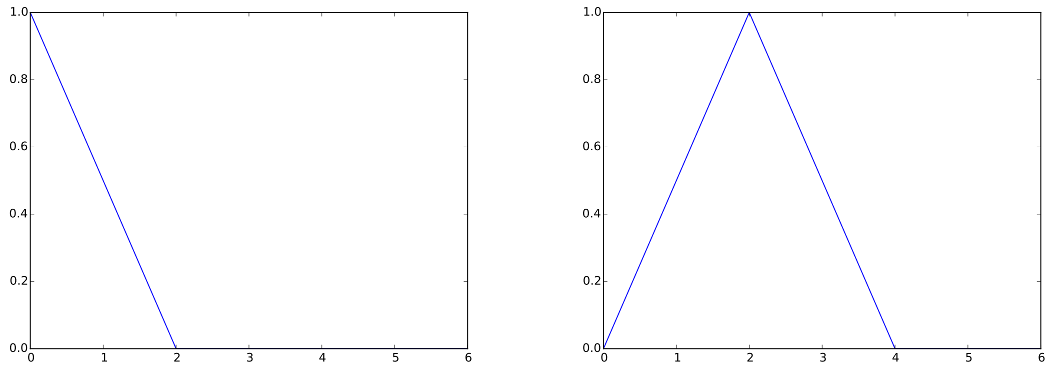

Figure 1: basis function for single variable scenarios

1 Single Variable Scenarios

Let ξ be the uncertain demand $\in [l, u]$ and $\{x_i, 0 \le i \le n : x_0 = l, x_n = u\}$ be breakpoints. Define

$$L_0(\xi) = \begin{cases} 1 - (\xi - l)/(x_1 - l) & \xi \in [l, x_1] \\ 0 & \xi \in (x_1, u] \end{cases}$$

$$L_1(\xi) = \begin{cases} (\xi - l)/(x_1 - l) & \xi \in [l, x_1] \\ 1 - (\xi - x_1)/(x_2 - x_1) & \xi \in (x_1, x_2] \\ 0 & \xi \in (x_2, u] \end{cases}$$

as shown in figure 1 and so on. Actually, we have

$$L_i(\xi) = \begin{cases} 0 & \xi \in [0, x_{i-1}] \\ (\xi - x_{i-1})/(x_i - x_{i-1}) & \xi \in (x_{i-1}, x_i] \\ 1 - (\xi - x_i)/(x_{i+1} - x_i) & \xi \in (x_i, x_{i+1}] \\ 0 & \xi \in (x_{i+1}, u] \end{cases}$$

for $0 \le i \le n$.Define lifting operator

$$L(\xi) = \begin{pmatrix} L_0(\xi) \\ L_1(\xi) \\ \dots \\ L_n(\xi) \end{pmatrix}$$

It is not hard to get corresponding retraction operator

$$R \circ L(\xi) = \sum_{i=0}^{n} x_i L_i(\xi) = \xi$$

If we are going to solve original problem following where $\xi, \mathbf{c}(\xi)$ and $\mathbf{x}(\xi)$ are all vectors.

$$\begin{aligned} \min \quad & \mathbb{E}_{\xi}\mathbf{c}^T\mathbf{x}(\xi) \\ s.t. \quad & Z(\mathbf{x}(\xi)) \geq 0 \\ & \forall \xi \in \Xi\{(\xi_i)_{len} : l_i \leq \xi_i \leq u_i\} \end{aligned}$$

We have to consider about defining L again,that is

$$L^i(\xi) = \begin{pmatrix} L_0(\xi_i) \\ L_1(\xi_i) \\ \cdots \\ L_n(\xi_i) \end{pmatrix}$$

and

$$\tilde{L}(\xi) = \begin{pmatrix} L^1(\xi) \\ L^2(\xi) \\ \cdots \\ L^{len}(\xi) \end{pmatrix}$$

$$\tilde{R} \circ \tilde{L}(\xi) = \begin{pmatrix} R^1(L^1(\xi)) \\ R^2(L^2(\xi)) \\ \cdots \\ R^{len}(L^{len}(\xi)) \end{pmatrix}$$

Approaching the original problem with lifting variables, we will have below equation

$$\begin{aligned} \min \quad & \mathbb{E}_{R(\xi)}\mathbf{c}^T X\xi \\ s.t. \quad & ZX\xi \geq 0 \\ & \forall \xi \in \tilde{L}(\Xi) \end{aligned}$$

