Parametric Gröbner bases

Geometry & applications

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Introduction

1 Preliminaries

This project will assume familiarity with ring theory, multivariate polynomials over fields. A familiarity with Gröbner bases will be beneficial, but we will introduce the necesary notations and definitions. Let R be a Noetherian, commutative ring and $X = (x_1, x_2, \dots, x_n)$ be an ordered collection of symbols. We denote the ring of polynomials in these variables $R[X]$. Given two (disjoint) sets of variables X and Y, we will use $R[X, Y]$ to mean $R[X \cup Y]$, which is isomorphic to $R[X][Y]$. A monomial is a product of variables and a term is a monomial times a coefficient. We denote a monomial as X^v for some $v \in \mathbb{N}^n$. For a polynomial

$$f = \sum_{v \in \mathbb{N}^n} a_v X^v$$

we denote the coefficient of the term $t = a_v X^v$ by $\mathrm{coef}(f, X^v)$.

1.1 · Definition (Monomial order, leading term). A *monomial order* is a well-order[a] $<$ on the set of monomials satisfying that $u < v \implies wu < wv$.

Given a monomial order $<$ and a polynomial $f \in R[X]$, the *leading term* of f is the term with the largest monomial w.r.t. $<$ and is denoted by $\mathrm{lt}_<(f)$. If $\mathrm{lt}_<(f) = a \cdot m$ for some monomial m and $a \in R$, then we denote $\mathrm{lm}_<(f) = m$ and $\mathrm{lc}_<(f) = a$. If $<$ is clear from context, it will be omitted.

[a] A total order, for which any chain $a > b > c > \dots$ must be finite.

These definitions naturally extend to sets of polynomials, so given a set of polynomials $F \subset k[X]$, we denote $\mathrm{lm}_<(F) := \{\mathrm{lm}_<(f) \mid f \in F\}$. With this, we can give the definition of a Gröbner basis.

1.2 · Definition (Gröbner basis). Let $G \subset R[X]$ be a finite set of polynomials and $<$ be a monomial order. We say G is a *Gröbner basis* if $\langle \mathrm{lt}_<(G) \rangle = \langle \mathrm{lt}_<(\langle G \rangle) \rangle$.

Note, that if R is a field, then it is enough that $\langle \mathrm{lm}_<(G) \rangle = \langle \mathrm{lm}_<(\langle G \rangle) \rangle$. We say G is a Gröbner basis for an ideal I if G is a Gröbner basis and $\langle G \rangle = I$. We will also have to use an alternative description of Gröbner bases.

1.3 · Definition (Reduction modulo). Let $f, g \in R[X]$ be polynomials and $<$ be a term order. We say f *reduces modulo* g if $\mathrm{lt}(g) \mid \mathrm{lt}(f)$, since in that case $\mathrm{lt}(\mathrm{lc}(g) \cdot f - p \cdot \mathrm{lc}(f) \cdot g) < \mathrm{lt}(f)$ where $\mathrm{lm}(f) = p \cdot \mathrm{lm}(g)$. We say a polynomial reduces modulo a set of polynomials if it reduces modulo any polynomial in the set. We say a polynomial *reduces to zero* if there is a chain of reductions that end in the zero polynomial.

1.4 · Theorem. *Let $G \subset R[X]$. Then G is a Gröbner basis if and only if every polynomial in $\langle G \rangle$ reduces to 0 modulo G.*

Proof. A good exercise. ☐

A Gröbner basis need not be unique. Indeed, given a Gröbner basis G, we can add any element of $\langle G\rangle$ to G and it is still a Gröbner basis. However, reduced Gröbner bases are unique.

1.5 · Definition (Reduced Gröbner basis). A Gröbner basis G is called *reduced* if, for all $g \in G$, g is a monic polynomial (i.e. $\mathrm{lc}(g) = 1$) and the only term of g in $\mathrm{lt}(I)$ is $\mathrm{lt}(g)$.

1.6 · Theorem. *Let $I \subset k[X]$ be an ideal in a polynomial ring over a field. Then there is a unique reduced Gröbner basis of I.*

It is worth noting, that the second condition of reduced Gröbner bases is equivalent to saying that every term of g is irreducible modulo G, except for its leading coefficient.

2 Definitions and initial results

The purpose of this project is to study parametric Gröbner bases, so let's introduce those. The bare concept is rather simple.

2.1 · Definition (Parametric Gröbner basis). Let k and k_1 be fields, U and X be sets of variables and $F \subset k[X, U]$ be a finite set of polynomials. A *parametric Gröbner basis* is a finite set of polynomials $G \subset k[X, U]$ such that $\sigma(G)$ is a Gröbner basis of $\langle\sigma(F)\rangle$ for any ring homomorphism $\sigma : k[U] \to k_1$.

We call such a $\sigma : k[U] \to k_1$ a *specialization*. By the linearity of σ, all such ring homomorphisms can be characterized by their image of U. Thus, we can identify $\{\sigma : k[U] \to k_1 \mid \sigma \text{ is a ring hom.}\}$ with the affine space k_1^m when U has m elements. For $\alpha \in k_1^m$ we'll denote the corresponding map

$$\sigma_\alpha(u_i) = \alpha_i \quad \text{for } u_i \in U$$

extended linearly.

When we work with these parametric Gröbner bases, it will be more convenient to have a bit more information attached to them, namely which elements are required for which σ. Since σ is described by an $\alpha \in k_1^m$, we can restrict them using subsets of k_1^m.

2.2 · Definition (Vanishing sets & locally closed sets). Let $E \subset k[X]$. Then the *vanishing set* of E is $V(E) := \{v \in k^n \mid e(v) = 0 \ \ \forall e \in E\}$.

A *locally closed set* is a set of the form $V(E) \setminus V(N)$ for two subsets E and N of $k[X]$.

2.3 · Definition (Gröbner system). Let A be a locally closed set and $F, G \subset k[X, U]$ be finite sets. Then (A, G) is called a *segment of a Gröbner system for F* if $\sigma_\alpha(G)$ is a Gröbner basis of $\langle\sigma_\alpha(F)\rangle$ for all $\alpha \in A$. A set $\{(A_1, G_1), \dots, (A_t, G_t)\}$ is called a *Gröbner system* if each (A_i, G_i) is a segment of a Gröbner system.

We call the locally closed sets A_i for the *conditions* on a segment.

A Gröbner system $\{(A_1, G_1), \dots, (A_t, G_t))\}$ is called *comprehensive*, if $\bigcup_{i=1}^{t} A_i = k_1^{|U|}$. We also say a Gröbner system is *comprehensive on* $L \subset k_1^{|U|}$ if $\bigcup_{i=1}^{t} A_i = L$.

We will sometimes call a triple (E, N, G) for a segment of a Gröbner system. By this we mean that $(V(E) \setminus V(N), G)$ is a segment of a Gröbner system.

2.4 · Example. Let $X = \{x, y\}$ and $U = \{u\}$ and consider the polynomials $f(x, y, u) = ux^2 + x$ and $g(x, y, u) = xy + 1$. When $u \neq 0$, a Gröbner basis of $\langle f, g \rangle$ could be $(y - u, ux + 1)$, whatever u may be. TODO

2.5 · Definition (Leading coefficient w.r.t. variables). Let $f \in k[U][X]$. Then the leading term of f is denoted $\mathrm{lt}_U(f)$, the leading coefficient is $\mathrm{lc}_U(f)$ and the leading monomial is $\mathrm{lm}_U(f)$. These notations are also used when $f \in k[X, U]$, just viewing f as a polynomial in $k[U][X]$.

Note that $\mathrm{lc}_U(f) \in k[U]$, i.e. the leading term is a polynomial in $k[U]$ times a monomial in X.

From this point, we assume that the monomial order on $k[X, U]$ satisfies $X^{\nu_1} > U^{\nu_2}$ for all $\nu_1 \in \mathbb{N}^{|X|}$ and $\nu_2 \in \mathbb{N}^{|U|}$. This monomial order restricts to a monomial order on $k[X]$, denoted by $<_X$. Note that this assumption is not too restrictive, as we're usually only interested in a certain monomial order on the variables, since the parameters will be specialized away anyway. Thus for a given monomial order $<_X$, we can construct a suitable monomial order on $k[X, U]$, by using $<_X$ and breaking ties with any monomial order on $k[U]$.

2.1 A useful criterion

In this section we will prove a criterion to decide when a Gröbner basis G of an ideal $\langle F \rangle$ maps to a Gröbner basis $\sigma(G)$ if the ideal $\langle \sigma(F) \rangle$. This is theorem 3.1 in [1].

2.6 · Lemma. *Let G be a Gröbner basis of an ideal $\langle F \rangle \subset R[X]$ w.r.t. $<$, let $\sigma : R \to K$ be a ring homomorphism to a field K and set $G_\sigma = \{g \in G \mid \sigma(\mathrm{lc}(g)) \neq 0\} = \{g_1, g_2, \dots, g_l\} \subset R[X]$. Then $\sigma(G_\sigma)$ is a Gröbner basis of the ideal $\langle \sigma(F) \rangle$ w.r.t. $<_X$ if and only if $\sigma(g)$ is reducible to 0 modulo $\sigma(G_\sigma)$ for every $g \in G$.*

Proof. First, we prove "$\Longrightarrow$": Suppose $\sigma(G_\sigma)$ is a Gröbner basis of $\langle \sigma(F) \rangle$. Since $\sigma(g) \in \langle \sigma(F) \rangle$, we get that $\sigma(g)$ reduces to zero modulo any Gröbner basis of $\langle \sigma(F) \rangle$ by theorem 1.4, in particular $\sigma(G_\sigma)$.

Next, we prove "$\Longleftarrow$": Assume that $\sigma(g)$ is reducible to 0 modulo G_σ for every $g \in G$ and let $f \in \langle F \rangle$ such that $\sigma(f) \neq 0$. It's enough to show that

$$\exists\, h \in \langle F \rangle \;:\; \sigma(\mathrm{lc}(h)) \neq 0 \wedge \mathrm{lm}(h) \mid \mathrm{lm}(\sigma(f)).$$

Indeed, since G is a Gröbner basis of $\langle F \rangle$, that implies there is some $g \in G$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(h)$ and $\mathrm{lm}(h) = \mathrm{lm}(\sigma(h)) \mid \mathrm{lm}(\sigma(f))$. Furthermore, since $\mathrm{lc}(g) \mid \mathrm{lc}(h)$, we have that $\sigma(\mathrm{lc}(g)) \neq 0$, hence $\mathrm{lt}(\sigma(g)) \mid \mathrm{lt}(\sigma(f))$. Thus, if the above holds for any f, then $\sigma(G)$ is a Gröbner basis of $\langle \sigma(F) \rangle$. We prove this claim by induction on $<_X$.

The base case is when $\mathrm{lm}(f) = 1$, which means $f \in R$. Since we assumed $\sigma(f) \neq 0$, we have $\mathrm{lm}(\sigma(f)) = \mathrm{lm}(f)$ and $\sigma(\mathrm{lc}(f)) \neq 0$.

Now, the induction step. Let $f \in \langle F \rangle$ with $\sigma(\mathrm{lc}(f)) \neq 0$ and assume that every $f' \in \langle F \rangle$ with $\mathrm{lm}(f') < \mathrm{lm}(f)$ we have $\exists\, h \in \langle F \rangle : \sigma(\mathrm{lc}(h)) \neq 0 \wedge \mathrm{lm}(h) \mid \mathrm{lm}(\sigma(f'))$. If $\sigma(\mathrm{lc}(f)) \neq 0$, we can simply use $h = f$, so consider the case when $\sigma(\mathrm{lc}(f)) = 0$. If there is some $\sigma(g) \in G_\sigma$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(f)$, then we can reduce f by g to get $f' = \mathrm{lc}(g) \cdot f - \mathrm{lc}(f) \cdot \frac{\mathrm{lm}(f)}{\mathrm{lm}(g)} g$. Then $\mathrm{lm}(\sigma(f')) = \mathrm{lm}(\sigma(f))$ since $\sigma(\mathrm{lc}(f)) = 0$ and $\mathrm{lm}(f') < \mathrm{lm}(f)$, so the assertion holds by the induction hypothesis.

On the other hand, if there is no such $\sigma(g) \in G_\sigma$, then we must have some $g \in G \setminus G_\sigma$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(f)$. However, we can't simply reduce by g, since the factor $\mathrm{lc}(g)$ is zero under σ. Instead, we can find a subset $\{g_{j_1}, \ldots, g_{j_r}\} \subset G \setminus G_\alpha$ such that

$$\mathrm{lm}(f) = \sum_{i=1}^{r} c_i \frac{\mathrm{lm}(f)}{\mathrm{lm}(g_{j_i})} \mathrm{lm}(g_{j_i}).$$

Since each of the $\sigma(g_{j_i})$ are reducible to 0 modulo G_σ, we can find some $h_i \in \langle F \rangle$ and $b_i \in R \setminus \ker(\sigma)$ such that $\sigma(b_i g_{j_i}) = \sigma(h_i)$ and $\mathrm{lm}(\sigma(h_i)) = \mathrm{lm}(\sigma(g_{j_i})) > \mathrm{lm}(g_{j_i})$ for each $i \in \{1, \ldots, r\}$. Let $b = \prod_{i=1}^{r} b_i$, which is non-zero, then

$$f' = bf - \sum_{i=1}^{r} c_i \frac{b}{b_i} \frac{\mathrm{lm}(f)}{\mathrm{lm}(g_{j_i})} (b_i g_{j_i} - h_i)$$

is a new polynomial with

$$\sigma(f') = \sigma(bf) - \sum_{i=1}^{r} \sigma\left(c_i \frac{b}{b_i} \frac{\mathrm{lm}(f)}{\mathrm{lm}(g_{j_i})} \right) (\sigma(b_i g_{j_i}) - \sigma(h_i)) = \sigma(bf)$$

hence $\mathrm{lm}(\sigma(f')) = \mathrm{lm}(\sigma(f))$ but also $\mathrm{lm}(f') < \mathrm{lm}(f)$ since $\mathrm{lm}(g_{j_i}) > \mathrm{lm}(h_i)$. Thus the conclusion follows from the induction hypothesis. □

We will use a consequence of this lemma, which uses a test that is much easier to check. We use the above lemma with $R = k[U]$.

