CHAPTER 7

SOURCE FUNCTIONS IN INFINITE MEDIA

7.1. General.

The material in this section is intimately linked with two applications. The first is primarily mathematical and is the development of representation theorems in which the elastic wave motion is expressed in terms of volume and surface integrals involving the source function and its derivatives. This will be discussed in detail in Chapter 8.

The second application is more physical in nature, in that we will try to find a mathematical model for the source mechanism of earthquakes. Postulated mechanisms are force systems, stress differences, and the collapse or expansion of cavities among others. A discussion of source mechanisms for earthquakes will be put off until Chapter 28, however.

The material in this Chapter is among the most complicated in the book, rather more than one would like. However, even more detail is necessary to give a rigorous development. Both Achenbach (1973, Chapter 3) and Eringen and Suhubi (1975, Chapter 5) give a more detailed description than we do here. They also develop source functions from several other points of view. These other formulations may prove useful at times and the reader should be aware of them.

7.2. One Dimension.

One cannot usefully speak of sources in only one dimension. However, the wavefields appropriate here are plane waves, and they may be thought of as the limiting form of spherical or cylindrical waves as the source region is removed to very great distance. In this case, the near-field terms have long since decayed. Consequently, plane waves are very special, and the resulting phenomena from their use do not fully represent the physical situation. The relationships derived using plane waves will most likely remain true, but quite often are interfered with or overridden by additional effects when the source function remains at a distance close to the region of interest.

The simplification afforded by plane wave, in spite of the above obstacles, makes them useful in obtaining partial answers. This is especially true in problems where elastic waves are scattered from surface or volume irregularities. In some scattering problems, we will find two plane wave expansions of special importance. The first expands a plane wave into cylindrical coordinates. In this case the plane wave is going in the direction of **k** with velocity $\underline{v}$ (where $k = \omega/v$), and the x-axis is chosen to lie in the direction of the projection of **k** onto the horizontal plane.

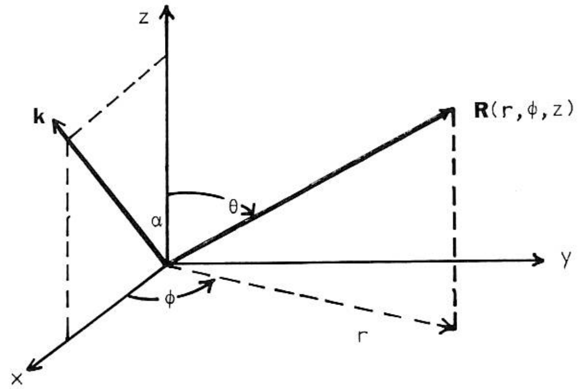

Figure 7-1. Plane Wave geometry.

For example, a plane P-wave can be represented in cartesian coordinates (where $x = r\cos\phi$, $y = r\sin\phi$), as

$$\Phi = e^{ik_o\left[r\sin\alpha\cos\phi + z\cos\alpha\right] - i\omega t} \tag{7-1}$$

$$= \left[\sum_{n=-\infty}^{\infty} i^n J_n(k_o r\sin\alpha)e^{in\phi}\right] e^{ik_o z\cos\alpha - i\omega t},$$

(Stratton, 1941, page 372). Likewise, for a plane P-wave going in the z-direction, we have the following expansion in spherical coordinates (ibid. p. 409):

$$\Phi = e^{i(k_o R\cos\theta - \omega t)} = e^{-i\omega t}\sum_{n=0}^{\infty} i^n (2n+1)\, j_n(k_o R)\, P_n(\cos\theta). \tag{7-2}$$

7.3. Two-Dimensional Point Sources.

Two-dimensional problems arise in two ways. The first is when you have a line-source in three dimensions and the axial direction may be ignored. In elastodynamics, SH motion would be parallel to the axis of the line-source and the greater part of the P-SV motion would lie in a plane perpendicular to that axis. This P-SV motion would then be in a state of plane strain (Sokolnikoff, 1956, p. 250). The second arises when elastic waves are propagating in thin sheets of material (not flexural waves). Here the stresses normal to the faces of the sheets vanish on the faces and are very small on the interior. The stress vector lies largely in the plane of the sheet. Such motion is in a state of plane stress (ibid, p. 253). For motion in a thin sheet, SH and SV are defined with respect to the edges, not the faces.

