Linear Decision Rule Approach

Zhan Lin

1 Original Problem

Consider a network system consisting of m resources, with capacity levels $\mathbf{c} = (c_1, \ldots, c_m)^T$, and n products, with corresponding prices denoted by $\mathbf{v} = (v_1, \ldots, v_n)^T$. Each products needs at most one unit of each resource. Let $A = (a_{ij})$ be the resource coefficient matrix, where $a_{ij} = 1$ if product j uses one unit of resource i and $a_{ij} = 0$ otherwise. Define $\xi = (1, \xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,n}, \ldots, \xi_{t,1}, \xi_{t,2}, \ldots, \xi_{t,n})^T$ where $\xi_{t,j}$ is demand of product j in period t. Assuming observing τ periods of history,let ξ^t be the observed history demands, ξ_t be the demand at period t and

$$\mathbf{x_t}(\xi^{t-1}, \xi_t) = \left(x_{t,1}(\xi^{t-1}, \xi_{t,1}), x_{t,2}(\xi^{t-1}, \xi_{t,2}), \ldots, x_{t,n}(\xi^{t-1}, \xi_{t,n})\right)^T$$

be the booking limits in period t. Realisation of ξ is limited to Ξ. The optimality equations can be expressed as

$$\begin{aligned}
\max_{\mathbf{x_t}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^{T} \mathbf{v}^T \mathbf{x_t} \left(\xi^{t-1}, \xi_t\right) \right) \\
s.t. \quad & \sum_{t=1}^{T} A\mathbf{x_t} \left(\xi^{t-1}, \xi_t\right) \leq \mathbf{c} \\
& \mathbf{x_t}(\xi^{t-1}, \xi_t) \geq 0 \\
& \forall \xi \in \Xi, t = 1, \ldots, T
\end{aligned} \tag{1}$$

It's important to notice that actually $x_{t,j}(\xi^{t-1}, \xi_t) = u(\xi^{t-1}) \bigwedge \xi_t$.If making decisions without ξ^{t-1}, we will have $x_{t,j}(\xi_t) = u \bigwedge \xi_t$ and equation 2 .

$$\begin{aligned}
\max_{\mathbf{x_t}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^{T} \mathbf{v}^T \mathbf{x_t} \left(\xi_t\right) \right) \\
s.t. \quad & \sum_{t=1}^{T} A\mathbf{x_t} \left(\xi_t\right) \leq \mathbf{c} \\
& \mathbf{x_t}(\xi_t) \geq 0 \\
& \forall \xi \in \Xi, t = 1, \ldots, T
\end{aligned} \tag{2}$$

Equation 2 can be reformulated as the following equation 3 according to Preservation of Structural Properties in Optimization with Decisions Truncated by Random Variables and Its Applications.

$$
\begin{aligned}
\max_{\mathcal{X}_\mathbf{t}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^{T} \mathbf{v}^T \mathcal{X}_\mathbf{t} \left(\xi_t \right) \right) \\
s.t. \quad & \mathcal{X}_t(\xi_t) \leq \xi_t \\
& \{ \mathcal{X}_\mathbf{t}(\xi_t) = (\mathcal{X}_{t,1}(\xi_{t,1}), \mathcal{X}_{t,2}(\xi_{t,2}) \ldots, \mathcal{X}_{t,j}(\xi_{t,j}))^T \} \in \mathcal{F} \\
& \forall \xi \in \Xi, t = 1, \ldots, T
\end{aligned}
\tag{3}
$$

where $\mathcal{F} = \{\{f(\xi_t)\} : f(\xi_t) = \mathbf{u}_t \bigwedge \xi_t, u_t \in \mathbb{R}^+, \sum_{t=1}^{T} A\mathbf{u}_\mathbf{t} \leq \mathbf{c}\}$.

2 Linear Decision Rule Approach via Lifting

2.1 Piecewise Linear Continuous Decision Rules with Axial Segmentation

Define

$$L(\xi) = (L_{t,j}(\xi_{t,j}))_{t \times j \times r, 1}$$

where

$$L_{t,j}(x) = (L_{t,j,d}(x))_{r \times 1}$$

and

$$
L_{t,j,d}(x) = \begin{cases}
x & r = 1 \\
\min\{x, z_1^{t,j}\} & r > 1, d = 1 \\
\max\{\min\{x, z_d^{t,j}\} - z_{d-1}^{t,j}, 0\} & r > 1, d = 2, \ldots, r-1 \\
\max\{x - z_{d-1}^{t,j}, 0\} & r > 1, d = r
\end{cases}
$$

as show in figure 1.