2.7 · Lemma. *Let $G = \{g_1, g_2, \ldots, g_k\}$ be a Gröbner basis of an ideal $\langle F \rangle$ in $k[X, U]$ w.r.t $<$ and let $\alpha \in k_1^m$. If $\sigma_\alpha(\mathrm{lc}_U(g)) \neq 0$ for each $g \in G \setminus k[U]$, then $\sigma_\alpha(G)$ is a Gröbner basis of $\langle \sigma_\alpha(F) \rangle$.*

Proof. First note that since $X^{\nu_1} > U^{\nu_2}$, any Gröbner basis of $\langle F \rangle \subset k[X,U]$ is also a Gröbner basis of $\langle F \rangle \subset k[U][X]$. Let $G_\alpha = \{\sigma_\alpha(g) \mid \sigma_\alpha(\mathrm{lc}_U(g)) \neq 0\}$. If there is any $g \in G$, such that $\sigma_\alpha(g) \in k_1 \setminus \{0\}$, then $g \in G \cap k[U]$ since $\sigma_\alpha(\mathrm{lc}_U(g)) \neq 0$ for all $g \in G \setminus K[U]$. Furthermore, since $g \in \langle F \rangle$, we get that $\langle \sigma_\alpha(F) \rangle = k_1[X]$ and $\sigma_\alpha(G)$ is a Gröbner basis.

If there is no such g, then $\alpha \in V(G \cap k[U])$. Take any $g \in G$. If $\sigma_\alpha(g) \in G_\alpha$, then $\mathrm{lt}(\sigma_\alpha(g)) = a \cdot \mathrm{lm}_U(g)$ for some $a \in k_1$ since $X^{\nu_1} > U^{\nu_2}$. Thus the monomial of its leading term is preserved by σ_α, so $\sigma_\alpha(g)$ is reducible to 0 modulo G_α, since it's leading term is divisible by its own leading term.

On the other hand, if $\sigma_\alpha(g) \notin G_\alpha$, then we must have $g \in G \cap k[U]$. Since $\alpha \in V(G \cap k[U])$ then $\sigma_\alpha(g) = 0$, so is immediately reducible to zero. Thus $\sigma_\alpha(G)$ is a Gröbner basis of $\langle \sigma_\alpha(F) \rangle$ by lemma 2.6. □

3 Computing Gröbner systems

With lemma 2.7 in mind, we can start constructing Gröbner systems. Let G be a reduced Gröbner basis of an ideal $\langle F \rangle \subset k[X,U]$, and let $H = \{\mathrm{lc}_U(g) \mid g \in G \setminus k[U]\}$. Then $\left(k_1^m \setminus \bigcup_{h \in H} V(h), G\right)$ is a segment of a Gröbner system. Thus, to make a Gröbner system, we need to find segments covering $\bigcup_{h \in H} V(h) = V(\mathrm{lcm}(H))$.

If we take G to be a reduced Gröbner basis, then $h \notin \langle F \rangle$ for any $h \in H$ since then the corresponding leading term would be divisible by a leading term in G. This is not allowed when G is reduced. Hence, we can find a Gröbner basis G_1 of $F \cup \{h\}$, which will then form a segment $(V(h) \setminus \bigcup_{h_1 \in H_1} V(h_1), G_1)$ where $H_1 = \{\mathrm{lc}_U(g) \mid g \in G_1\}$. Since $k[X,U]$ is Noetherian, this will eventually stop, forming a Gröbner system.

This gives us the ingredients for a simple algorithm for computing Gröbner systems, Algorithm 1.

Algorithm 1: $\mathtt{CGS}_{\mathtt{simple}}$, an algorithm for computing comprehensive Gröbner systems on $V(S)$

INPUT: Two finite sets $F \subset k[X,U]$, $S \subset k[U]$
OUTPUT: A finite set of triples (E,N,G), each forming a segment of a comprehensive Gröbner system on $V(S)$.

if $\exists g \in S \cap (k \setminus \{0\})$ **then**
  return $\varnothing$;
else
  $G \leftarrow$ **groebner**$(F \cup S)$;
  $H \leftarrow \{\mathrm{lc}_U(g) \mid g \in G \setminus k[U]\}$;
  $h \leftarrow \mathrm{lcm}(H)$;
  return $\{(S, \{h\}, G)\} \cup \bigcup_{h' \in H} \mathtt{CGS}_{\mathtt{simple}}(G \cup \{h'\}, S \cup \{h'\})$
end

3.1 · Theorem. *Let $F \subset k[X,U]$ and $S \subset k[U]$ be finite sets of polynomials. Then* $\mathbf{CGS_{simple}(F, S)}$ *terminates and the output $\mathscr{H}$ is a comprehensive Gröbner system on $V(S)$.*

Proof. First, we prove termination. Let F and S be inputs to $\mathbf{CGS_{simple}}$, let G be the reduced Gröbner basis of $F \cup S$ and let $H = \{\mathrm{lc}_U(g) \mid g \in G \setminus k[U]\}$. Take any $h \in H$. Since G is reduced, $h \notin \langle F \cup S \rangle$, since then its leading term would be divisible by an element in G, but that cannot be the case. Indeed, since $h \in k[U]$, it cannot be reduced by any $g \in G \setminus k[U]$ (as $X^{\nu_1} > U^2$, so the leading terms of $G \setminus k[U]$ must contain a variable from

X), and if it was reducible by a $p \in G \cap k[U]$, then that p would also reduce one of the elements of $G \setminus k[U]$, which is not allowed when G is reduced. Thus $\langle F \cup S \rangle \subsetneq \langle F \cup S \cup \{h\} \rangle$. Since this is the case at every recursive call, each successive call to $\mathbf{CGS_{simple}}$ will have a strictly greater ideal $\langle F \cup S \rangle$. Since $k[X, U]$ is Noetherian, this must stop eventually. Note also, that since F stays constant, this means that $\langle S \rangle \subsetneq \langle S \cup \{h\} \rangle$.

Next, we prove that if $(E, N, G) \in \mathscr{H}$, then $(V(E) \setminus V(N), G)$ is a segment of a Gröbner system. By the algorithm, $N = \mathrm{lcm}(H)$, where $H = \{\mathrm{lc}_U(g) \mid g \in G \setminus k[U]\}$ as before, for G being the reduced Gröbner basis of $\langle F \cup S \rangle$. Hence, for any $\alpha \in V(E) \setminus V(N)$, we have that $\sigma_\alpha(\mathrm{lc}_U(g)) \neq 0$ for every $g \in G \setminus k[U]$. Thus $\sigma_\alpha(G)$ is a Gröbner basis of $\langle \sigma_\alpha(F \cup S) \rangle$ by lemma 2.7. Also, $E = S$, so $\sigma_\alpha(S) = 0$. Hence $\langle \sigma_\alpha(F \cup S) \rangle = \langle \sigma_\alpha(F) \rangle$, so $\sigma_\alpha(G)$ is a Gröbner basis of $\langle \sigma_\alpha(F) \rangle$.

Finally, we need to prove that

$$\bigcup_{(E,N,G) \in \mathscr{H}} V(E) \setminus V(N) = V(S).$$

Note, that since $V(\mathrm{lcm}(H)) = \bigcup_{h \in H} V(H)$, we have the following:

$$\begin{aligned} V(S) &= (V(S) \setminus V(\mathrm{lcm}(H))) \cup \bigcup_{h \in H} V(h) \\ &= (V(S) \setminus V(\mathrm{lcm}(H))) \cup \bigcup_{h \in H} V(S \cup \{h\}) \end{aligned}$$

Inductively, the recursive calls to $\mathbf{CGS_{simple}}$ will compute Gröbner systems covering $\bigcup_{h \in H} V(S \cup \{h\})$. The base case is when $\langle S \rangle = k[U]$. In that case, $V(S) = \emptyset$, so $\emptyset$ is a comprehensive Gröbner system on $V(S)$. □

Note that in the implementation, we use $G \setminus S$ instead of G for the Gröbner segments. This has no impact on the validity of the segments, it just removes elements, which would specialize to 0 on that segment anyway.

3.1 Parametric Gröbner bases

We now move on to the problem of computing parametric Gröbner bases, which is the problem which Weispfenning tackled in his original article [4]. Recall the definition of parametric Gröbner bases from definition 2.1

3.2 · Definition (Faithful Gröbner system). A Gröbner system $\{(A_1, G_1), \ldots, (A_t, G_t)\}$ of an ideal $\langle F \rangle$ is called *faithful* if $G_i \subset \langle F \rangle$ for all i.

3.3 · Corollary. *Let $\mathscr{G} = \{(A_1, G_1), \ldots, (A_t, G_t)\}$ be a faithful comprehensive Gröbner system of an ideal $\langle F \rangle$. Then $\bigcup_{(A,G) \in \mathscr{G}} G$ is a parametric Gröbner basis of $\langle F \rangle$.*

Proof. Let σ_α be a specialization. Since $\mathscr{G}$ was comprehensive, there is some l such that $\alpha \in A_l$. Then $\sigma_\alpha(G_l)$ is a Gröbner basis of $\langle \sigma_\alpha(F) \rangle$, so $\langle \mathrm{lt}(\sigma_\alpha(G_l)) \rangle = \langle \mathrm{lt}(\sigma_\alpha(\langle F \rangle)) \rangle$. Since for all i we have that $\langle \sigma_\alpha(G_i) \rangle \subset \langle \sigma_\alpha(F) \rangle$, we have that $\langle \mathrm{lt}(\sigma_\alpha(G_i)) \rangle = \langle \mathrm{lt}(\sigma_\alpha(\langle F \rangle)) \rangle$, so $\sum_{i=1}^t \langle \mathrm{lt}(\sigma_\alpha(G_i)) \rangle = \langle \mathrm{lt}(\sigma_\alpha(\langle F \rangle)) \rangle$, thus $\sigma_\alpha\left(\bigcup_{(A,G) \in \mathscr{G}} G\right)$ is a Gröbner basis for $\langle \sigma_\alpha(F) \rangle$. □

The path to computing parametric Gröbner bases seem clear. We simply need to modify the segments of a comprehensive Gröbner system to be faithful, then we're done. While this is surpisingly easy to implement, proving that the way we do it works is a little more cumbersome.

3.2 Computing faithful segments

We follow the path laid out by [2], and introduce a new variable t and extend the monomial order such that $t^n > X^{v_1} > U^{v_2}$ for all $n \in \mathbb{N}$ and vectors v_1, v_2. In the CGS algorithm we added leading coefficients h to a set $S \subset k[U]$, and computed reduced Gröbner bases of $\langle F \cup S \rangle$ to produce the segments. However, this "mixes up" the original ideal with the added leading coefficients. We need a way to seperate them. We do this by replacing $F \cup S$ with $t \cdot F \cup (1-t) \cdot S$, where t is a new auxilliary variable that does not occur in F or S. Here we use the convention, that for a polynomial a and a set of polynomials F, $a \cdot F := \{a \cdot f \mid f \in F\}$. Note, that this need not be an ideal.

In this way we can seperate the original ideal from the added polynomials by specializing away t. That is the content of this first lemma.

3.4 · Lemma. *Let $F, S \subset k[X,U]$ be finite sets and let $g \in \langle t \cdot F \cup (1-t) \cdot S \rangle_{k[t,X,U]}$. Then $g(0,X,U) \in \langle S \rangle_{k[X,U]}$ and $g(1,X,U) \in \langle F \rangle_{k[X,U]}$.*

Proof. By assumption, we can find $f_1, \dots, f_n \in F$, $s_1, \dots, s_m \in S$ and $q_1, \dots, q_n, p_1, \dots, p_m \in k[t,X,U]$ such that

$$g = \sum_{i=1}^{n} t\, q_i\, f_i + \sum_{j=1}^{m} (t-1) p_j\, s_j.$$

By linearity of the evaluation map, we get that

$$g(0,X,U) = \sum_{j=1}^{m} p_j(0,X,U)\, s_j(X,U) \in \langle S \rangle_{k[X,U]}$$

and

$$g(1,X,U) = \sum_{i=1}^{n} q_i(1,X,U)\, f_i(X,U) \in \langle F \rangle_{k[X,U]}.$$

□

We're going to need these two specializations a lot, so we'll give them names. Let $\sigma^0(f) = f(0,X,U)$ and $\sigma^1(f) = f(1,X,U)$. We also need that Gröbner bases are preserved under σ^1. While that is not true in general, the following is good enough for our uses.

