The Spectral Sequence of a Double Complex

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Contents

Spectral sequences are a tool which greatly simplifies homological algebra. It is though to learn the subject for the first time. The main hurdle is the jungle of indices and symbols. I have included many pictures to make reading the document as painless as possible. The main reference for the text are Ravi Vakils expository notes *Spectral sequences: friend or foe?* [1].

1 What is a spectral sequence?

Fix an abelian category from which we take the objects and arrows to build our spectral sequence. A spectral sequence is a book. It starts on page zero and has countable infinitely many pages. On each page r we find a two dimensional array of objects $E_r^{*,*}$ and we find differentials $d_r^{*,*}$ pointing in directions which depend on the page number. The rth page of a spectral sequence without differentials is drawn in figure 1. The first upper index p in $E_r^{p,q}$ denotes the horizontal position, and

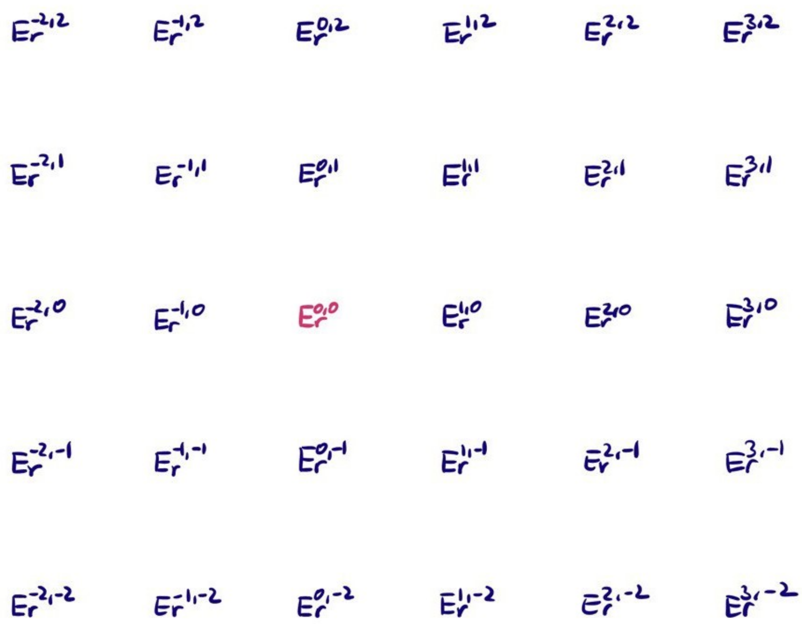

Figure 1: rth page of a spectral sequence without differentials

the index q the vertical position. Every object on every page has exactly on differential which points into it, and one differential which points out of it. The differentials will be labelled by the position of their domain. That is $d_r^{p,q}$ is a morphism on the rth page which starts in $E_r^{p,q}$. The codomain of the rth differential is $E_r^{p+r,q+1-r}$. To visualise what this means we need to look at the diagonals. For each n and for each page number r we call the objects $E_r^{p,q}$ for which $n = p + q$ the nth diagonal of that page. If you fix a position (p, q) in the grid, let the page number vary like in a flip book, and concentrate on how the arrow head of $d_r^{p,q}$ moves with increasing r, then you will see that it slides down the $(p + q + 1)$th diagonal. I have tried to draw this in figure 2. Figure 3 shows pictures of the first three pages of a spectral sequence. The zeroth and the first diagonal are marked in colour.

The differentials must square to zero. The objects on the $(r + 1)$th page are the homology of the rth page with respect to the differentials on the rth page. A spectral sequence is bounded, if there are only finitely many non-zero objects on each diagonal of page zero. All our spectral sequences will be bounded.

