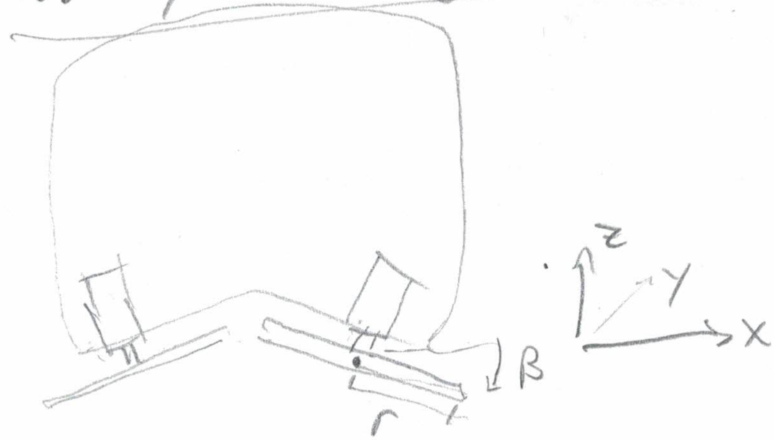
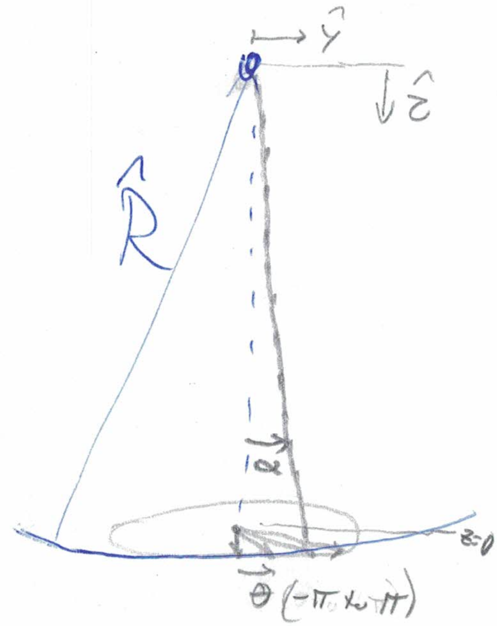
2017-03-21

Passively Balancing robot





Points on wheel : Side view :

$$z = -r\cos\theta\,\sin\beta$$

$$y = r\sin\theta \rightarrow \theta = \sin^{-1}\frac{y}{r}$$

$$\therefore z = r\cos\left(\sin^{-1}\left(\tfrac{y}{r}\right)\right)\sin\beta \rightarrow \text{small ANGLE } \theta:\ \theta \approx \frac{y}{r}$$

$$\therefore z \approx r\cos\left(\frac{y}{r}\right)\sin\beta$$

$$\left[z \approx r\sin\beta\right]$$

Effective Radius of Curvature :

$$\hat{R} = \frac{\left(1 + \left(\frac{dz}{dy}\right)^2\right)^{3/2}}{\frac{d^2z}{dy^2}}$$

$$\frac{dz}{dy} = r\sin\beta\left(-\sin\left(\tfrac{y}{r}\right)\right)\cdot\frac{1}{r} = -\sin\frac{y}{r}\sin\beta$$

$$\text{small Ang.: } \frac{dz}{dy} \simeq -\frac{y}{r}\sin\beta$$

$$\frac{d^2z}{dy^2} \approx -\frac{1}{r}\sin\beta \qquad R = \frac{\left(1 + \frac{y^2}{r^2}\sin^2\beta\right)^{3/2}}{-\frac{1}{r}\sin\beta}$$

or for any angle θ; find equivalent angle α

$$y = r\sin\theta = R\sin\alpha = \hat{y}$$

$$z = -r\cos\theta\sin\beta = -r\sin\beta + R - R\cos\alpha$$

$$\text{or } \hat{z} = R\cos\alpha = R - r\sin\beta + r\cos\theta\sin\beta$$

small angles: $y \approx R\alpha$

$$\therefore \frac{dy}{d\alpha} = R$$

consider $\dfrac{\hat{y}}{\hat{z}} = \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{r\sin\theta}{R + r\sin\beta(\cos\theta - 1)}$