And actually $\tilde{R}(\cdot)$ could be replaced by a matrix $\hat{R}$. Now we are going to discuss about $L(\Xi)$.As we want to utilize duality arguments, so we need to replace $\tilde{L}(\Xi)$ with $conv\tilde{L}(\Xi)$. It could be no better if we don't have to add more constraints in LP during lifting.As we all know,

$$conv\tilde{L}(\Xi) = \prod_{i=1}^{len} convL^i(\Xi)$$

So we only need to care about singe variable lifting $L^i(\xi) = L(\xi_i)$ again. That is why I call it single variable scenarios.For the sake of simplicity synonym, I will omit index i on both $L^i(\cdot)$ and ξ_i now.As for $convL(\xi)$,actually we have

$$convL(\Xi) = conv\{\mathbf{e_0}, \cdots, \mathbf{e_n}\}$$

where $\{\mathbf{e_i}\}$ are standard basis in euclidian space. It is because that for any realisation $\xi \in \Xi$, at most two elements of ξ are non-zero. Let ith and $i+1$th element be the non-zero. Benefiting from piecewise linear basis functions, in fact, we have

$$\xi = \xi_i \mathbf{e_i} + \xi_{i+1}\mathbf{e_{i+1}}$$

$$\xi_i + \xi_{i+1} = 1, \xi_i \geq 0, \xi_{i+1} \geq 0$$

Therefore,

$$conv L(\Xi) \subseteq conv\{\mathbf{e_0}, \cdots, \mathbf{e_n}\}$$

In the meantime,

$$\{\mathbf{e_0}, \cdots, \mathbf{e_n}\} \subseteq conv L(\Xi)$$

So we can tell that

$$conv L(\Xi) = conv\{\mathbf{e_0}, \cdots, \mathbf{e_n}\}$$

What's more,as $conv\{\mathbf{e_0}, \cdots, \mathbf{e_n}\}$ are corner points of the feasible region, we can keep constraints unchanged.

2 Multivariate Scenarios

Same as single variable scenarios, let (ξ, η) still be the uncertain demand,but now we have to do triangulation on 2-d mesh grid. Let $\{(x_m, y_m), 0 \leq m \leq n\}$ be points on triangular meshs. As shown in figure 2, piecewise linear function $L_i(\xi, \eta)$ now is defined by

$$L_i(x_i, y_i) = 1$$

$$L_i(\hat{x}, \hat{y}) = 0, \forall (\hat{x}, \hat{y}) \in \{(x_m, y_m), 0 \leq m \leq n\} \backslash \{(x_i, y_i)\}$$

Also, we can define

$$L(\xi, \eta) = \begin{pmatrix} L_0(\xi, \eta) \\ L_1(\xi, \eta) \\ \cdots \\ L_n(\xi, \eta) \end{pmatrix}$$

with retraction operator

$$R \circ L(\xi, \eta) = \sum_{i=0}^{n} (x_i, y_i) L_i(\xi, \eta) = (\xi, \eta)$$

Now consider about applying it on

$$\begin{aligned} \min \quad & \mathbb{E}_\xi \mathbf{c}^T \mathbf{x}(\xi) \\ s.t. \quad & Z(\mathbf{x}(\xi)) \geq 0 \\ & \forall \xi \in \Xi\{(\xi_i)_{len} : l_i \leq \xi_i \leq u_i\} \end{aligned}$$