A subquotient of an object in an abelian category is a quotient of a subobject. In an abelian category a subquotient of a subquotient of an object A is always a subquotient of A. To see that this is true,

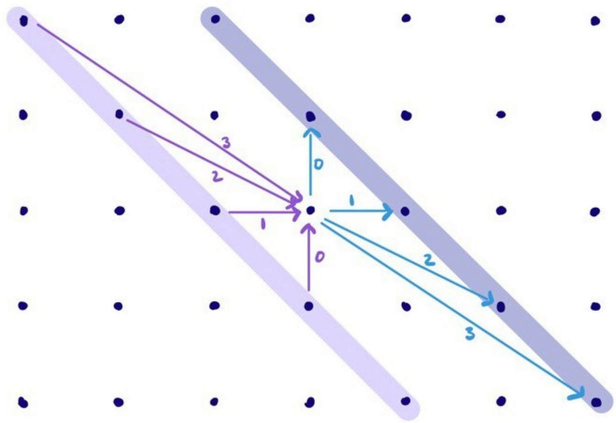

Figure 2: Incoming and outgoing differentials for fixed p, q and varying r

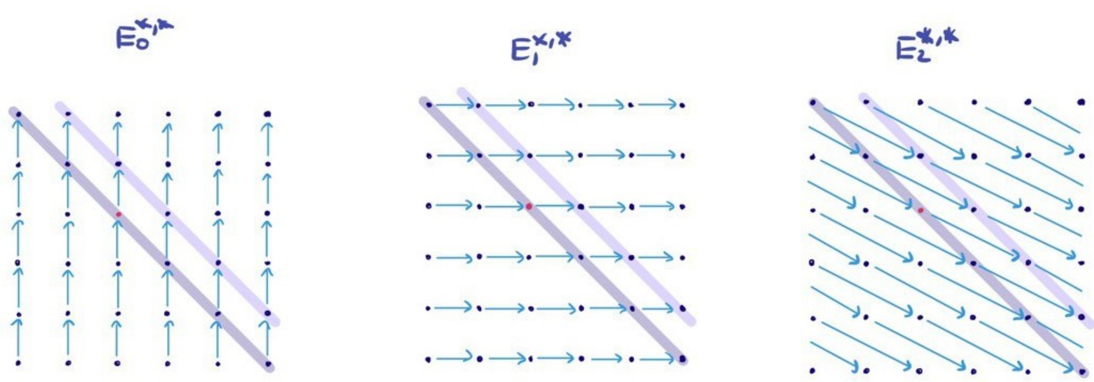

Figure 3: The first three pages of a spectral sequence.

let us draw the situation as in the diagram on the left below.

$$\begin{tikzcd}
A & \\
L \arrow[u, tail] \arrow[r, two heads, "q"] & M \\
& N \arrow[u, tail] \arrow[r, two heads] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
& A & \\
& L \arrow[u, tail] \arrow[r, two heads, "q"] & M \\
& q^{-1}N \arrow[u, tail] \arrow[r, two heads] & N \arrow[u, tail] \arrow[r, two heads] & K
\end{tikzcd}$$

We can take pullback of N along q to see that K is a quotient of the subobject $q^{-1}N \rightarrowtail A$ of A. We have used that the pullback of an epimorphism in an abelian category is always and epimorphism.

The object $E_{r+1}^{p,q}$ of a spectral sequence is the homology group of $E_r^{p,q}$ with respect to the differentials d_r and thus a subquotient of $E_r^{p,q}$. But $E_r^{p,q}$ is a subquotient of $E_{r-1}^{p,q}$, and by induction we see that each $E_r^{p,q}$ is a subquotient of $E_0^{p,q}$. We can find decreasing chain of subobjects $A_*^{p,q}$ of $E_0^{p,q}$ and an increasing chain of subobjects $B_*^{p,q}$ such that for each r it holds that $B_r^{p,q} \subset A_r^{p,q}$ and $E_r^{p,q} = A_r^{p,q}/B_r^{p,q}$. It is clear that if $E_*^{p,q}$ is zero at page r, then it is zero at all following pages. The objects $E_*^{p,q}$ (if we think of them as modules over a ring, which is without loss of generality by the Mitchell-Freyd embedding theorem) get smaller and smaller with increasing page numbers. We take less and divide by more when we increase the page number.