Scalar Media — Time Harmonic Source. We wish to find a solution of the inhomogeneous scalar wave equation;

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2} = -2\pi\delta(x)\delta(z)e^{-i\omega t} = -\frac{\delta(r)}{r}e^{-i\omega t}, \tag{7-3}$$

SPECFEM2D implements the plane strain P-SV approximation, and thus the source is a line source in the third direction (orthogonal to the plane).

where the factor -2π is inserted for normalization, $r = (x^2 + z^2)^{\frac{1}{2}}$, $\delta(x)$, $\delta(z)$ are Dirac δ-functions, and the time variation of the source is given by the factor $e^{-i\omega t}$. If we set

$$\Phi(x,z,t) = \bar{\Phi}(x,z)e^{-i\omega t}, \tag{7-4}$$

then (7-3) becomes

$$\frac{\partial^2\bar{\Phi}}{\partial x^2} + \frac{\partial^2\bar{\Phi}}{\partial z^2} + k_o^2\bar{\Phi} = -2\pi\delta(x)\delta(z), \tag{7-5}$$

where $k_o = \omega/v$. We then take a Fourier transform with respect to $\underline{x}$, i.e.,

$$\bar{\bar{\Phi}}(k,z) = \int_{-\infty}^{\infty} \bar{\Phi}(x,z)e^{-ikx}dx \quad ; \quad \bar{\Phi} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\bar{\Phi}}(k,z)e^{ikx}dk; \tag{7-6}$$

giving

$$-k^2\bar{\bar{\Phi}} + \frac{\partial^2\bar{\bar{\Phi}}}{\partial z^2} + k_o^2\bar{\bar{\Phi}} = -2\pi\delta(z) \quad . \tag{7-7}$$

Lastly, we take a two-sided Laplace transform with respect to $\underline{z}$, i.e.,

$$\bar{\bar{\bar{\Phi}}}(k,p) = \int_{-\infty}^{\infty} \bar{\bar{\Phi}}(k,z)e^{-pz}dz; \quad \bar{\bar{\Phi}} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\bar{\bar{\bar{\Phi}}}(k,p)e^{pz}dp, \tag{7-8}$$

giving

$$\{p^2 - (k^2 - k_o^2)\}\,\bar{\bar{\bar{\Phi}}} = -2\pi. \tag{7-9}$$

Inverting with respect to $\underline{p}$ gives

$$\bar{\bar{\Phi}} = \frac{\pi e^{-(k^2-k_o^2)^{\frac{1}{2}}|z|}}{(k^2-k_o^2)^{\frac{1}{2}}} \quad . \tag{7-10}$$

Inverting with respect to $\underline{k}$, we obtain

$$\bar{\Phi}(x,z) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-(k^2-k_o^2)^{\frac{1}{2}}|z|}e^{ikx}dk}{(k^2-k_o^2)^{\frac{1}{2}}} = \int_{0}^{\infty}\frac{e^{-(k^2-k_o^2)^{\frac{1}{2}}|z|}\operatorname{Cos} kx\,dk}{(k^2-k_o^2)^{\frac{1}{2}}}$$

$$= \frac{i\pi}{2}H_o^{(1)}(k_or), \tag{7-11}$$

where $H_o^{(1)}(k_or)$ is the Hankel function of the first kind (Morse and Feshbach, 1953, page 823) with properties

$$H_o^{(1)}(k_or) = J_o(k_or) + iN_o(k_or) \xrightarrow[r\to\infty]{} \frac{2}{(\pi k_or)^{\frac{1}{2}}}e^{i(k_or-\pi/4)} \quad . \tag{7-12}$$

On the other hand, we could have reversed the Fourier and Laplace transforms obtaining

$$\frac{\partial^2\bar{\bar{\Phi}}}{\partial x^2} - k^2\bar{\bar{\Phi}} + k_o^2\bar{\bar{\Phi}} = -2\pi\delta(x), \tag{7-13}$$

and

$$\{p^2 - (k^2 - k_o^2)\}\,\bar{\bar{\bar{\Phi}}} = -2\pi. \tag{7-14}$$

Then inverting with respect to $\underline{p}$ we have

$$\bar{\bar{\Phi}} = \frac{\pi e^{-(k^2-k_o^2)^{\frac{1}{2}}|x|}}{(k^2-k_o^2)^{\frac{1}{2}}} \quad . \tag{7-15}$$

Finally inverting with respect to $\underline{k}$, we get

$$\bar{\Phi}(x,z) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-(k^2-k_o^2)^{\frac{1}{2}}|x|}e^{ikz}dk}{(k^2-k_o^2)^{\frac{1}{2}}} = \int_{0}^{\infty}\frac{\operatorname{Cos} kz\, e^{-(k^2-k_o^2)^{\frac{1}{2}}|x|}dk}{(k^2-k_o^2)^{\frac{1}{2}}}. \tag{7-16}$$

It should be pointed out that (7-11) and (7-16) are just two plane-wave expansions for the same function, even though they look quite different.

