2D spectral-elements formulation in cartesian coordinates

Introduction : assumptions, conventions, weak form

We will mainly adopt the notations of (Komatitsch & Tromp 1999). We shall start from the wave equation for a 3D inhomogeneous medium Ω :

$$\rho\ddot{\boldsymbol{u}} = \boldsymbol{\nabla}\cdot\boldsymbol{T} + \boldsymbol{f} \tag{1}$$

With $\boldsymbol{\nabla}\cdot\boldsymbol{T}$ refering to the divergence of the stress tensor field $\boldsymbol{T}$. In elastic solids $\boldsymbol{T}$ is linearly related to the displacement field gradient $\boldsymbol{\nabla u}$ (Hooke's law)

$$\boldsymbol{T} = \boldsymbol{c} : \boldsymbol{\nabla u}$$

Means : each component of the stress tensor $\boldsymbol{T}$ is a linear combination of each component of the displacement gradient $\boldsymbol{\nabla u}$. The elastic properties of the earth model are contained into the fourth-order elastic tensor $\boldsymbol{c}$.
In elastic isotropic media :

$$\forall i,j \in \{x,y,z\}^2 \quad T_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} = \lambda\delta_{ij}(\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) + 2\mu\epsilon_{ij}$$

That is :

$$\begin{Bmatrix} T_{xx}\\ T_{yy}\\ T_{zz}\\ T_{xy}\\ T_{yz}\\ T_{zx}\end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ \epsilon_{xy}\\ \epsilon_{yz}\\ \epsilon_{zx}\end{Bmatrix}$$

$\boldsymbol{\epsilon}$ being the strain tensor. By definition :

$$\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla}^{\top}\boldsymbol{u})$$

This relation does not depend on the coordinate system (if it remains orthogonal). Nevertheless the value of $\boldsymbol{\nabla u}$ expresses differently depending on each coordinate system. Let M be a point of Ω and $M \mapsto \boldsymbol{w}(M)$ an arbitrary test function. Let us suppose that Ω has a free surface $\partial\Omega$ with unit normal $\boldsymbol{n}(x,y,z)$ and an artificial boundary Γ. On the free surface $\boldsymbol{T}\cdot\boldsymbol{n} = 0$. We obtain the week formulation from the wave equation by dotting the momentum equation (1) with an the test function $\boldsymbol{w}$ and integrating by part over the model volume Ω :

$$\underbrace{\int_\Omega \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^3\boldsymbol{x}}_{\text{mass integral}} = \underbrace{-\int_\Omega \boldsymbol{\nabla w} : \boldsymbol{T}\,\mathrm{d}^3\boldsymbol{x}}_{\text{stiffness integral}} + \underbrace{\int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^3\boldsymbol{x}}_{\text{source integral}} + \underbrace{\int_\Gamma \boldsymbol{w}\cdot\boldsymbol{T}\cdot\boldsymbol{n}\,\mathrm{d}^2\boldsymbol{x}}_{\text{0 for the moment}}$$

(The formula for integration by parts can be extended to functions of several variables. Instead of an interval one needs to integrate over an n-dimensional set. Also, one replaces the derivative with a partial derivative.
More specifically, suppose Ω is an open bounded subset of $\mathbb{R}^n$ with a piecewise smooth boundary Γ. If u and u are two continuously differentiable functions on the closure of Ω, then the formula for integration by parts is

$$\int_\Omega \frac{\partial u}{\partial x_i} v\, d\Omega = \int_\Gamma uv\,\nu_i\, d\Gamma - \int_\Omega u\frac{\partial v}{\partial x_i}\, d\Omega$$

where $\hat{\nu}$ is the outward unit surface normal to Γ, ν_i is its i-th component, and i ranges from 1 to n. By replacing v in the above formula with v_i and summing over i gives the vector formula :

$$\int_\Omega \nabla u\cdot\mathbf{v}\, d\Omega = \int_\Gamma u(\mathbf{v}\cdot\hat{\nu})\, d\Gamma - \int_\Omega u\,\nabla\cdot\mathbf{v}\, d\Omega$$

where v is a vector-valued function with components $v_1,\ldots,v_n$)
We will write the displacement vector : $\boldsymbol{u} = u_x(x,z,t)\boldsymbol{i} + u_y(x,z,t)\boldsymbol{j} + u_z(x,z,t)\boldsymbol{k}$

The strain tensor expresses : $\forall i,j\in\{x,y,z\}^2 \quad \epsilon_{ij} = \dfrac{1}{2}(\partial_i u_j + \partial_j u_i)$, that is : $\begin{Bmatrix} \epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ \epsilon_{xy}\\ \epsilon_{yz}\\ \epsilon_{zx}\end{Bmatrix} = \begin{bmatrix} \partial_x & 0 & 0\\ 0 & \partial_y & 0\\ 0 & 0 & \partial_z\\ \frac{1}{2}\partial_y & \frac{1}{2}\partial_x & 0\\ 0 & \frac{1}{2}\partial_z & \frac{1}{2}\partial_y\\ \frac{1}{2}\partial_z & 0 & \frac{1}{2}\partial_x \end{bmatrix} \begin{Bmatrix} u_x\\ u_y\\ u_z\end{Bmatrix}$

At this point we will choose the plane-strain convention : we suppose an infinite medium along y and that the important loads are in the $x-z$ plane and do not change with y:

$$\frac{\partial}{\partial y} = 0$$

That leads to :

$$\begin{Bmatrix} \epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ \epsilon_{xy}\\ \epsilon_{yz}\\ \epsilon_{zx}\end{Bmatrix} = \begin{bmatrix} \partial_x & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \partial_z\\ 0 & \frac{1}{2}\partial_x & 0\\ 0 & \frac{1}{2}\partial_z & 0\\ \frac{1}{2}\partial_z & 0 & \frac{1}{2}\partial_x \end{bmatrix} \begin{Bmatrix} u_x\\ u_y\\ u_z\end{Bmatrix}$$

We see here that the plane-strain convention implies $\epsilon_{yy} = 0$ (Note : dans le manuel de specfem2d on inverse definition et consequence je crois).
Consequently :

$$\begin{Bmatrix} T_{xx}\\ T_{yy}\\ T_{zz}\\ T_{xy}\\ T_{yz}\\ T_{zx}\end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}\\ \epsilon_{zz}\\ \epsilon_{xy}\\ \epsilon_{yz}\\ \epsilon_{zx}\end{Bmatrix}$$

We see here that the terms depending on u_y (ϵ_{xy} and ϵ_{yz}) are not coupled with the other components of the displacement.
For now we will adopt the notations : $(x,z) \leftrightarrow (1,2)$. For example : $A_{12} \equiv A_{xy}$
The test functions $M \mapsto \boldsymbol{w}(M)$ expresses$(x,z) \mapsto \boldsymbol{w}(x,z) = w_x(x,z)\boldsymbol{i} + w_y(x,z)\boldsymbol{j} + w_z(x,z)\boldsymbol{k} \equiv (w_x,w_y,w_z)$. We have then :

$$\boldsymbol{\nabla w} : \boldsymbol{T} = \sum_{i,j=1}^{3} T_{ij}\boldsymbol{\nabla w}_{ji} = \sum_{i,j=1}^{3} T_{ij}\partial_j w_i = T_{xx}\frac{\partial w_x}{\partial x} + T_{zz}\frac{\partial w_z}{\partial z} + T_{xy}\frac{\partial w_y}{\partial x} + T_{yz}\frac{\partial w_y}{\partial z} + T_{xz}\left(\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z}\right)$$

The wave equation becomes :

$$\begin{aligned} & \int_\Omega \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^2\boldsymbol{x} && = && -\int_\Omega \boldsymbol{\nabla w} : \boldsymbol{T}\,\mathrm{d}^2\boldsymbol{x} + \int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^2\boldsymbol{x} \\ \iff & \int_\Omega \rho\left(w_x\ddot{u}_x + w_y\ddot{u}_y + w_z\ddot{u}_z\right)\,\mathrm{dxdz} && = && -\int_\Omega \boldsymbol{\nabla w} : \boldsymbol{T}\,\mathrm{d}^2\boldsymbol{x} + \int_\Omega \left(w_x f_x + w_y f_y + w_z f_z\right)\,\mathrm{dxdz} \end{aligned} \tag{2}$$

The stiffness integral reads :

$$\begin{aligned} \int_\Omega \boldsymbol{\nabla w} : \boldsymbol{T}\,\mathrm{d}^2\boldsymbol{x} &= \int_\Omega \left[(\lambda+2\mu)\,\epsilon_{xx} + \lambda\epsilon_{zz}\right]\frac{\partial w_x}{\partial x} + \left[\lambda\epsilon_{xx} + (\lambda+2\mu)\,\epsilon_{zz}\right]\frac{\partial w_z}{\partial z} + \left[2\mu\epsilon_{xy}\right]\frac{\partial w_y}{\partial x} + \left[2\mu\epsilon_{yz}\right]\frac{\partial w_y}{\partial z} + \left[2\mu\epsilon_{zx}\right]\left(\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z}\right)\,\mathrm{dxdz} \\ &= \int_\Omega \left[(\lambda+2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z}\right]\frac{\partial w_x}{\partial x} + \left[\lambda\frac{\partial u_x}{\partial x} + (\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]\frac{\partial w_z}{\partial z} + \mu\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]\left(\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z}\right)\,\mathrm{dxdz} \\ &\quad + \int_\Omega \left(\mu\frac{\partial u_y}{\partial x}\frac{\partial w_y}{\partial x} + \mu\frac{\partial u_y}{\partial z}\frac{\partial w_y}{\partial z}\right)\,\mathrm{dxdz} \end{aligned}$$

As the relation (2) must hold for any test function $\boldsymbol{w} = (w_x, w_y, w_z)$ it must hold for test functions of the form $\boldsymbol{w} = (w_x, 0, w_z)$:

$$\underbrace{\int_\Omega \rho\left(w_x\ddot{u}_x + w_z\ddot{u}_z\right)\,\mathrm{dxdz} = -\int_\Omega \left[(\lambda+2\mu)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_z}{\partial z}\right]\frac{\partial w_x}{\partial x} + \left[\lambda\frac{\partial u_x}{\partial x} + (\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]\frac{\partial w_z}{\partial z} + \mu\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]\left(\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z}\right)\,\mathrm{dxdz} + \int_\Omega \left(w_x f_x + w_z f_z\right)\,\mathrm{dxdz}}_{\text{2D P-SV equation}}$$

And for test functions of the form $\boldsymbol{w} = (0, w_y, 0)$:

$$\underbrace{\int_\Omega \rho w_y\ddot{u}_y\,\mathrm{dxdz} = -\int_\Omega \left(\mu\frac{\partial u_y}{\partial x}\frac{\partial w_y}{\partial x} + \mu\frac{\partial u_y}{\partial z}\frac{\partial w_y}{\partial z}\right)\,\mathrm{dxdz} + \int_\Omega w_y f_y\,\mathrm{dxdz}}_{\text{2D SH equation}}$$

These two equations are totally independent. On the following we will focus on the first one i.e the 2D P-SV equation. Consequently we study displacements of the form $\boldsymbol{u} = u_x(x,z,t)\boldsymbol{i} + u_z(x,z,t)\boldsymbol{k} \equiv (u_x, u_z)$, sources $\boldsymbol{f} = f_x(x,z,t)\boldsymbol{i} + f_z(x,z,t)\boldsymbol{k} \equiv (f_x, f_z)$ and test functions $\boldsymbol{w}(x,z) = w_x(x,z)\boldsymbol{i} + w_z(x,z)\boldsymbol{k} \equiv (w_x, w_z)$. Note : we could have obtained this result from the strong formulation as well (maybe it is clearer?). The 2D P-SV momentum equation (1) reads :

$$\int_\Omega \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^2\boldsymbol{x} = -\int_\Omega \sum_{i,j=1}^{2} T_{ij}\boldsymbol{\nabla w}_{ji}\,\mathrm{d}^2\boldsymbol{x} + \int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^2\boldsymbol{x}$$

Mapping

We subdivide the model volume Ω into a number of non-overlapping hexahedral elements Ω_e , $e = 1,\ldots,n_e$ such that $\Omega = \bigcup_e \Omega_e$. As the result of this subdivision, the artificial boundary Γ would be similarly represented by a number of 1D elements Γ_b, $b = 1,\ldots,n_b$ such that $\Gamma = \bigcup_b \Gamma_b$ but we do not consider it for now.

