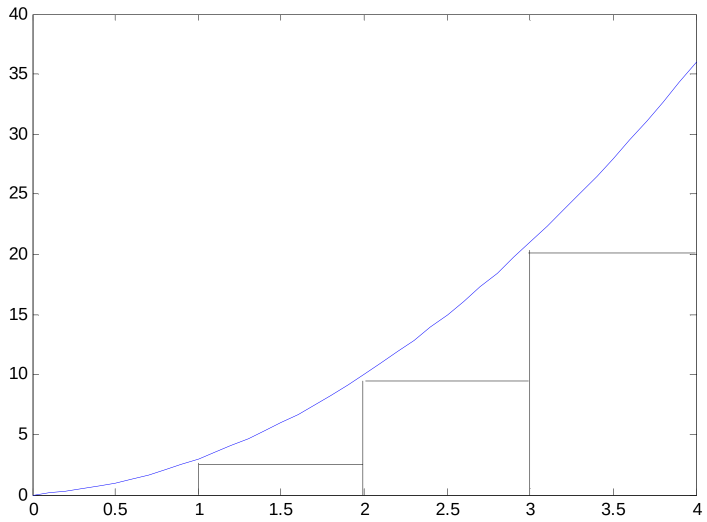
微積分先修班期末考解答

1.



$$\Delta x = \frac{4-0}{4} = 1$$

x	0	1	2	3
$f(x)$	0	3	10	21

$Sum = 0\cdot 1 + 3\cdot 1 + 10\cdot 1 + 21\cdot 1 = 34$

2.

$f(x) = x^4 \sin x$ is increasing

If $0 \le x \le 1$, then $0 \le x^4 \le 1$ and $0 \le \sin x \le 1$

$0 \le x^4 \sin x \le x^4$

$$0 \le \int_0^1 x^4 \sin x dx \le \int_0^1 x^4 dx$$

So $0 \le \int_0^1 x^4 \sin x dx \le 0.2$

3.

$$\int_1^5 f'(x)dx = f(5) - f(1)$$

$$f(5) - 2 = 9$$

$$f(5) = 11$$

Ans: $f(5) = 11$

4.

$$\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^6+\left(\frac{2}{n}\right)^6+\left(\frac{3}{n}\right)^6+.............+\left(\frac{n}{n}\right)^6\right]$$

$$=\lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^6$$

$$=\int_0^1 x^6 dx = \frac{1}{7}$$

Ans: $\frac{1}{7}$

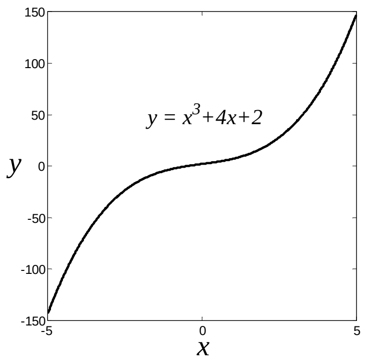
5.

(1) $f(x) = x^3 + 4x + 2$,

$f'(x) = 3x^2 + 4 > 0 \ \ \forall x \in R$.

Thus, $f(x)$ is increasing on the whole real line.

Hence, it is one-to-one and has an inverse.



(2) $(f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$.

$f(1) = 1^3 + 4\cdot 1 + 2 = 7$.

$f'(x) = 3x^2 + 4$, $f'(1) = 3\cdot 1^2 + 4 = 7$.

$(f^{-1})'(7) = (f^{-1})'(f(1)) = \dfrac{1}{f'(1)} = \dfrac{1}{7}$.

6.

(a) $y' = e^{\tan 2x}\cdot(\tan 2x)' = 2\sec^2 2x \cdot e^{\tan 2x}$.

(b) $y' = \dfrac{1}{\sec^2 x}\cdot(\sec^2 x)' = \dfrac{1}{\sec^2 x}\cdot 2\sec x\cdot(\sec x)' = \dfrac{2\sec x\cdot(\sec x\cdot\tan x)}{\sec^2 x} = 2\tan x.$

7.