Lastly, adding in the time term, we have

$$\Phi(x,z,t) = \frac{i\pi}{2}H_o^{(1)}(k_or)e^{-i\omega t} \quad . \tag{7-17}$$

Scalar Media — Impulsive Source. In this case, we wish to find a solution of the inhomogeneous scalar wave equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2} = -2\pi\delta(x)\delta(z)\delta(t) = -\frac{\delta(r)\delta(t)}{r} \quad . \tag{7-18}$$

Taking a Laplace transform with respect to time, we obtain

$$\frac{\partial^2\bar{\Phi}}{\partial x^2} + \frac{\partial^2\bar{\Phi}}{\partial z^2} - \frac{s^2}{v^2}\bar{\Phi} = -2\pi\delta(x)\delta(z), \tag{7-19}$$

where

$$\bar{\Phi} = \int_{0}^{\infty}\Phi(x,z,t)e^{-st}dt. \tag{7-20}$$

In order to simplify what is to come, we shall take a slightly modified Fourier transform with respect to $\underline{x}$, i.e.,

$$\bar{\bar{\Phi}}(q,z,s) = \int_{-\infty}^{\infty}\bar{\Phi}(x,z,s)e^{-isqx/v}dx \; , \tag{7-21}$$

with inverse

$$\bar{\Phi} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\bar{\Phi}}(q,z,s)e^{isqx/v}d(sq/v) \; . \tag{7-22}$$

This gives

$$-(sq/v)^2\bar{\bar{\Phi}} + \partial^2\bar{\bar{\Phi}}/\partial z^2 - (s/v)^2\bar{\bar{\Phi}} = -2\pi\delta(z). \tag{7-23}$$

Finally, taking a two-sided Laplace transform with respect to $\underline{z}$, we have

$$\{p^2 - (s/v)^2(q^2+1)\}\,\bar{\bar{\bar{\Phi}}} = -2\pi \; , \tag{7-24}$$

where

$$\bar{\bar{\bar{\Phi}}} = \int_{-\infty}^{\infty}\bar{\bar{\Phi}}(q,z,s)e^{-pz}dz \quad .$$

Inverting with respect to $\underline{p}$, we have

$$\bar{\bar{\Phi}} = (\pi v/s)e^{-(s/v)(q^2+1)^{\frac{1}{2}}|z|}(q^2+1)^{-\frac{1}{2}} \quad . \qquad (7\text{-}25)$$

Inverting with respect to $\underline{q}$, we obtain

$$\bar{\Phi}(x,z,s) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-(s/v)(q^2+1)^{\frac{1}{2}}|z|}(q^2+1)^{-\frac{1}{2}}e^{isqx/v}\,dq = K_o(sr/v). \qquad (7\text{-}26)$$

The expression (7-26) is just the integral representation of the Macdonald function $K_o(sr/v)$. Now the Macdonald and Hankel functions of zero order are related by

$$K_o(x) \equiv (i\pi/2)H_o^{(1)}(ix) \xrightarrow[x\to\infty]{} (\pi/2x)^{\frac{1}{2}}e^{-x}, \qquad (7\text{-}27)$$

(Stratton, 1941, pp. 390-391). Then using the well-known correspondence, $s \longleftrightarrow -i\omega$, relating Laplace and Fourier transforms, one can show that (7-26) corresponds to (7-11) and (7-16).

To return to the time domain, we would then have to write

$$\Phi = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \bar{\Phi}(x,z,s)e^{st}\,ds, \qquad (7\text{-}28)$$

i.e., Φ would be represented by a double integral over the two variables $\underline{q}$ and $\underline{s}$, or one integral over $\underline{s}$ with the kernel $K_o(sr/v)$. How to evaluate such an integral is not immediately obvious; however, a special technique allows us to obtain the solution to (7-18) as well as more complex problems. This method is known as the <u>Cagniard</u> - <u>deHoop</u> <u>Transformation</u>. This technique (deHoop, 1960) involves the following change of variable:

$$\cos\theta\,(q^2+1)^{\frac{1}{2}} - iq\sin\theta = \tau = vt/r, \qquad (7\text{-}29)$$

where $r\cos\theta = z$, $r\sin\theta = x$, and τ is the reduced time variable.

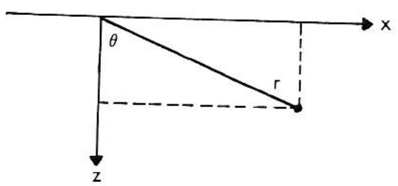

Figure 7-2. Two-dimensional coordinate systems. Note that the $r - \theta$ system is not standard cylindrical coordinates.

