The general problem of elastic wave propagation in multilayered anisotropic media

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Exact analytical treatment of the interaction of harmonic elastic waves with n-layered anisotropic plates is presented. Each layer of the plate can possess up to as low as monoclinic symmetry and thus allowing results for higher symmetry materials such as orthotropic, transversely isotropic, cubic, and isotropic to be obtained as special cases. The wave is allowed to propagate along an arbitrary angle from the normal to the plate as well as along any azimuthal angle. Solutions are obtained by using the transfer matrix method. According to this method formal solutions for each layer are derived and expressed in terms of wave amplitudes. By eliminating these amplitudes the stresses and displacements on one side of the layer are related to those of the other side. By satisfying appropriate continuity conditions at interlayer interfaces a global transfer matrix can be constructed which relates the displacements and stresses on one side of the plate to those on the other. Invoking appropriate boundary conditions on the plates outer boundaries a large variety of important problems can be solved. Of these mention is made of the propagation of free waves on the plate and the propagation of waves in a periodic media consisting of a periodic repetition of the plate. Confidence is the approach and results are confirmed by comparisons with whatever is available from specialized solutions. A variety of numerical illustrations are included.

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INTRODUCTION

Studies of the propagation of elastic waves in layered media have long been of interest to researchers in the fields of geophysics, acoustics, and electromagnetics. Applications of these studies include such technologically important areas as earthquake prediction, underground fault mapping, oil and gas exploration, architectural noise reduction, and the recently evolving concern of the analysis and design of advanced fibrous and layered composite materials. Common to all of these studies is the degree of the interactions between the layers, which manifest themselves in the form of reflection and transmission agents and hence give rise to geometric dispersion. These interactions depend, among many factors, upon the properties, direction of propagation, frequency and number, and nature of the interfacial conditions. Extensive review of works on this subject until the mid 60's has been reported in the literature as is evidenced from the book by Ewing *et al.*[1] and up to the early 80's by Brekhovskikh.[2] For more recent works on the general subject of wave propagation in layered media we refer the reader to Refs. 3–5 as representative references.

Typically a layered medium consists of two or more material components attached at their interface in some fashion. A body made up of an arbitrary number of different material components and whose outer boundaries are either free or supported by semi-infinite media constitutes a general layered medium. Often the above definition is relaxed to include semi-infinite solids, single-layer plates, and two semi-infinite solids in contact as degenerate cases of layered media.

Most of the available literature on layered media is restricted to the study of situations where the individual material layers are isotropic. Generally speaking, for wave propagation in such media, solutions are obtained by expressing the displacements and stresses in each layer in terms of its wave potential amplitudes. By satisfying appropriate interfacial conditions, characteristic equations are constructed that involve the amplitudes of all layers; this constitutes the direct approach. The degree of complication in the algebraic manipulation of the analysis will thus depend upon the number of layers. For relatively few layers the direct approach is appropriate. However, as the number of layers increases the direct approach becomes cumbersome, and one may resort to the alternative transfer (propagator) matrix technique introduced originally by Thomson[6] and somewhat later on by Haskell[7] and Gilbert and Backus.[8] According to this technique one constructs the propagation matrix for a stack of an arbitrary number of layers by extending the solution from one layer to the next while satisfying the appropriate interfacial continuity conditions.

To narrow down our discussion of problems relating to the interaction of elastic waves with periodic media we note that Rytov[9] utilized the direct approach method and derived some analytical expressions for characteristic equations of a periodic array of two isotropic layers. However, Rytov was only able to present solutions for propagation either along or normal to the layers. Nayfeh[10] derived an exact expression for the characteristic equation of waves propagating normal to a periodic array of an arbitrary number of isotropic layers. Sve[11] extended the results of Rytov to any oblique incidence and derived the characteristic equation for the periodic array of two isotropic layers in the form of the vanishing determinant of an 8×8 matrix. Schoenberg[12] utilized the matrix transfer method and presented results for oblique incidence on an alternating fluid–isotropic solid medium. Gilbert[13] discussed the utility of the propagator matrix formalism of Gilbert and Backus[8] to the study of wave propagation in stratified media and obtained explicit expressions for the simple case of a periodically layered fluid.

For cases involving periodic anisotropic media we mention that Yamada and Nemat-Nasser[14] extended the results of Sve to the case of orthotropic layers. This resulted, due to the added coupling of the horizontally polarized component of the wave, to the vanishing of the determinant of a 12×12 matrix. In a recently held special symposium on wave propagation in structural composites[5], several papers were presented on guided waves in laminated anisotropic plates as well as on periodically laminated anisotropic media. The papers presented include, either individually or collectively, the most comprehensive surveys of the relevant literature. The most relevant works for the present work are given by Braga and Herrmann,[15] Ting and Chadwick,[16] and Nayfeh *et al.*[17] Braga and Hermann[15] used the propagator matrix method and presented results for a periodic array of an arbitrary number of orthotropic layers. Their work is restricted however to the case of propagation along layer interfaces (no oblique incidence) and also for propagation along an axis of symmetry of each layer. This implies that their layering is restricted such that the symmetry axes of all layers coincide. Thus, their model does not account for the coupling between the in-plane motion (*SH*) and that of the sagittal plane. Ting and Chadwick[16] also used the propagator matrix approach (in conjunction with a formalism developed for steady plane motions of anisotropic bodies by Stroh[18]) and derived a characteristic equation for harmonic waves in periodically anisotropic media. Their analysis was carried initially for waves traveling along the layering and then they outlined how it can be generalized to an arbitrary direction of propagation in the sagittal plane.

Nayfeh *et al.*,[17] with the help of linear orthogonal transformations, were able to derive exact analytical expressions for the reflection coefficient from a fluid-loaded arbitrarily oriented multilayered orthotropic plate. The approach used in Ref. 17 was introduced in their earlier works that dealt with single layer anisotropic plates.[19,20] The use of the linear transformation, which facilitates and leads to execution ease of the analysis, was motivated by the important observation that the wave vectors of the incident and reflected waves all lie in the same plane.[17,19–21] The analysis was therefore conducted in a coordinate system formed by incident and reflected planes rather than by material axes.

In this paper we utilize combinations of the linear transformation approach and the transfer matrix method and extend the results of Ref. 17 to the study of the interaction of free harmonic waves with multilayered anisotropic media. Our solutions will be general and include results pertaining to several special cases. Of these we mention: (a) dispersion characteristics for a multilayered plate consisting of an arbitrary number of arbitrarily oriented anisotropic layers; (b) dispersion of an infinite medium built from repetition of the multilayered plate (the resulting medium will thus be a periodic one with respect to the individual components of the plate); and (c) slowness results for either homogeneous or periodic media. It is obvious that the layered plate constitutes the repeating cell of the infinite medium.

Besides the advantages gained by the use of the linear transformation approach, another important feature of our analysis concerns the manner in which the oblique propagation direction is introduced and the way it modifies the criterion necessary to insure periodicity. If we designate the angle θ (measured from the normal to the interfaces) to define the propagation direction, then this will lead to an explicit dependence of the characteristic equations upon θ. Confidence in our results is established by comparing with the limited available numerical examples of the special case models of Refs. 11 and 14.

