114252 - Project T

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March 15, 2015

Abstract

In lots of branches of Physics, the approach for solving the dynamics of a system is to use Perturbation theory. In Quantum Chromodynamics, Perturbation theory cannot be used because the characterized values aren't small enough. The relationship between QCD and anti-de Sitter spacetimes, which follows from Juan Maldacena's 1997 paper, let us solve the gravitational dynamics in asymptotically anti-de Sitter spacetime, and conclude about the properties and behavior of the quantum system which is governed by the strong force. The project's goal is to numerically solve the gravitational dynamics, and as a hallmark of numerical accuracy, find the first quasi-normal mode of the black hole.

1 Presentation of the problem

The physical system is an asymptotically anti-de Sitter spacetime. The metric is given by the expression

$$g_{\mu\nu} = \begin{pmatrix} -2A(t,r) & 0 & 0 & 1 \\ 0 & e^{-B(t,r)}\Sigma^2(t,r) & 0 & 0 \\ 0 & 0 & e^{B(t,r)}\Sigma^2(t,r) & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

For a scalar function, $f(t,r)$, we define

$$\dot{f}(t,r) \triangleq A(t,r)\,\partial_r f(t,r) + \partial_t f(t,r) \tag{1.2}$$

$$\ddot{f}(t,r) \triangleq A(t,r)\,\partial_r \dot{f}(t,r) + \partial_t \dot{f}(t,r). \tag{1.3}$$

Using the above definitions, the Einstein equations of motion are

$$\frac{1}{2}\Sigma(t,r)\left[\partial_r B(t,r)\right]^2 + 2\left[\partial_{rr}\Sigma(t,r)\right]^2 = 0 \tag{1.4}$$

$$\Sigma^2(t,r)\left[-6\Sigma^2(t,r) + \dot{\Sigma}(t,r)\,\partial_r\Sigma(t,r)\right] + 4\Sigma^3(t,r)\,\partial_r\dot{\Sigma}(t,r) = 0 \tag{1.5}$$

$$4\Sigma^4(t,r)\,\partial_r\dot{B}(t,r) + 4\Sigma^3(t,r)\left[\dot{\Sigma}(t,r)\,\partial_r B(t,r) + \dot{B}(t,r)\,\partial_r\Sigma(t,r)\right] = 0 \tag{1.6}$$

$$-4\Sigma^2(t,r)\,\dot{\Sigma}(t,r)\,\partial_r\Sigma(t,r) + \Sigma^4(t,r)\left[\dot{B}(t,r)\,\partial_r B(t,r) + 2\partial_{rr}A(t,r)\right] = 0 \tag{1.7}$$

$$\dot{B}^2(t,r)\,\Sigma^3(t,r) + 4\Sigma^2(t,r)\,\ddot{\Sigma}(t,r) - 4\Sigma^2(t,r)\,\dot{\Sigma}(t,r)\,\partial_r A(t,r) = 0 \tag{1.8}$$

with the boundary condition

$$\dot{\Sigma}(t,r_h) = 0 \tag{1.9}$$

at $r = r_h$ - the apparent horizon location. This boundary condition will help us determine where is the location of the apparent horizon at any given time.

We seek a solution for $t \geq 0$ and $r \in [r_h, \infty]$, where $r = \infty$ relates to the limit there. A general black hole solution is given by

$$\Sigma(t,r) = r \quad , \quad B(t,r) = 0 \quad , \quad A(t,r) = \frac{1}{2}r^2 - \frac{{r_0}^3}{2}r^{-1} \tag{1.10}$$

where $r_0 > 0$. (1.10) is the late time solution one should expect to get after a sufficient amount of time.

We would like to solve the system dynamics with an initial perturbation, which has an series expansion near $r = \infty$ of the form

$$B(0,r) = \sum_{i=0}^{\infty} c_i r^{-i}. \tag{1.11}$$

2 Approach for solving

Although equations (1.4)-(1.8) seems a bit complicated, there is an algorithm for solving them in a defined order such that in every step we'll have only one unknown function at hand.

Assume we know $B(t_0, r)$, then equation (1.4) can be solved for $\Sigma(t_0, r)$. With that knowledge, equation (1.5) can be solved for $\dot{\Sigma}(t_0, r)$. Than equation (1.6) can be solved for $\dot{B}(t_0, r)$, and finally equation (1.7) can be solved for $A(t_0, r)$. Equation (1.8) is satisfied automatically for $t = t_0$.

