2D spectral-elements formulation in cartesian coordinates (acoustic medium)

Introduction : assumptions, conventions, weak form

We shall start from the wave equation for a 3D inhomogeneous fluid medium Ω without source:

$$\rho \ddot{\boldsymbol{u}} = \boldsymbol{\nabla} \cdot \boldsymbol{T} \tag{1}$$

With $\boldsymbol{\nabla} \cdot \boldsymbol{T}$ refering to the divergence of the stress tensor field $\boldsymbol{T}$. In acoustic isotropic media :

$$\forall i,j \in \{x,y,z\}^2 \quad T_{ij} = \lambda \delta_{ij} \epsilon_{kk} = \lambda \delta_{ij}(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})$$

(We could introduce the bulk modulus here : $\kappa = \lambda + \frac{2}{3}\mu = \lambda$)

$\boldsymbol{\epsilon}$ being the strain tensor, by definition : $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla u} + \boldsymbol{\nabla}^\top \boldsymbol{u})$ which in cartesian coordinate is written : $\forall i,j \in \{x,y,z\}^2 \quad \epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

So $\epsilon_{kk} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$

To summarize :

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = \begin{pmatrix} \lambda\epsilon_{kk} & 0 & 0 \\ 0 & \lambda\epsilon_{kk} & 0 \\ 0 & 0 & \lambda\epsilon_{kk} \end{pmatrix} = \begin{pmatrix} \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u} & 0 & 0 \\ 0 & \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u} & 0 \\ 0 & 0 & \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u} \end{pmatrix} \equiv - \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

We also remark that :

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = -\boldsymbol{\nabla} P$$

It could be useful to express $\rho\ddot{\boldsymbol{u}}$ as a gradient! As Chaljub Valette 2004 we introduce a scalar potential χ of $\rho\boldsymbol{u}$:

$$\rho \boldsymbol{u} = \boldsymbol{\nabla} \chi$$

The wave equation becomes :

$$\boldsymbol{\nabla} \ddot{\chi} = -\boldsymbol{\nabla} P$$

It implies then :

$$\ddot{\chi}(x,y,z,t) = -P(x,y,z,t) + P_0(t)$$

We decide $P_0(t) = 0$.

$$\ddot{\chi} = -P = \lambda \boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\lambda}{\rho} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \chi$$

And the wave equation expresses :

$$\frac{1}{\lambda}\ddot{\chi} = \frac{1}{\rho} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \chi \tag{2}$$

These relations does not depend on the coordinate system (if it remains orthogonal).
Let M be a point of Ω and $M \mapsto w(M)$ an arbitrary test function. Let us suppose that Ω has a free surface $\partial\Omega$ with normal $\boldsymbol{n}(x,y,z)$ and an artificial boundary Γ with a solid medium. We obtain the week formulation from the wave equation by dotting the momentum equation (2) with an the test function w and integrating by part over the model volume Ω :

$$\underbrace{\int_\Omega w \frac{1}{\lambda} \ddot{\chi} \, \mathrm{d}^2\boldsymbol{x}}_{\text{mass integral}} = -\underbrace{\int_\Omega \frac{1}{\rho} \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} \chi \, \mathrm{d}^2\boldsymbol{x}}_{\text{stiffness integral}} + \underbrace{\int_\Gamma w \boldsymbol{n} \cdot \dot{\boldsymbol{u}} \, \mathrm{d}x}_{\text{coupling integral}}$$

$$\int_\Omega w \frac{1}{\lambda} \ddot{\chi} \, \mathrm{d}^2\boldsymbol{x} = -\int_\Omega \frac{1}{\rho} \boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} \chi \, \mathrm{d}^2\boldsymbol{x} + \int_\Gamma w \boldsymbol{n} \cdot \dot{\boldsymbol{u}} \, \mathrm{d}x$$

At this point we will choose the plane-strain convention : we suppose an infinite medium along y and that the important loads are in the $x-z$ plane and do not change with y.

Hence for example $P = -\lambda \left[\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right]$

The test functions $M \mapsto w(M)$ expresses$(x,z) \mapsto w(x,z)$. We have then :

$$\boldsymbol{\nabla} w \cdot \boldsymbol{\nabla} \chi = \frac{\partial w}{\partial x}\frac{\partial \chi}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial \chi}{\partial z}$$

The wave equation becomes :

$$\int_\Omega w \frac{1}{\lambda} \ddot{\chi} \, \mathrm{d}^2\boldsymbol{x} = -\int_\Omega \frac{1}{\rho} \left(\frac{\partial w}{\partial x}\frac{\partial \chi}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial \chi}{\partial z} \right) \mathrm{d}^2\boldsymbol{x} \tag{3}$$

Mapping

We subdivide the model volume Ω into a number of non-overlapping hexahedral elements Ω_e , $e = 1, \ldots, n_e$ such that $\Omega = \bigcup_e \Omega_e$. As the result of this subdivision, the artificial boundary Γ would be similarly represented by a number of 1D elements Γ_b, $b = 1, \ldots, n_b$ such that $\Gamma = \bigcup_b \Gamma_b$.

