Linear Decision Rule Approach

Zhan Lin

1 Original Problem

Consider a network system consisting of m resources, with capacity levels $\mathbf{c} = (c_1, \ldots, c_m)^T$, and n products, with corresponding prices denoted by $\mathbf{v} = (v_1, \ldots, v_n)^T$. Each products needs at most one unit of each resource. Let $A = (a_{ij})$ be the resource coefficient matrix, where $a_{ij} = 1$ if product j uses one unit of resource i and $a_{ij} = 0$ otherwise. Define $\xi = (1, \xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,n}, \ldots, \xi_{t,1}, \xi_{t,2}, \ldots, \xi_{t,n})^T$ where $\xi_{t,j}$ is demand of product j in period t. Assuming observing τ periods of history,let ξ^t be the observed history demands, ξ_t be the demand at period t and

$$\mathbf{x_t}(\xi^{t-1}, \xi_t) = \big(x_{t,1}(\xi^{t-1}, \xi_{t,1}), x_{t,2}(\xi^{t-1}, \xi_{t,2}), \ldots, x_{t,n}(\xi^{t-1}, \xi_{t,n})\big)^T$$

be the booking limits in period t. Realisation of ξ is limited to Ξ. The optimality equations can be expressed as

$$\begin{aligned}
\max_{\mathbf{x_t}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^{T} \mathbf{v}^T \mathbf{x_t} \left(\xi^{t-1}, \xi_t\right) \right) \\
s.t. \quad & \sum_{t=1}^{T} A \mathbf{x_t} \left(\xi^{t-1}, \xi_t\right) \leq \mathbf{c} \\
& \mathbf{x_t}(\xi^{t-1}, \xi_t) \geq 0 \\
& \forall \xi \in \Xi, t = 1, \ldots, T
\end{aligned} \tag{1}$$

It's important to notice that actually $x_{t,j}(\xi^{t-1}, \xi_t) = u(\xi^{t-1}) \bigwedge \xi_t$ and equation 1 can be reformulated as the following equation 2 according to Preservation of Structural Properties in Optimization with Decisions Truncated by Random Variables and Its Applications.

$$
\begin{aligned}
\max_{\mathcal{X}_{\mathbf{t}}} \quad & \mathbb{E}_{\xi}\left(\sum_{t=1}^{T} \mathbf{v}^{T} \mathcal{X}_{\mathbf{t}}\left(\xi_{t}\right)\right) \\
s.t. \quad & \sum_{t=1}^{T} A \mathcal{X}_{\mathbf{t}}\left(\xi_{t}\right) \leq \mathbf{c} \\
& 0 \leq \mathcal{X}_{t}(\xi_{t}) \leq \xi_{t} \\
& \mathcal{X}_{\mathbf{t}}(\xi_{t}) = (\mathcal{X}_{t,1}(\xi_{t,1}), \mathcal{X}_{t,2}(\xi_{t,2}) \ldots, \mathcal{X}_{t,j}(\xi_{t,j}))^{T} \in \mathcal{F}(\xi^{t-1}) \\
& \forall \xi \in \Xi, t = 1, \ldots, T
\end{aligned}
\tag{2}
$$

where $\mathcal{F}(\xi^{t-1}) = \{f(\xi_t) : f(\xi_t) = u(\xi^{t-1}) \bigwedge \xi_t\}$.

2 Linear Decision Rule Approach via Lifting

2.1 Primal Problem

Approach original problem with

$$x_{t,j}(\xi^{t-1}, \xi_{t,j}) = (X_t P_t L(\xi))_j + \tilde{X}_{t,j} Q_{t,j} L(\xi)$$

where

$$L_{t,j}(x) = (L_{t,j,d}(x))_{r_{t,j} \times 1}$$

with

$$
L_{t,j,d}(x) = \begin{cases}
x & r_{t,j} = 1 \\
\min\{x, z_1^{t,j}\} & r_{t,j} > 1, d = 1 \\
\max\{\min\{x, z_d^{t,j}\} - z_{d-1}^{t,j}, 0\} & r_{t,j} > 1, d = 2, \ldots, r_{t,j} - 1 \\
\max\{x - z_{d-1}^{t,j}, 0\} & r_{t,j} > 1, d = r_{t,j}
\end{cases}
$$