- **Surface elements (for the moment we just have surface elements)**

Points $\boldsymbol{x} = (x,z)$ within each hexahedral element Ω_e may be uniquely related to points $\boldsymbol{\xi} = (\xi,\eta)$, $-1 \le \xi,\eta \le 1$ in a reference square Λ based upon the invertible mapping

$$\boldsymbol{x}(\boldsymbol{\xi}) = \sum_{a=1}^{n_a} \boldsymbol{x}_a N_a(\boldsymbol{\xi})$$

The n_a anchors nodes $\boldsymbol{x}_a = \boldsymbol{x}(\xi_a,\eta_a)$ and shape functions $N_a(\boldsymbol{\xi})$ define the geometry of an element Ω_e (here we should add more detail for a paper). A surface element dxdz within a given element Ω_e is related to a surface element in the reference square by the relation :

$$\mathrm{dxdz} = |\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta$$

Where $\mathcal{J}_e$ is the surface Jacobian. For $(x,z)\in\Omega_e$ it expresses :

$$\mathcal{J}_e = \left|\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta} - \frac{\partial x}{\partial\eta}\frac{\partial z}{\partial\xi}\right|$$

With of course $\dfrac{\partial x}{\partial\xi} = \displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\xi}x_a$, $\dfrac{\partial z}{\partial\xi} = \displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\xi}z_a$, $\dfrac{\partial x}{\partial\eta} = \displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\eta}x_a$ and $\dfrac{\partial z}{\partial\eta} = \displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\eta}z_a$. Moreover for any function $(x,z)\in\Omega_e \mapsto f(x,z)$ we have the three identities :

$$\begin{pmatrix}\dfrac{\partial f}{\partial\xi}\\ \dfrac{\partial f}{\partial\eta}\end{pmatrix} = \underbrace{\begin{pmatrix}\dfrac{\partial x}{\partial\xi} & \dfrac{\partial z}{\partial\xi}\\ \dfrac{\partial x}{\partial\eta} & \dfrac{\partial z}{\partial\eta}\end{pmatrix}}_{J_e} \times \begin{pmatrix}\dfrac{\partial f}{\partial x}\\ \dfrac{\partial f}{\partial z}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial\xi} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial\xi}\\ \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial\eta} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial\eta}\end{pmatrix},\ \begin{pmatrix}\dfrac{\partial f}{\partial x}\\ \dfrac{\partial f}{\partial z}\end{pmatrix} = \underbrace{\begin{pmatrix}\dfrac{\partial\xi}{\partial x} & \dfrac{\partial\eta}{\partial x}\\ \dfrac{\partial\xi}{\partial z} & \dfrac{\partial\eta}{\partial z}\end{pmatrix}}_{J_e^{-1}} \times \begin{pmatrix}\dfrac{\partial f}{\partial\xi}\\ \dfrac{\partial f}{\partial\eta}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial f}{\partial\xi}\dfrac{\partial\xi}{\partial x} + \dfrac{\partial f}{\partial\eta}\dfrac{\partial\eta}{\partial x}\\ \dfrac{\partial f}{\partial\xi}\dfrac{\partial\xi}{\partial z} + \dfrac{\partial f}{\partial\eta}\dfrac{\partial\eta}{\partial z}\end{pmatrix}$$

and $$\begin{pmatrix}\dfrac{\partial f}{\partial x}\\ \dfrac{\partial f}{\partial z}\end{pmatrix} = \underbrace{\frac{1}{\mathcal{J}_e}\begin{pmatrix}\dfrac{\partial z}{\partial\eta} & -\dfrac{\partial z}{\partial\xi}\\ -\dfrac{\partial x}{\partial\eta} & \dfrac{\partial x}{\partial\xi}\end{pmatrix}}_{J_e^{-1}} \times \begin{pmatrix}\dfrac{\partial f}{\partial\xi}\\ \dfrac{\partial f}{\partial\eta}\end{pmatrix} = \frac{1}{\mathcal{J}_e}\begin{pmatrix}\dfrac{\partial f}{\partial\xi}\dfrac{\partial z}{\partial\eta} - \dfrac{\partial f}{\partial\eta}\dfrac{\partial z}{\partial\xi}\\ -\dfrac{\partial f}{\partial\xi}\dfrac{\partial x}{\partial\eta} + \dfrac{\partial f}{\partial\eta}\dfrac{\partial x}{\partial\xi}\end{pmatrix}$$

That supplies :
$$\begin{aligned} \frac{\partial x}{\partial\xi} &= \mathcal{J}_e\frac{\partial\eta}{\partial z} & \frac{\partial z}{\partial\xi} &= -\mathcal{J}_e\frac{\partial\eta}{\partial x} \\ \frac{\partial x}{\partial\eta} &= -\mathcal{J}_e\frac{\partial\xi}{\partial z} & \frac{\partial z}{\partial\eta} &= \mathcal{J}_e\frac{\partial\xi}{\partial x} \end{aligned}$$

Note : Something would have to be said about the Jacobian (it must not be singular)

We can then write :

$$
\begin{aligned}
\boldsymbol{\nabla w}:\boldsymbol{T} &= T_{xx}\frac{\partial w_x}{\partial x}+T_{zz}\frac{\partial w_z}{\partial z}+T_{xz}\left(\frac{\partial w_z}{\partial x}+\frac{\partial w_x}{\partial z}\right)\\
&= T_{xx}\left[\frac{\partial w_x}{\partial \xi}\partial_x\xi+\frac{\partial w_x}{\partial \eta}\partial_x\eta\right]+T_{zz}\left[\frac{\partial w_z}{\partial \xi}\partial_z\xi+\frac{\partial w_z}{\partial \eta}\partial_z\eta\right]+T_{xz}\left[\frac{\partial w_x}{\partial \xi}\partial_z\xi+\frac{\partial w_x}{\partial \eta}\partial_z\eta+\frac{\partial w_z}{\partial \xi}\partial_x\xi+\frac{\partial w_z}{\partial \eta}\partial_x\eta\right]\\
&= \frac{\partial w_x}{\partial \xi}\underbrace{[T_{xx}\partial_x\xi+T_{xz}\partial_z\xi]}_{F_{11}}+\frac{\partial w_z}{\partial \xi}\underbrace{[T_{zx}\partial_x\xi+T_{zz}\partial_z\xi]}_{F_{21}}+\frac{\partial w_x}{\partial \eta}\underbrace{[T_{xx}\partial_x\eta+T_{xz}\partial_z\eta]}_{F_{12}}+\frac{\partial w_z}{\partial \eta}\underbrace{[T_{xz}\partial_x\eta+T_{zz}\partial_z\eta]}_{F_{22}}\\
&= \sum_{i,k=1}^{2}\left(\sum_{j=1}^{2}T_{ij}\partial_j\xi_k\right)\frac{\partial w_i}{\partial \xi_k}\\
&\equiv \sum_{i,k=1}^{2}F_{ik}\frac{\partial w_i}{\partial \xi_k}
\end{aligned}
$$

Where we have posed : $\xi_1=\xi$ and $\xi_2=\eta$. Note : Je n'avais pas fait ca pour pas qu'il y ait de confusion avec les points GLL.
After splitting the 2D P-SV wave equation becomes :

$$
\forall e=1\ldots n_e \qquad \int_{\Omega^e}\boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^2\boldsymbol{x}=-\int_{\Omega^e}\sum_{i,j=1}^{2}T_{ij}\boldsymbol{\nabla w}_{ji}\,\mathrm{d}^2\boldsymbol{x}+\int_{\Omega^e}\boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^2\boldsymbol{x}
$$

Note : something has to be said about the test functions before being able to split the equation.
Then we make the substitution :

$$
\forall e=1\ldots n_e \qquad \int_{\Lambda}\boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta=-\int_{\Lambda}\sum_{i,j=1}^{2}T_{ij}\boldsymbol{\nabla w}_{ji}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta+\int_{\Lambda}\boldsymbol{w}\cdot\boldsymbol{f}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta
$$

Representation of functions on the elements (GLL interpolation)

- **Surface elements (for the moment we just have surface elements)**

We interpolate the functions with Lagrange polynomials of degree N with GLL points $(\xi_\alpha,\eta_\beta)_{\alpha,\beta=0\ldots N}$ as collocation points (here we should add more detail for a paper). For any function $f:(x,z)\mapsto f(x,z)$ on Ω_e:

$$
\forall(\xi,\eta)\in\Lambda \quad f(\boldsymbol{x}(\xi,\eta))\approx\sum_{\alpha=0}^{N}\sum_{\beta=0}^{N}f(\xi_\alpha,\eta_\beta)\ell_\alpha(\xi)\ell_\beta(\eta)=\sum_{\alpha,\beta=0}^{N}f^{\alpha\beta}\ell_\alpha(\xi)\ell_\beta(\eta)
$$

and :

$$
\forall(\xi,\eta)\in\Lambda
$$

$$
\begin{aligned}
\frac{\partial f}{\partial \xi}(\boldsymbol{x}(\xi,\eta)) &= \sum_{\alpha,\beta=0}^{N}f^{\alpha\beta}\frac{\partial \ell_\alpha(\xi)}{\partial \xi}\ell_\beta(\eta) &&= \sum_{\alpha,\beta=0}^{N}f^{\alpha\beta}\ell'_\alpha(\xi)\ell_\beta(\eta)\\
\frac{\partial f}{\partial \eta}(\boldsymbol{x}(\xi,\eta)) &= \sum_{\alpha,\beta=0}^{N}g^{\alpha\beta}\ell_\alpha(\xi)\frac{\partial \ell_\beta(\eta)}{\partial \eta} &&= \sum_{\alpha,\beta=0}^{N}f^{\alpha\beta}\ell_\alpha(\xi)\ell'_\beta(\eta)
\end{aligned}
$$