- **Boundary elements**

Each linear boundary element Γ_b is isomorphous to a line and can be mapped onto the reference interval $[-1, 1]$:

$$\begin{aligned} \forall x \in \Gamma_b \, \forall \xi \in [-1,1] \quad x(\xi) &= F(\xi) \\ &= \sum_{a=1}^{n_a} N_a(\xi) x_a \end{aligned}$$

The N_a are called shape functions and define the mapping. They usually are Lagrange polynomials of degree $n_a - 1$. x_a are the anchor nodes $\forall a \in 1 \ldots n_a \quad x(\xi_a) = x_a$. On this 1-D case we have got two shape functions N_1 and N_2 (which are Lagrange polynomials of degree 1) and two anchor nodes $x_1 = X_b$ and $x_2 = X_{b+1}$ corresponding to $\xi_1 = -1$ and $\xi_2 = 1$.

$$\forall x \in \Gamma_b \, \forall \xi \in [-1,1] \quad x(\xi) = (X_{b+1} - X_b)\frac{\xi + 1}{2} + X_b \;\; = \Delta_e \frac{\xi+1}{2} + X_b$$

Thus $\frac{dx}{d\xi} = \frac{X_{b+1} - X_b}{2}$ and $\frac{d\xi}{dx} = \frac{2}{X_{b+1} - X_b}$

- **Surface elements**

Points $\boldsymbol{x}=(x,z)$ within each hexahedral element Ω_e may be uniquely related to points $\boldsymbol{\xi}=(\xi,\eta)$, $-1\leq\xi,\eta\leq 1$ in a reference square Λ based upon the invertible mapping

$$\boldsymbol{x}(\boldsymbol{\xi})=\sum_{a=1}^{n_a}\boldsymbol{x}_a N_a(\boldsymbol{\xi})$$

The n_a anchors nodes $\boldsymbol{x}_a=\boldsymbol{x}(\xi_a,\eta_a)$ and shape functions $N_a(\boldsymbol{\xi})$ define the geometry of an element Ω_e (here we should add more detail for a paper). A surface element dxdz within a given element Ω_e is related to a surface element in the reference square by the relation :

$$\mathrm{dxdz}=|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta$$

Where $\mathcal{J}_e$ is the surface Jacobian. For $(x,z)\in\Omega_e$ it expresses :

$$\mathcal{J}_e=\left|\frac{\partial x}{\partial\xi}\frac{\partial z}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial z}{\partial\xi}\right|$$

With of course $\dfrac{\partial x}{\partial\xi}=\displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\xi}x_a$, $\dfrac{\partial z}{\partial\xi}=\displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\xi}z_a$, $\dfrac{\partial x}{\partial\eta}=\displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\eta}x_a$ and $\dfrac{\partial z}{\partial\eta}=\displaystyle\sum_{a=1}^{n_a}\frac{\partial N_a}{\partial\eta}z_a$. Moreover for any function $(x,z)\in\Omega_e\mapsto f(x,z)$ we have the three identities :

$$\begin{pmatrix}\frac{\partial f}{\partial\xi}\\ \frac{\partial f}{\partial\eta}\end{pmatrix}=\underbrace{\begin{pmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial z}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial z}{\partial\eta}\end{pmatrix}}_{J_e}\times\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial z}\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial x}\frac{\partial x}{\partial\xi}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial\xi}\\ \frac{\partial f}{\partial x}\frac{\partial x}{\partial\eta}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial\eta}\end{pmatrix},\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial z}\end{pmatrix}=\underbrace{\begin{pmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\xi}{\partial z} & \frac{\partial\eta}{\partial z}\end{pmatrix}}_{J_e^{-1}}\times\begin{pmatrix}\frac{\partial f}{\partial\xi}\\ \frac{\partial f}{\partial\eta}\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial x}\\ \frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial z}+\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial z}\end{pmatrix}$$

and $$\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial z}\end{pmatrix}=\underbrace{\frac{1}{\mathcal{J}_e}\begin{pmatrix}\frac{\partial z}{\partial\eta} & -\frac{\partial z}{\partial\xi}\\ -\frac{\partial x}{\partial\eta} & \frac{\partial x}{\partial\xi}\end{pmatrix}}_{J_e^{-1}}\times\begin{pmatrix}\frac{\partial f}{\partial\xi}\\ \frac{\partial f}{\partial\eta}\end{pmatrix}=\frac{1}{\mathcal{J}_e}\begin{pmatrix}\frac{\partial f}{\partial\xi}\frac{\partial z}{\partial\eta}-\frac{\partial f}{\partial\eta}\frac{\partial z}{\partial\xi}\\ -\frac{\partial f}{\partial\xi}\frac{\partial x}{\partial\eta}+\frac{\partial f}{\partial\eta}\frac{\partial x}{\partial\xi}\end{pmatrix}$$