3 Series expansion discussion

We expand $\Sigma(t,r)$, $B(t,r)$ and $A(t,r)$ near $r = \infty$

$$\Sigma(t,r) = \sum_{i=-1}^{\infty} s_i(t) r^{-i} \tag{3.1}$$

$$B(t,r) = \sum_{i=1}^{\infty} b_i(t) r^{-i} \tag{3.2}$$

$$A(t,r) = \sum_{i=-2}^{\infty} a_i(t) r^{-i} \tag{3.3}$$

After substituting (3.1)-(3.3) in (1.4)-(1.8), we get

$$a_{-2}(t) = \frac{1}{2}, \tag{3.4}$$

$$a_1{}'(t) = 0 \tag{3.5}$$

and that $s_{-1}(t)$, $s_0(t)$ and $b_3(t)$ are free, time dependent, functions which we have the freedom to choose. All other coefficients in (3.1)-(3.3) depend on $a_{-2}(t)$, $a_1(t)$, $s_{-1}(t)$, $s_0(t)$ and $b_3(t)$.

4 Equation modifications

Before we can talk about the time evolution, we should note that equations (1.4)-(1.7) can't be solved numerically just like that and we need to address the issues.

4.1 Avoiding the singularity

We don't know the location of the apparent horizon and the location is not fixed, as the perturbation falls into the black hole and changes the black hole's properties. We consider the following transformation

$$r \longmapsto r + \lambda(t). \tag{4.1}$$

If we define $\hat{r}_h$ as our guess to the apparent horizon's location at $t = 0$, $\hat{r}_h + \lambda(t)$ is the apparent horizon's location at arbitrary time. This transformation let us solve the equations without the need to change the grid every time (as we need it to be contained in $[r_h, \infty]$). That way we stay far enough from the singularity.

4.2 Divergent functions

Because we would like to solve for very big values of r, and even find the limit there, we should find non divergent functions to solve for. To overcome the divergence at $r=\infty$, we expand $\Sigma(t,r)$, $B(t,r)$, $\dot{\Sigma}(t,r)$, $\dot{B}(t,r)$ and $A(t,r)$ near $r=\infty$ and introduce the rescaled functions

$$\Sigma(t,r)=r+\lambda(t)+\sum_{i=1}^{4}s_i(t)\,r^{-i}+\sigma(t,r)\,r^{-5} \tag{4.2}$$

$$B(t,r)=\sum_{i=1}^{2}b_i(t)\,r^{-i}+\beta(t,r)\,r^{-3} \tag{4.3}$$

$$\dot{\Sigma}(t,r)=\sum_{i=-2}^{2}\tilde{s}_i(t)\,r^{-i}+\tilde{\sigma}(t,r)\,r^{-3} \tag{4.4}$$

$$\dot{B}(t,r)=\sum_{i=0}^{1}\tilde{b}_i(t)\,r^{-i}+\tilde{\beta}(t,r)\,r^{-2} \tag{4.5}$$

$$A(t,r)=\frac{r^2}{2}+\sum_{i=-1}^{2}a_i(t)\,r^{-i}+\alpha(t,r)\,r^{-3} \tag{4.6}$$

while applying (3.4) to the leading term of $A(t,r)$ and (4.1) to the leading term of $\Sigma(t,r)$ (effectively setting $s_0(t)\equiv\lambda(t)$). We note that from now on, the apparent horizon's "location" lies on $r=\hat{r}_h$ ($\lambda(t)$ let us address the location in that way).

We find the unknowns $s_i(t)$, $b_i(t)$ and $a_i(t)$ by substitute expansions (4.2)-(4.6) in equations (1.4)-(1.8). The equations constrain these functions to be

$$s_1(t)=0,\ s_2(t)=0,\ s_3(t)=0,\ s_4(t)=0, \tag{4.7}$$

$$b_1(t)=0,\ b_2(t)=0, \tag{4.8}$$

$$a_{-1}(t)=\lambda(t)\,,\ a_0(t)=\frac{1}{2}\lambda^2(t)-\lambda'(t)\,,\ a_2(t)=-a_1(t)\,\lambda(t)\,. \tag{4.9}$$

$\tilde{s}_i(t)$ and $\tilde{b}_i(t)$ can be obtained by substituting the relevant terms in (1.2).