The inverse of this transformation is

$$q(\tau) = i\tau\sin\theta + \cos\theta\,(\tau^2-1)^{\frac{1}{2}}; \qquad (7\text{-}30)$$

and for later use we also compute

$$\frac{dq}{d\tau} = i\sin\theta + \frac{\tau\cos\theta}{(\tau^2-1)^{\frac{1}{2}}} = \frac{(q^2+1)^{\frac{1}{2}}}{(\tau^2-1)^{\frac{1}{2}}} \quad , \qquad (7\text{-}31)$$

the last expression coming from solving for $(q^2+1)^{\frac{1}{2}}$ from (7-29) while substituting (7-30) for $\underline{q}$. Taking account of the symmetry of the real and imaginary parts of $\exp\{isqx/v\}$, we can write (7-26) as

$$\bar{\Phi} = \mathrm{Re}\left[\int_o^{\infty}\frac{e^{-(s/v)(q^2+1)^{\frac{1}{2}}|z| + isqx/v}}{(q^2+1)^{\frac{1}{2}}}\,dq\right] \quad , \qquad (7\text{-}32)$$

where Re $\left[\quad\right]$ stands for the real part of the bracketed expression. We can now write this in terms of the new variable "τ" and obtain

$$\bar{\Phi}(x,z,s) = \mathrm{Re}\left[\int_?^? \frac{e^{-st}}{(q^2+1)^{\frac{1}{2}}}\,\frac{dq}{d\tau}\,\frac{v}{r}\,dt\right] \qquad (7\text{-}33)$$

$$= \mathrm{Re}\left[\int_?^? \frac{e^{-st}}{(\tau^2-1)^{\frac{1}{2}}}\,\frac{v}{r}\,dt\right] \quad ,$$

where we have used (7-31). Equation (7-33) can now be recognized as the Laplace Transform of the function

$$\mathrm{Re}\left[\frac{1}{(\tau^2-1)^{\frac{1}{2}}}\,\frac{v}{r}\right]$$

looked at as a function of the time variable "τ". However, we have to look at a few details before we can say that this identification is valid and place proper limits on the integral. First of all, we want to look at the path $\underline{q}$ takes as we let the variable τ run from 0 to ∞. For $\tau = 0$, we have that $q = -i\cos\theta$ where the sign has been chosen in (7-30) to satisfy (7-29). The variable $\underline{q}$ then moves up the imaginary axis to $q = i\sin\theta$, and then branches out into the first quadrant along a hyperbola as defined by (7-30) and along an asymptote at an angle θ as in Figure 7-3(a). Inasmuch as the singularities of (7-32) are branch points at $q = \pm i$, we see that the original path can be deformed into the dashed line path as in Figure 7-3(b). However, on the vertical segment from 0 to $i\sin\theta$ we see that the integrand of (7-32) has no real part. Consequently the limits on (7-33) may be written

$$\bar{\Phi}(x,z,s) = \mathrm{Re}\left[\int_{r/v}^{\infty}\frac{e^{-st}}{(\tau^2-1)^{\frac{1}{2}}}\,\frac{v}{r}\,dt\right] \quad . \qquad (7\text{-}34)$$

By inspection we have that

$$\Phi = \frac{1}{(t^2-r^2/v^2)^{\frac{1}{2}}}H(t-r/v) \quad , \qquad (7\text{-}35)$$

where H is the Heaviside Unit Step Function defined by

$$\begin{aligned} H(x) &= 1, \quad x > 0 \\ &= \tfrac{1}{2}, \quad x = 0 \\ &= 0, \quad x < 0 \,. \end{aligned} \qquad (7\text{-}36)$$

There is a sharp wavefront associated with the response to a delta-function source, but in two dimensions we also have a tail associated with the waveform in contrast to the delta-function which has zero width.

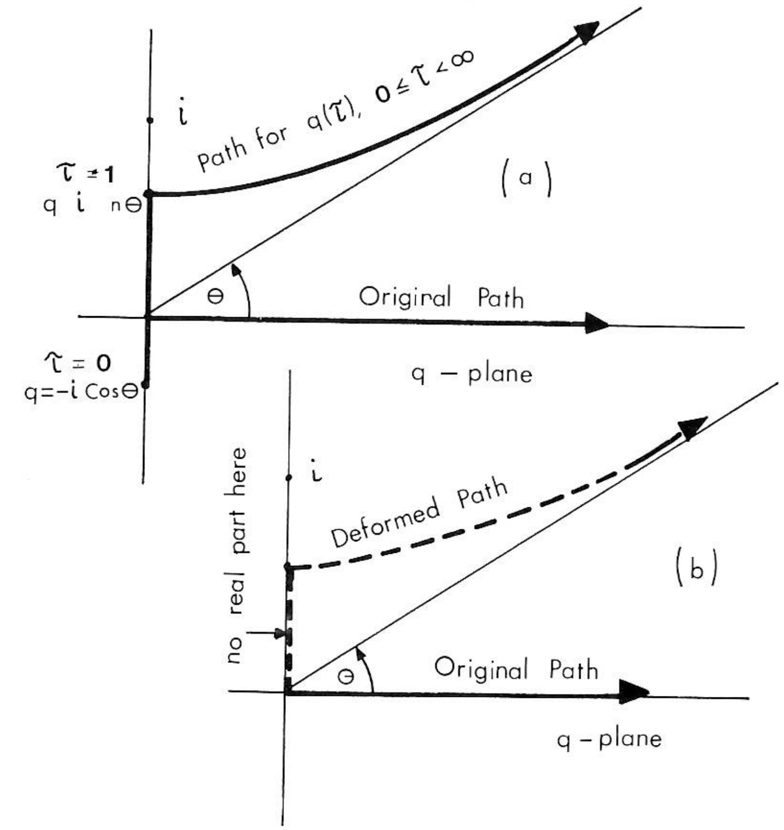

Figure 7-3(a). The relationship between the original path of integration in (7-32) and the path which $\underline{q}$ takes as $\underline{\tau}$ varies between zero and infinity.