The ℓ_i are the Lagrange polynomials defined on the colocation points of the Gauss-Lobatto-Legendre quadrature. The ξ_i,η_i are the GLL points. It is of paramount importance to note that $\ell_i(\xi_j)=\delta_{ij}$.
We evaluate the integrals with the quadrature :

$$
\int_{\Lambda}f(\boldsymbol{x}(\xi,\eta))\,\mathrm{d}\xi\mathrm{d}\eta\approx\sum_{i,j=0}^{N}\omega_i\omega_j f^{ij}
$$

Derivation of the algebraic system

We begin by calculating the displacement gradients on the GLL points $(\xi_\sigma,\eta_\nu)_{\sigma,\nu=0\ldots N}$ of element Ω_e:

$$
\begin{aligned}
\partial_i u_j(\boldsymbol{x}(\xi_\sigma,\eta_\nu),t) &= \frac{\partial u_j}{\partial \xi}\partial_i\xi+\frac{\partial u_j}{\partial \eta}\partial_i\eta\\
&= \sum_{\alpha,\beta=0}^{N}u_j^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\ell_\beta(\eta_\nu)\partial_i\xi+\sum_{\alpha,\beta=0}^{N}u_j^{\alpha\beta}\ell_\alpha(\xi_\sigma)\ell'_\beta(\eta_\nu)\partial_i\eta\\
&= \sum_{\alpha=0}^{N}u_j^{\alpha\nu}\ell'_\alpha(\xi_\sigma)\partial_i\xi+\sum_{\beta=0}^{N}u_j^{\sigma\beta}\ell'_\beta(\eta_\nu)\partial_i\eta\\
&= \left[\sum_{\alpha=0}^{N}u_j^{\alpha\nu}\ell'_\alpha(\xi_\sigma)\right]\partial_i\xi+\left[\sum_{\alpha=0}^{N}u_j^{\sigma\alpha}\ell'_\alpha(\eta_\nu)\right]\partial_i\eta
\end{aligned}
$$

We need the four terms $\partial_x\xi$,$\partial_z\xi$,$\partial_x\eta$,$\partial_z\eta$. Then we calculate the four elements of the stress tensor on these GLL points :

$$
\begin{aligned}
T_{xx} &= (\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_z}{\partial z}\\
T_{zz} &= \lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z}\\
T_{xz} &= \mu\left[\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right]\\
T_{zx} &= T_{xz}
\end{aligned}
$$

We calculate the four matrix elements on the GLL points $F_{ik}=\sum_{j=1}^{2}T_{ij}\partial_j\xi_k$. With these notations the elemental stiffness integral reads :

$$
\int_{\Lambda}\sum_{i,j=1}^{2}T_{ij}\boldsymbol{\nabla w}_{ji}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta \quad = \quad \int_{\Lambda}\sum_{i,k=1}^{2}F_{ik}\frac{\partial w_i}{\partial \xi_k}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta
$$

The GLL interpolation tells :

$$\begin{aligned}
\frac{\partial w_i}{\partial \xi_1}(\xi_\sigma,\eta_\nu) &= \frac{\partial w_i}{\partial \xi}(\xi_\sigma,\eta_\nu &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell_\alpha(\xi_\sigma)\ell'_\beta(\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu) \\
\frac{\partial w_i}{\partial \xi_2}(\xi_\sigma,\eta_\nu) &= \frac{\partial w_i}{\partial \eta}(\xi_\sigma,\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\ell_\beta(\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu}
\end{aligned}$$

We use then the quadrature rule to calculate the elemental stiffness integral :

$$\begin{aligned}
\int_\Lambda \sum_{i,j=1}^{2} T_{ij}\boldsymbol{\nabla}\boldsymbol{w}_{ji}\ \ |\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta &= \int_\Lambda \sum_{i,k=1}^{2} F_{ik}\frac{\partial w_i}{\partial \xi_k}\ \ |\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta \\
&= \sum_{\sigma,\nu=0}^{N}\omega_\sigma\omega_\nu\,|\mathcal{J}_e^{\sigma\nu}|\sum_{i=1}^{2}\left(F_{i1}^{\sigma\nu}\frac{\partial w_i}{\partial \xi}(\xi_\sigma,\eta_\nu)+F_{i2}^{\sigma\nu}\frac{\partial w_i}{\partial \eta}(\xi_\sigma,\eta_\nu)\right) \\
&= \sum_{\sigma,\nu=0}^{N}\omega_\sigma\omega_\nu\,|\mathcal{J}_e^{\sigma\nu}|\sum_{i=1}^{2}\left(F_{i1}^{\sigma\nu}\sum_{\alpha,\beta=0}^{N}w_i^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu}+F_{i2}^{\sigma\nu}\sum_{\alpha,\beta=0}^{N}w_i^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)\right) \\
&= \sum_{i=1}^{2}\sum_{\alpha,\beta=0}^{N}w_i^{\alpha\beta}\left(\omega_\beta\sum_{\sigma=0}^{N}\omega_\sigma\left|\mathcal{J}_e^{\sigma\beta}\right|F_{i1}^{\sigma\beta}\ell'_\alpha(\xi_\sigma)+\omega_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\alpha\nu}\right|F_{i2}^{\alpha\nu}\ell'_\beta(\eta_\nu)\right) \\
&= \sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}B_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}B_z^{\alpha\beta}
\end{aligned}$$

Note : the sum $\sum_{\alpha,\beta=0}^{N}$ has been forgotten in (Komatitsch & Tromp 1999).

We follow the same reasoning for the elemental mass integral :

$$\begin{aligned}
\int_\Lambda \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\ |\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta &= \sum_{\alpha,\beta=0}^{N}\rho^{\alpha\beta}\left(w_x^{\alpha\beta}\ddot{u}_x^{\alpha\beta}+w_z^{\alpha\beta}\ddot{u}_z^{\alpha\beta}\right)\left|\mathcal{J}_e^{\alpha\beta}\right|\omega_\alpha\omega_\beta \\
&= \sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}\omega_\alpha\omega_\beta\rho^{\alpha\beta}\left|\mathcal{J}_e^{\alpha\beta}\right|\ddot{u}_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}\omega_\alpha\omega_\beta\rho^{\alpha\beta}\left|\mathcal{J}_e^{\alpha\beta}\right|\ddot{u}_z^{\alpha\beta} \\
&= \sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}A_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}A_z^{\alpha\beta}
\end{aligned}$$

Moreover as $\boldsymbol{f}=(f_x,f_z)=(\delta(x-x_s)f_x(t),\delta(z-z_s)f_z(t))$ the elemental source integral can be written :

$$\begin{aligned}
\int_\Lambda \boldsymbol{w}\cdot\boldsymbol{f}\ |\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta &= (w_x(\xi_s,\eta_s)f_x(t)+w_z(\xi_s,\eta_s)f_z(t))\,|\mathcal{J}_e|\,(\xi_s,\eta_s) \\
&= \sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}\delta_{\alpha s}\delta_{s\beta}f_x(t)\left|\mathcal{J}_e^{\alpha\beta}\right|+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}\delta_{\alpha s}\delta_{s\beta}f_z(t)\left|\mathcal{J}_e^{\alpha\beta}\right| \\
&= \sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}C_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}C_z^{\alpha\beta}
\end{aligned}$$

As the relation : $\sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}A_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}A_z^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}B_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}B_z^{\alpha\beta}=\sum_{\alpha,\beta=0}^{N}w_x^{\alpha\beta}C_x^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N}w_z^{\alpha\beta}C_z^{\alpha\beta}$ must hold for any test function $\boldsymbol{w}=(w_x,w_z)$, we can conclude :

$$\forall\alpha,\beta=0\ldots N\quad\forall e=0\ldots n_e-1\quad\left\{\begin{array}{lcll} A_x^{\alpha\beta}+B_x^{\alpha\beta} &=& C_x^{\alpha\beta} & \quad(\boldsymbol{w}=(\delta_{\alpha\beta},0)) \\ A_z^{\alpha\beta}+B_z^{\alpha\beta} &=& C_z^{\alpha\beta} & \quad(\boldsymbol{w}=(0,\delta_{\alpha\beta}))\end{array}\right.$$

We have then :

$$\begin{aligned}
&\forall\alpha,\beta=0\ldots N\quad\forall e=0\ldots n_e-1\quad\forall i\in\{x,z\} \\
\omega_\alpha\omega_\beta\rho^{\alpha\beta}\left|\mathcal{J}_e^{\alpha\beta}\right|\ddot{u}_i^{\alpha\beta}+\sum_{j\in\{x,z\}}\left[\sum_{I,J=0}^{N}K_{ij}^{\alpha\beta IJ}u_j^{IJ}\right] &= \delta_{\alpha s}\delta_{s\beta}f_i(t)\left|\mathcal{J}_e^{\alpha\beta}\right| \\
M^{\alpha\beta}\ddot{u}_i^{\alpha\beta}+\sum_{j\in\{x,z\}}\left[\sum_{I,J=0}^{N}K_{ij}^{\alpha\beta IJ}u_j^{IJ}\right] &= F_i^{\alpha\beta}
\end{aligned}$$

Or in tensorial form for each element e :

$$\left\{\begin{array}{lcl} \boldsymbol{M}\odot\ddot{\boldsymbol{u}}_x+\boldsymbol{K}_{xx}\cdot\boldsymbol{u}_x+\boldsymbol{K}_{xz}\cdot\boldsymbol{u}_z &=& \boldsymbol{F}_x \\ \boldsymbol{M}\odot\ddot{\boldsymbol{u}}_z+\boldsymbol{K}_{zx}\cdot\boldsymbol{u}_x+\boldsymbol{K}_{zz}\cdot\boldsymbol{u}_z &=& \boldsymbol{F}_z\end{array}\right.$$

(We have defined $(\boldsymbol{A}\odot\boldsymbol{B})_{ij}=A_{ij}B_{ij}$ and $(\boldsymbol{A}\cdot\boldsymbol{B})_{ij}=\sum_{k,l=0}^{N}A_{ijkl}B_{kl}$). Hence, after assembly :

$$\left\{\begin{array}{lcl} \ddot{\boldsymbol{u}}_x &=& \frac{1}{\boldsymbol{M}^g}\left(\boldsymbol{F}_x^g(t)-\boldsymbol{F}_{x,int}^g(t)\right) \\ \ddot{\boldsymbol{u}}_z &=& \frac{1}{\boldsymbol{M}^g}\left(\boldsymbol{F}_z^g(t)-\boldsymbol{F}_{z,int}^g(t)\right)\end{array}\right.$$

