

Figure 1: basis function for single variable scenarios

1 Single Variable Scenarios

Let ξ be the uncertain demand $\in [l, u]$ and $\{x_i, 0 \leq i \leq n : x_0 = l, x_n = u\}$ be breakpoints. Define

$$L_0(\xi) = \begin{cases} 1 - (\xi - l)/(x_1 - l) & \xi \in [l, x_1] \\ 0 & \xi \in (x_1, u] \end{cases}$$

$$L_1(\xi) = \begin{cases} (\xi - l)/(x_1 - l) & \xi \in [l, x_1] \\ 1 - (\xi - x_1)/(x_2 - x_1) & \xi \in (x_1, x_2] \\ 0 & \xi \in (x_2, u] \end{cases}$$

as shown in figure 1 and so on. Actually, we have

$$L_i(\xi) = \begin{cases} 0 & \xi \in [0, x_{i-1}] \\ (\xi - x_{i-1})/(x_i - x_{i-1}) & \xi \in (x_{i-1}, x_i] \\ 1 - (\xi - x_i)/(x_{i+1} - x_i) & \xi \in (x_i, x_{i+1}] \\ 0 & \xi \in (x_{i+1}, u] \end{cases}$$

for $0 \leq i \leq n$. Define lifting operator

$$L(\xi) = \begin{pmatrix} L_0(\xi) \\ L_1(\xi) \\ \dots \\ L_n(\xi) \end{pmatrix}$$

It is not hard to get corresponding retraction operator

$$R \circ L(\xi) = \sum_{i=0}^n x_i L_i(\xi) = \xi$$

If we are going to solve original problem following where ξ , $\mathbf{c}(\xi)$ and $\mathbf{x}(\xi)$ are all vectors.

$$\begin{aligned} \min \quad & \mathbb{E}_{\xi} \mathbf{c}^T \mathbf{x}(\xi) \\ \text{s.t.} \quad & Z(\mathbf{x}(\xi)) \geq 0 \\ & \forall \xi \in \Xi \{(\xi_i)_{len} : l_i \leq \xi_i \leq u_i\} \end{aligned}$$

We have to consider about defining L again, that is

$$L^i(\xi) = \begin{pmatrix} L_0(\xi_i) \\ L_1(\xi_i) \\ \dots \\ L_n(\xi_i) \end{pmatrix}$$

and

$$\begin{aligned} \tilde{L}(\xi) &= \begin{pmatrix} L^1(\xi) \\ L^2(\xi) \\ \dots \\ L^{len}(\xi) \end{pmatrix} \\ \tilde{R} \circ \tilde{L}(\xi) &= \begin{pmatrix} R^1(L^1(\xi)) \\ R^2(L^2(\xi)) \\ \dots \\ R^{len}(L^{len}(\xi)) \end{pmatrix} \end{aligned}$$

Approaching the original problem with lifting variables, we will have below equation

$$\begin{aligned} \min \quad & \mathbb{E}_{R(\xi)} \mathbf{c}^T X \xi \\ \text{s.t.} \quad & ZX \xi \geq 0 \\ & \forall \xi \in \tilde{L}(\Xi) \end{aligned}$$

And actually $\tilde{R}(\cdot)$ could be replaced by a matrix \hat{R} . Now we are going to discuss about $L(\Xi)$. As we want to utilize duality arguments, so we need to replace $\tilde{L}(\Xi)$ with $\text{conv} \tilde{L}(\Xi)$. It could be no better if we don't have to add more constraints in LP during lifting. As we all know,

$$\text{conv} \tilde{L}(\Xi) = \prod_{i=1}^{len} \text{conv} L^i(\Xi)$$

So we only need to care about single variable lifting $L^i(\xi) = L(\xi_i)$ again. That is why I call it single variable scenarios. For the sake of simplicity synonym, I will omit index i on both $L^i(\cdot)$ and ξ_i now. As for $\text{conv} L(\xi)$, actually we have