It is obvious that the linear retraction operator corresponding to $L_{t,j}$ is

$$R_{t,j}(y) = \sum_{d=1}^{r} y_d = e^T y$$

2.2 Primal Problem

Approach equation 3 with

$$x_{t,j}(\xi_{t,j}) = \tilde{X}_{t,j} Q_{t,j} L(\xi)$$

where $\tilde{X}$ and $Q_{t,j}$ are both matrixes meanwhile $Q_{t,j}$ restricts $L(\xi)$ to $L(\xi_{t,j})$.

According to [1], with $\Xi = \{\xi : \xi_1 = 1, l_{t,j} \leq \xi_{t,j} \leq r_{t,j}\}$ and $\Xi' = L(\Xi)$,

$$\text{conv}\Xi' = \left\{ \xi' : \xi'_1 = 1, V_{t,j}^{-1}(1, \xi'_{t,j})^T_{(r+1)\times 1} \geq 0 \right\}$$

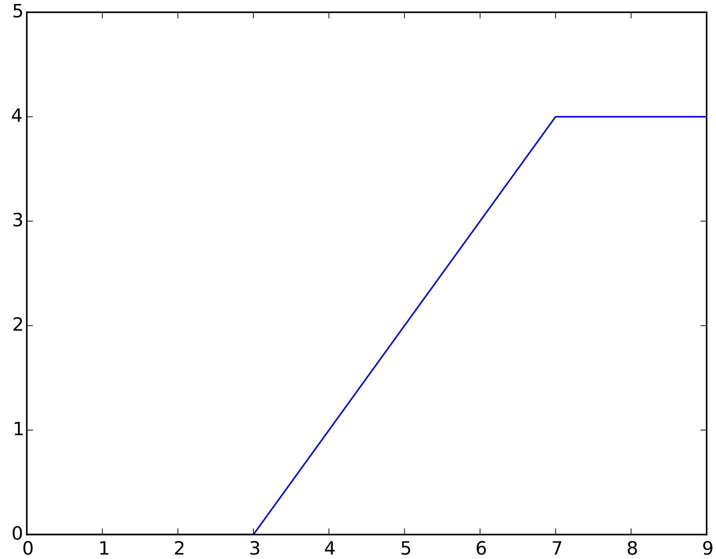

Figure 1: piecewise linear function

$$
V_{t,j}^{-1} = \begin{pmatrix}
\frac{z_1^{t,j}}{z_1^{t,j}-l_{t,j}} & -\frac{1}{z_1^{t,j}-l_{t,j}} & & & & \\
-\frac{l_{t,j}}{z_1^{t,j}-l_{t,j}} & \frac{1}{z_1^{t,j}-l_{t,j}} & -\frac{1}{z_2^{t,j}-z_1^{t,j}} & & & \\
 & & \frac{1}{z_2^{t,j}-z_1^{t,j}} & \ddots & & \\
 & & & \ddots & -\frac{1}{z_{r-1}^{t,j}-z_{r-2}^{t,j}} & \\
 & & & & \frac{1}{z_{r-1}^{t,j}-z_{r-2}^{t,j}} & -\frac{1}{u_{t,j}-z_{r-1}^{t,j}} \\
 & & & & & \frac{1}{u_{t,j}-z_{r-1}^{t,j}}
\end{pmatrix}
$$

Then we obtain the primal problem equation 4

$$
\begin{aligned}
\max_{\tilde{X}_{t,k}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^{T} \mathbf{v}^T (\sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k}) \xi \right) \\
\text{s.t.} \quad & \sum_{t=1}^{T} A (\sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k}) \xi \leq 2\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^{T} A (\sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k}) \xi \right) \\
& \tilde{X}_{t,k} Q_{t,k} \xi \leq e^T Q_{t,k} \xi \\
& \tilde{X}_{t,k} Q_{t,k} \xi \geq 0 \\
& \forall \xi \in \text{conv} \Xi', t = 1, \ldots, T, k = 1, \ldots, j
\end{aligned} \tag{4}
$$

The correction $\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^{T} A (\sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k}) \xi \right)$ arises from inaccurately approaching $\mathbf{x}(\xi^{t-1}, \xi_t)$.Also, there are $O(n^2 t^2 r)$ variables and $O(t^2 n^2 r + tnmr)$ constraints in total.

The solution $\tilde{X}_{t,j}(\xi_{t,j})$ of equation 4 has same structure with either figure 2 or figure 3.The proof is trivial.