(b). The relationship between the original path and the deformed path in (7-33) which leads to (7-34). The deformed path is sometimes known as the Cagniard path.

Some final remarks on the Cagniard-deHoop technique. It is necessary to be able to invert the transformation (7-29) in analytical form to obtain a closed form of the solution. Secondly, it is necessary that the Laplace transform variable $\underline{s}$ not appear in the integrand of expressions such as (7-26) except in the exponential terms or as a multiplicative factor of the whole integrand. That is, the integrand (save the exponential factor) must be homogeneous in $\underline{s}$. This occurs rather naturally in most elastodynamic problems without any loss mechanism. However, if a dissipative term appears in (7-18), for example, one would have a different exponential expression after following through the analysis to an equivalent of (7-26). One would also have the variable $\underline{s}$ appearing in the integrand. The path on which $\underline{\tau}$ was real could still be found, but the time function could not be separated out from the integrand as we

did in going from (7-34) to (7-35). This means that the power of the Cagniard-deHoop technique is lost for wave-propagation problems in lossy media. However, the effect of dissipation can be determined from plane-wave problems where solutions are possible.

<u>Elastic Media — Line Source of Dilatation</u>. Here Φ can be any of the scalar source functions discussed above. Then we have

$$\mathbf{u}^P = \nabla\Phi = \mathbf{e}_r \frac{\partial\Phi}{\partial r} . \qquad (7\text{-}37)$$

<u>Elastic Media — Line Source of Shear</u>. In this case, Ψ can be any one of the scalar sources discussed above. Then we have

$$\mathbf{u}^S = \nabla\times\mathbf{e}_r\Psi_r \equiv 0 . \qquad (7\text{-}38)$$

$$\mathbf{u}^S = \nabla\times\mathbf{e}_\theta\Psi_\theta = \mathbf{e}_z \frac{1}{r}\frac{\partial}{\partial r}(r\Psi_\theta) . \qquad \text{AXIAL SHEAR} \qquad (7\text{-}39)$$

$$\mathbf{u}^S = \nabla\times\mathbf{e}_z\Psi_z = \mathbf{e}_\theta \frac{\partial\Psi_z}{\partial r} . \qquad \text{TORSIONAL SHEAR} \qquad (7\text{-}40)$$

Here the form of (7-38) tells us that a radial component of the shear vector potential is the non-contributory one. Thus (7-37), (7-39), and (7-40) are our three independent potentials. For <u>axial shear</u> and <u>torsional shear</u> as defined above, coordinates are standard cylindrical.

<u>Elastic Media — Line Force</u>. The elastic equation of motion (5-9) for a line force acting in an infinite medium can be written as

$$\frac{\lambda + 2\mu}{\rho}\nabla\nabla\cdot\mathbf{w} - \frac{\mu}{\rho}\nabla\times\nabla\times\mathbf{w} - \frac{\partial^2\mathbf{w}}{\partial t^2} = -\mathbf{e}_z\frac{f(t)\delta(r)}{\rho 2\pi r} , \qquad (7\text{-}41)$$

where $r = (x^2 + z^2)^{\frac{1}{2}}$, and $\mathbf{e}_z$ is a unit vector in the direction of the force $f(t)$. We will return again to the special two-dimensional coordinate system of Figure 7-2. If we take a Fourier transform of both sides of this equation, we obtain

$$v_p^2\nabla\nabla\cdot\mathbf{W} - v_s^2\nabla\times\nabla\times\mathbf{W} + \omega^2\mathbf{W} = -\mathbf{e}_z\frac{F(\omega)\delta(r)}{\rho 2\pi r} , \qquad (7\text{-}42)$$

where

$$\mathbf{W}(x,z,\omega) = \int_{-\infty}^{\infty}\mathbf{w}(x,z,t)e^{i\omega t}\,dt \quad ; \quad F(\omega) = \int_{-\infty}^{\infty}f(t)e^{i\omega t}\,dt .$$