Alt: Guess:

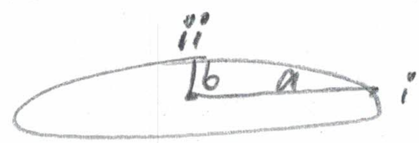
$R = r$ when $\beta = \frac{\pi}{2}$

$R = \infty$ when $\beta = 0$

$$\boxed{\text{Guess } R = \frac{r}{\sin\beta}}$$

From Wikipedia: "Radius of Curvature"

For ellipse with major axis $2a$
and minor axis $2b$,



curvature

at i: $R = \frac{b^2}{a}$

ii: $R = \frac{a^2}{b}$

---

here $b = r\sin\beta$

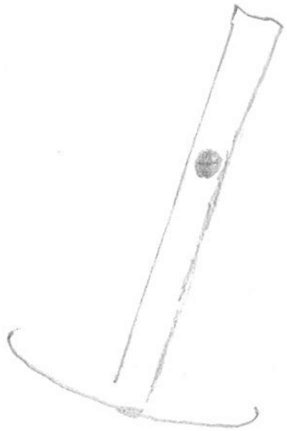
$a = r$

$$\therefore \boxed{R = \frac{r^2}{r\sin\beta} = \frac{r}{\sin\beta}}$$

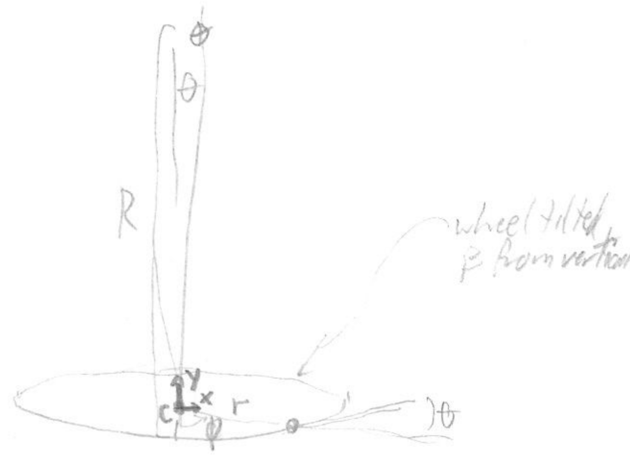
In practice r is fixed (eg. 30mm) and R is a design spec (eg 200mm)

Therefore $\boxed{\beta = \sin^{-1}\left(\frac{r}{R}\right)}$ is solved for (eg. 8.6°)

2019-05-08



MAPPING of θ to ϕ



point rel to point C: $x = r \sin\phi$

$Y = r\cos\phi\cos\beta$

$$\tan\theta = \frac{dY}{dx} = \frac{\frac{\partial Y}{\partial\phi}\,d\phi}{\frac{\partial x}{\partial\phi}\,d\phi}$$

$$\cancel{\tan\theta = \frac{dY}{dx} = \frac{r\cos\phi\,d\phi}{r\sin\phi\cos\beta\,d\phi}}$$

$$\cancel{\tan\theta = \cot\phi\sec\beta}$$

$$\cancel{\therefore \theta = \tan^{-1}(\cot\phi\sec\beta)}$$

$$\cancel{\text{or } \phi =}$$

or $\cot\theta = \tan\phi\cos\beta$

$$\boxed{\tan^{-1}(\cot\theta\sec\beta) = \phi}$$

$$\tan\theta = \frac{dY}{dx} = \frac{\frac{\partial Y}{\partial\phi}\,d\phi}{\frac{\partial x}{\partial\phi}\,d\phi}$$