We say that the objects at position p, q stabilise as the page number goes to infinity if there is some page number r_0 such that the objects $E_r^{p,q}$ do not change anymore (up to isomorphism) for all pages r which come after r_0. This is the same as saying that $A_{r_0}^{p,q} = A_{r_0+1}^{p,q} = A_{r_0+2}^{p,q} = \ldots$ and $B_{r_0}^{p,q} = B_{r_0+1}^{p,q} = B_{r_0+2}^{p,q} = \ldots$. We denote the object to which $E_*^{p,q}$ converges by $E_\infty^{p,q}$. We say that a spectral sequence stabilises if each sequence $E_*^{p,q}$ is eventually stable. A bounded spectral sequence always stabilises. Let us see, why: Fix a position p, q at the diagonal $n = p + q$. The $(n+1)$th and $(n-1)$th diagonal directly above and directly below the diagonal n contain only finitely many non-zero objects by assumptions. The tail of the differential which points into the position p, q slides upwards on the $(n-1)$th diagonal with increasing r, and the head of the differential which points out of position p, q slides down the $(n+1)$th diagonal. Thus the codomain of the arrow which goes out of p, q and the domain of the arrow which points into p, q are eventually zero and stay zero forever. But from this point on to obtain $E_{r+1}^{p,q}$ we have to take the homology of

$$0 \longrightarrow E_r^{p,q} \longrightarrow 0$$

which is just $E_r^{p,q}$. Thus $E_*^{p,q}$ is eventually stable. An example for a bounded spectral sequence is a first-quadrant sequences. Here all non-zero modules lie in the first quadrant of the plane. Look at figure 4 for a sketch of the zeroth page. Another example for a bounded spectral sequence is one where the non-zero objects lie on a vertical strip. Look again at figure 4 to see a picture of such a sequence. Note that in the strip sequence in figure 4 all $E_*^{p,q}$ will already stabilise on the third page, since there are only three non-zero objects at every diagonal. We have that $E_3^{*,*}$ is already the page $E_\infty^{*,*}$ at infinity. On the other hand in the case of a first quadrant spectral sequence the speed of convergence depends on the position. $E_*^{0,0}$ is stable after two steps, but $E_*^{0,2}$ can change up until page five. There may be no page E_r on which looks like the page at infinity E_∞.

A morphism of spectral sequence $f : E \to W$ consists of a collection of morphisms $E_r^{p,q} \to W_r^{p,q}$ which commute with the differentials of the two sequences, and which are such that the maps from the $(r+1)$th page of E to the $(r+1)$th page of W are the homology of the maps between the rth pages. That means in particular that f_{r+1} is completely determined by f_r. A spectral functor is a functor from some abelian category into a category of spectral sequences. We will construct a spectral functor from the category of double complexes to the category of spectral sequences in the next section.

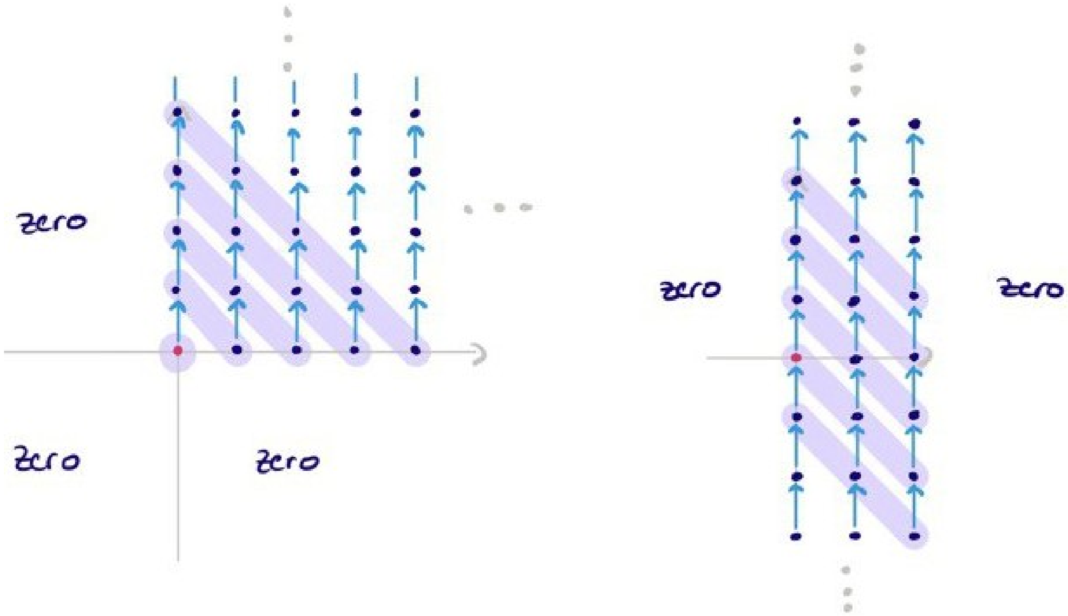

Figure 4: 0th page of a first-quadrant and a strip-spectral sequence.

