + (\eta\lambda + \xi\mu)I, \end{aligned}$$

which belongs to A^L and

$$\begin{aligned} (L_x + \lambda I)(L_y + \mu I) &= L_x(L_y + \mu I) + \lambda I(L_y + \mu I) \\ &= L_{xy} + \mu L_x + \lambda L_y + \lambda L_y + \lambda\mu I \\ &= L_{xy + \mu x + \lambda y} + (\lambda\mu I), \end{aligned}$$

which also belongs to A^L. Now we observe that

(1) The map $x \mapsto L_x$ is an isometry between complete metric spaces, so its image is closed in $B(A)$,
(2) $\{\lambda I;\ \lambda \in \mathbb{C}\}$ is a finite dimensional subspace of $B(A)$,

and since A^L is the sum of $\{L_x;\ x \in A\}$ and $\{\lambda I;\ \lambda \in \mathbb{C}\}$ we apply [6, Chapter III, Proposition 4.3] to conclude that A^L is closed, hence is complete.

Checking that the involution which we defined satisfies the necessary properties is very easy, so we omit the details and this just leaves us with the task of checking that the C*-identity is satisfied. By Lemma 4.19 it is sufficient to check that the inequality $\|L_x + \lambda I\|_{\text{op}}^2 \leq \|(L_x + \lambda I)^*(L_x + \lambda I)\|_{\text{op}}$, holds for every $L_x + \lambda I \in A^L$. We do this by direct computation,

$$\begin{aligned} \|L_x + \lambda I\|_{\text{op}}^2 &= \sup_{\|y\| \leq 1} \|xy + \lambda y\|_A^2 \\ &= \sup_{\|y\| \leq 1} \|(xy + \lambda y)^*(xy + \lambda y)\|_A \\ &= \sup_{\|y\| \leq 1} \|y^*x^*xy + \overline{\lambda} y^*xy + \lambda y^*x^*y + |\lambda|^2 y^*y\|_A \\ &= \sup_{\|y\| \leq 1} \|y^*(L_{x^*} + \overline{\lambda} I)(xy + \lambda y)\|_A \\ &= \sup_{\|y\| \leq 1} \|y^*(L_x + \lambda I)^*(L_x + \lambda I)y\|_A \\ &\leq \sup_{\|y\| \leq 1} \|(L_x + \lambda I)^*(L_x + \lambda I)y\|_A \\ &\leq \|(L_x + \lambda I)^*(L_x + \lambda I)\|_{\text{op}}, \end{aligned}$$

and this completes the proof that A^L is a C*-algebra. Finally we consider the map $\widetilde{A} \to A^L$ given by $(x, \lambda) \mapsto L_x + \lambda I$. Simple calculations proves that the map is an algebra homomorphism and it is equally easy to verify that it is a *-homomorphism. Moreover the map is clearly surjective and if we suppose that $L_x + \lambda I = 0$ for some $(x, \lambda) \in \widetilde{A}$ then we must have $xy + \lambda y = 0$ for every $y \in A$. Then if $\lambda \neq 0$ we may replace y by $-\frac{y}{\lambda}$ to give

$$y\left(-\frac{x}{\lambda}\right) = 1_A,$$

which implies that $-\frac{x}{\lambda}$ is a right unit[7] in A. Applying the involution to the above gives us

$$\left(-\frac{x}{\lambda}\right)^* y^* = 1_A,$$

which shows that $\left(-\frac{x}{\lambda}\right)^*$ is a left unit in A and hence that A is unital, which is of course a contradiction. So it must be the case that $\lambda = 0$ and of course the equality $xy = 0$ for all $y \in A$ holds if and only if $x = 0$, thus proving that $(x, \lambda) \mapsto L_x + \lambda I$ is injective and hence is an *-isomorphism.

By defining $\|(x, \lambda)\| = \|L_x + \lambda I\|_{\text{op}}$ for all $(x, \lambda) \in \widetilde{A}$, it follows that $\widetilde{A}$ is a C*-algebra. $\square$

Example 4.36. Suppose that X is a non-compact locally compact Hausdorff space. Recall that we may identify $C_0(X)$ with the maximal ideal $I(\{\infty\}) \subseteq C(\widehat{X})$ of continuous functions on $\widehat{X}$ which are zero at the point ∞. We define a map $\varphi : \widetilde{C_0(X)} \to C(\widehat{X})$ by

$$\varphi((f, \lambda)) = f + \lambda,$$

where on the right hand side we are implicitly identifying $f \in C_0(X)$ with the corresponding function in $I(\{\infty\})$ and we are writing λ for the function which takes constant value λ. This is a linear map, and if $(f, \lambda), (g, \mu) \in \widetilde{C_0(X)}$ then

$$\varphi((f, \lambda)(g, \mu)) = fg + \lambda g + \mu f + \lambda\mu = (f + \lambda)(g + \mu),$$

so φ is a homomorphism and

$$\varphi((f, \lambda)^*) = \overline{f} + \overline{\lambda},$$

so it is in fact a *-homomorphism. We observe that for $(f, \lambda) \in \widetilde{C_0(X)}$,

$$\varphi((f, \lambda)) = 0 \iff f = -\lambda,$$

and because $f \in C_0(X)$, this is the case if and only if $f = 0$. If $f \in C(\widehat{X})$, then the function $f - f(\infty)$ belongs to $C(\widehat{X})$ and satisfies $(f - f(\infty))(\infty) = 0$. Proposition 1.20 tells us that the restriction of $f - f(\infty)$ to X belongs to $C_0(X)$ and so

$$\varphi((f - f(\infty), f(\infty)) = f - f(\infty) + f(\infty) = f,$$

so φ is a bijection. Once we have read through the next section we shall see that this *-isomorphism is automatically isometric.

4.5. ***-homomorphisms Between C*-algebras.** Now that we have dealt with the (slightly messy) business of unitizing a C*-algebra, we can present the following results which highlight the very pleasing automatic continuity which *-homomorphisms between C*-algebras posses.

Proposition 4.37. *Let $\varphi : A \to B$ be a unital *-homomorphism between unital C*-algebras, then for every $x \in A$ we have $\sigma_B(\varphi(x)) \subseteq \sigma_A(x)$ and moreover φ is continuous with $\|\varphi\|_{op} \leq 1$.*

Proof. First assume that $x \in A$ is invertible, then $\varphi(x)\varphi(x^{-1}) = 1_B = \varphi(x^{-1})\varphi(x)$ and hence $\varphi(x)$ is invertible in B. Now let x be an arbitrary element of A and take $\lambda \in \text{Res}_A(x)$, then we have

$$\lambda\varphi(1_A) - \varphi(x) = \varphi(\lambda 1_A - x) \in \text{Inv}(B),$$

so $\lambda \in \text{Res}_B(\varphi(x))$. The inclusion $\text{Res}_A(x) \subseteq \text{Res}_B(\varphi(x))$ is of course equivalent to $\sigma_B(\varphi(x)) \subseteq \sigma_A(x)$, which proves our first statement and gives us the inequality $r_B(\varphi(x)) \leq r_A(x)$.

[7]An element r in a unital algebra A is called a *right unit* if $xr = x$ for every $x \in A$. Similarly we define a *left unit* and its easy to see that if both a left and right unit exist then they are equal, so that A is unital.

With $x \in A$ still arbitrary, now we consider the self adjoint elements $x^*x \in A$ and $\varphi(x)^*\varphi(x) \in B$. Using a combination of the C*-identity, Lemma 4.30, multiplicativity of homomorphisms and the previous paragraph we make the following estimate,

$$\|\varphi(x)\|^2 = \|\varphi(x)^*\varphi(x)\| = r_B(\varphi(x^*x)) \le r_A(x^*x) = \|x^*x\| = \|x\|^2,$$

and this completes the proof. □

By using the untization procedure which we detailed in the previous section we shall now strengthen this proposition.

Corollary 4.38. *Let $\varphi : A \to B$ be a *-homomorphism between any two C*-algebras A and B. Then φ is continuous and satisfies $\|\varphi\|_{op} \le 1$.*

Proof. First assume that A has a unit. We consider the closure of $\varphi(A)$ in B, which is a C*-subalgebra of B and contains $\varphi(A)$ as a dense subset. The element $\varphi(1_A)$ is easily checked to be a unit in $\varphi(A)$ and it is not difficult to show that $\varphi(1_A)$ is also the unit of $\overline{\varphi(A)}$. It follows that $\varphi : A \to \overline{\varphi(A)}$ is a unital *-homomorphism between unital C*-algebras, so the previous proposition now guarantees that φ satisfies $\|\varphi\|_{\mathrm{op}} \le 1$.

Now assume that A has no unit. If necessary we pass to the unitization of B, in order to assume that B has a unit. We define a map $\widetilde{\varphi} : \widetilde{A} \to B$ by

$$\widetilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda 1_B,$$

for each $(x, \lambda) \in \widetilde{A}$, which is a unital *-homomorphism and hence is continuous by the previous proposition. Moreover $\widetilde{\varphi}$ extends φ, so for any $x \in A$ we have that

$$\|\varphi(x)\|_B = \|\widetilde{\varphi}((x, 0))\|_{\widetilde{B}} \le \|(x, 0)\|_{\widetilde{A}} = \|x\|_A,$$

so once more φ satisfies $\|\varphi\|_{\mathrm{op}} \le 1$. □

Corollary 4.39. *If $\varphi : A \to B$ is a *-isomorphism of C*-algebras, then then φ is isometric.*

Proof. We apply Corollary 4.38 to φ, giving $\|\varphi(x)\|_B \le \|x\|_A$ for all $x \in A$. Then apply the same corollary to φ^{-1}, giving the inequality

$$\|\varphi^{-1}(y)\|_A \le \|y\|_B,$$

for all $y \in B$. We note that every $x \in A$ may be written as $\varphi^{-1}(y)$ for some $y \in B$, so it follows that

$$\|x\|_A = \|\varphi^{-1}(y)\|_A \le \|y\|_B = \|\varphi(x)\|_B.$$

Combining these estimates gives the result. □

Remark 4.40. Suppose that $\varphi : A \to B$ is an injective *-homomorphism of C*-algebras. If we just view A as a *-algebra we see that $\varphi : A \to \varphi(A)$ is a *-isomorphism. Naively we may now attempt to apply Corollary 4.39 to $\varphi : A \to \varphi(A)$ and deduce that φ is isometric. However the problem with this conclusion is that we do not know whether or not the image of φ is closed. It will turn out that an injective *-homomorphism of C*-algebras is automatically isometric but we must do some more work before we can prove this.

Example 4.41. Let X and Y be compact Hausdorff spaces and let $h : Y \to X$ be a continuous map. We define a map $\phi_h : C(X) \to C(Y)$ by

$$\phi_h(f) = f \circ h,$$

for every $f \in C(X)$. One easily checks that ϕ_h is a *-homomorphism, for example multiplicativity holds because for $f, g \in C(X)$ and any $x \in X$ we have

$$\phi_h(fg)(x) = ((fg) \circ h)(x) = (f \circ h)(x)(g \circ h)(x) = \phi_h(f)(x)\phi_h(g)(x).$$

If we assume further that h is a homeomorphism, then it turns out that ϕ_h is an isomorphism. Injectivity is because $f, g \in C(X)$ satisfy $\phi_h(f) = \phi_h(g)$ if and only if $f \circ h = g \circ h$, and bijectivity of h means that this happens if and only if $f = g$. Surjectivity is because for $f \in C(Y)$, the function $f \circ h^{-1}$ belongs to $C(X)$ and $\phi_h(f \circ h^{-1}) = f$. By Corollary 4.39 the map ϕ_h is an isometric *-isomorphism.

5. **The Gelfand-Naimark Theorems**

Two papers [9] [10] published in the 1940's by Israel Gelfand and Mark Naimark presented a pair of revolutionary results regarding C*-algebras. The first of these is that any commutative C*-algebra is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space, and the second is that any C*-algebra (commutative or not) is *-isomorphic to a C*-subalgebra of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. In this chapter we aim to prove both of these results, starting with the theorem concerning commutative C*-algebras.

5.1. **The Commutative Gelfand-Naimark Theorem.**

Assumption 5.1. In this section, we shall assume that all C*-algebras are commutative.

Here we shall revisit the theory which we developed in Chapter 3. Recall that if A is a commutative Banach algebra, then a character on A is an algebra homomorphism $\varphi : A \to \mathbb{C}$ and we denote by $\Omega(A)$ the set of all characters on A. Shortly we shall show that characters on C*-algebras are always *-homomorphisms, and that the character space of a C*-algebra is always non-empty. From these results and our previous efforts it is surprisingly easy to prove that the Gelfand Representation of a C*-algebra is always a *-isomorphism.

Lemma 5.2. *Let A be a C*-algebra and let φ be a character on A. Then $\varphi(x^*) = \overline{\varphi(x)}$ for every $x \in A$.*

Proof. For $x \in A$, we write $x = a+bi$ for the self adjoint elements $a = \frac{1}{2}(x+x^*)$, $b = \frac{1}{2i}(x-x^*) \in A$. Then $\varphi(a) \in \sigma(a)$ and $\varphi(b) \in \sigma(b)$, so by Theorem 4.29 $\varphi(a)$ and $\varphi(b)$ are real numbers. Therefore,

$$\varphi(x^*) = \varphi(a - bi) = \varphi(a) - i\varphi(b) = \overline{\varphi(a) + i\varphi(b)} = \overline{\varphi(x)}.$$

□

When we were looking at commutative Banach algebras earlier, we saw that the character space of a unital commutative Banach algebra is always non-empty, so it follows immediately that the character space of a unital commutative C*-algebra is non empty. However in the non unital case, we gave an example to show that the character space could be empty. Now we are in the setting of C*-algebras, this is no longer the case.

Lemma 5.3. *If A is a non-unital C*-algebra, then $\Omega(A) \neq \emptyset$.*

Proof. We let $0 \neq x \in A$ be any self adjoint element. By Lemma 4.30, $r(x) = \|x\|$ and by Theorem 3.11 we have

$$\sigma_{\tilde{A}}(x) = \{\varphi(x);\, \varphi \in \Omega(A)\} \cup \{0\}.$$

$\sigma_{\tilde{A}}(x)$ is a compact set, so its image under the continuous map $\theta : \mathbb{C} \to \mathbb{R}$, where $\theta(\lambda) = |\lambda|$ is compact and hence θ attains its maximum which must be equal to $r(x)$. Moreover $0 = \theta(0) \neq \|x\| = r(x)$, so there must be a character $\varphi \in \Omega(A)$ with $|\varphi(x)| = \|x\| \neq 0$. □

Theorem 5.4. (The Commutative Gelfand-Naimark Theorem)
Let A be a C-algebra. Then the Gelfand representation*

$$\begin{gathered}\Gamma : A \to C_0(\Omega(A)) \\ x \mapsto \widehat{x}\end{gathered}$$

*is an isometric *-isomorphism.*

Proof. Recall that in Theorem 3.18 we proved that the Gelfand representation is a norm-decreasing homomorphism which additionally satisfies $\|\Gamma(x)\|_\infty = \|\hat{x}\|_\infty = r(x)$ for each $x \in A$. Fixing our choice of $x \in A$, we use Lemma 5.2 to see that

$$\Gamma(x^*)(\varphi) = \varphi(x^*) = \overline{\varphi(x)} = \Gamma(x)^*(\varphi),$$

for all $\varphi \in \Omega(A)$, so Γ is a *-homomorphism and furthermore

$$\|\Gamma(x)\|_\infty^2 = \|\Gamma(x)^*\Gamma(x)\|_\infty = \|\Gamma(x^*x)\|_\infty = r(x^*x) = \|x^*x\|_A = \|x\|_A^2,$$

so that Γ is an isometry. Now we make three observations.

(i) Since A and $C_0(\Omega(A))$ are complete metric spaces and Γ is an isometry, $\Gamma(A)$ is a closed and hence is a C*-subalgebra of $C_0(\Omega(A))$.
(ii) If $\varphi, \tau \in \Omega(A)$ satisfy $\Gamma(x)(\varphi) = \Gamma(x)(\tau)$, for all $x \in A$, then $\varphi(x) = \tau(x)$ for all $x \in A$ so that $\varphi = \tau$. Hence $\Gamma(A)$ separates the points of $\Omega(A)$.
(iii) For any $\tau \in \Omega(A)$, τ is not the zero map so we may find $x \in A$ with $\varphi(x) \neq 0$. Therefore $\Gamma(x) \in \Gamma(A)$ satisfies $\Gamma(x)(\varphi) \neq 0$.

We take these observations and appeal to the Stone-Weierstrass theorem [6, Chapter V, Corollary 8.3] to see that $\Gamma(A) = C_0(\Omega(A))$. □

The commutative Gelfand Naimark Theorem is quite an exciting result. Not only is its statement surprising but it also pops up frequently in proofs, where it will allow us reduce many problems about C*-algebras to problems about $C_0(X)$ and $C(X)$. The relationships between C*-algebraic and topological concepts which are a consequence of the commutative Gelfand Naimark Theorem are often referred to as non-commutative topology. For the time being we present the following two examples as simple applications of our latest Theorem.