2.5D spectral-elements formulation (curvilinear cylindrical coordinates)

Introduction : assumptions, conventions, weak form

We shall start from a 3D formulation. I recall the weak form of the 3D momentum equation :

$$\underbrace{\int_\Omega \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^3\boldsymbol{x}}_{\text{mass integral}} = -\underbrace{\int_\Omega \boldsymbol{\nabla w}:\boldsymbol{T}\,\mathrm{d}^3\boldsymbol{x}}_{\text{stiffness integral}} + \underbrace{\int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^3\boldsymbol{x}}_{\text{source integral}} + \underbrace{\int_\Gamma \boldsymbol{w}\cdot\boldsymbol{T}\cdot\boldsymbol{n}\,\mathrm{d}^2\boldsymbol{x}}_{\text{0 for the moment}}$$

Now we will write the displacement vector : $(r,\theta,z)\mapsto \boldsymbol{u} = u_r(r,\theta,z,t)\boldsymbol{r} + u_\theta(r,\theta,z,t)\boldsymbol{\theta} + u_z(r,\theta,z,t)\boldsymbol{k}$. The test functions : $(r,\theta,z)\mapsto \boldsymbol{w}(r,\theta,z) = w_r(r,\theta,z,t)\boldsymbol{r} + w_\theta(r,\theta,z,t)\boldsymbol{\theta} + w_z(r,\theta,z,t)\boldsymbol{k}$ now belongs to the subspace of Sobolev space $H_1^1(\Omega)$ of the functions that cancel on the axis. (Bernardi and p.12 of Alexandre Fournier phd tesis, we would have to introduce these spaces). The wave equation reads :

$$\begin{aligned} &\int_\Omega \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^3\boldsymbol{x} &=& -\int_\Omega \boldsymbol{\nabla w}:\boldsymbol{T}\,\mathrm{d}^3\boldsymbol{x} + \int_\Omega \boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^3\boldsymbol{x} \\ \iff &\int_\Omega \rho\left(w_r\ddot{u}_r + w_\theta\ddot{u}_\theta + w_z\ddot{u}_z\right)2\pi r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z &=& -\int_\Omega \boldsymbol{\nabla w}:\boldsymbol{T}\,2\pi r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z + \int_\Omega \left(w_r f_r + w_\theta f_\theta + w_z f_z\right)2\pi r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \end{aligned} \tag{3}$$

We still consider an elastic isotropic media, the stress tensor still expresses :

$$\begin{Bmatrix} T_{rr}\\ T_{\theta\theta}\\ T_{zz}\\ T_{r\theta}\\ T_{\theta z}\\ T_{zr}\end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \epsilon_{r\theta}\\ \epsilon_{\theta z}\\ \epsilon_{zr}\end{Bmatrix}$$

Nevertheless now the strain tensor expresses :

$$\begin{Bmatrix} \epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \epsilon_{r\theta}\\ \epsilon_{\theta z}\\ \epsilon_{zr}\end{Bmatrix} = \begin{bmatrix} \partial_r & 0 & 0\\ \frac{1}{r} & \frac{1}{r}\partial_\theta & 0\\ 0 & 0 & \partial_z\\ \frac{1}{2r}\partial_\theta & \frac{1}{2}\left(\partial_r - \frac{\cdot}{r}\right) & 0\\ 0 & \frac{1}{2}\partial_z & \frac{1}{2r}\partial_\theta\\ \frac{1}{2}\partial_z & 0 & \frac{1}{2}\partial_r \end{bmatrix} \begin{Bmatrix} u_r\\ u_\theta\\ u_z\end{Bmatrix}$$

At this point we will choose the 2.5D convention : we suppose an axisymmetric geometry and that the important loads are not along $\boldsymbol{\theta}$ and do not change with θ:

$$\frac{\partial}{\partial\theta} = 0$$

That leads to :

$$\begin{Bmatrix} \epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \epsilon_{r\theta}\\ \epsilon_{\theta z}\\ \epsilon_{zr}\end{Bmatrix} = \begin{bmatrix} \partial_r & 0 & 0\\ \frac{1}{r} & 0 & 0\\ 0 & 0 & \partial_z\\ 0 & \frac{1}{2}\left(\partial_r - \frac{\cdot}{r}\right) & 0\\ 0 & \frac{1}{2}\partial_z & 0\\ \frac{1}{2}\partial_z & 0 & \frac{1}{2}\partial_r \end{bmatrix} \begin{Bmatrix} u_r\\ u_\theta\\ u_z\end{Bmatrix}$$

We have also :

$$\boldsymbol{\nabla w} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{1}{r}\frac{\partial w_r}{\partial\theta} - \frac{w_\theta}{r} & \frac{\partial w_r}{\partial z}\\ \frac{\partial w_\theta}{\partial r} & \frac{1}{r}\frac{\partial w_\theta}{\partial\theta} + \frac{w_r}{r} & \frac{\partial w_\theta}{\partial z}\\ \frac{\partial w_z}{\partial r} & \frac{1}{r}\frac{\partial w_z}{\partial\theta} & \frac{\partial w_z}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & -\frac{w_\theta}{r} & \frac{\partial w_r}{\partial z}\\ \frac{\partial w_\theta}{\partial r} & \frac{w_r}{r} & \frac{\partial w_\theta}{\partial z}\\ \frac{\partial w_z}{\partial r} & 0 & \frac{\partial w_z}{\partial z} \end{pmatrix}$$

For now we will adopt the notations : $(r,z)\leftrightarrow(1,2)$. For example : $A_{12}\equiv A_{rz}$

$$\begin{aligned} \boldsymbol{\nabla w}:\boldsymbol{T} &= T_{rr}\boldsymbol{\nabla w}_{rr} + T_{\theta\theta}\boldsymbol{\nabla w}_{\theta\theta} + T_{zz}\boldsymbol{\nabla w}_{zz} + T_{r\theta}\left(\boldsymbol{\nabla w}_{\theta r} + \boldsymbol{\nabla w}_{r\theta}\right) + T_{\theta z}\left(\boldsymbol{\nabla w}_{z\theta} + \boldsymbol{\nabla w}_{\theta z}\right) + T_{zr}\left(\boldsymbol{\nabla w}_{rz} + \boldsymbol{\nabla w}_{zr}\right)\\ &= T_{rr}\frac{\partial w_r}{\partial r} + T_{\theta\theta}\frac{w_r}{r} + T_{zz}\frac{\partial w_z}{\partial z} + T_{r\theta}\left(\frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r}\right) + T_{\theta z}\frac{\partial w_\theta}{\partial z} + T_{zr}\left(\frac{\partial w_r}{\partial z} + \frac{\partial w_z}{\partial r}\right)\\ &= \left[(\lambda+2\mu)\epsilon_{rr} + \lambda\epsilon_{\theta\theta} + \lambda\epsilon_{zz}\right]\frac{\partial w_r}{\partial r} + \left[\lambda\epsilon_{rr} + (\lambda+2\mu)\epsilon_{\theta\theta} + \lambda\epsilon_{zz}\right]\frac{w_r}{r} + \left[\lambda\epsilon_{rr} + \lambda\epsilon_{\theta\theta} + (\lambda+2\mu)\epsilon_{zz}\right]\frac{\partial w_z}{\partial z}\\ &\quad +2\mu\epsilon_{r\theta}\left(\frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r}\right) + 2\mu\epsilon_{\theta z}\frac{\partial w_\theta}{\partial z} + 2\mu\epsilon_{zr}\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z}\right)\\ &= \left[(\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}\right]\frac{\partial w_r}{\partial r} + \left[\lambda\frac{\partial u_r}{\partial r} + (\lambda+2\mu)\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}\right]\frac{w_r}{r}\\ &\quad + \left[\lambda\frac{\partial u_r}{\partial r} + \lambda\frac{1}{r}u_r + (\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]\frac{\partial w_z}{\partial z} + \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\left(\frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r}\right) + \mu\frac{\partial u_\theta}{\partial z}\frac{\partial w_\theta}{\partial z} + \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z}\right) \end{aligned}$$

As the relation (3) must hold for any test function $\boldsymbol{w} = (w_r, w_\theta, w_z)$ it must hold for test functions of the form $\boldsymbol{w} = (w_r, 0, w_z)$:

$$\underbrace{\begin{aligned} &\int_\Omega \rho\left(w_r\ddot{u}_r + w_z\ddot{u}_z\right)r\mathrm{d}r\mathrm{d}z = \\ &-\int_\Omega \left[(\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}\right]\frac{\partial w_r}{\partial r} + \left[\lambda\frac{\partial u_r}{\partial r} + (\lambda+2\mu)\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}\right]\frac{w_r}{r} + \left[\lambda\frac{\partial u_r}{\partial r} + \lambda\frac{1}{r}u_r + (\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]\frac{\partial w_z}{\partial z} + \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z}\right)r\mathrm{d}r\mathrm{d}z \\ &+\int_\Omega \left(w_r f_r + w_z f_z\right)r\mathrm{d}r\mathrm{d}z \end{aligned}}_{\text{2.5D P-SV equation}}$$

$$\begin{aligned} \int_\Omega \rho\left(w_r\ddot{u}_r + w_z\ddot{u}_z\right)r\mathrm{d}r\mathrm{d}z &= -\int_\Omega \left[(\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}\right]\frac{\partial w_r}{\partial r}r\mathrm{d}r\mathrm{d}z\\ &\quad -\int_\Omega \left[\lambda\frac{\partial u_r}{\partial r} + \lambda\frac{1}{r}u_r + (\lambda+2\mu)\frac{\partial u_z}{\partial z}\right]\frac{\partial w_z}{\partial z}r\mathrm{d}r\mathrm{d}z\\ &\quad -\int_\Omega \left[\lambda\frac{\partial u_r}{\partial r} + (\lambda+2\mu)\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}\right]\frac{w_r}{r}r\mathrm{d}r\mathrm{d}z\\ &\quad -\int_\Omega \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]\left(\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z}\right)r\mathrm{d}r\mathrm{d}z\\ &\quad +\int_\Omega \left(w_r f_r + w_z f_z\right)r\mathrm{d}r\mathrm{d}z\\ &\quad +\int_\Gamma \boldsymbol{w}\cdot\boldsymbol{T}\cdot\boldsymbol{n}\,\mathrm{d}x \end{aligned}$$

And for test functions of the form $\boldsymbol{w} = (0, w_y, 0)$:

$$\underbrace{\int_\Omega \rho w_\theta \ddot{u}_\theta \,\mathrm{rdrdz} = -\int_\Omega \mu \left(\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \left(\frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r} \right) + \frac{\partial u_\theta}{\partial z}\frac{\partial w_\theta}{\partial z} \right) r\mathrm{drdz} + \int_\Omega w_\theta f_\theta \,\mathrm{rdrdz}}_{\text{2.5D P-SH equation}}$$

These two equations are totally independent. On the following we will focus on the first one i.e the 2.5D P-SV equation. Consequently we study displacements of the form $\boldsymbol{u} = u_r(r,z,t)\boldsymbol{i} + u_z(r,z,t)\boldsymbol{k} \equiv (u_r, u_z)$, sources $\boldsymbol{f} = f_r(r,z,t)\boldsymbol{i} + f_z(r,z,t)\boldsymbol{k} \equiv (f_r, f_z)$ and test functions $\boldsymbol{w}(r,z) = w_r(r,z)\boldsymbol{i} + w_z(r,z)\boldsymbol{k} \equiv (w_r, w_z)$. Note : we could have obtained this result from the strong formulation as well (maybe it is clearer?). The 2.5D P-SV momentum equation (1) reads :

$$\int_\Omega \boldsymbol{w} \cdot \rho \ddot{\mathbf{u}} \,\mathrm{rdrdz} = -\int_\Omega \sum_{i,j=1}^{2} T_{ij} \boldsymbol{\nabla} \boldsymbol{w}_{ji} \,\mathrm{rdrdz} + \int_\Omega \boldsymbol{w} \cdot \boldsymbol{f} \,\mathrm{rdrdz}$$

Mapping

We subdivide the model volume Ω into a number of non-overlapping hexahedral elements Ω_e , $e = 1, \ldots, n_h \times n_v$, such that $\Omega = \bigcup_e \Omega_e$. As the result of this subdivision, the artificial boundary Γ would be similarly represented by a number of 1D elements Γ_b, $b = 1, \ldots, n_b$ such that $\Gamma = \bigcup_b \Gamma_b$ but we don't consider it for now. The n_v 2D elements along the axis need to be distinguished.