That supplies : $$\begin{array}{lllll}\frac{\partial x}{\partial\xi} & = & \mathcal{J}_e\frac{\partial\eta}{\partial z} \qquad \frac{\partial z}{\partial\xi} & = & -\mathcal{J}_e\frac{\partial\eta}{\partial x}\\ \frac{\partial x}{\partial\eta} & = & -\mathcal{J}_e\frac{\partial\xi}{\partial z} \qquad \frac{\partial z}{\partial\eta} & = & \mathcal{J}_e\frac{\partial\xi}{\partial x}\end{array}$$

Note : Something would have to be said about the Jacobian (it must not be singular)
We can then write :

$$\begin{aligned}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi &= \frac{\partial w}{\partial x}\frac{\partial\chi}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial\chi}{\partial z}\\ &= \left[\frac{\partial w}{\partial\xi}\partial_x\xi+\frac{\partial w}{\partial\eta}\partial_x\eta\right]\left[\frac{\partial\chi}{\partial\xi}\partial_x\xi+\frac{\partial\chi}{\partial\eta}\partial_x\eta\right]+\left[\frac{\partial w}{\partial\xi}\partial_z\xi+\frac{\partial w}{\partial\eta}\partial_z\eta\right]\left[\frac{\partial\chi}{\partial\xi}\partial_z\xi+\frac{\partial\chi}{\partial\eta}\partial_z\eta\right]\\ &= \frac{\partial w}{\partial\xi}\left(\partial_x\xi\left[\frac{\partial\chi}{\partial\xi}\partial_x\xi+\frac{\partial\chi}{\partial\eta}\partial_x\eta\right]+\partial_z\xi\left[\frac{\partial\chi}{\partial\xi}\partial_z\xi+\frac{\partial\chi}{\partial\eta}\partial_z\eta\right]\right)+\frac{\partial w}{\partial\eta}\left(\partial_x\eta\left[\frac{\partial\chi}{\partial\xi}\partial_x\xi+\frac{\partial\chi}{\partial\eta}\partial_x\eta\right]+\partial_z\eta\left[\frac{\partial\chi}{\partial\xi}\partial_z\xi+\frac{\partial\chi}{\partial\eta}\partial_z\eta\right]\right)\\ &= \frac{\partial w}{\partial\xi}\left(\partial_x\xi\frac{\partial\chi}{\partial x}+\partial_z\xi\frac{\partial\chi}{\partial z}\right)+\frac{\partial w}{\partial\eta}\left(\partial_x\eta\frac{\partial\chi}{\partial x}+\partial_z\eta\frac{\partial\chi}{\partial z}\right)\end{aligned}$$

- **Splitting and substitution**

_For elements on the boundary, after splitting the 2D P-SV wave equation becomes :

$$\forall e=1\ldots n_e \qquad \int_{\Omega^e}w\frac{1}{\lambda}\ddot{\chi}\,\mathrm{d}^2\boldsymbol{x}=\int_{\Omega^e}\frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\,\mathrm{d}^2\boldsymbol{x}+\int_{\Gamma^e}w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,\mathrm{dx}$$

Note : something has to be said about the test functions before being able to split the equation.
Then we make the substitution :

$$\forall e=1\ldots n_e \qquad \int_\Lambda w\frac{1}{\lambda}\ddot{\chi}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta=\int_\Lambda\frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta+\int_{-1}^{1}w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,\frac{dx}{d\xi}\mathrm{d}\xi$$

_For elements with no boundary, after splitting the 2D P-SV wave equation becomes :

$$\forall e=1\ldots n_e \qquad \int_{\Omega^e}w\frac{1}{\lambda}\ddot{\chi}\,\mathrm{d}^2\boldsymbol{x}=\int_{\Omega^e}\frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\,\mathrm{d}^2\boldsymbol{x}$$

Note : something has to be said about the test functions before being able to split the equation.
Then we make the substitution :

$$\forall e=1\ldots n_e \qquad \int_\Lambda w\frac{1}{\lambda}\ddot{\chi}\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta=\int_\Lambda\frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\;|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta$$

Representation of functions on the elements (GLL interpolation)