I. FORMULATION OF THE PROBLEM

Consider a plate consisting of an arbitrary number n of monoclinic layers rigidly bonded at their interfaces and stacked normal to the x_3 axis of a global orthogonal Cartesian system $x_i = (x_1,x_2,x_3)$ as illustrated in Fig. 1. Hence the plane of each layer is parallel to the x_1-x_2 plane which is also chosen to coincide with the bottom surface of the layered plate. To maintain generality we assume each layer to be arbitrarily oriented in the x_1-x_2 plane. In order to be able to describe the relative orientation of the layers, we assign for each layer k, $k = 1,2,...,n$, a local cartesian coordinate $(x_i')_k$ such that its origin is located in the bottom plane of the layer with $(x_3')_k$ normal to it. Thus layer k extends from $0\leqslant (x_3')_k \leqslant d^{(k)}$, where d^k is its thickness. According to this notation the total thickness of the layered plate d equals to the sum of the thicknesses of its layers and hence, the plate occupies the region $0\leqslant x_3 \leqslant d$. Equivalently, the orientation of the k th layer in the x_i space can be described by a rotation of an angle ϕ_k between $(x_1')_k$ and x_1. Hence, once all orientation angles ϕ_k are specified the geometry of the plate will be defined. Without any loss in generality we shall assume that a plane wave propagates in the x_1-x_3 plane at an arbitrary angle θ measured from the normal x_3.

In this section we follow the analytical procedure of Refs. 15–17 in order to construct a transfer matrix for each layer k. This matrix relates the displacements $(u_i)_k$ and stresses $(\sigma_{ij})_k$ of one face of the plate k to those of the other. With respect to the primed coordinate system $(k_i')_k$ and using the standard summation convention on repeated indices the elastic field equations of layer k are given by the momentum equation

$$\frac{\partial \sigma_{ij}'}{\partial x_j'} = \rho' \frac{\partial^2 u_i'}{\partial t^2} \tag{1}$$

and, from the general constitutive relations for anisotropic media,

$$\sigma_{ij}' = c_{ijkl}' e_{kl}' \tag{2}$$

by the specialized expanded matrix form of monoclinic media

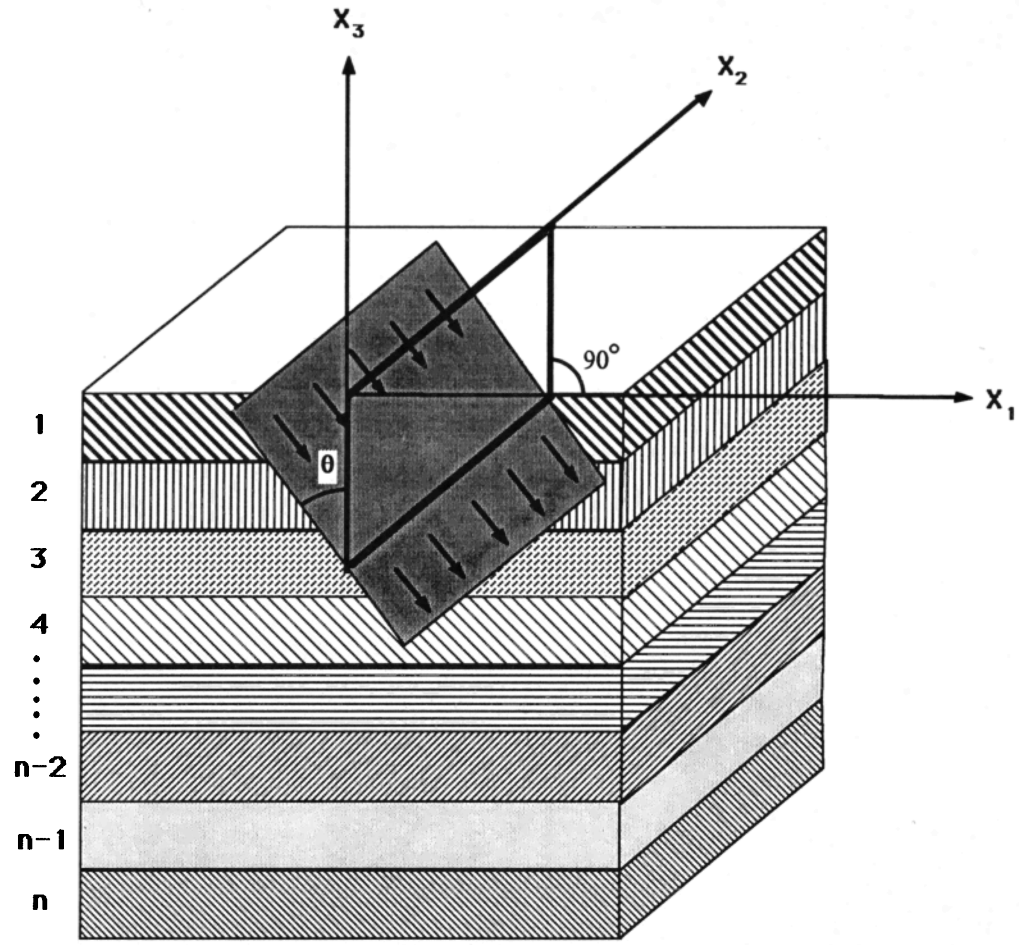

FIG. 1. Model geometry.

$$\begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{23} \\ \sigma'_{13} \\ \sigma'_{12} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & 0 & 0 & C'_{16} \\ C'_{12} & C'_{22} & C'_{23} & 0 & 0 & C'_{26} \\ C'_{13} & C'_{23} & C'_{33} & 0 & 0 & C'_{36} \\ 0 & 0 & 0 & C'_{44} & C'_{45} & 0 \\ 0 & 0 & 0 & C'_{45} & C'_{55} & 0 \\ C'_{16} & C'_{26} & C'_{36} & 0 & 0 & C'_{66} \end{bmatrix} \begin{bmatrix} e'_{11} \\ e'_{22} \\ e'_{33} \\ \gamma'_{23} \\ \gamma'_{13} \\ \gamma'_{12} \end{bmatrix}, \tag{3}$$

where we used the contracting subscript notations $1 \to 11$, $2 \to 22$, $3 \to 33$, $4 \to 23$, $5 \to 13$, and $6 \to 12$ to relate c'_{ijkl} to C'_{pq} ($i,j,k,l = 1,2,3$ and $p,q = 1,2,...,6$). Here σ'_{ij}, e'_{ij}, and u'_i are the components of stress, strain, and displacement, respectively, and ρ' and c'_{ijkl} are the material density and elastic constants, respectively. In Eq. (3), $\gamma'_{ij} = 2e'_{ij}$ (with $i \neq j$) defines the engineering shear strain components.