3.5 · Lemma. *Let $F \subset k[X,U]$, $S \subset k[U]$ be finite sets with $V(S) \subset V(\langle F \rangle \cap k[U])$ and let G be the reduced Gröbner basis of $\langle t \cdot F \cup (1-t) \cdot S \rangle$. Let also*

$$H = \{\mathrm{lc}_U(g) \mid g \in G,\ \mathrm{lt}(g) \notin k[X,U],\ \mathrm{lc}_{X,U}(g) \notin k[U]\}.$$

Then $\sigma_\alpha(\sigma^1(G))$ is a Gröbner basis of $\langle \sigma_\alpha(F) \rangle$ for any $\alpha \in V(S) \setminus V(\mathrm{lcm}(H))$.

Proof. First note, that $\mathrm{lt}(g) \notin k[X,U]$ means that the leading term of g contains the variable t and since t dominates the other variables, this means that $g \in k[t,X,U] \setminus k[X,U]$. Also, any polynomial in G has degree at most 1 in t, again since t dominates the other variables. For any polynomial $g \in G$ we can therefor write $g = t\, g^t + g_t$ where $g_t = \sigma^0(g)$ and $g^t = \sigma^1(g) - \sigma^0(g)$.

Let $\alpha \in V(S) \setminus V(\mathrm{lcm}(H))$. By lemma 3.4 we have that $\langle \sigma^1(G)\rangle = \langle F\rangle$ and thus $\langle \sigma_\alpha(\sigma^1(G))\rangle = \langle \sigma_\alpha(F)\rangle$ for any specialization σ_α. Thus we only need to show that $\sigma_\alpha(\sigma^1(G))$ is a Gröbner basis for itself.

Let $G' = \{g \in G \mid \mathrm{lt}(g) \notin k[X,U],\ \mathrm{lc}_{X,U}(g) \notin k[U]\}$. Then $\sigma_\alpha(\mathrm{lc}_U(g)) \neq 0$ for any $g \in G'$ since $\alpha \notin V(\mathrm{lcm}(H))$. We will show later, that if $g \in G \setminus G'$ then $\sigma_\alpha(g) = 0$. Thus $\sigma_\alpha(G) = \sigma_\alpha(G') \cup \{0\}$. By lemma 2.7 this means that both $\sigma_\alpha(G)$ and $\sigma_\alpha(G')$ are Gröbner bases in $k_1[t,X]$.

Now we only need to show, that $\sigma_\alpha(\sigma^1(G'))$ is a Gröbner basis in $k_1[X]$. For any $g \in G'$ we have that $\sigma_\alpha(g) = \sigma_\alpha(t \cdot g^t) + \sigma_\alpha(g_t)$. Since $g_t = \sigma^0(g) \in \langle S\rangle$ by lemma 3.4 and $\alpha \in V(S)$, we have that $\sigma_\alpha(g_t) = 0$, thus $\sigma_\alpha(g) = \sigma_\alpha(t \cdot g^t)$. This means that $\sigma_\alpha(G') = \sigma_\alpha(\{t \cdot g^t \mid g \in G'\})$. Since t divides every polynomial, and thus term, in that ideal, divisibility of leading terms is independent of t. Thus $\sigma_\alpha(\sigma^1(G'))$ is a Gröbner basis.

To finish the proof, we need to prove the assertion that if $g \in G \setminus G'$ then $\sigma_\alpha(g) = 0$. If $g \in G \setminus G'$, then either $\mathrm{lt}(g) \in k[X,U]$ or $\mathrm{lc}_{X,U}(g) \in k[U]$. In the first case, since t dominates the other variables, g cannot contain t as a variable. Thus $g = \sigma^0(g) \in \langle S\rangle_{k[X,U]}$ by lemma 3.4. Since $\alpha \in V(S)$, $\sigma_\alpha(g) = 0$. On the other hand, if $\mathrm{lt}(g) \notin k[X,U]$ but $\mathrm{lc}_{X,U}(g) \in k[U]$, we note that $g^t = \mathrm{lc}_{X,U}(g)$. Since $g^t = \sigma^1(g) - \sigma^0(g)$, we get from lemma 3.4 that $g^t \in \langle F\rangle + \langle S\rangle = \langle F \cup S\rangle$. Since we also had $g^t \in k[U]$, we have $g^t \in \langle F \cup S\rangle \cap k[U]$. But by assumption $V(S) \subset V(\langle F\rangle \cap k[U])$, thus $\alpha \in V(S) \cap V(\langle F\rangle \cap k[U]) = V(\langle F \cup S\rangle \cap k[U])$. Hence, $\sigma_\alpha(g^t) = 0$. But we proved earlier that for any $g \in G$ we have $\sigma_\alpha(g_t) = 0$, so as $\sigma_\alpha(g) = t \cdot \sigma_\alpha(g^t) + \sigma_\alpha(g_t) = 0$, we are done. □

This lemma is a generalization of lemma 2.7, and as such, it leads us to an algorithm for computing comprehensive, faithful Gröbner systems, at least on the vanishing set of some $S \subset k[U]$. We compute the reduced Gröbner basis of $\langle t \cdot F \cup (1-t) \cdot S\rangle$, which gives a faithful Gröbner segment on $V(S) \setminus V(\mathrm{lcm}(H))$, where $H = \{\mathrm{lc}_U(g) \mid g \in G,\ \mathrm{lt}(g) \notin k[X,U],\ \mathrm{lc}_{X,U}(g) \notin k[U]\}$. Then, we recursively compute faithful Gröbner segments on each $V(h)$ for $h \in H$, by adding h to S.

3.6 · Theorem. *Let $F \subset k[X,U]$ and $S \subset k[U]$ be finite and assume $V(S) \subset V(\langle F\rangle \cap k[U])$. Then* $\mathbf{CGB}_{\mathbf{aux}}(F,S)$ *terminates, and the result is a faithful, comprehensive Gröbner system on $V(S)$ for F.*

Proof. We first show termination. Let G be the reduced Gröbner basis of $\langle t \cdot F \cup (1-t) \cdot S\rangle$, and let $h \in \{\mathrm{lc}_U(g) \mid g \in G,\ \mathrm{lt}(g) \notin k[X,U],\ \mathrm{lc}_{X,U}(g) \notin k[U]\}$. Let $g \in G$ be the element such that $\mathrm{lc}_U(g) = h$. By assumption, g is of the form $h \cdot t \cdot X^\nu + g'$ for some vector ν and $g' \in k[X,U]$. If $g \in \langle S\rangle$, then $(1-t) \cdot h \in \langle G\rangle$, by the construction of G. This means that $\mathrm{lt}((1-t) \cdot h) = \mathrm{lt}(t \cdot h)$ is divisible by some leading term of G, and since the leading

Algorithm 2: $\mathbf{CGB_{aux}}$

INPUT: $F \subset k[X,U]$ and $S \subset k[U]$, two finite sets such that $V(S) \subset V(\langle F\rangle \cap k[U])$
OUTPUT: A finite set of triples (E,N,G) forming a comprehensive, faithful Gröbner system on $V(S)$

if $1 \in \langle S\rangle$ **then**
  return $\varnothing$;
else
  $G \leftarrow \mathbf{groebner}(t \cdot F \cup (1-t) \cdot S)$;
  $H \leftarrow \{\mathrm{lc}_U(g) \mid g \in G,\ \mathrm{lt}(g) \notin k[X,U],\ \mathrm{lc}_{X,U}(g) \notin k[U]\}$;
  $h \leftarrow \mathrm{lcm}(H)$;
  return $\{(S,\{h\},\sigma^1(G))\} \cup \bigcup_{h' \in H} \mathbf{CGB_{aux}}(F, S \cup \{h'\})$;
end

term of g doesn't divide it, $\mathrm{lt}(t \cdot h)$ must be divisible by some leading term of $G \setminus \{g\}$. But this implies that the leading term of g is divisible by some leading term in $G \setminus \{g\}$, which is not allowed as G is a *reduced* Gröbner basis. Thus $\langle S\rangle \subsetneq \langle S \cup \{h\}\rangle$. Since $k[t,X,U]$ is Noetherian, we can only expand this ideal finitely many times. Thus the algorithm terminates.

Next, observe that the precondition $V(S) \subset V(\langle F\rangle \cap k[U])$ always hold if it held initially, as $V(S') \subset V(S)$ for any $S' \supset S$. Apply this to $S' = S \cup \{h\}$.

If $(S,\{h\},G)$ is in the output of $\mathbf{CGB_{aux}}(F,S)$, then $(V(S) \setminus V(h), G)$ is a segment of a Gröbner system by lemma 3.5. It is also faithful by lemma 3.4.

Finally, we need to show that $V(S) = \bigcup_{E,N,G} \in \mathbf{CGB_{aux}(F,S)} V(E) \setminus V(N)$. Let $H = \{\mathrm{lc}_U(g) \mid g \in G,\ \mathrm{lt}(g) \notin k[X,U],\ \mathrm{lc}_{X,U}(g) \notin k[U]\}$ and $h = \mathrm{lcm}(H)$. Then

$$\begin{aligned} V(S) &= (V(S) \setminus V(h)) \cup \bigcup_{h' \in H} V(h') \\ &= (V(S) \setminus V(h)) \cup \bigcup_{h' \in H} V(S \cup \{h'\}) \end{aligned}$$

By induction, the recursive calls to $\mathbf{CGB_{aux}}$ computes segments covering each $V(S \cup \{h'\})$. The base case is when $S \cup \{h'\} = k[U]$, but in this case $V(S \cup \{h'\}) = \varnothing$, and $\varnothing$ is a comprehensive Gröbner system on $\varnothing$. □

The only thing left is to figure out what to do with that $V(S)$. With the **CGS** algorithm we could choose $S = \varnothing$, then $V(S) = k_1^{|U|}$, but that doesn't work here, as it violates the assumption that $V(S) \subset V(\langle F\rangle \cap k[U])$. However, we can choose S to be a set of generators of the ideal $\langle F\rangle \cap k[U]$. Then $S \subset \langle F\rangle$ and $\langle \sigma_\alpha(S)\rangle$ is either zero or $k_1[X]$, depending whether $\alpha \in V(S)$ or not. Hence, $(k^{|U|} \setminus V(S), S)$ is a faithful segment of a Gröbner system.

Algorithm 3: CGB

INPUT: $F \subset k[X, U]$ a finite set of polynomials
OUTPUT: $G \subset k[U, X]$ a comprehensive Gröbner basis of F
$S \leftarrow$ **groebner**$(F) \cap k[U]$;
$\mathscr{H} \leftarrow$ $\mathbf{CGB_{aux}}(F, S)$;
return $S \cup \bigcup_{(E,N,G)\in\mathscr{H}} G$;

3.7 · Theorem. *Let $F \subset k[X, U]$ be a finite set of polynomials. Then* **CGB**(F) *terminates and the output is a parametric Gröbner basis of $\langle F\rangle$.*

Proof. **CGB** doesn't loop, and every subroutine it calls terminates, so it terminates. Since S is a set of generator of the ideal $\langle F\rangle \cap k[U]$, we have that $V(S) = V(\langle F\rangle \cap k[U])$, so by theorem 3.6, $\mathscr{H}$ is a faithful, comprehensive Gröbner system on $V(S)$. Since $\langle \sigma_\alpha(S)\rangle$ is either 0 or $k_1[X]$, $(k^{|U|} \setminus V(S), S)$ is a segment of a faithful, comprehensive Gröbner system. Hence

$$\{(V(\emptyset) \setminus V(S), S)\} \cup \mathscr{H}$$

is a faithful, comprehensive Gröbner system for $\langle F\rangle$. By corollary 3.3 we get that $S \cup \bigcup_{(E,N,G)\in\mathscr{H}} G$ is a parametric Gröbner basis for $\langle F\rangle$. □

4 Geometric description & Gröbner covers

In this section, we develop a geometric description of Gröbner systems. We follow the development of [5] quite closely, albeit with a slightly different focus. The description makes heavy use of terms from mordern algebraic geometry, specifically the language of sheaves. However, in section 4.6, we relate this abstract description to the **CGS** algorithm, which hopefully will provide a translation into more concrete terms. We also provide worked examples throughout, to relate the abstract concepts to the more classical setting.

We will now work over a Noetherian, commutative, reduced (with no nil-potent elements) ring A, which in concrete cases can be thought of as $k[U]$, the polynomial ring over the parameters. We let $\mathrm{Spec}(A)$ be the set of prime ideals in A, equipped with the Zariski topology, where the closed sets are of the form $\mathbf{V}(I) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid I \subset \mathfrak{p}\}$. Note that maximal ideals are prime ideals, and in the case when $A = k[U]$, ideals on the form $\langle u_1 - \alpha_1, \dots, u_n - \alpha_n\rangle$ are maximal. Note also, that there is a natural bijection between $\mathrm{Spec}(A/I)$ and $\mathbf{V}(I)$, which we will use implicitly. Given a closed set $Y \subset \mathrm{Spec}(A)$, there is a unique radical ideal $\mathbf{I}(Y) := \bigcap\{I \mid I \subset \mathfrak{p} \;\; \forall \mathfrak{p} \in Y\}$ such that $Y = \mathbf{V}(\mathbf{I}(Y))$.