$$= \frac{r\sin\phi\cos\beta\,d\phi}{r\cos\phi\,d\phi}$$

$$\tan\theta = \tan\phi\cos\beta$$

$\cancel{\therefore \tan\theta =}$

$$\therefore \theta = \tan^{-1}(\tan\phi\cos\beta)$$

$$\text{and } \phi = \tan^{-1}(\tan\theta\sec\beta)$$

RADIUS of CURVATURE $\cancel{R = \frac{dx}{d\theta}}$

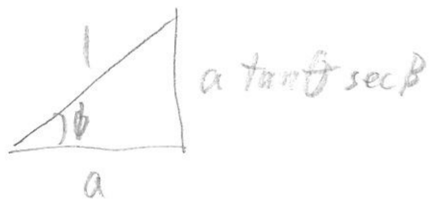
NOT True

$\cancel{\sin(\tan^{-1}(\tan}$ where $x = r\sin\phi$

$$\frac{dx}{d\theta} = \frac{dx}{d\phi}\frac{\partial\phi}{\partial\theta}$$

$$\frac{dx}{d\theta} = r\cos\phi\,\frac{d}{d\theta}\left(\tan^{-1}(\tan\theta\sec\beta)\right)$$

Consider $\sin\phi$ given $\tan\phi = \tan\theta \sec\beta$



$$1^2 = a^2 + a^2 \tan^2\theta \sec^2\beta$$

$$1^2 = a^2 (1 + \tan^2\theta \sec^2\beta)$$

$$a^2 = \frac{1}{1 + \tan^2\theta \sec^2\beta} = \frac{\cos^2\beta}{\cos^2\beta + \tan^2\theta}$$ (by mult $\frac{\cos^2\beta}{\cos^2\beta}$)

$$a = \sqrt{\frac{\cos^2\beta}{\cos^2\beta + \tan^2\theta}} = \frac{\cos\beta}{\sqrt{\cos^2\beta + \tan^2\theta}}$$

$$\therefore \sin\phi = a \tan\theta \sec\beta = a \frac{\tan\theta}{\cos\beta} = \frac{\cos\beta}{\sqrt{\cos^2\beta + \tan^2\theta}} \frac{\tan\theta}{\cos\beta}$$

$$\therefore \boxed{\sin\phi = \frac{\tan\theta}{\sqrt{\cos^2\beta + \tan^2\theta}}}$$? check and re-derive

$$\therefore \boxed{x = r\sin\phi = \frac{r\tan\theta}{\sqrt{\cos^2\beta + \tan^2\theta}}}$$

$\cancel{R = \frac{dx}{d\theta}}$ NOT true

Then $\cos\phi = a = \frac{\cos\beta}{\sqrt{\cos^2\beta + \tan^2\theta}}$

and $y = -r\cos\phi\cos\beta$

$$\boxed{y = -r\frac{\cos^2\beta}{\sqrt{\cos^2\beta + \tan^2\theta}}}$$

$$R = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

from Wolfram Alpha

$$\frac{dx}{d\theta} = \frac{r\cos^2\beta\,\sec^2\theta}{(\cos^2\beta + \tan^2\theta)^{3/2}} \qquad \text{for } x = \frac{r\tan\theta}{\sqrt{\cos^2\beta + \tan^2\theta}}$$

$$\frac{dy}{d\theta} = \frac{r\cos^2\beta\,\tan\theta\,\sec^2\theta}{(\cos^2\beta + \tan^2\theta)^{3/2}} \qquad \text{for } y = \frac{-r\cos^2\beta}{\sqrt{\cos^2\beta + \tan^2\theta}}$$

$$\left(\frac{dx}{d\theta}\right)^2 = \frac{r^2\cos^4\beta\,\sec^4\theta}{(\cos^2\beta + \tan^2\theta)^3}$$

$$\left(\frac{dy}{d\theta}\right)^2 = \frac{r^2\cos^4\beta\,\tan^2\theta\,\sec^4\theta}{(\cos^2\beta + \tan^2\theta)^3}$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \frac{r^2\cos^4\beta\,\sec^4\theta}{(\cos^2\beta + \tan^2\theta)^3}\underbrace{(1 + \tan^2\theta)}_{=\sec^2\theta}$$

$$= \frac{r^2\cos^4\beta\,\sec^6\theta}{(\cos^2\beta + \tan^2\theta)^3}$$

$$\therefore \boxed{R = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{\frac{r^2\cos^4\beta\,\sec^6\theta}{(\cos^2\beta + \tan^2\theta)^3}} = \frac{r\cos^2\beta\,\sec^3\theta}{(\cos^2\beta + \tan^2\theta)^{3/2}}}$$

checks:

for $\theta = 0$

$R = \frac{r}{\cos\beta}$ ✓

for $\theta = \frac{\pi}{2}$

$R = \frac{\infty}{\infty}$ ✓

but limit is $r\cos^2\beta$

Then where is $R = r$? (For what angle θ)?