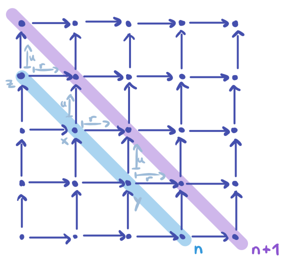

Figure 5: The differential of the total complex

2 The spectral sequence of a double complex

A double complex M is a two dimensional array of objects with upward and rightward pointing differentials. We will denote the differentials by u and r respectively - u stands for up and r stands for right. We require that the differentials square to zero, and that they anticommute. In equations:

$$u^2 = 0, \qquad r^2 = 0, \qquad ur + ru = 0\,.$$

We can associate to a double complex M its total complex TM. The nth entry of TM is the sum $(TM)^n = \bigoplus_{p+q=n} M^{p,q}$ of objects on the nth diagonal of M. Omitting inclusion maps in the notation, the nth differential of TM is induced by the maps $u + r : M^{p,q} \to (TM)^{n+1}$. The anticommutativity the differentials of M implies that the differentials of TM square to zero. We may view the elements of $(TM)^n$ as finite formal linear combinations of elements from the diagonal n of M. To compute the differential of such a formal combination, you take each of its components, apply u and r to it to push it to the next diagonal and recombine the results. Figure 5 shows a sketch of the process. The differential of the formal combination $z + x + y$ is $uz + (rz + ux) + (rx + uy) + ry$. We denote the homology of the total complex by HM.

Next let us describe how the spectral sequence E associated to the a double complex M looks like. The zeroth page of E is just M with the right pointing differentials removed. To get the first page we take the homology of the objects on the zeroth page with respect to the upward pointing differentials. The differentials on the first page of E must point to the right. There is only one reasonable choice.

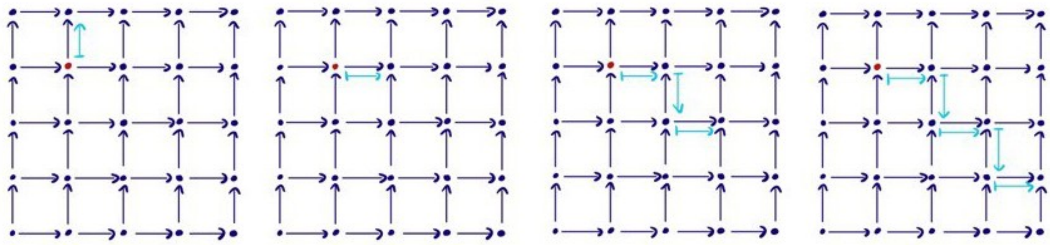

Figure 6: The spectral sequence associated to M.

We let d_1 be the maps induced by the right pointing differentials r of the double complex. The objects on the second page of E are the homology groups of the objects on the first page which we have already defined. The hard part is to define the differentials. It is not so obvious how to do that. For simplicity, let us assume we work in a category of modules. Here is another description of the differential $d_1^{p,q}$. Take an element h in $E_1^{p,q}$. It is an equivalence class of elements in $E_0^{p,q} = M^{p,q}$. In other words there is a subobject of $M^{p,q}$ which surjects onto $E_1^{p,q}$. Choose an element x on the first page such that $[x] = h$. Apply r to x. Check that rx is in the kernel of u. Push rx back to the page $E_1^{p,q}$. The equivalence class $[rx]$ is $d_1^{p,q}h$. Note that the recipe involves one choice at the beginning, but because we divide by the image of u at the end the resulting class does not depend on that choice. The procedure generalises. The objects $E_2^{p,q}$ is the homology group of a homology group. So its elements are classes of classes. It is not helpful to think like this. Rather you should remember that there is surjection from (a subobject of) the previous page onto $E_2^{p,q}$ and hence a way to lift elements from page 2 to page 1. Take some $h \in E_2^{p,q}$. Choose x on page 1 such that $[x] = h$ and then choose y on page 0 such that $[y] = x$. Now we are on page zero, so we can use r and u to walk on the page. Apply r to y to go one step to the right. Go one step down: choose some z such that $uz = y$. Apply r to z. Check that rz is in the kernel of u. It is, so you are allowed to push rz to page 1. Show that $[rz]$ is in the kernel of d_1. Define $d_2 h := [[rz]]$. It is a messy diagram chase to show that d_2 is well defined. In general, if you are on the rth page, have some class $h \in E_r^{p,q}$ and you want to define what $d_r h$ is, lift h all the way up to page 0, use r and u to walk in a zigzag to the position to which d_r points and then push the result down to page r where you came from. Picture 6 shows a cartoon of the procedure for the first few pages. It is clearly an unpleasant job to check that the differentials are well defined. There is a an increasing amount of choice for large r, and we would need to show that the result is independent of that choice. It is already a mess for d_2. A detailed proof can be found in the notes by Ravi Vakil. A similar description of the differential may be found on page 162 and following of Bott and Tu's book *Differential Forms in Algebraic Geometry* [2].