Example 5.5. Here we shall show that not every Banach *-algebra is a C*-algebra, a counter example which we promised to deliver earlier on. Let A be the unital commutative Banach algebra which we defined in Example 3.7. We equip A with the involution inherited from $\mathbb{M}_n(\mathbb{C})$ and recalling that the norm on A is the operator norm on $B(\mathbb{C}^2)$ we see that the involution is isometric, thus A is a Banach *-algebra. Now we recall from Remark 3.19 that for this algebra, the Gelfand representation $\Gamma : A \to C(\Omega(A))$ fails to be injective. From this we deduce that A is not a C*-algebra under our choices of norm and involution.

Example 5.6. Suppose that A is a unital commutative C*-algebra and that $x \in A$ is a non-trivial projection. By the commutative Gelfand-Naimark Theorem, $\Gamma : A \to C(\Omega(A))$ is an isometric *-isomorphism and using the properties of Γ we may easily verify that $\Gamma(x)$ is also a non-trivial projection in $C(\Omega(A))$. Let $f := \Gamma(x)$, then because f is a projection, it satisfies $f^2 - f = 0$. Once we factorise this as $f(f-1) = 0$ we see that the only values f can take are zero and one. By continuity, the sets $E := f^{-1}(\{0\})$ and $F := f^{-1}(\{1\})$ are open, disjoint and satisfy $E \cup F = \Omega(A)$. Moreover if either E or F is empty, then f would be a trivial projection, thus we conclude that $\Omega(A)$ is disconnected.

Conversely, assume that $\Omega(A)$ is disconnected, say with $\Omega(A) = E \cup F$ for disjoint and non-empty open subsets $E, F \subseteq \Omega(A)$. We consider the indicator function of the set E, denoted by χ_E. The assumption that E and F are non empty means that χ_E cannot take a constant value of zero or one on $\Omega(A)$. If $U \subseteq \mathbb{C}$ is any open set then the preimage $\chi_E^{-1}(U)$ must be one of the four sets

$$\emptyset,\ E,\ F \text{ or } \Omega(A).$$

But all of these sets are open in $\Omega(A)$, so we see that $\chi_E \in C(\Omega(A))$. Because $\Gamma : A \to C(\Omega(A))$ is an isometric *-isomorphism, there is an element $x \in A$ for which $\Gamma(x) = \chi_E$. Since Γ is a *-homomorphism we see that

$$\Gamma(x^*) = \overline{\chi_E} = \chi_E = \Gamma(x),$$

and

$$\Gamma(x^2) = \chi_E^2 = \chi_E = \Gamma(x).$$

Because Γ is injective it follows that $x^2 = x = x^*$, so combined with our previous observation that $\chi_E \neq 0$ and $\chi_E \neq 1$, we have shown that x is a non-trivial projection in A.

5.2. **The Continuous Functional Calculus and Positive Elements.** As we have noted previously, we shall use the Gelfand-Naimark Theorem many times. Here we shall use it to develop the continuous functional calculus, a way in which we can give meaning to the (currently nonsensical) expression $f(x)$, for an element x of a C*-algebra and a continuous function $f \in C(\sigma(x))$. Using this functional calculus we will be able to define a partial order on the self adjoint elements of a C*-algebra and will will briefly mention the concept of an approximate unit.

Theorem 5.7. *Let A be a unital C*-algebra and let $x \in A$ be a normal element. Then there is a unique unital *-homomorphism*

$$\varphi_x : C(\sigma(x)) \to A,$$

which satisfies the following properties,

(i) φ_x *is an isometry,*
(ii) $\varphi_x(\iota) = x$, *where* $\iota : \sigma(x) \hookrightarrow \mathbb{C}$ *is the inclusion map,*
(iii) $\varphi_x(C(\sigma(x)) = C^*(\{1_A, x\})$.

Proof. To keep things a bit tidier we let $B = C^*(\{1_A, x\})$, then B is a commutative unital C*-algebra and the Gelfand representation $\Gamma : B \to C(\Omega(B))$ is an isometric *-isomorphism.

From Theorem 3.20 we know that $\widehat{x} : \Omega(B) \to \sigma(x)$ is a homeomorphism and thus it follows from Example 4.41 that

$$\phi_{\widehat{x}} : C(\sigma(x)) \to C(\Omega(B))$$
$$f \mapsto f \circ \widehat{x}$$

is an isometric *-isomorphism. Now we shall consider the composition

$$C(\sigma(x)) \xrightarrow{\phi_{\widehat{x}}} C(\Omega(B)) \xrightarrow{\Gamma^{-1}} B,$$

which is a unital *-isomorphism and is therefore isometric by Corollary 4.39. We define φ_x to be the composition of this map with the inclusion $B \hookrightarrow A$. Then we have

$$\varphi(\iota) = \Gamma^{-1}(\phi_{\widehat{x}}(\iota)) = \Gamma^{-1}(\iota \circ \widehat{x}) = \Gamma^{-1}(\widehat{x}) = x,$$

and

$$\varphi(1) = \Gamma^{-1}(\phi_{\widehat{x}}(1)) = \Gamma^{-1}(1) = 1_B = 1_A.$$

It is immediate from its construction that the image of φ_x is B, so all that remains to be proved is that φ_x is unique. For this we note that the set $\{1, \iota\} \subseteq C(\sigma(x))$ separates the points of $\sigma(x)$, so by the Stone-Weierstrass Theorem [6, Chapter V, Theorem 8.1], the *-algebra generated[8] by $\{1, \iota\}$ is dense in $C(\sigma(x))$. However, all unital *-homomorphisms which satisfy the properties in the statement of the theorem must agree on this *-algebra and hence agree on $C(\sigma(x))$. This proves uniqueness of φ_x and hence completes our proof. □

Remark 5.8. We shall refer to the homomorphism φ_x in the theorem as the *continuous functional calculus at* x, and we shall denote the image of an element $x \in A$ under φ_x by $f(x)$. It is possible to prove an analogous result for non-unital C*-algebras, but we will not require this for our purposes.

When we first encounter isometries of metric spaces we learn that whilst isometries are injective, the converse fails in general. However in our C*-algebraic setting, the converse is true, and is a consequence of the continuous functional calculus.

Proposition 5.9. *Let $\varphi : A \to B$ be an injective *-homomorphism between C*-algebras, then φ is isometric.*

Proof. First lets note that if we can prove that $\|\varphi(y)\|_B = \|y\|_A$ for $y \in A$ self adjoint, then the result follows because

$$\|\varphi(x)\|_B^2 = \|\varphi(x^*x)\|_B = \|x^*x\|_A = \|x\|_A^2,$$

for an arbitrary $x \in A$. Just like we did in Corollary 4.38, first we assume that A has a unit, then if necessary we replace B by $\overline{\varphi(A)}$ in order to assume further that B is unital and that φ is a unital homomorphism.

Now let $x \in A$ be any self adjoint element and use Proposition 4.37 to see that $\sigma_B(\varphi(x)) \subseteq \sigma_A(x)$. By Lemma 4.30 and Proposition 4.37 again, we know that

$$r_B(\varphi(x)) = \|\varphi(x)\|_B \leq \|x\|_A = r_A(x).$$

So if we can prove that $\sigma_B(\varphi(x)) = \sigma_A(x)$ then we will have shown that $r_B(\varphi(x)) = r_A(x)$ and hence the result. Toward a contradiction we assume that $\sigma_B(\varphi(x)) \subsetneq \sigma_A(x)$ and take

[8]The *-algebra generated by a set S is precisely the set of polynomials in $S \cup S^*$.

$\lambda \in \sigma_A(x) \setminus \sigma_B(\varphi(x))$. As a topological space, $\sigma_A(x)$ is compact and Hausdorff and we apply Urysohn's Lemma to the closed sets $\sigma_B(\varphi(x))$ and $\{\lambda\}$ to find a continuous function $f \in C(\sigma_A(x))$ with $f|_{\sigma_B(\varphi(x))} = 0$ and $f(\lambda) = 1$.

Since self adjoint elements are normal, we may consider the continuous functional calculus at x and deduce that $f(x) \neq 0$ because $f \neq 0$. Similarly we may consider the continuous functional calculus at $\varphi(x)$, but in this case we have that $f(\varphi(x)) = 0$ because $f = 0$ on $\sigma_B(\varphi(x))$. Finally we note that $\varphi(f(x)) = f(\varphi(x))$. This is true because $\varphi(p(x)) = p(\varphi(x))$ certainly holds for any polynomial $p \in C(\sigma_A(x))$, then one can argue by density of the polynomials in $C(\sigma(x))$ and continuity of φ to extend to functions in $C(\sigma(x))$. But now this is a problem, because when we combine with the injectivity of φ, we have the contradiction

$$0 \neq \varphi(f(x)) = f(\varphi(x)) = 0.$$

Now we deal with the case that A is non-unital. Again, just like in Corollary 4.38 we pass to the unitization of A and may assume that B is unital. Then define $\tilde{\varphi} : \tilde{A} \to B$ by

$$\widetilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda 1_B,$$

which is a unital *-homomorphism which extends φ. Once we have checked that $\widetilde{\varphi}$ is also injective, we apply the first part of the proof to conclude that $\widetilde{\varphi}$ is an isometry, from which it follows that

$$\|\varphi(x)\|_A = \|\widetilde{\varphi}((x, 0))\|_B = \|(x, 0)\|_{\widetilde{A}} = \|x\|_A.$$

$\square$

We shall now discuss positive elements of C*-algebras, a particular type of self adjoint element and as we shall see shortly, they allow us to define a partial order on the set of self adjoint elements. Positive elements are useful, because they can be used to show that every C*-algebra admits an approximate unit, which is a generalisation of the units possessed by unital algebras. We shall avoid using approximate units in the later sections, so we will not go into much detail on this subject.

Definition 5.10. Let A be a C*-algebra. An element $x \in A$ is called *positive* if x is self adjoint and $\sigma(x) \subseteq \mathbb{R}^{\geq 0}$.

Example 5.11. Suppose that a matrix $A \in \mathbb{M}_2(\mathbb{C})$ is positive in the sense of Definition 5.10. Then A is a hermitian matrix with positive eigenvalues, so is a positive definite matrix.

Example 5.12. Consider the function $f \in C_0(\mathbb{C})$ which is defined by

$$f(z) = \frac{1}{1 + |z|}.$$

Since f only takes values in $\mathbb{R}^{\geq 0}$, we have $\sigma(f) \subseteq \mathbb{R}^{\geq 0}$ and clearly $f^* = f$. Therefore f is positive.

Lemma 5.13. *Let A be a unital C*-algebra. Then an element $x \in A$ is positive if and only if $x = y^2$ for some self adjoint element $y \in A$.*

Proof. If $x = y^2$ for some self adjoint element $y \in A$, then $x^* = y^* y^* = y^2 = x$ so that x is self adjoint. By Theorem 4.29, $\sigma(y) \subseteq \mathbb{R}$ and by the spectral mapping property for polynomials,

$$\sigma(x) = \sigma(y^2) = \{\lambda^2;\, \lambda \in \sigma(x)\},$$

which implies that $\sigma(x) \subseteq \mathbb{R}^{\geq 0}$, so x is positive.

Conversely, if x is positive then $\sigma(x) \subseteq \mathbb{R}^{\geq 0}$, so $\sqrt{\cdot} : \sigma(x) \to \mathbb{R}$ is a continuous function. Using the continuous functional calculus at x we define $y = \sqrt{x} \in A$ and the properties of the continuous functional calculus at x then show that $y^* = y$ and $y^2 = x$. $\square$

Remark 5.14. We can actually strengthen this lemma in two ways, both of which may be found in [15, Theorem 2.2.1]. First we can prove that positive elements in non unital algebras have a square root, and second we can prove that the square root of a positive element is unique.

Example 5.15. Let $\mathcal{H}$ be a Hilbert space and consider the C*-algebra $B(\mathcal{H})$. Recall that an operator $T \in B(\mathcal{H})$ is said to be a *positive operator* if

$$\langle \nu, T\nu \rangle \geq 0,$$

for every vector $\nu \in \mathcal{H}$. Now suppose that the operator $T \in B(\mathcal{H})$ is positive in the sense of Definition 5.10, then Lemma 5.13 tells us that there is a self adjoint operator $S \in B(\mathcal{H})$ which satisfies $T = S^*S$. For each $\nu \in \mathcal{H}$ we have

$$\langle \nu, T\nu \rangle = \langle \nu, S^*S\nu \rangle = \langle S\nu, S\nu \rangle = \|\nu\|^2 \geq 0.$$

So that T is a positive operator. The converse to this also holds; if $T \in B(\mathcal{H})$ is a positive operator, then T is positive in the sense of Definition 5.10, see [1, Proposition 6.32].

With our next proposition, we can show that the sum of positive elements is again positive. This is the final thing we need in order to define the partial order on self adjoint elements.

Proposition 5.16. *Let A be a unital C*-algebra, then for a self adjoint element $x \in A$ the following are equivalent,*

(i) $\sigma(x) \subseteq \mathbb{R}^{\geq 0}$,
(ii) For some $\lambda \in \mathbb{R}$ *with* $\|x\| \leq \lambda$ *we have* $\|x - \lambda 1_A\| \leq \lambda$,
(iii) For all $\lambda \in \mathbb{R}$ *with* $\|x\| \leq \lambda$ *we have* $\|x - \lambda 1_A\| \leq \lambda$.

Proof. We consider the commutative C*-subalgebra $B := \mathrm{C}^*(\{1_A, x\})$ and the continuous functional calculus at x gives an isometric *-isomorphism between B and $C(\sigma(x))$.

Claim 5.17. *If X is a compact Hausdorff space, then for a function $f \in C(X)$ which satisfies $f = \overline{f}$, the following are equivalent*

(i) $f(X) \subseteq \mathbb{R}^{\geq 0}$,
(ii) For some $\lambda \in \mathbb{R}$ *with* $\|f\|_\infty \leq \lambda$ *we have* $\|f - \lambda\|_\infty \leq \lambda$,
(iii) For all $\lambda \in \mathbb{R}$ *with* $\|f\|_\infty \leq \lambda$ *we have* $\|f - \lambda\|_\infty \leq \lambda$.

Proof. First note that the implication (iii)⇒(ii) is obvious. To show that (ii)⇒(i) we take $x \in X$ and $\lambda \in \mathbb{R}$ with $\|f\|_\infty \leq \lambda$, then by assumption

$$|f(x) - \lambda| \leq \|f - \lambda\|_\infty \leq \lambda.$$

Expansion of the absolute value gives $-\lambda \leq f(x) - \lambda \leq \lambda$ from which it follows that $f(X) \subseteq \mathbb{R}^{\geq 0}$.

We finish by showing (i)⇒(iii), so take $\lambda \in \mathbb{R}$ which satisfies $\|f\|_\infty \leq \lambda$. Then for $x \in X$ we have

$$-\lambda \leq f(x) - \lambda \leq 0,$$

where the first inequality is because $0 \leq f(x)$ and the second is because $f(x) \leq \|f\|_\infty$. Since $0 \leq \|f\|_\infty \leq \lambda$ it now follows that $|f(x) - \lambda| \leq \lambda$ and therefore $\|f - \lambda\|_\infty \leq \lambda$, which proves our claim. □

If we now apply the claim, then we see that $\sigma_B(x) \subseteq \mathbb{R}^{\geq 0}$ is equivalent to points (ii) and (iii) in the statement of the lemma, so if it is true that $\sigma_B(x) = \sigma_A(x)$ then we are done. Thankfully this assertion is correct and Murphy supplies us with a proof [15, Theorem 2.1.11]. □

Corollary 5.18. *If A is a C*-algebra then for any pair of positive elements $x, y \in A$, their sum is also a positive element of A.*

Proof. First assume that A is non-unital and let $x \in A$ be a self adjoint element, then the spectrum of x is by definition the spectrum of x as viewed as an element of $\widetilde{A}$, so x is positive if and only if $(x, 0)$ is positive in $\widetilde{A}$. So if $y \in A$ is another positive element, then if $(x + y, 0) \in \widetilde{A}$ is positive it follows that $x + y \in A$ is positive. Thus it suffices for us to prove the statement in the case that A is unital.

Henceforth, we assume that A is unital and take $x, y \in A$. Then applying the previous proposition gives the inequalities

$$\|x - \|x\|\| \leq \|x\|,$$
$$\|y - \|y\|\| \leq \|y\|,$$

which combined with the triangle inequality yield

$$\|(x+y) - (\|x\| + \|y\|)\| \leq \|x\| + \|y\|.$$

So $x+y$ is also positive. □

As promised, we shall now briefly discuss the notion of an approximate unit. An approximate unit is a net of self adjoint elements which generalises the concept of a unit. We shall say precisely what we mean by this statement shortly. If A is a C*-algebra, then we shall define a relation $\preceq$ on the set of self adjoint elements of A by

$$y \preceq x \iff x - y \text{ is positive.}$$

We check that this is a partial order, so let $x, y, z \in A$ be self adjoint elements, then

(i) $x - x = 0$ is obviously positive.

(ii) If $x - y$ and $y - x$ are positive, their spectra satisfy

$$\sigma(x-y), \sigma(y-x) \subseteq \mathbb{R}^{\geq 0}.$$

We use the spectral mapping property for polynomials to see that

$$\sigma(x-y) = \sigma(-(y-x)) = \{-\lambda;\, \lambda \in \sigma(y-x)\},$$

from which it follows that $\sigma(x-y) = \sigma(y-x) = \{0\}$. Because $x-y$ is self adjoint we have the equality

$$0 = r(x-y) = \|x-y\|,$$

which of course implies that $x = y$.