Specializations are now given by prime ideals (elements of $\mathrm{Spec}(A)$). Given a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, let $A_\mathfrak{p}$ denote the localization of A by $\mathfrak{p}$, which is the set of fractions of the form $\frac{f}{g}$ where $f \in A$ and $g \notin \mathfrak{p}$. The residue field at $\mathfrak{p}$ is then $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$, and there is a canonical map $A \to A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ given by $a \mapsto \frac{a}{1} + \mathfrak{p}_\mathfrak{p}$. The specialization $\sigma_\mathfrak{p} : A[X] \to k(\mathfrak{p})[X]$ is this canonical map, applied to each coefficient. If $A = k[U]$

and $\mathfrak{p}$ is a maximal ideal $\langle u_1 - \alpha_1, \dots, u_n - \alpha_n \rangle$, then $\sigma_\mathfrak{p}$ is simply the evaluation of the parameters at $(\alpha_1, \dots, \alpha_n)$.

Given an open subset $U \subset \operatorname{Spec}(A)$, there is a ring of regular functions on U. Let $\mathfrak{a} = \mathbf{I}(\overline{U})$, then a regular function f is a function from U to $\coprod_{\mathfrak{p} \in U} (A/\mathfrak{a})_\mathfrak{p}$ which is locally a fraction and $f(\mathfrak{p}) \in (A/\mathfrak{a})_\mathfrak{p}$. This means, that any $\mathfrak{p} \in U$ there is an open $\mathfrak{p} \in U' \subset U$ and $p, q \in A/\mathfrak{a}$ such that $f(\mathfrak{p}') = \frac{p}{q} \in (A/\mathfrak{a})_{\mathfrak{p}'}$ for every $\mathfrak{p}' \in U'$. Note that this means $s \notin \mathfrak{p}'$.

4.1 · Example. In classical terms, we can think of regular functions as functions, which can locally be written as fractions of polynomials. For example, on $\mathbf{V}(ad - bc) \setminus \mathbf{V}(a, b) \subset \mathbb{C}^4$, there is a regular function f given by $\frac{c}{a}$ when $a \neq 0$ and $\frac{d}{b}$ when $b \neq 0$. Even though $\mathbf{V}(ad - bc) \setminus \mathbf{V}(a, b)$ isn't open in $\mathbb{C}^4$, we can see $\mathbf{V}(ad - bc)$ as a topological subspace of $\mathbb{C}^4$ in which $\mathbf{V}(ad - bc) \setminus \mathbf{V}(a, b)$ is open.

Moving from $\mathbb{C}^4$ to $\operatorname{Spec}(\mathbb{C}[a, b, c, d])$, we can identify $\mathbf{V}(ad-bc)$ with $\operatorname{Spec}(\mathbb{C}[a, b, c, d]/\langle ad-bc\rangle)$, so we can equivalently see f as a regular function on $\operatorname{Spec}(\mathbb{C}[a, b, c, d]/\langle ad - bc\rangle) \setminus \mathbf{V}(\langle a, b\rangle)$. This means, for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathbb{C}[a, b, c, d]/\langle ad - bc\rangle)$ which doesn't contain $\langle a, b\rangle$, f assigns it an element of $(\mathbb{C}[a, b, c, d]/\langle ad - bc\rangle)_\mathfrak{p}$. In this case, whenever $\mathfrak{p} \not\supset \langle a\rangle$, $f(\mathfrak{p}) = \frac{c}{a}$ and whenever $\mathfrak{p} \not\supset \langle b\rangle$, $f(\mathfrak{p}) = \frac{d}{b}$. When $\mathfrak{p}$ is a maximal ideal, this is equivalent to saying that when $\sigma_\mathfrak{p}$ doesn't evaluate a to 0, then $f(\mathfrak{p}) = \frac{c}{a}$, and when $\sigma_\mathfrak{p}(b) \neq 0$, then $f(\mathfrak{p}) = \frac{d}{b}$. Since we work in $\mathbb{C}[a, b, c, d]/\langle ad - bc\rangle$, these two fractions agree whenever $\sigma_\mathfrak{p}(a) \neq 0 \neq \sigma_\mathfrak{p}(b)$. We are sure that we never have $\sigma_\mathfrak{p}(a) = \sigma_\mathfrak{p}(b) = 0$ since $\langle a, b\rangle \not\subset \mathfrak{p}$ by assumption.

Similarly to this example, we will often work with regular functions on a locally closed set $S = Y \cap U$, denoted by $\mathscr{O}_Y(U)$ or $\mathscr{O}_S$. We will make good use of the following result about $\mathscr{O}_Y(U)$.

4.2 · Lemma. *An element of $\mathscr{O}_Y(U)$ is uniquely determined by its images in $k(\mathfrak{p})$ for each $\mathfrak{p} \in Y \cap U$.*

Proof. Let $\mathfrak{a} = \mathbf{I}(Y)$ and let $\rho_\mathfrak{p} : \mathscr{O}_Y(U) \to (A/\mathfrak{a})_\mathfrak{p}/(\mathfrak{p}/\mathfrak{a})_\mathfrak{p}$ be the map given by $\rho_\mathfrak{p}(f) = f(\mathfrak{p}) + (\mathfrak{p}/\mathfrak{a})_\mathfrak{p}$. Let $f \in \mathscr{O}_Y(U)$. It is enough to prove that $(\forall \mathfrak{p} \in Y \cap U : \rho_\mathfrak{p}(f) = 0) \implies f = 0$, so assume $f(\mathfrak{p}) \in (\mathfrak{p}/\mathfrak{a})_\mathfrak{p}$ for any $\mathfrak{p} \in Y \cap U$. Then $f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A/\mathfrak{a})} \mathfrak{p} = \sqrt{\langle 0\rangle} \subset A/\mathfrak{a}$, so if $A/\mathfrak{a}$ has no nil-potent elements, then $\sqrt{\langle 0\rangle} = \langle 0\rangle$ and thus $f = 0$. Since $\mathfrak{a}$ was radical, this follows from the assumption that A has no nil-potent elements. □

Given a locally closed set $S = Y \cap U \subset \operatorname{Spec}(A)$ take the radical ideal $\mathfrak{a} = \mathbf{I}(\overline{S})$, and consider the polynomial ring $(A/\mathfrak{a})[X]$. Let $I \subset A[X]$ be an ideal, and let $\overline{I}$ denote its image in $(A/\mathfrak{a})[X]$. Then we can consider the regular functions in $\overline{I}$ on S, which we denote by $\mathscr{I}_S$ or $\mathscr{I}_Y(U)$, and is given by functions f, which can be described locally as fractions $f(\mathfrak{p}) = \frac{p}{q}$ where $p \in \overline{I}$ and $q \in (A/\mathfrak{a}) \setminus \mathfrak{p}$. In this light, we can also see $\mathscr{I}_S$ as an ideal in the polynomial ring $\mathscr{O}_S[X]$, which is how we'll use it most of the time.

In an abuse of notation, for a $\mathfrak{p} \in \operatorname{Spec}(A/\mathfrak{a})$, we denote the map $\mathscr{I}_S \to k(\mathfrak{p}) = (A/\mathfrak{a})_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$

given by mapping $\frac{p}{q} \in \mathscr{I}_S$ to $\frac{\sigma_{\mathfrak{p}}(p)}{\sigma_{\mathfrak{p}}(q)}$ by $\sigma_{\mathfrak{p}}$. We can see $\mathscr{O}_S$ as a subring of $\mathscr{O}_S[X]$, so $\sigma_{\mathfrak{p}}$ also denotes the evaluation of an element in $\mathscr{O}_S$ at $\mathfrak{p}$.

The idea is to describe segments og Gröbner systems, not as point-sets in $k^{|U|}$ with a set of polynomials, but as point-sets in $\operatorname{Spec}(k[U])$ with a set of regular functions. These functions can be evaluated at a maximal ideal, giving a fraction of two polynomials, which can then be specialized at the same maximal ideal, giving a polynomial in $k[X]$. Using regular functions instead of polynomials will allow us to describe not only a Gröbner basis, but the reduced Gröbner basis of a whole segment.

4.3 · Example. Consider the ideal $I = \langle ax + cy, bx + dy\rangle \subset \mathbb{C}[a, b, c, d][x, y]$ with a term order such that $x > y$ as well as the subset $S = Y \cap U$ where $Y = \mathbf{V}(ad - bc)$ and $U = \mathbb{C}[a, b, c, d] \setminus \mathbf{V}(a, b)$. For any specialization where $ad - bc = 0$ and $a \neq 0$, we can divide the first polynomial by a and reduce the second polynomial with it:

$$bx + dy - b\left(x + \frac{c}{a}y\right) = \left(d - \frac{bc}{a}\right)y = 0$$

Hence the reduced Gröbner basis is $\{x + \frac{c}{a}y\}$. Similarly, if $b \neq 0$, then $\{x + \frac{d}{b}y\}$ is the reduced Gröbner basis. Let's see how we can describe this using regular functions. The star of the show will be the regular function $f \in \mathscr{O}_Y(U)$ from example 4.1 given by $f(\mathfrak{p}) = \frac{c}{a}$ if $\mathfrak{p} \not\supset \langle a\rangle$ and $f(\mathfrak{p}) = \frac{d}{b}$ if $\mathfrak{p} \not\supset \langle b\rangle$.

Consider now the polynomial $P = x+f\cdot y \subset {}_Y(U)[x, y]$, and let $\mathfrak{m} \in \operatorname{Spec}(\mathbb{C}[a, b, c, d]/\mathbf{V}(ad-bc))$ be a maximal ideal, which doesn't contain $\langle a, b\rangle$. This is equivalent to $\mathfrak{m}$ being a maximal ideal in $\mathbb{C}[a, b, c, d]$ of the form $\langle a-m_1, b-m_2, c-m_3, d-m_3\rangle$ with the condition that $m_1m_4-m_2m_3 = 0$ and m_1 and m_2 not both being zero. Then $f(\mathfrak{m}) = x+\frac{c}{a}y$ if $m_1 \neq 0$ and $f(\mathfrak{m}) = x + \frac{d}{b}x$ if $m_2 \neq 0$.

Hence

$$\sigma_{\mathfrak{m}}(P) = \begin{cases} x + \frac{m_3}{m_1}y & m_1 \neq 0 \\ x + \frac{m_4}{m_2}y & m_2 \neq 0 \end{cases}$$

Notice, for any such choice of $m_1, \ldots, m_4$, $\sigma_{\mathfrak{m}}(P)$ is indeed the reduced Gröbner basis of $\sigma_{\mathfrak{m}}(I) \subset \mathbb{C}[x, y]$. Lasly, we can write $P = (ax+cy)/a \in I_{\mathfrak{p}}$ when $a \neq 0$ and $P = (bx+dy)/b$ when $b \neq 0$. Hence $P \in \mathscr{I}_Y(U)$.

4.1 Parametric sets

Parametric Gröbner bases are nice for applications because we have a single object, which is easily translated into a Gröbner basis for any given specialization. However, that translation may include zeros and redundant elements. In particular, there is no way in general to produce a "parametric reduced Gröbner basis", i.e. a Gröbner basis which specializes to the reduced Gröbner basis of $\sigma(\langle G\rangle)$ for any specialization σ. Hence, we might want to find the maximal segments, where we can find such a parametric reduced

Gröbner basis. This is the following definition.

4.4 · Definition (Parametric set). Let $I \subset A[X]$ be an ideal and let $S \subset \operatorname{Spec}(A)$ be locally closed. We say S is a *parametric set for I* if there is a finite set $G \subset \mathscr{I}_S$ such that

1. $\sigma_\mathfrak{p}(G)$ is the reduced Gröbner basis of $\langle\sigma_\mathfrak{p}(I)\rangle$ for each $\mathfrak{p} \in S$.

2. For any $g \in G$ and $\mathfrak{p}, \mathfrak{p}' \in S$, we have $\langle \operatorname{lt}(\sigma_\mathfrak{p}(g))\rangle = \langle \operatorname{lt}(\sigma_{\mathfrak{p}'}(g))\rangle$.

Reduced Gröbner bases are supposed to be unique, and indeed that's also the case for the set G in the definition of parametric sets. To prove this, we'll first need a lemma.

4.5 · Lemma. *Let $Y \subset \operatorname{Spec}(A)$ be a closed set and $f, g \in \mathscr{I}_Y$. If $\sigma_\mathfrak{p}(f) = \sigma_\mathfrak{p}(g)$ for all $\mathfrak{p} \in Y$, then $f = g$.*

Proof. By linearity of $\sigma_\mathfrak{p}$, we can assume without loss of generality that $f = 0$. We can see g as a polynomial with coefficients in $\mathscr{O}_Y(Y)$. Then $\sigma_\mathfrak{p}(g) = 0$ means that every coefficient of g lies in $\mathfrak{p}_\mathfrak{p}$. Since this hold for every $\mathfrak{p} \in Y$, $g = 0$ by lemma 4.2 □

4.6 · Theorem. *Let $S \subset \operatorname{Spec}(A)$ be a parametric set for an ideal I and let $G \subset \mathscr{I}_Y$ be the finite set such that $\sigma_\mathfrak{p}(G)$ is the reduced Gröbner basis of $\langle\sigma_\mathfrak{p}(I)\rangle$ for every $\mathfrak{p} \in S$. Then G is unique and every $g \in G$ is monic (has invertible leading coefficient) with* $\operatorname{lm}(g) = \operatorname{lm}(\sigma_\mathfrak{p}(g))$ *for every $\mathfrak{p} \in Y$.*

Proof. Let $F \subset \mathscr{I}_Y$ be a finite set satisfying the two conditions for Y to be a parametric set. For any fixed $f \in F$ and $\mathfrak{p} \in Y$, there is then a $g \in G$ such that $\sigma_\mathfrak{p}(f) = \sigma_\mathfrak{p}(g)$. Since $\operatorname{lm}(\sigma_\mathfrak{p}(f))$ and $\operatorname{lm}(\sigma_\mathfrak{p}(g))$ is independent of $\mathfrak{p}$, we have $\operatorname{lm}(\sigma_\mathfrak{p}(f)) = \operatorname{lm}(\sigma_\mathfrak{p}(g))$ for all $\mathfrak{p} \in Y$. Since $\sigma_\mathfrak{p}(F) = \sigma_\mathfrak{p}(G)$ is a reduced Gröbner basis, there can only be one polynomial with that leading monomial. Hence $\sigma_\mathfrak{p}(f) = \sigma_\mathfrak{p}(g)$ for all $\mathfrak{p} \in Y$, so $f = g$ by lemma 4.5. Thus $F \subset G$, and since the situation is symmetric, $F = G$.