Main Theorem. Let M be a bounded complex and let E be the spectral sequence of M. Then E has a page at infinity E_∞. The objects $E_\infty^{p,q}$ at the page at infinity are related to the homology of the total complex of M. Fix a diagonal $p + q = n$ and let $E_\infty^{a,n-a}$ be the left most object in the diagonal which is non-zero. Let $E_\infty^{b,n-b}$ be the right most object on the diagonal which is non-zero. There is a filtration of the total homology of $H^n(TM)$ which looks like this:

$$E_\infty^{a,n-a} \xhookrightarrow{E_\infty^{a+1,n-a-1}} \; ? \; \xhookrightarrow{E_\infty^{a+2,n-a-2}} \; ? \qquad \dots \qquad ? \xhookrightarrow{E_\infty^{b,n-b}} H^n(TM)$$

The quotient of successive subobjects of the filtration are depicted above the arrows. Most subobjects of the filtration are depicted as ?, because we will see that it does not really matter what they are.

For example assume we start with a first quadrant double complex M. Then we get a first quadrant spectral sequence E from M. The fourth diagonal already stabilises after five steps, and at the page at infinity the 5th diagonal looks as drawn in figure 7. In this case the main theorem tells us that

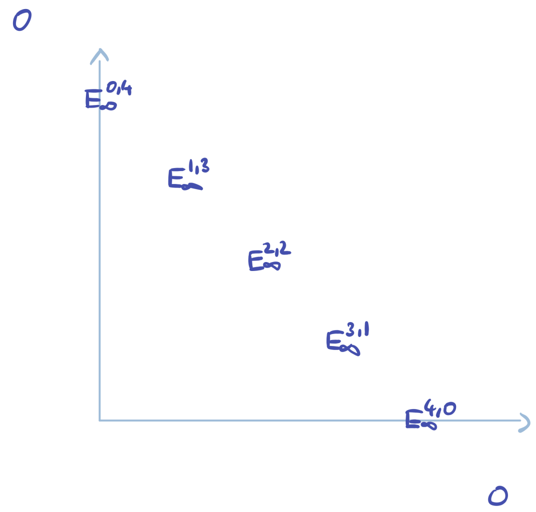

Figure 7: The fourth diagonal of the page at infinity of a first quadrant spectral sequence.

there is a filtration of $H^4(TM)$ of the following form.

$$E_\infty^{0,4} \overset{E_\infty^{1,3}}{\hookrightarrow} ? \overset{E_\infty^{2,2}}{\hookrightarrow} ? \overset{E_\infty^{3,1}}{\hookrightarrow} ? \overset{E_\infty^{4,0}}{\hookrightarrow} H^4(TM)\,.$$

In the examples which we will see today the information which we will extract from the spectral sequence will not come from the filtration! The total homology of all double complexes which we consider today will be zero. This implies that all the objects on the page E_∞ at infinity must be zero. That is all we need to know to reprove every diagram lemma of last semester. Here is an important observation: If we transpose a double complex M then we get a double complex M^t with entries $(M^t)^{p,q} = M^{q,p}$. The upward pointing differentials of M are the right pointing differentials of M^t. The total complex of M^t is equal to the total complex of M, so in particular the total homology of a double complex does not change under transposition. Let's see why this is important by doing some examples.

3 Applications

Instead of proving new theorems, we like show how spectral sequences can be used to prove theorems you already know. Nearly always when you see a statement in a homological algebra books which involves two dimensional diagrams there is a simple way to prove it with a spectral sequence argument.