- **1D elements**

We interpolate the functions with Lagrange polynomials of degree N with GLL points $(\xi_\alpha)_{\alpha=0\ldots N}$ as collocation points. For any function $f:\xi\mapsto f(\xi)$ on $\Lambda=[-1,1]$:

$$\forall\xi\in[-1,1]\quad f(\boldsymbol{x}(\xi))=\sum_{\alpha=0}^{N}f(\xi_\alpha)\ell_\alpha(\xi)=\sum_{\alpha=0}^{N}f^\alpha\ell_\alpha(\xi)$$

and :

$$\forall\xi\in[-1,1]\quad \frac{\partial f}{\partial\xi}(\boldsymbol{x}(\xi))=\sum_{\alpha=0}^{N}f(\xi_\alpha)\frac{d\ell_\alpha}{d\xi}(\xi)=\sum_{\alpha=0}^{N}f^\alpha\ell'_\alpha(\xi)$$

The ℓ_α are the Lagrange polynomials defined on the colocation points of the Gauss-Lobatto-Legendre quadrature. The ξ_α are the GLL points. It is of paramount importance to note that $\ell_i(\xi_j) = \delta_{ij}$.

We evaluate the integrals with the quadrature :

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi \approx \sum_{i=0}^{N} \omega_i f^i$$

- **Surface elements**

We interpolate the functions with Lagrange polynomials of degree N with GLL points $(\xi_\alpha, \eta_\beta)_{\alpha,\beta=0\dots N}$ as collocation points (here we should add more detail for a paper). For any function $f : (x, z) \mapsto f(x, z)$ on Ω_e:

$$\forall(\xi,\eta) \in \Lambda \quad f(\boldsymbol{x}(\xi,\eta)) \approx \sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} f(\xi_\alpha,\eta_\beta)\ell_\alpha(\xi)\ell_\beta(\eta) = \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta}\ell_\alpha(\xi)\ell_\beta(\eta)$$

and :

$$\forall(\xi,\eta) \in \Lambda$$
$$\begin{aligned}
\frac{\partial f}{\partial \xi}(\boldsymbol{x}(\xi,\eta)) &= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta}\frac{\partial \ell_\alpha(\xi)}{\partial \xi}\ell_\beta(\eta) &&= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta}\ell'_\alpha(\xi)\ell_\beta(\eta)\\
\frac{\partial f}{\partial \eta}(\boldsymbol{x}(\xi,\eta)) &= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta}\ell_\alpha(\xi)\frac{\partial \ell_\beta(\eta)}{\partial \eta} &&= \sum_{\alpha,\beta=0}^{N} f^{\alpha\beta}\ell_\alpha(\xi)\ell'_\beta(\eta)
\end{aligned}$$

The ℓ_i are the Lagrange polynomials defined on the colocation points of the Gauss-Lobatto-Legendre quadrature. The ξ_i, η_i are the GLL points. It is of paramount importance to note that $\ell_i(\xi_j) = \delta_{ij}$.

We evaluate the integrals with the quadrature :

$$\int_{\Lambda} f(\boldsymbol{x}(\xi,\eta))\,\mathrm{d}\xi\mathrm{d}\eta \approx \sum_{i,j=0}^{N} \omega_i\omega_j f^{ij}$$

Derivation of the algebraic system

We begin by calculating the potential gradient on the GLL points $(\xi_\sigma, \eta_\nu)_{\sigma,\nu=0\dots N}$ of element Ω_e:

$$\begin{aligned}
\partial_i\chi &= \frac{\partial \chi}{\partial \xi}\partial_i\xi + \frac{\partial \chi}{\partial \eta}\partial_i\eta\\
&= \sum_{\alpha,\beta=0}^{N} \chi^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\ell_\beta(\eta_\nu)\partial_i\xi + \sum_{\alpha,\beta=0}^{N} \chi^{\alpha\beta}\ell_\alpha(\xi_\sigma)\ell'_\beta(\eta_\nu)\partial_i\eta\\
&= \sum_{\alpha=0}^{N} \chi^{\alpha\nu}\ell'_\alpha(\xi_\sigma)\partial_i\xi + \sum_{\beta=0}^{N} \chi^{\sigma\beta}\ell'_\beta(\eta_\nu)\partial_i\eta\\
&= \left[\sum_{\alpha=0}^{N} \chi^{\alpha\nu}\ell'_\alpha(\xi_\sigma)\right]\partial_i\xi + \left[\sum_{\alpha=0}^{N} \chi^{\sigma\alpha}\ell'_\alpha(\eta_\nu)\right]\partial_i\eta
\end{aligned}$$

We need the four terms $\partial_x\xi$,$\partial_z\xi$,$\partial_x\eta$,$\partial_z\eta$.