Since σ'_{ij}, e'_{kl}, and c'_{ijkl} are tensors and since we are conducting our analysis in the global x_i coordinate, any orthogonal transformation of the primed to the nonprimed coordinates, i.e., $(x'_i)_k$ to x_i, they transform according to

$$\sigma_{mn} = \beta_{mi}\beta_{nj}\sigma'_{ij}, \tag{4a}$$

$$e_{op} = \beta_{ok}\beta_{pl}e'_{kl}, \tag{4b}$$

$$c_{mnop} = \beta_{mi}\beta_{nj}\beta_{ok}\beta_{pl}c'_{ijkl}, \tag{4c}$$

where β_{ij} is the cosines of the angle between x'_i and x_j, respectively. For a rotation of angle ϕ in the x'_1-x'_2 plane, the transformation tensor β_{ij} reduces to

$$\beta_{ij} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

which, if applied to Eq. (2) through the relation of Eq. (3) yields the known constitutive relations:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}, \tag{6}$$

where the transformation relations between the C_{pq} and C'_{pq} entries are listed in Appendix A. Notice that no matter what rotational angle ϕ is used, the zero entries in Eq. (3) will remain zero in (6). In fact, even if the matrix of Eq. (3) is particularized to orthotropic media, its transformed matrix will resemble that of monoclinic media. In terms of the rotated coordinate system x_i, the momentum equation transforms to

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}. \qquad (7)$$

II. ANALYSIS

Substituting from Eq. (6) into Eq. (7) results in a system of three coupled equations for the displacements u_1, u_2, and u_3. If we now identify the plane of incidence to be the x_1x_3, as in Fig. 1 then for an angle of incidence θ, we propose a solution for the displacements u_i in the form

$$(u_1,u_2,u_3) = (1,V,W)\,U \exp[i\xi(x_1 \sin\theta + \alpha x_3 - ct)], \qquad (8)$$

where ξ is the wave number, c is the phase velocity ($=\omega/\xi$), ω is the circular frequency, α is still an unknown parameter, and V and W are ratios of the displacement amplitudes of u_2 and u_3 to that of u_1, respectively. Notice that, although solutions (8) are explicitly independent of x_2, an implicit dependence is contained in the transformation. Furthermore, notice the nonvanishing of the transverse displacement component u_2 in Eq. (8). This choice of solutions leads to the three coupled equations

$$K_{mn}(\alpha)U_n = 0, \quad m,n = 1,2,3, \qquad (9a)$$

where the summation convention is implied, K_{mn} is symmetric, namely $K_{mn} = K_{nm}$, and

$$\begin{aligned}
K_{11} &= C_{11}\sin^2\theta - \rho c^2 + C_{55}\alpha^2,\\
K_{12} &= C_{16}\sin^2\theta + C_{45}\alpha^2,\\
K_{13} &= (C_{13} + C_{55})\alpha\sin\theta,\\
K_{22} &= C_{66}\sin^2\theta - \rho c^2 + C_{44}\alpha^2,\\
K_{23} &= (C_{36} + C_{45})\alpha\sin\theta,\\
K_{33} &= C_{55}\sin^2\theta - \rho c^2 + C_{33}\alpha^2.
\end{aligned} \qquad (9b)$$

The existence of nontrivial solutions for U_1, U_2, and U_3 demands the vanishing of the determinant in Eq. (9a), and yields the sixth-degree polynomial equation

$$\alpha^6 + A_1\alpha^4 + A_2\alpha^2 + A_3 = 0, \qquad (10)$$

relating α to c, where the coefficients A_1, A_2, and A_3 are given in Appendix B. Equation (10) admits six solutions for α (having the properties)

$$\alpha_2 = -\alpha_1, \quad \alpha_4 = -\alpha_3, \quad \alpha_6 = -\alpha_5. \qquad (11)$$

For each α_q, $q = 1,2,...,6$, we can use the relations (9) and express the displacement ratios $V_q = U_{2q}/U_{1q}$ and $W_q = U_{3q}/U_{1q}$ as

$$V_q = \frac{K_{11}(\alpha_q)K_{23}(\alpha_q) - K_{13}(\alpha_q)K_{12}(\alpha_q)}{K_{13}(\alpha_q)K_{22}(\alpha_q) - K_{12}(\alpha_q)K_{23}(\alpha_q)}, \qquad (12)$$

$$W_q = \frac{K_{11}(\alpha_q)K_{23}(\alpha_q) - K_{12}(\alpha_q)K_{13}(\alpha_q)}{K_{12}(\alpha_q)K_{33}(\alpha_q) - K_{23}(\alpha_q)K_{13}(\alpha_q)}. \qquad (13)$$

Combining Eqs. (12) and (13) with the stress–strain relations (6), and using superposition, we finally write the formal solutions for the displacements and stresses in the expanded matrix form,

$$\begin{bmatrix} u_1\\ u_2\\ u_3\\ \sigma_{33}\\ \sigma_{13}\\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ V_1 & V_1 & V_3 & V_3 & V_5 & V_5\\ W_1 & -W_1 & W_3 & -W_3 & W_5 & -W_5\\ D_{11} & D_{11} & D_{13} & D_{13} & D_{15} & D_{15}\\ D_{21} & -D_{21} & D_{23} & -D_{23} & D_{25} & -D_{25}\\ D_{31} & -D_{31} & D_{33} & -D_{33} & D_{35} & -D_{35} \end{bmatrix} \times \begin{bmatrix} U_{11}E_1\\ U_{12}E_2\\ U_{13}E_3\\ U_{14}E_4\\ U_{15}E_5\\ U_{16}E_6 \end{bmatrix}, \qquad (14)$$

where

$$\begin{aligned}
E_q &= e^{i\xi\alpha_q x_3},\\
D_{1q} &= i\xi(C_{13}\sin\theta + C_{36}\sin\theta V_q + C_{33}\alpha_q W_q),\\
D_{2q} &= i\xi\,[C_{55}(\alpha_q + W_q\sin\theta) + C_{45}\alpha_q V_q],\\
D_{3q} &= i\xi\,[C_{45}(\alpha_q + W_q\sin\theta) + C_{44}\alpha_q V_q],\\
&\quad q = 1,2,...,6.
\end{aligned} \qquad (15)$$

Notice that the specific relations in the entries of the square matrix of Eq. (14), such as $W_2 = -W_1$ and $V_6 = V_5$, for examples, can be seen by inspection of the ratios (12) and (13) in conjunction with the restrictions (11).

Equation (14) can be used to relate the displacements and stresses at $(x_3')_k = 0$ to those at $(x_3')_k = d^{(k)}$. This is done by specializing (14) to these two locations, eliminating the common amplitudes $U_{11},...,U_{16}$ and getting

$$P_k^+ = A_k P_k^-, \quad k = 1,2,...,n, \qquad (16a)$$

where

$$P_k^\pm = \{[u_1,u_2,u_3,\sigma_{33},\sigma_{13},\sigma_{23}]^T_\pm\}_k \qquad (16b)$$

defines the variables column specialized to the upper and lower surfaces of the layer, k, respectively, and

$$A_k = X_k D_k X_k^{-1}, \qquad (17)$$

where X_k is the 6×6 square matrix of Eq. (14) and D_k is a 6×6 diagonal matrix whose entries are E_q.

The matrix A_k constitutes the transfer matrix for the monoclinic layer k. It allows the wave to be incident on the layer at an arbitrary angle θ from the normal x_3 or equivalently $(x_3')_k$ and at any azimuthal angle ϕ. By applying the above procedure for each layer followed by invoking the continuity of the displacement and stress components (16b) at the layer interfaces, we can finally relate the displacements and stresses at the top of the layered plate, $x_3 = d$, to those at its bottom, $x_3 = 0$, via the transfer matrix multiplication

$$A = A_n A_{n-1} \cdots A_1 \qquad (18)$$

resulting in

$$P^+ = AP^-, \qquad (19)$$

where now P^+ and P^- are the displacement and stress column vectors at the top, $x_3 = d$, and lower, $x_3 = 0$, of the total plate, respectively.

