COMPSCI 575/ MATH 513: Combinatorics and Graph Theory **Fall 2022**

Homework 6

Released 11/29/2022 Due 12/12/2022 11:59pm in Gradescope

Instructions. All work submitted must be your own **in presentation**. You may discuss homework with other students, but the writeup **must** be your own work. Copying, or collaboration so close that it looks like copying, are not allowed. A good practice is to divide your work into an "ideas phase" where you collaborate and a "writeup phase" where you work alone – enter the writeup phase with notes, but not written solutions. If you make use of a printed or online source for the homework, other than specific course materials such as the textbook or web site, please mention it in your writeup.

Submissions. Please submit a PDF file. Please assign pages to questions in Gradescope.
You may submit a scanned handwritten document, but a typed submission is preferred.

1. (Bóna 1.46) Prove that the number of northeastern lattice paths that start at $(0, 0)$ and end in (n, n) is equal to the number of northeastern lattice paths that start at $(0, 0)$, consist of $2n$ steps, and never return to the main diagonal after their initial point.

2. (Bóna 2.26) Prove, with a direct combinatorial argument, that for all positive integers n, we have $q(1) + q(2) + \cdots + q(n) \leq 2^n$, where $q(n)$ denotes the number of partitions of n into *distinct* parts.

3. (Bóna 2.32) Prove that the number of partitions of n in which the size of the average part is at most two is equal to the number of partitions of n in which the largest part is at least as large as the sum of all the other parts.

4. Let s_n be the number of ways to seat people at an unspecified number of round tables and arrange the tables in a line. Two seating arrangements are considered identical if each person has the same left neighbor in both, and each person sits at the same table in both. Find the generating function for the sequence s_n.

5. Show that $e^x/(1-x)^k$ is the exponential generating function counting the ways to choose a possibly empty subset of n distinct objects and place them into k different boxes, where the order in each box matters.

6. (Bóna 4.43) a) Find the number of permutations of length n whose longest cycle is of length $k > n/2$.
b) Solve the same problem for $n/3 < k \leq n/2$.

7. (Bóna 5.19) How many decreasing binary trees are there on n vertices with k right edges?

8. Suppose that in a set of $2n + 1$ people, among any three of them there exist two who know each other. Show that there are at least $n + 1$ people who each know at least n others.

9. Two players take turns removing from a pile of n stones. The number of stones taken at each move must be a power of two. Whoever moves last wins. Find the Grundy number for any n, and prove your result.

10. Show that no matter how the edges are directed in a bipartite graph, it always has a Grundy function.