$$R = \frac{r\cos^2\beta \sec^3\theta}{(\cos^2\beta + \tan^2\theta)^{3/2}} = r$$

$$\cos^2\beta \sec^3\theta = (\cos^2\beta + \tan^2\theta)^{3/2} \qquad \text{raise to } 2/3 \text{ power}$$

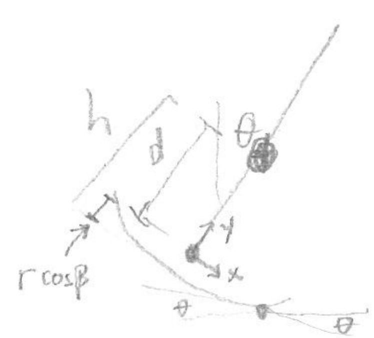
$$\cos^{4/3}\beta \sec^2\theta = \cos^2\beta + \tan^2\theta$$

= Big ugly mess!
solve numerically

$$\left.\begin{aligned} \text{for } \beta = 81^\circ &: \theta(R=r) \approx \pm 0.25 \text{ rad} \\ 80^\circ &: \theta(R=r) = \pm 0.265 \end{aligned}\right\} \approx 15 \text{ deg}$$

Note: This is of no consequence.

Of consequence is the angle when the COM is farther forward than the contact point



distance of COM ahead of wheel center $\boxed{P_{COMx} = d\sin\theta}$

position of contact point in Cartesian space:

$$\vec{p}_{cp} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{matrix} x(\theta) \\ y(\theta) \end{matrix}$$

$$\therefore p_{cp_x} = \cos\theta \, x(\theta) + \sin\theta \, y(\theta)$$

$$= \cos\theta\left(\frac{r\tan\theta}{\sqrt{\cos^2\beta + \tan^2\theta}}\right) + \sin\theta\left(\frac{-r\cos^2\beta}{\sqrt{\cos^2\beta + \tan^2\theta}}\right)$$

$$\therefore \boxed{p_{cp_x} = \frac{r\sin\theta}{\sqrt{\cos^2\beta + \tan^2\theta}}\left(1 - \cos^2\beta\right) = \frac{r\sin^2\beta\sin\theta}{\sqrt{\cos^2\beta + \tan^2\theta}}}$$

Solve $p_{cmx} = p_{cpx}$ to find θ_{lim}: max statically stable lean angle

$$d \cancel{\sin\theta_{lim}} = \frac{r \sin^2\beta \cancel{\sin\theta_{lim}}}{\sqrt{\cos^2\beta + \tan^2\theta_{lim}}}$$

$$\sqrt{\cos^2\beta + \tan^2\theta_{lim}} = \frac{r}{d} \sin^2\beta$$

$$\cos^2\beta + \tan^2\theta_{lim} = \frac{r^2}{d^2} \sin^4\beta$$

$$\tan^2\theta_{lim} = \frac{r^2}{d^2} \sin^4\beta - \cos^2\beta \quad \leftarrow ☆$$

$$\tan\theta_{lim} = \sqrt{\left(\frac{r}{d}\right)^2 \sin^4\beta - \cos^2\beta}$$

$$\theta_{lim} = \tan^{-1}\left(\sqrt{\frac{r^2}{d^2} \sin^4\beta - \cos^2\beta}\right)$$

at angle where ~~[illegible]~~ COM is directly vertical from contact point.

This is θ_{lim} for static stability.