(iii) If $x-y$ and $y-z$ are positive, then their sum is positive and $(x-y)+(y-z) = x-z$.

Definition 5.19. Let A be any C*-algebra then an *approximate unit* for A is a net (e_λ) of positive elements, contained in the closed unit ball of A which satisfies the following,

(i) If $\lambda \leq \delta$ then $e_\lambda \preceq e_\delta$,

(ii) For every $x \in A$, we have $\lim_\lambda e_\lambda x = x = \lim_\lambda x e_\lambda$.

Example 5.20. If A is a unital C*-algebra, then $(e_\lambda) = (1_A)$ is (unsurprisingly) an approximate unit for A.

Example 5.21. In this example we consider $C_0(\mathbb{R})$. Because $\mathbb{R}$ is not compact, $C_0(\mathbb{R})$ is not unital. For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{C}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq -(n+1), \\ n+1+x & \text{if } -(n+1) \leq x \leq -n, \\ 1 & \text{if } -n \leq x \leq n, \\ n+1-x & \text{if } n \leq n \leq x \leq n+1, \\ 0 & \text{if } n+1 \leq x, \end{cases}$$

For each $n \in \mathbb{N}$, the function f_n is clearly continuous and vanishes at infinity, so $f_n \in C_0(\mathbb{R})$. Moreover, f_n takes values in $[0,1]$, so is positive and the maximum value it can attain is 1. Therefore f_n is positive and contained in the closed unit ball of $C_0(\mathbb{R})$. It should also be clear that if $m \leq n$ then $f_m \leq f_n$.

If we let $f \in C_0(X)$ be any function, then in order to finish the proof that (f_n) is a approximate unit, we must show that $\lim_{n\to\infty} \|f - ff_n\|_\infty = 0$ and $\lim_{n\to\infty} \|f - f_n f\|_\infty = 0$. Here we shall only prove the first of these. For any given $n \in \mathbb{N}$ we have

$$(f - ff_n)(x) = \begin{cases} f(x) & \text{if } x \leq -(n+1), \\ -(n+x)f(x) & \text{if } -(n+1) \leq x \leq -n, \\ 0 & \text{if } -n \leq x \leq n, \\ (n-x)f(x) & \text{if } n \leq n \leq x \leq n+1, \\ f(x) & \text{if } n+1 \leq x. \end{cases}$$

A swift analysis of the different cases shows us that $|(f - ff_n)(x)| \leq |f(x)|$ for every $x \in \mathbb{R}$.

Now let $\epsilon > 0$ be arbitrary. $C_0(\mathbb{R})$ is an algebra so obviously the function $f - ff_n$ belongs to $C_0(\mathbb{R})$, thus we may find a compact subset $K \subseteq \mathbb{R}$ such that $|(f - ff_n)(x)| < \epsilon$ for all $x \in \mathbb{R} \setminus K$. Because K is compact, there is $N \in \mathbb{N}$ such that $K \subseteq [-N, N]$ and the work we have done above shows that $|(f - ff_N)(x)| = 0$ for $x \in [-N, N]$. Combined with the the inclusion $\mathbb{R} \setminus [-N, N] \subseteq \mathbb{R} \setminus K$, this shows that $|(f - ff_N)(x)| < \epsilon$ for every $x \in \mathbb{R}$. Now for any $n > N$, because $[-N, N] \subseteq [-n, n]$, it follows that $|(f - ff_n)(x)| < \epsilon$ and we deduce that ff_n converges uniformly to f, so $\lim_{n\to\infty} \|f - ff_n\|_\infty = 0$.

In fact every C*-algebra possesses an approximate unit, as is shown by Murphy [15, Theorem 3.1.1]. Using an approximate unit, it is possible to show that every closed ideal of a C*-algebra is self adjoint [15, Theorem 3.1.3] and that the quotient of a C*-algebra by a closed ideal is again a C*-algebra [15, Theorem 3.1.4].

5.3. **Representations of C*-algebras.** We find ourselves rapidly approaching the second Gelfand-Naimark Theorem and to give the proof of this we shall require some representation theory. This is quite a large topic and one may speak about representations of C*-algebras, Banach *-algebras, *-algebras and even algebras. So that we don't get sidetracked by dealing with these more general situations, we shall restrict our definitions and results to C*-algebras, but note that many carry over and remain true for more general algebras. If we want to read about the representation theory of more general algebras then we should consult [16].

Definition 5.22. let A be a C*-algebra. A *representation* of A is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$ and a non-zero *-homomorphism

$$\pi : A \to B(\mathcal{H}).$$

If the map π is injective, then we say that the representation is *faithful.* Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of A are called *unitarily equivalent* if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ that satisfies

$$U\pi_1(x) = \pi_2(x)U,$$

for all $x \in A$.

Remark 5.23. In light of Proposition 5.9, if a C*-algebra possesses a faithful representation $(\mathcal{H}, \pi)$, then $\pi(A)$ is a C*-subalgebra of $B(\mathcal{H})$. Hence the map $\pi : A \to \pi(A)$ is an isometric *-isomorphism, so we may view A as a C*-subalgebra of $B(\mathcal{H})$.

Example 5.24. If $\mathcal{H}$ is a Hilbert space and A is a C*-subalgebra of $B(\mathcal{H})$ then $(\mathcal{H}, \iota)$ is a representation of A, where $\iota : A \hookrightarrow B(\mathcal{H})$ is the inclusion map.

Example 5.25. Recall that $\mathbb{M}_2(\mathbb{C})$ may be viewed as $B(\mathbb{C}^2)$ and consider the map $\pi : C([0,1]) \to B(\mathbb{C}^2)$ defined for $f \in C([0,1])$ by

$$\pi(f) = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix}.$$

This is obviously a linear map. We take functions $f, g \in C([0,1])$ and see that

$$\pi(fg) = \begin{pmatrix} f(0)g(0) & 0 \\ 0 & f(1)g(1) \end{pmatrix} = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix} \begin{pmatrix} g(0) & 0 \\ 0 & g(1) \end{pmatrix},$$

so π is an algebra homomorphism and furthermore

$$\pi(f^*) = \begin{pmatrix} \overline{f(0)} & 0 \\ 0 & \overline{f(1)} \end{pmatrix} = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix}^*.$$

Hence π is a *-homomorphism and is therefore a representation of $C([0,1])$.

Note however that this representation is not faithful, to see this we only need to find distinct continuous functions which take the same values at 0 and 1. The identity function and $f(x) = x^2$ should do the trick.

Our next example is an important one. It tells us how we can take an arbitrary family of representations and combine them to give another representation. This example is practically always left as an exercise in textbooks so we shall present all of the details here.

Example 5.26. Let A be a C*-algebra and let $\{(\mathcal{H}_\lambda, \pi_\lambda)\}_{\lambda\in\Lambda}$ be a collection of representations of A, in this example we shall define the *direct sum* of these representations. First we define the direct sum of the Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda\in\Lambda}$, this is denoted by $\oplus_\lambda\mathcal{H}_\lambda$ and consists of all families $(\nu_\lambda) := \{\nu_\lambda\}_{\lambda\in\Lambda}$ such that $\nu_\lambda \in \mathcal{H}_\lambda$ for every $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 < \infty$.[9]

Claim 5.27. *$\oplus_\lambda\mathcal{H}_\lambda$ is a Hilbert space under the map $\langle\cdot,\cdot\rangle : \oplus_\lambda\mathcal{H}_\lambda \times \oplus_\lambda\mathcal{H}_\lambda \to \mathbb{C}$, which for $(\nu_\lambda), (\eta_\lambda) \in \oplus_\lambda\mathcal{H}_\lambda$ is defined by*

$$\langle(\nu_\lambda), (\eta_\lambda)\rangle = \sum_{\lambda\in\Lambda}\langle\nu_\lambda, \eta_\lambda\rangle_\lambda.$$

Proof. Note that (0) belongs to $\oplus_\lambda\mathcal{H}_\lambda$, so that $\oplus_\lambda\mathcal{H}_\lambda$ is a non-empty subset of the direct product of the $\mathcal{H}_\lambda$'s. Now we take $(\nu_\lambda), (\eta_\lambda) \in \oplus_\lambda\mathcal{H}_\lambda$ and note that for fixed $\lambda \in \Lambda$, we have

$$0 \leq (\|\nu_\lambda\|_\lambda - \|\eta_\lambda\|_\lambda)^2 = \|\nu_\lambda\|_\lambda^2 + \|\eta_\lambda\|_\lambda^2 - 2\|\nu_\lambda\|_\lambda\|\eta_\lambda\|_\lambda,$$

which implies the inequality

$$\begin{aligned}\|\nu_\lambda + \eta_\lambda\|_\lambda^2 &\leq (\|\nu_\lambda\|_\lambda + \|\eta_\lambda\|_\lambda)^2 \\ &= \|\nu_\lambda\|_\lambda^2 + \|\eta_\lambda\|_\lambda^2 + 2\|\nu_\lambda\|_\lambda\|\eta_\lambda\|_\lambda \\ &\leq 2\|\nu_\lambda\|_\lambda^2 + 2\|\eta_\lambda\|_\lambda^2.\end{aligned}$$

We now use this to show that

$$\sum_{\lambda\in\Lambda} \|\nu_\lambda + \eta_\lambda\|_\lambda^2 \leq 2\sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 + 2\sum_{\lambda\in\Lambda} \|\eta_\lambda\|_\lambda^2 < \infty,$$

and if $\mu \in \mathbb{C}$ then

$$\sum_{\lambda\in\Lambda} \|\mu\nu_\lambda\|_\lambda^2 = |\mu|^2\sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 < \infty.$$

So $(\nu_\lambda) + (\eta_\lambda)$ and $\mu(\nu_\lambda)$ belong to $\oplus_\lambda\mathcal{H}_\lambda$, thus proving that $\oplus_\lambda\mathcal{H}_\lambda$ is a vector space. To show that $\langle\cdot,\cdot\rangle$ is an inner product, we take $(\nu_\lambda), (\eta_\lambda) \in \oplus_\lambda\mathcal{H}_\lambda$, then

$$\sum_{\lambda\in\Lambda} |\langle\nu_\lambda, \eta_\lambda\rangle_\lambda| \leq \sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda\|\eta_\lambda\|_\lambda \leq \frac{1}{2}\sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 + \frac{1}{2}\sum_{\lambda\in\Lambda} \|\eta_\lambda\|_\lambda^2 < \infty,$$

which shows that the series defining $\langle\cdot,\cdot\rangle$ converges absolutely, hence converges by completeness of $\mathbb{C}$. For the actual properties of an inner product, linearity of $\langle\cdot,\cdot\rangle$ in its second argument is practically immediate from the definition of $\langle\cdot,\cdot\rangle$ and the fact that each $\langle\cdot,\cdot\rangle_\lambda$ is an inner product. It is easy to see that $\langle(\nu_\lambda), (\nu_\lambda)\rangle \geq 0$ and that

$$\langle(\nu_\lambda), (\nu_\lambda)\rangle = 0 \iff \nu_\lambda = 0 \text{ for all } \lambda \in \Lambda.$$

The final property is a consequence of the fact that if the unordered sum of complex numbers $\sum_{i\in I}\alpha_i$ converges, then $\sum_{i\in I}\overline{\alpha_i}$ converges to $\overline{\sum_{i\in I}\alpha_i}$. Finally we should prove completeness, so we take a Cauchy sequence $((\nu_\lambda^{(n)})_{\lambda\in\Lambda})_{n\in\mathbb{N}}$ in $\oplus_\lambda\mathcal{H}_\lambda$, then for $\epsilon > 0$ we may find $N \in \mathbb{N}$ such that

$$m, n > N \Rightarrow \sum_{\lambda\in\Lambda} \|\nu_\lambda^{(m)} - \nu_\lambda^{(n)}\|_\lambda^2 < \epsilon.$$

This implies that for fixed $\lambda \in \Lambda$, the sequence $(\nu_\lambda^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space $\mathcal{H}_\lambda$, so completeness will give us a limit $\nu_\lambda \in \mathcal{H}_\lambda$ and we shall show that $\lim_{n\to\infty}(\nu_\lambda^{(n)})_{\lambda\in\Lambda} =$

[9] We have imposed no restrictions on the indexing set Λ, so the summation here is an example of an *unordered sum* and defining what it means for such a sum to converge goes a similar way to how we define the convergence of infinite sums over $\mathbb{N}$, but with nets taking the place of sequences. If Λ is any indexing set, let I_Λ denote the set of all finite subsets of Λ and if we order I_Λ by inclusion I_Λ becomes a directed set. If X is a normed space and $\{x_\lambda\}_{\lambda\in\Lambda}$ is any collection of elements from X, then for any finite subset $F \in I_\Lambda$ we write $S_F = \sum_{n\in F} x_n \in X$. The assignment $F \mapsto S_F$ is then a net in X and we say that the collection $\{x_\lambda\}_{\lambda\in\Lambda}$ is *summable* with sum $x \in X$ if the net $(S_F)_{F\in I_\Lambda}$ converges to x and in this case we write $x = \sum_{\lambda\in\Lambda} x_\lambda$. Unpicking the definition of net-convergence, this means that for each $\epsilon > 0$, there is a finite subset $F_0 \in I_\Lambda$ such that for any other finite subset $F_0 \subseteq F \in I_\Lambda$ we have $\|x - \sum_{n\in F} x_n\| < \epsilon$.

$(\nu_\lambda)_{\lambda\in\Lambda}$. If $\epsilon > 0$ and $F \subseteq \Lambda$ is finite, then because our original sequence is Cauchy we may find $N \in \mathbb{N}$ so that

$$\sum_{i\in F} \|\nu_i^{(m)} - \nu_i^{(n)}\|_\lambda^2 < \epsilon,$$

whenever $m, n > N$. Because this is a finite sum, we may safely take the limit $m \to \infty$ to conclude that

$$\sum_{i\in F} \|\nu_i - \nu_i^{(n)}\|_\lambda^2 < \epsilon,$$

whenever $n > N$, and since this holds for any finite subset $F \subseteq \Lambda$ we deduce that

$$\sum_{\lambda\in\Lambda} \|\nu_\lambda - \nu_\lambda^{(n)}\|_\lambda^2 < \epsilon,$$

whenever $n > N$. This shows two things, the first is that if $n > N$, then $(\nu_\lambda) - (\nu_\lambda^{(n)}) \in \oplus_\lambda \mathcal{H}_\lambda$ so

$$(\nu_\lambda) = (\nu_\lambda) - (\nu_\lambda^{(n)}) + (\nu_\lambda^{(n)}) \in \oplus_\lambda \mathcal{H}_\lambda,$$

and the second is that $(\nu_\lambda^{(n)})_{\lambda\in\Lambda} \to (\nu_\lambda)_{\lambda\in\Lambda}$ as $n \to \infty$. □

In order to complete this example, we will need to define a *-homomorphism $\oplus_\lambda\pi_\lambda : A \to B(\oplus_\lambda\mathcal{H}_\lambda)$ so that we have a representation of A. We shall define our map $\oplus_\lambda\pi_\lambda$ as follows; for each $x \in A$, $\oplus_\lambda\pi_\lambda(x) : \oplus_\lambda\mathcal{H}_\lambda \to \oplus_\lambda\mathcal{H}_\lambda$ is the map which is given by

$$\oplus_\lambda\pi_\lambda(x)(\nu_\lambda) = (\pi_\lambda(x)\nu_\lambda)_{\lambda\in\Lambda},$$

for all $(\nu_\lambda) \in \oplus_\lambda\mathcal{H}_\lambda$.

Claim 5.28. *For each $x \in A$, the map $\oplus_\lambda\pi_\lambda$ belongs to $B(\oplus_\lambda\mathcal{H}_\lambda)$ and the correspondence $x \mapsto \oplus_\lambda\pi_\lambda(x)$ is a *-homomorphism.*

Proof. Fix $x \in A$, then the fact that $\oplus_\lambda\pi_\lambda(x)$ is linear follows easily from linearity of each $\pi_\lambda(x)$. For boundedness we take $(\nu_\lambda) \in \oplus_\lambda\mathcal{H}_\lambda$ with $\|(\nu_\lambda)\| = 1$ and then

$$\begin{aligned}
\| \oplus_\lambda \pi_\lambda(x)(\nu_\lambda)\|^2 &= \|(\pi_\lambda(x)\nu_\lambda)\|^2 \\
&= \sum_{\lambda\in\Lambda} \|\pi_\lambda(x)\nu_\lambda\|_\lambda^2 \\
&\leq \sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 \|\pi_\lambda(x)\|_{\text{op}}^2 \\
&\leq \sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 \|x\|_A^2 \\
&= \|x\|_A^2 \sum_{\lambda\in\Lambda} \|\nu_\lambda\|_\lambda^2 \\
&= \|x\|_A^2.
\end{aligned}$$

Here the first inequality is because each $\pi_\lambda(x)$ belongs to $B(\mathcal{H}_\lambda)$ and the second is because each π_λ is a *-homomorphism of C*-algebras. This proves both that $\oplus_\lambda\pi_\lambda(x)$ takes values in $\oplus_\lambda\mathcal{H}_\lambda$ and that $\oplus_\lambda\pi_\lambda(x)$ is bounded.