The GLL interpolation tells :

$$\begin{aligned}
\frac{\partial w}{\partial \xi}(\xi_\sigma,\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\ell_\beta(\eta_\nu) &&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu}\\
\frac{\partial w}{\partial \eta}(\xi_\sigma,\eta_\nu) &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\ell_\alpha(\xi_\sigma)\ell'_\beta(\eta_\nu) &&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)
\end{aligned}$$

We use then the quadrature rule to calculate the elemental stiffness integral :

$$\begin{aligned}
\int_{\Lambda}\frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\,|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta &= \int_{\Lambda}\frac{1}{\rho}\left[\frac{\partial w}{\partial \xi}\left(\partial_x\xi\frac{\partial \chi}{\partial x} + \partial_z\xi\frac{\partial \chi}{\partial z}\right) + \frac{\partial w}{\partial \eta}\left(\partial_x\eta\frac{\partial \chi}{\partial x} + \partial_z\eta\frac{\partial \chi}{\partial z}\right)\right]|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta\\
&= \sum_{\sigma,\nu=0}^{N}\omega_\sigma\omega_\nu\frac{|\mathcal{J}_e^{\sigma\nu}|}{\rho^{\sigma\nu}}\left[\sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu}\left(\partial_x\xi\frac{\partial \chi}{\partial x} + \partial_z\xi\frac{\partial \chi}{\partial z}\right) + \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)\left(\partial_x\eta\frac{\partial \chi}{\partial x} + \partial_z\eta\frac{\partial \chi}{\partial z}\right)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\omega_\sigma\frac{\left|\mathcal{J}_e^{\sigma\beta}\right|}{\rho^{\sigma\beta}}\ell'_\alpha(\xi_\sigma)\left(\partial_x\xi\frac{\partial \chi}{\partial x} + \partial_z\xi\frac{\partial \chi}{\partial z}\right) + \omega_\alpha\sum_{\nu=0}^{N}\omega_\nu\frac{|\mathcal{J}_e^{\alpha\nu}|}{\rho^{\alpha\nu}}\ell'_\beta(\eta_\nu)\left(\partial_x\eta\frac{\partial \chi}{\partial x} + \partial_z\eta\frac{\partial \chi}{\partial z}\right)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}B^{\alpha\beta}
\end{aligned}$$

We follow the same reasoning for the elemental mass integral :

$$\begin{aligned}
\int_{\Lambda} w\frac{1}{\lambda}\ddot{\chi}\,|\mathcal{J}_e|\,\mathrm{d}\xi\mathrm{d}\eta &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\omega_\alpha\omega_\beta\frac{1}{\lambda^{\alpha\beta}}\left|\mathcal{J}_e^{\alpha\beta}\right|\ddot{\chi}^{\alpha\beta}\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}A^{\alpha\beta}
\end{aligned}$$

The coupling integral reads :

$$\begin{aligned}
\textstyle\int_{-1}^{1} w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,\frac{dx}{d\xi}\mathrm{d}\xi &= \sum_{\alpha=0}^{N} w^{\alpha N}\omega_\alpha\left.\frac{dx}{d\xi}\right|^{\alpha}(n_x^\alpha\dot{u}_x^\alpha + n_z^\alpha\dot{u}_z^\alpha)\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\omega_\alpha\left.\frac{dx}{d\xi}\right|^{\alpha}(n_x^\alpha\dot{u}_x^\alpha + n_z^\alpha\dot{u}_z^\alpha)\,\delta_{\beta N}\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}C^{\alpha\beta}
\end{aligned}$$

As the relation : $\sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}A^{\alpha\beta} + \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}B^{\alpha\beta} = C^{\alpha\beta}$ must hold for any test function $w(x,z)$, we can conclude :

$$\forall\alpha,\beta = 0\dots N \quad \forall e = 0\dots n_e - 1 \quad A^{\alpha\beta} + B^{\alpha\beta} = 0 \quad (w = \delta_{\alpha\beta})$$

We have then :

$$\forall\alpha,\beta = 0\dots N \quad \forall e = 0\dots n_e - 1$$

$$\begin{aligned} \omega_\alpha\omega_\beta \frac{1}{\lambda^{\alpha\beta}} \left|\mathcal{J}_e^{\alpha\beta}\right| \ddot{\chi}^{\alpha\beta} + \left[\omega_\beta \sum_{\sigma=0}^{N} \omega_\sigma \frac{\left|\mathcal{J}_e^{\sigma\beta}\right|}{\rho^{\sigma\beta}} \ell'_\alpha(\xi_\sigma) \left(\partial_x\xi \frac{\partial\chi}{\partial x} + \partial_z\xi\frac{\partial\chi}{\partial z}\right) + \omega_\alpha \sum_{\nu=0}^{N} \omega_\nu \frac{|\mathcal{J}_e^{\alpha\nu}|}{\rho^{\alpha\nu}} \ell'_\beta(\eta_\nu) \left(\partial_x\eta\frac{\partial\chi}{\partial x} + \partial_z\eta\frac{\partial\chi}{\partial z}\right)\right] &= \omega_\alpha \left.\frac{dx}{d\xi}\right|^{\alpha} (n_x^\alpha \dot{u}_x^\alpha + n_z^\alpha \dot{u}_z^\alpha)\,\delta_{\beta N} \\ M^{\alpha\beta}\ddot{\chi}^{\alpha\beta} + \left[\sum_{I,J=0}^{N} K^{\alpha\beta IJ}\chi^{IJ}\right] &= C^{\alpha\beta} \end{aligned}$$