III. PROPERTIES OF THE TRANSFER MATRIX

The global transfer matrix A has several properties which, if exploited, can ease the execution of the analysis and lead to simple analytical representation of the results. Before we proceed to list and discuss these properties, we now indicate that such propertis are also characteristics of the transfer matrices of the individual layers. In fact, since Eq. (19) holds for any number of layers n, then it holds for a single layer in particular [see Eq. (16)], and thus A can be represented by A_k for $k = 1,2,...,n$. Accordingly, we hypothesize that any general property of A_k is also a property of A.

With this we now concentrate on listing and discussing properties of the individual transfer matrix A_k.

(a) $\det(A_k) = 1.$ (20)

This property can be easily proven by employing the well-known result that the determinant of the product is equal to the product of the determinants, which together with the relation (17), implies that

$$\det(A_k) = \det(X_k)\det(X_k^{-1})\det(D_k) = \det(X_k X_k^{-1})\det(D_k) = \det(D_k). \quad (21)$$

This conclusion can also be arrived at by noting that A_k and D_k are similar and hence their determinants are equal.[22] Since D_k is diagonal, its determinant is equal to the product of its entries which, by employing (11), is seen to be unity.

(b) As a consequence of their similarity, A_k and D_k also have the same eigenvalues. This means that the six possible eigenvalues (say $\lambda_q, q = 1,2,...,6$) of A_k are given by the diagonal elements of D_k. By inspection we see that these eigenvalues consist of three pairs with the entries of each pair being the inverse of each other. thus, if λ_1, λ_3, and λ_5 are eigenvalues of A_k, so are $\lambda_2 = 1/\lambda_1$, $\lambda_4 = 1/\lambda_3$, and $\lambda_6 = 1/\lambda_5$.

(c) The results of (b), and the fact that the eigenvalues of A_k^{-1} are the inverse of the corresponding eigenvalues of A_k, lead to the conclusion that A_k and A_k^{-1} have the same set of eigenvalues.

As a consequence of (a)–(c) and the definition (18) we conclude that:

(i) $\det(A) = \det(A_n)\det(A_{n-1}) \cdots \det(A_1) = 1.$ (22)

(ii) The eigenvalues of A^{-1} are equal to the eigenvalues of A. To show this, let us assume that the eigenvalue of A is σ. It follows then that the eigenvalue of A^{-1} is $1/\sigma$. By substituting from (17) into (18) and carrying the inverse of A we get

$$\det(X_n D_n X_n^{-1} \cdots X_2 D_2 X_2^{-1} X_1 D_1 X_1^{-1} - \sigma I) = 0, \quad (23a)$$

$$\det(X_1 D_1^{-1} X_1^{-1} X_2 D_2^{-1} X_2^{-1} \cdots X_n \times D_n^{-1} X_n^{-1} - \sigma^{-1} I) = 0. \quad (23b)$$

Now, using the fact that the products of the two equal rank square matrices $M_1 M_2$ and $M_2 M_1$ (although $M_1 M_2 \neq M_2 M_1$) have the same eigenvalues,[22] by cyclic permutation, we can rewrite the relation (23a) as

$$\det(X_1 D_1 X_1^{-1} X_2 D_2 X_2^{-1} \cdots X_n D_n X_n^{-1} - \sigma I) = 0. \quad (23c)$$

By inspecting (23b) and (23c), we conclude that (23c) can be obtained from (23b) by merely inverting the diagonal matrices D_k and the eigenvalue σ. Since the entries of D_k are made up of pairs that are inverse of each other, then it is obvious that D_k and D_k^{-1} have the same entries (eigenvalues). Thus we conclude that the eigenvalues σ_q, $q = 1,2,...,6$ and $1/\sigma_q$ constitute the same set.

(iii) The property described under (ii) can only imply that the σ_q consists of three pairs where the entries of each pair are the inverse of each other. In our subsequent analysis we choose to arrange these six eigenvalues as σ_1, $1/\sigma_1$, σ_3, $1/\sigma_3$, σ_5, and $1/\sigma_5$.

(iv) In the degenerate case where all layers are made up of the same material (but not necessarily have equal thicknesses), X_k and the six values α_q are the same for every k. Now, substituting from (17) into (18) and recognizing that $X_{i+1}^{-1} X_i = I$ (identity) for $i = 1,2,...,n$, the global matrix A collapses to

$$A = X_1 D X_1^{-1}, \quad (24)$$

where we used the fact that here $X_n = X_1$ and

$$D = D_n D_{n-1} \cdots D_1 \quad (25)$$

is a diagonal matrix whose entries are given by $\exp(i\xi\alpha_q d)$, $q = 1,2,...,6$ and d is the total thickness. Thus, we have shown that the global transfer matrix correctly reduces to the corresponding matrix of the single material plate when all layer properties are the same.

(v) A very important consequence of the above listed properties is the resulting relations that exist between the invarients of A. To this end, if we expand the characteristic equation $\det(A - \sigma I) = 0$, write it in terms of both the eigenvalues σ_q and invarients I_q of A, $q = 1,2,....,6$ format, and compare the resulting expressions, we conclude the symmetric relations

$$I_5 = I_1, \quad I_4 = I_2, \quad I_6 = 1. \quad (26)$$

The result $I_6 = 1$ also confirms the fact that $\det(A) = 1$.

Equation (19) will now be used to present solutions for a variety of situations. In the first, we consider a single cell medium, namely a free n-layered plate. The characteristic equation for such a situation is obtained by choosing $\theta = 0$ and invoking the stress-free upper and bottom surfaces in Eq. (19) that lead to the characteristic equation

$$\begin{vmatrix} A_{41} & A_{42} & A_{43} \\ A_{51} & A_{52} & A_{53} \\ A_{61} & A_{62} & A_{63} \end{vmatrix} = 0. \quad (27)$$

A second important situation is that of a periodic medium consisting of a repetition of the unit cell (plate). Here we generalize the classical Floquet periodicity condition to require

$$P^+ = P^- e^{i\xi d \cos\theta} \quad (28)$$

which is consistent with the formal solution (8). Combinations of (26) and (19) yields the characteristic equation

$$\det(A - Ie^{i\xi d\cos\theta}) = 0. \tag{29}$$

Equation (29) can also be expanded and written in terms of the invariants I_q of A which, after algebraic manipulation reduce to

$$\cos[3\xi d\cos\theta] - I_1\cos[2\xi d\cos\theta] + I_2\cos[\xi d\cos\theta] - I_3/2 = 0. \tag{30}$$

In terms of the individual entries A_{ij} of A the invariants I_q are given in Ref. 22 by

$$(-1)^m I_m = \sum_{i_1 < i \cdots < i_m} \Delta(i_1, i_2, \ldots, i_m), \tag{31}$$

where $\Delta(i_1, i_2, \ldots, i_m)$ is the mth determinant formed from the rows $i = i_1, i_2, \ldots, i_m$ and columns $j = i_1, i_2, \ldots, i_m$.

