# Derived Functors in the Context of Model Categories

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# 1 Revision: Localisation at the Class of Weak Equivalences

This section closely follows the exposition by Emily Riehl in [1] and [2]. We have learned in a previous talk: The homotopy category of a model category is the localisation of that category at its class of weak equivalences. This does characterise the homotopy category up to unique isomorphisms of categories. There is a more general method to construct the localisation of a category, which works for categories which do not admit a model structure.

**Definition 1.** (*Gabriel-Zisman localisation*) Let $\mathsf{K}$ be a category and $\mathcal{W}$ be a set of its morphisms. The Gabriel-Zisman category of fractions $\mathsf{K}[\mathcal{W}^{-1}]$ is a quotient of the free category on the directed graph obtained by adding backwards pointing copies $w^{-1}$ of the morphisms $w$ in $\mathcal{W}$ to the underlying graph of $\mathsf{K}$ modulo the following relations:

- Adjacent pairs of arrows from $\mathsf{K}$ may be composed.
- Adjacent pairs of arrows of the form $ww^{-1}$ and $w^{-1}w$ may be removed.
- All other relations generated by these.

The category of fractions comes with a functor $\iota : \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ which sends arrows in $\mathsf{K}$ to unary zig-zags pointing forward. Note that the relations above are precisely those relations which are needed so that $\iota$ is a functor which inverts $\mathcal{W}$.[1]

The localisation functor $\iota$ is the functor which inverts the morphisms of $\mathcal{W}$ in the most efficient way. That is whenever there is another functor $F : \mathsf{K} \to \mathsf{C}$ which sends morphisms in $\mathcal{W}$ to isomorphisms, then $F$ factors uniquely through $\iota$. There is a trick to strengthens the universal property of the localisation to one which involves natural transformations. The arrow category $\mathsf{Arr}(\mathsf{C})$ comes with two forgetful functors $\mathsf{Arr}(\mathsf{C}) \to \mathsf{C}$ called domain and codomain. Note that a natural transformation $\phi : F \Rightarrow G$ is the same as a functor $\phi : \mathsf{K} \to \mathsf{Arr}(\mathsf{C})$ into the arrow category such that postcomposing $\phi$ with domain yields $F$ and postcomposing $\phi$ with codomain yields $G$. Applying the universal property of the localisation to the arrow categories shows us that it satisfies the following universal property.

**Proposition 1.** Restriction along the localisation functor $\iota : \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ induces an isomorphism between the functor category $\mathsf{Fun}(\mathsf{K}[\mathcal{W}^{-1}], \mathsf{M})$ and the full subcategory of $\mathsf{Fun}(\mathsf{K}, \mathsf{M})$ consisting of those functors which invert $\mathcal{W}$. This can be expressed more concisely with help of the following diagram:

$$\begin{tikzcd} \mathsf{Fun}(\mathsf{K}[\mathcal{W}^{-1}], \mathsf{M}) \arrow[rr, "\iota^*"] \arrow[dr, "\cong"'] & & \mathsf{Fun}(\mathsf{K}, \mathsf{M}) \\ & \underset{\mathcal{W} \mapsto \cong}{\mathsf{Fun}}(\mathsf{K}, \mathsf{M}) \arrow[ur, hook] & \end{tikzcd}$$

[1] If you are careful you might want to know how this yields a well-defined category. Assume you have a category $\mathsf{C}$ and some relations $R$ which you would like to enforce by passing to a quotient. You need to give an equivalence relation on each hom-set such that composition in $\mathsf{C}$ induces a well-defined composition function on the equivalence classes. There is at least one such set of equivalence relations: set everything equal to the identity morphism. If you have any collection of such equivalence relations, then their hom-set wise intersection will again be an equivalence relation through which composition can factor. Hence it makes sense to speak of the smallest set of relations which contains $R$ and allows you to form a quotient category.

Remember how the homotopy category $\mathsf{HoK}$ of a model category $\mathsf{K}$ was defined. For each object $x$ we choose a cofibrant replacement $Qx$ together with a fibrant weak equivalence $q_x : Qx \to x$ by factoring the map from the initial object into $x$. Similarly we choose a fibrant replacement $r_x : x \to Rx$ for every object $x$. By definition $\mathsf{HoK}(x, y) = \mathsf{K}(RQx, RQy)/_{\sim}$, where $\sim$ denotes modulo homotopy. The localisation functor $\gamma : \mathsf{K} \to \mathsf{HoK}$ is the identity on objects. It acts on a morphism by first lifting it to a morphism between the fibrant replacement and then lifting the resulting map to a map between the fibrant-cofibrant replacements of the domain and codomain respectively. Such lifts always exist by the lifting axioms in a model category, and the resulting map does not depend on the choice of lifts up to homotopy because of the model-category-yoga we have done so far. Also, if two maps between cofibrant-fibrant objects are homotopic, then they are homotopic with respect to every cylinder-object and every path-object.

## 2 Derived Functors

The main result is Quillen adjunction theorem which allows to lift adjunctions between model categories to adjunctions between their homotopy categories. Under certain conditions the lifted adjunctions become adjoint equivalences, hence allow to show that the homotopy theories of different homotopical categories agree.

Not all functors preserve weak equivalences. The derived functors of a functor $F : \mathsf{K} \to \mathsf{N}$ are in some sense the best approximation to $F$ by a homotopical functors. The localisation of a model category does invert only the weak equivalences. Hence in the case of model categories $F$ is homotopical if and only if the composite of $F$ with the localisation of $\mathsf{N}$ factors through the homotopy category of $\mathsf{K}$.

$$\begin{tikzcd} \mathsf{K} \arrow[r, "F"] \arrow[d] & \mathsf{N} \arrow[d] \\ \mathsf{HoK} \arrow[r, dashed, "\exists"] & \mathsf{HoN} \end{tikzcd}$$

If $F$ is not a homotopical functor, then we can not extend its composition with the localisation as pictured in the diagram. The next best thing are left and right Kan-extensions. Here is the definition of a derived functor from Dwyer-Spalinski.

