

Linear Decision Rule Approach

Zhan Lin

1 Original Problem

Consider a network system consisting of m resources, with capacity levels $\mathbf{c} = (c_1, \dots, c_m)^T$, and n products, with corresponding prices denoted by $\mathbf{v} = (v_1, \dots, v_n)^T$. Each products needs at most one unit of each resource. Let $A = (a_{ij})$ be the resource coefficient matrix, where $a_{ij} = 1$ if product j uses one unit of resource i and $a_{ij} = 0$ otherwise. Define $\xi = (1, \xi_{1,1}, \xi_{1,2}, \dots, \xi_{1,n}, \dots, \xi_{t,1}, \xi_{t,2}, \dots, \xi_{t,n})^T$ where $\xi_{t,j}$ is demand of product j in period t . Assuming observing τ periods of history, let ξ^t be the observed history demands, ξ_t be the demand at period t and

$$\mathbf{x}_t(\xi^{t-1}, \xi_t) = (x_{t,1}(\xi^{t-1}, \xi_{t,1}), x_{t,2}(\xi^{t-1}, \xi_{t,2}), \dots, x_{t,n}(\xi^{t-1}, \xi_{t,n}))^T$$

be the booking limits in period t . Realisation of ξ is limited to Ξ . The optimality equations can be expressed as

$$\begin{aligned} \max_{\mathbf{x}_t} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T \mathbf{x}_t(\xi^{t-1}, \xi_t) \right) \\ \text{s.t.} \quad & \sum_{t=1}^T A \mathbf{x}_t(\xi^{t-1}, \xi_t) \leq \mathbf{c} \\ & \mathbf{x}_t(\xi^{t-1}, \xi_t) \geq 0 \\ & \forall \xi \in \Xi, t = 1, \dots, T \end{aligned} \tag{1}$$

It's important to notice that actually $x_{t,j}(\xi^{t-1}, \xi_t) = u(\xi^{t-1}) \wedge \xi_t$. If making decisions without ξ^{t-1} , we will have $x_{t,j}(\xi_t) = u \wedge \xi_t$ and equation 2 .

$$\begin{aligned} \max_{\mathbf{x}_t} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T \mathbf{x}_t(\xi_t) \right) \\ \text{s.t.} \quad & \sum_{t=1}^T A \mathbf{x}_t(\xi_t) \leq \mathbf{c} \\ & \mathbf{x}_t(\xi_t) \geq 0 \\ & \forall \xi \in \Xi, t = 1, \dots, T \end{aligned} \tag{2}$$

Equation 2 can be reformulated as the following equation 3 according to Preservation of Structural Properties in Optimization with Decisions Truncated by Random Variables and Its Applications.

$$\begin{aligned}
& \max_{\mathcal{X}_t} \quad \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T \mathcal{X}_t(\xi_t) \right) \\
& s.t. \quad \mathcal{X}_t(\xi_t) \leq \xi_t \\
& \quad \{\mathcal{X}_t(\xi_t) = (\mathcal{X}_{t,1}(\xi_{t,1}), \mathcal{X}_{t,2}(\xi_{t,2}) \dots, \mathcal{X}_{t,j}(\xi_{t,j}))^T\} \in \mathcal{F} \\
& \quad \forall \xi \in \Xi, t = 1, \dots, T
\end{aligned} \tag{3}$$

where $\mathcal{F} = \{f(\xi_t) : f(\xi_t) = \mathbf{u}_t \wedge \xi_t, u_t \in \mathbb{R}^+, \sum_{t=1}^T A \mathbf{u}_t \leq \mathbf{c}\}$.

2 Linear Decision Rule Approach via Lifting

2.1 Piecewise Linear Continuous Decision Rules with Axial Segmentation

Define

$$L(\xi) = (L_{t,j}(\xi_{t,j}))_{t \times j \times r, 1}$$

where

$$L_{t,j}(x) = (L_{t,j,d}(x))_{r \times 1}$$

and

$$L_{t,j,d}(x) = \begin{cases} x & r = 1 \\ \min\{x, z_1^{t,j}\} & r > 1, d = 1 \\ \max\{\min\{x, z_d^{t,j}\} - z_{d-1}^{t,j}, 0\} & r > 1, d = 2, \dots, r-1 \\ \max\{x - z_{d-1}^{t,j}, 0\} & r > 1, d = r \end{cases}$$

as show in figure 1.

It is obvious that the linear retraction operator corresponding to $L_{t,j}$ is

$$R_{t,j}(y) = \sum_{d=1}^r y_d = e^T y$$

2.2 Primal Problem

Approach equation 3 with

$$x_{t,j}(\xi_{t,j}) = \tilde{X}_{t,j} Q_{t,j} L(\xi)$$

where \tilde{X} and $Q_{t,j}$ are both matrixes meanwhile $Q_{t,j}$ restricts $L(\xi)$ to $L(\xi_{t,j})$.

