Grothendieck Universes January 3, 2025

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For a really nice discussion about the role of sets and classes in category theory, you can read in Peter Smith's book *Category Theory - A Gentle Introduction*.

A lot of category theory can't be expressed properly in ZFC, and there is no consensus about how foundations for categories shall look like. Nevertheless, the classical paradoxes of set theory can be reproduced in category theory if one is not careful. This is a problem, especially because category theory invites to form larger and larger collections. Size does matter in category theory. Here is an example: A well-known theorem by Freyd states that any small category which has all small limits must be a preorder. Size plays a role in many of the major results of category theory: The Yoneda-lemma, the adjoint functor theorems, the Mitchell-Freyd embedding. One way to deal with the insufficiencies of ZFC is to extend it to Neumann-Bernays-Gödel's set theory. In their system there is one more undefined primitive: classes. Every set is a class but not all classes are sets. The intended semantic is that classes are collections which are to big to be sets. For example, there is a class of all sets, a class of all topological spaces and a class of all groups. But there is no set of all sets[1]. Categories are allowed to have a proper class of objects. Those which have a set of objects are small, the other ones are large. This is a good solution for someone who wants to use category theory only as a language to organise their field[2], because it is a relative mild extension. Classes in this setting are in a sense virtual. They are a convenient way to talk about many thinks at once, but they are not objects of study themselves. Talk about them could be avoided without hurting the theory much. For example instead of saying that G is an element of the class of groups, one could say that G is a set together with an operation such that a bunch of axioms hold. Instead of saying that π_1 is a functor, one could construct $\pi_1(X, x)$ ad hoc when it is needed, and prove the statements which are proved with help of π_1[3]. The feeling that classes do not add anything new can be made precise: NBG is a conservative extensions of ZFC. Both systems prove the same statements about sets. Whenever you have a proposition that can be formulated within ZFC[4] and which has a proof in NBG, then logicians assure you that there is a proof of the same proposition in ZFC. A corollary is, that if NBG is inconsistent, then it is the fault of ZFC and not a consequence of the addition of classes.

Unfortunately NBG is not good enough if one wants to study categories the same way people study groups, sets and spaces. The term class has no good closure properties. Just as there is no set of all sets, there can't be a class of all classes. In NBG one is allowed to form the category Set, which has the proper class of all sets as objects, but there is no way to form the functor category $\mathsf{Fun}(\mathsf{Set}, \mathsf{Set})$. Poetically speaking, the need of mathematicians to treat many things as one, to turn the plural into singular, is given in for one more step in the hierarchy, but only one more[5]. Note that this is the same need which made mathematicians invent sets, and some do not like those either[6]. If we commit ourselves to sets, why not allow Grothendieck universes.

Definition 0.1 A Grothendieck universe is a set $\mathfrak{U}$ such that

[1]And in consequence there is no set of all sets which do not contain themselves.

[2]"This here is a product.", "π_1 is a functor.", "Studying matrices is the same as studying finite dimensional vector spaces.", etc.

[3]Brouwer's fixed point theorem say.

[4]That is does not involve the word class.

[5]See Peter Smith's book *Category Theory - A gentle introduction* section 2.5. for an extensive discussion of the matter.

[6]N J Wildberger - *Why infinite sets don't exist.* - https://youtu.be/XKy_VTBq0yk

- $\mathfrak{U}$ is transitive. If $x \in \mathfrak{U}$ and $y \in x$ then $y \in \mathfrak{U}$.
- The empty set $\varnothing$ is in $\mathfrak{U}$.
- $\mathfrak{U}$ is closed under the following operations: Power-sets, subsets, products indexed over sets in $\mathfrak{U}$, sums indexed over sets in $\mathfrak{U}$, exponentials.
- If $f : a \to \mathfrak{U}$ is a function and $a \in \mathfrak{U}$, then $\bigcup_{i\in a} f(i) \in \mathfrak{U}$.