IV. HIGHER SYMMETRY MATERIALS

The results of Eqs. (27) and (30) obtained for the monoclinic material case can, under restricted conditions, also hold for higher symmetry ones such as orthotropic, transversely isotropic, and cubic. Recognizing that these classes of materials are different from monoclinic materials in that each possess two orthogonal principal axes in the planes of the layers, Eqs. (27) and (30) apply if the wave propagates along directions other than these principal axes. This is due to the fact that coupling of the SH wave field equations with those of the sagittal plane wave will persist. As was pointed out in an earlier work,[21] which deals with single orthotropic and monoclinic plates, uncoupling of these equations occurs, on the other hand, for propagation along axes of rotational symmetry. The implication here is that for the present problem this results in simplified versions of Eqs. (27) and (30). Here, we also recognize that results for all symmetry classes higher than orthotropic (which include transversely isotropic, cubic, and isotropic) are contained as special cases of the corresponding solutions for the orthotropic case. Thus, once the characteristic equations for propagation either off or along principal axes are derived for orthotropic symmetry, the corresponding results for the higher symmetry materials can be obtained by merely applying the appropriate restrictions on their properties. For this reason, we now derive results for orthotropic symmetry.

Thus, for off-principal-axis propagation, one needs only to assure that further appropriate reductions in the number of nonzero elastic constants are exploited in Eqs. (27) and (30). If x_1' and x_2' are chosen to coincide with the in-plane principal axes for orthotropic symmetry then

$$C_{16}' = C_{26}' = C_{36}' = C_{45}' = 0. \tag{32}$$

Results for material possessing transverse isotropy, whose x_1' axis is normal to the plane of isotropy, can be easily obtained from Eq. (3) by noting the additional conditions imposed by symmetry, namely,

$$C_{33}' = C_{22}', \quad C_{13}' = C_{12}', \quad C_{55}' = C_{66}',$$
$$C_{22}' - C_{23}' = 2C_{44}' \tag{33}$$

on the nine constants describing the orthotropic case. Finally, the presence of cubic symmetry requires the further restrictions

$$C_{11}' = C_{22}' = C_{33}', \quad C_{12}' = C_{13}' = C_{23}',$$
$$C_{44}' = C_{55}' = C_{66}'. \tag{34}$$

V. PROPAGATION ALONG AXES OF ROTATIONAL SYMMETRY

Returning to the case of orthotropic symmetry, we note that for each layer k the axes $(x_1')_k$ and $(x_2')_k$ coincide with the azimuthal angles $\phi = 0°$ and $\phi = 90°$, respectively. If the wave happens to propagate along an inplane axis of symmetry of layer k (namely for $\phi_k = 0°$ or $90°$) then Eqs. (27) and (30), strictly speaking, can only hold in the limit. This is of course due to the presence of the superfluous coupling [as implied by (10)] between the equations describing the horizontally polarized wave (SH) and those belonging to the sagittal plane. For such situations the appropriate transfer matrix does not contain this nonexistent coupling. In fact, the analysis reduces to a two-dimensional and hence simpler one.

The corresponding transfer matrix for a stack of 0°,90° orthotropic layers have been recently derived in order to study the reflection from a fluid-loaded laminated plate of unidirectional fibrous composites.[23] For completeness we here outline the analysis leading to the derivation of the transfer matrix for propagation along axes of symmetry. To this end substituting from Eq. (32), which particularizes the constitutive relations (3) to orthotropic media, into Appendix A and inspecting the resulting entries leads to the conclusion that for propagation along rotational symmetry axes, the matrix elements C_{16}, C_{26}, C_{36}, and C_{45} also vanish. This simplification of the elastic stiffness matrix has implications for the analysis commencing at Eq. (9b). Of greatest importance is the fact that K_{13} and K_{23} in Eq. (9b) will vanish. This result means that the SH wave motions uncouple from those of the sagittal plane. As a consequence Eq. (10) leads, for $\phi = 0$, to the simpler solution for the α_q

$$\alpha_{1,3} = [-B \pm (B^2 - 4AC)^{1/2}]/2A,$$
$$\alpha_2 = -\alpha_1, \quad \alpha_4 = -\alpha_3, \tag{35}$$
$$\alpha_5 = -\alpha_6 = [(\rho c^2 - C_{66}\sin^2\theta)/C_{44}], \tag{36}$$

with

$$A = C_{33}C_{55},$$
$$B = (C_{11}\sin^2\theta - \rho c^2)C_{33} + (C_{55}\sin^2\theta - \rho c^2)C_{55} - (C_{13} + C_{55})^2\sin^2\theta,$$
$$C = (C_{11}\sin^2\theta - \rho c^2)(C_{55}\sin^2\theta - \rho c^2). \tag{37}$$

Notice that α_5 and α_6 of Eq. (36) correspond to the (SH) motion while those of Eq. (35) correspond to the sagittal plane waves. Nayfeh recently studied the interaction of the (SH) elastic waves with multilayered media[24] as a prelude to the present general case and hence will not be pursued further.

As for the sagittal plane motion we notice that for each $\alpha_q, q = 1,2,3,4$ the displacements and stress amplitudes reduce to

$$W_q = \frac{\rho c^2 - C_{11}\sin^2\theta - C_{55}\alpha_q^2}{(C_{13}+C_{55})\alpha_q \sin\theta}, \quad (38)$$

$$D_{1q} = i\xi(C_{13}\sin\theta + C_{33}\alpha_q W_q),$$

$$D_{2q} = i\xi C_{55}(\alpha_q + W_q \sin\theta). \quad (39)$$

Accordingly, formal solutions for propagation along an axis of symmetry of an orthotropic material are given by

$$\begin{bmatrix} u_1 \\ u_3 \\ \sigma_{33} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ W_1 & -W_1 & W_3 & -W_3 \\ D_{11} & D_{11} & D_{13} & D_{13} \\ D_{21} & -D_{21} & D_{23} & -D_{23} \end{bmatrix} \begin{bmatrix} U_{11}E_1 \\ U_{12}E_2 \\ U_{13}E_3 \\ U_{14}E_4 \end{bmatrix}, \quad (40a)$$

where

$$E_q = e^{i\xi\alpha_q x_3}, \quad q = 1,2,3,4. \quad (40b)$$

Once again, Eq. (40) can be used to relate the displacements and stresses at $x_3^{(k)} = 0$ to those at $x_3^{(k)} = d^{(k)}$ for such a restricted direction of propagation. This can be done by specializing Eq. (40) to $x_3^{(k)} = 0$ and to $x_3^{(k)} = d^{(k)}$ and eliminating the common amplitude column of U_{11}, U_{12}, U_{13}, and U_{14} resulting in an equation similar to (27) with X_k now given by the 4×4 matrix of Eq. (40) and D_k is again the 4×4 diagonal matrix whose entries are given as $e^{i\xi\alpha_q d^{(k)}}$, $q = 1,2,3,4$ for the α_q defined in Eq. (35). Hence the global transfer matrix is again constructed by multiplication of the individual material transfer matrices. The properties of the resulting 4×4 matrix (here referred to as A' to avoid confusing it with A of the monoclinic case) are identical with those of the 6×6 matrix A of (18), except for the fact that A' has only two pairs of eigenvalues rather than three.