**Definition 2.** Assume $\mathsf{K}$ is a model category, $\mathsf{N}$ is a category and $F : \mathsf{K} \to \mathsf{N}$ is a functor. Then the left derived functor of $F$ is a representation of $\mathrm{Nat}(\gamma^*-, F)$. Here $\gamma^*$ denotes restricting along the localisation $\gamma : \mathsf{K} \to \mathsf{HoK}$. In more detail, a left derived functor for $F$ consists of both a functor $\mathbb{L}F : \mathsf{HoK} \to \mathsf{N}$ and an isomorphism

$$\mathrm{Nat}(H, \mathbb{L}F) \cong \mathrm{Nat}(H \circ \gamma, F)\,.$$

natural in $H$. In category theory this is called a right Kan-extension of $F$ along $\gamma$.

The above isomorphism is an isomorphism between hom-functors of functor categories. Yoneda's lemma states that the data of a transformation $\mathrm{Nat}(-, \mathbb{L}F) \Rightarrow \mathrm{Nat}(- \circ \gamma, F)$ is the same as an element $t$ of $\mathrm{Nat}(\mathbb{L}F \circ \gamma, F)$. The natural transformation of hom-sets corresponding to $t$ is an isomorphism if and only if $t$ is universal in the following sense:

Given any functor $H : \mathsf{HoK} \to \mathsf{N}$ together with a natural transformation $s : H\gamma \Rightarrow F$ there is a unique transformation $v : H \Rightarrow \mathbb{L}F$ such that

$$\begin{tikzcd} G\gamma \arrow[d, Rightarrow, "v\gamma"'] \arrow[dr, Rightarrow, "s"] & \\ (\mathbb{L}F)\gamma \arrow[r, Rightarrow, "t"'] & F \end{tikzcd}$$

commutes. This is the same as saying that $t$ is terminal in the category of elements of $\mathrm{Nat}(- \circ \gamma, F)$. The first two chapters of Emily Riehl's book *Categories in Context* are a really good reference if you feel uncomfortable with these kinds of manipulation. Derived functors do not exist in general, but sometimes they do. Here is an criteria. A good alternative reference for the proof is the paper by Dwyer-Spalinski [3] we are reading.

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**Proposition 2.** Assume $\mathsf{K}$ is a model category and $F : \mathsf{K} \to \mathsf{N}$ is a functor which sends weak equivalences between cofibrant objects to isomorphisms. Then $F$ has a left derived functor. The proof describes its construction.

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PROOF Remember that we have chosen a cofibrant replacement $q_x : Qx \to x$ for each object $x$. The objects $Qx$ are cofibrant and the maps $q_x$ are fibrant equivalences. A functor out of the homotopy category is the same as a functor out of $\mathsf{K}$ which inverts the weak equivalences. We define such a functor $F'$ by the following recipe: Given a morphism $f : x \to y$ in $\mathsf{K}$, chose a lift $\tilde{f}$ of $f$ as in

$$\begin{tikzcd} Qx \arrow[r, "\tilde{f}"] \arrow[d, "q_x"] & Qy \arrow[d, "q_y"] \\ x \arrow[r, "f"] & y \end{tikzcd} \tag{2.1}$$

The image of $f$ under our new functor $F'$ is $F'f = F\tilde{f}$. We will suggestively write $FQ$ instead of $F'$ even though it might not be possible to lift in a functorial way with our choice of cofibrant replacements.[2] We will check below that this yields a well-defined functor which inverts weak equivalences. Hence it factors through the homotopy category. The resulting functor $\mathbb{L}F$ is our candidate for a right Kan-extension of $F$. This diagram summarises its definition:

$$\begin{tikzcd} \mathsf{K} \arrow[r, "FQ"] \arrow[d, "\gamma"'] & \mathsf{C} \\ \mathsf{HoK} \arrow[ur, "\mathbb{L}F"'] & \end{tikzcd}$$

The missing piece of data is a natural transformation $t : \mathbb{L}F \circ \gamma \Rightarrow F$. Since $\mathbb{L}F \circ \gamma$ is $FQ$ we can set $t_x = Fq_x$. Now to the details. We need to check that $FQ$ is a well-defined functor which inverts weak equivalences, and that the induced functor $\mathbb{L}F$ together with the transformation $t$ is a right Kan-extension of $F$.

Assume both $a$ and $b$ lift $f$ to a map between cofibrant replacements as in diagram 2.1. Then $a$ and $b$ are right homotopic. Since $Qy$ is cofibrant we can find a homotopy $H$ from

[2]Most model categories, including all of those we have seen so far, admit functorial factorisation. In this case we have a cofibrant replacement functor $Q$ and we can literally write $FQ$.

$a$ to $b$ with respect to a very good path object path$(Qy)$. Its structure maps are named in the following diagram.

$$\begin{tikzcd} Qy \arrow[r, tail, "\sim", "w"'] \arrow[rr, bend left, "\text{diagonal}"] & \mathrm{path}(Qy) \arrow[r, "(p_0,p_1)"'] & Qy \times Qy \end{tikzcd}$$