Or in tensorial form for each element e :

$$\boldsymbol{M} \odot \ddot{\boldsymbol{\chi}} + \boldsymbol{K}\cdot\boldsymbol{\chi} = \boldsymbol{0}$$

(We have defined $(\boldsymbol{A}\odot\boldsymbol{B})_{ij} = A_{ij}B_{ij}$ and $(\boldsymbol{A}\cdot\boldsymbol{B})_{ij} = \sum_{k,l=0}^{N} A_{ijkl}B_{kl}$). Hence, after assembly :

$$\ddot{\boldsymbol{\chi}} = \frac{\boldsymbol{F}^g_{x,int}(t)}{\boldsymbol{M}^g}$$

2.5D spectral-elements formulation in curvilinear cylindrical coordinates (acoustic medium)

Introduction : assumptions, conventions, weak form

We shall start from a 3D formulation. I recall the weak form of the 3D momentum equation :

$$\underbrace{\int_\Omega w\frac{1}{\lambda}\ddot{\chi}\,\mathrm{d}^3\boldsymbol{x}}_{\text{mass integral}} \quad = \quad \underbrace{-\int_\Omega \frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\,\mathrm{d}^3\boldsymbol{x}}_{\text{stiffness integral}} + \underbrace{\int_\Gamma w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,\mathrm{dx}}_{\text{coupling integral}}$$

The test functions : $(r,\theta,z)\mapsto w(r,\theta,z)$ now belongs to the subspace of Sobolev space $H^1_1(\Omega)$ of the functions that cancel on the axis. (Bernardi and p.12 of Alexandre Fournier phd tesis, we would have to introduce these spaces). The wave equation reads :

$$\begin{aligned} &\int_\Omega w\frac{1}{\lambda}\ddot{\chi}\,\mathrm{d}^3\boldsymbol{x} &&= -\int_\Omega \frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\,\mathrm{d}^3\boldsymbol{x} + \int_\Gamma w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,2\pi r\mathrm{dr} \\ \iff &\int_\Omega w\frac{1}{\lambda}\ddot{\chi}\,2\pi r\mathrm{drd}\theta\mathrm{dz} &&= -\int_\Omega \frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\,2\pi r\mathrm{drd}\theta\mathrm{dz} + \textstyle\int_\Gamma w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,2\pi r\mathrm{dx} \end{aligned} \tag{4}$$

We still consider an acoustic isotropic media, the stress tensor still expresses :

$$\forall i,j \in \{x,y,z\}^2 \quad T_{ij} = \lambda\delta_{ij}\epsilon_{kk} = \lambda\delta_{ij}(\epsilon_{11}+\epsilon_{22}+\epsilon_{33})$$

Nevertheless now $\boldsymbol{\nabla}f = \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r}\frac{\partial f}{\partial\theta} \\ \frac{\partial f}{\partial z}\end{pmatrix}$ consequently : $\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi = \frac{\partial w}{\partial r}\frac{\partial\chi}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial\theta}\frac{\partial\chi}{\partial\theta} + \frac{\partial w}{\partial z}\frac{\partial\chi}{\partial z}$. Moreover $\left\{\begin{array}{lcl} u_r &=& \frac{1}{\rho}\frac{\partial\chi}{\partial r} \\ u_\theta &=& \frac{1}{\rho r}\frac{\partial\chi}{\partial\theta} \\ u_z &=& \frac{1}{\rho}\frac{\partial\chi}{\partial z}\end{array}\right.$

At this point we will choose the 2.5D convention : we suppose an axisymetric geometry and that the important loads are not along $\boldsymbol{\theta}$ and do not change with θ:

$$\frac{\partial}{\partial\theta} = 0$$

That leads to : $\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi = \frac{\partial w}{\partial r}\frac{\partial\chi}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial\chi}{\partial z}$ and $\left\{\begin{array}{lcl} u_r &=& \frac{1}{\rho}\frac{\partial\chi}{\partial r} \\ u_\theta &=& 0 \\ u_z &=& \frac{1}{\rho}\frac{\partial\chi}{\partial z}\end{array}\right.$ which is exactly the same than in cartesian coordinates with $r \longleftrightarrow x$.