(a)

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+x}\left(\to\frac{0}{0}\right)$$

$$\overset{L'H}{=}\lim_{x\to 0}\frac{\sin x}{2x+1} = \frac{0}{1} = 0$$

(b).

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) = \lim_{x\to 1^+}\frac{x\ln x-(x-1)}{(x-1)\ln x}\left(\to\frac{0}{0}\right)$$

$$\overset{L'H}{=}\frac{\ln x + x\cdot\frac{1}{x}-1}{(x-1)\cdot\frac{1}{x}+\ln x} = \lim_{x\to 1^+}\frac{\ln x}{1-\frac{1}{x}+\ln x}\left(\to\frac{0}{0}\right)$$

$$\overset{L''H}{=}\lim_{x\to 1^+}\frac{\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}$$

8.

(a)

Let $u = 1 + \sin x, du = \cos x dx$

$$x = 0 \to \frac{\pi}{2} \Rightarrow u = 1 \to 2$$

$$\int_1^2 \frac{1}{u} du = \ln|u|\Big|_1^2 = \ln 2$$

(b)

$$\int \sin^2 x \cdot \cos^4 x \cdot \cos x dx = \int \sin^2 \cdot (1 - \sin^2 x)^2 d(\sin x) = \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$$

(c)

$$f(x) = \frac{\sin x}{1 + x^2}, f(-x) = \frac{-\sin x}{1 + x^2} = -f(x)$$

$\Rightarrow f(x)$ is odd function.

$$\int_{-1}^1 \frac{\sin x}{1 + x^2} dx = 0$$

(d)

$$\int_1^4 x^{\frac{3}{2}} \ln x dx$$

$u = \ln x \qquad dv = x^{\frac{3}{2}} dx$

$du = \frac{1}{x} dx \qquad v = \frac{2}{5} x^{\frac{5}{2}}$

$$\int x^{\frac{3}{2}} \ln x dx = \ln x \cdot \frac{2}{5} x^{\frac{5}{2}} - \frac{2}{5} \int x^{\frac{3}{2}} dx = \frac{2}{5} x^{\frac{5}{2}} \ln x - \frac{2}{5} \cdot \frac{2}{5} x^{\frac{5}{2}} + c = \frac{2}{5} x^{\frac{5}{2}} \ln x - \frac{4}{25} x^{\frac{5}{2}} + c$$

$$\int_1^4 x^{\frac{3}{2}} \ln x dx = \left(\frac{2}{5} x^{\frac{5}{2}} \ln x - \frac{4}{25} x^{\frac{5}{2}}\right)\Big|_1^4 = \frac{64}{5}\ln 4 - \frac{124}{25}$$

(e)

$$\frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

$A = 1, B = -1, C = 0$

$$\int \frac{1}{x(x^2 + 1)} dx = \int \frac{1}{x} dx - \int \frac{x}{x^2 + 1} dx = \ln|x| - \frac{1}{2}\ln|x^2 + 1| + c$$

9.

(a) If $p \neq 1$

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t x^{-p} dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_{x=1}^{x=t} = \lim_{t\to\infty} \frac{1}{1-p}\left(\frac{1}{t^{p-1}} - 1\right)$$

(i) If $p > 1$, then $p - 1 > 0$, so $t \to \infty$

$$\Rightarrow t^{p-1} \to \infty$$

$$\Rightarrow \frac{1}{t^{p-1}} \to 0$$

Therefore $\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1}$ if $p > 1$

$\Rightarrow$ *convergent.*

(ii) If $p < 1$, then $p - 1 < 0$, so $t \to \infty$

$$\Rightarrow t^{p-1} \to 0$$

$$\Rightarrow \frac{1}{t^{p-1}} \to \infty$$

Therefore $\int_1^{\infty} \frac{1}{x^p} dx$ is *divergent.*

(b) If $p = 1$

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{a\to\infty} \int_1^a \frac{1}{x} dx = \lim_{a\to\infty}\big[\ln|x|\big]_1^a = \lim_{a\to\infty}(\ln a - \ln 1) = \lim_{a\to\infty} \ln a = \infty$$

$\Rightarrow$ *divergent.*

$\Rightarrow p > 1$ *convergent*, $p \leq 1$ *divergent.*

10.