3.1 The five lemma

We can prove the five lemma with a spectral sequence argument. The statement is that if in the commutative diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & E' \arrow[d] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

the rows are exact and the outer four vertical maps are isomorphisms, then so is the map $C' \to C$. We have to draw the diagram upside down and replace some of the arrows by their negative to get a double complex. Imagine that A' sits in the origin, and that outside of the diagram everything is zero.

$$\begin{tikzcd}
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E \\
A' \arrow[r] \arrow[u] & B' \arrow[r] \arrow[u, "-"] & C' \arrow[r] \arrow[u] & D' \arrow[r] \arrow[u, "-"] & E' \arrow[u]
\end{tikzcd}$$

Write M for the double complex above. We like to compute the spectral sequence of M. But first we will compute the spectral sequence of the transpose M^t of M. The diagram below shows the transpose M^t and the first two pages of its spectral sequence.

$$\begin{tikzcd}
E' \arrow[r] & E & & E' & E & & ? \arrow[r] & ? \\
D' \arrow[r] \arrow[u] & D \arrow[u] & & D' \arrow[u] & D \arrow[u] & & 0 \arrow[r] & 0 \\
C' \arrow[r] \arrow[u] & C \arrow[u] & & C' \arrow[u] & C \arrow[u] & & 0 \arrow[r] & 0 \\
B' \arrow[r] \arrow[u] & B \arrow[u] & & B' \arrow[u] & B \arrow[u] & & 0 \arrow[r] & 0 \\
A' \arrow[r] \arrow[u] & A \arrow[u] & & A' \arrow[u] & A \arrow[u] & & ? \arrow[r] & ?
\end{tikzcd}$$

Note that most of the objects are already zero at page one. This is because we assumed that the rows of the diagram we started with are exact. We can now use the main theorem of the last section to conclude that the second and third total homology group of M^t has a filtration which consists all of zeros. In consequence $H^2(TM) = H^3(TM) = 0$. This is what we need when we do the same computation but with M instead on M^t. Let us denote the arrow $C' \to C$ by γ. The zeroth page of the spectral sequence of M looks like this.

$$\begin{tikzcd}
A & B & C & D & E \\
A' \arrow[u] & B' \arrow[u] & C' \arrow[u, "\gamma"] & D' \arrow[u] & E' \arrow[u]
\end{tikzcd}$$

To get the first page, we take homology. Since all maps except γ are already known to be isomorphisms, we see that the first page must look like this.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \operatorname{coker}\gamma \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \ker\gamma \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

But note that the sequence is already stable. We have reached the page at infinity. The main theorem now tells us that $H^2(TM) = \ker\gamma$ and $H^3(TM) = \operatorname{coker}\gamma$. Thus the cokernel and kernel of γ are zero, and we have shown that γ is an isomorphism. The proof makes it easy to see how we can weaken the assumptions. We only need that $A' \to A$ is epi and that $E' \to E$ is monic.

3.2 The nine lemma

Assume $0 \to A \to B \to C \to 0$ is a short exact sequence of chain complexes. If two of the three complexes A, B and C are exact, then so is the remaining third. To see this is true, let us for example assume that A and B are exact. The other cases are similar. We replace the differentials of B by their negative and draw all the data into one big diagram which is a double complex.

$$
\begin{tikzcd}
{} & {} & {} \\
A^3 \arrow[u] \arrow[r] & B^3 \arrow[u] \arrow[r] & C^3 \arrow[u] \\
A^2 \arrow[u] \arrow[r] & B^2 \arrow[u] \arrow[r] & C^2 \arrow[u] \\
A^1 \arrow[u] \arrow[r] & B^1 \arrow[u] \arrow[r] & C^1 \arrow[u] \\
A^0 \arrow[u] \arrow[r] & B^0 \arrow[u] \arrow[r] & C^0 \arrow[u] \\
A^{-1} \arrow[u] \arrow[r] & B^{-1} \arrow[u] \arrow[r] & C^{-1} \arrow[u] \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}
$$

Denote this double complex by M. We imagine that A^0 sits in the origin. By computing the spectral sequence of the transpose M^t of M as in the first example we immediately see that the total homology of M is zero in every degree! So let us now compute the spectral sequence of M. The diagram below shows the 0th, first and second page with some of the differentials.