- **Surface elements (for the moment we just have surface elements)**

Points $\boldsymbol{r} = (r, z)$ within each hexahedral non-axial element Ω_e may be uniquely related to points $\boldsymbol{\xi} = (\xi, \eta)$, $-1 \le \xi, \eta \le 1$ in a reference square Λ based upon the invertible mapping

$$\boldsymbol{r}(\boldsymbol{\xi}) = \sum_{a=1}^{n_a} \boldsymbol{r}_a N_a(\boldsymbol{\xi})$$

The $a = 1, \ldots, n_a$ anchors $\boldsymbol{r}_a = \boldsymbol{r}_e(\xi_a, \eta_a)$ and shape functions $N_a(\boldsymbol{\xi})$ define the geometry of an element Ω_e. A surface element $\mathrm{d}^2\boldsymbol{x} = 2\pi r\mathrm{drdz}$ within a given element Ω_e is related to a surface element in the reference square by the relation :

$$2\pi r\mathrm{drdz} = 2\pi r(\xi,\eta)\,|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta$$

Where J_e is the surface Jacobian. For $(r,z) \in \Omega_e$ it expresses :

$$\mathcal{J}_e = \left| \frac{\partial r}{\partial \xi}\frac{\partial z}{\partial \eta} - \frac{\partial r}{\partial \eta}\frac{\partial z}{\partial \xi} \right|$$

With of course $\dfrac{\partial r}{\partial \xi} = \sum_{a=1}^{n_a} \dfrac{\partial N_a}{\partial \xi} r_a$, $\dfrac{\partial z}{\partial \xi} = \sum_{a=1}^{n_a} \dfrac{\partial N_a}{\partial \xi} z_a$, $\dfrac{\partial r}{\partial \eta} = \sum_{a=1}^{n_a} \dfrac{\partial N_a}{\partial \eta} r_a$ and $\dfrac{\partial z}{\partial \eta} = \sum_{a=1}^{n_a} \dfrac{\partial N_a}{\partial \eta} z_a$. Moreover for any function $(r,z) \in \Omega_e \mapsto f(r,z)$ we have the three identities :

$$\begin{pmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial r}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial r}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \end{pmatrix}}_{J_e} \times \begin{pmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial \xi} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial \eta} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial \eta} \end{pmatrix}, \begin{pmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial \xi}{\partial r} & \dfrac{\partial \eta}{\partial r} \\ \dfrac{\partial \xi}{\partial z} & \dfrac{\partial \eta}{\partial z} \end{pmatrix}}_{J_e^{-1}} \times \begin{pmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial r} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial r} \\ \dfrac{\partial f}{\partial \xi}\dfrac{\partial \xi}{\partial z} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial \eta}{\partial z} \end{pmatrix}$$

and
$$\begin{pmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \underbrace{\frac{1}{\mathcal{J}_e} \begin{pmatrix} \dfrac{\partial z}{\partial \eta} & -\dfrac{\partial z}{\partial \xi} \\ -\dfrac{\partial r}{\partial \eta} & \dfrac{\partial r}{\partial \xi} \end{pmatrix}}_{J_e^{-1}} \times \begin{pmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{pmatrix} = \frac{1}{\mathcal{J}_e} \begin{pmatrix} \dfrac{\partial f}{\partial \xi}\dfrac{\partial z}{\partial \eta} - \dfrac{\partial f}{\partial \eta}\dfrac{\partial z}{\partial \xi} \\ -\dfrac{\partial f}{\partial \xi}\dfrac{\partial r}{\partial \eta} + \dfrac{\partial f}{\partial \eta}\dfrac{\partial r}{\partial \xi} \end{pmatrix}$$

That supplies :
$$\begin{array}{llllll} \dfrac{\partial r}{\partial \xi} & = & \mathcal{J}_e \dfrac{\partial \eta}{\partial z} & \qquad \dfrac{\partial z}{\partial \xi} & = & -\mathcal{J}_e \dfrac{\partial \eta}{\partial r} \\ \dfrac{\partial r}{\partial \eta} & = & -\mathcal{J}_e \dfrac{\partial \xi}{\partial z} & \qquad \dfrac{\partial z}{\partial \eta} & = & \mathcal{J}_e \dfrac{\partial \xi}{\partial r} \end{array}$$

Note : Something would have to be said about the Jacobian (it must not be singular)
We can then write :

$$\begin{aligned} \boldsymbol{\nabla} \boldsymbol{w} : \boldsymbol{T} &= T_{rr}\frac{\partial w_r}{\partial r} + T_{zz}\frac{\partial w_z}{\partial z} + T_{zr}\left(\frac{\partial w_r}{\partial z} + \frac{\partial w_z}{\partial r}\right) + T_{\theta\theta}\frac{w_r}{r} \\ &= T_{rr}\left[\frac{\partial w_r}{\partial \xi}\partial_r \xi + \frac{\partial w_r}{\partial \eta}\partial_r \eta\right] + T_{zz}\left[\frac{\partial w_z}{\partial \xi}\partial_z \xi + \frac{\partial w_z}{\partial \eta}\partial_z \eta\right] + T_{rz}\left[\frac{\partial w_r}{\partial \xi}\partial_z \xi + \frac{\partial w_r}{\partial \eta}\partial_z \eta + \frac{\partial w_z}{\partial \xi}\partial_r \xi + \frac{\partial w_z}{\partial \eta}\partial_r \eta\right] + T_{\theta\theta}\frac{w_r}{r} \\ &= \frac{\partial w_r}{\partial \xi}\underbrace{[T_{rr}\partial_r \xi + T_{rz}\partial_z \xi]}_{F_{11}} + \frac{\partial w_z}{\partial \xi}\underbrace{[T_{zr}\partial_r \xi + T_{zz}\partial_z \xi]}_{F_{21}} + \frac{\partial w_r}{\partial \eta}\underbrace{[T_{rr}\partial_r \eta + T_{rz}\partial_z \eta]}_{F_{12}} + \frac{\partial w_z}{\partial \eta}\underbrace{[T_{rz}\partial_r \eta + T_{zz}\partial_z \eta]}_{F_{22}} + T_{\theta\theta}\frac{w_r}{r} \\ &= \sum_{i,k=1}^{2}\left(\sum_{j=1}^{2} T_{ij}\partial_j \xi_k\right)\frac{\partial w_i}{\partial \xi_k} + T_{\theta\theta}\frac{w_r}{r} \\ &\equiv \sum_{i,k=1}^{2} F_{ik}\frac{\partial w_i}{\partial \xi_k} + T_{\theta\theta}\frac{w_r}{r} \end{aligned}$$

$$\left\{ \begin{array}{lll} F_{11} & = & T_{rr}\dfrac{\partial \xi}{\partial r} + T_{rz}\dfrac{\partial \xi}{\partial z} \\ F_{12} & = & T_{rr}\dfrac{\partial \eta}{\partial r} + T_{rz}\dfrac{\partial \eta}{\partial z} \\ F_{21} & = & T_{zr}\dfrac{\partial \xi}{\partial r} + T_{zz}\dfrac{\partial \xi}{\partial z} \\ F_{22} & = & T_{rz}\dfrac{\partial \eta}{\partial r} + T_{zz}\dfrac{\partial \eta}{\partial z} \end{array} \right.$$

Where we have posed : $\xi_1 = \xi$ and $\xi_2 = \eta$.
After splitting the 2.5D P-SV wave equation becomes :

$$\forall e = 1 \dots n_e \qquad \int_{\Omega^e} \boldsymbol{w} \cdot \rho \ddot{\boldsymbol{u}} \, \mathrm{rdrdz} = -\int_{\Omega^e} \sum_{i,j=1}^{2} T_{ij} \boldsymbol{\nabla} \boldsymbol{w}_{ji} \, \mathrm{rdrdz} + \int_{\Omega^e} \boldsymbol{w} \cdot \boldsymbol{f} \, \mathrm{rdrdz}$$

Note : something has to be said about the test functions before being able to split the equation.
Then we make the substitution :

$$\forall e = 1 \dots n_e \qquad \int_{\Lambda} \boldsymbol{w} \cdot \rho \ddot{\boldsymbol{u}} \, |\mathcal{J}_e| \, \mathrm{r d}\xi \mathrm{d}\eta = -\int_{\Lambda} \sum_{i,j=1}^{2} T_{ij} \boldsymbol{\nabla} \boldsymbol{w}_{ji} \, |\mathcal{J}_e| \, \mathrm{r d}\xi \mathrm{d}\eta + \int_{\Lambda} \boldsymbol{w} \cdot \boldsymbol{f} \, |\mathcal{J}_e| \, \mathrm{r d}\xi \mathrm{d}\eta$$

Representation of functions on the elements (GLL interpolation)