(a)

$$\int_2^{\infty} \frac{1}{x\ln x} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x} dx \quad \left(u = \ln x,\, du = \frac{1}{x} dx;\, x = 2 \Rightarrow u = \ln 2,\, x = t \Rightarrow u = \ln t\right)$$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u} du = \lim_{t\to\infty}\big[\ln|u|\big]_{\ln 2}^{\ln t} = \lim_{t\to\infty}\big(\ln t - \ln(\ln 2)\big) = \infty$$

$\Rightarrow$ *divergent.*

(b)

$$\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \quad (u = \ln x,\ \ du = \frac{1}{x}dx \ \ ;\ \ dv = \frac{1}{\sqrt{x}}dx,\ \ v = 2\sqrt{x})$$

$$= \lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_t^4 - \int_t^4 2\sqrt{x}\cdot\frac{1}{x}dx\right)$$

$$= \lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_t^4 - \int_t^4 \frac{2}{\sqrt{x}}dx\right)$$

$$= \lim_{t\to 0^+}\left(\left(4\ln 4 - 2\sqrt{t}\ln t\right) - \left[4\sqrt{x}\right]_t^4\right)$$

$$= \lim_{t\to 0^+}\left(\left(4\ln 4 - 2\sqrt{t}\ln t\right) - \left(8 - 4\sqrt{t}\right)\right)$$

$$= 4\ln 4 - 0 - 8 + 0$$

$$= 4\ln 4 - 8$$

<Note>

$$\lim_{t\to 0^+} 2\sqrt{t}\ln t \quad (0\cdot\infty)$$

$$= \lim_{t\to 0^+}\frac{2\ln t}{\frac{1}{\sqrt{t}}} \quad (\frac{\infty}{\infty})$$

(by $L'H\hat{o}pital's\ Rule$)

$$= \lim_{t\to 0^+}\frac{\frac{2}{t}}{-\frac{1}{2}t^{\frac{-3}{2}}}$$

$$= \lim_{t\to 0^+} -4\sqrt{t}$$

$$= 0$$

11.

$x^2 = 2x - x^2$

$\Rightarrow x^2 = x$

$\Rightarrow$ 兩拋物線交點(0,0) (1,1)

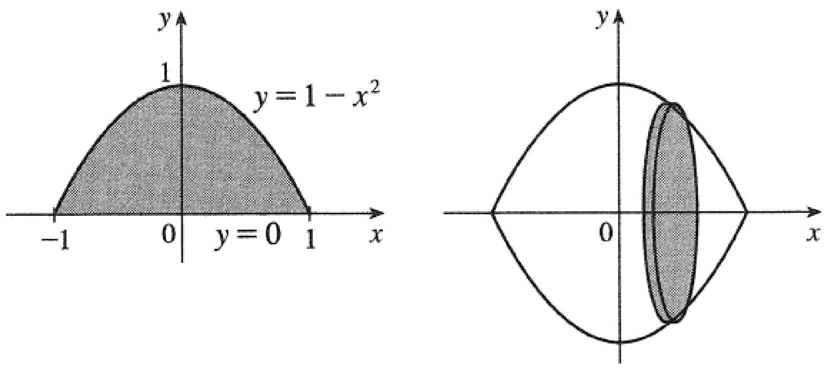
$$\int_0^1 (2x - x^2 - x^2)dx$$

$$= \int_0^1 (2x - 2x^2)dx$$

$$= \left[x^2 - \frac{2}{3}x^3 \right]_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

12.



$1 - x^2 = 0 \Rightarrow x = \pm 1$

$$\int_{-1}^1 (1 - x^2)^2 \pi dx$$

$$= \pi \int_{-1}^1 (1 - 2x^2 + x^4)dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x \right]_{-1}^1$$

$$= \pi[(\frac{1}{5} - \frac{2}{3} + 1) - (-\frac{1}{5} + \frac{2}{3} - 1)]$$

$$= \frac{16}{15}\pi$$