Proof. As we can see from constraints, defining $\tilde{X}_{t,k}(\xi_{t,k}) = \tilde{X}_{t,k} Q_{t,k} \xi$, $\tilde{X}_{t,k}(\xi_{t,k}) \leq \xi_{t,k}$. Therefore the angle

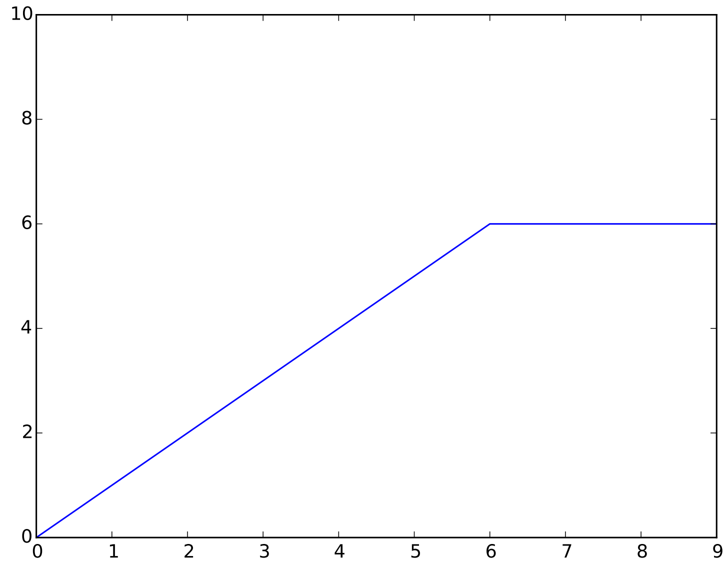

Figure 2: $z_k^{t,j}$ happened to be a break point

at $(0,0)$ must less than $45°$. On the other hand, if we keep $\tilde{X}_{t,k}(\xi_{t,k})$ no longer growing after the maximum point, it will never violate constraints as it doesn't need more resource than the maximum point which could be achieved. So $\tilde{X}_{t,k}(\xi_{t,k})$ must be flatten after the break point. It also hint us that we could learn booking limits from the break point. $\square$

However, the problem is that we cannot adjust booking limits dynamically with information before the clock. Elements of $\tilde{X}_{t,j}$ are not able to do changes when information is provided. Another way to approach the process is adding items $X_t P_t L(\xi)$ as shown below. P_t will restrict $L(\xi)$ to $L(\xi^{t-1})$. In that case, we need to set low bound of $\xi_{t,j}$ greater than 0. There will be $O(nmrt\tau + n^2t^2r)$ variables and $O(t^2n^2r + tnmr)$ constraints in total.

$$
\begin{aligned}
\max_{X,\tilde{X}_{t,k}} \quad & \mathbb{E}_\xi \left(\sum_{t=1}^{T} \mathbf{v}^T (X_t P_t + \sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k}) \xi \right) \\
\text{s.t.} \quad & \sum_{t=1}^{T} A(X_t P_t + \sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k})\xi \leq 2\mathbf{c} - \mathbb{E}_\xi \left(\sum_{t=1}^{T} A(X_t P_t + \sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k})\xi \right) \\
& (X_t P_t + \sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k})\xi \leq e^T Q_{t,k} \xi \\
& (X_t P_t + \sum_{k=1}^{j} \tilde{X}_{t,k} Q_{t,k})\xi \geq 0 \\
& \forall \xi \in \text{conv}\Xi', t = 1, \ldots, T
\end{aligned} \tag{5}
$$

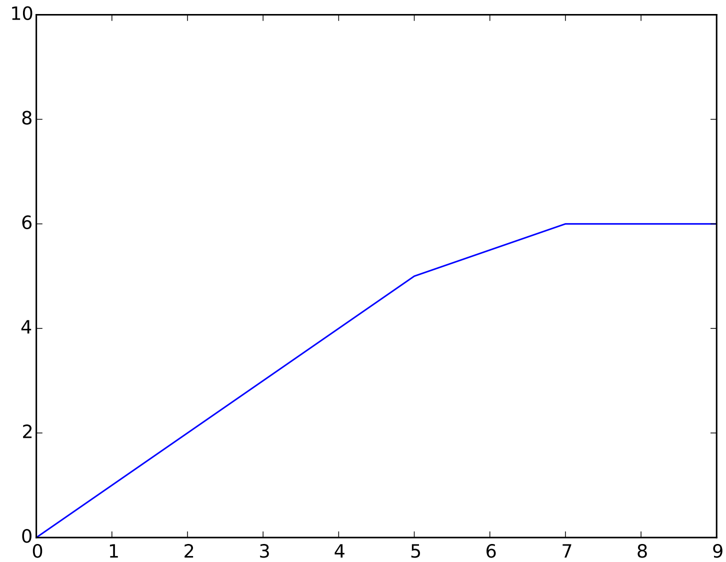

Figure 3: every $z_k^{t,j}$ could not be a break point

3 Computational Results

As we have shown above, $\tilde{X}_{t,j}(\xi^{t-1}, \xi_{t,j})$ should equal $u_{t,j}(\xi^{t-1}) \bigwedge \xi_{t,j}$. Benefiting from piecewise linear function, the structure is preserved with linear decision rule. Therefore, it's convenient to implement the policy as we can get u from $\tilde{X}_{t,j}$. Results are computed at servers locating at USTC with 96 Intel Xeon CPU E5-4657L v2 2.4GHz and 256GB memory. Gurobi will make use of 32 cpu to solve the LP problem parallelly.

