

# Linear Decision Rule Approach

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## 1 Original Problem

Consider a network system consisting of  $m$  resources, with capacity levels  $\mathbf{c} = (c_1, \dots, c_m)^T$ , and  $n$  products, with corresponding prices denoted by  $\mathbf{v} = (v_1, \dots, v_n)^T$ . Each products needs at most one unit of each resource. Let  $A = (a_{ij})$  be the resource coefficient matrix, where  $a_{ij} = 1$  if product  $j$  uses one unit of resource  $i$  and  $a_{ij} = 0$  otherwise. Define  $\xi = (1, \xi_{1,1}, \xi_{1,2}, \dots, \xi_{1,n}, \dots, \xi_{t,1}, \xi_{t,2}, \dots, \xi_{t,n})^T$  where  $\xi_{t,j}$  is demand of product  $j$  in period  $t$ . Assuming observing  $\tau$  periods of history, let  $\xi^t$  be the observed history demands,  $\xi_t$  be the demand at period  $t$  and

$$\mathbf{x}_t(\xi^t, \xi_t) = (x_{t,1}(\xi^t, \xi_{t,1}), x_{t,2}(\xi^t, \xi_{t,2}), \dots, x_{t,n}(\xi^t, \xi_{t,n}))^T$$

be the booking limits in period  $t$ . Realisation of  $\xi$  is limited to  $\Xi$ . The optimality equations can be expressed as

$$\begin{aligned} \max \quad & \mathbb{E}_\xi \left( \sum_{t=1}^T \mathbf{v}^T \mathbf{x}_t(\xi^t, \xi_t) \right) \\ s.t. \quad & \sum_{t=1}^T A \mathbf{x}_t(\xi^t, \xi_t) \leq \mathbf{c} \\ & \mathbf{x}_t(\xi^t, \xi_t) \leq \xi_t \\ & \mathbf{x}_t(\xi^t, \xi_t) \geq 0 \\ & \forall \xi \in \Xi, t = 1, \dots, T \end{aligned} \tag{1}$$

## 2 Linear Decision Rule Approach via Lifting

### 2.1 Primal Problem

Approach original problem with

$$x_{t,j}(\xi^t, \xi_{t,j}) = (X_t L_t(\xi^t))_j + \tilde{X}_{t,j} L_{t,j}(\xi_{t,j}) = (X_t P_t L(\xi))_j + \tilde{X}_{t,j} Q_{t,j} L(\xi)$$

where

$$L_{t,j}(x) = (L_{t,j,d}(x))_{r_{t,j} \times 1}$$

with

$$L_{t,j,d}(x) = \begin{cases} x & r_{t,j} = 1 \\ \min\{x, z_1^{t,j}\} & r_{t,j} > 1, d = 1 \\ \max\{\min\{x, z_d^{t,j}\} - z_{d-1}^{t,j}, 0\} & r_{t,j} > 1, d = 2, \dots, r_{t,j} - 1 \\ \max\{x - z_{d-1}^{t,j}, 0\} & r_{t,j} > 1, d = r_{t,j} \end{cases}$$

and

$$L(\xi^t) = (L_{t,j}(\xi_{t,j}))_{\tau \times j, 1}$$

It is obvious that the linear retraction operator corresponding to  $L_{t,j}$  is

$$R_{t,j}(y) = \sum_{d=1}^{r_{t,j}} y_d = e^T y$$

According to [1], with  $\Xi = \{\xi : \xi_1 = 1, l_{t,j} \leq \xi_{t,j} \leq r_{t,j}\}$ ,

$$\text{conv}\Xi' = L(\Xi) = \left\{ \xi' : \xi'_1 = 1, V_{t,j}^{-1}(1, \xi'_{t,j})_{(d+1) \times 1}^T \geq 0 \right\}$$

$$V_{t,j}^{-1} = \begin{pmatrix} \frac{z_1^{t,j}}{z_1^{t,j} - l_{t,j}} & -\frac{1}{z_1^{t,j} - l_{t,j}} & & & & \\ -\frac{l_{t,j}}{z_1^{t,j} - l_{t,j}} & \frac{1}{z_1^{t,j} - l_{t,j}} & -\frac{1}{z_2^{t,j} - z_1^{t,j}} & & & \\ & & \frac{1}{z_2^{t,j} - z_1^{t,j}} & \ddots & & \\ & & & \ddots & -\frac{1}{z_{r_{t,j}-1}^{t,j} - z_{r_{t,j}-2}^{t,j}} & \\ & & & & \frac{1}{z_{r_{t,j}-1}^{t,j} - z_{r_{t,j}-2}^{t,j}} & -\frac{1}{u_{t,j} - z_{r_{t,j}-1}^{t,j}} \\ & & & & & \frac{1}{u_{t,j} - z_{r_{t,j}-1}^{t,j}} \end{pmatrix}$$

Then we obtain the primal problem

$$\begin{aligned}
\max \quad & \mathbb{E}_\xi \left( \sum_{t=1}^T \mathbf{v}^T (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \right) \\
\text{s.t.} \quad & \sum_{t=1}^T A(X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \leq \mathbf{c} \\
& (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \leq e^T p_t \xi \\
& (X_t P_t + \sum_{k=1}^j \tilde{X}_{t,k} Q_{t,k}) \xi \geq 0 \\
& \forall \xi \in \text{conv} \Xi', t = 1, \dots, T
\end{aligned} \tag{2}$$

There are  $O(nmrtr + n^2 t^2 r)$  variables and  $O(t^2 n^2 r + tnmr)$  constraints in total.

### 3 Computational Results

As we can see from Preservation of Structural Properties in Optimization with Decisions Truncated by Random Variables and Its Applications,  $\tilde{X}_{t,j}(\xi^t, \xi_{t,j})$  should behave  $u \wedge \xi_{t,j}$ . Benefiting from piecewise linear function, the structure is preserved with linear decision rule. Therefore, it's convenient to implement the policy as we can get  $u$  from  $\tilde{X}_{t,j}$ . What's more, rounding can matter a lot to benefits and some simple rounding policy is tested including ceiling, flooring, rounding,  $\text{floor}(x)+1$ . Results are computed at servers locating at USTC with 96 Intel Xeon CPU E5-4657L v2 2.4GHz and 256GB memory. Gurobi will make use of 32 cpu to solve the LP problem parallely.

### References

- [1] Georghiou, A., Wiesemann, W. and Kuhn, D., 2015. Generalized decision rule approximations for stochastic programming via liftings. Mathematical Programming, 152(1-2), pp.301-338.

	DLP-alloc	SLP-alloc	round	ceil
Example 1 without re-solving	401980	415410	411651	421684
Example 2 without re-solving	583630	595620	590245	604264
Example 1 with re-solving	409740	421894		
Example 2 with re-solving	594021	604859		

Table 1: Summary

	floor	floor(x)+1	Wait-and-see
Example 1 without re-solving	374129	422066	432730
Example 2 without re-solving	560178	591747	623530
Example 1 with re-solving			432730
Example 2 with re-solving			623530

Table 2: Summary

Case	Parameters	Computational Time/s	Optimal Objective	Best Simulation Result
1	$t = 10, \tau = 0, r = 7$	22.25	415305	421684
2	$t = 10, \tau = 0, r = 7$	29.49	596130	604264
2	$t = 10, \tau = 5, r = 7$	27280.45	599866	601999
2	$t = 5, \tau = 5, r = 5$	18.94	557856	562202

Table 3: Summary