To see that every $g \in G$ is monic, we observe that since $\sigma_\mathfrak{p}(g)$ is an element of a reduced Gröbner basis, it's leading coefficient is 1 for all $\mathfrak{p} \in Y$. Since $\operatorname{lm}(\sigma_{\mathfrak{p}'}(g)) = \operatorname{lm}(\sigma_\mathfrak{p}(g))$ for all $\mathfrak{p}, \mathfrak{p}' \in S$, we have $\sigma_\mathfrak{p}(\operatorname{lc}(g)) \neq 0$ for all $\mathfrak{p} \in S$. Thus $1 = \operatorname{lc}(\sigma_\mathfrak{p}(g)) = \sigma_\mathfrak{p}(\operatorname{lc}(g))$, hence $\operatorname{lc}(g) = 1$ by lemma 4.2. And since $\sigma_\mathfrak{p}(1) = 1$ for any $\mathfrak{p}$, we get that $\operatorname{lm}(g) = \operatorname{lm}(\sigma_\mathfrak{p}(g))$.□

In light of this theorem, for a parametric set S, we will call its uniquely determined set of polynomials for its reduced Gröbner basis. In certain ways, they are even more well-behaved than classical reduced Gröbner bases, which the following proposition will show.

4.7 · Proposition. *Let $S \subset \operatorname{Spec}(A)$ be a parametric set for an ideal I and let $S' \subset S$ be locally closed. Then S' is also parametric, and there is a canonical map $\mathscr{I}_S \to \mathscr{I}_{S'}$ which maps the reduced Gröbner basis of S to the reduced Gröbner basis of S'.*

Proof. To construct the canonical map, let $\mathfrak{a} = \mathbf{I}(\overline{S})$, $\mathfrak{a}' = \mathbf{I}(\overline{S'})$. Let $\overline{I}$ and $\overline{I'}$ be the images of I in $(A/\mathfrak{a})[X]$ and $(A/\mathfrak{a}')[X]$ respectively. Since $\overline{S} \subset \overline{S'}$, we get $\mathfrak{a} \subset \mathfrak{a}'$ and thus an inclusion map $\iota : A/\mathfrak{a} \to A/\mathfrak{a}'$. This extends to $\phi : \overline{I} \to \overline{I'}$, which we can localize for every $\mathfrak{p} \in S'$, giving $\phi_\mathfrak{p} : \overline{I}_\mathfrak{p} \to \overline{I'}_\mathfrak{p}$. Then the map

$$(g \in \mathscr{I}_S) \mapsto (\mathfrak{p} \mapsto \phi_\mathfrak{p}(g(\mathfrak{p})))$$

is well-defined since it agrees on every open set, and gives us the desired map, call it $\Phi : \mathscr{I}_S \to \mathscr{I}_{S'}$.

Since ϕ_p was just the localization of an inclusion, we get that $\sigma_\mathfrak{p}(\phi_\mathfrak{p}(g)) = \sigma_\mathfrak{p}(g)$ for any g in $\overline{I}_\mathfrak{p}$. Thus we also have $\sigma_\mathfrak{p}(\Phi(g)) = \sigma_\mathfrak{p}(g)$ for any $g \in \mathscr{I}_S$. Thus, by lemma 4.5 $\Phi(G) = G'$ where G and G' are the reduced Gröbner bases for S and S' respectively. □

4.2 Monic ideals and the reduced Gröbner basis of $\mathscr{I}_S$

Another pleasant surprise is that the unique reduced Gröbner basis of a parametric set for an ideal I, is actually the reduced Gröbner basis of the ideal $\mathscr{I}_S \subset \mathscr{O}_S[X]$. Since a reduced Gröbner basis consists of monic polynomials, this will imply that $\mathscr{I}_S$ is a monic ideal. In fact, that is a sufficient condition for S to be a parametric set. This subsection will be spent proving this, as well as some lemmas which will be useful later.

4.8 · Definition (Monic ideal). An ideal $I \subset A[X]$ is called *monic* if, for every $m \in \mathrm{lm}(I)$, there is a monic $f \in I$ with $\mathrm{lm}(f) = m$.

We will use without proof that reduced Gröbner bases exists for monic ideals. If the base ring is a field, then every ideal is monic.

4.9 · Proposition. *Let $I \subset A[X]$ be an ideal. Then there exists a unique reduced Gröbner basis of I if and only if I is monic.*

Before we prove the main content, we need two lemmas. First, for any localized polynomial, we can represent it by a fraction of a polynomial with the same terms.

4.10 · Lemma. *Let $I \subset A[X]$ be an ideal, $\mathfrak{p} \in \mathrm{Spec}(A)$ and $f \in I_\mathfrak{p}$. Then there exists a $P \in I$ and $Q \in A \setminus \mathfrak{p}$ such that $f = \frac{P}{Q} \in I_\mathfrak{p}$ and $\mathrm{coef}(f,t) = 0 \implies \mathrm{coef}(P,t) = 0$.*

Proof. By definition of $I_\mathfrak{p}$, there is some $p \in I$ and $Q \in A \setminus \mathfrak{p}$ such that $f = \frac{P}{Q}$. If $\mathrm{coef}(f,t) = 0$, then $\mathrm{coef}(P,t)/Q = 0$. Hence there is a $Q_t \in A \setminus \mathfrak{p}$ such that $\mathrm{coef}(P,t) \cdot Q_t = 0 \in A$. Then

$$f = \frac{P \cdot \prod_t Q_t}{Q \cdot \prod_t Q_t}$$

satisfies what we want. □

Secondly, when we embed polynomials in $\mathscr{I}_S$, we preserve their leading monomial.

4.11 · Lemma. *Let $S \subset \mathrm{Spec}(A)$ be a locally closed set and $\mathfrak{a} = \mathbf{I}(\overline{Y})$. Let $I \subset A[X]$ be an ideal, let $\overline{I} \subset (A/\mathfrak{a})[X]$ be its image in $(A/\mathfrak{a})[X]$, let $P \in \overline{I}$. Then the leading monomial of $\frac{P}{1} \in \mathscr{I}_S \subset \mathscr{O}_S[X]$ is equal to the leading monomial of P.*

Proof. We will show that there is a $\mathfrak{p} \in S$ with $\mathrm{lc}(P) \notin \mathfrak{p}$. Indeed, if that was the case, then $\mathrm{lc}(P) \in \mathfrak{p}$ for every $\mathfrak{p} \in S$, which would imply $\sigma_\mathfrak{p}(\mathrm{lc}(P)) = 0$ for every $\mathfrak{p} \in S$. Thus $\mathrm{lc}\left(\frac{P}{1}\right) = 0$ since elements of $\mathscr{O}_S$ are determined by $\sigma_\mathfrak{p}$ by lemma 4.2.

So assume for a contradiction that $\mathrm{lc}(P) \in \mathfrak{p}$ for all $\mathfrak{p} \in S$. Then $S \subset W := \mathbf{V}(\mathrm{lc}(P)) = \{\mathfrak{p} \in \mathbf{V}(\mathfrak{a}) \mid \mathrm{lc}(P) \in p\}$. Since W is closed and $S \subset W \subset \overline{S}$, we get that $W = \mathbf{V}(\mathfrak{a})$, thus $\mathrm{lc}(P) \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$. But since $\mathfrak{a}$ is radical and so $A/\mathfrak{a}$ has no nil-potents, this means

$$\mathrm{lc}(P) \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A/\mathfrak{a})} \mathfrak{p} = \sqrt{\langle 0 \rangle} = 0$$

hence $\mathrm{lc}(P) = 0$, which is a contradiction. □

4.12 · Theorem. *Let $I \subset A[X]$ be an ideal and $S \subset \mathrm{Spec}(A)$ be a locally closed set. Then*

1. *S is parametric for I if and only if $\mathscr{I}_S$, when seen as a ideal in $\mathscr{O}_S[X]$ is monic.*
2. *In the above case, the reduced Gröbner of $\mathscr{I}_S$ is equal to the reduced Gröbner basis for the parametric set S.*

Proof. For the first implication, assume S is parametric for I and let $G \subset \mathscr{I}_S$ be its reduced Gröbner basis. First, we show that $\mathscr{I}_S$ is monic, so let $f \in \mathscr{I}_S$. Then there is some $\mathfrak{p} \in S$ such that $\mathrm{lc}(f) \notin \mathfrak{p}$, i.e. $\sigma_{\mathfrak{p}}(\mathrm{lc}(f)) \neq 0$, since otherwise $\mathrm{lc}(f) = 0$ by lemma 4.2. Since $\sigma_{\mathfrak{p}}(G)$ is a Gröbner basis for $\langle \sigma_{\mathfrak{p}}(\mathscr{I}_S) \rangle$, there is some $g \in G$ where $\mathrm{lm}(\sigma_{\mathfrak{p}}) \mid \mathrm{lm}(\sigma_{\mathfrak{p}}(f))$. Since $\mathrm{lm}(g) = \mathrm{lm}(\sigma_{\mathfrak{p}}(g))$ by theorem 4.6 and $\mathrm{lm}(f) = \mathrm{lm}(\sigma_{\mathfrak{p}}(f))$, we get $\mathrm{lm}(g) \mid \mathrm{lm}(f)$. Since g is monic, every leading monomial of $\mathscr{I}_S$ is found as the leading monomial of a monic polynomial, so $\mathscr{I}_S$ is monic.

For the other implication, assume $\mathscr{I}_S$ is monic, let $G = \{g_1, \dots, g_n\}$ denote its unique reduced Gröbner basis and let $f \in \mathscr{I}_S$. By the division algorithm we can write

$$f = \sum_{i=1}^{n} f_i g_i$$

with $\mathrm{lm}(f_i)\,\mathrm{lm}(g_i) \leq \mathrm{lt}(f)$ and $\mathrm{coef}(f_i, m) \in \langle \mathrm{coef}(f, m') \mid m' \geq m\,\mathrm{lt}(g_i) \rangle \subset A/\,\mathbf{I}(S)$ for all monomials m. The last condition may be unfamiliar if you're used to work over fields, but it simply states that the coefficients of each f_i "comes from" coefficients in f. In other words, we don't use different g_i to reduce another g_j, we only use the g_is to reduce f.

The last condition gives us, for any $\mathfrak{p} \in S$ that if $\mathrm{lm}(f_i)\,\mathrm{lm}(g_i) > \mathrm{lm}(\sigma_{\mathfrak{p}}(f))$, then $\sigma_{\mathfrak{p}}(\mathrm{lc}(f_i)) \in \langle 0 \rangle$, thus $\sigma_{\mathfrak{p}}(\mathrm{lc}(f_i)) = 0$. Since this holds for every other term of f_i as well, we get that $\mathrm{lm}(\sigma_{\mathfrak{p}}(f_i))\,\mathrm{lm}(\sigma_{\mathfrak{p}}(g_i)) \leq \mathrm{lm}(\sigma_{\mathfrak{p}})(f)$. Since $\sigma_{\mathfrak{p}}$ is linear so $\sigma_{\mathfrak{p}}(f) = \sum_{i=1}^{n} \sigma_{\mathfrak{p}}(f_i)\sigma_{\mathfrak{p}}(g_i)$, there must be some g_i for which $\mathrm{lm}(\sigma_{\mathfrak{p}})(g_i) \mid \mathrm{lm}(\sigma_{\mathfrak{p}}(f))$. Since every element of $\langle \sigma_{\mathfrak{p}}(I) \rangle$ is a scalar multiple of $\sigma_{\mathfrak{p}}(f)$ for some $f \in \mathscr{I}_S$, we get that $\sigma_{\mathfrak{p}}(G)$ is a Gröbner basis of $\langle \sigma_{\mathfrak{p}}(I) \rangle$. Since every $g \in G$ is monic, $\sigma_{\mathfrak{p}}(g)$ is also monic, and $\sigma_{\mathfrak{p}}(G)$ is reduced because G is. Thus, $\sigma_{\mathfrak{p}}(G)$ is the reduced Gröbner basis of $\sigma_{\mathfrak{p}}(I)$ for every $\mathfrak{p} \in S$, so S is parametric. Furthermore, since G was defined to be the reduced Gröbner basis of $\mathscr{I}_S$, the second assertion follows immediately. □

This theorem gives us, that the parametric Gröbner basis, which was defined as specialising to a reduced Gröbner basis in all points, lifts to a reduced Gröbner basis of $\mathscr{I}_S$. The next theorem is a local test, to determine parametricity.