$$
\begin{tikzcd}
{} & {} & {} \\
A^2 \arrow[u] & B^2 \arrow[u] & C^2 \arrow[u] \\
A^1 \arrow[u] & B^1 \arrow[u] & C^1 \arrow[u] \\
A^0 \arrow[u] & B^0 \arrow[u] & C^0 \arrow[u] \\
A^{-1} \arrow[u] & B^{-1} \arrow[u] & C^{-1} \arrow[u] \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & ? \\
0 \arrow[r] & 0 \arrow[r] & ? \\
0 \arrow[r] & 0 \arrow[r] & ? \\
0 \arrow[r] & 0 \arrow[r] & ?
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[drr] & & \\
0 \arrow[drr] & 0 & ? \\
0 \arrow[drr] & 0 & ? \\
0 \arrow[drr] & 0 & ? \\
0 & 0 & ?
\end{tikzcd}
$$

We can see that the sequence is stable already on page E_1. The page 1 is the page at infinity. But because the total homology is zero, it must be the case that all the unknown objects ? on the page 1 are zero (because they are part of a filtration of the total homology groups by the main theorem). But the ? groups are precisely the homology groups of the chain complex C, hence C is exact as claimed.

3.3 A SES of chain complexes induces a LES in homology

This is the first example where we will actually use some of the differentials of the spectral sequence. We like to show that a short exact sequence of chain complexes induces a long exact sequence in homology. So let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes. We write f

for the map $A \to B$ and g for the map $B \to C$. We replace the differentials of B by their negative to obtain a double complex as in the previous example, which we denote by M. Computing the spectral sequence of the transpose M^t shows that the total homology of M vanishes. This is where we use the assumption that $0 \to A \to B \to C \to 0$ is exact. Now let us compute the spectral sequence of M. Page zero and page one look like this:

$$\begin{tikzcd}
{} & {} & {} \\
A^2 \arrow[u] & B^2 \arrow[u] & C^2 \arrow[u] \\
A^1 \arrow[u] & B^1 \arrow[u] & C^1 \arrow[u] \\
A^0 \arrow[u] & B^0 \arrow[u] & C^0 \arrow[u] \\
A^{-1} \arrow[u] & B^{-1} \arrow[u] & C^{-1} \arrow[u] \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
H^2A \arrow[r, "H^2f"] & H^2B \arrow[r, "H^2g"] & H^2C \\
H^1A \arrow[r, "H^1f"] & H^1B \arrow[r, "H^1g"] & H^1C \\
H^0A \arrow[r, "H^0f"] & H^0B \arrow[r, "H^0g"] & H^0C \\
H^{-1}A \arrow[r, "H^{-1}f"] & H^{-1}B \arrow[r, "H^{-1}g"] & H^{-1}C
\end{tikzcd}$$

The page E_1 is basically the long exact sequence in homology, but without the connecting morphisms. How does the third page look like? We have to take homology. I have only drawn the differentials which we are interested in.

$$\begin{tikzcd}
\ker(H^2f) \arrow[drr] & ? & \operatorname{coker}(H^2g) \\
\ker(H^1f) \arrow[drr] & ? & \operatorname{coker}(H^1g) \\
\ker(H^0f) \arrow[drr] & ? & \operatorname{coker}(H^0g) \\
\ker(H^{-1}f) & ? & \operatorname{coker}(H^{-1}g)
\end{tikzcd}$$

First note that the objects marked with the symbol ? are already stable. Since the total homology is zero, they must vanish. This shows the exactness of the long exact sequence at the positions H^kB. Next note that the other objects will become stable on page 3. Hence everything which is not yet zero on the page E_2 above must be annihilated in the process of taking homology. In particular this shows that all the differentials $d_2^{0,q} : \ker(H^qf) \to \operatorname{coker}(H^{q-1}g)$ from the second page which we have drawn must be isomorphisms! We get connecting morphisms $H^qC \to H^{q+1}A$ by composing $H^qC \twoheadrightarrow \operatorname{coker}(H^qg) \cong \ker(H^{q+1}f) \rightarrowtail H^{q+1}A$ and it is clear that the resulting long sequence is exact. If one sits down and carefully works through the definition of the differentials on the second page of a spectral sequence as described in the second chapter, then one sees that the connecting morphism we have here is up to a sign exactly the connecting morphism one defines in the usual proof of the theorem. As an exercise, you can now prove the snake lemma by a spectral sequence argument.