and

$$L(\xi^{t-1}) = (L_{\tilde{t},j}(\xi_{\tilde{t},j}))_{\tau \times j, 1}$$

It is obvious that the linear retraction operator corresponding to $L_{t,j}$ is

$$R_{t,j}(y) = \sum_{d=1}^{r_{t,j}} y_d = e^T y$$

According to [1], with $\Xi = \{\xi : \xi_1 = 1, l_{t,j} \le \xi_{t,j} \le r_{t,j}\}$,

$$\mathrm{conv}\Xi' = L(\Xi) = \left\{\xi' : \xi'_1 = 1, V_{t,j}^{-1}(1, \xi'_{t,j})^T_{(d+1)\times 1} \ge 0\right\}$$

$$V_{t,j}^{-1} = \begin{pmatrix} \frac{z_1^{t,j}}{z_1^{t,j} - l_{t,j}} & -\frac{1}{z_1^{t,j} - l_1^{t,j}} & & & & \\ -\frac{l_{t,j}}{z_1^{t,j} - l_{t,j}} & \frac{1}{z_1^{t,j} - l_{t,j}} & -\frac{1}{z_2^{t,j} - z_1^{t,j}} & & & \\ & & \frac{1}{z_2^{t,j} - z_1^{t,j}} & \ddots & & \\ & & & \ddots & -\frac{1}{z_{r_{t,j}-1}^{t,j} - z_{r_{t,j}-2}^{t,j}} & \\ & & & & \frac{1}{z_{rt,j-1}^{t,j} - z_{r_{t,j}-2}^{t,j}} & -\frac{1}{u_{t,j} - z_{r_{t,j}-1}^{t,j}} \\ & & & & & \frac{1}{u_{t,j} - z_{r_{t,j}-1}^{t,j}} \end{pmatrix}$$

Then we obtain the primal problem

$$\begin{aligned} \max \quad & \mathbb{E}_\xi \left(\textstyle\sum_{t=1}^T \mathbf{v}^T (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k})\xi\right) \\ \text{s.t.} \quad & \textstyle\sum_{t=1}^T A(X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k})\xi \le 2\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^T A(X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k})\xi\right) \\ & (X_t P_t + \textstyle\sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k})\xi \le e^T p_t \xi \\ & (X_t P_t + \textstyle\sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k})\xi \ge 0 \\ & \forall \xi \in \mathrm{conv}\Xi', t = 1, \dots, T \end{aligned} \tag{3}$$

The correction $\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^T A(X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k})\xi\right)$ arises from inaccurately approaching $\mathbf{x}(\xi^{t-1}, \xi_t)$.

There are $O(nmrt\tau + n^2t^2r)$ variables and $O(t^2n^2r + tnmr)$ constraints in total.

3 Computational Results

As we can see from Preservation of Structural Properties in Optimization with Decisions Truncated by Random Variables and Its Applications, $\tilde{X}_{t,j}(\xi^{t-1}, \xi_{t,j})$ should behave $u \bigwedge \xi_{t,j}$. Benefiting from piecewise linear function, the structure is preserved with linear decision rule. Therefore, it's convenient to implement the policy as we can get u from $\tilde{X}_{t,j}$. What's more, rounding can matter a lot

	DLP-alloc	SLP-alloc	round	ceil
Example 1 without re-solving	401980	415410	411651	421684
Example 2 without re-solving	583630	595620	590245	604264
Example 1 with re-solving	409740	421894		
Example 2 with re-solving	594021	604859		

Table 1: Summary

	floor	floor(x)+1	Wait-and-see
Example 1 without re-solving	374129	422066	432730
Example 2 without re-solving	560178	591747	623530
Example 1 with re-solving			432730
Example 2 with re-solving			623530

Table 2: Summary

Case	Parameters	Computational Time/s	Optimal Objective	Best Simulation Result
1	$t = 10, \tau = 0, r = 7$	22.25	415305	421684
2	$t = 10, \tau = 0, r = 7$	29.49	596130	604264
2	$t = 10, \tau = 5, r = 7$	27280.45	599866	601999
2	$t = 5, \tau = 5, r = 5$	18.94	557856	562202

Table 3: Summary

to benefits and some simple rounding policy is tested including ceiling, flooring, rounding, floor(x)+1. Results are computed at servers locating at USTC with 96 Intel Xeon CPU E5-4657L v2 2.4GHz and 256GB memory.Gurobi will make use of 32 cpu to solve the LP problem parallelly.

References

[1] Georghiou, A., Wiesemann, W. and Kuhn, D., 2015. Generalized decision rule approximations for stochastic programming via liftings. Mathematical

Programming, 152(1-2), pp.301-338.