which is the **complex Fourier series** for $u(t)$.

Exactly the same methodology applies to a periodic function of a spatial coordinate x. In that case, the terminology is to say **wavenumber** $k_M = 2\pi M/L$ in place of angular frequency ω_M.

7.4 Discrete Fourier Transform (DFT) and FFT

Let $u_j, j = 1, \ldots, N$ be a sequence of N possibly complex values. The **Discrete Fourier Transform** (DFT) of this sequence is the sequence $\hat{u}_m, m = 1, \ldots, N$, where

$$\hat{u}_m = \sum_{j=1}^{N} u_j e^{-2\pi i(m-1)(j-1)/N} \tag{7.4.1}$$

The inverse discrete Fourier transform (IDFT) is

$$u_j = \frac{1}{N}\sum_{m=1}^{N} \hat{u}_m e^{2\pi i(m-1)(j-1)/N} \tag{7.4.2}$$

The FFT is a fast algorithm for computing the discrete Fourier transform for data lengths $N = 2^p$, taking $O(N \log_2 N)$ flops as compared with $O(N^2)$ flops for doing the computation directly using the above formulas. Versions of the FFT that are nearly as efficient also apply for $N = 2^p 3^q 5^r$.

To show that the IDFT really is the inverse of the DFT, we substitute eqn. (7.4.1) into (7.4.2), after changing the summation index in the former to J:

$$\begin{aligned} u_j &\overset{?}{=} \frac{1}{N}\sum_{m=1}^{N} e^{2\pi i(m-1)(j-1)/N} \sum_{J=1}^{N} u_J e^{-2\pi i(m-1)(J-1)/N} \\ &= \frac{1}{N}\sum_{m=1}^{N}\sum_{J=1}^{N} u_J e^{2\pi i(m-1)(j-J)/N} \\ &= \frac{1}{N}\sum_{J=1}^{N} u_J \sum_{m=1}^{N} e^{2\pi i(m-1)(j-J)/N} \end{aligned} \tag{7.4.3}$$

The inner sum over m is a geometric series with ratio $\exp[2\pi i(m-1)(j-J)/N]$. If $J = j$, each term is 1 so the series sums to N. If $J \neq j$, the sum is:

$$\begin{aligned} \sum_{m=1}^{N} e^{2\pi i(m-1)(j-J)/N} &= \frac{\exp\left[\frac{2\pi i}{N}N(j-J)\right]-1}{\exp\left[\frac{2\pi i}{N}(j-J)\right]-1} \\ &= 0 \ (J \neq j) \end{aligned}$$

Thus, the only term in the outer sum over J that contributes is from $J = j$, for which the inner sum is N, and we indeed find that the RHS of (7.4.3) is equal to u_j as claimed.