Linear Decision Rule Approach

Zhan Lin

1 Original Problem

Consider a network system consisting of m resources, with capacity levels $\mathbf{c} = (c_1, \ldots, c_m)^T$, and n products, with corresponding prices denoted by $\mathbf{v} = (v_1, \ldots, v_n)^T$. Each products needs at most one unit of each resource. Let $A = (a_{ij})$ be the resource coefficient matrix, where $a_{ij} = 1$ if product j uses one unit of resource i and $a_{ij} = 0$ otherwise. Define $\xi = (1, \xi_{1,1}, \xi_{1,2}, \ldots, \xi_{1,n}, \ldots, \xi_{t,1}, \xi_{t,2}, \ldots, \xi_{t,n})^T$ where $\xi_{t,j}$ is demand of product j in period t. Let ξ^t be the observed history demands, ξ_t be the demand at period t and

$$\mathbf{x_t}(\xi^t, \xi_t) = \big(x_{t,1}(\xi^t, \xi_{t,1}), x_{t,2}(\xi^t, \xi_{t,2}), \ldots, x_{t,n}(\xi^t, \xi_{t,n})\big)^T$$

be the booking limits in period t. Realisation of ξ is limited to Ξ. The optimality equations can be expressed as

$$\begin{array}{ll} \max & \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T \mathbf{x_t} \left(\xi^t, \xi_t\right) \right) \\ s.t. & \sum_{t=1}^T A\mathbf{x_t} \left(\xi^t, \xi_t\right) \leq \mathbf{c} \\ & \mathbf{x_t}(\xi^t, \xi_t) \leq \xi_t \\ & \forall \xi \in \Xi, t = 1, \ldots, T \end{array} \tag{1}$$

2 Linear Decision Rule Approach

2.1 Primal Problem

Approach original problem with

$$\mathbf{x_{t,j}}(\xi^t, \xi_{t,j}) = X_t\xi^t + \tilde{\mathbf{X}}_{\mathbf{t}}^T \left(\mathbf{1}_{\xi\in(l_1,r_1)}, \mathbf{1}_{\xi\in(l_2,r_2)}, \mathbf{1}_{\xi\in(l_3,r_3)}, \ldots, \mathbf{1}_{\xi\in(l_k,r_k)}\right)^T \xi_t$$

, $\Xi = \{\xi : W\xi \leq h\}$, and $p_t\xi = \xi_t$ where X_t, P_t , W and p_t are all matrix. Then we obtain the primal problem

$$\begin{array}{ll} \max & \mathbb{E}_\xi \left(\sum_{t=1}^T \mathbf{v}^T X_t P_t \xi\right) \\ s.t. & \sum_{t=1}^T A X_t P_t \xi \leq \mathbf{c} \\ & X_t P_t \xi \leq p_t \xi \\ & \forall \xi \in \Xi = \{\xi : W\xi \leq h\}, t = 1, \ldots, T \end{array} \tag{2}$$

where

$$W = \begin{pmatrix} 1 & & & \\ -1 & & & \\ & 1 & & \\ & -1 & & \\ & & \vdots & \\ & & & 1 \\ & & & -1 \end{pmatrix}$$

and

$$h = (1, -1, q_{1,1,sup}, -q_{1,1,inf}, q_{1,2,sup}, -q_{1,2,inf}, \ldots, q_{t,n,sup}, -q_{t,n,inf})^T$$

in which $q_{t,j,p}$ is the p percentile of $\xi_{t,j}$.

There are $nmt\tau + 2nmt + n^2t^2$ variables and $m + 2nt$ constraints in total.

2.2 Duality

Firstly we transform equation(1) into a tighter formulation.

$$\begin{array}{ll} \max & \mathbb{E}_{\xi}\left(\sum_{t=1}^{T}\mathbf{v}^{T}\mathbf{x_t}\left(\xi^{t}\right)\right) \\ s.t. & \sum_{t=1}^{T}\tilde{A}\mathbf{x_t}\left(\xi^{t}\right)\leq\tilde{\mathbf{c}}_{t}\left(\xi\right) \\ & \mathbf{x_t}\left(\xi^{t}\right)\geq 0 \\ & \forall\xi\in\Xi, t=1,\ldots,T \end{array} \tag{3}$$

Then it has a duality.

$$\begin{array}{ll} \min & \mathbb{E}_{\xi}\left(\sum_{t=1}^{T}\tilde{\mathbf{c}}_{t}\left(\xi\right)^{T}\mathbf{y_t}\left(\xi^{t}\right)\right) \\ s.t. & \sum_{t=1}^{T}\tilde{A}^{T}\mathbf{y_t}\left(\xi^{t}\right)\geq\mathbf{v}^{T} \\ & \mathbf{y_t}\left(\xi^{t}\right)\geq 0 \\ & \forall\xi\in\Xi, t=1,\ldots,T \end{array} \tag{4}$$

We can apply same approach and obtain

$$\begin{array}{ll} \min & \mathbb{E}_{\xi}\left(\sum_{t=1}^{T}\left(\mathbf{c}^{T},\left(p_t\xi\right)^{T}\right)\mathbf{y_t}\left(\xi^{t}\right)\right) \\ s.t. & \sum_{t=1}^{T}\tilde{A}^{T}Y_tP_t\xi\geq\mathbf{v}^{T} \\ & Y_tP_t\xi\geq 0 \\ & \forall\xi\in\Xi\left\{\xi:W\xi\leq h\right\}, t=1,\ldots,T \end{array} \tag{5}$$

3 Numerical Results

Rounding can matter a lot to benefits. Therefore, I tried genetic algorithm to get a not bad policy. However, it sometimes offer a worse policy than simply ceil. What's more, please notice that all results obtained now are under constant policies.

3.1 Parameters

The upper bound and lower bound of ξ in Ξ play important roles in optimizing. With strict bounds, booking limits will be under rigorous limitations. Therefore

	DLP-alloc	SLP-alloc	LDR	Wait-and-see
First Example without re-solving	401980	415410	419831	432730
Second Example without re-solving	583630	595620	601212	432730
First Example with re-solving	409740	421894		623530
Second Example with re-solving	594021	604859		623530

Table 1: Summary

we need to set lower bound of the primal problem a bit higher. In the first example of Re-Solving Stochastic Programming Models for Airline Revenue, upper bound = 80 percentile and lower bound = 60 percentile. In the second example, upper bound = 90 percentile and lower bound = 70 percentile. Results are given at table 1. The results of data from approximate linear programming are not listed here. But in most cases, comparing with approximate linear programming, linear decision rules will only result in 5 percentile loss.