According to [1], with $\Xi = \{\xi : \xi_1 = 1, l_{t,j} \leq \xi_{t,j} \leq r_{t,j}\}$ and $\Xi' = L(\Xi)$,

$$\text{conv} \Xi' = \left\{ \xi' : \xi'_1 = 1, V_{t,j}^{-1}(1, \xi'_{t,j})_{(r+1) \times 1}^T \geq 0 \right\}$$

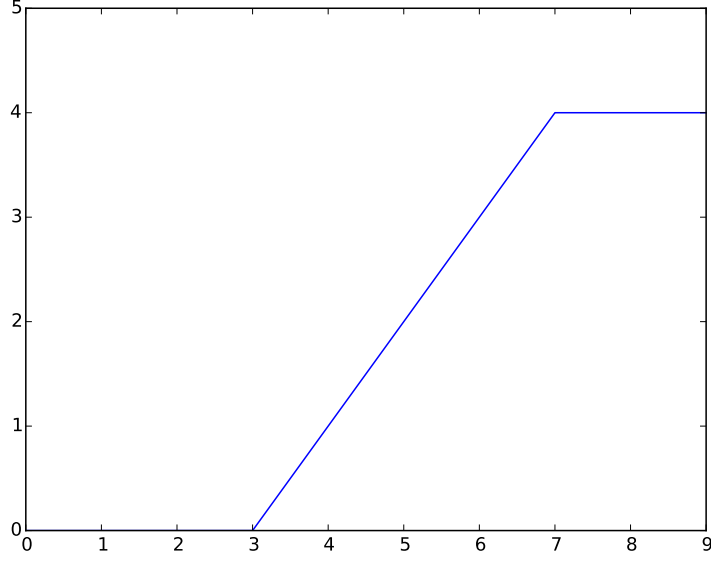


Figure 1: piecewise linear function

$$V_{t,j}^{-1} = \begin{pmatrix} \frac{z_1^{t,j}}{z_1^{t,j} - l_{t,j}} & -\frac{1}{z_1^{t,j} - l_{t,j}} & & & & \\ -\frac{l_{t,j}}{z_1^{t,j} - l_{t,j}} & \frac{1}{z_1^{t,j} - l_{t,j}} & -\frac{1}{z_2^{t,j} - z_1^{t,j}} & & & \\ & \frac{1}{z_2^{t,j} - z_1^{t,j}} & \frac{1}{z_2^{t,j} - z_1^{t,j}} & \ddots & & \\ & & \ddots & \ddots & -\frac{1}{z_{r-1}^{t,j} - z_{r-2}^{t,j}} & \\ & & & & \frac{1}{z_{r-1}^{t,j} - z_{r-2}^{t,j}} & -\frac{1}{u_{t,j} - z_{r-1}^{t,j}} \\ & & & & & \frac{1}{u_{t,j} - z_{r-1}^{t,j}} \end{pmatrix}$$

Then we obtain the primal problem equation 4

$$\begin{aligned} \max_{\tilde{X}_{t,k}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T \left(\sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k} \right) \xi \right) \\ \text{s.t.} \quad & \sum_{t=1}^T A \left(\sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k} \right) \xi \leq 2\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^T A \left(\sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k} \right) \xi \right) \\ & \tilde{X}_{t,k} Q_{t,k} \xi \leq e^T Q_{t,k} \xi \\ & \tilde{X}_{t,k} Q_{t,k} \xi \geq 0 \\ & \forall \xi \in \text{conv} \Xi', t = 1, \dots, T, k = 1, \dots, j \end{aligned} \tag{4}$$

The correction $\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^T A \left(\sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k} \right) \xi \right)$ arises from inaccurately approaching $\mathbf{x}(\xi^{t-1}, \xi_t)$. Also, there are $O(n^2 t^2 r)$ variables and $O(t^2 n^2 r + tnmr)$ constraints in total.

The solution $\tilde{X}_{t,j}(\xi_{t,j})$ of equation 4 has same structure with either figure 2 or figure 3. The proof is trivial.

