CSE525 Monsoon 2020	Ashima Garg (PhD19003)
Homework 14 Problem 1	Pramil Panjwani (PhD19008)

(a) Prove that $convert1()$ is a valid reduction from $3SUM$ to $PARTITION$ or show a counterexample.

Solution: ***Lemma:***If A is *YES* instance of algorithm $3SUM$, then set B is *YES* instance of algorithm $PARTITION$
Lemma:If A is *NO* instance of algorithm $3SUM$, then set B is *NO* instance of algorithm $PARTITION$
The counterexample to show that $convert1()$ is not a valid reduction is:
If the given input set A to the $3SUM$ has all the positive numbers, let a, b, c, be all positive numbers given as input to $3SUM$. Then, $convert1(A)$, inserts all the numbers of the set A to the output set along with the sum of the numbers in the set A. That is, the output set of $convert1$ has numbers a, b, c, and $a+b+c$. Let's call this set as B.

By the definition of $3SUM$, it returns $True$ only if there are 3 numbers in the input set that has sum equal to 0. Since, all the positive numbers are greater than 0, (i.e. $a > 0, b > 0, c > 0$), then $a+b+c = 0$ can never hold $True$. Therefore, this is a "NO" instance of the problem $3SUM$.

Whereas, set B has numbers a, b, c, $a+b+c$. The $PARTITION$ divides the input set B into two subsets with equal sums. The elements of the set B can be divided into two subsets of equal sum with first set having elements a, b, c and the second subset element has element $a+b+c$. Therefore, this is a "YES" instance of $PARTITION$.

For set A which is *NO* instance of algorithm $3SUM$, set B is *YES* instance of algorithm $PARTITION$
$\therefore$ $convert1$ is not a valid reduction from $3SUM$ to $PARTITION$

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Homework 14 Problem 2	Pramil Panjwani (PhD19008)

(b) Prove that $convert2()$ is a valid reduction of $3SUM$ to $EQUALDIFF$ or show a counterexample.

Solution: $3SUM$: Given three sets A,B,C containing numbers between 0 and 1 (and not equal to 0 and 1), is there a number $a \in A$, $b \in B$, and $c \in C$ such that $a + b = c$?

$EQUALDIFF$: Given two sets P and Q of numbers, are there $p1$ and $p2$ in P and $q1$ and $q2$ in Q such that $p1 - p2 = q1 - q2$?

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def convert2(A, B, C) : ⟨⟨three arrays A,B,C containing numbers between 0 and 1 (and not equal to 0 and 1)⟩⟩
  P = {} and Q = A
  for i=1 ... n: ⟨⟨ n denotes the number of elements in C⟩⟩
      add 100i to P
      add 100i - ci + 3 to P // ci is the i-th element of C
  for every b in B:
      add 3-b to Q
  return (P, Q)
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convert2 adds elements of the form $100i$ and $100i - ci + 3$ to set P. In set Q, it adds all the elements of A, and $3 - b$. Let $p1$ and $p2$ be two numbers in set P. Let $q1$ and $q2$ be two numbers in set Q.

Let $i, j \in 1 \dots n$. Then, $p1 - p2$ can be $100(j - i)$ (difference between $100i$ and $100j$) , or $ci - 3$ (difference between $100i$ and $100i - ci + 3$) and $100(j - i) + cj - ci$ (difference between $100i - ci + 3$ and $100j - cj + 3$).

Lemma:If $I2$ is *YES* instance of algorithm $PARTITION$, then set $I1$ is *YES* instance of algorithm $3SUM$

Let m be the number of elements in B. Let $i, k \in 1 \dots m$. Then, elements of $I2$ can be:

- $ai - ak$ (difference between elements from set A) ,
- $3 - bi - ak = 3 - (bi + ak)$ (difference between $3 - bi$ and ak), or
- $ak - 3 + bi = (ak + bi) - 3$ (difference between ak and $3 - bi$).

For a *YES* instance of *EQUALDIFF*,

$$p1 - p2 = q1 - q2 \tag{1}$$
$$p1 - p2 = cj - 3 \tag{2}$$
$$q1 - q2 = ak + bi - 3 \tag{3}$$
$$cj - 3 = ak + bi - 3 \tag{4}$$
$$cj = ak + bi \qquad \dots from(3, 4) \tag{5}$$

Thus, if $p1 - p2 = q1 - q2$ holds for $I2$, then $I1$ $c = a + b$ holds for sure.

Lemma:If $I1$ is *YES* instance of algorithm $3SUM$, then set $I2$ is *YES* instance of algorithm $PARTITION$

Conversely, if $a + b = c$ holds in $3SUM$, then *convert2* will add numbers of the form $100i - ci + 3$ and $100i$ in P and all the elements of A and $3 - b$ in Q. Thus, differences of the form explained above will surely exist between the numbers. Let $a + b = c$ holds true. Corresponding to a, b, c that satisfies this assignment, there will surely exist numbers $100j$, $100j - cj + 3$ in P that gives the difference $cj - 3$ and ak, $3 - bi$ in Q that gives the difference $ak + bj - 3$ from the numbers between set Q. Therefore, $p1$, $q1$, $p2$, $q2$ exists such that $p1 = 100j$, $p2 = 100j - cj + 3$, $q1 = ak$ and $q2 = 3 - bj$ and $\therefore$ $p1 - p2 = q1 - q2$ holds true.

$\therefore$ *convert2* is a valid reduction from $3SUM$ to $EQUALDIFF$. ■