$$\text{conv} L(\Xi) = \text{conv} \{\mathbf{e}_0, \dots, \mathbf{e}_n\}$$

where $\{\mathbf{e}_i\}$ are standard basis in euclidian space. It is because that for any realisation $\xi \in \Xi$, at most two elements of ξ are non-zero. Let i th and $i + 1$ th element be the non-zero. Benefiting from piecewise linear basis functions, in fact, we have

$$\xi = \xi_i \mathbf{e}_i + \xi_{i+1} \mathbf{e}_{i+1}$$

$$\xi_i + \xi_{i+1} = 1, \xi_i \geq 0, \xi_{i+1} \geq 0$$

Therefore,

$$\text{conv}L(\Xi) \subseteq \text{conv}\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$$

In the meantime,

$$\{\mathbf{e}_0, \dots, \mathbf{e}_n\} \subseteq \text{conv}L(\Xi)$$

So we can tell that

$$\text{conv}L(\Xi) = \text{conv}\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$$

What's more, as $\text{conv}\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$ are corner points of the feasible region, we can keep constraints unchanged.

2 Multivariate Scenarios

Same as single variable scenarios, let (ξ, η) still be the uncertain demand, but now we have to do triangulation on 2-d mesh grid. Let $\{(x_m, y_m), 0 \leq m \leq n\}$ be points on triangular meshes. As shown in figure 2, piecewise linear function $L_i(\xi, \eta)$ now is defined by

$$L_i(x_i, y_i) = 1$$

$$L_i(\hat{x}, \hat{y}) = 0, \forall (\hat{x}, \hat{y}) \in \{(x_m, y_m), 0 \leq m \leq n\} \setminus \{(x_i, y_i)\}$$

Also, we can define

$$L(\xi, \eta) = \begin{pmatrix} L_0(\xi, \eta) \\ L_1(\xi, \eta) \\ \dots \\ L_n(\xi, \eta) \end{pmatrix}$$

with retraction operator

$$R \circ L(\xi, \eta) = \sum_{i=0}^n (x_i, y_i) L_i(\xi, \eta) = (\xi, \eta)$$

Now consider about applying it on

$$\begin{aligned} \min \quad & \mathbb{E}_\xi \mathbf{c}^T \mathbf{x}(\xi) \\ \text{s.t.} \quad & Z(\mathbf{x}(\xi)) \geq 0 \\ & \forall \xi \in \Xi \{(\xi_i)_{len} : l_i \leq \xi_i \leq u_i\} \end{aligned}$$

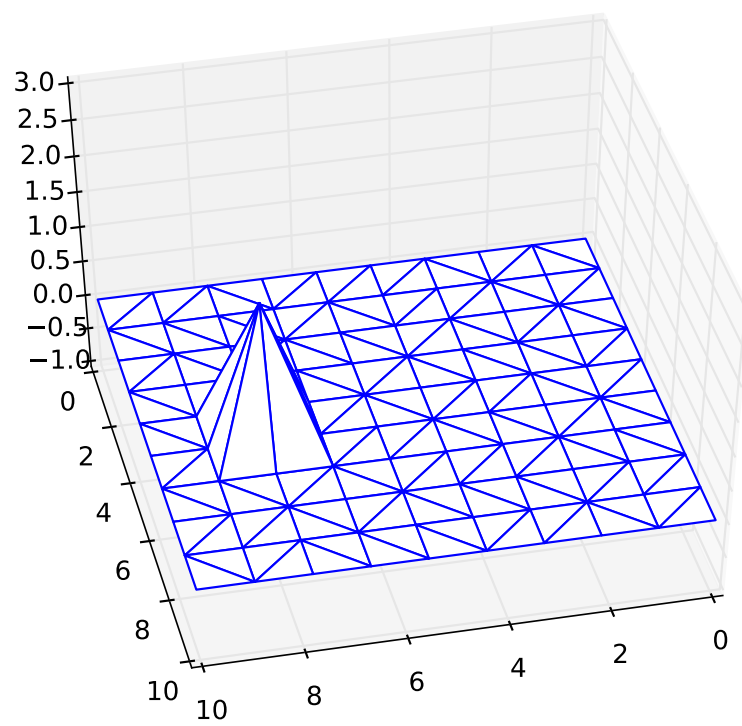


Figure 2: basis functions for multivariate scenarios

If we are attempting to approach with $x(\sum_{i=1}^{t-1} \xi_i, \xi_t)$. We should design lifting demands like