Basically, $\mathfrak{U}$ is closed under all of the usual set theoretic constructions, except the existence of an infinite set. If a Grothendieck universe exists which has ω as an element, then it is a model for the formal language ZFC[7]. All mathematics which can be done within ZFC sits happily inside that Grothendieck universe. You can construct all the topological spaces and groups you know without ever leaving the universe. Consequently ZFC can not prove the existence of a Grothendieck universe which contains ω, because that would mean that ZFC can prove its own consistency. We are thus in need of an additional axiom: Every set is an element of some Grothendieck universe. In this setting we do not have to use classes to talk about the classical categories Top, Grp. We can model them all inside some fixed universe $\mathfrak{U}$ (with $\omega \in \mathfrak{U}$). In the setting of Grothendieck universes every category is small (has a set of objects) because there are no classes. Here is our definition of a category.

Definition 0.2 A category $\mathcal{C}$ consists of two sets $\mathrm{ob}(\mathcal{C})$ and $\mathrm{mor}(\mathcal{C})$ together with functions

$$\begin{tikzcd} \mathrm{mor}(\mathcal{C}) \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{codom}"'] & \mathrm{ob}(\mathcal{C}) \arrow[l, "\mathrm{id}" description] \end{tikzcd}$$

and a partially defined composition function $\circ : \mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ such that the usual axioms hold.

The advantage is that we never have to worry when we form functor categories or do other constructions which are impossible within the NBG framework due to size issues. Here are some disadvantages: we can no longer speak unqualified about the category of sets, the category of groups and of all the other classical categories we are interested in. Instead we have different versions of them relative to Grothendieck universes. There is no category of all sets. Instead there is a category $\mathsf{Set}_{\mathfrak{U}}$ of all sets $x \in \mathfrak{U}$ contained inside a fixed Grothendieck universe $\mathfrak{U}$. Similarly there are categories $\mathsf{Grp}_{\mathfrak{U}}$, $\mathsf{Top}_{\mathfrak{U}}$, relative to Grothendieck universes $\mathfrak{U}$ which contain ω. It is inelegant to have many versions of what is essentially the same category. Sometimes it does not matter which category of sets we use. For example Yoneda's lemma for a category $\mathcal{C}$ holds with respect to any category of sets $\mathsf{Set}_{\mathfrak{U}}$ which is large enough to contain all the arrow sets $\mathcal{C}(c, c')$ of $\mathcal{C}$. In this case we will write Ens to denote an unspecified but large enough category of sets.

In practice the use of Grothendieck universes works as follows. We fix one universe $\mathfrak{U}_0$ with infinity once and for all. We imagine that all the classical categories we like to discuss sit inside $\mathfrak{U}_0$. Unqualified terms such as *locally small*, *small*, *large*, *complete*, etc. always refer to this universe. When we are in trouble, then we pass to a higher universe $\mathfrak{U}_1$ with $\mathfrak{U}_0 \in \mathfrak{U}_1$. This

[7]See https://plato.stanford.edu/entries/model-theory/ for an explanation what *model* means in this context. I use it in a vague philosophically charged way. There is precise definitions of this term in mathematical logic, but it depends on an a prior understanding of what a set is. This makes it problematic when discussing foundations. In my understanding, to give a model is to give semantics to an axiomatic system in a formal language. For example there is a first order language of groups with axioms such as $\forall x \forall y \forall z : (x \cdot y) \cdot z = x \cdot (y \cdot z)$. A model for this language is an actual group. But if you are not a Platonist, then there are no real groups floating around. So the only way to give a group is to construct it in another formal system, in this case some set theory. Hence we might say: A model for a formal system is an interpretation of that system in another formal system. A Grothendieck universe $\mathfrak{U}$, with $\omega \in \mathfrak{U}$ to have the axiom of infinity, is a model for ZFC, because we can reinterpret sets in ZFC as elements of $\mathfrak{U}$ and make sense of every construction a ZFC user makes as happening inside $\mathfrak{U}$.

is the analogue to $\mathfrak{U}_0 : \mathfrak{U}_1$ in homotopy type theory, only that we work within a material set theory.