Utilizing A', the corresponding characteristic equations for the free waves on a single layered plate and on the periodic media for propagation along an axis of symmetry of each layer are given, respectively, by

$$\begin{vmatrix} A'_{31} & A'_{32} \\ A'_{41} & A'_{42} \end{vmatrix} = 0 \quad (41)$$

and

$$\cos[2\xi d \cos\theta] - I'_1 \cos[\xi d \cos\theta] + I'_2/2 = 0. \quad (42)$$

VI. DISCUSSIONS AND NUMERICAL ILLUSTRATION

In this section, we illustrate the analytical results (27), (30), (41), and (42) with a limited selection of numerical examples. While the cases we present here are certainly typical, they are by no means exhaustive of the variety of the phenomenology contained in the analysis. Once the number of layers, their properties, and geometric stacking are specified, we present our numerical results in two categories. In the first, we demonstrate the variations of phase velocity c with angle of incidence θ for specified frequencies F and orientation angles ϕ; this is effectively a form of demonstration of the dependence of wave front (inverse of slowness) curves with frequency for specified orientation angles. In the second, we present phase velocity dispersion curves plotted as functions of the product of frequency and unit cell thickness, namely Fd, for specified angles of incidence θ. The properties of a representative orthotropic material that we used in our calculations are given in GPa by $C'_{11} = 128$, $C'_{12} = 7$, $C'_{13} = 6$, $C'_{22} = 72$, $C'_{23} = 5$, $C'_{33} = 32$, $C'_{44} = 18$, $C'_{55} = 12.25$, $C'_{66} = 8$, and $\rho = 2$ g/cm^3. Here, different layers can be constructed from this chosen material by assigning appropriate rotational angles. This choice is not restrictive and has the advantage of saving space by not having to list different material properties. Thus, for examples a combination of 0°, 90°, 60°, and −60° layup constitutes a four-layered cell whereas a combination of 0°, 0°, 0°, and 0° cells defines a single homogeneous material. Without any loss in generality the thickness d of the representative unit cell is kept constant, and its constituents (layers) are assigned volume fractions adding to unity.

To show the extent of generality in the results, we now discuss the case in which all layers are the same. This is expected to undoubtedly result in a description of the behavior of single homogeneous anisotropic materials. As was discussed earlier, the global transfer matrix for such a situation collapses to the form given in (24). Using this matrix, together with the fact that for this case D and A are similar, dictates that the characteristic Eq. (29) admits the solution

$$\alpha_q^2 = \cos^2\theta, \quad q = 1,2,...,6, \quad (43)$$

which also specializes the formal solution (8) to the one appropriate for the single homogeneous medium. With reference to Eq. (10) and for a fixed θ, the results (43) admit three roots for the phase velocity c corresponding to one quasilongitudinal and two quasishear motions. Thus, for a variable θ, Eq. (43) describes the variation of the three phase velocities with the incident angle and hence constitute wave front curves. For this specialized single medium case, these curves will be independent of frequency, however.

For an isotropic material, for example, Eq. (10) uncouples and gives

$$\alpha_1^2 = c^2/c_L^2 - \sin^2\theta; \quad \alpha^2_{3,5} = c^2/c_T^2 - \sin^2\theta, \quad (44)$$

where c_L and c_T are the longitudinal and shear wave speeds in the medium respectively. Thus, combination of Eqs. (43) and (44) gives the roots $c = c_L$ and $c = c_T$ yielding two concentric spherical wave front curves as is expected. For the anisotropic case, however, the three solutions will be coupled resulting in nonspherical wave fronts.

For the layered media case, the situation is much more complicated due to the dependence of the phase velocities, not only on the individual layer properties, but most importantly on the wave number ξ or frequency F (more precisely on the parameter Fd). However, for a fixed frequency, we can construct wave front curves and hence, by varying the frequency in a discrete manner, demonstrate a frequency-dependent "dispersive" character of the wave fronts. For $Fd = 0$ MHz mm, the curves will thus constitute the wave fronts for an effective homogenized medium whose properties are volume fraction weighted properties of the individual layers.

Conventional dispersion curves in the forms of variations of wave velocities with wavenumber can also be constructed using either Eq. (29) or (30). This is done, however, for fixed values of θ. Here we mention that further confidence in our analytical and computational procedure was established by reproducing the numerical results of Refs. 6 and 9 that constitute special cases of the present work. Sample examples, which demonstrate the dependence of such curves on the number of layers and their orientations, are given for the representative angle, $\theta = 45°$, in Figs. 2–5. They correspond to (0°, 90°); (0°, 90°, 45°, −45°); (60°, 0°, −60°); and (0°, 15°, 30°, 45°, 60°, 75°, 90°), periodic media layup configurations, respectively. Note once again that all of these plates have the same thickness d. In these figures the phase velocity c is given by km/s and the wave number ξ by mm^{-1}.

Close examination of these figures reveals several interesting features. At the zero wave-number limit, each figure displays three values of wave speeds corresponding to one quasilongitudinal and two quasishear. It is obvious that the largest value corresponds to the quasilongitudinal mode. At relatively low values of the wave number, little change is seen to take place in these values. As ξ increases, other higher-order modes appear; one of these seems to be associated with a rapid change in the slope of the quasilongitudinal mode.

Furthermore, the isotropiclike behavior (suggested by the closeness of the two quasitransverse modes of Figs. 4 and 5 as compared with those of Fig. 3), is worth commenting upon. It is consistent with the static prediction of the quasi-isotropy of the (0°, 90°, 45°, −45°) and (60°, 0°, −60°) layups.

In Fig. 6(a)–(c) we depict, for the selected values $Fd = 0$,2, and 4 MHz mm, wave front curves in the K_1-K_2 plane where $K_1 = c \sin \theta$ and $K_2 = c \cos \theta$, using a (60°, 0°, −60°) layup periodic medium as a representative case. These curves demonstrate the inverse of the slowness curves as functions of frequency and hence display and demonstrate wavefront dispersion behavior. The complicated features

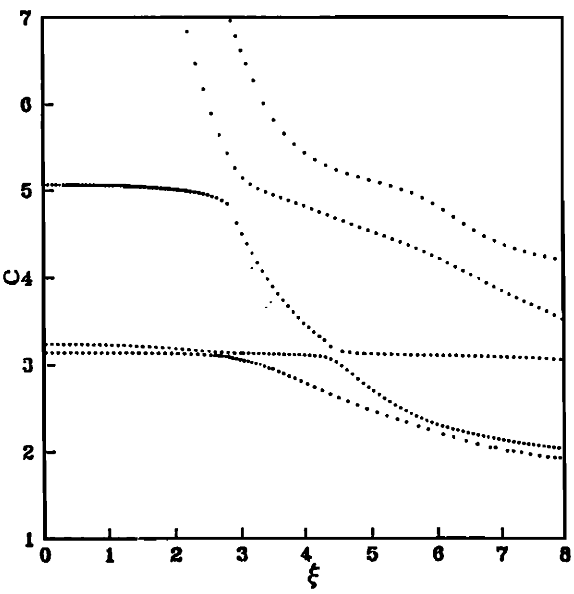

FIG. 3. Same as Fig. 2 with (0°, 90°, 45°, −45°) layup.

shown in Fig. 6(c) and to a lesser degree in 6(b) are due to multivalued behavior shown in Fig. 4 especially at $Fd = 4$ MHz mm brought about by the presence of the higher-order modes. Notice in contrast that the "clean" behavior displayed in Fig. 6(a) reflects the variations of effective wave speed c (namely at $Fd = 0$) with the angle of incidence.