4.13 · Theorem. *Let $S \subset \operatorname{Spec}(A)$ be locally closed, let $\mathfrak{a} = \mathbf{I}(\overline{S})$ and let $\overline{I}$ be the image of I in $(A/\mathfrak{a})[X]$. Then S is parametric if and only if $\overline{I}_\mathfrak{p}$ is monic for every $\mathfrak{p} \in S$ and $\mathfrak{p} \mapsto \operatorname{lm}(\overline{I}_\mathfrak{p})$ is constant on S. Furthermore, in this case $\operatorname{lm}(\mathscr{I}_S) = \operatorname{lm}(\overline{I}_\mathfrak{p})$ for all $\mathfrak{p} \in S$.*

Proof. For the first implication, assume S is parametric and let $G \subset \mathbf{I}_S$ be its reduced Gröbner basis. Fix some $\mathfrak{p} \in S$ and let $\frac{P}{Q} \in \overline{I}_\mathfrak{p}$. By lemma 4.10 we can assume $\operatorname{lm}(P) = \operatorname{lm}\left(\frac{P}{Q}\right)$. By lemma 4.11 the leading monomial P is preserved when we embed it in $\mathscr{I}_S$. Hence $\operatorname{lm}\left(\frac{P}{Q}\right) \in \operatorname{lm}(\mathscr{I}_S)$, and since the image of G in $\overline{I}_\mathfrak{p}$ is monic, it is a reduced Gröbner basis of $\overline{I}_\mathfrak{p}$. Hence $\mathbf{I}_\mathfrak{p}$ is monic and it's leading monomials are constant with $\operatorname{lm}(\overline{I}_\mathfrak{p}) = \operatorname{lm}(\mathscr{I}_S)$.

For the other implication, assume $\overline{I}_\mathfrak{p}$ is monic for every $\mathfrak{p} \in S$, and $\operatorname{lm}(\overline{I}_\mathfrak{p}) = \operatorname{lm}(\overline{I}_{\mathfrak{p}'})$ for all $\mathfrak{p}, \mathfrak{p}' \in S$. Let $\{t_1, \dots, t_n\}$ be a minimal set of generators of the monomial ideal $\operatorname{lm}(\overline{I}_\mathfrak{p})$ (which is independent of $\mathfrak{p}$). For each $\mathfrak{p} \in S$, let $g_i(\mathfrak{p})$ denote the element of the reduced Gröbner basis of $\overline{I}_\mathfrak{p}$ with $\operatorname{lm}(g_i(\mathfrak{p})) = t_i$. Then g_i is a function $(\mathfrak{p} \in \operatorname{Spec}(S)) \to \overline{I}_\mathfrak{p}$, and so is potentially an element of $\mathscr{I}_S$. We just need that it locally can be described by the same fraction. Fix a $\mathfrak{p} \in S$ and find $P/Q = g_i(\mathfrak{p}) \in \overline{I}_\mathfrak{p}$ such that $\operatorname{lm}(P) = \operatorname{lm}(g_i(\mathfrak{p}))$, which exists by lemma 4.10. Also by lemma 4.10, we may assume that $\operatorname{coef}(P, m) = 0$ for all $m \in \operatorname{lm}(\overline{I}_\mathfrak{p}) \setminus t_i$, since that is the case for $g_i(\mathfrak{p})$ because it comes from a reduced Gröbner basis. Because $g_i(\mathfrak{p})$ is monic, we have $\operatorname{lc}(P)/Q = 1$. Consider the open set $U = \{\mathfrak{p}' \in S \mid Q \notin \mathfrak{p}'\}$, which is an open neighborhood of $\mathfrak{p}$. Then $g_i(\mathfrak{p}') = P/Q \in \overline{I}_{\mathfrak{p}'}$ for all $\mathfrak{p}' \in U$ since $P/Q \in \overline{I}_{\mathfrak{p}'}$ is monic and has leading monomial t_i and $\operatorname{coef}(P/Q, m) = 0$ for all $m \in \operatorname{lm}(\overline{I}_{\mathfrak{p}'})$, which is the defining properties of $g_i(\mathfrak{p}')$. Thus $g_i \in \mathscr{I}_S$.

This makes the set $G = \{g_1, \dots, g_n\} \subset \mathbf{I}_S$ a good candidate for a Gröbner basis of $\mathscr{I}_S$, which would make S parametric by theorem 4.12 because the g_i are monic. So take an $f \in \mathscr{I}_S$. By lemma 4.2 there is a $\mathfrak{p} \in S$ such that $\sigma_\mathfrak{p}(\operatorname{lc}(f)) \neq 0$. Letting $\overline{f}$ denote the image of f in $\overline{I} \subset (A/\mathfrak{a})[X]$ and $\overline{f}_\mathfrak{p}$ its image in $\overline{I}_\mathfrak{p}$, this implies that $\operatorname{lc}(\overline{f}) \neq 0$, hence $\operatorname{lm}(f) = \operatorname{lm}(\overline{f}) = \operatorname{lm}(\overline{f}_\mathfrak{p})$. Thus $\operatorname{lm}(\mathscr{I}_S) = \operatorname{lm}(\overline{I}_\mathfrak{p}) = \operatorname{lm}(\overline{f}_\mathfrak{p})$, so $\operatorname{lm}(\mathscr{I}_S) = \operatorname{lm}(\overline{I}_\mathfrak{p}) = \operatorname{lm}(G)$. Thus $\mathscr{I}_S$ is monic, so S is parametric by theorem 4.12. $\square$

This theorem allows us to characterize the leading monomials of $\mathscr{I}_S$.

4.14 · Corollary. *Let $I \subset A[X]$ be an ideal, $S \subset \operatorname{Spec}(A)$ be parametric for I, $\mathfrak{a} = \mathbf{I}(\overline{S})$ and let $\overline{I}$ be the image of I in $(A/\mathfrak{a})[X]$. Then $\operatorname{lm}(\mathscr{I}_S) = \operatorname{lm}(\overline{I})$.*

Proof. Let $m \in \operatorname{lm}(\mathscr{I}_S)$ and $\mathfrak{p} \in S$. Theorem 4.13 gives us that $\overline{I}_\mathfrak{p} \subset (A/\mathfrak{a})_\mathfrak{p}[X]$ is monic with $\operatorname{lm}(\overline{I}_\mathfrak{p}) = \operatorname{lm}(\mathscr{I}_S)$. So take some $P/Q \in \overline{I}_\mathfrak{p}$ with $\operatorname{lm}(P/Q) = m$. By lemma 4.10 we can take P/Q such that $\operatorname{lm}(P) = m$. Hence $\operatorname{lm}(\mathscr{I}_S) \subset \operatorname{lm}(\overline{I})$.

For the reverse inclusion, let $P \in \overline{I}$. By lemma 4.11 the element $P/1 \in \mathscr{I}_S$ has $\operatorname{lm}(P/1) = \operatorname{lm}(P)$, so $\operatorname{lm}(\overline{I}) \subset \operatorname{lm}(\mathscr{I}_S)$. $\square$

4.3 An aside on flatness

It is proven in [5] that if S is parametric for an ideal I, then the canonical morphism ϕ : $\operatorname{Spec}(A[X](I)) \to \operatorname{Spec}(A)$ is flat over S. However, the flatness of ϕ has no dependence on the monomial order on I, while the parametricity of S does. Thus we have the stronger proposition, that ϕ is flat over S if there is any monomial order, such that S is parametric for I. For example, the ideal $I = \langle ux + y \rangle \subset A[x, y]$ where $A = k[u]$, we have that $\operatorname{Spec}(A)$ is parametric if $y > x$, but not if $x > y$. So flatness of ϕ doesn't capture fully the parametricity of S.

Consider instead the familiy of rings $\mathcal{O}_{\{\mathfrak{p}\}}/\mathcal{I}_{\{\mathfrak{p}\}}$ indexed by closed points $\mathfrak{p} \in S$ for some locally closed set $S \subset \operatorname{Spec}(A)$. We wish to show that S is parametric if and only is this familiy is a flat

4.4 The singular ideal

In the last section, we showed that a locally closed set S is parametric for an in deal I if and only if $\mathcal{I}_S$ is a monic ideal in $\mathcal{O}_S[X]$. Given a locally closed set, we can use this to find the maximal parametric subset of S. This maximal set is closely linked to the concept of a *lucky* prime ideal. Here, we will only include what we need. For a more in-depth discussion, see [5].

4.15 · Definition (Lucky prime). A prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ is called *lucky* if $\operatorname{lc}(I, m) \not\subset \mathfrak{p}$ for all $m \in \operatorname{lm}(I)$.

4.16 · Definition (Singular ideal). Let $I \subset A[X]$ be an ideal and let M be the (unique) minimal set of generators of $\langle \operatorname{lm}(I) \rangle$. The *singular ideal* of I is the radical ideal

$$\mathbf{J}(I) = \sqrt{\prod_{m \in M} \operatorname{lc}(I, m)}$$

where $\operatorname{lc}(I, m) = \langle \{\operatorname{lc}(g) \mid g \in I \wedge \operatorname{lm}(g) = m\} \rangle$.

We have the following connection between lucky primes and the singular ideal.

4.17 · Lemma. *Let $I \subset A[X]$ be an ideal, then a prime $\mathfrak{p} \in \operatorname{Spec}(A)$ is lucky if and only if $\mathbf{J}(I) \not\subset \mathfrak{p}$, i.e. $\mathfrak{p} \notin \mathbf{V}(\mathbf{J}(I))$.*

Proof. Let M be the unique minimal set of generators of $\langle \operatorname{lm}(I) \rangle$. For the first implication, let $p \in \operatorname{Spec}(A)$ be lucky. For each $m \in M$, let $f_m \in I$ have $\operatorname{lm}(f) = m$. Since $\mathfrak{p}$ is lucky, we can choose the f_m such that $\operatorname{lc}(f_m) \notin \mathfrak{p}$ for every $m \in M$. Since $\mathfrak{p}$ is prime, we thus have $\prod_{m \in M} \operatorname{lc}(f_m) \notin \mathfrak{p}$. Hence $\mathbf{J}(I) \not\subset \mathfrak{p}$.

The reverse implication we prove by contraposition, so assume that $\mathfrak{p}$ is unlucky. $\mathfrak{p}$ being unlucky means there is some $m \in \operatorname{lm}(I)$ with $\operatorname{lc}(I, m) \subset \mathfrak{p}$. Now, there is some $m' \in M$ with $m' | m$. We have $\operatorname{lc}(I, m') \subset \operatorname{lc}(I, m)$, thus there is some $m' \in M$ with $\operatorname{lc}(I, m') \subset \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, this gives $\prod_{m \in M} \operatorname{lc}(I, m) \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this gives that $\sqrt{\prod_{m \in M} \operatorname{lc}(I, m)} \subset \mathfrak{p}$ and we are done. □

If we have a Gröbner basis of I, then $\mathbf{J}(I)$ is particularly easy to compute.

4.18 · Proposition. *Let $I \subset A[X]$ be an ideal, let G be a Gröbner basis for I and let M be the minimal set of generators of* $\mathrm{lm}(I)$*. Then*

$$\mathbf{J}(I) = \sqrt{\prod_{m \in G} \langle \mathrm{lc}(g) \mid g \in G, \mathrm{lm}(g) = m\rangle}$$

Proof. This follows from the equality

$$\mathrm{lc}(I, m) = \langle \mathrm{lc}(g) \mid g \in G, \mathrm{lm}(g) = m\rangle \quad \text{for all } m \in M$$

A generator c on the left side is the leading coefficient of a polynomial $f \in I$ with leading monomial m. Since G is a Gröbner basis, there is some $g \in G$ with $\mathrm{lt}(g) \mid \mathrm{lt}(f)$. By the minimality of M, we have $\mathrm{lm}(g) = \mathrm{lm}(f) = m$, thus $\mathrm{lc}(g) \mid \mathrm{lc}(f) = c$, so $\mathrm{lc}(I, m) \subset \langle \mathrm{lc}(g) \mid g \in G, \mathrm{lm}(g) = m\rangle$.

On the other hand, each generator on the right side is by definition a generator on the left side. ☐

4.19 · Example. Consider again the ideal $I = \langle ax + cy, bx + dy\rangle \subset A = \mathbb{C}[a, b, c, d][x, y]$ with a term order such that $x > y$. A Gröbner basis of I can be found by computing a reduced Gröbner basis of I in $\mathbb{C}[x, y, a, b, c, d]$ and is given by

$$G = \{ax + cy, bx + dy, (ad - bc)y\}.$$

The minimal set of generators of $\mathrm{lm}(I)$ is $M = \{x, y\}$, so by proposition 4.18 we find that

$$\mathbf{J}(I) = \sqrt{\langle a, b\rangle\langle ad - bc\rangle} = \langle ad - bc\rangle.$$

For any $\mathfrak{p} \in A \setminus \mathbf{V}(ad - bc)$, we have $ad - bc \notin \mathfrak{p}$, so $\frac{(ad-bc)y}{ad-bc} \in \mathscr{I}_A(\mathbf{V}(ad - bc))$. Hence, we get the reduced Gröbner basis $\{x, y\}$ for the ideal $\sigma_{\mathfrak{p}}(I)$.

Clearly, the leading monomial ideal of I will remain unchanged, if we specialize with a point away from the singular ideal, as illutrated above. However, it is not enough to have the function $\mathfrak{p} \mapsto \mathrm{lm}(\sigma_{\mathfrak{p}}(I))$ be constant on $\mathrm{Spec}(A)$. The leading monomials might stay the same, even though some leading coefficients of I vanishes.