Now we let $\oplus_\lambda\pi_\lambda : A \to B(\oplus_\lambda\mathcal{H}_\lambda)$ be the map which sends $x \mapsto \oplus_\lambda\pi_\lambda(x)$. The fact that for each $\lambda \in \Lambda$ the map $\pi_\lambda : A \to B(\mathcal{H}_\lambda)$ is a *-homomorphism may be used to easily show that $\oplus_\lambda\pi_\lambda$ is in fact a *-homomorphism and is therefore a representation of A. □

We want to define a three special types of representation, but before we can do this we need to give a few supplementary definitions. We shall make use of the following notation in a few places; if $(\mathcal{H}, \pi)$ is a representation of a C*-algebra A and $\xi \in \mathcal{H}$ is any vector, then we write

$$\pi(A)\xi = \{\pi(x)\xi;\, x \in A\}.$$

Because π is a homomorphism, we can easily show that $\pi(A)\xi$ is a vector subspace of $\mathcal{H}$.

Definition 5.29. If $(\mathcal{H}, \pi)$ is a representation of a C*-algebra A, then a linear subspace $\mathcal{K} \subseteq \mathcal{H}$ is said to be an *invariant subspace* for $\pi(A)$ if $\pi(x)\mathcal{K} \subseteq \mathcal{K}$ for every $x \in A$.

Example 5.30. For any representation $(\mathcal{H}, \pi)$ of a C*-algebra A, the subspaces $\{0\}, \mathcal{H} \subseteq \mathcal{H}$ are invariant for $\pi(A)$.

Example 5.31. Suppose that $(\mathcal{H}, \pi)$ is a representation of a C*-algebra A and let $\mathcal{K} \subseteq \mathcal{H}$ be an invariant subspace for $\pi(A)$. If $\eta \in \mathcal{K}^{\perp}$ then for any $x \in A$ we have

$$\langle \nu, \pi(x)\eta \rangle = \langle \pi(x^*)\nu, \eta \rangle = 0,$$

for all $\nu \in \mathcal{K}$, because $\mathcal{K}$ is an invariant subspace. From this we deduce that $\pi(x)\eta \in \mathcal{K}^{\perp}$ and so $\mathcal{K}$ is also an invariant subspace for $\pi(A)$.

We can now give our definition regarding the different types of representation.

Definition 5.32. If A is a C*-algebra, then a representation $(\mathcal{H}, \pi)$ of A is said to be

(i) *Non-degenerate* if for every vector $\eta \in \mathcal{H}$, $\pi(x)\eta = 0$ for all $x \in A$ implies $\eta = 0$,
(ii) *Cyclic* if there exists a vector $\eta \in \mathcal{H}$ such that $\pi(A)\eta$ is dense in $\mathcal{H}$, in this case the vector ν is called a *cyclic vector*,
(iii) *Topologically irreducible* if the only closed invariant subspaces for $\pi(A)$ are $\{0\}$ and $\mathcal{H}$.

Unsurprisingly there are links between these definitions, for example assume that $(\mathcal{H}, \pi)$ is a cyclic representation of a C*-algebra A, with cyclic vector $x \in \mathcal{H}$. Then for any $\nu \in \mathcal{H}$, if $\pi(x)\nu = 0$ holds for every $x \in A$, then for every $x \in A$ we have

$$\langle \nu, \pi(x)\xi \rangle = \langle \pi(x)^*\nu y, \xi \rangle = \langle 0, \xi \rangle = 0.$$

Because the set $\pi(A)\xi$ is dense in $\mathcal{H}$, it now follows that $\nu = 0$ so that $(\mathcal{H}, \pi)$ is non-degenerate.

Example 5.33. [6, Exercise 1, p.253]
Let A be a unital C*-algebra and let $(\mathcal{H}, \pi)$ be a representation of A where π is a non-unital homomorphism. We define an operator $P : \mathcal{H} \to \mathcal{H}$ by $P := \pi(1_A)$, then observe that

$$P^2 = \pi(1_A)^2 = \pi(1_A) = P,$$

and

$$P^* = \pi(1_A)^* = \pi(1_A) = P,$$

because π is a *-homomorphism. Hence P is a projection. The set $\mathcal{K} := P(\mathcal{H}) \subseteq \mathcal{H}$ is therefore a closed subspace and we claim that it is invariant for $\pi(A)$. Indeed for any $x \in A$ and $\kappa \in \mathcal{K}$, by construction there is $\nu \in \mathcal{H}$ so that $\kappa = P(\nu)$. Then,

$$\pi(x)\kappa = \pi(x)P(\nu) = \pi(x)\pi(1_A)\nu = \pi(x1_A)\nu = \pi(1_A)\pi(x)\nu = P(\pi(x)\nu),$$

so because $\pi(x)\nu$ belongs to $\mathcal{H}$ it follows that $\pi(x)\kappa \in \mathcal{K}$.

If we now define $\pi|_{\mathcal{K}} : A \to B(\mathcal{K})$ by $\pi|_{\mathcal{K}}(x) = \pi(x)|_{\mathcal{K}}$ for each $x \in A$, then $(\mathcal{K}, \pi|_{\mathcal{K}})$ is a representation of A.

Lemma 5.34. *A representation $(\mathcal{H}, \pi)$ of a unital C*-algebra A is non-degenerate if and only if π is a unital homomorphism.*

Proof. Suppose that $(\mathcal{H}, \pi)$ is non-degenerate, then because π is an algebra homomorphism we have $\pi(1_A) = \pi(1_A 1_A) = \pi(1_A)\pi(1_A)$, so for any $\eta \in \mathcal{H}$ we have

$$\pi(1_A)\pi(1_A)\eta = \pi(1_A)\eta,$$

which rearranges to give $\pi(1_A)(\pi(1_A)\eta - \eta) = 0$. Then for any $x \in A$ we apply $\pi(x) \in B(\mathcal{H})$ to each side and obtain

$$\pi(x)(\pi(1_A)\eta - \eta) = 0.$$

By assumption, this implies that $\pi(1_A)\eta - \eta = 0$ or rather $\pi(1_A)\eta = \eta$ and this of course shows that $\pi(1_A)$ =id.

Conversely, we assume that $\pi(1_A) = \text{id}$, then if $\eta \in \mathcal{H}$ satisfies $\pi(x)\eta = 0$ for every $x \in A$, in particular $\pi(1_A)\eta = 0$. But we are assuming that $\pi(1_A)\eta = \eta$ which shows that $\eta = 0$ and we are done. □

Example 5.35. If $(\mathbb{C}^2, \pi)$ is the representation of $C([0,1])$ which we defined in Example 5.25, then because $\pi(1) = I_2$ it follows that this representation is non-degenerate.

Recall the definition of the commutant from Example 4.6. If we have a Hilbert space $\mathcal{H}$ and a subset $M \subseteq B(\mathcal{H})$ which is closed under taking adjoints, then it follows from the aforementioned example that M' is a C*-subalgebra of $B(\mathcal{H})$.

Example 5.36. Consider $B(\mathcal{H})'$, where $\mathcal{H}$ is any Hilbert space. If an operator $T \in B(\mathcal{H})$ is a multiple of the identity operator, then T must commute with every operator in $B(\mathcal{H})$, so $\mathbb{C} \cdot \mathrm{id} \subseteq B(\mathcal{H})'$. We claim that the opposite inclusion also holds.

For each pair of vectors $\nu, \eta \in \mathcal{H}$, we define $\phi_{\nu,\eta} : \mathcal{H} \to \mathcal{H}$ by

$$\phi_{\nu,\eta}(\xi) = \langle \nu, \xi \rangle \eta,$$

for every $\xi \in \mathcal{H}$ and can easily check that this is a bounded linear operator. If $T \in B(\mathcal{H})'$, then $T\phi_{\nu,\eta} = \phi_{\nu,\eta}T$ for every $\nu, \eta \in \mathcal{H}$, so

$$\langle \nu, \xi \rangle T\eta = \langle \nu, T\xi \rangle \eta,$$

for every $\xi \in \mathcal{H}$. Now we fix a vector $0 \neq \nu \in \mathcal{H}$ and put $\xi = \nu$, the equality above shows that

$$T\eta = \frac{\langle \nu, T\nu \rangle}{\langle \nu, \nu \rangle}\eta,$$

for every $\eta \in \mathcal{H}$. Since our choice of ν was arbitrary, we conclude that for each non-zero $\nu \in \mathcal{H}$, the quantity $\frac{\langle \nu, T\nu \rangle}{\langle \nu, \nu \rangle}$ is a constant, say $t \in \mathbb{C}$. This in turn means that $T\eta = t\eta$ for every $\eta \in \mathcal{H}$, so that $T \in \mathbb{C} \cdot \mathrm{id}$, giving the inclusion $B(\mathcal{H})' \subseteq \mathbb{C} \cdot \mathrm{id}$.

We could say a great deal more about representations, but for the sake of space we shall just conclude this section by presenting the following theorem. This gives alternative criteria for a representation to be topologically irreducible.

Theorem 5.37. *Let $(\mathcal{H}, \pi)$ be a representation of a C*-algebra A. Then the following are equivalent,*

(i) $(\mathcal{H}, \pi)$ is topologically irreducible,
(ii) $\pi(A)' = \mathbb{C} \cdot id$,
(iii) $\pi(A)'$ contains no non-trivial projections from $B(\mathcal{H})$.

Proof. The implication (i)⇒(ii) is the hardest, so we shall cover this first. The proof we shall employ makes use of the commutative Gelfand-Naimark Theorem.

Remark 5.38. One could argue that our proof is a bit long-winded. Indeed many shorter proofs are given in textbooks, for example see the proofs by Murphy [15, Theorem 5.1.5(1)] or Kadison and Ringrose [11, Theorem 5.4.1]. However these shorter proofs require knowledge of von Neumann algebras, which is a topic we have not covered. Additionally, our proof provides another example of how we can use the commutative Gelfand-Naimark theorem to prove a result about arbitrary C*-algebras.

Toward a contradiction, assume that $\pi(A)' \neq \mathbb{C}\cdot\mathrm{id}$. Pick a self adjoint operator $0 \neq T \in \pi(A)'$ and define

$$B := \mathrm{C}^*(\{\mathrm{id}, T\}) \subseteq \pi(A)'.$$

Then B is a commutative C*-subalgebra of $\pi(A)'$ and by the commutative Gelfand-Naimark Theorem is isometrically *-isomorphic to $C_0(\Omega(B))$.

We now need a couple of supplementary results, which are consequences of Urysohn's Lemma.

Claim 5.39. *If X is a compact Hausdorff space which contains at least two distinct points, then there exist non-zero functions $f, g \in C(X)$ such that $fg = gf = 0$.*

Proof. We take distinct points $x, y \in X$ and disjoint open neighbourhoods $U, V \subseteq X$ of x and y respectively. We consider the disjoint closed sets $E := X \setminus U$ and $F := \{x\}$ and use Urysohn's Lemma to find $f \in C(X)$ which satisfies

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ 1 & \text{if } x \in F. \end{cases}$$

Similarly we consider the disjoint closed sets $G := X \setminus V$ and $H := \{y\}$ and apply Urysohn's Lemma again to find $g \in C(X)$ with $g|_G = 0$ and $g(y) = 1$. It is clear that f and g satisfy $fg = gf = 0$. $\square$

Claim 5.40. *If X is a locally compact Hausdorff space which contains at least two distinct points, then there exist non-zero functions $f, g \in C_0(X)$ such that $fg = gf = 0$.*

Proof. If X is compact then we are done by our previous claim. If X is not compact then we consider the one-point compactification of X, which will also contain at least two distinct points and is compact. Let $x, y \in X$ be distinct points and take disjoint open neighbourhoods $U, V \subseteq \widehat{X}$ of x and y respectively. Proceeding as we did in the previous claim, we use Urysohn's Lemma to find $f, g \in C(\widehat{X})$ with

$$f(t) = \begin{cases} 0 & \text{if } t \in (X \setminus U) \cup \{\infty\} \\ 1 & \text{if } t = x, \end{cases}$$

and

$$g(t) = \begin{cases} 0 & \text{if } t \in (X \setminus V) \cup \{\infty\} \\ 1 & \text{if } t = y. \end{cases}$$

We recall from Example 2.24 that $C_0(X)$ may be identified as the maximal ideal of $C(\widehat{X})$ of functions which are zero at the point $\infty \in \widehat{X}$. Under this identification, our f and g correspond to a pair of non-zero functions in $C_0(X)$ whose product is zero. $\square$

Returning to the main proof, we first note that $\Omega(B)$ shall contain at least two characters, then we use the supplementary results to find non-zero operators $S, R \in B$ with $RS = SR = 0$. If we take $\eta \in \ker(R)$, then for any $x \in A$ we have

$$(R\pi(x))\eta = (\pi(x)R)\eta = \pi(x)0 = 0,$$

because $R \in \pi(A)'$ and $R\eta = 0$. This implies that $\pi(x)\eta \in \ker(R)$ and therefore $\ker(R)$ is a closed invariant subspace for $\pi(A)$. Since $R \neq 0$ we see that $\ker(R) \neq \mathcal{H}$. Moreover, since the composition RS is the zero operator, we have

$$\mathcal{H} = \ker(RS) = S^{-1}(\ker(R)),$$

which implies that $S(\mathcal{H}) = \ker(R)$ and because $S \neq 0$ we must also have $\ker(R) \neq \{0\}$. Hence π is not topologically irreducible because $\ker R$ is a non-trivial closed invariant subspace for $\pi(A)$.

The other implications are substantially less work. If $\pi(A)' = \mathbb{C} \cdot \mathrm{id}$ then clearly we cannot have any non-trivial projections in $\pi(A)'$, because such operators would not be multiples of the identity operator, so the implication (ii)$\Rightarrow$(iii) is proved.

Finally, assume that $\pi(A)'$ contains no non-trivial projections and let $\mathcal{K} \subseteq \mathcal{H}$ be a closed invariant subspace for $\pi(A)$. Then $\mathcal{K}^\perp$ is also a closed invariant subspace for $\pi(A)$ and we let P denote the orthogonal projection onto $\mathcal{K}$. For any $x \in A$, we have

$$(P\pi(x))\xi = \pi(x)\xi = (\pi(x)P)\xi,$$

for all $\xi \in \mathcal{K}$, and

$$(P\pi(x))\eta = 0 = (\pi(x)P)\eta,$$

for all $\eta \in \mathcal{K}^\perp$. So P belongs to the commutant of $\pi(A)$, which by assumption means that P must be trivial. Therefore it must be the case that either $\mathcal{K} = \{0\}$ and $\mathcal{K}^\perp = \mathcal{H}$ or $\mathcal{K} = \mathcal{H}$ and $\mathcal{K}^\perp = \{0\}$. $\square$

As an application of this Theorem, we present the following.

Corollary 5.41. *Let A be a commutative C*-algebra and let $(\mathcal{H}, \pi)$ be a topologically irreducible representation of A, with $\mathcal{H} \neq \{0\}$. Then $\mathcal{H}$ is one dimensional.*

Proof. Since $(\mathcal{H}, \pi)$ is topologically irreducible, the previous theorem tells us that $\pi(A)' = \mathbb{C} \cdot \mathrm{id}$. A is commutative, so it follows that $\pi(A)$ is also commutative and therefore $\pi(A) \subseteq \pi(A)' = \mathbb{C} \cdot \mathrm{id}$. By irreducibility, the only closed subspaces which are invariant under $\pi(A)$ are $\{0\}$ and $\mathcal{H}$. However the fact that $\pi(A) \subseteq \mathbb{C} \cdot \mathrm{id}$ implies that every closed subspace of $\mathcal{H}$ will be invariant for $\pi(A)$, which in turn implies that $\mathcal{H}$ is one dimensional. $\square$

Remark 5.42. There is another notion of irreducibility for representations, which we shall now discuss briefly. A representation $(\mathcal{H}, \pi)$ is said to be *algebraically irreducible* if the only invariant subspaces (closed or otherwise) for $\pi(A)$ are $\{0\}$ and $\mathcal{H}$. Obviously if a representation is algebraically irreducible, then it is topologically irreducible, but remarkably the converse holds for C*-algebras, which is a consequence of Kadison's Transitivity Theorem [15, Theorems 5.2.2 and 5.2.3].

5.4. **Positive Linear Functionals and States.** The final ingredient we need before we tackle the Gelfand Naimark Theorem is the notion of a state. The results which we shall prove here are all for positive linear functionals on C*-algebras which have an unit, however using approximate units these can be generalised in an appropriate manner to non-unital algebras.

Definition 5.43. Let A be a C*-algebra. A linear functional $\omega : A \to \mathbb{C}$ is called *positive* if

$$\omega(x^*x) \geq 0,$$

for every $x \in A$. A *state* on A is a positive linear functional $\omega : A \to \mathbb{C}$ which additionally satisfies $\|\omega\|_{\mathrm{op}} = 1$ and we shall write $\mathcal{S}(A)$ for the set of all states of A.

Remark 5.44. It is important to note that we make no assumptions about the continuity of general positive linear functionals.

Observe that if A is a unital C*-algebra and $x \in A$ is a positive element, then by Lemma 5.13 $x = y^2 = y^*y$ for some self adjoint $y \in A$. Therefore if $\omega : A \to \mathbb{C}$ is a positive linear functional, then $\omega(x) = \omega(y^*y) \geq 0$.