Proof. As we can see from constraints, defining $\tilde{X}_{t,k}(\xi_{t,k}) = \tilde{X}_{t,k} Q_{t,k} \xi$, $\tilde{X}_{t,k}(\xi_{t,k}) \leq \xi_{t,k}$. Therefore the angle

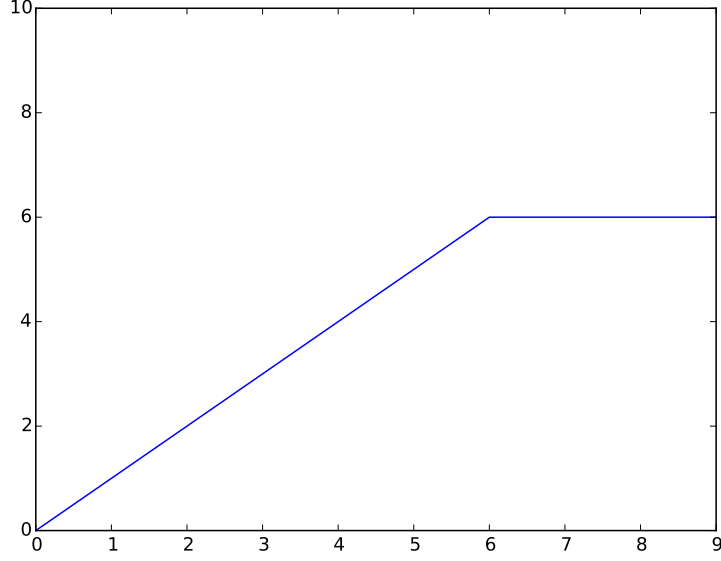


Figure 2: $z_k^{t,j}$ happened to be a break point

at $(0,0)$ must less than 45° . On the other hand, if we keep $\tilde{X}_{t,k}(\xi_{t,k})$ no longer growing after the maximum point, it will never violate constraints as it doesn't need more resource than the maximum point which could be achieved. So $\tilde{X}_{t,k}(\xi_{t,k})$ must be flatten after the break point. It also hint us that we could learn booking limits from the break point. \square

However, the problem is that we cannot adjust booking limits dynamically with information before the clock. Elements of $\tilde{X}_{t,j}$ are not able to do changes when information is provided. Another way to approach the process is adding items $X_t P_t L(\xi)$ as shown below. P_t will restrict $L(\xi)$ to $L(\xi^{t-1})$. In that case, we need to set low bound of $\xi_{t,j}$ greater than 0. There will be $O(nmrt\tau + n^2t^2r)$ variables and $O(t^2n^2r + tnmr)$ constraints in total.

$$\begin{aligned}
& \max_{X, \tilde{X}_{t,k}} \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \right) \\
& \text{s.t.} \quad \sum_{t=1}^T A(X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \leq 2\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^T A(X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \right) \\
& \quad (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \leq e^T Q_{t,k} \xi \\
& \quad (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \geq 0 \\
& \quad \forall \xi \in \text{conv} \Xi', t = 1, \dots, T
\end{aligned} \tag{5}$$

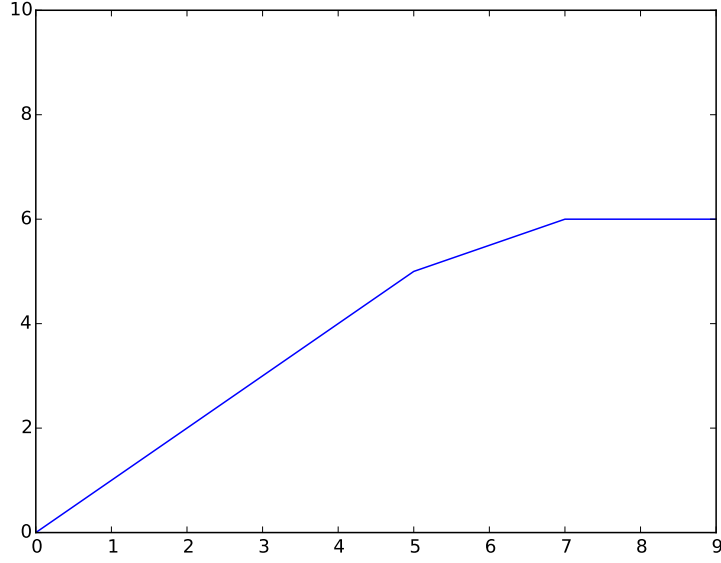


Figure 3: every $z_k^{t,j}$ could not be a break point

3 Computational Results

As we have shown above, $\tilde{X}_{t,j}(\xi^{t-1}, \xi_{t,j})$ should equal $u_{t,j}(\xi^{t-1}) \wedge \xi_{t,j}$. Benefiting from piecewise linear function, the structure is preserved with linear decision rule. Therefore, it's convenient to implement the policy as we can get u from $\tilde{X}_{t,j}$. Results are computed at servers locating at USTC with 96 Intel Xeon CPU E5-4657L v2 2.4GHz and 256GB memory. Gurobi will make use of 32 cpu to solve the LP problem parallelly.

