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Homework 19 Problem 1 — Pramil Panjawani (PhD19008)

Solution: Let $PART(S)$ be the decision problem that returns True if given a set S of numbers it can be divided into two sets A and B of equal sum.

```
def Reduce(G):
    if PART(S) = False: return NULL ⟨⟨Equal Partition does not exists.⟩⟩
    ⟨⟨Statement 1: Equal partition in Set S already exists ⟩⟩
    n = Size(S)
    Create two empty sets A = [], B = []
    Create S′ = copy of S
    x = S′[n] ⟨⟨Pick x as the last elements of S′⟩⟩
    S′[n] = 0 ⟨⟨Remove x from S′⟩⟩
    Add x to A.
    for i = n − 1 to 1: ⟨⟨Decide for every element of S′ in which set to add it.⟩⟩
        y = S′[i]
        S′[0] = x + y ⟨⟨Step 1⟩⟩
        if PART(S′) = False:
        ⟨⟨If x and y belong to same set, then there doesn't exist equal partition in S,⟩⟩
        ⟨⟨which is contradictory to Statement 1, therefore add y to opposite set of x⟩⟩
            Add y to set B.
            S′[i] = y ⟨⟨Add y back to set S′⟩⟩
        elif PART(S′) = True:
        ⟨⟨If x and y belong to same set, then there exist equal partition in S, therefore add y to same set of x⟩⟩
            Add y to set A
        S′[0] = 0. ⟨⟨Remove sentinel x + y added in Step 1⟩⟩
```

■

Time Complexity: Let time complexity of PART(S) be $O(T_S)$. The loop runs for every element in S, thus time complexity is $O(n) * O(T_S) = O(nT_S)$.

Claim: Let x and y are replaced with $x + y$ in set S' =copy of S. If S can be equally partitioned, then S' can also be equally partitioned.

Proof: Let x and y be elements of S and total sum of elements in S be s:

where $S[1\dots n] = [S_1, S_2 \dots S_n]$ and $\sum_{i=1}^{n} S_i = s$
$A[1\dots n1] = [A_1, A_2 \dots A_{n1}]$
$B[1\dots n-n1] = [B_1, B_2 \dots B_{n-n1}]$
and S' is a copy of S
Initially $\sum A = \sum B = s/2$

Case1: Initially $\sum_1^{n1} A = A_1 + A_2 + \cdots + A_{n1} = s/2$
Let x and y be two elements of type $A_i \in A$. x and y belong to same partition(say A)
now replace x, y with $x + y$ in A

since x, y belong to same partition after replacement in S'
$\sum_1^{n1} A = s/2 - x - y + (x + y) = s/2$

old $\sum A =$ new $\sum A$

Case2: x and y belong to different partitions
Initially $\sum A = \sum B = s/2$
now replace x, y with $x + y$ in S
since x,y belong to different partition
on removing $\sum A = s/2 - x$ and new $\sum B = s/2 - y$
on adding x+y to A
$\sum A = s/2 - x + (x + y)$ and new $\sum B = s/2 - y$
$\sum A = s/2 + y$ and new $\sum B = s/2 - y$
Now new $\sum A \neq \sum B$

If this holds, then x and y can be added to the same set. Else, x and y should be in different sets.

So we keep changing element y and if we get equal partitions. If PART(S') gives true for a particular value for x and y we keep them in same partition else we move one to partition A and other to B.