3.4 Tor_R is balanced

Remember that there are two ways to derive the bifunctor $\otimes_R : \mathtt{Mod}_R \times {}_R\mathtt{Mod} \to \mathtt{Ab}$. We can take projective resolutions in the first variable, and we can take projective resolutions in the second variable. The two functors we obtain are isomorphic. Let us first recall the basic definitions and facts from homological algebra. Fix two objects A and B from $\texttt{Mod}_R$ and ${}_R\texttt{Mod}$ respectively and let $P \to A$ and $Q \to B$ be projective resolutions. We can draw a big diagram which looks like this:

$$
\begin{tikzcd}
{} \arrow[r] & P_2 \otimes_R B \arrow[r] & P_1 \otimes_R B \arrow[r] & P_0 \otimes_R B \arrow[r] & A \otimes_R B \\
{} \arrow[r] & P_2 \otimes_R Q_0 \arrow[r] \arrow[u] & P_1 \otimes_R Q_0 \arrow[r] \arrow[u] & P_0 \otimes_R Q_0 \arrow[r] \arrow[u] & A \otimes_R Q_0 \arrow[u] \\
{} \arrow[r] & P_2 \otimes_R Q_1 \arrow[r] \arrow[u] & P_1 \otimes_R Q_1 \arrow[r] \arrow[u] & P_0 \otimes_R Q_1 \arrow[r] \arrow[u] & A \otimes_R Q_1 \arrow[u] \\
{} \arrow[r] & P_2 \otimes_R Q_2 \arrow[r] \arrow[u] & P_1 \otimes_R Q_2 \arrow[r] \arrow[u] & P_0 \otimes_R Q_2 \arrow[r] \arrow[u] & A \otimes_R Q_2 \arrow[u] \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}
$$

Let's say that $P_0 \otimes_R Q_0$ sits at the origin. Also we need to put minus signs in every odd numbered column, so that the diagram really is a double complex with anticommuting differentials. We denote it by M. Each P_i is projective and in particular flat. This means that all the columns, except the right most, are exact. Let us look at the first view pages of the spectral sequence of M. The diagram below shows the pages E_0 and E_1.

$$
\begin{tikzcd}
P_1 \otimes_R B & P_0 \otimes_R B & A \otimes_R B \\
P_1 \otimes_R Q_0 \arrow[u] & P_0 \otimes_R Q_0 \arrow[u] & A \otimes_R Q_0 \arrow[u] \\
P_1 \otimes_R Q_1 \arrow[u] & P_0 \otimes_R Q_1 \arrow[u] & A \otimes_R Q_1 \arrow[u] \\
P_1 \otimes_R Q_2 \arrow[u] & P_0 \otimes_R Q_2 \arrow[u] & A \otimes_R Q_2 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] & H^0(A \otimes_R Q) \\
0 \arrow[r] & 0 \arrow[r] & H^1(A \otimes_R Q) \\
0 \arrow[r] & 0 \arrow[r] & H^2(A \otimes_R Q)
\end{tikzcd}
$$

We see that the sequence is already stable at page E_1. In particular we find that $H^n(TM) \cong H^n(A \otimes_R Q)$. Doing the same computation we the transpose M^t of M shows us that $H^n(TM^t) \cong H^n(P \otimes_R B)$ and in consequence we find that $H^n(A \otimes_R Q) \cong H^n(P \otimes_R B)$. We have just shown that Tor_R is balanced. This way of showing it is sufficient if you only want that there is an isomorphism, but it does not help if you want to know that the isomorphism is a natural transformation and that it commutes with the connecting morphisms of the long exact sequences. A better way of showing that Tor_R is balanced can be found in the book *A course in homological algebra* by Stammbach and Hilton [3]. As an exercise you can now show that $\mathrm{Ext}_{\mathcal{A}}$ is balanced whenever $\mathcal{A}$ is an abelian category with enough projectives and injectives.

References

[1] Ravi Vakil. Spectral sequences: Friend or foe?, 2008.

[2] Raoul Bott, Loring W Tu, et al. *Differential forms in algebraic topology*, volume 82. Springer, 1982.

[3] Peter J Hilton and Urs Stammbach. *A course in homological algebra*, volume 4. Springer Science & Business Media, 2012.