References

- [1] Georghiou, A., Wiesemann, W. and Kuhn, D., 2015. Generalized decision rule approximations for stochastic programming via liftings. *Mathematical Programming*, 152(1-2), pp.301-338.

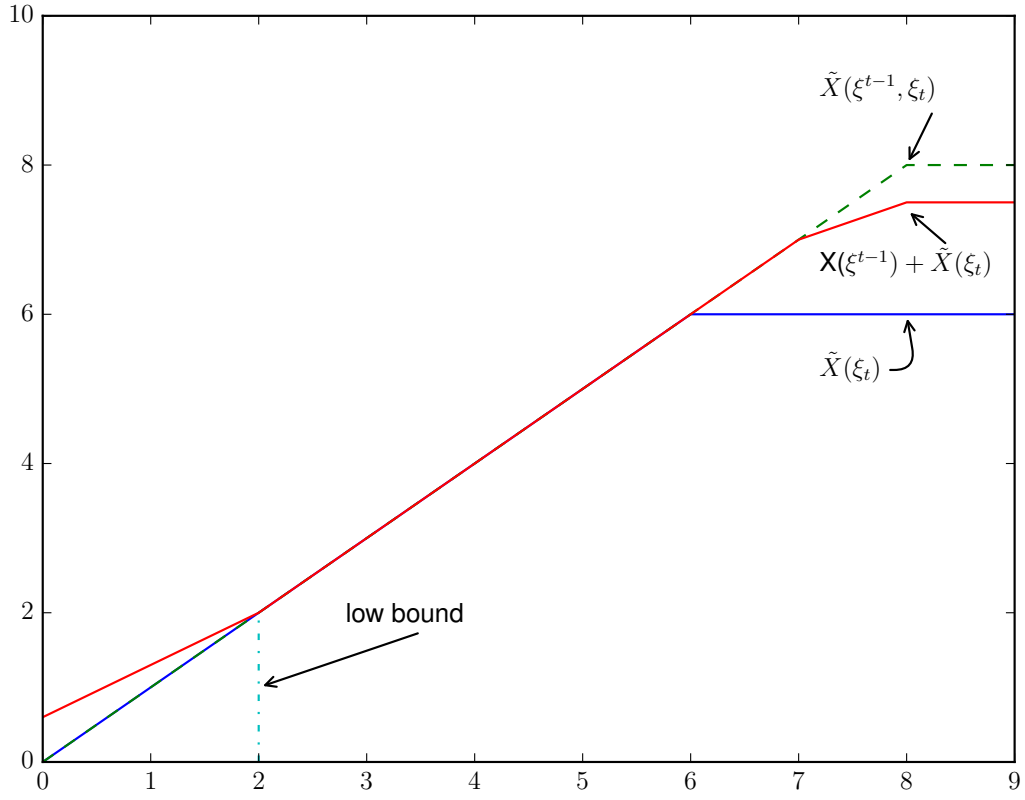


Figure 4: shape of functions

	DLP-alloc	SLP-alloc	ceil	objective value
Example 1 without re-solving	401980	415410	413227	409255
Example 1 with re-solving	409740	421894		
reduction of ALP			18844	18159

Table 1: Only depend on ξ_t

	floor(x)+1	floor(x)+2	floor(x)+3	Wait-and-see/truth value
Example 1 without re-solving	417761	420294	414305	432730
Example 1 with re-solving				432730
reduction of ALP	19414	19038	18474	20411

Table 2: Only depend on ξ_t

	DLP-alloc	SLP-alloc	ceil	objective value
Example 1 without re-solving	401980	415410	413477	410787
Example 1 with re-solving	409740	421894		
reduction of ALP			18903	18155

Table 3: Depend both on ξ^{t-1} and ξ_t

	floor(x)+1	floor(x)+2	floor(x)+3	Wait-and-see/truth value
Example 1 without re-solving	418139	420341	414183	432730
Example 1 with re-solving				432730
reduction of ALP	19430	18893	18425	20411

Table 4: Depend both on ξ^{t-1} and ξ_t

Case	Parameters	Solving LP Time/s	Total Time/s
reduction of ALP(Only depend on ξ_t)	$t = 5, \tau = 0, r = 10$	6.3	86.7
reduction of ALP(Depend on both ξ^{t-1} and ξ_t)	$t = 5, \tau = 5, r = 10$	41.07	137.83
Example 1(Only depend on ξ_t)	$t = 5, \tau = 0, r = 10$	20.72	208.81
Example 1(Depend on both ξ^{t-1} and ξ_t)	$t = 5, \tau = 5, r = 10$	204.82	435.97

Table 5: Computational Time