The structure map $w$ is a cofibrant weak equivalences. It follows that the path object is itself cofibrant. By assumption $F$ sends weak equivalences between cofibrant objects to isomorphisms. Hence $Fw$ is an isomorphism. By definition of an cylinder object it holds that

$$Fp_0 \circ Fw = F(\mathrm{id}) = Fp_1 \circ Fw\,.$$

Now since $Fw$ is an isomorphism we can cancel it and find that $Fp_0 = Fp_1$. The calculation

$$Fa = Fp_0 \circ FH = Fp_1 \circ FH = Fb$$

shows that $Fa = Fb$ as claimed. $FQ$ is a well-defined function between morphism sets. It is not hard to see that the assignment is functorial and preserves identities. To get a lift of $f \circ g$ you can compose a lift of $f$ with a lift of $g$, and the identity of $Qx$ lifts the identity of $x$. Next we check that $t = Fq$ is a natural transformation. Given a morphism $f : x \to y$ in $\mathsf{K}$ we need to show that the diagram

$$\begin{tikzcd} FQx \arrow[r, "F\tilde{f}"] \arrow[d, "Fq_x"'] & FQy \arrow[d, "Fq_y"] \\ Fx \arrow[r, "Ff"] & Fy \end{tikzcd}$$

commutes where $\tilde{f}$ is a lift of $f$. But this follows directly when we apply $F$ to the diagram which defines $\tilde{f}$. (Look at diagram 2.1). Next we check that $FQ$ factors through the homotopy category, that is inverts weak equivalences. If $w$ is a weak equivalence, then its lift between cofibrant replacements is also a weak equivalence because of 2-out-of-3. $F$ sends weak equivalences between cofibrant objects to isomorphisms by assumption, hence $FQw$ is an isomorphism. $FQ$ factors and we get a functor $\mathbb{L}F$. The final step is to show that $t : \mathbb{L}F \circ \gamma \Rightarrow F$ satisfies the universal property of a right Kan-extension. Let $H$ be any other functor $G : \mathsf{HoK} \to \mathsf{N}$ and $s : H\gamma \Rightarrow F$ a natural transformation. We need to show that there is a unique transformation $v : H \Rightarrow \mathbb{L}F$ such that

$$\begin{tikzcd} H\gamma \arrow[d, Rightarrow, "v\gamma"'] \arrow[dr, Rightarrow, bend left, "s"] & \\ (\mathbb{L}F)\gamma \arrow[r, Rightarrow, "t"'] & F \end{tikzcd}$$

commutes. Here we can use 2-categorical version of the universal property of $\gamma$ to simplify. If we can find a unique transformation $v' : H\gamma \Rightarrow (\mathbb{L}F)\gamma$ which fits in the place where $v\gamma$ sits, then we know that $v'$ is the restriction of one and only transformation $v$ along $\gamma$ and we are done. So we are reduced to show that

$$\begin{tikzcd} H\gamma \arrow[d, Rightarrow, "\exists! v"'] \arrow[dr, Rightarrow, bend left, "\forall s"] & \\ FQ \arrow[r, Rightarrow, "t"'] & F \end{tikzcd}$$

and can avoid dealing with the inner working of $\mathsf{HoK}$. Now assume that $v$ exists and fix an object $x$. By naturality the diagram

$$\begin{tikzcd} H\gamma Qx \arrow[r, "v_{Qx}"] \arrow[d, "H\gamma q_x"] & FQQx \arrow[r, "t_{Qx}"] \arrow[d, "FQq_x"] & FQx \arrow[d, "Fq_x"] \\ H\gamma x \arrow[r, "v_x"] & FQx \arrow[r, "t_x"] & Fx \end{tikzcd}$$

must commute. Now make a view observations: $t_{Qx} = Fq_{Qx}$ is an isomorphism because $q_{Qx}$ is a weak equivalence between cofibrant objects. $H\gamma q_x$ is an isomorphism because $\gamma q_x$ is an isomorphism. The composition of the top row is $s_{Qx}$. Using those we compute that if $v_x$ exists, it must be given by the formula

$$\begin{aligned} v_x &= FQq_x \circ v_{Qx} \circ (H\gamma q_x)^{-1} = FQq_x \circ t_{Qx}^{-1} \circ t_{Qx} \circ v_{Qx} \circ (H\gamma q_x)^{-1} \\ &= FQq_x \circ t_{Qx}^{-1} \circ s_{Qx} \circ (H\gamma q_x)^{-1} . \end{aligned}$$

Using that $q_{Qx}$ is a lift of $q_x$ we note that $FQq_x \circ t_{Qx}^{-1} = id$. We can simplify the equation. If $v$ exists, then it must be given by the following forumla:

$$v_x = s_{Qx} \circ (H\gamma q_x)^{-1} .$$

To finish the proof we can take this formula as a definition of $v_x$ and show that $v$ is a natural transformation which satisfies $t \circ v = s$. To see that $v$ is natural simply note that

$$\begin{tikzcd} H\gamma x \arrow[r, "H\gamma f"] \arrow[d, "(H\gamma q_x)^{-1}"'] & H\gamma y \arrow[d, "(H\gamma q_y)^{-1}"] \\ H\gamma Qx \arrow[r, "H\gamma Qf"] \arrow[d, "s_{Qx}"] & H\gamma Qy \arrow[d, "s_{Qy}"] \\ FQx \arrow[r, "FQf"] & FQy \end{tikzcd}$$

commutes if $f$ is any morphism and $Qf$ any lift of $f$. The equation $t \circ v = s$ is satisfied by construction. □

This was a really long proof. I would like to recall the main points, so that we don't get lost in details. If $F$ inverts weak equivalences between cofibrant objects, then a right Kan-extension $\mathbb{L}F, t$ of $F$ along the localisation exists. $\mathbb{L}F$ is defined by factoring $FQ$ through the homotopy category. The universal transformation is $t = Fq$. Given $s : H\gamma \Rightarrow F$ the unique $v : H \Rightarrow \mathbb{L}F$ such that $t \circ v\gamma = s$ is given by the formula

$$v_{\gamma x} = s_{Qx} \circ (H\gamma q_x)^{-1} .$$

We will use that formula later. It is an advantage that we have an explicit description of the derived functor. We get the following corollary for free.

**Corollary 1.** If the total derived functor of $F$ is constructed as in the last proposition, then $\mathbb{L}F, t$ is an absolute Kan-extension of $F$.