4.20 · Example. Consider the ideal $I = \langle u^2x - u, ux^2 - x\rangle \subset \mathbb{C}[u][x]$. Here, we have $\mathrm{lm}(\sigma_{\mathfrak{p}}(I)) = \{x\}$ for all $\mathfrak{p} \in \mathrm{Spec}(\mathbb{C}[u])$, but $\mathrm{Spec}(\mathbb{C}[u])$ is not parametric for I. Indeed $I_{\langle u\rangle}$ is not monic, since we can't divide by u in $\mathbb{C}[u]_{\langle u\rangle}$, so $\mathrm{Spec}(\mathbb{C}[u])$ is not parametric for I by theorem 4.13.

The generators given above turns out to be a Gröbner basis of I:

$$G = \{u^2x - u, ux^2 - x\}$$

which means that the minimal set of generators of $\operatorname{lm}(I)$ is $M = \{x\}$, hence

$$\mathbf{J}(I) = \sqrt{\langle u^2 \rangle} = \langle u \rangle.$$

Considering the two cases, we see that

$$\langle \sigma_{\mathfrak{p}}(I) \rangle = \begin{cases} \langle \sigma_{\mathfrak{p}}(u)x - 1 \rangle & \sigma_{\mathfrak{p}}(u) \neq 0 \\ \langle x \rangle & \sigma_{\mathfrak{p}}(u) = 0 \end{cases}$$

which should make it clear why there is no parametric Gröbner basis for I on all of $\mathbb{C}[u]$.

As seen in this example, the singular ideal captures something more subtle than just the leading monomials staying unchanged. In fact, the singular ideal expresses exactly the points, that prevents a set from being parametric.

4.21 · Theorem. *Let $I \subset A[X]$ be an ideal, let $Z \subset \operatorname{Spec}(A)$ be closed and $\mathfrak{a} = \mathbf{I}(Z)$ and let $\bar{I}$ be the image of I in $(A/\mathfrak{a})[X]$. Then*

1. *$Z_{gen} := Z \setminus \mathbf{V}(\mathbf{J}(\bar{I}))$ is parametric for I with $\operatorname{lm}(\mathscr{I}_{Z_{gen}}) = \operatorname{lm}(\bar{I})$.*

2. *If $Y \subset Z$ is parametric for I with $\operatorname{lm}(\mathscr{I}_Y) = \operatorname{lm}(\bar{I})$, then $Y \subset Z_{gen}$.*

Proof. First, let's show that Z_{gen} is parametric. It is locally closed, so we just need to show that $\mathscr{I}_{Z_{gen}}$ has a reduced Gröbner basis. Let $m \in \operatorname{lm}(\mathscr{I}_{Z_{gen}})$. Let $f \in \mathscr{I}_{Z_{gen}}$ and for each $\mathfrak{p} \in Z_{gen}$ let $P_{\mathfrak{p}} \in \bar{I}$ and $Q_{\mathfrak{p}} \in (A/\mathfrak{a}) \setminus \mathfrak{p}$ such that $f(\mathfrak{p}) = P_{\mathfrak{p}}/Q_{\mathfrak{p}} \in \bar{I}_{\mathfrak{p}}$, with $\operatorname{coef}(f, m) = 0 \implies \operatorname{coef}(P_{\mathfrak{p}}, m) = 0$ for all monomials m. Then $\operatorname{lm}(f) = \operatorname{lm}(P_{\mathfrak{p}})$, so $\operatorname{lm}(P_{\mathfrak{p}}) = \operatorname{lm}(P_{\mathfrak{p}'})$ for all $\mathfrak{p}, \mathfrak{p}' \in Z_{gen}$. By possibly multiplying with a generator of $\mathfrak{p}$, we can assume $\operatorname{lc}(P_{\mathfrak{p}}) \in \mathfrak{p}$ for all $\mathfrak{p} \in Z_{gen}$.

Now, we need to produce a monic polynomial f' with the same leading monomial as f. Since for each $\mathfrak{p} \in Z_{gen}$ we have $\mathfrak{p} \notin \mathbf{V}(\mathbf{J}(\bar{I}))$, we can find some $P \in \bar{I}$

Now, we need to produce a monic polynomial f' with the same leading monomial as f. Take a finite cover $\{U_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}$ of Z_{gen} such that $f(\mathfrak{p}') = \frac{P_{\mathfrak{p}}}{Q_{\mathfrak{p}}}$ for every $\mathfrak{p}' \in U_{\mathfrak{p}}$. Let $d = \prod_{\mathfrak{p} \in \mathfrak{P}} \operatorname{lc}(P'_{\mathfrak{p}})$ and let $d_{\mathfrak{p}} = d / \operatorname{lc}(P'_{\mathfrak{p}})$. Since the $\mathfrak{p}$ are prime, we have $d \notin \mathfrak{p}$ for any $\mathfrak{p} \in \mathfrak{P}$. Thus $\operatorname{lc}(d_{\mathfrak{p}} P'_{\mathfrak{p}}) \notin \mathfrak{p}$. Also

have $\operatorname{lc}(P) \notin \mathfrak{p}$, which gives $\operatorname{lc}(P)\, Q_{\mathfrak{p}} \notin \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. Hence

$$f'(\mathfrak{p}) = \frac{P_{\mathfrak{p}}}{\operatorname{lc}(P_{\mathfrak{p}})\, Q_{\mathfrak{p}}}$$

is a monic polynomial in $\mathscr{I}_{Z_{gen}}$ with $\operatorname{lm}(f) = \operatorname{lm}(f')$. So $\mathscr{I}_{Z_{gen}}$ is a monic ideal in $\mathscr{O}_{Z_{gen}}[X]$, and so Z_{gen} is parametric by theorem 4.12.

Now, to show that Z_{gen} is maximal, let $Y \subset Z$ be parametric and assume $\operatorname{lm}(\mathscr{I}_Y) = \operatorname{lm}(\bar{I})$. Let $\mathfrak{b} = \mathbf{I}(\bar{Y})$ and let $G \subset \mathscr{I}_Y$ be the reduced Gröbner basis of $\mathscr{I}_Y$. Fix a $\mathfrak{p} \in Y$ and a $g \in G$.

By lemma 4.10 we find a $P/Q = g(\mathfrak{p})$ with $\mathrm{lm}(P) = \mathrm{lm}(g(\mathfrak{p}))$. Since $\mathrm{lm}(P) = \mathrm{lm}(g(\mathfrak{p})) = \mathrm{lm}(g) = \mathrm{lm}(\sigma_{\mathfrak{p}}(g))$, we have $\mathrm{lc}(P) \notin \mathfrak{p}$. Since $Y \subset Z$, that $\mathfrak{p}$ is also in Z. Furthermore, since $Y \subset Z$, we have $\mathfrak{a} \subset \mathfrak{b}$, so P is the image of some $P' \in \bar{I} \subset (A/\mathfrak{a})[X]$ in $(A/\mathfrak{b})[X]$. Thus $\mathrm{lc}(P)$ is the image of $\mathrm{lc}(P')$ in $A/\mathfrak{b}$. This means $\mathrm{lc}(P') \notin \mathfrak{p}$, hence $\mathbf{J}(\bar{I}) \not\subset \mathfrak{p}$. Since $\mathfrak{p}$ was arbitrary, $Y \cap \mathbf{V}(\mathbf{J}(\bar{I})) = \emptyset$, so $Y \subset Z_{gen}$. □

4.5 The projective case

Let $I \subset A[X]$ be an ideal. In the affine case we've seen that, even though $\mathrm{lm}(\sigma_{\mathfrak{p}}(I))$ is constant over all $\mathfrak{p}$ in some locally closed set S, that does not mean that S is parametric. Thus, it is quite difficult to give a "canonical" cover of $\mathrm{Spec}(A)$ with parametric sets. If I is homogenous, we are in luck.

4.22 · Theorem. *Let $I \subset A[X]$ be a homogenous ideal and $\mathfrak{p} \in \mathrm{Spec}(A)$. Then $\mathfrak{p}$ is lucky for I if and only if $\mathrm{lm}(\sigma_{\mathfrak{p}}(I)) = \mathrm{lm}(I)$.*

Proof. By theorem 4.21, we have the first implication. For the reverse implication, assume that $\mathrm{lm}(\sigma_{\mathfrak{p}}(I)) = \mathrm{lm}(I)$ and assume for a contradiction that $\mathfrak{p}$ is unlucky for I, i.e. there is some $m \in \mathrm{lm}(I)$ with $\mathrm{lc}(I, m) \subset \mathfrak{p}$. Since there are only finitely many monomials with the same degree as m, we can assume that for every m' with $\deg(m') = \deg(m)$, we have $\mathrm{lc}(I, m') \subset \mathfrak{p} \implies m' < m$. Since by assumption $\mathrm{lm}(I) = \mathrm{lm}(\sigma_{\mathfrak{p}}(I))$, we can find a $P \in I$ with $\mathrm{lm}(\sigma_{\mathfrak{p}}(P)) = m$, and since I is homogenous, we can assume that P is homogenous by lemma A.3. Because $<$ is a well-order, we can take P to have minimal leading monomial, i.e. if $P' \in I$ with $\mathrm{lm}(\sigma_{\mathfrak{p}}(P')) = m$ then $\mathrm{lm}(P) < \mathrm{lm}(P')$.

Since $\mathrm{lc}(I, m) \subset P$, we have $\mathrm{lt}(P) \gneq m$, and because $\deg(\mathrm{lt}(P)) = m$, we have $\mathrm{lc}(I, \mathrm{lm}(P)) \not\subset \mathfrak{p}$ since we assumed m to be maximal among the monomials of its degree. Therefor we can find some $Q \in I$ with $\mathrm{lm}(Q) = m = \mathrm{lm}(P)$ and $\mathrm{lc}(Q) \notin \mathfrak{p}$. Now, we can construct a new polynomial

$$P' = \mathrm{lc}(Q)P - \mathrm{lc}(P)Q$$

which has $\mathrm{lm}(P') < \mathrm{lm}(P)$. However, see that $\mathrm{coef}(P, m') \in \mathfrak{p}$ for every $m' > m$ and $\mathrm{lc}(P) \in \mathfrak{p}$.Hence, we have $\mathrm{coef}(P', m') \in \mathfrak{p}$ for every $m' > m$ since the corresponding terms on both sides of the subtraction has coefficients in p. Hence $\mathrm{lm}(\sigma_{\mathfrak{p}}(P')) \leq m$. But $\mathrm{lc}(q) \notin \mathfrak{p}$ and $\mathrm{coef}(P, m) \notin \mathfrak{p}$, so $\mathrm{lc}(Q)\,\mathrm{coef}(P, m) \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime. But $\mathrm{lc}(P) \in \mathfrak{p}$, so $\mathrm{coef}(P', m) \notin \mathfrak{p}$, thus $\mathrm{lc}(\sigma_{\mathfrak{p}}(P')) = m$. However, this contradicts the minimality of P. □

We are now ready for the grand finale in the projective case, namely that partitioning $\mathrm{Spec}(A)$ with respect to $\mathrm{lm}(\sigma_{\mathfrak{p}}(I))$ gives a canonical partition into (maximal) parametric sets. Specifically, if we partition $\mathrm{Spec}(A)$ be the equivalence relation $\mathfrak{p} \sim \mathfrak{p}'$ exactly when $\mathrm{lm}(\sigma_{\mathfrak{p}}(I)) = \mathrm{lm}(\sigma_{\mathfrak{p}'}(I))$, then the equivalence classes are parametric sets. Since the leading monomials of a parametric set must remain constant, these equivalence classes are maximal and disjoint, giving us the most natural and canonical Göbner cover.

Before we can prove this theorem, we need a technical lemma.

4.23 · Lemma. *Let $S_1, S_2, \dots, S_n \subset \operatorname{Spec}(A)$ be locally closed sets and let $C = \bigcup_{i=1}^n S_i$. Then the closure of C can be written uniquely as a union of irreducible closed sets, where none is contained in another:*

$$\overline{C} = Z_1 \cup Z_2 \cup \cdots \cup Z_m.$$

Furthermore, for each $i \in \{1, 2, \dots, m\}$ there is a j such that $Z_i \cap S_j \neq \emptyset$.

Proof. The unique decomposition is a standard theorem, see f.ex. proposition 3.6.15 in [3].

For the second part, fix an $i \in \{1, 2, \dots, m\}$ and find a j such that $Z_i \cap \overline{S_j} \neq \emptyset$. By applying proposition 3.6.15 in [3] again, we can split $\overline{S_j}$ into irreducible closed sets, and find one which intersects non-emptily with Z_i. Hence we can assume that $\overline{S_j}$ is irreducible.

Since $\overline{S_j}$ is irreducible, we must have $\overline{S_j} \subset Z_i$. If that was not the case, then

$$\overline{S_j} = (\overline{S_j} \cap Z_i) \cup (\overline{S_j} \cap \bigcup_{i' \neq i} Z_{i'})$$

and thus $\overline{S_j}$ would not be irreducible. Hence, $S_j \subset \overline{S_j} \subset Z_i$ as wanted. □

We're now ready to prove the main theorem.

4.24 · Theorem. *Let $I \subset A[X]$ be a homogenous ideal and let $S \subset \operatorname{Spec}(A)$ be locally closed. Then the equivalence classes of $S/\sim$ by the equivalence relation described above are parametric sets for I.*

Proof. By proposition 4.7, we can assume $S = \operatorname{Spec}(A)$. Indeed, if we prove that an equivalence class $Y \subset \operatorname{Spec}(A)$ is parametric, then $S \cap Y$ is a locally closed subset of Y. Thus $S \cap Y$ is parametric by Proposition 4.7. Since every equivalence of $S/\sim$ is of the form $S \cap Y$ for some equivalence class Y of $\operatorname{Spec}(A)/\sim$, this gives us what we want.