To further show the versatility of the analyses we also generate, using the characteristic Eq. (27), the dispersion curves of Fig. 7 for free waves on a finite thickness multi-

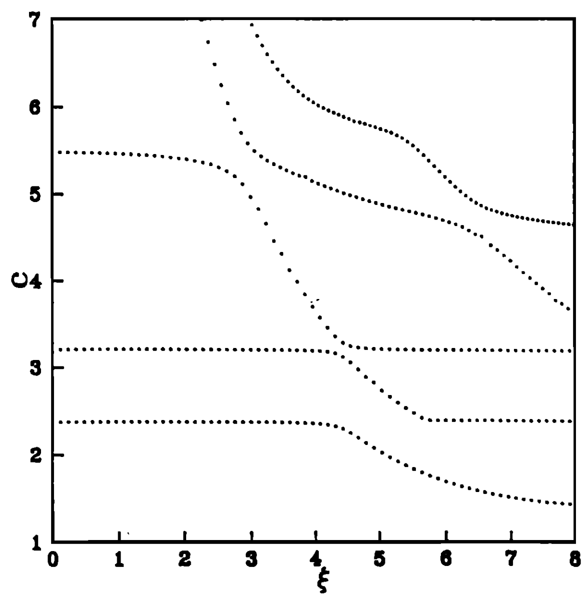

FIG. 2. Variation of phase velocity c with wavenumber ξ for angle of incidence $\theta = 45°$; (0°, 90°) layup.

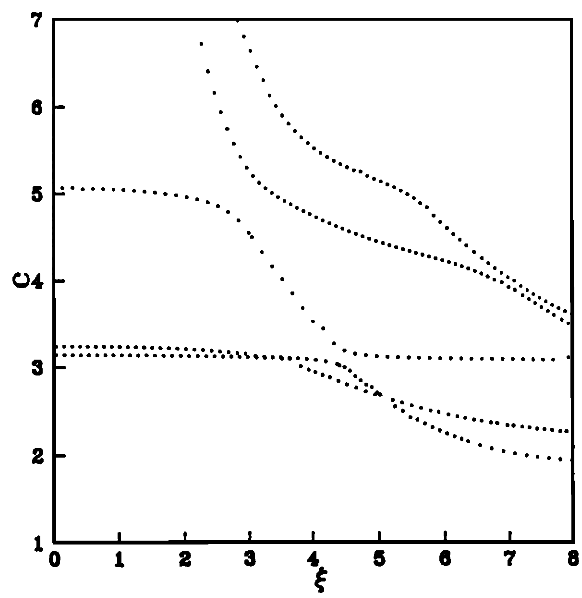

FIG. 4. Same as Fig. 2 with (60°, 0°, −60°) layup.

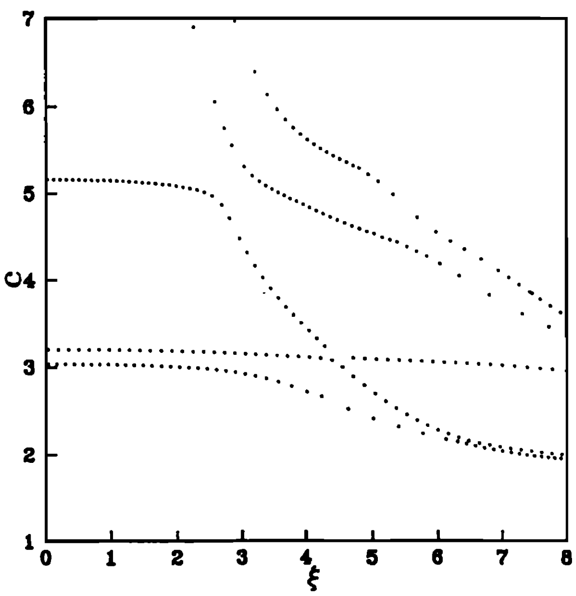

FIG. 5. Same as Fig. 2 with (0°, 15°, 30°, 45°, 60°, 75°, 90°) layup.

layered plate consisting of (−60°, 0°, 60°) layup. The curves displayed on this figure are typical of free waves in anisotropic plates.[17]

VII. CONCLUSION

We have derived analytical expressions that are easily adaptable to numerical illustrations of the interaction of elastic waves with multilayered anisotropic media. A plate consisting of an arbitrary number of layers each possessing as low as monoclinic symmetry is chosen as a representative cell of the medium. Waves are allowed to propagate along arbitrary directions in both azimuthal as well as incidence planes. Characteristic equations for a variety of physical systems are discussed. These include the cases of propagation of free harmonic waves in a multilayered plates and in periodic media constructed from a repetition of the layered plate. Results in the forms of dispersion curves are given for several representative layering. Wave front curves for fixed frequencies are also included to demonstrate their dispersive behaviors.