Example 5.45. Let $\mathcal{H}$ be a Hilbert space and for each $\eta \in \mathcal{H}$ with $\|\eta\| = 1$ define a map $\omega_\eta : B(\mathcal{H}) \to \mathbb{C}$ by

$$\omega_\eta(T) = \langle \eta, T\eta \rangle,$$

for $T \in B(\mathcal{H})$. Linearity of ω_η follows from the linearity of inner products in their second argument, and for $T \in B(\mathcal{H})$ we have

$$\omega_\eta(T^*T) = \langle \eta, T^*T\eta \rangle = \langle T\eta, T\eta \rangle = \|T\eta\|^2 \geq 0.$$

Therefore ω_η is a positive linear functional. To conclude this example, if $T \in B(\mathcal{H})$ has $\|T\|_{\mathrm{op}} = 1$ then

$$|\omega_\eta(T)| = |\langle \eta, T\eta \rangle| \leq \|\eta\|\|T\eta\| \leq \|T\|_{\mathrm{op}} = 1,$$

so that ω_η is bounded. Moreover $\omega_\eta(id) = \langle \eta, \eta \rangle = 1$, so $\|\omega_\eta\|_{\mathrm{op}} = 1$ and thus ω_η is a state on $B(\mathcal{H})$.

Example 5.46. For some fixed $n \in \mathbb{N}$, define $\omega : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ by

$$\omega(A) = \sum_{k=1}^{n} a_{kk},$$

for all $A = (a_{ij}) \in \mathbb{M}_n(\mathbb{C})$. This ω returns the *trace* of a given matrix. We recall from linear algebra that the trace of a matrix is equivalently given by the sum of its eigenvalues. For any $A \in \mathbb{M}_n(\mathbb{C})$ the matrix A^*A is positive definite, and thus has positive real eigenvalues, so $\omega(A^*A) \geq 0$ and ω is a positive linear functional.

Lemma 5.47. *If A is a unital C*-algebra and ω is a positive linear functional, then*

(i) $\omega(x^*) = \overline{\omega(x)}$, *for all* $x \in A$, *(ii)* $|\omega(y^*x)|^2 \leq \omega(x^*x)\omega(y^*y)$*, for all* $x, y \in A$.

Proof. For the first statement, we take any $x, y \in A$ and $\lambda \in \mathbb{C}$. Then by positivity and linearity of ω we have

$$0 \leq \omega((\lambda x + y)^*(\lambda x + y)) = |\lambda|^2\omega(x^*x) + \lambda\omega(y^*x) + \overline{\lambda}\omega(x^*y) + \omega(y^*y),$$

which implies that $\lambda\omega(y^*x) + \overline{\lambda}\omega(x^*y) \in \mathbb{R}$. Putting $y = 1_A$ and $\lambda = i$ shows that

$$i\omega(x) - i\omega(x^*) \in \mathbb{R},$$

and therefore the real parts of $\omega(x)$ and $\omega(x^*)$ are equal. Now we put $y = 1_A$ and $\lambda = 1$ which shows that

$$\omega(x) + \omega(x^*) \in \mathbb{R},$$

so the imaginary parts of $\omega(x)$ and $\omega(x^*)$ are opposite. Altogether this shows that $\omega(x^*) = \overline{\omega(x)}$. For the second statement, we again take $x, y \in A$ and $\lambda \in \mathbb{C}$ and use the above to get the following inequality

$$0 \leq |\lambda|^2\omega(x^*x) + \lambda\omega(y^*x) + \overline{\lambda}\ \overline{\omega(y^*x)} + \omega(y^*y) = |\lambda|^2\omega(x^*x) + 2\mathrm{Re}(\lambda\omega(y^*x)) + \omega(y^*y),$$

which we rearrange to give $-2\mathrm{Re}(\lambda\omega(y^*x)) \leq |\lambda|^2\omega(x^*x) + \omega(y^*y)$. If $\omega(x^*x) = 0$ then the fact that the last inequality holds for every $\lambda \in \mathbb{C}$ means that we must have $\omega(y^*x) = 0$ and the result follows. If $\omega(x^*x) \neq 0$ then we take $z \in \mathbb{T}$ which satisfies $z\omega(y^*x) = |\omega(y^*x)|$, then let $\lambda = -z\frac{\omega(y^*y)^{\frac{1}{2}}}{\omega(x^*x)^{\frac{1}{2}}}$ to obtain

$$2\frac{\omega(y^*y)^{\frac{1}{2}}}{\omega(x^*x)^{\frac{1}{2}}}|\omega(y^*x)| \leq \frac{|\omega(y^*y)^{\frac{1}{2}}|^2}{|\omega(x^*x)^{\frac{1}{2}}|^2}\omega(x^*x) + \omega(y^*y) = 2\omega(y^*y).$$

Rearranging this gives

$$|\omega(y^*x)| \leq \omega(x^*x)^{\frac{1}{2}}\omega(y^*y)^{\frac{1}{2}},$$

and we are done. □

Theorem 5.48. *If A is a unital C*-algebra, then a linear functional $\omega : A \to \mathbb{C}$ is positive if and only if ω is bounded and $\omega(1_A) = \|\omega\|_{op}$.*

Proof. First we assume that ω is positive. Let $x \in A$ be any element and write $x = a + ib$ for the self adjoint elements $a = \frac{1}{2}(x + x^*)$ and $b = \frac{1}{2i}(x - x^*)$. Self adjointness and Lemma 5.47(i) shows that $\omega(a) = \omega(a^*) = \overline{\omega(a)}$ and similarly for $\omega(b)$, thus implying that $\omega(a),\ \omega(b) \in \mathbb{R}$. Using our decomposition of x, we have

$$\mathrm{Re}(\omega(x)) = \mathrm{Re}(\omega(a) + i\omega(b)) = \omega(a).$$

The element $\|x\|1_A - a \in A$ is self adjoint and by the spectral mapping property for polynomials,

$$\sigma(\|x\|1_A - a) = \{\|x\| - \lambda;\ \lambda \in \sigma(a)\}.$$

For $\lambda \in \sigma(a)$, the inequality $|\lambda| \leq r(a) = \|a\|$ may be expanded to give

$$-\|a\| \leq \lambda \leq \|a\|,$$

which we rearrange to see that $0 \leq \|a\| - \lambda$. Now we recall that $a = \frac{1}{2}(x + x^*)$, so that $\|a\| \leq \|x\|$ and in fact $0 \leq \|x\| - \lambda$. From this it follows that $\sigma(\|x\|1_A - a) \subseteq \mathbb{R}^{\geq 0}$ and so $\|x\|1_A - a$ is positive. We are assuming that ω is positive, so $0 \leq \omega(\|x\|1_A - a)$ and we rearrange to see that $\omega(a) \leq \omega(\|x\|1_A)$. We conclude the first part of our proof by taking $\lambda \in \mathbb{T}$ such that $\lambda\omega(x) = |\omega(x)|$, then we make the following calculation,

$$|\omega(x)| = \mathrm{Re}(|\omega(x)|) = \mathrm{Re}(\omega(\lambda x)) = \omega(\lambda a) \leq \omega(\|\lambda x\|1_A) = \|x\|\omega(1_A),$$

in which we have used our previous work and the fact that $|\lambda| = 1$. This proves that $\|\omega\|_{\mathrm{op}} \leq \omega(1_A)$ and because $|\omega(1_A)| = \omega(1_A)$ it follows thaat that $\|\omega\|_{\mathrm{op}} = \omega(1_A)$.

Before commencing the proof of the opposite implication, we note that it suffices for us to check that if $\omega(1_A) = \|\omega\|_{\mathrm{op}} = 1$, then ω is a state. This is because if ω is bounded with

$\omega(1_A) = \|\omega\|_{\text{op}}$, then the norm of the functional $\frac{\omega}{\|\omega\|_{\text{op}}}$ is equal to one. So if $\frac{\omega}{\|\omega\|_{\text{op}}}$ is positive, then

$$\frac{\omega(x^*x)}{\|\omega\|_{\text{op}}} \geq 0,$$

for every $x \in A$ and we multiply by $\|\omega\|_{\text{op}}$ to see that ω is positive.

Lets assume now that ω is a bounded linear functional satisfying $\omega(1_A) = \|\omega\|_{\text{op}} = 1$. We take $x \in A$ and if $x^*x = 0$, then $\omega(x^*x) = 0$, so we may assume that $x^*x \neq 0$. Then we write

$$\omega(x^*x) = a + bi,$$

for some $a, b \in \mathbb{R}$. For the sake of neatness, we let $y := x^*x$. Now let ϵ be any real number satisfying $\epsilon \leq \frac{1}{\|y\|}$, the spectral mapping property for polynomials shows that

$$\sigma(1_A - \epsilon y) = \{1 - \epsilon\lambda;\ \lambda \in \sigma(y)\}.$$

For $\lambda \in \sigma(y)$, it is not difficult to check that the inequality $1 - \epsilon\lambda \geq 0$ is true and that $\sigma(1_A - \epsilon y) \subseteq [0, 1]$. Now we make the estimate

$$1 - \epsilon a \leq \sqrt{(1 - \epsilon a)^2 + b^2} = |(1 - \epsilon a) - bi| = |1 - \epsilon(a + bi)|.$$

We recall that $\omega(y) = a + bi$, so

$$1 - \epsilon a \leq |1 - \epsilon\omega(y)| = |\omega(1_A - \epsilon y)| \leq \|1_A - \epsilon y\|,$$

with the final inequality following from our boundedness assumption. Since $1_A - \epsilon y$ is self adjoint, we have

$$\|1_A - \epsilon y\| = r(1_A - \epsilon y) \leq 1,$$

and hence deduce that

$$1 - \epsilon a \leq 1,$$

implying that $a \geq 0$. All that remains now is for us to show that $b = 0$. Fix any $n \in \mathbb{N}$, then

$$\|y + in1_A\|^2 = \|(y)^2 + n^2 1_A\| \leq \|y\|^2 + n^2,$$

by the C*-identity and triangle inequality. We combine with our assumption that ω is bounded to get

$$|\omega(y + in1_A)|^2 \leq (\|y\|^2 + n^2).$$

Expanding the left hand side, we see that

$$|\omega(y + in1_A)|^2 = |\omega(y) + in|^2 = a^2 + (b + n)^2 = a^2 + b^2 + n^2 + 2bn,$$

and we deduce that

$$a^2 + b^2 + 2bn \leq \|y\|^2.$$

Because this must hold for every $n \in \mathbb{N}$, it follows that b must be zero, hence $\omega(y) = \omega(x^*x) = a \geq 0$ and ω is positive.

□

Corollary 5.49. *If A is a unital C*-algebra and ω is a positive linear functional on A, then for every $x, y \in A$ we have*

$$\omega(y^*x^*xy) \leq \|x\|^2\omega(y^*y).$$

Proof. Given $x, y \in A$ we consider the linear functional $\psi : A \to \mathbb{C}$ which is defined for $z \in A$ by

$$\psi(z) = \omega(y^*zy).$$

Since $\psi(z^*z) = \omega(y^*z^*zy) = \omega((zy)^*zy)$, it follows that ψ is positive. Therefore $\|\psi\|_{\text{op}} = \psi(1_A) = \omega(y^*y)$ and we have

$$\omega(y^*x^*xy) = \psi(x^*x) \leq \|\psi\|_{\text{op}}\|x^*x\| = \omega(y^*y)\|x\|^2.$$

□

Example 5.50. Let A be a unital C*-algebra and suppose that $(\mathcal{H}, \pi)$ is a cyclic representation of A, with cyclic vector $\xi \in \mathcal{H}$. Then we define a map $\omega_\pi : A \to \mathbb{C}$ by

$$\omega_\pi(x) = \langle \xi, \pi(x)\xi \rangle,$$

for all $x \in A$. This is a linear map and for any $x \in A$ we have

$$\omega_\pi(x^*x) = \langle \xi, \pi(x)^*\pi(x)\xi \rangle = \langle \pi(x)\xi, \pi(x)\xi \rangle = \|\pi(x)\xi\|^2 \geq 0,$$

so that ω_π is a positive linear functional. If we assume further that $\|\xi\| = 1$ then the inequality

$$|\omega_\pi(x)| \leq \|\xi\|\|\pi(x)\xi\| \leq \|x\|_A,$$

holds for every $x \in A$, thus showing that $\|\omega_\pi\|_{\text{op}} \leq 1$. We combine this with the fact that cyclic representations are non-degenerate to see that $\|\omega\|_{\text{op}} = \omega(1_A) = 1$ and so ω is a state.

We shall see in the next section that every cyclic representation of a unital C*-algebra arises in this manner.

Theorem 5.51. *If A is a C*-algebra and $0 \neq x \in A$ is self adjoint, then there exists a state ω on A with $|\omega(x)| = \|x\|$.*

Proof. If necessary, we pass to the unitization of A and may assume that A is unital. We take a self adjoint element $x \in A$ and consider the commutative C*-subalgebra $B := \mathrm{C}^*(\{1_A, x\}) \subseteq A$. Because x is self adjoint, we know that $\|x\| = r(x)$ and by our earlier work on the Gelfand transform, we know that

$$r(x) = \sup\{|\widehat{x}(\varphi)|;\ \varphi \in \Omega(B)\}.$$

Since $\widehat{x}$ is a continuous function defined on the compact set $\Omega(B)$, there must be a character $\tau \in \Omega(B)$ so that the supremum is attained. So we have

$$\|x\| = r(x) = |\widehat{x}(\tau)| = |\tau(x)|,$$

and we recall that since τ is a character on a unital algebra, $\|\tau\|_{\text{op}} = 1$. Using the Hahn-Banach Theorem, we find an extension ω of τ to A which satisfies $\|\omega\|_{\text{op}} = 1$. Moreover since τ is a character and ω extends τ, we must have that $\omega(1_A) = 1$ and hence by Theorem 5.48, ω is a state.

We finish the proof by noting that if we had to use the unitization of A, then the restriction $\omega|_A$ of ω to A is also positive and satisfies $\|\omega|_A\|_{\text{op}} \leq 1$. But since $|\omega|_A(x)| = |\omega(x)| = \|x\|$ for every $x \in A$, it follows that $\|\omega|_A\|_{\text{op}} = 1$ so that $\omega|_A$ is a state on A. $\square$

5.5. **The Gelfand-Naimark Theorem.** This section is what the preceding chapters have all been leading up to. We shall shortly prove the Gelfand-Naimark Theorem, which is arguably one of the most important results in the theory of C*-algebras. It asserts that every C*-algebra is *-isomorphic to a C*-algebra of bounded operators on some Hilbert space, which is quite remarkable when we consider our initial definition of a C*-algebra. The proof of this theorem relies on the Gelfand-Naimark-Segal construction, which we henceforth call the GNS construction.

Definition 5.52. Let A be a C*-algebra and let ω be a state on A. A *GNS pair* for ω is a pair (π, ξ) where $(\mathcal{H}, \pi)$ is a representation of A on some Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a cyclic vector which satisfies

$$\omega(x) = \langle \xi, \pi(x)\xi \rangle,$$

for all $x \in A$.

The GNS construction tells us how we can take a state on a unital C*-algebra and produce a representation of the algebra. In textbooks, the details of this constriction are often left as an exercise so we shall provide full details here.

Theorem 5.53. (The GNS Construction)
If A is a unital C-algebra, then every state ω on A has a GNS pair. Moreover any two GNS pairs for ω are unitary equivalent.*

Proof. As the name suggests, we shall give a step by step method for constructing a GNS pair for each state, so let ω be any state on A.

Step 1: *Show that* $\mathcal{N}_\omega := \{x \in A;\, \omega(x^*x) = 0\}$ *is a left ideal of* A.
With $\mathcal{N}_\omega$ as above, take any $x, y \in \mathcal{N}_\omega$ and any $\lambda \in \mathbb{C}$, then by Lemma 5.47(ii) we have that $|\omega(x^*y)|^2 = |\omega(y^*x)|^2 = 0$, so it follows that

$$\omega((x+\lambda y)^*(x+\lambda y)) = \lambda\omega(x^*y) + \overline{\lambda}\omega(y^*x) = 0,$$

therefore $\mathcal{N}_\omega$ is a linear subspace of A. Now take $x \in A$ and $n \in \mathcal{N}_\omega$, Corollary 5.49 gives the inequality

$$\omega((xn)^*xn) = \omega(n^*x^*xn) \leq \|x\|^2\omega(n^*n) = 0,$$

which combined with the fact that ω is positive proves that $\mathcal{N}_\omega$ is a left ideal of A. We shall consider the quotient vector space $A/\mathcal{N}_\omega$.

Step 2: *Define an inner product on* $A/\mathcal{N}_\omega$.
We will write x_ω for the coset $x + \mathcal{N}_\omega \in A/\mathcal{N}_\omega$ and define a map $\langle \cdot, \cdot \rangle : A/\mathcal{N}_\omega \to \mathbb{C}$ by

$$\langle x_\omega, y_\omega \rangle = \omega(x^*y),$$

for all $x_\omega, y_\omega \in A/\mathcal{N}_\omega$. Checking that this satisfies the properties required of an inner product is easy, so we omit the details but we shall check that the map is well defined. Suppose that cosets $x_\omega, \widetilde{x}_\omega, y_\omega, \widetilde{y}_\omega \in A/\mathcal{N}_\omega$ satisfy $x_\omega = \widetilde{x}_\omega$ and $y_\omega = \widetilde{y}_\omega$, then we note that by definition $x - \widetilde{x} \in \mathcal{N}_\omega$ and $y - \widetilde{y} \in \mathcal{N}_\omega$. Using this observation we see that

$$\langle x_\omega, y_\omega \rangle - \langle \widetilde{x}_\omega, y_\omega \rangle = \omega((x-\widetilde{x})^*y) = 0,$$

and similarly $\langle \widetilde{x}_\omega, y_\omega \rangle - \langle \widetilde{x}_\omega, \widetilde{y}_\omega \rangle = 0$, so we have

$$\langle x_\omega, y_\omega \rangle = \langle \widetilde{x}_\omega, y_\omega \rangle = \langle \widetilde{x}_\omega, \widetilde{y}_\omega \rangle,$$

We form the completion of this inner product space and denote the resulting Hilbert space by $\mathcal{H}_\omega$.