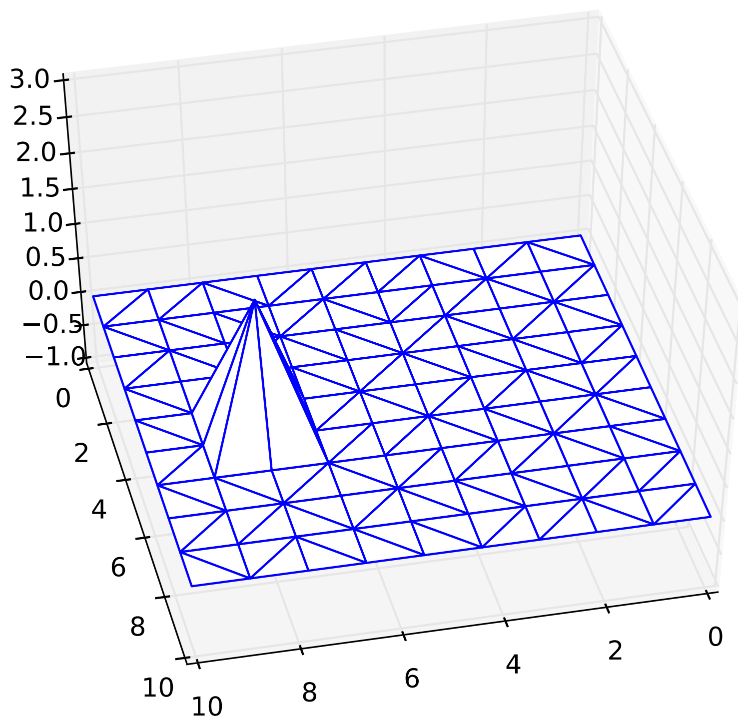

Figure 2: basis functions for multivariate scenarios

If we are attempting to approach with $x(\sum_{i=1}^{t-1}\xi_i,\xi_t)$.We should design lifting demands like

$$\tilde{\xi}=(\xi_1,\xi_2,\cdots,\xi_T,L(\xi_1,\xi_2),L(\xi_1+\xi_2,\xi_3),\cdots,L(\sum_{i=1}^{T-1}\xi_i,\xi_T))^T$$

that is

$$\begin{aligned}\tilde{\xi}\in\Xi'=\quad&\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&L(\sum_{i=1}^{t}\xi_i',\xi_{t+1}')=\xi_{T+t}',1\le t\le T-1\}\end{aligned}$$

The left problem is how to get $conv\Xi'$.In fact,

$$\begin{aligned}\Xi'=\quad&\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&\xi_{T+t}'\in\bigcup_{i,j,k}conv\{\mathbf{e_i},\mathbf{e_j},\mathbf{e_k}\}(i,j,k\text{ are neighbourhood points}),\\&\sum_{i=1}^{t}\xi_i'=(R(\xi_{T+t}'))_1,\xi_{t+1}=(R(\xi_{T+t}'))_2,1\le t\le T-1\}\end{aligned}$$

Let

$$\begin{aligned}A=\quad&\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&\xi_{T+t}'\in\bigcup_{i,j,k}conv\{\mathbf{e_i},\mathbf{e_j},\mathbf{e_k}\}(i,j,k\text{ are neighbourhood points}),1\le t\le T-1\}\end{aligned}$$

$$\begin{aligned}B=\quad&\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&\sum_{i=1}^{t}\xi_i'=(R(\xi_{T+t}'))_1,\xi_{t+1}=(R(\xi_{T+t}'))_2,1\le t\le T-1\}\end{aligned}$$

We have $conv(A\bigcap B)=convA\bigcap B$

Proof. Assuming it is false, there will be a element ξ in $convA\bigcap B$ but not in $conv(A\bigcap B)$. As $\xi\in convA$, so we can let $\xi=\lambda_1\xi_1+\lambda_2\xi_2(\lambda_1\ge0,\lambda_2>0,\lambda_1+\lambda_2\le1)$,where $\xi_1\in A\bigcap B$ and $\xi_2\in A\backslash(A\bigcap B)$. But as we can see, the constraints in B is linear. So ξ mustn't belong to B otherwise $\xi_2=(\xi-\lambda_1\xi_1)/\lambda_2\in B$. □

In other words, we have

$$\begin{aligned}conv\Xi'=\quad&conv\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&\xi_{T+t}'\in\bigcup_{i,j,k}conv\{\mathbf{e_i},\mathbf{e_j},\mathbf{e_k}\}(i,j,k\text{ are neighbourhood points}),1\le t\le T-1\}\\&\bigcap\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&\sum_{i=1}^{t}\xi_i'=(R(\xi_{T+t}'))_1,\xi_{t+1}=(R(\xi_{T+t}'))_2,1\le t\le T-1\}\\&=\{(\xi_1',\xi_2',\cdots,\xi_T',\xi_{T+1}',\xi_{T+2}',\cdots,\xi_{T+T-1}')^T:l_i\le\xi_i'\le u_i,1\le i\le T,\\&\xi_{T+t}'\in conv\{\mathbf{e_0},\mathbf{e_1},\cdots,\mathbf{e_n}\},\sum_{i=1}^{t}\xi_i'=(R(\xi_{T+t}'))_1,\xi_{t+1}=(R(\xi_{T+t}'))_2,1\le t\le T-1\}\end{aligned}$$

Therefore, we won't change constraints in the end. Also,it is possible for us to approach original problem with $x(\max\{\xi_1,\xi_2,\cdots,\xi_{t-1}\},\xi_t)$.