For non-axial elements

$$\begin{aligned} \int_\Lambda \frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\;|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta &= \int_\Lambda \frac{1}{\rho}\left[\frac{\partial w}{\partial\xi}\left(\partial_r\xi\frac{\partial\chi}{\partial r}+\partial_z\xi\frac{\partial\chi}{\partial z}\right) + \frac{\partial w}{\partial\eta}\left(\partial_r\eta\frac{\partial\chi}{\partial r}+\partial_z\eta\frac{\partial\chi}{\partial z}\right)\right]\;|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta \\ &= \sum_{\sigma,\nu=0}^{N}\omega_\sigma\omega_\nu\frac{|\mathcal{J}_e^{\sigma\nu}|\,r^{\sigma\nu}}{\rho^{\sigma\nu}}\left[\sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\ell'_\alpha(\xi_\sigma)\delta_{\beta\nu}\left(\partial_r\xi\frac{\partial\chi}{\partial r}+\partial_z\xi\frac{\partial\chi}{\partial z}\right) + \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)\left(\partial_r\eta\frac{\partial\chi}{\partial r}+\partial_z\eta\frac{\partial\chi}{\partial z}\right)\right] \\ &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\omega_\sigma\frac{\left|\mathcal{J}_e^{\sigma\beta}\right|}{\rho^{\sigma\beta}}r^{\sigma\beta}\ell'_\alpha(\xi_\sigma)\left(\partial_r\xi\frac{\partial\chi}{\partial r}+\partial_z\xi\frac{\partial\chi}{\partial z}\right) + \omega_\alpha\sum_{\nu=0}^{N}\omega_\nu\frac{|\mathcal{J}_e^{\alpha\nu}|}{\rho^{\alpha\nu}}r^{\alpha\nu}\ell'_\beta(\eta_\nu)\left(\partial_r\eta\frac{\partial\chi}{\partial r}+\partial_z\eta\frac{\partial\chi}{\partial z}\right)\right] \\ &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}B^{\alpha\beta} \end{aligned}$$

$$\begin{aligned} \int_\Lambda w\frac{1}{\lambda}\ddot{\chi}\;|\mathcal{J}_e|\,r\mathrm{d}\xi\mathrm{d}\eta &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\omega_\alpha\omega_\beta\frac{1}{\lambda^{\alpha\beta}}\left|\mathcal{J}_e^{\alpha\beta}\right| r^{\alpha\beta}\ddot{\chi}^{\alpha\beta} \\ &= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}A^{\alpha\beta} \end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1} w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,\frac{dr}{d\xi} r\mathrm{d}\xi &= \sum_{\alpha=0}^{N} w^{\alpha N}\omega_\alpha \left.\frac{dr}{d\xi}\right|^{\alpha} r^{\alpha}\left(n_r^\alpha \dot{u}_r^\alpha + n_z^\alpha \dot{u}_z^\alpha\right)\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}\omega_\alpha \left.\frac{dx}{d\xi}\right|^{\alpha} r^{\alpha}\left(n_r^\alpha \dot{u}_r^\alpha + n_z^\alpha \dot{u}_z^\alpha\right)\delta_{\beta N}\\
&= \sum_{\alpha,\beta=0}^{N} w^{\alpha\beta}C^{\alpha\beta}
\end{aligned}$$

For axial elements

$$\begin{aligned}
\partial_i\chi(\boldsymbol{r}(\overline{\xi}_\sigma,\eta_\nu),t) &= \frac{\partial\chi}{\partial\xi}\partial_i\xi + \frac{\partial\chi}{\partial\eta}\partial_i\eta\\
&= \sum_{\alpha,\beta=0}^{N}\chi^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\ell_\beta(\eta_\nu)\partial_i\xi + \sum_{\alpha,\beta=0}^{N}\chi^{\overline{\alpha}\beta}\ell_\alpha(\overline{\xi}_\sigma)\ell'_\beta(\eta_\nu)\partial_i\eta\\
&= \sum_{\alpha=0}^{N}\chi^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\partial_i\xi + \sum_{\beta=0}^{N}\chi^{\overline{\sigma}\beta}\ell'_\beta(\eta)\partial_i\eta\\
&= \left[\sum_{\alpha=0}^{N}\chi^{\overline{\alpha}\nu}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\right]\partial_i\xi + \left[\sum_{\alpha=0}^{N}\chi^{\overline{\sigma}\alpha}\ell'_\alpha(\eta)\right]\partial_i\eta
\end{aligned}$$