Step 3: *For each* $x \in A$*, construct a bounded linear map* $\pi_\omega(x) : A/\mathcal{N}_\omega \to A/\mathcal{N}_\omega$.
We take $x \in A$ and define $\pi_\omega(x)$ by $\pi_\omega(x)y_\omega = xy + \mathcal{N}_\omega$ for all $y_\omega \in A/\mathcal{N}_\omega$. First we should first check that $\pi_\omega(x)$ is well defined, so assume that $y_\omega, \widetilde{y}_\omega \in A/\mathcal{N}_\omega$ are such that $y - \widetilde{y} \in A/\mathcal{N}_\omega$, then we have

$$\pi_\omega(x)y_\omega - \pi_\omega(x)\widetilde{y}_\omega = x(y-\widetilde{y}) + A/\mathcal{N}_\omega.$$

We notice that $x(y-\widetilde{y}) \in A/\mathcal{N}_\omega$ because $\mathcal{N}_\omega$ is a left ideal and because $y - \widetilde{y} \in A/\mathcal{N}_\omega$, therefore $\pi_\omega(x)y_\omega = \pi_\omega(x)\widetilde{y}_\omega$. We now take $y_\omega, z_\omega \in A/\mathcal{N}_\omega$ and $\lambda, \mu \in \mathbb{C}$, then

$$\begin{aligned}\pi_\omega(x)(\lambda y_\omega + \mu z_\omega) &= (\lambda xy + \mu xz) + \mathcal{N}_\omega \\ &= \lambda(xy + \mathcal{N}_\omega) + \mu(xz + \mathcal{N}_\omega) \\ &= \lambda\pi_\omega(x)y_\omega + \mu\pi_\omega(x)z_\omega,\end{aligned}$$

so $\pi_\omega(x)$ is linear and for boundedness we have

$$\|\pi_\omega(x)y_\omega\|^2_{A/\mathcal{N}_\omega} = \omega((xy)^*xy) = \omega(y^*x^*xy) \leq \|x\|_A^2\omega(y^*y) = \|x\|_A^2\|y_\omega\|^2_{A/\mathcal{N}_\omega},$$

where the inequality is due to Corollary 5.49. Because $\pi_\omega(x)$ is continuous, we may extend it uniquely to a bounded linear operator on $\mathcal{H}_\omega$, which we shall also denote by $\pi_\omega(x)$. Seeing as our initial choice of x was arbitrary, this step gives us an operator $\pi_\omega(x) \in B(\mathcal{H}_\omega)$ for each $x \in A$.

Step 4: *Show that the map* $x \mapsto \pi_\omega(x)$ *is a representation of* A *on* $\mathcal{H}_\omega$.
We let $\pi_\omega : A \to B(\mathcal{H}_\omega)$ denote the map $x \mapsto \pi_\omega(x)$ and we must check that this is a *-homomorphism. To show that π_ω is a homomorphism, let $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$, then the

relations

$$\pi_\omega(\lambda x + \mu y) = \lambda \pi_\omega(x) + \mu \pi_\omega(y),$$
$$\pi_\omega(xy) = \pi_\omega(x)\pi_\omega(y),$$

both hold if we restrict the operators to $A/\mathcal{N}_\omega$ and we argue by continuity of each $\pi_\omega(x)$ that these relations also hold on the completion $\mathcal{H}_\omega$. Moreover for $x \in A$ and $y_\omega, z_\omega \in A/\mathcal{N}_\omega$ we have

$$\langle y_\omega, \pi(x) z_\omega \rangle = \omega(y^* x z) = \langle \pi_\omega(x^*) y_\omega, z_\omega \rangle,$$

and once again we argue by continuity that this also holds on $\mathcal{H}_\omega$.

Step 5: *The vector $\xi_\omega = 1_A + \mathcal{N}_\omega \in \mathcal{H}_\omega$ is cyclic, with $\|\xi\|_\omega = 1$ and (π_ω, ξ_ω) is a GNS pair for ω.*
We take any $x \in A$, then $\pi_\omega(x)\xi_\omega = x_\omega$, giving the inclusion

$$A/\mathcal{N}_\omega \subseteq \pi_\omega(A)\xi_\omega \subseteq \mathcal{H}_\omega.$$

It follows once we take closures that $\pi_\omega(A)\xi_\omega$ is dense in $\mathcal{H}_\omega$ so that ξ_ω is a cyclic vector. For the second statement, we simply compute the norm of ξ_ω,

$$\|\xi_\omega\| = \sqrt{\langle 1_A + \mathcal{N}_\omega, 1_A + \mathcal{N}_\omega \rangle} = \omega(1_A^* 1_A)^{\frac{1}{2}} = \omega(1_A)^{\frac{1}{2}} = 1,$$

and for the final statement we take any $x \in A$ and compute

$$\langle \xi_\omega, \pi_\omega(x)\xi_\omega \rangle = \langle 1_A + \mathcal{N}_\omega, x + \mathcal{N}_\omega \rangle = \omega(1_A^* x) = \omega(x).$$

Step 6: *Any two GNS pairs for ω are unitarily equivalent.*
Suppose that $(\tau_\omega, \eta_\omega)$ is another GNS pair for ω, with corresponding representation $(\mathcal{K}, \tau_\omega)$. We define a map $u : \pi_\omega(A)\xi_\omega \to \tau_\omega(A)\eta_\omega$ by

$$u(\pi_\omega(x)\xi_\omega) = \tau_\omega(x)\eta_\omega,$$

for $x \in A$. Then u is a surjective linear map which for any $x \in A$ satisfies

$$\begin{aligned}
\langle \pi_\omega(x)\xi_\omega, \pi_\omega(x)\xi_\omega \rangle &= \langle \xi_\omega, \pi_\omega(x)^* \pi_\omega(x)\xi_\omega \rangle \\
&= \langle \xi_\omega, \pi_\omega(x^* x)\xi_\omega \rangle \\
&= \omega(x^* x) \\
&= \langle \eta_\omega, \tau_\omega(x^* x)\eta_\omega \rangle \\
&= \langle \tau_\omega(x)\eta_\omega, \tau_\omega(x)\eta_\omega \rangle,
\end{aligned}$$

so that u is an isometry and we may now extend u uniquely to a unitary operator $U : \mathcal{H} \to \mathcal{K}$. We see that because $\xi_\omega = \pi_\omega(1_A)\xi_\omega \in \pi_\omega(A)\xi_\omega$, we have

$$U(\xi_\omega) = u(\pi_\omega(1_A)\xi_\omega) = \tau_\omega(1_A)\eta_\omega = \eta_\omega,$$

and using this we see that for any $x \in A$

$$U(\pi_\omega(x)\xi_\omega) = \tau_\omega(x)\eta_\omega = \tau_\omega(x)U(\eta_\omega).$$

Since $\pi_\omega(A)\xi_\omega$ is dense in $\mathcal{H}$ we now argue that $U\pi_\omega(x) = \tau_\omega(x)U$ for any $x \in A$. □

Remark 5.54. The GNS construction has a generalisation for states on non-unital C*-algebras, which we do not cover.

Definition 5.55. If A is a unital C*-algebra and $\omega : A \to \mathbb{C}$ is a state, then if $(\mathcal{H}_\omega, \pi_\omega)$ is the representation of A created by the GNS construction, we refer to $(\mathcal{H}_\omega, \pi_\omega)$ as a *GNS representation* of A.

Example 5.56. [2, Exercise 1.6.A(a)]
Let $\mathcal{H}$ be a Hilbert space and for a unit vector $\eta \in \mathcal{H}$ let ω_η be the state we defined in Example 5.45. Because $B(\mathcal{H})$ is a unital C*-algebra, we may apply the GNS construction to produce a GNS pair (π_η, ξ_η) for ω_η with corresponding GNS representation $(\mathcal{H}_\eta, \pi_\eta)$. Now consider the identity representation $\pi_{id} : B(\mathcal{H}) \to B(\mathcal{H})$. Because ω_η is defined by

$$\omega_\eta(T) = \langle \eta, T\eta \rangle,$$

for each $T \in B(\mathcal{H})$, if we can prove that η is a cyclic vector for π_{id} then (π_{id}, η) will also be a GNS pair for ω_η. It will then follow by step 6 of the GNS construction that π_{id} and π_η are unitarily equivalent. Example 5.36 tells us that $B(\mathcal{H})' = \mathbb{C} \cdot \mathrm{id}$, so by Theorem 5.37 the identity representation is irreducible. $\overline{\pi_{id}(B(\mathcal{H}))\eta}$ is a closed invariant subspace for π_{id}, so by irreducibility is either $\{0\}$ or $\mathcal{H}$. It obviously cannot be the zero subspace, so (π_{id}, η) is also a GNS pair for ω_η and hence unitarily equivalent to the GNS representation $(\mathcal{H}_\eta, \pi_\eta)$.

Definition 5.57. If A is a unital C*-algebra then its *universal representation* is the representation

$$(\oplus_\omega \mathcal{H}_\omega, \oplus_\omega \pi_\omega),$$

where we have taken the direct sum of all the GNS representations as ω runs through all the states of A.

Theorem 5.58. (The Gelfand-Naimark Theorem)
If A is any C-algebra, then A possesses a faithful representation. Consequently, every C*-algebra is isometrically *-isomorphic to a C*-subalgebra $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Proof. Assume first that A is unital and consider the universal representation $(\oplus_\omega \mathcal{H}_\omega, \oplus_\omega \pi_\omega)$ of A. Suppose that $\oplus_\omega \pi_\omega(x) = 0$ for some $0 \neq x \in A$, so

$$\oplus_\omega \pi_\omega(x)(\nu_\omega) = (\oplus_\omega \pi_\omega(x)\nu_\omega)_{\omega \in \mathcal{S}(A)} = (0),$$

for all $(\nu_\omega) \in \oplus_\omega \mathcal{H}_\omega$. The element $x^*x \in A$ is self adjoint (and non-zero) so there exists a state τ on A which satisfies

$$|\tau(x^*x)| = \|x^*x\| = \|x\|^2.$$

Since τ is a state, it has a GNS pair (π_τ, ξ_τ) and therefore

$$\tau(x^*x) = \langle \xi_\tau, \pi_\tau(x^*x)\xi_\tau)\rangle = \langle \xi_\tau, \pi_\tau(x)^*\pi_\tau(x)\xi_\tau)\rangle = 0,$$

because $\oplus_\omega \pi_\omega(x) = 0$. This shows that

$$0 = \tau(x^*x) = \|x^*x\| = \|x\|^2,$$

which of course is equivalent to $x = 0$, proving that the universal representation of A is faithful.

If A is non unital, then the preceding paragraph tells us that $\widetilde{A}$ has a faithful representation which we denote by $(\mathcal{H}, \pi)$. The composition

$$A \xhookrightarrow{\varphi} \widetilde{A} \xrightarrow{\pi} B(\mathcal{H}),$$

where $\varphi : A \to \widetilde{A}$ is the canonical embedding is then a faithful representation of A. □

As an application of the Gelfand Naimark Theorem, we present the following example, which concerns matrices whose entries are from a C*-algebra.

Example 5.59. Let A be an C*-algebra and let $\mathbb{M}_n(A)$ denote the set of $n \times n$ matrices with entries from A. If we define scalar multiplication, matrix addition and matrix multiplication exactly as we do for $\mathbb{M}_n(\mathbb{K})$ then $\mathbb{M}_n(A)$ is an algebra and is in fact a *-algebra under the involution

$$(a_{ij})^* = (a_{ij}^*),$$

for $(a_{ij}) \in \mathbb{M}_n(A)$. Observe that if B is another C*-algebra and $\varphi : A \to B$ is a *-homomorphism, then there is an obvious map $\mathbb{M}_n(A) \to \mathbb{M}_n(B)$ where we simply apply φ to each entry in a given matrix. This is also a *-homomorphism.

We want to show that $\mathbb{M}_n(A)$ is in fact a C*-algebra. For this, we first note that by the Gelfand-Naimark Theorem, A has a faithful representation which we denote by $(\mathcal{H},\pi)$. For $X \in \mathbb{M}_n(B(\mathcal{H}))$ we let $\tau(X) : \mathcal{H}^n \to \mathcal{H}^n$ be the map given by

$$\tau(X)(\nu_1,\dots,\nu_n) = \left(\sum_{k=1}^{n} X_{1k}(\nu_k),\dots,\sum_{k=1}^{n} X_{nk}(\nu_k)\right),$$

for $(\nu_1,\dots,\nu_n) \in \mathcal{H}^n$.

Claim 5.60. *The map $\tau : \mathbb{M}_n(B(\mathcal{H})) \to B(\mathcal{H}^n)$ is a *-isomorphism, where for each $X \in \mathbb{M}_n(B(\mathcal{H}))$, $\tau(X)$ is the map described above.*

Proof. Take any matrix $X \in \mathbb{M}_n(B(\mathcal{H}))$. We will first show that $\tau(X)$ belongs to $B(\mathcal{H}^n)$, indeed if $(\nu_1,\dots,\nu_n) \in \mathcal{H}^n$ satisfies $\|(\nu_1,\dots,\nu_n)\|_{\mathcal{H}^n} = \sum_{k=1}^n \|\nu_k\|_{\mathcal{H}} = 1$, then in particular $\|\nu_k\|_{\mathcal{H}} \leq 1$ for each $1 \leq k \leq n$. Moreover we have

$$\begin{aligned}
\|\tau(X)(\nu_1,\dots,\nu_n)\|_{\mathcal{H}^n} &= \sum_{j=1}^{n} \left\| \sum_{k=1}^{n} X_{jk}\nu_k \right\|_{\mathcal{H}} \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{n} \|X_{jk}(\nu_k)\|_{\mathcal{H}} \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{n} \|X_{jk}\|_{\text{op}}\|\nu_k\|_{\mathcal{H}} \\
&\leq \sum_{j=1}^{n}\sum_{k=1}^{n} \|X_{jk}\|_{\text{op}},
\end{aligned}$$

which shows that $\tau(X)$ is bounded, thus really does belong to $B(\mathcal{H}^n)$. One may easily check that τ is a *-homomorphism, and for the bijectivity of τ first observe that

$$\tau(X) = 0 \iff X_{jk} = 0 \text{ for all } 1 \leq j,k \leq n \iff X = 0,$$

so τ is injective. Then for any operator $(X_1,\dots,X_n) \in B(\mathcal{H}^n)$ we let X denote the matrix whose i^{th} row consists solely of the operator X_i, then

$$\tau(X) = (X_1,\dots,X_n),$$

so τ is surjective. □

Claim 5.61. $\mathbb{M}_n(B(\mathcal{H}))$ *is a C*-algebra.*

Proof. We define a norm $\|\cdot\|$ on $\mathbb{M}_n(B(\mathcal{H}))$ by $\|X\| = \|\tau(X)\|_{\text{op}}$. That this is a submultiplicative norm which satisfies the C*-identity follows from the properties of τ and the operator norm. For example

$$\|X^*X\| = \|\tau(X)^*\tau(X)\|_{\text{op}} = \|\tau(X)\|_{\text{op}} = \|x\|^2,$$

holds for every $X \in \mathbb{M}_n(B(\mathcal{H}))$ and

$$0 = \|X\| = \|\tau(X)\|_{\text{op}} \iff \tau(X) = 0 \iff X = 0,$$

because τ is injective. To prove completeness, we make the observation that for $X \in \mathbb{M}_n(B(\mathcal{H}))$, the estimates made to show that $\tau(X)$ is bounded imply that

$$\|X_{jk}\|_{\text{op}} \leq \|\tau(X)\|_{\text{op}} \leq \sum_{j=1}^{n}\sum_{k=1}^{n} \|X_{jk}\|_{\text{op}},$$

for all $1 \leq j,k \leq n$. So if $(X^{(n)})$ is a Cauchy sequence in $\mathbb{M}_n(B(\mathcal{H}))$, then for any $\epsilon > 0$, we may find $N \in \mathbb{N}$ such that whenever $m,n > N$ the inequality

$$\epsilon > \|X^{(n)} - X^{(m)}\| = \|\tau(X^{(n)} - X^{(m)})\|_{\text{op}} \geq \|X_{jk}^{(n)} - X_{jk}^{(m)}\|_{\text{op}},$$

holds for every $1 \leq i, j \leq n$. Therefore each of the sequences $(X_{jk}^{(n)})$ is Cauchy in $B(\mathcal{H})$ and is therefore convergent. The matrix formed in the obvious manner from the limits of these sequences is easily checked to be the limit of the initial sequence. $\square$

Returning to the main problem, let $\varphi_\pi : \mathbb{M}_n(A) \to \mathbb{M}_n(B(\mathcal{H}))$ denote the *-homomorphism which is induced by the map $\pi : A \to B(\mathcal{H})$. Since π is injective, it follows that φ_π is injective and we define a norm on $\mathbb{M}_n(A)$ by

$$\|x\| = \|\varphi_\pi(x)\|,$$

for any $x \in \mathbb{M}_n(A)$. Proving that $\mathbb{M}_n(A)$ is now easy, if $x \in \mathbb{M}_n(A)$ then the inequalities

$$\|x_{jk}\|_A = \|\varphi_\pi(x)_{jk}\| \leq \|\tau(\varphi_\pi(x))\|_{\mathrm{op}} \leq \sum_{j=1}^{n}\sum_{k=1}^{n} \|\varphi_\pi(x)_{jk}\| = \sum_{j=1}^{n}\sum_{k=1}^{n} \|x_{jk}\|_A,$$

hold for every $1 \leq j, k \leq n$, and we can now simply mimic the proof that we gave for Claim 5.61.