The method of solving (7-42) is definitely not intuitive. A number of <u>ad hoc</u> steps are introduced to achieve the solution. Although the computation to follow is not completely straightforward, it is relatively simple. I also think it lends to better understanding of the physics involved with the equivoluminal and irrotational parts of the solution. First of all we write

$$\mathbf{W} = \nabla(\nabla\cdot\mathbf{A}_p) - \nabla\times(\nabla\times\mathbf{A}_s) , \qquad (7\text{-}43)$$

i.e., we define a scalar potential $\nabla\cdot\mathbf{A}_p$ and a vector potential $\nabla\times\mathbf{A}_s$. In two dimensions we have that

$$-\mathbf{e}_z \frac{F\delta(r)}{\rho 2\pi r} = -\mathbf{e}_z \frac{F}{2\pi\rho}\nabla^2(\ln r) \tag{7-44}$$

$$= \nabla\nabla\cdot\left[-\mathbf{e}_z \frac{F \ln r}{2\pi\rho}\right] - \nabla\times\nabla\times\left[-\mathbf{e}_z \frac{F \ln r}{2\pi\rho}\right] ,$$

where we have used the vector identity

$$\nabla^2\mathbf{A} \equiv \nabla\nabla\cdot\mathbf{A} - \nabla\times\nabla\times\mathbf{A} .$$

That this is true may be seen by evaluating the following:

$$\frac{1}{2\pi}\int \nabla^2(\ln r)dS = \frac{1}{2\pi}\int \frac{\partial}{\partial r}(\ln r)dl = \frac{1}{2\pi}\int \frac{r d\theta}{r} = 1 = \int \frac{\delta(r)dS}{2\pi r}$$

per unit thickness of slab as in Figure 7-4. Substituting (7-43) and (7-44)

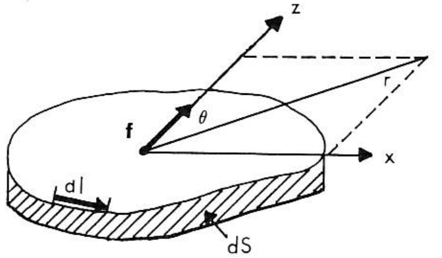

Figure 7-4. Two-dimensional geometry. The force **f** lies in the plane of the slab along the z-axis.

into (7-42), we obtain

$$v_p^2\nabla\nabla\cdot\left[\nabla\nabla\cdot\mathbf{A}_p + k_p^2\mathbf{A}_p + \mathbf{e}_z\frac{F\ln r}{2\pi\rho v_p^2}\right] + v_s^2\,\nabla\times\nabla\times\left[\nabla\times\nabla\times\mathbf{A}_s - k_s^2\mathbf{A}_s - \mathbf{e}_z\frac{F\ln r}{2\pi\rho v_s^2}\right] = 0.$$

We can add the term $-\nabla\times\nabla\times\mathbf{A}_p$ to the first bracketed portion and the term $-\nabla\nabla\cdot\mathbf{A}_s$ to the second in that they do not change the value of the whole expression. Adding these two terms, we then have

$$v_p^2\nabla\nabla\cdot\left[\nabla\nabla\cdot\mathbf{A}_p - \nabla\times\nabla\times\mathbf{A}_p + k_p^2\mathbf{A}_p + \mathbf{e}_z\frac{F\ln r}{2\pi\rho v_p^2}\right] \tag{7-45}$$

$$+ v_s^2\nabla\times\nabla\times\left[\nabla\times\nabla\times\mathbf{A}_s - \nabla\nabla\cdot\mathbf{A}_s - k_s^2\mathbf{A}_s - \mathbf{e}_z\frac{F\ln r}{2\pi\rho v_s^2}\right] = 0 .$$

We then have a solution if

$$\nabla^2\mathbf{A}_p + k_p^2\mathbf{A}_p = -\mathbf{e}_z\frac{F\ln r}{2\pi\rho v_p^2} , \tag{7-46}$$

and

$$\nabla^2\mathbf{A}_s + k_s^2\mathbf{A}_s = -\mathbf{e}_z\frac{F\ln r}{2\pi\rho v_s^2} . \tag{7-47}$$

We can then write $\mathbf{A}_p = \mathbf{e}_z A_p$ and $\mathbf{A}_s = \mathbf{e}_z A_s$, and we will still have a solution if

$$\frac{\partial^2 A_p}{\partial x^2} + \frac{\partial^2 A_p}{\partial z^2} + k_p^2 A_p = -\frac{F\ln r}{2\pi\rho v_p^2} , \tag{7-48}$$

and

$$\frac{\partial^2 A_s}{\partial x^2} + \frac{\partial^2 A_s}{\partial z^2} + k_s^2 A_s = -\frac{F\ln r}{2\pi\rho v_s^2} , \tag{7-49}$$

where we have written out the components of the Laplacian operator ∇^2. We have reduced the problem of (7-42) to two identical scalar equations, one for A_p and one for A_s. There are still some difficulties however.