From (4.3), (4.8) and the discussion in section 3 we understand that knowledge of $\beta(t,r)$ is equivalent of knowing $b_3(t)$. So, from the discussion in section 3 we can deduce that if we know $a_{-2}(t_0)$, $a_1(t_0)$, $s_{-1}(t_0)$, $s_0(t_0)$ and $\beta(t_0,r)$ for arbitrary time $t=t_0$, we can find $\sigma(t_0,r)$, $\tilde{\sigma}(t_0,r)$, $\tilde{\beta}(t_0,r)$ and $\alpha(t_0,r)$, by solving the equations which are being obtained after substitute (4.2)-(4.6) in (1.4)-(1.7). Specifically, this means that $a_{-2}(0)$, $a_1(0)$, $s_{-1}(0)$, $s_0(0)$ and $\beta(0,r)$ are the initial conditions of the problem.

5 Time evolution

We need to evolve $\sigma(t,r)$, $\tilde{\sigma}(t,r)$, $\tilde{\beta}(t,r)$ and $\alpha(t,r)$ (I'll address them as the rescaled functions) in time. As mentioned in subsection 4.2, the time dependent functions which determine the spatial solution for any time t are $a_{-2}(t)$, $a_1(t)$, $s_{-1}(t)$, $s_0(t)$ and $\beta(t,r)$ (from now I'll address them as the time condition functions). So we conclude that we should evolve those 5 time condition functions in time if we want to evolve the rescaled functions in time. The goal here is to get ODEs/PDEs on the time condition functions.

5.1 $\mathbf{a_{-2}(t)}$, $\mathbf{a_1(t)}$ and $\mathbf{s_{-1}(t)}$

We formerly got (3.4) which is true for all t. Another early result is (3.5), which means that $a_1(t)=a_1(0)$ - it is constant as well. Finally, in (4.2) we chose $s_{-1}(t)=1$.

5.2 $\mathbf{s_0}\left(\mathbf{t}\right)$

In (4.2) we chose $s_0\left(t\right) \equiv \lambda\left(t\right)$. We take a time derivative of (1.9) to get

$$\partial_t \dot{\Sigma}\left(t, r_h\right) = 0. \tag{5.1}$$

Using (1.2)-(1.9) we get

$$\partial_t \dot{\Sigma}\left(t, r_h\right) = \cdots = 6A\left(t, r_h\right) + \dot{B}^2\left(t, r_h\right) = 0. \tag{5.2}$$

Substitute (4.5)-(4.6) in (5.2) to get

$$\lambda'\left(t\right) = \frac{3{r_h}^6 + 6{r_h}^3 a_1\left(t\right) + 6r_h\alpha\left(t, r_h\right) + \tilde{\beta}^2\left(t, r_h\right) 6{r_h}^5\lambda\left(t\right) - 6{r_h}^2 a_1\left(t\right)\lambda\left(t\right) + 3{r_h}^4\lambda^2\left(t\right)}{6{r_h}^4} \tag{5.3}$$

5.3 $\beta\left(\mathbf{t}, \mathbf{r}\right)$

After substituting (4.3), (4.5) and (4.6) in

$$\dot{B}\left(t, r\right) = A\left(t, r\right)\partial_r B\left(t, r\right) + \partial_t B\left(t, r\right), \tag{5.4}$$

we get

$$\begin{aligned}\partial_t \beta\left(t, r\right) =& r\tilde{\beta}\left(t, r\right) \\ &- \left[\frac{r^2}{2} + \frac{a_1\left(t\right)}{r} + \frac{\alpha\left(t, r\right)}{r^3} + r\lambda\left(t\right) - \frac{a_1\left(t\right)\lambda\left(t\right)}{r^2} + \frac{1}{2}\left(\lambda^2\left(t\right) - 2\lambda'\left(t\right)\right)\right]\left[-\frac{3\beta\left(t, r\right)}{r} + \partial_r\beta\left(t, r\right)\right].\end{aligned} \tag{5.5}$$

It is obvious that (5.5) can't be used to find the time derivative at $r = \infty$. To find $\lim_{r\to\infty}\partial_t\beta\left(t, r\right)$, we expand (5.4) near $r = \infty$ and get

$$\partial_t\beta\left(t, r\right) = -\left[2\partial_r\beta\left(t, r\right) + \partial_r\tilde{\beta}\left(t, r\right)\right]r^2 + \left[\frac{3}{2}\beta\left(t, r\right) + \tilde{\beta}\left(t, r\right)\right]r + 3\beta\left(t, r\right)\lambda\left(t\right) + \mathcal{O}\left(r^{-1}\right). \tag{5.6}$$