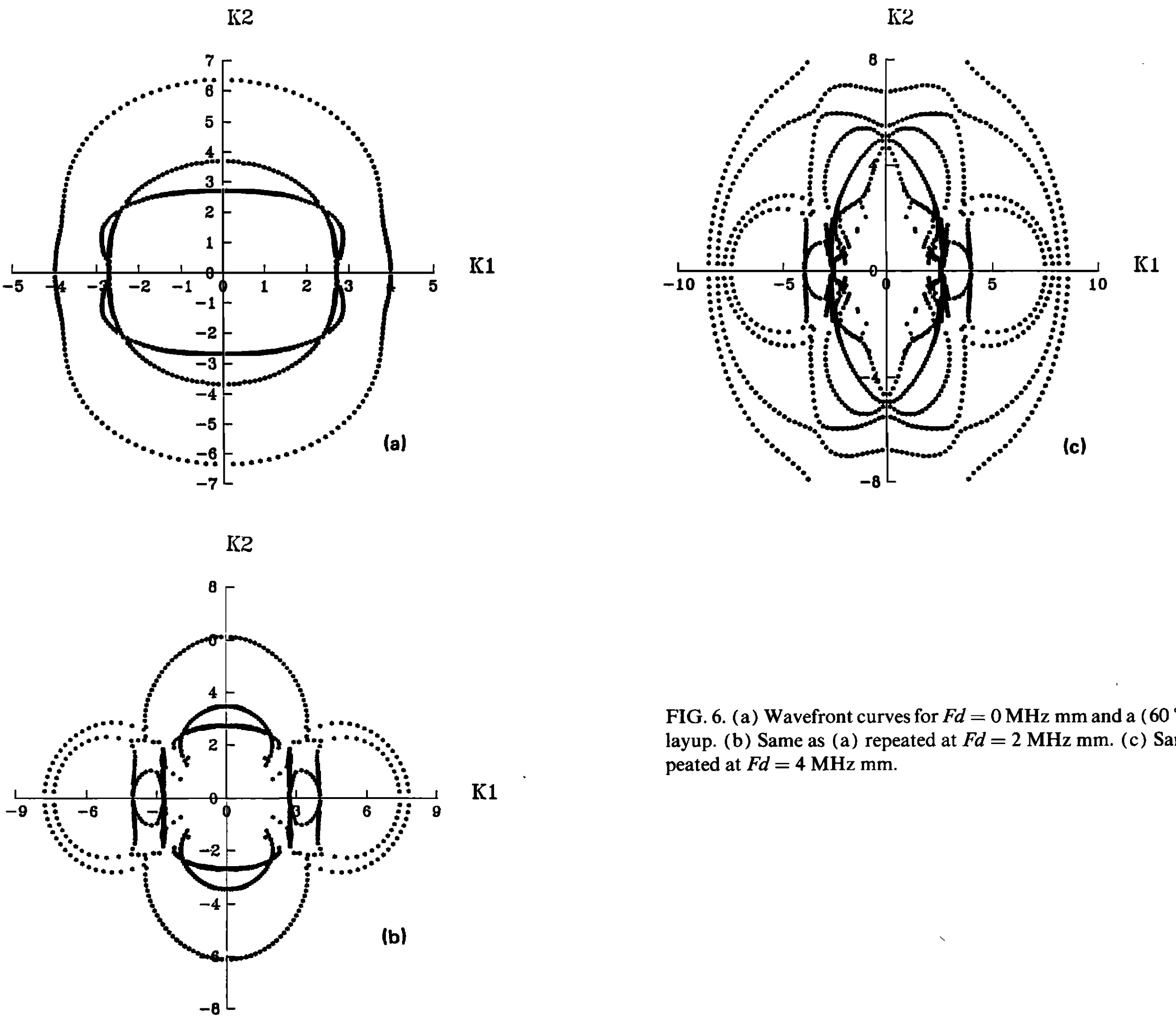

FIG. 6. (a) Wavefront curves for $Fd = 0$ MHz mm and a (60°, 0°, −60°) layup. (b) Same as (a) repeated at $Fd = 2$ MHz mm. (c) Same as (a) repeated at $Fd = 4$ MHz mm.

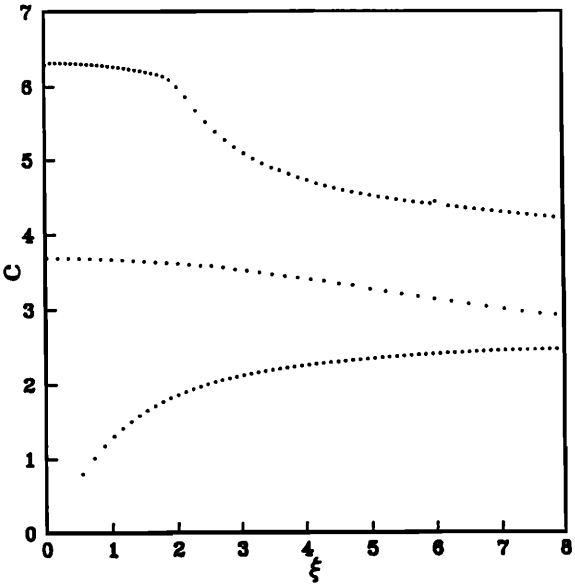

FIG. 7. Variation of phase velocity c with wave number ξ for a (60°, 0°, −60°) layup free plate.

ACKNOWLEDGMENT

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APPENDIX A

Combination of the transformation matrix (5) with the constitutive relations (3) yields the following transformed properties:

$$
\begin{aligned}
C_{11} &= C'_{11}G^4 + C'_{22}S^4 + 2(C'_{12} + 2C'_{66})S^2G^2,\\
C_{12} &= (C'_{11} + C'_{22} - 4C'_{66})S^2G^2 + C'_{12}(S^4 + G^4),\\
C_{13} &= C'_{13}G^2 + C'_{23}S^2,\\
C_{16} &= (C'_{11} - C'_{12} - 2C'_{66})SG^3\\
&\quad + (C'_{12} - C'_{22} + 2C'_{66})GS^3,\\
C_{22} &= C'_{11}S^4 + 2(C'_{12} + 2C'_{66})S^2G^2 + C'_{22}G^4,\\
C_{23} &= C'_{23}G^2 + C'_{13}S^2,\\
C_{26} &= (C'_{11} - C'_{12} - 2C'_{66})GS^3\\
&\quad + (C'_{12} - C'_{22} + 2C'_{66})SG^3,\\
C_{33} &= C'_{33},\\
C_{36} &= (C'_{23} - C'_{13})SG,\\
C_{45} &= (C'_{44} - C'_{55})SG,\\
C_{44} &= C'_{44}G^2 + C'_{55}S^2,\\
C_{55} &= C'_{55}G^2 + C'_{44}S^2,\\
C_{66} &= (C'_{11} + C'_{22} - 2C'_{12} - 2C'_{66})S^2G^2\\
&\quad + C'_{16}(S^4 + G^4),
\end{aligned}
$$

where $G = \cos\phi$ and $S = \sin\phi$.

APPENDIX B

The various coefficients of Eq. (10) are given by

$$
\begin{aligned}
A_1 = {} & [(C_{11}C_{33}C_{44} - C_{13}^2C_{44} + 2C_{13}C_{36}C_{45} - 2C_{13}C_{44}C_{55} + C_{13}C_{45}^2 - 2C_{16}C_{33}C_{45} + C_{33}C_{55}C_{66} - C_{36}^2C_{55})\sin^2\theta\\
& - (C_{33}C_{44} + C_{33}C_{55} + C_{44}C_{55} - C_{45}^2)\rho c^2]/\Delta,\\
A_2 = {} & [(C_{11}C_{33}C_{66} - C_{11}C_{36}^2 - 2C_{11}C_{36}C_{45} + C_{11}C_{44}C_{55} - C_{11}C_{45}^2 + C_{13}^2C_{66} + 2C_{13}C_{16}C_{36} + 2C_{13}C_{16}C_{45}\\
& - 2C_{13}C_{55}C_{66} - C_{16}^2C_{33} + 2C_{16}C_{36}C_{55})\sin^4\theta - (C_{11}C_{33} + C_{11}C_{44} - C_{13}^2 - 2C_{13}C_{55} - 2C_{16}C_{45}\\
& + C_{33}C_{66} - C_{36}^2 - 2C_{36}C_{45} + C_{44}C_{55} - C_{45}^2 + C_{55}C_{66})\rho c^2\sin^2\theta + (C_{33} + C_{44} + C_{55})\rho^2c^4]/\Delta,\\
A_3 = {} & [(C_{11}C_{55}C_{66} - C_{16}^2C_{55})\sin^6\theta - (C_{11}C_{55} + C_{11}C_{66} + C_{16}^2 + C_{55}C_{66})\rho c^2\sin^4\theta\\
& + (C_{11} + C_{55} + C_{66})\rho^2c^4\sin^2\theta - \rho^3c^6]/\Delta,
\end{aligned}
$$

with

$$\Delta = C_{33}C_{44}C_{55} - C_{33}C_{45}^2.$$

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