Let $Y \subset \operatorname{Spec}(A)$ be an equivalence class and let M we the constant value of $\operatorname{lm}(\sigma_{\mathfrak{p}}(I))$ for any $\mathfrak{p} \in Y$. Let $Z = \overline{Y}$ be the closure of Y, let $\mathfrak{a} = \mathbf{I}(Z)$ and let $\overline{I}$ be the image of I in $(A/\mathfrak{a})[X]$. The goal is to show that $Y = \overline{Y} \setminus \mathbf{V}(\mathbf{J}(\overline{I}))$, which is parametric by theorem 4.21. Note that for any $f \in I$ and $\mathfrak{p} \in Y$, we have $\sigma_{\mathfrak{p}}(f) = \sigma_{\mathfrak{p}}(f + \mathfrak{a})$, hence $M = \operatorname{lm}(\sigma_{\mathfrak{p}}(I)) = \operatorname{lm}(\sigma_{\mathfrak{p}}(\overline{I}))$. Since $\overline{I}$ is also homogenous, by theorem 4.22 (applied to $\overline{I}$) and lemma 4.17, we have for all $\mathfrak{p} \in \overline{Y}$ that if $\operatorname{lm}(\overline{I}) = \operatorname{lm}(\sigma_{\mathfrak{p}}(I))$ then $\mathfrak{p} \notin \mathbf{V}(\mathbf{J}(\overline{I}))$. Since Y is exactly those $\mathfrak{p}$, where $\operatorname{lm}(\sigma_{\mathfrak{p}}(I)) = M$, we just need to show that $\operatorname{lm}(\overline{I}) = M$.

By lemma 4.23, we can write Z as a union of irreducible, closed sets:

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_n.$$

For each i, let $\overline{I}_i$ denote the image of I in $(A/\mathbf{I}(Z_i))[X]$ and let $S_i = Z_i \setminus \mathbf{V}(\mathbf{J}(\overline{I}_i))$. Notice that since $\mathbf{I}(Z) \subset \mathbf{I}(Z_i)$, we have that $\sigma_{\mathfrak{p}}(\overline{I}_i) = \sigma_{\mathfrak{p}}(\overline{I})$ for all $\mathfrak{p} \in Z_i \subset \overline{Y}$. Also, by theorem 4.21 we have that S_i is parametric with $\operatorname{lm}(\mathcal{I}_{S_i}) = \operatorname{lm}(\overline{I}_i)$ and by theorem 4.6 $\operatorname{lm}(\mathcal{I}_{S_i}) = \operatorname{lm}(\sigma_{\mathfrak{p}}(\overline{I}_i))$ for all $\mathfrak{p} \in S_i$. By the second part of lemma 4.23, there is some $\mathfrak{p} \in S_i \cap Y$, so $\operatorname{lm}(\sigma_{\mathfrak{p}}(\overline{I}_i)) = M$

for all $\mathfrak{p} \in S_i$. Hence,

$$M = \mathrm{lm}(\sigma_{\mathfrak{p}}(\overline{I})) = \mathrm{lm}(\sigma_{\mathfrak{p}}(\overline{I}_i)) = \mathrm{lm}(\mathscr{I}_{S_i}) = \mathrm{lm}(\overline{I}_i) \quad \text{for all } \mathfrak{p} \in S_i.$$

Now, we use this to show that $\mathrm{lm}(\overline{I}) = M$. Let $P \in \overline{I}$, and let $\overline{P_i}$ denote the image of P in $\overline{I}_i$. If there is an i such that $\mathrm{lm}(P) = \mathrm{lm}(\overline{P_i})$, then $\mathrm{lm}(P) \in \mathrm{lm}(\overline{I}_i) = M$. On the other hand, if $\mathrm{lm}(P) > \mathrm{lm}(\overline{P_i})$ for all i, then $\mathrm{lc}(P) \in \mathbf{I}(Z_1) \cap \cdots \cap \mathbf{I}(Z_n) = \mathfrak{a}$. Thus, $\mathrm{lc}(P) = 0$, which is not allowed. This gives $\mathrm{lm}(\overline{I}) \subset M$.

For the reverse inclusion, take an $m \in M$. Since $M = \mathrm{lm}(\overline{I}_1)$, we can find some $P \in \overline{I}$ such that $\mathrm{lm}(\overline{P_1}) = m$ ($\overline{P_1}$ being the image of P in $\mathbf{I}(Z_1)$ as before). This means $\mathrm{coef}(P, m) \notin \mathbf{I}(Z_1)$ but $\mathrm{coef}(P, m') \in \mathbf{I}(Z_1)$ for all $m' > m$. If $n = 1$, then $Z = Z_1$ and we are done, so assume $n > 1$ and find some $c \in \bigcap_{i=2}^n \mathbf{I}(Z_i) \setminus \mathbf{I}(Z_1)$. Such an element exist, because the $\mathbf{I}(Z_i)$'s are a minimal primary decomposition of $\mathbf{I}(Z)$, so by minimality $\mathbf{I}(Z_1) \not\supset \bigcap_{i=2}^n \mathbf{I}(Z_i)$. Consider now the polynomial cP, which has the property that $\mathrm{coef}(cP, m') \in \mathbf{I}(Z)$ for all $m' > m$. Furthermore, since $\mathbf{I}(Z_1)$ is a radical, primary ideal, it is prime, so $\mathrm{coef}(cP, m) \notin \mathbf{I}(Z_1)$. This gives $\mathrm{coef}(cP, m) \notin \mathbf{I}(Z)$. Thus every term in cP larger than m is zero, so $\mathrm{lm}(cP) = m$. Thus $M \subset \mathrm{lm}(\overline{I})$, which completes the proof. □

4.6 Relation to the CGS algorithm

The **CGS** algorithm can be seen as an algorithm that computes Gröbner covers. Indeed, by inspecting the construction, we see that if $(E, \{h\}, G)$ is a segment in the output of $\mathbf{CGS}(F, S)$, then $V(E) \setminus V(\{h\})$ is a parametric set.

4.25 · Theorem. *Let $F \subset k[X, U]$ and $S \subset k[U]$ be finite sets of polynomials and let $\mathscr{H} = \mathbf{CGS}(F, S)$. If $(E, \{h\}, G) \in \mathscr{H}$, then $V(E) \setminus V(\{h\})$ is a parametric set and*

$$\{\frac{g}{\mathrm{lc}_U(g)} \mid g \in G\} \subset \mathscr{O}_{k[U]}[X]$$

is its reduced Gröbner basis.

Proof. Since $h = \mathrm{lcm}(\{\mathrm{lc}_U(g) \mid g \in G \setminus k[U]\})$, □

5 Applications

5.1 Quantifier elimination

One of the first applications of parametric Gröbner bases was presented by its inventor Weispfenning [4] in the original article. It concerns the problem of computing a system of polynomial equations, whose solutions are equivalent to solutions to a set of logical expressions involving polynomial equations, con- and disjunctions, negations and existential quantifiers.

Sepcifically, we're given a formula $\exists x_1, \dots, x_n : \phi(U, x_1, \dots, x_n)$ where ϕ is a combination using $\wedge$ and $\vee$ of polynomial equalities and inequalities in $k[U, X]$. If k_1 is an extension field of k, then that formula determines a partioning of $k_1^{|U|}$, namely those values of U where the formula is true and those where it isn't. Our goal is to find a system of polynomial equations in $k[U]$ that is satisfied in exactly the same points.

First, we need to normalize the logical expressions, to fit a format we can work with.

5.1 · Definition (Positive, primitive formula). A logical formula φ is called *positive and primitive* if it only involves polynomial equalities in $k[X]$, conjunctions and existential quantifiers.

5.2 · Lemma. *Let ϕ be a logical formula involving polynomial equalities, conjunctions, disjunctions, negations and existential quantifiers. Then there exists a finite set of positive, primitive formula $\varphi_1, \varphi_2, \dots, \varphi_r$ such that $\phi \iff (\varphi_1 \vee \dots \vee \varphi_r)$.*

Proof. Using standard logical rules, we can find $\phi_1, \dots, \phi_r$ containing only polynomial equalities, conjunction, negation and existential quantifiers such that

$$\phi \iff \bigvee_{i=1}^{r} \phi_r.$$

Using De Morgans law and distributivity we can assume that negations are at the lowest level of the formulas. Thus, we can see the ϕ_i's as existstential formulas containing conjunctions of polynomial equations and inequations.

Now, to eliminate the inequalities, we use the following trick:

$$f(X) \neq 0 \iff \exists t : f(X) \cdot t - 1 = 0. \qquad \square$$

Thus we can solve each of the positive, primitive formulas independently, and see if any of them are satifiable.

5.3 · Theorem. *Let $F \subset k[U, X]$ be a finite set of polynomials over an algebraically closed field and let G be a parametric Gröbner basis of F. For a polynomial $f \in k[U][X]$, let*

$C(f) \subset k[U]$ denote the set of coefficients of non-constant terms in f. Then

$$\left(\exists x_1, \dots, x_n : \bigwedge_{f \in F} f(U, x_1, \dots, x_n) = 0\right) \iff \bigwedge_{g \in G} \left(g(U, 0, \dots, 0) = 0 \vee \bigvee_{c \in C(g)} c(U) \neq 0\right)$$

in any extension field $k_1 \supset k$.

Proof. Let $\alpha \in k_1^{|U|}$. Then the question of whether $\exists x_1, \dots, x_n : \bigwedge_{f \in F} f(U, x_1, \dots, x_n) = 0$ is satisfied in $U = \alpha$ is equivalent to whether $\langle \sigma_\alpha(F) \rangle$ has a common zero, i.e. if $V(\langle \sigma_\alpha(F) \rangle) \neq \emptyset$.

For the first implication, assume $\exists x_1, \dots, x_n : \bigwedge_{f \in F} f(U, x_1, \dots, x_n) = 0$ is satisfied at some $\alpha \in k_1^{|U|}$. Let $\beta \in k_1^{|X|}$ be a vector of $(x_1, \dots, x_n)$ such that $f(\alpha, \beta) = 0$ for all $f \in F$. Then, since all $g \in G$ are also in $\langle F \rangle$, we get $g(\alpha, \beta) = 0 \ \forall g \in G$. Hence, if $g(\alpha, 0, \dots, 0) \neq 0$, then there has to be some non-constant term in g, which is also non-zero at α.

For the other implication, assume every $g \in G$ has zero constant term or some non-zero non-constant term, when viewed as a polynomial in $k[U][X]$. Assume for a contradiction that $V(\langle \sigma_\alpha(F) \rangle) = \emptyset$. By the weak Nullstellensatz we get that $1 \in \langle \sigma_\alpha(F) \rangle$. Since G is a parametric Gröbner basis, there is some $g \in G$ such that $\mathrm{lt}(\sigma_\alpha(g)) \mid 1$. Thus $\sigma_\alpha(g)$ is a constant polynomial with non-zero constant term, contradicting the assumption. $\square$

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A Miscellaneous results

In this section, we prove results that we need in the main text, but don't fit in the flow if the text. These are well-known results, which nevertheless aren't usually covered in introductionary algebra courses. Hence, we present them here.

A.1 Reduced Gröbner bases

A.2 The nilradical

The nilradical is the ideal of all nilpotent elements of a ring. It is widely used in the study of general rings. In our case, where the base ring is assumed to have no nilpotents, it is zero, but we still need a different characterization of it.

A.1 · Definition (Nilradical). Let A be a commutative ring. Then the ideal

$$\sqrt{\langle 0\rangle} = \{a \in A \mid \exists n \in \mathbb{N} \,:\, A^n = 0\}$$

is called the *nilradical.*

A.2 · Theorem. *Let A be a commutative ring, and let* $\operatorname{Spec}(A)$ *be the set of prime ideals of A. Then*

$$\sqrt{\langle 0\rangle} = \bigcap_{\mathfrak{p}\in\operatorname{Spec}(A)} \mathfrak{p}$$

Proof. First, a quick induction proof gives that every nilpotent element is in every $\mathfrak{p} \in \operatorname{Spec}(A)$. Indeed, $0 \in \mathfrak{p}$ and if $a^n = 0 \in \mathfrak{p}$, then either a or a^{n-1} is in p, since $\mathfrak{p}$ is prime. By induction, $a \in \mathfrak{p}$.

For the converse inclusion, □

A.3 Homogenous ideals

Here, we present a basic lemma about homogenous ideals.

A.3 · Lemma. *Let $I \subset A[X]$ be a homogenous ideal and let $f \in I$. Writing*

$$f = \sum_i f_i$$

where each f_i is homogenous, each $f_i \in I$.

Proof. Let $\{g_1, \dots, g_n\} \subset I$ be a finite set of homogenous generators of I, and let $f \in I$. Then we can write

$$f = \sum_{i=0}^{n} h_i g_i$$

for some $h_i \in A[X]$. Consider a single term of this sum, which we can write as

$$h_i g_i = \sum_j h_{i,j} g_i, \quad \text{where } h_i = \sum_j h_{i,j}.$$

Each term of this sum is homogenous and $h_{i,j}g_i \in I$. Since

$$f = \sum_{i,j} h_{i,j}g_i$$

is a sum of homogenous polynomials, and each term of the sum is homogenous and in I, each homogenous component of f is in I. $\square$