Proof A right Kan-extension $\mathbb{L}F, t$ is absolute iff whenever we have a third functor $G : \mathsf{C} \to \mathsf{D}$, then $G(\mathbb{L}F), Gt$ is a right Kan-extension of $GF$ along the localisation. In other words a Kan-extension is absolute iff it is preserved by postcomposing with any functor. Say $\mathbb{L}F, t$ is constructed as above and we are given any $G$. The $GF$ sends weak equivalences between cofibrant objects to isomorphisms, hence by the proposition has a right Kan-extension $\mathbb{L}(GF), GFq$. Here $\mathbb{L}(GF)$ is obtained by factoring $GFQ$ through the homotopy category. Both $\mathbb{L}(GF)$ and $G(\mathbb{L}F)$ are factorisation of $GFQ$. Hence $\mathbb{L}(GF) = G(\mathbb{L}F)$ by the universal property of localisation. Also $GFq = Gt$. In other words $G(\mathbb{L}F), Gt$ is a right Kan-extension of $GF$ along the localisation as claimed. $\square$

Next imagine a situation in which also $\mathsf{N}$ is a model category and $F$ sends weak equivalences between cofibrant objects only to weak equivalences and not to isomorphisms. Then we can not apply the above proposition to $F$, but we can apply it to the composition of $F$ with the localisation of $\mathsf{N}$. This motivates the definition of total derived functors.

**Definition 3.** Let $F : \mathsf{K} \to \mathsf{N}$ be a functor between model categories. The total left derived functor $\mathbf{L}F$ of $F$ is the right Kan-extension of $\delta F$ along $\gamma$. Here $\delta$ denotes the localisation of $\mathsf{N}$ and $\gamma$ is the localisation of $\mathsf{K}$ as before.

The main aim of this text is to proof the Quillen adjunction theorem. Given an adjunction $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ between model categories we like to left derive $F$ and right derive $G$ to get an adjunction $\mathbf{L}F : \mathsf{HoM} \rightleftarrows \mathsf{HoN} : \mathbf{R}G$ between the homotopy categories. Total derived functors do not always exist. Functors to which we can apply our proposition have special names.

**Definition 4.** (*Quillen functors*) A functor between model categories is left Quillen if it preserves cofibrations, acyclic cofibrations, and cofibrant objects. It is right Quillen if it preserves fibrations, acyclic fibrations, and fibrant objects.

It is not directly clear that proposition 2 guaranties us the existence of the total derived functor $\mathbb{L}F$ of a left Quillen functor $F$. Remember, in order to apply proposition 2 we need to know that $F$ sends weak equivalences between cofibrant objects to weak equivalences. This is the content of Ken Brown's lemma.

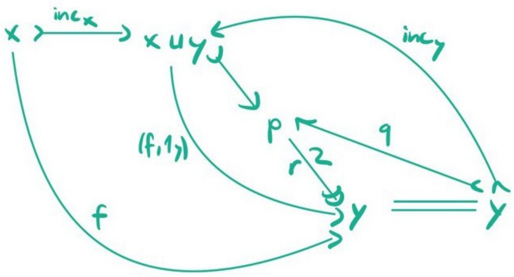
**Lemma 1.** (*Ken Brown's lemma*)

(i) Every morphism $f$ between cofibrant objects in a model category may be factored as a cofibration followed by trivial fibration which has a trivial cofibration as a right invers.

$$\begin{tikzcd}
& P \arrow[dr, two heads, "r"', "\sim"] & \\
X \arrow[ur, tail, "j"] \arrow[r, "f"'] & Y \arrow[r, equal] & Y \arrow[ul, tail, bend right, "q"', "\sim"]
\end{tikzcd}$$

(ii) Let $F : \mathsf{M} \to \mathsf{N}$ be a functor between model categories. If $F$ carries cofibrant weak equivalences to weak equivalences, then $F$ carries all weak equivalences between cofibrant objects to weak equivalences.

Proof For a proof of the first part look at the following diagram.




$p$ is obtained by factoring the cograph $(f, 1_y)$ as a cofibration followed by a fibrant weak equivalence. $q$ is a weak equivalence by 2-out-of-3. The inclusions into the coproduct are cofibrant because $x$ and $y$ are cofibrant objects. For the second part assume $f$ is a weak equivalence between cofibrant objects. Factor $f$ as in part (i). $j$ is a weak equivalence by 2-out-of-3 and $p$ is cofibrant. Apply the functor $F$ to the factorisation. $Fq$ and $Fj$ are weak equivalences by assumption, since $q$ and $j$ are cofibrant weak equivalences between cofibrant objects. $F1_y$ is a weak equivalence. Hence $Fr$ is a weak equivalence and in consequence $Ff$ is a weak equivalence by 2-out-of-3. $\square$

Ken Brown's lemma and its dual tell us that left Quillen functors have total left derived functors and right Quillen functors have total right derived functors. Next we define what a Quillen adjunction is.

**Definition 5.** (*Quillen adjunction*) Assume $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ is an adjunction between model categories. The following conditions are equivalent.

(i) $F$ is left Quillen.

(ii) $G$ is right Quillen.

(iii) $F$ preserves cofibrations and $G$ preserves fibrations.

(iv) $F$ preserves trivial cofibrations and $G$ preserves trivial fibrations.

If one and hence all of the condition hold, then the adjunction is a Quillen adjunction.