- **Non axial surface elements (GLL interpolation)**

We interpolate the functions with Lagrange polynomials of degree N with GLL points $(\xi_\alpha, \eta_\beta)_{\alpha,\beta=0\dots N}$ as collocation points (here we should add more detail for a paper). For any function $f : (r, z) \mapsto f(r, z)$ on Ω_e:

$$\forall (\xi, \eta) \in \Lambda \quad f(\boldsymbol{r}(\xi, \eta)) \approx \sum_{\alpha=0}^{N} \sum_{\beta=0}^{N} f(\xi_\alpha, \eta_\beta) \ell_\alpha(\xi) \ell_\beta(\eta) = \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta} \ell_\alpha(\xi) \ell_\beta(\eta)$$

and :

$$\forall (\xi, \eta) \in \Lambda$$

$$\begin{aligned}
\frac{\partial f}{\partial \xi}(\boldsymbol{r}(\xi, \eta)) &= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta} \frac{\partial \ell_\alpha(\xi)}{\partial \xi} \ell_\beta(\eta) &= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta} \ell'_\alpha(\xi) \ell_\beta(\eta) \\
\frac{\partial f}{\partial \eta}(\boldsymbol{r}(\xi, \eta)) &= \sum_{\alpha,\beta=0}^{N} g^{\alpha\beta} \ell_\alpha(\xi) \frac{\partial \ell_\beta(\eta)}{\partial \eta} &= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta} \ell_\alpha(\xi) \ell'_\beta(\eta)
\end{aligned}$$

The ℓ_i are the Lagrange polynomials defined on the colocation points of the Gauss-Lobatto-Legendre quadrature. The ξ_i, η_i are the GLL points. It is of paramount importance to note that $\ell_i(\xi_j) = \delta_{ij}$.
We evaluate the integrals with the quadrature :

$$\int_{\Lambda} f(\boldsymbol{r}(\xi, \eta)) \, \mathrm{d}\xi \mathrm{d}\eta \approx \sum_{i,j=0}^{N} \omega_i \omega_j f^{ij}$$

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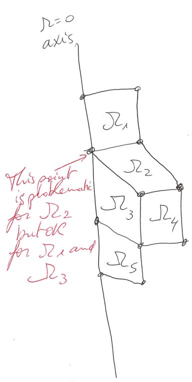
- **Axial surface elements (GLL+GLJ interpolation)**



In the direction η nothing different has to be done. We interpolate the functions with Lagrange polynomials of degree N with GLL points as collocation points. In the direction ξ we use a Gauss-Lobatto-Jacobi quadrature ($\alpha = 0, \beta = 1$). Any function is reconstructed using a set of polynomial defined by :

$$\overline{P}_N(\xi) = \frac{P_N(\xi) + P_{N+1}(\xi)}{1 + \xi}$$

P_n being the Legendre polynomial of degree N.

Here, we define Gauss-Lobatto-Jacobi (GLJ) points $\overline{\xi}_i$ as the zeroes of $(1 - \xi^2) \dfrac{d\overline{P}_N}{d\xi}(\xi)$.

Then we can compute basis functions $\overline{l}_i(\xi)$ and their derivatives (see Nissen-Meyer GJI 2007b, !! One error $\partial_\xi \overline{l}_i(\overline{\xi}_I) = \dfrac{1}{\overline{P}_N(\overline{\xi}_i)(1 - \overline{\xi}_i)}$!!).

For any function $f : (r, z) \mapsto f(r, z)$ on $\overline{\Omega}_e$ we can write :

$$\forall (\xi, \eta) \in \Lambda \quad f(\boldsymbol{r}(\xi, \eta)) = \sum_{\alpha=0}^{N} \sum_{\beta=0}^{N} f(\overline{\xi}_\alpha, \eta_\beta) \overline{\ell}_\alpha(\xi) \ell_\beta(\eta) = \sum_{\alpha,\beta=0}^{N} f^{\overline{\alpha}\beta} \overline{\ell}_\alpha(\xi) \ell_\beta(\eta)$$

and :

$\forall(\xi,\eta) \in \Lambda$

$$\begin{aligned}
\frac{\partial f}{\partial \xi}(\boldsymbol{r}(\xi,\eta)) &= \sum_{\alpha,\beta=0}^{N} f^{\overline{\alpha}\beta}\frac{\partial \overline{\ell}_\alpha(\xi)}{\partial \xi}\ell_\beta(\eta) &= \sum_{\alpha,\beta=0}^{N} f^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\xi)\ell_\beta(\eta) \\
\frac{\partial f}{\partial \eta}(\boldsymbol{r}(\xi,\eta)) &= \sum_{\alpha,\beta=0}^{N} f^{\overline{\alpha}\beta}\overline{\ell}_\alpha(\xi)\frac{\partial \ell_\beta(\eta)}{\partial \eta} &= \sum_{\alpha,\beta=0}^{N} f^{\overline{\alpha}\beta}\overline{\ell}_\alpha(\xi)\ell'_\beta(\eta)
\end{aligned}$$

It is of paramount importance to note that $\overline{\ell}_i(\overline{\xi}_j) = \ell_i(\xi_j) = \delta_{ij}$.

We evaluate the integrals with a Gauss-Lobatto-Jacobi quadrature. The $\overline{\xi}_i$ are the GLJ points and the $\overline{\omega}_j$ are the associated weights :

$$\int_\Lambda f(\boldsymbol{x}(\xi,\eta))\,\mathrm{d}\xi\mathrm{d}\eta \approx \sum_{i,j=0}^{N} \overline{\omega}_i\omega_j \frac{f(\boldsymbol{x}(\overline{\xi}_i,\eta_j))}{\overline{\xi}_i+1} = \sum_{i,j=0}^{N} \overline{\omega}_i\omega_j \frac{f^{\overline{i}j}}{\overline{\xi}_i+1}$$

Derivation of the algebraic system for non-axial elements

We begin by calculating the displacement gradients on the GLL points $(\xi_\sigma,\eta_\nu)_{\sigma,\nu=0\ldots N}$ of element Ω_e:

$$\begin{aligned}
\partial_i u_j(\boldsymbol{r}(\xi_\sigma,\eta_\nu),t) &= \frac{\partial u_j}{\partial \xi}\partial_i\xi + \frac{\partial u_j}{\partial \eta}\partial_i\eta \\
&= \sum_{\alpha,\beta=0}^{N} u_j^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\ell_\beta(\eta_\nu)\partial_i\xi + \sum_{\alpha,\beta=0}^{N} u_j^{\alpha\beta}\ell_\alpha(\xi_\sigma)\ell'_\beta(\eta_\nu)\partial_i\eta \\
&= \sum_{\alpha=0}^{N} u_j^{\alpha\nu}\ell'_\alpha(\xi_\sigma)\partial_i\xi + \sum_{\beta=0}^{N} u_j^{\sigma\beta}\ell'_\beta(\eta_\nu)\partial_i\eta \\
&= \left[\sum_{\alpha=0}^{N} u_j^{\alpha\nu}\ell'_\alpha(\xi_\sigma)\right]\partial_i\xi + \left[\sum_{\alpha=0}^{N} u_j^{\sigma\alpha}\ell'_\alpha(\eta_\nu)\right]\partial_i\eta
\end{aligned}$$

We need the four terms $\partial_r\xi$,$\partial_z\xi$,$\partial_r\eta$,$\partial_z\eta$. Then we calculate the five elements of the stress tensor on these GLL points :

$$\begin{aligned}
T_{rr} &= (\lambda+2\mu)\frac{\partial u_r}{\partial r} + \lambda\frac{\partial u_z}{\partial z} + \lambda\frac{1}{r}u_r \\
T_{zz} &= \lambda\frac{\partial u_r}{\partial r} + (\lambda+2\mu)\frac{\partial u_z}{\partial z} + \lambda\frac{1}{r}u_r \\
T_{rz} &= \mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] \\
T_{zr} &= T_{xz} \\
T_{\theta\theta} &= \lambda\frac{\partial u_r}{\partial r} + (\lambda+2\mu)\frac{1}{r}u_r + \lambda\frac{\partial u_z}{\partial z}
\end{aligned}$$

We calculate the four matrix elements on the GLL points $F_{ik} = \sum_{j=1}^{2} T_{ij}\partial_j\xi_k$. With these notations the elemental stiffness integral reads :

$$\int_\Lambda \sum_{i,j=1}^{2} T_{ij}\boldsymbol{\nabla}\boldsymbol{w}_{ji}\;|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta = \int_\Lambda \left(\sum_{i,k=1}^{2} F_{ik}\frac{\partial w_i}{\partial \xi_k} + T_{\theta\theta}\frac{w_r}{r}\right)|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta$$

The GLL interpolation tells :

$$\begin{aligned}
\frac{\partial w_i}{\partial \xi_1}(\xi_\sigma,\eta_\nu) &= \frac{\partial w_i}{\partial \xi}(\xi_\sigma,\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\ell_\beta(\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu} \\
\frac{\partial w_i}{\partial \xi_2}(\xi_\sigma,\eta_\nu) &= \frac{\partial w_i}{\partial \eta}(\xi_\sigma,\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell_\alpha(\xi_\sigma)\ell'_\beta(\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)
\end{aligned}$$

(and of course : $w_r(\xi_\sigma,\eta_\nu) \equiv w_r^{\sigma\nu} = \sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}\delta_{\alpha\sigma}\delta_{\beta\nu}$)

We use then the quadrature rule to calculate the elemental stiffness integral :

$$\begin{aligned}
\int_\Lambda \sum_{i,j=1}^{2} T_{ij}\boldsymbol{\nabla}\boldsymbol{w}_{ji}\;|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta &= \int_\Lambda \left(\sum_{i,k=1}^{2} F_{ik}\frac{\partial w_i}{\partial \xi_k} + T_{\theta\theta}\frac{w_r}{r}\right)|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta \\
&= \int_\Lambda |\mathcal{J}_e|\left(r\sum_{i,k=1}^{2} F_{ik}\frac{\partial w_i}{\partial \xi_k} + T_{\theta\theta}w_r\right)\mathrm{d}\xi\mathrm{d}\eta \\
&= \sum_{\sigma,\nu=0}^{N} \omega_\sigma\omega_\nu\left|\mathcal{J}_e^{\sigma\nu}\right|\left[r^{\sigma\nu}\sum_{i=1}^{2}\left(F_{i1}^{\sigma\nu}\sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu} + F_{i2}^{\sigma\nu}\sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)\right) + T_{\theta\theta}^{\sigma\nu}\sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}\delta_{\alpha\sigma}\delta_{\beta\nu}\right] \\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\omega_\sigma\left|\mathcal{J}_e^{\sigma\beta}\right| r^{\sigma\beta}F_{11}^{\sigma\beta}\ell'_\alpha(\xi_\sigma) + \omega_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\alpha\nu}\right| r^{\alpha\nu}F_{12}^{\alpha\nu}\ell'_\beta(\eta_\nu) + \omega_\alpha\omega_\beta\left|\mathcal{J}_e^{\alpha\beta}\right|T_{\theta\theta}^{\alpha\beta}\right] \\
&\quad + \sum_{\alpha,\beta=0}^{N} w_z^{\alpha\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\omega_\sigma\left|\mathcal{J}_e^{\sigma\beta}\right| r^{\sigma\beta}F_{21}^{\sigma\beta}\ell'_\alpha(\xi_\sigma) + \omega_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\alpha\nu}\right| r^{\alpha\nu}F_{22}^{\alpha\nu}\ell'_\beta(\eta_\nu)\right] \\
&= \sum_{\alpha,\beta=0}^{N} w_x^{\alpha\beta}B_x^{\alpha\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\alpha\beta}B_z^{\alpha\beta}
\end{aligned}$$

We follow the same reasoning for the mass integral :

$$\begin{aligned}
\int_\Lambda \boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta &= \sum_{\sigma,\beta=0}^{N}\rho^{\alpha\beta}\left(w_r^{\alpha\beta}\ddot{u}_r^{\alpha\beta}+w_z^{\alpha\beta}\ddot{u}_z^{\alpha\beta}\right)r_e^{\alpha\beta}\left|\mathcal{J}_e^{\alpha\beta}\right|\omega_\alpha\omega_\beta\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}\omega_\alpha\omega_\beta\rho^{\alpha\beta}r_e^{\alpha\beta}\left|\mathcal{J}_e^{\alpha\beta}\right|\ddot{u}_r^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N} w_z^{\alpha\beta}\omega_\alpha\omega_\beta\rho^{\alpha\beta}r_e^{\alpha\beta}\left|\mathcal{J}_e^{\alpha\beta}\right|\ddot{u}_z^{\alpha\beta}\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}A_r^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N} w_z^{\alpha\beta}A_z^{\alpha\beta}
\end{aligned}$$

Moreover the source integral is zero.