$$\begin{aligned}
\int_\Lambda \frac{1}{\rho}\boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}\chi\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta &= \int_\Lambda \frac{1}{\rho}\left[\frac{\partial w}{\partial\xi}\left(\partial_r\xi\frac{\partial\chi}{\partial r} + \partial_z\xi\frac{\partial\chi}{\partial z}\right) + \frac{\partial w}{\partial\eta}\left(\partial_r\eta\frac{\partial\chi}{\partial r} + \partial_z\eta\frac{\partial\chi}{\partial z}\right)\right]\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta\\
&= \sum_{\sigma,\nu=0}^{N}\overline{\omega}_\sigma\omega_\nu\frac{\left|\mathcal{J}_e^{\overline{\sigma}\nu}\right|}{\rho^{\overline{\sigma}\nu}}\frac{r^{\overline{\sigma}\nu}}{1+\overline{\xi}_\sigma}\left[\sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\delta_{\beta\nu}\left(\partial_r\xi\frac{\partial\chi}{\partial r} + \partial_z\xi\frac{\partial\chi}{\partial z}\right) + \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}\delta_{\alpha\sigma}\ell'_\beta(\eta_\nu)\left(\partial_r\eta\frac{\partial\chi}{\partial r} + \partial_z\eta\frac{\partial\chi}{\partial z}\right)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}\left[\omega_\beta\sum_{\sigma=0}^{N}\overline{\omega}_\sigma\frac{\left|\mathcal{J}_e^{\overline{\sigma}\beta}\right|}{\rho^{\overline{\sigma}\beta}}\frac{r^{\overline{\sigma}\beta}}{1+\overline{\xi}_\sigma}\overline{\ell}'_\alpha(\overline{\xi}_\sigma)\left(\partial_r\xi\frac{\partial\chi}{\partial r} + \partial_z\xi\frac{\partial\chi}{\partial z}\right) + \overline{\omega}_\alpha\sum_{\nu=0}^{N}\omega_\nu\frac{\left|\mathcal{J}_e^{\overline{\alpha}\nu}\right|}{\rho^{\overline{\alpha}\nu}}\frac{r^{\overline{\alpha}\nu}}{1+\overline{\xi}_\alpha}\ell'_\beta(\eta_\nu)\left(\partial_r\eta\frac{\partial\chi}{\partial r} + \partial_z\eta\frac{\partial\chi}{\partial z}\right)\right]\\
&= \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}B^{\overline{\alpha}\beta}
\end{aligned}$$

$$\begin{aligned}
\int_\Lambda w\frac{1}{\lambda}\ddot{\chi}\ |\mathcal{J}_e|\, r\mathrm{d}\xi\mathrm{d}\eta &= \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}\omega_\alpha\omega_\beta\frac{1}{\lambda^{\overline{\alpha}\beta}}\left|\mathcal{J}_e^{\overline{\alpha}\beta}\right|\frac{r^{\overline{\alpha}\beta}}{1+\overline{\xi}_\alpha}\ddot{\chi}^{\overline{\alpha}\beta}\\
&= \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}A^{\overline{\alpha}\beta}
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1} w\boldsymbol{n}\cdot\dot{\boldsymbol{u}}\,\frac{dr}{d\xi} r\mathrm{d}\xi &= \sum_{\alpha=0}^{N} w^{\overline{\alpha}N}\overline{\omega}_\alpha \left.\frac{dr}{d\xi}\right|^{\overline{\alpha}}\frac{r^{\overline{\alpha}}}{1+\overline{\xi}_\alpha}\left(n_r^{\overline{\alpha}}\dot{u}_r^{\overline{\alpha}} + n_z^{\overline{\alpha}}\dot{u}_z^{\overline{\alpha}}\right)\\
&= \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}\overline{\omega}_\alpha \left.\frac{dx}{d\xi}\right|^{\overline{\alpha}}\frac{r^{\overline{\alpha}}}{1+\overline{\xi}_\alpha}\left(n_r^{\overline{\alpha}}\dot{u}_r^{\overline{\alpha}} + n_z^{\overline{\alpha}}\dot{u}_z^{\overline{\alpha}}\right)\delta_{\beta N}\\
&= \sum_{\alpha,\beta=0}^{N} w^{\overline{\alpha}\beta}C^{\overline{\alpha}\beta}
\end{aligned}$$

With $\dfrac{r^{0\beta}}{1+\overline{\xi}_0} \equiv \dfrac{\partial r}{\partial\xi}(\overline{\xi}_0,\eta_\beta)$

There is a point if we need to calculate the pressure on the axis : $P = \lambda\left[\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r}\right] = \lambda\left[\frac{\partial\rho^{-1}\frac{\partial\chi}{\partial r}}{\partial r} + \frac{\partial\rho^{-1}\frac{\partial\chi}{\partial z}}{\partial z} + \frac{1}{\rho r}\frac{\partial\chi}{\partial r}\right] = \lambda\left[\left(\frac{\partial\rho^{-1}}{\partial r} + \frac{1}{\rho r}\right)\frac{\partial\chi}{\partial r} + \frac{1}{\rho}\frac{\partial^2\chi}{\partial r^2} + \right.$

But do we need it?