Proof Assume $F$ is left Quillen. We like to show that then $G$ is right Quillen. Let $g : n \to n'$ be a fibration. $Gg$ is a fibration in $\mathsf{M}$ if and only if $Gg$ has the right lifting property with respect to all cofibrant weak equivalences. But due to adjunction the lifting problem on the left of

$$\begin{tikzcd} m \arrow[r, "a"] \arrow[d, "f"'] & Gn \arrow[d, "Gg"] \\ m' \arrow[r, "b"'] \arrow[ur, dashed, "\exists"] & Gn' \end{tikzcd} \qquad \begin{tikzcd} Fm \arrow[r, "\overline{a}"] \arrow[d, "Ff"'] & n \arrow[d, "g"] \\ Fm' \arrow[r, "\overline{b}"'] \arrow[ur, dashed, "\exists"] & n' \end{tikzcd}$$

is equivalent to the lifting problem at the right. The lifting problem on the right has a solution since $F$ preserves cofibrant weak equivalences. In a similar way one shows that $G$ preserves fibrant weak equivalences. They are characterised by right lifting against all cofibrations. To see that $G$ preserves fibrant objects note that $G$ preserves all limits and hence terminal objects. This finishes (i) $\Rightarrow$ (ii). The other implications are similar. $\square$

Now we are ready to state our main theorem. A Quillen adjunction between model categories lifts to an adjunction between the homotopy categories. We will prove the theorem in a way which is completely formal and allows us to not look into the construction of the homotopy categories. This elegant proof is due to Georges Maltsiniotis [4]. There is also an English version of their paper.

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**Theorem 1.** *(Quillen adjuctions lift)*

If $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ is an adjunction between model categories such that the total derived functors $\mathbf{L}F$ and $\mathbf{R}G$ exist and are absolute Kan-extensions, then $\mathbf{L}F$ is left adjoint to $\mathbf{R}G$. In particular this does apply to Quillen adjunctions.

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Proof The total derived functors $\mathbf{L}F$ and $\mathbf{R}G$ exist by Ken Brown's lemma. They are constructed as in proposition 2, hence are absolute Kan-extensions. Let $\mu : \mathsf{M} \to \mathsf{HoM}$ and $\nu : \mathsf{N} \to \mathsf{HoN}$ be the localisation of $\mathsf{M}$ and $\mathsf{N}$ respectively. Then $\mathbf{L}F$ is obtained by factoring $\nu FQ$ through $\mathsf{HoM}$. The universal natural transformation of $\mathbf{L}F$ is $\nu Fq$. Similarly $\mathbf{R}G$ is induced by $\mu GR$ and comes equipped with a universal transformation $\mu Gr : \mu G \Rightarrow (\mathbf{R}G)\nu$.

The idea of the proof is to construct unit $\overline{\eta}$ and counit $\overline{\varepsilon}$ of $\mathbf{L}F \dashv \mathbf{R}G$ from unit $\eta$ and counit $\varepsilon$ of the adjunction $F \dashv G$. We need a transformation $\overline{\eta} : 1 \Rightarrow \mathbf{R}G \circ \mathbf{L}F$, in other words we need an element of $\mathrm{Nat}(1, \mathbf{R}G \circ \mathbf{L}F)$. Here it comes in handy that $\mathbf{L}F$ is an absolute Kan extension. This implies that $\mathbf{R}G \circ \mathbf{L}F, (\mathbf{R}G)\nu Fq$ is a right Kan-extension of $(\mathbf{R}G)\nu F$ along $\mu$. In other words

$$\mathrm{Nat}(1, \mathbf{R}G \circ \mathbf{L}F) \cong \mathrm{Nat}(\mu, \mathbf{R}G \circ \nu \circ F)\,.$$

This means in order to produce $\overline{\eta}$ we need only produce a transformation $\mu \Rightarrow (\mathbf{R}G)\nu F$. We can do that with the data which we have. Consider

$$\mu \xRightarrow{\mu\eta} \mu GF \xRightarrow{\mu GrF} (\mathbf{R}G)\nu F\,.$$

We define $\overline{\eta}$ as the image of $\mu GrF \circ \mu\eta$ under the isomorphism $\mathrm{Nat}(1, \mathbf{R}G \circ \mathbf{L}F) \cong \mathrm{Nat}(\mu, (\mathbf{R}G)\nu F)$. This means $\overline{\eta}$ is the unique transformation such that

$$\begin{tikzcd}
\mu \arrow[r, Rightarrow, "\mu\eta"] \arrow[d, Rightarrow, "\overline{\eta}\mu"] & \mu GF \arrow[d, Rightarrow, "\mu GrF"] \\
(\mathbf{R}G \circ \mathbf{L}F)\mu \arrow[r, Rightarrow, "(\mathbf{R}G)\nu Fq"'] & (\mathbf{R}G)\nu F
\end{tikzcd}$$

commutes. Dually we can define $\overline{\varepsilon}$ as the unique transformation $\overline{\varepsilon} : \mathbf{L}F \circ \mathbf{R}G \Rightarrow 1$ such that the diagram

$$\begin{tikzcd}
\nu & \nu FG \arrow[l, Rightarrow, "\nu\varepsilon"'] \\
(\mathbf{L}F \circ \mathbf{R}G)\nu \arrow[u, Rightarrow, "\overline{\varepsilon}\nu"] & (\mathbf{L}F)\mu G \arrow[l, Rightarrow, "(\mathbf{L}F)\mu Gr"] \arrow[u, Rightarrow, "\nu FqG"']
\end{tikzcd}$$

commutes. This is actually enough to prove that $\overline{\eta}$ and $\overline{\varepsilon}$ satisfy the triangle identities. Hence they are the unit-counit pair of an adjunction. Details can be found in the original paper [4]. $\square$

If we go back to the proof of proposition 2, then we can get an actual formula for the transformations $\overline{\eta}$ and $\overline{\varepsilon}$. $\overline{\eta}_{\mu x}$ is equal to the composition

$$\overline{\eta}_{\mu x} = \mu x \xrightarrow{(\mu q_x)^{-1}} \mu Qx \xrightarrow{\mu\eta_{Qx}} \mu GFQx \xrightarrow{\mu Gr_{FQx}} \mu GRFQx$$

and $\overline{\varepsilon}_{\nu x}$ is the composition

$$\overline{\varepsilon}_{\nu y} = \nu FQGRy \xrightarrow{\nu Fq_{GRy}} \nu FGRy \xrightarrow{\nu\varepsilon_{Ry}} \nu Ry \xrightarrow{(\nu r_y)^{-1}} \nu y\,.$$

We won't need those formulas though. There is a simple condition when the induced (co)unit pair becomes isomorphisms and the induced adjunction is an adjoint equivalence.