At first glance, it seems (5.6) does diverge at $r = \infty$, but one can show that

$$\lim_{r\to\infty}\tilde{\beta}\left(t, r\right) = -\frac{3}{2}\beta\left(t, r\right),\ \partial_r\beta\left(t, r\right) \sim r^{-2},\ \partial_r\tilde{\beta}\left(t, r\right) \sim r^{-2}, \tag{5.7}$$

and with the fact that this system of equations will be solved for a finite parameter

$$\rho \triangleq 1 - \frac{2r_h}{r}, \tag{5.8}$$

(5.6) is finite and can be used to find $\lim_{r\to\infty}\partial_r\beta\left(t, r\right)$.

6 Results

When introducing a perturbation to the black hole, this perturbation decays in time. This perturbation can be described by a superposition of what we call the black hole quasi-normal modes. A quasi-normal mode is an oscillation which have a decaying part as well. As a hallmark of the numerical accuracy, we would like to find the first quasi-normal mode frequency and fit the numerical data to the theoretical function.

I chose the initial conditions

$$\beta\left(0, r\right) = 5e^{-\left(\frac{\frac{1}{r} - \frac{1}{4}}{0.15}\right)^2} \quad , \quad a_1\left(t\right) = -\frac{1}{2} \quad , \quad \hat{r}_h = 1,$$

where $\lambda\left(0\right)$ was found to satisfy (1.9). The Chebyshev grid I used was of 33 grid points and the time integration algorithm was a predictor-corrector method where the four-step Adams-Bashforth method is used as the predictor and the three-step Adams-Moulton is used as the corrector. The predictor-corrector method was initialized with fourth-order Runge-Kutta method and used a time-step of $\Delta t = 10^{-3}$.

The theoretical first quasi-normal mode's frequency is

$$\omega_1 = 2.66385 + 1.84942i. \tag{6.1}$$

In figure 6.1 we can see a very good fit between the numerical and the theoretical values.

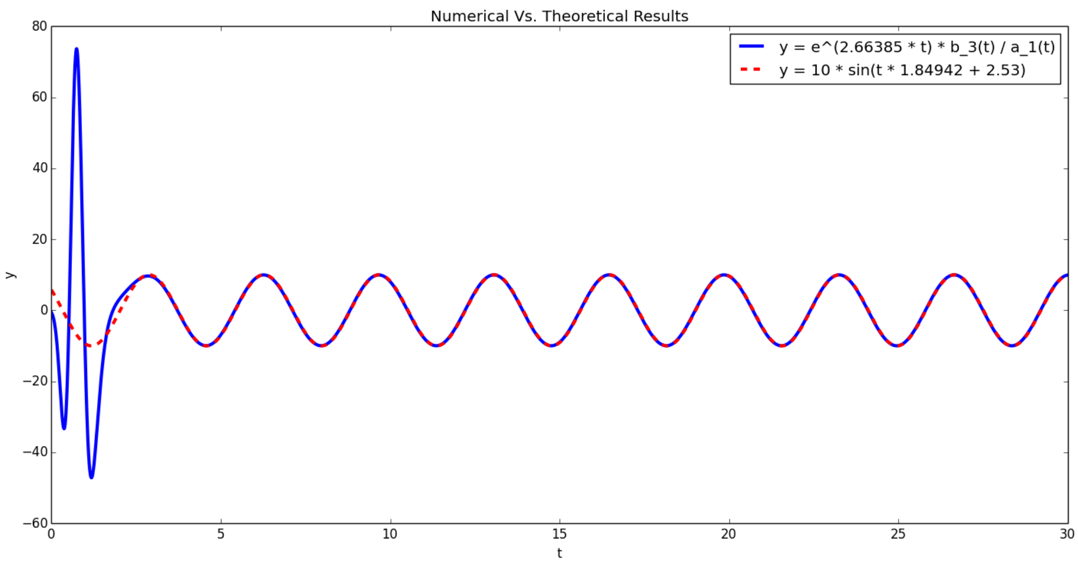

Figure 6.1: A fit between $e^{2.66385t} \cdot \frac{b_3(t)}{a_1(t)}$ (where $b_3(t) = \lim_{r\to\infty} \beta(t, r)$) and $\sin(1.84942t + 2.53)$.