Check for $d=4r$: for $\beta = 81°$: $\theta_{lim} = 0.185$ rad $\approx 10.6°$
$80°$: $\theta_{lim} = 0.167$ rad $\approx 9.6°$
$90°$: $\theta_{lim} = \tan^{-1}\left(\frac{r}{d}\right)$ ✓ $\Rightarrow 0.245$ rad $\approx 14.0°$

~~So, angle where static~~

Note: Speed limit for sudden stop is defined by energy conservation:

Moving Upright:

$E = T + V$

$= \frac{1}{2}mv^2 + 0$

After stops and rises to "edge" where height is max:

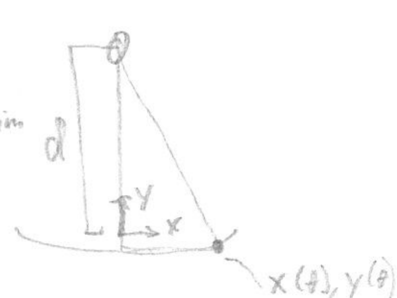
$E = T + V$

$= 0 + mg\underbrace{\Delta y_{com}}_{\text{change in height of COM}}$

& This max height is the same configuration as the static limit!

height when upright $= h = d + r\cos\beta \; (= y_0)$

height at static limit $(y_{max}) = \| [x(\theta), d - y(\theta)] \|_{\theta_{lim}}$



$\Delta y_{com} = y_{max} - y_0$

$y_{max} = \sqrt{x(\theta_{lim})^2 + (d - y(\theta_{lim}))^2}$

$= \sqrt{\frac{r^2\tan^2\theta_{lim}}{\cos^2\beta + \tan^2\theta_{lim}} + d^2 - \frac{2d(-r\cos^2\beta)}{\sqrt{\cos^2\beta + \tan^2\theta_{lim}}} + \frac{r^2\cos^4\beta}{\cos^2\beta + \tan^2\theta_{lim}}}$

all $\tan^2\theta_{lim}$ can be substituted with Eq. ★

$$\tan^2\theta_{lim} = \frac{r^2}{d^2}\sin^4\beta - \cos^2\beta$$

$$\therefore Y_{MAX} = \sqrt{\frac{r^2\left(\frac{r^2}{d^2}\sin^4\beta - \cos^2\beta\right)}{\cos^2\beta + \left(\frac{r^2}{d^2}\sin^4\beta - \cos^2\beta\right)} + d^2 + \frac{2dr\cos^2\beta}{\sqrt{\cancel{\cos^2\beta} + \frac{r^2}{d^2}\sin^4\beta - \cancel{\cos^2\beta}}} + \frac{r^2\cos^4\beta}{\cancel{\cos^2\beta} + \frac{r^2}{d^2}\sin^4\beta - \cancel{\cos^2\beta}}}$$

$$Y_{MAX} = \sqrt{\frac{\frac{r^4}{d^2}\sin^4\beta - \cos^2\beta\, r^2}{\frac{r^2}{d^2}\sin^4\beta} + d^2 + \frac{2dr\cos^2\beta}{\sqrt{\frac{r^2}{d^2}\sin^4\beta}} + \frac{r^2\cos^4\beta}{\frac{r^2}{d^2}\sin^4\beta}}$$

$$Y_{MAX} = \sqrt{\frac{\frac{r^4}{d^2}\sin^4\beta - r^2\cos^2\beta + r^2\sin^4\beta + r^2\cos^4\beta + 2r^2\sin^2\beta\cos^2\beta}{\frac{r^2}{d^2}\sin^4\beta}}$$

