Carcione, Seismic modeling in viscoelastic media, Geophysics vol. 58, p. 110-120 (1993)

with infinite values at the low- and high-frequency limits. A combination of several relaxation mechanisms can model any quality factor function versus frequency where the fitting parameters are the relaxation times.

The theory, developed in Carcione et al. (1988c), circumvents the convolutional relation between the stress and strain tensors by the introduction of the memory variables. In the 3-D case, the resulting wave equation is solved for the displacement field, one memory variable for each dissipation mechanism related to the dilatational wave, and five memory variables for each mechanism related to the shear wave. In the 2-D case, two memory variables are used for each shear relaxation mechanism. The problem is solved in the time domain by a new time integration method based on an optimum polynomial interpolation of the evolution operator (Tal-Ezer et al., 1990). This method is especially designed to solve wave propagation in linear viscoelastic media and greatly improves the spectral technique used in Carcione et al. (1988c). The spatial derivative terms are computed by means of the Fourier pseudospectral method. Similar approaches based on finite difference in time (a Taylor expansion of the evolution operator) and space were given in Day and Minster (1984), and in Emmerich and Korn (1987) for the viscoacoustic wave equation.

In the earth, there are cases where the impedance contrast is very weak but the contrast in attenuation is significant, i.e., if one of the materials is very unconsolidated or has fluid-filled pores. To simulate this situation, I present an example of waves impinging on a plane interface separating an elastic material of a viscoelastic medium with similar elastic moduli and density but different quality factors (Q interface). A second example displays a common shot time section in a medium that includes highly anelastic lens-shaped bodies. Then, I compute the seismic response to a single shot of a complex structure containing a gas cap in an anticlinal fold, a typical trap in exploration geophysics. Finally, I consider examples of wave simulation in 3-D homogeneous and inhomogeneous structures. The algorithm is tested against the analytical solution, which is based on a 3-D viscoelastic Green's function derived from the correspondence principle.

EQUATION OF MOTION

The time-domain equation of motion of an $n-D$ viscoelastic medium is formed with the following equations (Carcione, 1987; Carcione et al., 1988c):

1) The linearized equations of momentum conservation:

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i, \qquad i = 1, \ldots, n,$$

where $\mathbf{x} = (x_1, x_2, x_3) \equiv (x, y, z)$ is the position vector, $\sigma_{ij}(\mathbf{x}, t)$ are the stress components, $u_i(\mathbf{x}, t)$ are the displacements, $\rho(\mathbf{x})$ denotes the density, and $f_i(\mathbf{x}, t)$ are the body forces, t being the time variable. Repeated indices imply summation and a dot above a variable indicates time differentiation.

2) The stress-strain relations:

$$\sigma_{xx} = (\lambda_u + 2\mu_u)\frac{\partial u_x}{\partial x} + \lambda_u\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \left(\lambda_u + \frac{2}{n}\mu_u\right)\sum_{\ell=1}^{L_1} e_{1\ell} + 2\mu_u \sum_{\ell=1}^{L_2} e_{11\ell}, \tag{2a}$$

$$\sigma_{yy} = (\lambda_u + 2\mu_u)\frac{\partial u_y}{\partial y} + \lambda_u\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + \left(\lambda_u + \frac{2}{n}\mu_u\right)\sum_{\ell=1}^{L_1} e_{1\ell} + 2\mu_u \sum_{\ell=1}^{L_2} e_{22\ell}, \tag{2b}$$

$$\sigma_{zz} = (\lambda_u + 2\mu_u)\frac{\partial u_z}{\partial z} + \lambda_u\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \left(\lambda_u + \frac{2\mu_u}{n}\right)\sum_{\ell=1}^{L_1} e_{1\ell} - \frac{2}{n}\mu_u \sum_{\ell=1}^{L_2} (e_{11\ell} + e_{22\ell}), \tag{2c}$$