6. The C*-algebra of Canonical Commutation Relations

The main purpose of this chapter is to present a theorem due to Joseph Slawny. Without going into the details, his theorem states that to every (non-degenerate) real symplectic vector space, we can assign a unital C*-algebra in a non-trivial manner, and the C*-algebra is unique up to *-isomorphism.

6.1. Prelude: Completions and Extensions.

Before we get into the meat of this chapter, we present a few supplementary results. We begin with a brief discussion of how we may complete a *-algebra to give a C*-algebra. Before we do this, lets recap how to form the completion of a normed space. We shall be deliberately brief here since more information may easily be found in textbooks, for example [11, Theorem 1.5.1].

If we have a normed vector space A, then we may gather together all of the the Cauchy sequences in A, introduce the the equivalence relation

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} \|x_n - y_n\|_A = 0,$$

and then define appropriate operations to make the corresponding quotient space into a vector space, denoted by $\overline{A}$. The norm of A may be extended to $\overline{A}$ by continuity and there is also a linear isometry $\iota : A \to \overline{A}$ with $\iota(A)$ dense in $\overline{A}$. The space $\overline{A}$ is called the Banach space completion of A.

If we started with an incomplete normed algebra, then the above procedure may be used to give a Banach space and we can then extend the multiplication of A to $\overline{A}$ by continuity, giving us a Banach algebra. In this situation we call $\overline{A}$ the Banach algebra completion of A.

Unsurprisingly there is a similar procedure in which we can complete a *-algebra to a C*-algebra, but we must insist that the norm is more than merely submultiplicative.

Definition 6.1. Let A be an *-algebra. We shall call a norm $\|\cdot\| : A \to \mathbb{R}^{\geq 0}$ a *C*-norm* if it satisfies

(i) $\|xy\| \leq \|x\|\|y\|$,
(ii) $\|x^*x\| = \|x\|^2$,

for all $x, y \in A$. We do not insist that A be complete under a given C*-norm, so A may not be a C*-algebra. Consequently Corollary 4.33 fails for C*-norms and we can have many different C*-norms on a *-algebra.

Remark 6.2. Let $\|\cdot\|$ be a C*-norm on a *-algebra A. Looking back to the proof of Lemma 4.17 we see that the proof may be simply copied verbatim to show that $\|x^*\| = \|x\|$ for every $x \in A$.

Example 6.3. Suppose that A is a *-algebra, B is a C*-algebra and $\varphi : A \to B$ is an injective *-homomorphism. Define $\|\cdot\| : A \to \mathbb{R}^{\geq 0}$ by

$$\|x\| = \|\varphi(x)\|_B,$$

for every $x \in A$. Using the fact that φ is an algebra homomorphism, we can show that $\|\cdot\|$ is a submultiplicative norm on A, where injectivity of φ is used to show that $\|x\| = 0$ if and only if $x = 0$. The C*-identity holds because φ is a *-homomorphism and B is a C*-algebra. Thus $\|\cdot\|$ is a C*-norm on A.

If we have a *-algebra which is equipped with an incomplete C*-norm, then upon forming the Banach algebra completion we may extend the involution by continuity and are left with a C*-algebra. As well as completions, we need to speak briefly about extending a *-homomorphism from a dense *-subalgebra of a C*-algebra. The proof of the following lemma is very straightforward, but we present it nevertheless.

Lemma 6.4. *Let A and B be C*-algebras, let $\mathcal{D} \subseteq A$ be a dense *-subalgebra and assume that $\phi : \mathcal{D} \to B$ is a *-homomorphism. If ϕ is continuous with respect to $\|\cdot\|_A$ then ϕ extends uniquely to a *-homomorphism $\Phi : A \to B$.*

Proof. We note that if such an extension exists, it will certainly be unique. Our continuity assumption gives us a constant $M \in \mathbb{R}^{>0}$ such that $\|\phi(x)\|_B \leq M\|x\|_A$ for any $x \in \mathcal{D}$. By standard results, ϕ extends to a map $\Phi : A \to B$ which satisfies $\|\Phi(x)\|_B \leq M\|x\|_A$, for every $x \in A$. Take $x, y \in A$ and sequences $(x_n), (y_n)$ in $\mathcal{D}$ which converge to x and y respectively. By continuity of multiplication and involution in a C*-algebra, we have that $x_n y_n \to xy$ and $x_n^* \to x^*$. In addition we take scalars $\lambda, \mu \in \mathbb{C}$ and then we have

$$\Phi(\lambda x + \mu y) = \lim_{n\to\infty} \phi(\lambda x_n + \mu y_n) = \lambda \lim_{n\to\infty} \phi(x_n) + \mu \lim_{n\to\infty} \phi(y_n) = \lambda\Phi(x) + \mu\Phi(y),$$

so Φ is linear,

$$\Phi(xy) = \lim_{n\to\infty} \phi(x_n y_n) = \lim_{n\to\infty} \phi(x_n)\phi(y_n) = \Phi(x)\Phi(y),$$

so Φ is an algebra homomorphism,

$$\Phi(x_n^*) = \lim_{n\to\infty} \phi(x_n^*) = \lim_{n\to\infty} \phi(x_n)^* = \Phi(x)^*,$$

so Φ is a *-homomorphism. All of the equalities above follow from either the definition of Φ, the fact that ϕ is a $*$-homomorphism or continuity of multiplication and involution in B. □

6.2. **Canonical Commutation Relations.** In order to properly motivate the final part of this chapter, we shall now provide a rapid introduction to canonical commutation relations. For brevity, we shall henceforth abbreviate canonical commutation relations simply to CCR.

The mathematics of quantum mechanics is formulated using the language of Hilbert spaces and the linear operators which act upon them. If p and q are the self adjoint operators representing the position and momentum of a particle, then they are required to satisfy the commutation relation

$$pq - qp = i\hbar.$$

One of the challenges that arises is to find operators which satisfy these relations. There has been much work done in investigating what properties such operators will carry, indeed there is an entire textbook [19] devoted to the subject. What we shall demonstrate now is one unavoidable restriction that is imposed on any pair of operators which satisfy the above relation; they cannot both be bounded linear operators on a Hilbert space.

Lemma 6.5. *Let A be a unital algebra. Then for any $x, y \in A$ we have*

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}.$$

Proof. Take any $\lambda \in \sigma(xy) \cup \{0\}$, obviously if $\lambda = 0$ then $\lambda \in \sigma(yx) \cup \{0\}$ so we may safely assume that $\lambda \neq 0$. Toward a contradiction, assume that $\lambda 1_A - yx \in \mathrm{Inv}(A)$, with inverse $z \in A$. Now observe that

$$\begin{aligned}(\lambda 1_A - xy)(\lambda^{-1} + \lambda^{-1}xzy) &= 1_A + xzy - \lambda^{-1}xy - \lambda^{-1}xyxzy \\ &= \lambda^{-1}x(\lambda 1_A - yx)zy + 1_A - \lambda^{-1}xy \\ &= \lambda^{-1}xy + 1_A - \lambda^{-1}xy \\ &= 1_A.\end{aligned}$$

A similar computation shows that $\lambda^{-1} + \lambda^{-1}xzy$ is also a left inverse of $\lambda 1_A - xy$ and this contradicts the assumption that $\lambda \in \sigma(xy)$, therefore $\sigma(xy) \cup \{0\} \subseteq \sigma(yx) \cup \{0\}$. For the opposite inclusion we simply switch the roles of x and y. □

Proposition 6.6. *If A is a unital Banach algebra, then the equation*

$$xy - yx = 1_A,$$

cannot be satisfied by any elements $x, y \in A$.

Proof. Suppose that $x, y \in A$ satisfy $xy - yx = 1_A$, and we shall try to derive a contradiction. The spectral mapping property for polynomials shows that

$$\sigma(xy) = \sigma(1_A + yx) = \{1 + \lambda;\, \lambda \in \sigma(yx)\}.$$

If either $0 \in \sigma(xy) \cap \sigma(yx)$ or $0 \notin \sigma(xy) \cap \sigma(yx)$, then the previous lemma tells us that $\sigma(xy) = \sigma(yx)$. We use this fact to see that

$$\sigma(xy) = \{1 + \lambda;\, \lambda \in \sigma(xy)\},$$

which implies that

$$\sigma(xy) = \{n + \lambda;\, \lambda \in \sigma(xy)\},$$

for every $n \in \mathbb{N}$. In particular this shall hold for any natural number $k > \|xy\|$ and so we may find a complex number $\lambda = a + ib$ such that $k + \lambda \in \sigma(xy)$. Examining the absolute value of k, we have

$$|k + \lambda| = \sqrt{(a+k)^2 + b^2} \geq \sqrt{(a+k)^2} = |a + k| \geq k > \|xy\|,$$

which implies that $k + \lambda$ lies outside of the closed ball $\overline{B}_{\|xy\|}(0)$ which is a contradiction since $\sigma(xy) \subseteq \overline{B}_{\|xy\|}(0)$.

If 0 belongs to precisely one of $\sigma(xy)$ or $\sigma(yx)$, then we may assume without loss of generality that $0 \in \sigma(yx)$. Another application of the previous lemma shows that $\sigma(yx) = \sigma(xy) \cup \{0\}$ and it follows that

$$\sigma(xy) = \{1 + \lambda;\, \lambda \in \sigma(xy) \cup \{0\}\} = \{1 + \lambda;\, \lambda \in \sigma(xy)\} \cup \{1\}.$$

It follows that for any $n \in \mathbb{N}$,

$$\sigma(xy) = \{1 + \lambda;\, \lambda \in \sigma(xy)\} \cup \{1, \ldots, n\},$$

so if $k \in \mathbb{N}$ satisfies $k > \|xy\|$ then just like we did in the above, we may derive a contradiction.

□

One way in which this problem can be "avoided" is due to Weyl. By exponentiating operators p and q which satisfy $pq - qp = i\hbar$, one obtains operators which satisfy the so called Weyl form of the CCR and remarkably these operators are bounded. We do not have the time to go into the details of this argument, and for more information we should see [19, Chapter 4.2] and [20, Chapter VIII.4].

6.3. **Symplectic Forms and Weyl Systems.** In this section we shall closely follow the presentation given by Bär and Becker [4, Section 6] and will expand on a number of the arguments which they give. As we stated at the start of this chapter, our main goal is to prove Slawny's Theorem.

Definition 6.7. Let V be a real vector space. A map $\sigma : V \times V \to \mathbb{R}$ is called a *symplectic form* on V if for all $u, v, w \in V$ and all $\lambda, \mu \in \mathbb{C}$,

(i) $\sigma(u, v) = -\sigma(v, u)$,
(ii) $\sigma(u, \lambda v + \mu w) = \lambda\sigma(u, v) + \mu\sigma(u, w)$,
(iii) $\sigma(u, v) = 0$ for all $v \in V$ implies $u = 0$.

The pair (V, σ) shall be called a *symplectic vector space.* Condition (iii) means that σ is *non-degenerate* and omitting this point gives the definition of a degenerate symplectic form.

Remark 6.8. The constructions in this chapter may be generalised to cover degenerate symplectic forms, as is done in [14] but we shall not be pursuing this.

Assumption 6.9. From here onwards, we shall assume that all symplectic forms are non-degenerate and (V, σ) shall denote a real symplectic vector space (of arbitrary dimension).

Definition 6.10. A *Weyl system* of (V, σ) is a pair (A, W) where A is a unital C*-algebra and $W : V \to A$ is a map which satisfies

(i) $W(v)^* = W(-v)$,
(ii) $W(u)W(v) = e^{-\frac{i}{2}\sigma(u,v)}W(u + v)$,

for all $u, v \in V$. Observe that these properties and the properties of σ show that $W(0) = 1_A$ and that $W(v)$ is unitary for every $v \in V$.

Of course this definition is only of any use to us if Weyl systems actually exist, so we shall prove the existence of Weyl systems before we go any further.

Theorem 6.11. *If (V,σ) is any real symplectic vector space, then there exists a Weyl system of (V,σ).*

Proof. Consider the Hilbert space $\mathcal{H} = L^2(V)$, where the functions in $\mathcal{H}$ are square-integrable with respect to the counting measure. Then every $f \in \mathcal{H}$ satisfies

$$\|f\|_2^2 = \int_V |f|^2 d\mu = \sum_{v\in V} |f(v)|^2 < \infty,$$

and finiteness of this sum means that $f(v) = 0$ for all but countably many $v \in V$. Now we let $A = B(\mathcal{H})$ and define the map $W : V \to A$ as follows; for each $u \in V$, $W(u) : \mathcal{H} \to \mathcal{H}$ is the map which for $f \in \mathcal{H}$ is given by

$$(W(u)f)(v) = e^{\frac{i}{2}\sigma(u,v)} f(u+v),$$

for all $v \in V$. First we should check that for each $u \in V$, $W(u)$ really belongs to $B(\mathcal{H})$. Fixing $u \in V$, it is certainly obvious that $W(u)$ is linear and if we take any $f \in \mathcal{H}$, then

$$\|W(u)f\|_2^2 = \sum_{v\in V} |(W(u)f)(v)|^2 = \sum_{v\in V} \left|e^{\frac{i}{2}\sigma(u,v)}\right|^2 |f(u+v)|^2 = \sum_{v\in V} |f(u+v)|^2 < \infty.$$

So $W(u)$ really does take values in $\mathcal{H}$. Moreover if $f \in \mathcal{H}$ satisfies $\|f\|_2 = 1$, then

$$\|W(u)f\|_2^2 = \sum_{v\in V} |f(u+v)|^2 = \sum_{v\in V} |f(u+v)|^2 = 1,$$

so $W(u)$ is bounded, with $\|W(u)\|_{\text{op}} = 1$. To check that the pair (A, W) satisfies property (i) of a Weyl system, we take any $u \in V$ and any $f, g \in \mathcal{H}$. Then

$$\langle f, W(u)g\rangle = \sum_{v\in V} \overline{f(v)} e^{\frac{i}{2}\sigma(u,v)} g(u+v).$$

Now make the substitution $w = u + v$ to obtain

$$\begin{aligned}
\langle f, W(u)g\rangle &= \sum_{w\in V} e^{\frac{i}{2}\sigma(u,w-u)} \overline{f(w-u)} g(w) \\
&= \sum_{w\in V} \overline{e^{-\frac{i}{2}\sigma(u,w-u)}}\, \overline{f(w-u)} g(w) \\
&= \sum_{w\in V} \overline{e^{\frac{i}{2}\sigma(-u,w-u)} f(w-u)} g(w) \\
&= \sum_{w\in V} \overline{e^{\frac{i}{2}\sigma(-u,w)} f(w-u)} g(w) \\
&= \langle W(-u)f, g\rangle.
\end{aligned}$$

Which shows that $W(u)^* = W(-u)$. To check property (ii) of a Weyl system, we take any $u, v \in V$ and consider the composition $W(u)W(v)$. If $f \in \mathcal{H}$ is an arbitrary function, then for $w \in V$ we have

$$\begin{aligned}
((W(u)W(v))f)(w) &= e^{\frac{i}{2}\sigma(u,w)} (W(u)f)(u+w) \\
&= e^{\frac{i}{2}\sigma(u,w)} e^{\frac{i}{2}\sigma(v,u+w)} f(u+v+w) \\
&= e^{\frac{i}{2}\sigma(u+v,w)} e^{\frac{i}{2}\sigma(v,u)} f(u+v+w) \\
&= e^{-\frac{i}{2}\sigma(u,v)} e^{\frac{i}{2}\sigma(u+v,w)} f(u+v+w) \\
&= e^{-\frac{i}{2}\sigma(u,v)} (W(u+v)f)(w),
\end{aligned}$$

and this shows that $W(u)W(v) = e^{-\frac{i}{2}\sigma(u,v)} W(u+v)$. $\square$

Remark 6.12. Notice that we did not place any assumptions about the continuity of the map $W : V \to A$ in a Weyl system. It turns out that W is never continuous, and in fact $\|W(u) - W(v)\| = 2$ for all $u, v \in V$ with $u \neq v$ [4, Proposition 7.2].

Definition 6.13. If (A, W) is a Weyl system of (V, σ), then we shall let $W(V)$ denote the set $\{W(v);\, v \in V\}$ and let $\mathrm{sp}(W(V))$ denote the complex linear span of $W(V)$.

Remark 6.14. If (A, W) is a Weyl system of (V, σ), then we may write an element $x \in \mathrm{sp}(W(V))$ as

$$x = \sum_{v \in V} \lambda_v W(V),$$

for scalars $\{\lambda_v\}_{v \in V} \subseteq \mathbb{C}$, of which all but finitely many are zero. We will use this notation and the usual notation for finite sums interchangeably in the following.