(mult $\frac{d^2}{r^2}$)

$$Y_{MAX} = \sqrt{\frac{r^2\sin^4\beta - d^2\cos^2\beta + d^2\sin^4\beta + d^2\cos^4\beta + 2d^2\sin^2\beta\cos^2\beta}{\sin^4\beta}}$$

$$Y_{MAX} = \frac{d}{\sin^2\beta}\sqrt{\frac{r^2}{d^2}\sin^4\beta - \cos^2\beta + \sin^4\beta + \cos^4\beta + 2\sin^2\beta\cos^2\beta}$$

$$Y_{MAX} = \frac{d}{\sin^2\beta}\sqrt{\frac{r^2}{d^2}\sin^4\beta - \cos^2\beta + \underbrace{\left(\sin^2\beta + \cos^2\beta\right)^2}_{=1^2}}$$

$$= \frac{d}{\sin^2\beta}\sqrt{\frac{r^2}{d^2}\sin^4\beta + \underbrace{-\cos^2\beta + 1}_{=\sin^2\beta}}$$

$$= \frac{d}{\sin^2\beta}\sqrt{\frac{r^2}{d^2}\sin^4\beta + \sin^2\beta}$$

$$= \frac{d}{\sin\beta}\sqrt{\frac{r^2}{d^2}\sin^2\beta + 1}$$

$$= \frac{r^2}{d^2}\frac{d^2}{r^2}$$

$$Y_{MAX} = \frac{r}{\sin\beta}\sqrt{\sin^2\beta + \frac{d^2}{r^2}}$$

$$\boxed{Y_{MAX} = \sqrt{r^2 + \frac{d^2}{\sin^2\beta}} = \sqrt{r^2 + d^2\csc^2\beta}}$$

MAXIMUM value of R is $\frac{r}{\cos\beta}$ (upright)

What min value of β is needed for $R \geq h$?

(i.e. static limit)

$$h = d + r\cos\beta$$

Find β s.t. $R \geq h$

$$R \geq h$$

$$\frac{r}{\cos\beta} \geq d + r\cos\beta$$

$$r \geq d\cos\beta + r\cos^2\beta$$

$$0 \geq r\cos^2\beta + d\cos\beta - r$$

Quad Formula $\therefore$ $\boxed{\underbrace{\cos\beta_{lim}}_{\text{MAX value of } \cos\beta} = \frac{-d \pm \sqrt{d^2+4r^2}}{2r}}$

$\therefore$ Min value (magnitude) of β

consider for $d=4r$: $\cos\beta_{lim} = \frac{-4r \pm \sqrt{16r^2+4r^2}}{2r} = -2 \pm \sqrt{5} = \begin{matrix} 0.2361 \\ -4.2361 \end{matrix}$ not feasible

$\boxed{\therefore \beta_{min} \approx 1.3325 \text{ rad} = 76.35°}$

Next Questions

- Compute MAX speed that's o.k. for
  instantaneous slip.
  - ↳ Finish calcs already underway

- Compute MAX allowable speed deviation as a function
  of speed.

- Consider the friction model:
  - Friction force opposes slip direction
  - Slip direction is complex:
    depends on
    - $\vec{V}_{com}$
    - $\vec{\Omega}_{body}$
    - $\vec{\omega}_{wheels}$ (each wheel)
    - θ

  * Lower friction may be better for stability
    ↳ but zero won't move
    ↳ So there may be an optimum. How to define it?

- Consider steering – including steering at speed.
- Design + Testing

2019-05-08

Stability: Inv. Pend: sanity check

$$\Sigma M_o = \underset{md^2}{I}\ddot{\theta} = +mgd\sin\theta$$

$$md^2\ddot{\theta} = mdg\sin\theta + \tau \approx mgd\theta + \tau$$

$$\ddot{\theta} = \frac{g}{d}\theta + \frac{\tau}{md^2}$$

State Space

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ \frac{g}{d} & 0\end{bmatrix}}_{A}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{md^2}\end{bmatrix}\tau$$

$$\text{eig}: \det|\lambda I - A| = 0$$

$$\det\begin{pmatrix}\lambda & -1\\ -\frac{g}{d} & \lambda\end{pmatrix} = 0$$

$$\lambda^2 - \frac{g}{d} = 0$$

$$\therefore \lambda = \pm\sqrt{\frac{g}{d}}$$

✓

LaPlace:

$$s^2\Theta = \frac{g}{d}\Theta + \frac{1}{md^2}T$$

$$\Theta\left(s^2 + -\frac{g}{d}\right) = \frac{1}{md^2}T$$

$$\therefore \frac{\Theta}{T} = \frac{1}{md^2\left(s^2 - \frac{g}{d}\right)}$$

$$\text{poles}: s^2 - \frac{g}{d} = 0$$

$$s = \pm\sqrt{\frac{g}{d}}$$

existence of pole at $+\sqrt{\frac{g}{d}}$ implies unstable

(positive real part)