**Corollary 2.** (*Quillen equivalence*) Assume the Quillen adjunction $F \dashv G$ satisfies the following extra hypothesis:

- $Fx \to y$ is a weak equivalence if and only if the adjunct $x \to Gy$ is a weak equivalence.

Then $\mathbf{L}F \dashv \mathbf{R}G$ is an adjoint equivalence (unit and counit are isomorphisms).

Proof Remember that $\overline{\eta}$ by definition satisfies the equation

$$(\mathbf{R}G)\nu Fq_x \circ \overline{\eta}_{\mu x} = \mu Gr_{Fx} \circ \mu\eta_x\,.$$

Assume $x$ is a cofibrant object. Then $q_x$ is a weak equivalence between cofibrant objects, so $Fq_x$ is a weak equivalence and $\nu Fq_x$ is an isomorphism. $\mu Gr_{Fx} \circ \mu\eta_x = \mu(Gr_{Fx} \circ \eta_x)$ is $\mu$ applied to the adjoint of $r_{Fx}$. Now $r_{Fx}$ is a weak equivalence, hence so is $Gr_{Fx} \circ \eta_x$ by assumption and $\mu(Gr_{Fx} \circ \eta_x)$ is an isomorphism. This shows that $\overline{\eta}_{\mu x}$ is an isomorphism if $x$ is cofibrant. Every object in $\mathsf{HoM}$ is isomorphic to a cofibrant object. Thus $\overline{\eta}_{\mu x}$ is an isomorphism for all $x$. In a similar way one checks that $\overline{\varepsilon}$ is a natural isomorphism. □

## 3 Connections to Homological Algebra

In this section we will record some connections to homological algebra. Recall that the category $\mathsf{Ch}_{\geq}$ of chain complexes concentrated in non-negative degree has a model structure. Weak equivalences are the maps which induce isomorphisms on homology, cofibrations are monomorphisms with degreewise projective cokernel and fibrations are chain maps which are epi in (strictly) positive degree. The most important step is to connect homotopy in the model categorical sense to the classical notion of chain homotopy. This is possible when we consider morphisms between fibrant-cofibrant objects. First we need a cylinder object.

**Lemma 2.** (*Cylinder object*) Assume $Q$ is a cofibrant chain complex (levelwise projective). Then there is a cylinder object $\mathrm{cyl}(Q)$ of $Q$ defined as follows. $\mathrm{cyl}(Q)_n = Q_n \oplus Q_{n-1} \oplus Q_n$ and the differentials are given by the matrix

$$\begin{pmatrix} d & 1 & 0 \\ 0 & -d & 0 \\ 0 & -1 & d \end{pmatrix}\,.$$

This is a standard construction in homological algebra[3].

[3] Weibel, *An Introduction to Homological Algebra*, Chapter 1.5

Proof The structure map $(i_0, i_1) : Q \oplus Q \to \mathrm{cyl}(Q)$ is levelwise defined by $(a, b) \mapsto (a, 0, b)$. The structure map $p : \mathrm{cyl}(Q) \to Q$ maps levelwise as $(a, b, c) \mapsto a + c$. It is not hard to see that both maps are chain maps and that they factor the fold map. To see that $Q \oplus Q \to \mathrm{cyl}(Q)$ is a cofibration note that it is monic in every degree and the cokernel is $Q_{n-1}$ at level $n$, i.e. levelwise projective. It is clear that $\mathrm{cyl}(Q) \to Q$ is epi in all degrees, hence a fibration. $\mathrm{cyl}(Q) \to Q$ is a chain homotopy equivalence. The inverse is the map $i_0$. It is clear that $pi_0 = 1_Q$. $i_0 p$ is chain homotopy equivalent to the identity via the homotopy $s_n : \mathrm{cyl}(Q)_n \to \mathrm{cyl}(Q)_{n+1}$, $(a, b, c) \mapsto (0, -c, 0)$. Hence $p$ is a weak equivalence. I did learn about the cylinder object in an answer of Zehn Lin on stack exchange[5]. □

The cylinder object above is used in homological algebra because it has a very useful property. Two maps $f, g : Q \to A$ are chain homotopic if and only if there is $(f, s, g) : \mathrm{cyl}(Q) \to A$ such that $(f, s, g)i_0 = f$ and $(f, s, g)i_1 = g$, in other words if and only if they are homotopic with respect to the cylinder object $\mathrm{cyl}(Q)$ [6, p. 21]. All chain complexes are fibrant. If $A$ is also cofibrant, then $f$ and $g$ are homotopic (in the model categorical sense) if and only if they are homotopic with respect to every path and every cylinder object. As a consequence they are homotopic with respect to our special cylinder object, hence chain homotopic. In summary we have the following corollary.

**Corollary 3.** Assume $Q$ and $P$ are both fibrant. Then $f, g : P \to Q$ are homotopic in the model categorical sense if and only if they are chain homotopic.

**Example 1.** As an application we get a fancy proof of a standard result in homological algebra. The theorem is: Assume $f : M \to M'$ is a map of modules and $Q_* \to M$ and $Q'_* \to M$ are projective resolutions of $M$ and $M'$ respectively. Then there is a chain map $g : Q_* \to Q'_*$ which lifts $f$ in the sense that

$$\begin{tikzcd} Q_* \arrow[r, "g"] \arrow[d] & Q'_* \arrow[d] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

commutes when we view $M$ and $N$ as chain complexes concentrated in degree zero. The lift $g$ is unique up to chain homotopy.