As the relation : $\sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}A_r^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N} w_z^{\alpha\beta}A_z^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N} w_r^{\alpha\beta}B_r^{\alpha\beta}+\sum_{\alpha,\beta=0}^{N} w_z^{\alpha\beta}B_z^{\alpha\beta}=0$ must hold for any test function $\boldsymbol{w}=(w_r,w_z)$, we can conclude :

$$\forall\alpha,\beta=0\ldots N\quad\forall e=1\ldots n_e-1\quad\left\{\begin{array}{lcll} A_r^{\alpha\beta}+B_r^{\alpha\beta} &=& 0 & (\boldsymbol{w}=(\delta_{\alpha\beta},0))\\ A_z^{\alpha\beta}+B_z^{\alpha\beta} &=& 0 & (\boldsymbol{w}=(0,\delta_{\alpha\beta}))\end{array}\right.$$

And we obtain the following system for each non axial element e :

$$\left\{\begin{array}{lcl}\boldsymbol{M}\odot\ddot{\boldsymbol{u}}_r+\boldsymbol{K}_{rr}\cdot\boldsymbol{u}_r+\boldsymbol{K}_{rz}\cdot\boldsymbol{u}_z &=& 0\\ \boldsymbol{M}\odot\ddot{\boldsymbol{u}}_z+\boldsymbol{K}_{zr}\cdot\boldsymbol{u}_r+\boldsymbol{K}_{zz}\cdot\boldsymbol{u}_z &=& 0\end{array}\right.$$

Derivation of the algebraic system for axial elements

Let $f:(r,z)\mapsto f(r,z)$ be a differentiable function on $\overline{\Omega}_e$ such that $f(r,z)\underset{r\to 0}{\longrightarrow}0$, thanks to l'Hôpital's rule we can say : $\lim_{\xi\to\overline{\xi}_0}\frac{f(\boldsymbol{r}(\overline{\xi},\eta))}{r(\overline{\xi},\eta)}=\frac{\frac{\partial f}{\partial\xi}(\boldsymbol{r}(\overline{\xi}_0,\eta))}{\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)}=\left(\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)\right)^{-1}\sum_{\alpha,\beta=0}^{N}f^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_0)\ell_\beta(\eta)$.

$$\lim_{\xi\to\overline{\xi}_0}f(\boldsymbol{x}(\overline{\xi},\eta))=\frac{\partial f}{\partial\xi}(\boldsymbol{x}(\overline{\xi}_0,\eta))$$

So we set $\frac{f(\boldsymbol{r}(\overline{\xi}_0,\eta))}{r(\overline{\xi}_0,\eta)}=\left(\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)\right)^{-1}\frac{\partial f}{\partial\xi}(\boldsymbol{r}(\overline{\xi}_0,\eta))=\left(\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)\right)^{-1}\sum_{\alpha,\beta=0}^{N}f^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_0)\ell_\beta(\eta)$. And

$$\begin{aligned}
\frac{f(\boldsymbol{r}(\overline{\xi},\eta))}{r(\overline{\xi},\eta)} &= \left\{\begin{array}{ll}\frac{f(\boldsymbol{r}(\overline{\xi},\eta))}{r(\overline{\xi},\eta)} & \xi\neq\overline{\xi}_0\\ \left(\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)\right)^{-1}\sum_{\alpha,\beta=0}^{N}f^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_0)\ell_\beta(\eta) & \xi=\overline{\xi}_0\end{array}\right.\\
&= (1-\delta_{\xi=\overline{\xi}_0})\frac{f(\boldsymbol{r}(\overline{\xi},\eta))}{r(\overline{\xi},\eta)}+\delta_{\xi=\overline{\xi}_0}\left(\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)\right)^{-1}\sum_{\alpha,\beta=0}^{N}f^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_0)\ell_\beta(\eta)\\
&= \sum_{\alpha,\beta=0}^{N}f^{\overline{\alpha}\beta}\left(\frac{1}{r(\overline{\xi},\eta)}\overline{\ell}_\alpha(\xi)\ell_\beta(\eta)(1-\delta_{\xi=\overline{\xi}_0})+\delta_{\xi=\overline{\xi}_0}\left(\frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)\right)^{-1}\overline{\ell}'_\alpha(\overline{\xi}_0)\right)\ell_\beta(\eta)
\end{aligned}$$

We calculate the displacement gradients on the GLJ/GLL points $(\overline{\xi}_\sigma,\eta_\nu)_{\sigma,\nu=0\ldots N}$ of element $\overline{\Omega}_e$:

$$\begin{aligned}
\partial_i u_j(\boldsymbol{r}(\overline{\xi}_\sigma,\eta_\nu),t) &= \frac{\partial u_j}{\partial\xi}\partial_i\xi+\frac{\partial u_j}{\partial\eta}\partial_i\eta\\
&= \sum_{\alpha,\beta=0}^{N}u_j^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\ell_\beta(\eta_\nu)\partial_i\xi+\sum_{\alpha,\beta=0}^{N}u_j^{\overline{\alpha}\beta}\ell_\alpha(\overline{\xi}_\sigma)\ell'_\beta(\eta_\nu)\partial_i\eta\\
&= \sum_{\alpha=0}^{N}u_j^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\partial_i\xi+\sum_{\beta=0}^{N}u_j^{\overline{\sigma}\beta}\ell'_\beta(\eta)\partial_i\eta\\
&= \left[\sum_{\alpha=0}^{N}u_j^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\right]\partial_i\xi+\left[\sum_{\alpha=0}^{N}u_j^{\overline{\sigma}\alpha}\ell'_\alpha(\eta)\right]\partial_i\eta
\end{aligned}$$

We need the four terms $\partial_x\xi$,$\partial_z\xi$,$\partial_x\eta$,$\partial_z\eta$. Then we calculate the five elements of the stress tensor on these GLJ/GLL points :

$$\begin{aligned}
T_{rr} &= (\lambda+2\mu)\frac{\partial u_r}{\partial r}+\lambda\frac{\partial u_z}{\partial z}+\lambda\left\{\begin{array}{ll}\frac{u_r^{\overline{\sigma}\nu}}{r^{\overline{\sigma}\nu}} & \sigma\neq 0\\ \left(\frac{\partial r}{\partial\xi}\right)^{-1}\sum_{\alpha=0}^{N}u_r^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_0) & \sigma=0\end{array}\right.\\
T_{zz} &= \lambda\frac{\partial u_r}{\partial r}+(\lambda+2\mu)\frac{\partial u_z}{\partial z}+\lambda\left\{\begin{array}{ll}\frac{u_r^{\overline{\sigma}\nu}}{r^{\overline{\sigma}\nu}} & \sigma\neq 0\\ \left(\frac{\partial r}{\partial\xi}\right)^{-1}\sum_{\alpha=0}^{N}u_r^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_0) & \sigma=0\end{array}\right.\\
T_{rz} &= \mu\left[\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right]\\
T_{zr} &= T_{xz}\\
T_{\theta\theta} &= \lambda\frac{\partial u_r}{\partial r}+\lambda\frac{\partial u_z}{\partial z}+(\lambda+2\mu)\left\{\begin{array}{ll}\frac{u_r^{\overline{\sigma}\nu}}{r^{\overline{\sigma}\nu}} & \sigma\neq 0\\ \left(\frac{\partial r}{\partial\xi}\right)^{-1}\sum_{\alpha=0}^{N}u_r^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_0) & \sigma=0\end{array}\right.
\end{aligned}$$

We calculate the four matrix elements on the GLJ/GLL points $F_{ik}=\sum_{j=1}^{2}T_{ij}\partial_j\xi_k$. With these notations the elemental stiffness integral reads :

$$\int_\Lambda\sum_{i,j=1}^{2}T_{ij}\boldsymbol{\nabla}\boldsymbol{w}_{ji}\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta\ =\ \int_\Lambda\left(\sum_{i,k=1}^{2}F_{ik}\frac{\partial w_i}{\partial\xi_k}+T_{\theta\theta}\frac{w_r}{r}\right)|\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta$$

The GLJ/GLL interpolations tell :

$$
\begin{aligned}
\frac{\partial w_i}{\partial \xi_1}(\overline{\xi}_\sigma,\eta_\nu) &= \frac{\partial w_i}{\partial \xi}(\overline{\xi}_\sigma,\eta_\nu) = \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\ell_\beta(\eta_\nu) = \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\delta_{\beta\nu}\\
\frac{\partial w_i}{\partial \xi_2}(\overline{\xi}_\sigma,\eta_\nu) &= \frac{\partial w_i}{\partial \eta}(\overline{\xi}_\sigma,\eta_\nu) = \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\overline{\ell}_\alpha(\overline{\xi}_\sigma)\ell'_\beta(\eta_\nu) = \sum_{\alpha,\beta=0}^{N} w_i^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)
\end{aligned}
$$

Moreover we have seen that : $\frac{w_r^{\overline{\sigma}\nu}}{r^{\overline{\sigma}\nu}} = \begin{cases}\frac{w_r^{\overline{\sigma}\nu}}{r^{\overline{\sigma}\nu}} & \sigma\neq 0\\ \left(\frac{\partial r}{\partial\xi}\right)^{-1}\sum_{\alpha=0}^{N} w_r^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_0) & \sigma=0\end{cases} = \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}\underbrace{\begin{cases}\frac{1}{r^{\overline{\alpha}\beta}}\delta_{\alpha\sigma}\delta_{\beta\nu} & \sigma\neq 0\\ \left(\frac{\partial r}{\partial\xi}\right)^{-1}\overline{\ell}'_\alpha(\overline{\xi}_0)\delta_{\beta\nu} & \sigma=0\end{cases}}_{A^{\alpha\beta\nu\sigma}}$

and as $r(\xi=\overline{\xi}_0,\eta)=0$, thanks to l'Hôspital's rule we remark : $\lim_{\xi\to -1}\frac{r(\xi,\eta)}{\xi+1} = \frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta)$. So we set $R^{\overline{\sigma}\nu} = \begin{cases}\frac{r^{\overline{\sigma}\nu}}{\overline{\xi}_\sigma+1} & \sigma\neq 0\\ \frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta_\nu) & \sigma=0\end{cases}$.