---

contrast with ordinary pendulum:

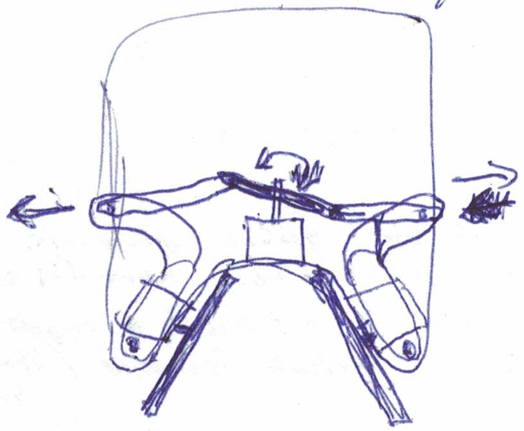
$$\ddot{\theta} = \frac{-g}{d}\theta + \frac{\tau}{md^2}$$

$$\therefore s = \pm\sqrt{-\frac{g}{d}} = \pm i\sqrt{\frac{g}{d}} \Rightarrow \text{zero real part.}$$

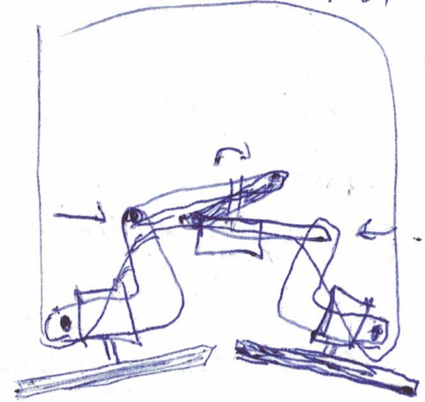
Not unstable.

WACKY ROBOT Ideas 2019-07-30

① Wobblebot with dynamic wheel tilt control



B: BALANCER



W: WOBBLER

- Start in W mode when off
- Start moving slowly
- Activate dynamic balance if (?)
  - target speed is above/approaches max ~~hard stop~~ E-stop speed
  - detect angles approaching stable limits
  - detect "The jitters"
  - Or simply "as speed increases"
  - or "as dynamic balance demands increase
    - ↳ eg. observe (ang accel • $\vec{\theta}$) >0 (same sign)
      - ↳ but this could be confused by motor action

possibly just increase P,I gains?

- Deactivate dynamic balance as speed lowers... and other opposites of
- "Dynamic Balance" = B mode • "Deactivated" = W mode.

Wobblebot

2019 08 18

{Cite ~~Gordon~~ GORDON McCOMB - Robot Bonanza}

~~Main~~ MOTIVATION:

* Most Upright BALANCING Robots ~~require~~ are ~~typically~~ highly dynamic machines
* Balancing ~~high aspect~~ machines with high COM (higher than wheel axis) requires active ~~control and~~ sensing and control to drive wheels under the COM
* This ~~causes such machines~~ active control requires high-power actuators capable ~~of~~ rapidly ~~continuously~~ righting the robot after any disturbance.
* ~~The risk~~ Such robots also require either manual launch from a standing position or ~~an~~ high-powered stand-up maneuvers to raise themselves from a lying-down position.
* They also create a safety risk ~~in that~~ because system malfunctions such as power failure ~~are~~ highly likely to cause the robot to ~~fall~~ topple over, at great risk to nearby people and objects.
* This paper ~~proposes~~ describes an alternative ~~design for~~ concept for two-wheeled balancing robots ~~with~~ high ~~centers of mass~~ COM of mass, which ~~includes a~~ includes a region of passive upright stability in which the robot will not fall even if it loses power. ~~We analyze the design~~ We analyze these ~~sta stability properties~~ origins and limits of

lean