Solving Harmonic Function using SVD

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1 Introduction

I will try to solve the Harmonic function using the SVD technique before going to more complicated PDEs.

2 Harmonic function

The harmonic function is given by the equation:

$$f(x, y) = 2\pi^2 \sin(\pi x)\sin(\pi y)$$

Find $U : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \\ \Omega &= (0,1) \times (0,1) \end{aligned}$$

3 Finite difference method

$$-\Delta u = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1}$$

The differential equation can be approximated as

$$-\Delta u = -\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{dx^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{dy^2}\right) \tag{2}$$

By assuming $dx = dy = h$, we can rewrite the Eq. (2) to be

$$-\Delta u = -\left(\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}\right) \tag{3}$$

At the boundary node, the equation will be

$$u_0 = 0 \tag{4}$$

For node inside the boundary, the equation will be

$$-\frac{1}{h^2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right) = f(x, y) \tag{5}$$

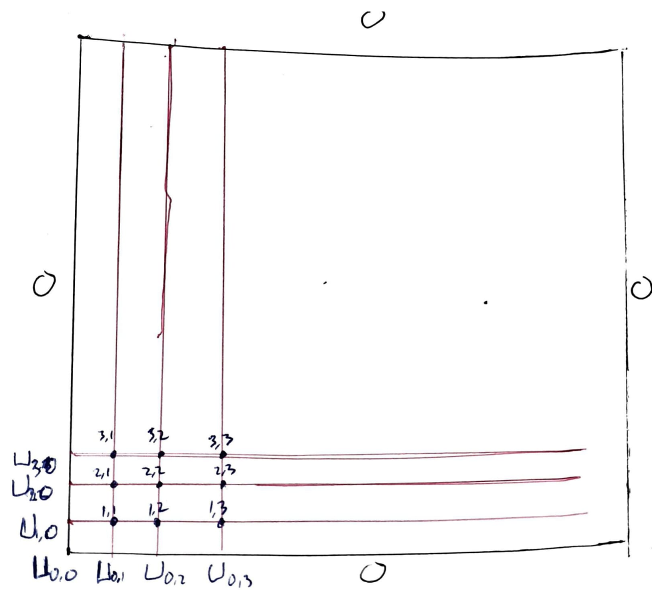

Figure 1: Discretization of the domain

4 Constructing the matrix

We will have a set of linear equations based on the size of h.

$$
-\frac{1}{h^2}\begin{bmatrix} 1 & \cdots & & & & & & & & 0 \\ \vdots & \ddots & & & & & & & & \\ & 1 & \cdots & 1 & -4 & 1 & \cdots & 1 & \cdots & \\ & & & & & & & & \ddots & \\ 0 & & & & & & & & \cdots & 1 \end{bmatrix}\begin{bmatrix} u_{00} \\ u_{01} \\ u_{02} \\ u_{03} \\ \vdots \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ \vdots \\ u_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f(x,y) \\ f(x,y) \\ f(x,y) \\ \vdots \\ 0 \end{bmatrix} \tag{6}
$$

This matrix has a diagonally dominant property.

5 Construct the python program