Proof Note that if we view $M$ and $M'$ as complexes concentrated in degree zero, then projective resolutions are precisely cofibrant replacements. The map $Q_* \to M$ is a weak equivalence by definition and fibrant since the chain complex $M$ is zero in positive degrees. Every object is fibrant in $\mathsf{Ch}_{\geq}$, so a cofibrant replacement is automatically fibrant. $Q_*$ and $Q'_*$ are cofibrant-fibrant replacements and $g$ is the result of lifting $f$ first to the fibrant replacement and then lifting the result to a map between cofibrant-fibrant replacements[4]. Hence by model-category results we know that $g$ exists and is unique up to the model-category homotopy relation. By the previous discussion it follows that $g$ is unique up to chain homotopy. □

How do the derived functors of homological algebra relate to the derived functors we have defined in the previous section? Here is an example.

[4]The last step involves doing nothing.

**Example 2.** The functor $N \otimes_R - : {}_R\mathsf{Mod} \to \mathsf{Ab}$ induces a functor $\mathsf{Ch}_{\geq}(R) \to \mathsf{Ch}_{\geq}(\mathbb{Z})$ by acting levelwise. Its total derived functor $\mathbf{L}F$ exists.

To see that this is true we only need to check that $N \otimes_R -$ sends weak equivalences between cofibrant complexes to weak equivalences. Assume $f : Q \to P$ is a weak equivalence and $P$ and $Q$ are both cofibrant. $P$ and $Q$ are also fibrant, because all complexes are fibrant in the particular model structure. From the theory of model categories we know that $f$ is a homotopy equivalences (homotopy in the sense of model categories). From our above discussion we know that this implies that $f$ is a chain homotopy equivalence. The property of being a chain homotopy equivalence is equational, hence preserved by the additive functor $N \otimes_R -$. It follows that $N \otimes f : N \otimes P_* \to N \otimes Q_*$ is a chain homotopy equivalence, and in consequence a weak equivalence. From our construction of total derived functors we see that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Mod}_R \arrow[r] \arrow[rrr, bend left, "\mathrm{Tor}_i^R(N,-)"] & \mathsf{Ch}_{\geq}(R) \arrow[r, "(N\otimes_R -)Q"] \arrow[d] & \mathsf{Ch}_{\geq}(\mathbb{Z}) \arrow[r, "H_i"] \arrow[d] & \mathsf{Ab} \\
& \mathsf{HoCh}_{\geq}(R) \arrow[r, "\mathbf{L}(N\otimes_R -)"] & \mathsf{HoCh}_{\geq}(\mathbb{Z}) \arrow[ur] &
\end{tikzcd}$$

Of course something similar works for all additive functors $F : \mathsf{A} \to \mathsf{B}$ between abelian categories. In general the following diagram commutes:

$$\begin{tikzcd}
\mathsf{A} \arrow[r] \arrow[rrr, bend left, "L_iF"] & \mathsf{Ch}_{\geq}(\mathsf{A}) \arrow[r, "FQ"] \arrow[d] & \mathsf{Ch}_{\geq}(\mathsf{B}) \arrow[r, "H_i"] \arrow[d] & \mathsf{B} \\
& \mathsf{HoCh}_{\geq}(\mathsf{A}) \arrow[r, "\mathbf{L}F"] & \mathsf{HoCh}_{\geq}(\mathsf{B}) \arrow[ur] &
\end{tikzcd}$$

where the upmost functor is the usual left derived functor of homological algebra.

# 4 Appendix: Grothendieck Universes

For a really nice discussion about the role of sets and classes in category theory, you can read in Peter Smith's book *Category Theory - A Gentle Introduction*[7].

A lot of category theory can't be expressed properly in ZFC, and there is no consensus about how foundations for categories shall look like. Nevertheless, the classical paradoxes of set theory can be reproduced in category theory if one is not careful. This is a problem, especially because category theory invites to form larger and larger collections. Size does matter in category theory. Here is an example: A well-known theorem by Freyd states that any small category which has all small limits must be a preorder. Size plays a role in many of the major results of category theory: The Yoneda-lemma, the adjoint functor theorems, the Mitchell-Freyd embedding. One way to deal with the insufficiencies of ZFC is to extend it to Neumann-Bernays-Gödel's set theory. In their system there is one more undefined primitive: classes. Every set is a class but not all classes are sets. The intended semantic is that classes are collections which are to big to be sets. For example, there is a class of all sets, a class of all topological spaces and a class of all groups. But there is no set of all sets[5]. Categories are allowed to have a proper class of objects. Those which have a set of objects are small, the other ones are large. This is a good solution for someone who wants to use category theory only as a language to organise their field[6], because it is a relative mild extension. Classes in this setting are in a sense virtual. They are a convenient way to talk about many thinks at once, but they are not objects of study themselves. Talk about them could be avoided without hurting the theory much. For example instead of saying that $G$ is an element of the class of groups, one could say that $G$ is a set together with an operation such that a bunch of axioms hold. Instead of saying that $\pi_1$ is a functor, one could construct $\pi_1(X, x)$ ad hoc when it is needed, and prove the statements which are proved with help of $\pi_1$[7]. The feeling that classes do not add anything new can be made precise: NBG is a conservative extensions of ZFC. Both systems prove the same statements about sets. Whenever you have a proposition that can be formulated within ZFC[8] and which has a proof in NBG, then logicians assure you that there is a proof of the same proposition in ZFC. A corollary is, that if NBG is inconsistent, then it is the fault of ZFC and not a consequence of the addition of classes.

Unfortunately NBG is not good enough if one wants to study categories the same way people study groups, sets and spaces. The term class has no good closure properties. Just as there is no set of all sets, there can't be a class of all classes. In NBG one is allowed to form the category $\mathsf{Set}$, which has the proper class of all sets as objects, but there is no way to form the functor category $\mathsf{Fun}(\mathsf{Set}, \mathsf{Set})$. Poetically speaking, the need of mathematicians to treat many things as one, to turn the plural into singular, is given in for one more step in the hierarchy, but only one more[9]. Note that this is the same need which made mathematicians invent sets, and some do not like those either[10]. If we commit ourselves to sets, why not allow Grothendieck universes.