References

[1] Georghiou, A., Wiesemann, W. and Kuhn, D., 2015. Generalized decision rule approximations for stochastic programming via liftings. Mathematical Programming, 152(1-2), pp.301-338.

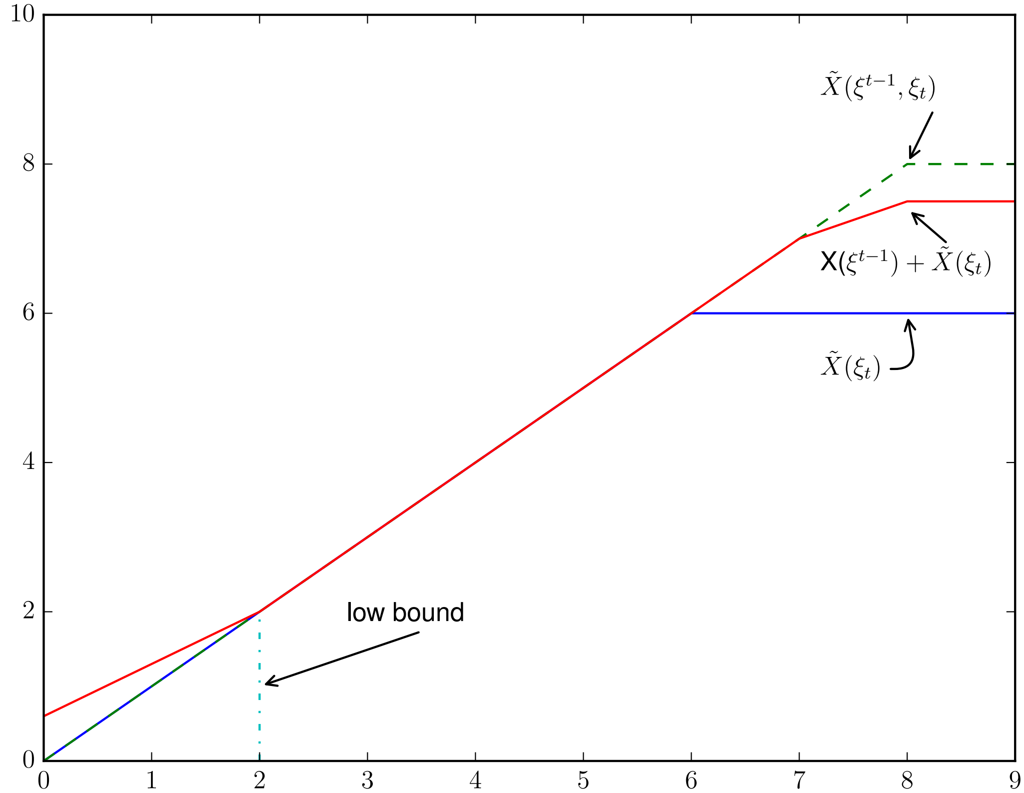

Figure 4: shape of functions

	DLP-alloc	SLP-alloc	ceil	objective value
Example 1 without re-solving	401980	415410	413227	409255
Example 1 with re-solving	409740	421894		
reduction of ALP			18844	18159

Table 1: Only depend on ξ_t

	floor(x)+1	floor(x)+2	floor(x)+3	Wait-and-see/truth value
Example 1 without re-solving	417761	420294	414305	432730
Example 1 with re-solving				432730
reduction of ALP	19414	19038	18474	20411

Table 2: Only depend on ξ_t

	DLP-alloc	SLP-alloc	ceil	objective value
Example 1 without re-solving	401980	415410	413477	410787
Example 1 with re-solving	409740	421894		
reduction of ALP			18903	18155

Table 3: Depend both on ξ^{t-1} and ξ_t

	floor(x)+1	floor(x)+2	floor(x)+3	Wait-and-see/truth value
Example 1 without re-solving	418139	420341	414183	432730
Example 1 with re-solving				432730
reduction of ALP	19430	18893	18425	20411

Table 4: Depend both on ξ^{t-1} and ξ_t

Case	Parameters	Solving LP Time/s	Total Time/s
reduction of ALP(Only depend on ξ_t)	$t = 5, \tau = 0, r = 10$	6.3	86.7
reduction of ALP(Depend on both ξ^{t-1} and ξ_t)	$t = 5, \tau = 5, r = 10$	41.07	137.83
Example 1(Only depend on ξ_t)	$t = 5, \tau = 0, r = 10$	20.72	208.81
Example 1(Depend on both ξ^{t-1} and ξ_t)	$t = 5, \tau = 5, r = 10$	204.82	435.97

Table 5: Computational Time