ceci OK car $+\sum \mu_r \varepsilon_{33\ell}$ mais $e_{11} + e_{22} + e_{33} = 0$ donc $e_{33} = -(e_{11} + e_{22})$

$$\sigma_{xy} = \mu_u\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \mu_u \sum_{\ell=1}^{L_2} e_{12\ell}, \tag{2d}$$

$$\sigma_{xz} = \mu_u\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \mu_u \sum_{\ell=1}^{L_2} e_{13\ell}, \tag{2e}$$

$$\sigma_{yz} = \mu_u\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \mu_u \sum_{\ell=1}^{L_2} e_{23\ell}, \tag{2f}$$

where

$$\lambda_u = \left(\lambda_r + \frac{2}{n}\mu_r\right) M_{u1} - \frac{2}{n}\mu_r M_{u2}, \tag{3a}$$

and

$$\mu_u = \mu_r M_{u2}, \tag{3b}$$

are the unrelaxed or high-frequency Lame constants, with λ_r and μ_r the relaxed or low-frequency Lame constants. $M_{u\nu}$, $\nu = 1, 2$ are relaxation functions evaluated at $t = 0$, with $\nu = 1$, the dilatational mode, and $\nu = 2$, the shear mode. For the general standard linear solid rheology, they are given by

$$M_{u\nu} = 1 - \frac{1}{L}\sum_{\ell=1}^{L_\nu}\left(1 - \frac{\tau_{\varepsilon\ell}^{(\nu)}}{\tau_{\sigma\ell}^{(\nu)}}\right) = +\frac{1}{L}\sum_{\ell=1}^{L}\frac{\tau_{\varepsilon\ell}}{\tau_{\sigma\ell}}, \qquad \nu = 1, 2, \tag{4}$$

Carcione 2007 eq 2.199

with $\tau_{\sigma\ell}^{(\nu)}$ and $\tau_{\varepsilon\ell}^{(\nu)}$ material relaxation times. The quantities $e_{1\ell}(\mathbf{x}, t)$ are memory variables related to the L_1 mechanisms which describe the anelastic characteristics of the dilatational wave, and $e_{11\ell}(\mathbf{x}, t)$, $e_{22\ell}(\mathbf{x}, t)$, $e_{12\ell}(\mathbf{x}, t)$, $e_{13\ell}(\mathbf{x}, t)$, and $e_{23\ell}(\mathbf{x}, t)$ are memory variables related to the L_2 mechanisms of the quasi-shear wave.

Remark from Dimitri Komatitsch : equation (2c) for sigma_zz is not correct, in 2D it should be:
sigma_zz = lambdaplus2mu_unrelaxed*duz_dz + lambda_unrelaxed*dux_dx +
(lambda_relaxed + mu_relaxed) * sum_of_e1 - 2 * mu_relaxed * sum_of_e11

In the 2D case, equation (2c) of Carcione 1993 should be:

$$\sigma_{zz} = (\lambda_u + 2\mu_u)\frac{\partial u_z}{\partial z} + \lambda_u \frac{\partial u_x}{\partial x} + (\lambda_r + \mu_r)\sum e_1 - 2\mu_r \sum e_{11} \tag{1}$$

i.e., n in equation (2c) is not at the right place.

trace de ε

3) The memory variable first-order equations in time:

$$\dot{e}_{1\ell} = \Theta\phi_{1\ell} - \frac{e_{1\ell}}{\tau_{\sigma\ell}^{(1)}}, \qquad \ell = 1, \ldots, L_1, \tag{5a}$$

e_{ij}

$$\dot{e}_{11\ell} = \left(\frac{\partial u_x}{\partial x} - \frac{\Theta}{n}\right)\phi_{2\ell} - \frac{e_{11\ell}}{\tau_{\sigma\ell}^{(2)}},$$

d_{ij} = deviatoric $_{ij} = \varepsilon_{ij} - \frac{1}{n}\theta\delta_{ij}$ (eq 3.6 de 2nd edition)