**Definition 6.** A Grothendieck universe is a set $\mathfrak{U}$ such that

- $\mathfrak{U}$ is transitive. If $x \in \mathfrak{U}$ and $y \in x$ then $y \in \mathfrak{U}$.
- The empty set $\emptyset$ is in $\mathfrak{U}$.
- $\mathfrak{U}$ is closed under the following operations: Power-sets, subsets, products indexed over sets in $\mathfrak{U}$, sums indexed over sets in $\mathfrak{U}$, exponentials.
- If $f : a \to \mathfrak{U}$ is a function and $a \in \mathfrak{U}$, then $\bigcup_{i \in a} f(i) \in \mathfrak{U}$.

Basically, $\mathfrak{U}$ is closed under all of the usual set theoretic constructions, except the existence of an infinite set. If a Grothendieck universe exists which has $\omega$ as an element, then it is a model for the formal language ZFC[11]. All mathematics which can be done within ZFC sits

[5] And in consequence there is no set of all sets which do not contain themselves.

[6] "This here is a product.", "$\pi_1$ is a functor.", "Studying matrices is the same as studying finite dimensional vector spaces.", etc.

[7] Brouwer's fixed point theorem say.

[8] That is does not involve the word class.

[9] See Peter Smith's book *Category Theory - A gentle introduction* section 2.5. for an extensive discussion of the matter.

[10] N J Wildberger - *Why infinite sets don't exist.* - https://youtu.be/XKy_VTBq0yk

[11] See https://plato.stanford.edu/entries/model-theory/ for an explanation what *model* means in this context. I use it in a vague philosophically charged way. There is precise definitions of this term in

happily inside that Grothendieck universe. You can construct all the topological spaces and groups you know without ever leaving the universe. Consequently ZFC can not prove the existence of a Grothendieck universe which contains $\omega$, because that would mean that ZFC can prove its own consistency. We are thus in need of an additional axiom: Every set is an element of some Grothendieck universe. In this setting we do not have to use classes to talk about the classical categories $\mathsf{Top}$, $\mathsf{Grp}$. We can model them all inside some fixed universe $\mathfrak{U}$ (with $\omega \in \mathfrak{U}$). In the setting of Grothendieck universes every category is small (has a set of objects) because there are no classes. Here is our definition of a category.

**Definition 7.** A category $\mathsf{C}$ consists of two sets $\mathrm{ob}(\mathsf{C})$ and $\mathrm{mor}(\mathsf{C})$ together with functions

$$\mathrm{mor}(\mathsf{C}) \begin{array}{c} \xrightarrow{\mathrm{dom}} \\ \xleftarrow{\mathrm{id}} \\ \xrightarrow[\mathrm{codom}]{} \end{array} \mathrm{ob}(\mathsf{C})$$

and a partially defined composition function $\circ : \mathrm{mor}(\mathsf{C}) \times \mathrm{mor}(\mathsf{C}) \to \mathrm{mor}(\mathsf{C})$ such that the usual axioms hold.

The advantage is that we never have to worry when we form functor categories or do other constructions which are impossible within the NBG framework due to size issues. Here are some disadvantages: we can no longer speak unqualified about the category of sets, the category of groups and of all the other classical categories we are interested in. Instead we have different versions of them relative to Grothendieck universes. There is no category of all sets. Instead there is a category $\mathsf{Set}_{\mathfrak{U}}$ of all sets $x \in \mathfrak{U}$ contained inside a fixed Grothendieck universe $\mathfrak{U}$. Similarly there are categories $\mathsf{Grp}_{\mathfrak{U}}$, $\mathsf{Top}_{\mathfrak{U}}$, relative to Grothendieck universes $\mathfrak{U}$ which contain $\omega$. It is inelegant to have many versions of what is essentially the same category. Sometimes it does not matter which category of sets we use. For example Yoneda's lemma for a category $\mathsf{C}$ holds with respect to any category of sets $\mathsf{Set}_{\mathfrak{U}}$ which is large enough to contain all the arrow sets $\mathsf{C}(c, c')$ of $\mathsf{C}$. In this case we will write $\mathsf{Ens}$ to denote an unspecified but large enough category of sets.

In practice the use of Grothendieck universes works as follows. We fix one universe $\mathfrak{U}_0$ with infinity once and for all. We imagine that all the classical categories we like to discuss sit inside $\mathfrak{U}_0$. Unqualified terms such as *locally small*, *small*, *large*, *complete*, etc. always refer to this universe. When we are in trouble, then we pass to a higher universe $\mathfrak{U}_1$ with $\mathfrak{U}_0 \in \mathfrak{U}_1$. This is the analogue to $\mathfrak{U}_0 : \mathfrak{U}_1$ in homotopy type theory, only that we work within a material set theory.

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mathematical logic, but it depends on an a prior understanding of what a set is. This makes it problematic when discussing foundations. In my understanding, to give a model is to give semantics to an axiomatic system in a formal language. For example there is a first order language of groups with axioms such as $\forall x \forall y \forall z : (x \cdot y) \cdot z = x \cdot (y \cdot z)$. A model for this language is an actual group. But if you are not a Platonist, then there are no real groups floating around. So the only way to give a group is to construct it in another formal system, in this case some set theory. Hence we might say: A model for a formal system is an interpretation of that system in another formal system. A Grothendieck universe $\mathfrak{U}$, with $\omega \in \mathfrak{U}$ to have the axiom of infinity, is a model for ZFC, because we can reinterpret sets in ZFC as elements of $\mathfrak{U}$ and make sense of every construction a ZFC user makes as happening inside $\mathfrak{U}$.

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