We would like to use transform methods to solve (7-48) and (7-49), but the singular nature of $\ln(r)$ prevents us from doing this directly. However, we can go at it indirectly for we have seen that

$$\nabla^2(\ln r) = \delta(r)/r = 2\pi\delta(x)\delta(z).$$

Taking a Fourier transform with respect to $\underline{x}$, and a two-sided Laplace transform with respect to $\underline{z}$, we have

$$(p^2 - k^2)\,\overline{\overline{\ln r}} = 2\pi ,$$

or

$$\overline{\overline{\ln r}} = \frac{2\pi}{p^2 - k^2} .$$

We now take the same transform pair of the left-hand side of (7-48) and obtain $(k_p^2 + p^2 - k^2)A_p$. Combining, we have

$$\overline{\overline{A}}_p = -\frac{F}{\rho v_p^2}\frac{1}{p^2 - k^2}\frac{1}{k_p^2 + p^2 - k^2} . \tag{7-50}$$

Expanding, we have

$$\overline{\overline{A}}_p = -\frac{F}{\rho\omega^2}\left[\frac{1}{p^2 - k^2} - \frac{1}{k_p^2 + p^2 - k^2}\right] .$$

But we recognize the first term in brackets as the double transform of $\ln(r/2)$ and the second term as the double transform of $-iH_o^{(1)}(k_p r)/4$. Hence we can write

$$A_p = \frac{-F}{2\pi\rho\omega^2}\left[\ln r + \frac{i\pi}{2}H_o^{(1)}(k_p r)\right] . \tag{7-51}$$

Similarly

$$A_s = \frac{-F}{2\pi\rho\omega^2}\left[\ln r + \frac{i\pi}{2}H_o^{(1)}(k_s r)\right] . \tag{7-52}$$

Now

$$\nabla\times\nabla\times\mathbf{A}_s = \nabla\nabla\cdot\mathbf{A}_s - \nabla^2\mathbf{A}_s$$

$$= \nabla\nabla\cdot\mathbf{A}_s + k_s^2\mathbf{A}_s + \mathbf{e}_z\frac{F\ln r}{2\pi\rho v_s^2}$$

$$= \nabla\nabla\cdot\mathbf{A}_s - \mathbf{e}_z\frac{iF}{4\rho v_s^2}H_o^{(1)}(k_s r) ,$$

and from (7-43) we finally obtain

$$\mathbf{W} = \nabla\nabla\cdot(\mathbf{A}_p - \mathbf{A}_s) + \mathbf{e}_z \frac{iF}{4\rho v_s^2} H_o^{(1)}(k_s r) \tag{7-53}$$

$$= -\frac{iF}{4\rho\omega^2}\left\{ \nabla\nabla\cdot \left[\mathbf{e}_z H_o^{(1)}(k_p r) - \mathbf{e}_z H_o^{(1)}(k_s r)\right] - \mathbf{e}_z k_s^2 H_o^{(1)}(k_s r)\right\} .$$

Making the correspondence between Laplace and Fourier transforms, we see that the transformed solution for an impulsive source would be

$$\mathbf{W} = \frac{F}{2\pi\rho}\left\{\frac{1}{s^2}\nabla\nabla\cdot\left[\mathbf{e}_z K_o(\frac{sr}{v_p}) - \mathbf{e}_z K_o(\frac{sr}{v_s})\right] + \frac{1}{v_s^2}\mathbf{e}_z K_o(\frac{sr}{v_s})\right\} , \tag{7-54}$$

where F is the strength of the impulse in the z-direction.

This expression can be analyzed further. Taking the divergence, we have

$$\mathbf{W} = \frac{F}{2\pi\rho}\left\{\frac{1}{s^2}\nabla\frac{\partial}{\partial z}\left[K_o(\frac{sr}{v_p}) - K_o(\frac{sr}{v_s})\right] + \frac{1}{v_s^2}\mathbf{e}_z K_o(\frac{sr}{v_s})\right\}$$

$$= \frac{F}{2\pi\rho}\left\{\frac{1}{s^2}\nabla\left[\frac{s\,\mathrm{Cos}\theta}{v_p}K_o'(\frac{sr}{v_p}) - \frac{s\,\mathrm{Cos}\theta}{v_s}K_o'(\frac{sr}{v_s})\right] + \frac{1}{v_s^2}\mathbf{e}_z K_o(\frac{sr}{v_s})\right\}$$

$$= \frac{F}{2\pi\rho}\left\{\begin{array}{l} \mathbf{e}_r \mathrm{Cos}\theta\left[v_p^{-2}K_o''(\frac{sr}{v_p}) - v_s^{-2}K_o''(\frac{sr}{v_s}) + v_s^{-2}K_o(\frac{sr}{v_s})\right] \\ + \mathbf{e}_\theta \mathrm{Sin}\theta\left[-v_p^{-2}(\frac{v_p}{sr})K_o'(\frac{sr}{v_p}) + v_s^{-2}(\frac{v_s}{sr})K_o'(\frac{sr}{v_s}) - v_s^{-2}K_o(\frac{sr}{v_s})\right]\end{array}\right\} .$$