We use then the quadrature rule to calculate the elemental stiffness integral :

$$
\begin{aligned}
\int_\Lambda \sum_{i,j=1}^{2} T_{ij}\boldsymbol{\nabla}\boldsymbol{w}_{ji}\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta &= \int_\Lambda\left(\sum_{i,k=1}^{2} F_{ik}\frac{\partial w_i}{\partial\xi_k} + T_{\theta\theta}\frac{w_r}{r}\right)\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta\\
&= \sum_{\sigma,\nu=0}^{N}\overline{\omega}_\sigma\omega_\nu\left|\mathcal{J}_e^{\overline{\sigma}\nu}\right| R^{\overline{\sigma}\nu}\left[\sum_{i=1}^{2}\left(F_{i1}^{\overline{\sigma}\nu}\sum_{\alpha,\beta=0}^{N} w_i^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\delta_{\beta\nu} + F_{i2}^{\overline{\sigma}\nu}\sum_{\alpha,\beta=0}^{N} w_i^{\overline{\alpha}\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)\right) + T_{\theta\theta}^{\overline{\sigma}\nu}\sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}A^{\alpha\beta\nu\sigma}\right]\\
&= \sum_{i=1}^{2}\sum_{\alpha,\beta=0}^{N} w_i^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}F_{i1}^{\overline{\sigma}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right| R^{\overline{\alpha}\nu}F_{i2}^{\overline{\alpha}\nu}\ell'_\beta(\eta_\nu) - \delta_{i1}\sum_{\sigma,\nu=0}^{N}\omega_\nu\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\nu}\right| R^{\overline{\sigma}\nu}T_{\theta\theta}^{\overline{\sigma}\nu}A^{\alpha\beta\nu\sigma}\right]\\
&= \sum_{i=1}^{2}\sum_{\alpha,\beta=0}^{N} w_i^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}F_{i1}^{\overline{\sigma}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right| R^{\overline{\alpha}\nu}F_{i2}^{\overline{\alpha}\nu}\ell'_\beta(\eta_\nu)\right.\\
&\qquad\left.+\delta_{i1}\left(\sum_{\nu=0}^{N}\omega_\nu\overline{\omega}_0\left|\mathcal{J}_e^{\overline{0}\nu}\right| R^{\overline{0}\nu}T_{\theta\theta}^{\overline{0}\nu}A^{\alpha\beta\nu 0} + \sum_{\nu=0}^{N}\sum_{\sigma=1}^{N}\omega_\nu\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\nu}\right| R^{\overline{\sigma}\nu}T_{\theta\theta}^{\overline{\sigma}\nu}A^{\alpha\beta\nu\sigma}\right)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}F_{11}^{\overline{\sigma}\beta}\ell'_\alpha(\overline{\xi}_\sigma) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right| R^{\overline{\alpha}\nu}F_{12}^{\overline{\alpha}\nu}\ell'_\beta(\eta_\nu)\right.\\
&\qquad\left.+\omega_\beta\left(\overline{\omega}_0\left|\mathcal{J}_e^{\overline{0}\beta}\right| R^{\overline{0}\beta}T_{\theta\theta}^{\overline{0}\beta}\left(\frac{\partial r}{\partial\xi}\right)^{-1}\overline{\ell}'_\alpha(\overline{\xi}_0) + \sum_{\sigma=1}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}T_{\theta\theta}^{\overline{\sigma}\beta}\frac{1}{r^{\overline{\sigma}\beta}}\delta_{\alpha\sigma}\right)\right]\\
&\quad+\sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}F_{21}^{\overline{\sigma}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right| R^{\overline{\alpha}\nu}F_{22}^{\overline{\alpha}\nu}\ell'_\beta(\eta_\nu)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}F_{11}^{\overline{\sigma}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right| R^{\overline{\alpha}\nu}F_{12}^{\overline{\alpha}\nu}\ell'_\beta(\eta_\nu)\right.\\
&\qquad\left.+\omega_\beta\left(\overline{\omega}_0\left|\mathcal{J}_e^{\overline{0}\beta}\right| T_{\theta\theta}^{\overline{0}\beta}\overline{\ell}'_\alpha(\overline{\xi}_0) + \begin{cases}\overline{\omega}_\alpha\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right| T_{\theta\theta}^{\overline{\alpha}\beta}\frac{1}{\overline{\xi}_\alpha+1} & \alpha\neq 0\\ 0 & \alpha=0\end{cases}\right)\right]\\
&\quad+\sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right| R^{\overline{\sigma}\beta}F_{21}^{\overline{\sigma}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right| R^{\overline{\alpha}\nu}F_{22}^{\overline{\alpha}\nu}\ell'_\beta(\eta_\nu)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w_x^{\overline{\alpha}\beta}B_x^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}B_z^{\overline{\alpha}\beta}
\end{aligned}
$$

We follow the same reasoning for the mass integral :

$$
\begin{aligned}
\int_{\Omega^e}\boldsymbol{w}\cdot\rho\ddot{\boldsymbol{u}}\,\mathrm{d}^2\boldsymbol{x} &= \sum_{\alpha,\beta=0}^{N}\rho^{\overline{\alpha}\beta}\left(w_r^{\overline{\alpha}\beta}\ddot{u}_r^{\overline{\alpha}\beta} + w_z^{\overline{\alpha}\beta}\ddot{u}_z^{\overline{\alpha}\beta}\right)\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right|\overline{\omega}_\alpha\omega_\beta\begin{cases}\frac{r^{\overline{\alpha}\beta}}{\overline{\xi}_\alpha+1} & \alpha\neq 0\\ \frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta_\beta) & \alpha=0\end{cases}\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}\ddot{u}_r^{\overline{\alpha}\beta}\overline{\omega}_\alpha\omega_\beta\rho^{\overline{\alpha}\beta}\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right|\begin{cases}\frac{r^{\overline{\alpha}\beta}}{\overline{\xi}_\alpha+1} & \alpha\neq 0\\ \frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta_\beta) & \alpha=0\end{cases} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}\ddot{u}_z^{\overline{\alpha}\beta}\overline{\omega}_\alpha\omega_\beta\rho^{\overline{\alpha}\beta}\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right|\begin{cases}\frac{r^{\overline{\alpha}\beta}}{\overline{\xi}_\alpha+1} & \alpha\neq 0\\ \frac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta_\beta) & \alpha=0\end{cases}\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}A_r^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}A_z^{\overline{\alpha}\beta}
\end{aligned}
$$

Moreover as $\boldsymbol{f}=(f_r,f_z)=\left(\frac{\delta(r)}{r},\delta(z-z_s)f(t)\right)$ the source integral can be written :

$$
\begin{aligned}
\int_{\Omega^e}\boldsymbol{w}\cdot\boldsymbol{f}\,\mathrm{d}^2\boldsymbol{x} &= (w_r(0,\eta_s)+w_z(0,\eta_s)f(t))\,|\mathcal{J}_e|\,(0,\eta_s)\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}\delta_{\alpha s}\delta_{0\beta}\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right| + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}\delta_{\alpha s}\delta_{s\beta}f(t)\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right|\\
&= \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}C_r^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}C_z^{\overline{\alpha}\beta}
\end{aligned}
$$

As the relation : $\sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}A_r^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}A_z^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}B_r^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}B_z^{\overline{\alpha}\beta} = \sum_{\alpha,\beta=0}^{N} w_r^{\overline{\alpha}\beta}C_r^{\overline{\alpha}\beta} + \sum_{\alpha,\beta=0}^{N} w_z^{\overline{\alpha}\beta}C_z^{\overline{\alpha}\beta}$ must hold for any test function $\boldsymbol{w}=(w_r,w_z)$, we can conclude :

$$\forall \alpha, \beta = 0 \dots N \quad \forall e = 1 \dots n_e - 1 \quad \left\{ \begin{array}{lcll} A_r^{\overline{\alpha}\beta} + B_r^{\overline{\alpha}\beta} & = & C_r^{\overline{\alpha}\beta} & \quad (\boldsymbol{w} = (\delta_{\alpha\beta}, 0)) \\ A_z^{\overline{\alpha}\beta} + B_z^{\overline{\alpha}\beta} & = & C_z^{\overline{\alpha}\beta} & \quad (\boldsymbol{w} = (0, \delta_{\alpha\beta})) \end{array} \right.$$

And we obtain the following system for each axial element e :

$$\left\{ \begin{array}{lcl} \overline{\boldsymbol{M}} \odot \ddot{\boldsymbol{u}}_r + \overline{\boldsymbol{K}_{rr}} \cdot \boldsymbol{u}_r + \overline{\boldsymbol{K}_{rz}} \cdot \boldsymbol{u}_z & = & \boldsymbol{F}_r \\ \overline{\boldsymbol{M}} \odot \ddot{\boldsymbol{u}}_z + \overline{\boldsymbol{K}_{zr}} \cdot \boldsymbol{u}_r + \overline{\boldsymbol{K}_{zz}} \cdot \boldsymbol{u}_z & = & \boldsymbol{F}_z \end{array} \right.$$

- **Conclusion**

$$\boxed{\left\{ \begin{array}{ll} \left\{ \begin{array}{lcl} \boldsymbol{M} \odot \ddot{\boldsymbol{u}}_r + \boldsymbol{K}_{rr} \cdot \boldsymbol{u}_r + \boldsymbol{K}_{rz} \cdot \boldsymbol{u}_z & = & 0 \\ \boldsymbol{M} \odot \ddot{\boldsymbol{u}}_z + \boldsymbol{K}_{zr} \cdot \boldsymbol{u}_r + \boldsymbol{K}_{zz} \cdot \boldsymbol{u}_z & = & 0 \end{array} \right. & \text{In non-axial elements} \\ \left\{ \begin{array}{lcl} \overline{\boldsymbol{M}} \odot \ddot{\boldsymbol{u}}_r + \overline{\boldsymbol{K}_{rr}} \cdot \boldsymbol{u}_r + \overline{\boldsymbol{K}_{rz}} \cdot \boldsymbol{u}_z & = & \boldsymbol{F}_r \\ \overline{\boldsymbol{M}} \odot \ddot{\boldsymbol{u}}_z + \overline{\boldsymbol{K}_{zr}} \cdot \boldsymbol{u}_r + \overline{\boldsymbol{K}_{zz}} \cdot \boldsymbol{u}_z & = & \boldsymbol{F}_z \end{array} \right. & \text{In axial elements} \end{array} \right.}$$