$$\tilde{\xi} = (\xi_1, \xi_2, \dots, \xi_T, L(\xi_1, \xi_2), L(\xi_1 + \xi_2, \xi_3), \dots, L(\sum_{i=1}^{T-1} \xi_i, \xi_T))^T$$

that is

$$\begin{aligned} \tilde{\xi} \in \Xi' = & \{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & L(\sum_{i=1}^t \xi'_i, \xi'_{t+1}) = \xi'_{T+t}, 1 \leq t \leq T-1\} \end{aligned}$$

The left problem is how to get $\text{conv}\Xi'$. In fact,

$$\begin{aligned} \Xi' = & \{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & \xi'_{T+t} \in \bigcup_{i,j,k} \text{conv}\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\} (i, j, k \text{ are neighbourhood points}), \\ & \sum_{i=1}^t \xi'_i = (R(\xi'_{T+t}))_1, \xi'_{t+1} = (R(\xi'_{T+t}))_2, 1 \leq t \leq T-1\} \end{aligned}$$

Let

$$\begin{aligned} A = & \{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & \xi'_{T+t} \in \bigcup_{i,j,k} \text{conv}\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\} (i, j, k \text{ are neighbourhood points}), 1 \leq t \leq T-1\} \\ B = & \{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & \sum_{i=1}^t \xi'_i = (R(\xi'_{T+t}))_1, \xi'_{t+1} = (R(\xi'_{T+t}))_2, 1 \leq t \leq T-1\} \end{aligned}$$

We have $\text{conv}(A \cap B) = \text{conv}A \cap B$

Proof. Assuming it is false, there will be a element ξ in $\text{conv}A \cap B$ but not in $\text{conv}(A \cap B)$. As $\xi \in \text{conv}A$, so we can let $\xi = \lambda_1 \xi_1 + \lambda_2 \xi_2$ ($\lambda_1 \geq 0, \lambda_2 > 0, \lambda_1 + \lambda_2 \leq 1$), where $\xi_1 \in A \cap B$ and $\xi_2 \in A \setminus (A \cap B)$. But as we can see, the constraints in B is linear. So ξ mustn't belong to B otherwise $\xi_2 = (\xi - \lambda_1 \xi_1) / \lambda_2 \in B$. \square

In other words, we have

$$\begin{aligned} \text{conv}\Xi' = & \text{conv}\{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & \xi'_{T+t} \in \bigcup_{i,j,k} \text{conv}\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\} (i, j, k \text{ are neighbourhood points}), 1 \leq t \leq T-1\} \\ & \cap \{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & \sum_{i=1}^t \xi'_i = (R(\xi'_{T+t}))_1, \xi'_{t+1} = (R(\xi'_{T+t}))_2, 1 \leq t \leq T-1\} \\ = & \{(\xi'_1, \xi'_2, \dots, \xi'_T, \xi'_{T+1}, \xi'_{T+2}, \dots, \xi'_{T+T-1})^T : l_i \leq \xi'_i \leq u_i, 1 \leq i \leq T, \\ & \xi'_{T+t} \in \text{conv}\{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n\}, \sum_{i=1}^t \xi'_i = (R(\xi'_{T+t}))_1, \xi'_{t+1} = (R(\xi'_{T+t}))_2, 1 \leq t \leq T-1\} \end{aligned}$$

Therefore, we won't change constraints in the end. Also, it is possible for us to approach original problem with $x(\max\{\xi_1, \xi_2, \dots, \xi_{t-1}\}, \xi_t)$.