Here we have used

$$\partial K_o/\partial z = s v_p^{-1}\mathrm{Cos}\theta K_o'(sr/v_p),$$

where

$$\mathbf{e}_z = \mathbf{e}_r\mathrm{Cos}\theta - \mathbf{e}_\theta\mathrm{Sin}\theta .$$

Also we note that

$$K_o'(\xi) = -K_1(\xi) \quad ; \quad K_o''(\xi) = K_o(\xi) - \xi^{-1}K_1(\xi) .$$

Continuing to work with the expression for **W**, we obtain

$$\mathbf{W} = \frac{F}{2\pi\rho}\left\{\begin{array}{l} \mathbf{e}_r\mathrm{Cos}\theta\left[v_p^{-2}K_o(\frac{sr}{v_p}) + (\frac{v_p}{sr})K_1(\frac{sr}{v_p}) - (\frac{v_s}{sr})K_1(\frac{sr}{v_s})\right] \\ + \mathbf{e}_\theta\mathrm{Sin}\theta\left[v_p^{-2}(\frac{v_p}{sr})K_1(\frac{sr}{v_p}) - v_s^{-2}K_o(\frac{sr}{v_s}) - v_s^{-2}(\frac{v_s}{sr})K_1(\frac{sr}{v_s})\right]\end{array}\right\} .$$

From Erdelyi et al (1954, Inverse transform #5.15(11)), we have that

$$(\frac{v}{sr})K_1(\frac{sr}{v}) \Longleftrightarrow \frac{v^2}{r^2}(t^2 - r^2/v^2)^{\frac{1}{2}}H(t - r/v) , \tag{7-55}$$

and from (7-35) and (7-26) above, we have that

$$K_o(\frac{sr}{v}) \Longleftrightarrow (t^2 - r^2/v^2)^{-\frac{1}{2}}H(t - r/v) . \tag{7-56}$$

Returning to the time dimension, we calculate that

$$\mathbf{W} = \frac{F}{2\pi\rho}\left\{\begin{array}{l} \mathbf{e}_r\mathrm{Cos}\theta\left[\frac{H(t - r/v_p)}{v_p^2(t^2 - r^2/v_p^2)^{\frac{1}{2}}} + \frac{(t^2 - r^2/v_p^2)^{\frac{1}{2}}}{r^2}H(t-r/v_p) - \frac{(t^2 - r^2/v_s^2)^{\frac{1}{2}}}{r^2}H(t-r/v_s)\right] \\ + \mathbf{e}_\theta\mathrm{Sin}\theta\left[\frac{(t^2 - r^2/v_p^2)^{\frac{1}{2}}}{r^2}H(t-r/v_p) - \frac{H(t-r/v_s)}{v_s^2(t^2 - r^2/v_s^2)^{\frac{1}{2}}} - \frac{(t^2 - r^2/v_s^2)^{\frac{1}{2}}}{r^2}H(t-r/v_s)\right]\end{array}\right\} \tag{7-57}$$

This agrees with a solution worked out by Eason, Fulton and Sneddon (1956) and expressed in cartesian coordinates.

This analysis shows us two things. To begin with, the first order term of w_r travels with velocity v_p, that is, at large distances the greatest part of the displacement motion associated with v_p (P-motion) is perpendicular to the wave front, i.e., it is a longitudinal wave. However, it is neither true that all P-motion is perpendicular to the wave front, nor true that all motion perpendicular to the wave front is P-motion. Similarly with S-motion. The first order term of w_θ travels with velocity v_s and consequently the greatest part of the S-motion is parallel to the wave front. However, some P-motion is parallel to the wave front and some S-motion is perpendicular to the wave front.

Secondly, we note that the response $(t^2-r^2/v^2)^{-\frac{1}{2}}$ is the characteristic impulse response in two-dimensions. We see that terms like $r^{-2}(t^2-r^2/v^2)^{\frac{1}{2}}$ not only fall off faster with distance, but are time-integrals of the first order P- and S-wave terms. This is a rather frequent occurrence in elastic wave theory. One finds that diffracted or near field terms fall off one or more powers of <u>r</u> faster than the direct waves and at the same time they are time-integrals of the direct wave time function.

<u>Elastic Media — Line Source of Stress.</u> In infinite media, the line source of stress and the line force are equivalent in that one chooses the stress to go to infinity in such a way that the product of stress times area (equals force) remains finite. Consequently one does not have to distinguish between them.

However, when a bounding surface is introduced, and forces act on this surface, it is quite often more convenient to build the source function into the boundary conditions rather than introduce it as a singularity into the differential equation. We will see an example of this in two dimensions in the discussion of Lamb's Problem (Chapter 11).

7.4. <u>Three-Dimensional Point Sources.</u>

<u>Scalar Media — Time Harmonic Source.</u> We wish to find a solution of the inhomogeneous scalar wave equation

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