If (A, W) is a Weyl system for (V, σ), then observe that the pair $(\mathrm{C}^*(W(V)), W)$ is also a Weyl system for (V, σ). As the next lemma will show, the C*-algebra $\mathrm{C}^*(W(V))$ has a very simple description.

Lemma 6.15. *Let (A, W) be a Weyl system for (V, σ), then the closure of $sp(W(V))$ with respect to $\|\cdot\|_A$ satisfies*

$$\overline{sp(W(V))} = C^*(W(V)).$$

Proof. First take any linear combination $x \in \mathrm{sp}(W(V))$, then each term in the expression of x obviously belongs to $\mathrm{C}^*(W(V))$, so x belongs to $\mathrm{C}^*(W(V))$ and upon taking the closure of $\mathrm{sp}(W(V))$ we see that $\overline{\mathrm{sp}(W(V))} \subseteq \mathrm{C}^*(W(V))$. Conversely, recall that the set of polynomials in $W(V) \cup W(V)^*$ is dense in $\mathrm{C}^*(W(V))$. These polynomials are finite linear combinations of elements of the form

$$x = \lambda W(u_1) \cdots W(u_k) W(u_{k+1})^* \cdots W(u_n)^*,$$

where $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. We use property (i) of a Weyl system to rewrite x as

$$\lambda W(u_1) \cdots W(u_k) W(-u_{k+1}) \cdots W(-u_n),$$

and then use property (ii) of a Weyl system to further reduce x to

$$\tilde{\lambda} W(\tilde{u}).$$

Therefore any polynomial in $W(V) \cup W(V)^*$ belongs to $\mathrm{sp}(W(V))$ and taking the closure of these sets shows that $\mathrm{C}^*(W(V)) \subseteq \overline{\mathrm{sp}(W(V))}$. □

Lemma 6.16. *If (A, W) is a Weyl system of (V, σ), then $sp(W(V))$ is a *-algebra. Moreover if (B, R) is another Weyl system of (V, σ) then there is a *-isomorphism*

$$\pi : sp(W(V)) \to sp(R(V)).$$

Proof. Consider two linear combinations $x, y \in \mathrm{sp}(W(V))$. By the properties of a Weyl system, the product of x and y will be a linear combination of terms of the form

$$\lambda \mu W(u) W(v) = \lambda \mu e^{-\frac{i}{2}\sigma(u,v)} W(u+v).$$

Therefore $\mathrm{sp}(W(V))$ is closed under multiplication, so is an algebra and of course it is also closed under the involution, so must be a *-algebra.

If (B, R) is another Weyl system, then we define the map π by

$$\pi\left(\sum_{v \in V} \lambda_v W(v)\right) = \sum_{v \in V} \lambda_v R(v),$$

for each $\sum_{v \in V} \lambda_v W(v) \in \mathrm{sp}(W(V))$. It is clear that π is a linear isomorphism, and for $u, v \in V$ we have

$$\pi(W(u)^*) = \pi(W(-u)) = R(-u) = R(u)^*,$$

and

$$\pi(W(u)W(v)) = \pi\left(e^{-\frac{i}{2}\sigma(u,v)} W(u+v)\right) = e^{-\frac{i}{2}\sigma(u,v)} R(u+v) = R(u)R(v).$$

So in fact π is a *-isomorphism of *-algebras. □

Lemma 6.17. *Let (A, W) be a Weyl system of (V, σ) and for $x \in sp(W(V))$ define*

$$\|x\|_{max} = \sup\{\|x\|;\ \|\cdot\| : sp(W(V)) \to \mathbb{R}^{\geq 0} \text{ is a } C^*\text{-norm}\}.$$

Then $\|\cdot\|_{max}$ is a C-norm on $sp(W(V))$.*

Proof. First lets make a couple of observations. The first is that $\|\cdot\|_A$ restricted to $\mathrm{sp}(W(V))$ is a C*-norm, so our supremum isn't taken over the empty set. The second is that we may define a submultiplicative norm $\|\cdot\|_0$ on $\mathrm{sp}(W(V))$ by

$$\left\|\sum_{v\in V} \lambda_v W(v)\right\|_0 = \sum_{v\in V} |\lambda_v|.$$

If we take any $0 \neq v \in V$ and put $x = W(0) + W(v) - W(-v)$, then it not hard to show that

$$\|x^* x\|_0 = 8,$$

whereas

$$\|x\|_0^2 = 9.$$

Hence the C*-identity fails and $\|\cdot\|_0$ is not a C*-norm on $\mathrm{sp}(W(V))$. However if $\|\cdot\|$ is any other C*-norm $\|\cdot\|$ on $\mathrm{sp}(W(V))$, then we have

$$\left\|\sum_{v\in V} \lambda_v W(v)\right\| \leq \sum_{v\in V} |\lambda_v| \|W(v)\| = \sum_{v\in V} |\lambda_v|,$$

by the triangle inequality and because the elements $W(v)$ are unitary. The other properties which we need to check are simply inherited from the C*-norm properties of the norms which we are taking the supremum. ☐

6.4. **Slawny's Theorem.** Forming the completion of $\mathrm{sp}(W(V))$ in the norm $\|\cdot\|_{\max}$ will give us a C*-algebra and with the next couple of results we aim to show that this completion contains no non-trivial closed ideals. These two results are given by Bär and Becker in one result [4, Lemma 10] and we follow the proofs which they give. The idea behind their proof is a very elegant one, but one could argue that their presentation is quite confusing and in fact hides the role played by the operator ϕ which we shall introduce shortly. By breaking their one lemma into two results, and expanding on many of the points which they make, we intend to clarify the argument so that it is easier to appreciate.

Lemma 6.18. *The map $\phi : sp(W(V)) \to \mathbb{C} \cdot W(0)$ defined by*

$$\phi\left(\sum_{v\in V} \lambda_v W(v)\right) = \lambda_0 W(0),$$

is a bounded linear operator, thus extends uniquely to a bounded linear operator $\overline{sp(W(V))}^{max} \to \mathbb{C} \cdot W(0)$ which we shall also denote by ϕ.

Proof. First observe that the map is clearly linear. Now note that $\mathbb{C} \cdot W(0)$ is a finite dimensional subspace of $\overline{sp(W(V))}^{\max}$, so is complete and this means that we only need to prove that ϕ is bounded. By Lemma 6.16 we may assume that (A, W) is the Weyl system we constructed in Theorem 6.11. Now define $f : V \to \mathbb{C}$ by

$$f(v) = \begin{cases} 1 & \text{if } v = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then f certainly belongs to $L^2(V)$. We now let $x = \sum_{v\in V} \lambda_v W(v) \in \mathrm{sp}(W(V))$, then $x \in B(L^2(V))$ and we may of course apply x to f giving a bounded linear operator $xf : V \to \mathbb{C}$. For $w \in V$, we have

$$(xf)(w) = \left(\sum_{v\in V} \lambda_v W(v) f\right)(w) = \sum_{v\in V} \lambda_v e^{\frac{i}{2}\sigma(v,w)} f(v+w) = \lambda_{-v} e^{\frac{i}{2}\sigma(-v,v)} = \lambda_{-v},$$

from which it follows that

$$\langle f, xf\rangle = \sum_{v\in V} \overline{f(v)}xf(v) = xf(0) = \lambda_0.$$

Therefore,

$$\begin{aligned}\|\phi(x)\|_{\max} &= |\lambda_0| \\ &= |\langle f, xf\rangle| \\ &\leq \|f\|_2\|xf\|_{\text{op}} \\ &= \|xf\|_{\text{op}} \\ &\leq \|x\|_{\text{op}}\|f\|_2 \\ &= \|x\|_{\text{op}},\end{aligned}$$

where we have made use of the Cauchy-Schwarz inequality and the fact that $\|f\|_2 = 1$. Once we observe that $\|x\|_0 := \|x\|_{\text{op}}$ defines a C*-norm on $\text{sp}(W(V))$ we can conclude that $\|\phi(x)\|_{\max} \leq \|x\|_{\max}$ so that ϕ is bounded. We use Lemma 6.4 to extend ϕ by continuity, giving a bounded linear operator

$$\phi : \overline{\text{sp}(W(V))}^{\max} \to \mathbb{C}\cdot W(0).$$

□

Example 6.19. Let (A, W) be a Weyl system of (V, σ) and consider a non-zero linear combination $x = \sum_{v\in V}\lambda_v W(v) \in \text{sp}(W(V))$. Pick any non-zero scalar λ_0 which appears in the expression of x and then consider the product

$$xW(-v_0) = \sum_{v\in V}\lambda_v e^{-\frac{i}{2}\sigma(v,-v_0)}W(v-v_0) = \lambda_0 W(0)\sum_{v\in V\setminus\{0\}}\widetilde{\lambda}_v W(\widetilde{v}).$$

Applying ϕ to this gives $\phi(xW(-v_0)) = \lambda_0 W(0)$.

Proposition 6.20. *Let (A, W) be a Weyl system of (V, σ) and let $\overline{sp(W(V))}^{max}$ denote the completion of $sp(W(V))$ with respect to $\|\cdot\|_{max}$. Then the only closed ideals of $\overline{sp(W(V))}^{max}$ are $\{0\}$ and $\overline{sp(W(V))}^{max}$.*

Proof. Let $I \subseteq \overline{\text{sp}(W(V))}^{\max}$ be a closed ideal. As vector spaces, $\mathbb{C}\cdot W(0)$ and $\mathbb{C}$ are isomorphic, from which it follows that the intersection $I\cap\mathbb{C}\cdot W(0)$ can only be $\{0\}$ or $\mathbb{C}\cdot W(0)$. If the intersection is $\mathbb{C}\cdot W(0)$, then $W(0)\in I$ and from this it follows that $I = \overline{\text{sp}(W(V))}^{\max}$, so we assume that $I\cap\mathbb{C}\cdot W(0) = \{0\}$.

Now take any $x\in I$, then because $I \subseteq \overline{\text{sp}(W(V))}^{\max}$ we can find a sequence (x_n) in $\text{sp}(W(V))$ which converges to x. By construction of ϕ, we have that $\phi(x) = \lim_{n\to\infty}\phi(x_n)$ and hence the sequence $(x_n - \phi(x_n))$ in $\text{sp}(W(V))$ converges to $x - \phi(x)$. So for any $k\in\mathbb{N}^{>0}$, it is possible to find a linear combination $y_k = \sum_{v\in V}\lambda_v W(v)\in\text{sp}(W(V))$ which satisfies

$$\|(x-\phi(x)) - (y_k - \phi(y_k))\|_{\max} < \frac{1}{k}.$$

Obviously $\phi(y_k) = \lambda_0 W(0)$ and so we can write this inequality as

$$\left\|(x-\phi(x)) - \sum_{v\in V\setminus\{0\}}\lambda_v W(v)\right\|_{\max} < \frac{1}{k},$$

and hence we express x as

$$x = \phi(x) + \sum_{j=1}^{n}\lambda_j W(v_j) + r,$$

where $v_j\neq 0$ for every $1\leq j\leq n$ and the remainder term $r\in\overline{\text{sp}(W(V))}^{\max}$ satisfies $\|r\|_{\max} < \frac{1}{k}$.

If we take an arbitrary vector $w \in V$, then because I is an ideal, the product $W(w)xW(-w)$ belongs to I. Remembering that $\phi(x) = \lambda_x W(0)$ for some $\lambda_x \in \mathbb{C}$, we see that $W(w)\phi(x)W(-w) = \phi(x)$ and so the product $W(w)xW(-w)$ has the expression

$$\phi(x) + \sum_{j=1}^{n} \lambda_j e^{-i\sigma(w,v_j)} W(v_j) + W(w)rW(-w),$$

where $\|W(w)rW(-w)\|_{\max} \leq \|r\|_{\max} < \frac{1}{k}$.

Because σ is non-degenerate, we can find a vector $w \in V$ so that $\sigma(w, v_n) \neq 0$. It then follows that we may choose a pair of vectors $w, t \in V$ such that $e^{-i\sigma(w,v_n)} = -e^{-i\sigma(t,v_n)}$. The elements $W(w)xW(-w)$ and $W(t)xW(-t)$ both belong to I and we first add them together, then rescale and relabel to give

$$\phi(x) + \sum_{j=1}^{n-1} \widetilde{\lambda}_j W(v_j) + \widetilde{r} \in I,$$

where $\|\widetilde{r}\|_{\max} \leq \frac{1}{k}$.

Repeating this argument another $n-1$ times allows us to eliminate every term in the summation $\sum_{j=1}^{n} \lambda_k W(v_j)$, leaving us with

$$\phi(x) + r_k \in I,$$

where $\|r_k\|_{\max} < \frac{1}{k}$. After we do this for each $k \in \mathbb{N}$, we get a sequence $(\phi(x)) + r_k)$ in I which obviously converges to $\phi(x)$. Because I is closed, it follows that $\phi(x)$ belongs to I and our assumption that $I \cap \mathbb{C} \cdot W(0) = \{0\}$ then means that $\phi(x) = 0$. If we consider any fixed $k \in \mathbb{N}$, then we now know that $r_k \in I$. Recalling how r_k is constructed, we notice that we may simply multiply r_k by appropriate elements of $\overline{\mathrm{sp}(W(V))}^{\max}$ in order to show that the original remainder term r belongs to I. This then means that

$$x - \sum_{k=1}^{n} \lambda_k W(v_k) \in I,$$

or rather that $y_k = \sum_{j=1}^{n} \lambda_j W(v_j) \in I$.

If $u \in W$ is any vector, then $y_k W(u) \in I$. Applying what we have just proved, it follows that $\phi(y_k W(u)) = 0$, so following Example 6.19 we can make careful choices of u in order to show that $\lambda_k = 0$ for each $1 \leq k \leq n$, thus proving that $y_k = 0$. We repeat this argument to show that $y_k = 0$ for every $k \in \mathbb{N}$ and see that $\|x\| < \frac{1}{k}$ holds for each $k \in \mathbb{N}$, thus implying that $x = 0$ and therefore $I = \{0\}$. □

Definition 6.21. Let (A, W) be a Weyl system of (V, σ). Let A_W denote the C*-subalgebra of A which is generated by $W(V)$, then we shall call the pair (A_W, W) a *CCR-representation* of (V, σ) and call A_W a *CCR-algebra* of (V, σ).

Our final theorem is Slawny's Theorem. The approach given in his original paper [22] uses methods from harmonic analysis, and this approach is also followed in textbooks by Petz [18, Theorem 2.1] and Warner [23, Theorem 16.2]. Whilst the approaches are quite different, the message to take home from Slawny's Theorem is that this doesn't matter; both approaches will produce the same C*-algebras.

Theorem 6.22. (Slawny's Theorem)
*Let (V, σ) be a symplectic vector space. Then there exists a CCR-representation of (V, σ). Moreover if (A_{W_1}, W_1) and (A_{W_2}, W_2) are both CCR-representations of (V, σ), then there exists an isometric *-isomorphism from A_{W_1} to A_{W_2}.*

Proof. The existence of a CCR-representation follows from Theorem 6.11. For the uniqueness, we consider the*-isomorphism π between the *-algebras $\mathrm{sp}(W_1(V))$ and $\mathrm{sp}(W_2(V))$. By Lemma 6.15, the set $\mathrm{sp}(W_2(V))$ is dense in A_{W_2} and the composition

$$\mathrm{sp}(W_1(V)) \xrightarrow{\pi} \mathrm{sp}(W_2(V)) \xhookrightarrow{\iota} A_{W_2},$$

(where ι denotes the inclusion map), is a *-homomorphism which is defined on a dense *-subalgebra of $\overline{\mathrm{sp}(W_1(V))}^{\max}$ whose image is $\mathrm{sp}(W_2(V))$. If we define $\|x\| := \|\pi(x)\|_{A_{W_2}}$ for $x \in \mathrm{sp}(W_1(V))$, then we have a C*-norm on $\mathrm{sp}(W_1(V))$ so of course have the bound $\|\pi(x)\|_{A_{W_2}} = \|x\| \leq \|x\|_{\max}$. This shows that the composition above is continuous and by Lemma 6.4 we may extend it to a *-homomorphism $\varphi : \overline{\mathrm{sp}(W_1(V))}^{\max} \to A_{W_2}$. Now we invoke Proposition 6.20 to deduce that φ is injective and hence isometric.

Since φ extends $\iota \circ \pi$, the image of φ contains $\mathrm{sp}(W_2(V))$ and because φ is an isometry into a complete metric space, its image must be closed. This implies that φ is surjective so is an isometric *-isomorphism.

Completing the proof is now easy. We note that the pair $(\overline{\mathrm{sp}(W_1(V))}^{\max}, W_1)$ is also Weyl system and hence a CCR-representation of (V, σ), so we carry out the procedure detailed above for the identity map between $\mathrm{sp}(W(V))$ and itself to give a *-isomorphism $\psi : \overline{\mathrm{sp}(W_1(V))}^{\max} \to A_{W_1}$. The composition

$$A_{W_1} \xrightarrow{\psi^{-1}} \overline{\mathrm{sp}(W_1(V))}^{\max} \xrightarrow{\varphi} A_{W_2},$$

is then the map which we wanted to find. □

Consequently, if we have any real symplectic vector space (V, σ), then we can talk about *the* CCR-algebra of (V, σ) and this is often denoted by $\mathrm{CCR}(V, \sigma)$.

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