$$\ell = 1, \ldots, L_2, \tag{5b}$$

$$\dot{e}_{22\ell} = \left(\frac{\partial u_y}{\partial y} - \frac{\Theta}{n}\right)\phi_{2\ell} - \frac{e_{22\ell}}{\tau_{\sigma\ell}^{(2)}},$$

$$\ell = 1, \ldots, L_2, \tag{5c}$$

$$\dot{e}_{12\ell} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)\phi_{2\ell} - \frac{e_{12\ell}}{\tau_{\sigma\ell}^{(2)}},$$

$$\ell = 1, \ldots, L_2, \tag{5d}$$

$$\dot{e}_{13\ell} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\phi_{2\ell} - \frac{e_{13\ell}}{\tau_{\sigma\ell}^{(2)}},$$

$$\ell = 1, \ldots, L_2, \tag{5e}$$

$$\dot{e}_{23\ell} = \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\phi_{2\ell} - \frac{e_{23\ell}}{\tau_{\sigma\ell}^{(2)}},$$

$$\ell = 1, \ldots, L_2, \tag{5f}$$

where

$$\Theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \quad = \text{trace de } \varepsilon \tag{6}$$

is the dilatation field, and

$$\phi_{\nu\ell} = \frac{1}{\tau_{\sigma\ell}^{(\nu)}}\left(1 - \frac{\tau_{\varepsilon\ell}^{(\nu)}}{\tau_{\sigma\ell}^{(\nu)}}\right) \times \frac{1}{\sum_{\ell=1}^{L} \frac{\tau_{\varepsilon\ell}}{\tau_{\sigma\ell}}}, \qquad \nu = 1, 2, \tag{7}$$

eq 2.330 de 3rd edition, idem que eq 2.293 du 2nd edition. Aussi equations 3.192 et 3.193

are the response function components evaluated at $t = 0$. The low-frequency or elastic limit ins obtained when $\tau_{\varepsilon\ell}^{(\nu)} \to \tau_{\sigma\ell}^{(\nu)}$, $\forall\ell$; thus, $M_{uv} \to 1$ and $\phi_{\nu\ell} \to 0$, and the memory variables vanish. On the other hand, at the high-frequency limit the system also behaves elastically, corresponding to the instantaneous response. As can be seen from the stress-strain equations, the mean stress depends only on the parameters and memory variables with index $\nu = 1$ which involve dilatational dissipation mechanisms. Similarly, the deviatoric stress components depend on the parameters and memory variables with index $\nu = 2$, involving shear mechanisms. The 3-D case is obtained with $n = 3$, the 2-D case with $n = 2$ and, say, $\partial/\partial y\,[\,\cdot\,] = 0$, $\sigma_{yy} = \sigma_{xy} = \sigma_{zy} = 0$, and $e_{22\ell} = e_{12\ell} = e_{23\ell} = 0$. The elastic case is obtained by taking $\tau_{\varepsilon\ell} = \tau_{\sigma\ell}$, $\forall\ell$ (low-frequency limit), or by zeroing the memory variables and taking the unrelaxed Lame constants as the elastic Lame constants (high-frequency limit). Viscoacoustic wave propagation is simply obtained by setting $\mu_r = 0$; the resulting equation can be written in terms of the dilatation 0, or in terms of the pressure $p = -\sigma_{xx} = -\sigma_{yy} = -\sigma_{zz}$. The system of equations (1), (2a, f) and (5a, f) is solved for the displacement field and memory variables by using a new spectral algorithm as a time marching scheme (Tal-Ezer et al., 1990). To balance time integration and spatial accuracies, the spatial derivatives are computed by means of the Fourier pseudospectral method.

et c'est la trace de 2ε

e = (eq 1.11 de 2nd edition)

eq 1.3 de 2nd editions: $e_{ij} = 2\varepsilon_{ij}$

eq 1.2 de 2nd edition $e_{ij} = 2\varepsilon_{ij} = \partial_j u_i + \partial_i u_j$

2-D WAVE PROPAGATION

Q interface

This example considers wave propagation across an interface separating media with different quality factors but similar elastic moduli. The left half-space is elastic and the right half-space is viscoelastic (see Figure 4). The viscoelastic medium has almost constant quality factors in the seismic exploration band, as can be seen in Figure 1 where the bulk (Q_k), compressional (Q_P), and shear (Q_S) quality factors are plotted. Relaxation times are $\tau_{\varepsilon 1}^{(1)} = 0.0334$ s, $\tau_{\sigma 1}^{(1)} = 0.0303$ s, $\tau_{\varepsilon 1}^{(2)} = 0.0352$ s, $\tau_{\sigma 1}^{(2)} = 0.0287$ s, $\tau_{\varepsilon 2}^{(1)} = 0.0028$ s, $\tau_{\sigma 2}^{(1)} = 0.0025$ s, $\tau_{\varepsilon 2}^{(2)} = 0.0029$ s, and $\tau_{\sigma 2}^{(2)} = 0.0024$ s. The group and phase velocities are displayed in Figure 2a and 2b for P- and S-waves, respectively; they indicate strong wave dispersion. Expressions for the quality factors and wave velocities in viscoelastic media can be found in Carcione et al. (1988c).

The compressional-and shear-wave velocities of the elastic medium are chosen in such a way as to minimize the normal *PP* and *SS* reflection coefficients at the central frequency of the source (25 Hz) whose spectra is plotted in Figure 1 with a dotted line. As stated by the correspondence principle (Bland, 1960), the reflection and transmission coefficients for an interface in attenuating media may be obtained from their analogues in elastic media by merely substituting the elastic velocities for the complex anelastic velocities. Assuming constant density in the whole space, the normal incidence reflection coefficient simply becomes $R(x) = (V - x)/(V + x)$, where V is the P-wave (S-wave)

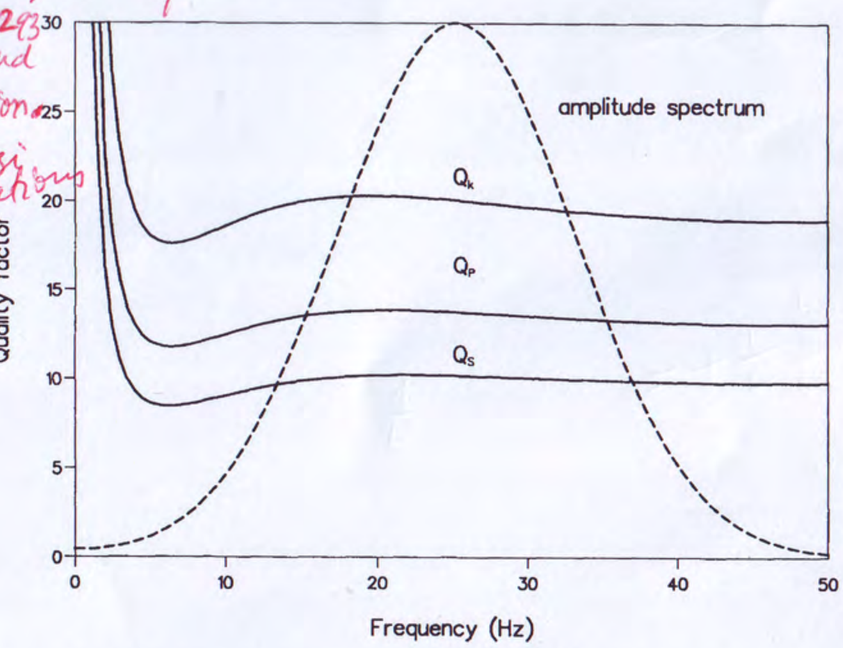

FIG. 1. P-wave, S-wave, and bulk quality factors for the viscoelastic medium of the Q interface. The dashed line represents the amplitude spectrum of the source.