The Discrete Fourier Transform

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Discrete Fourier Transform

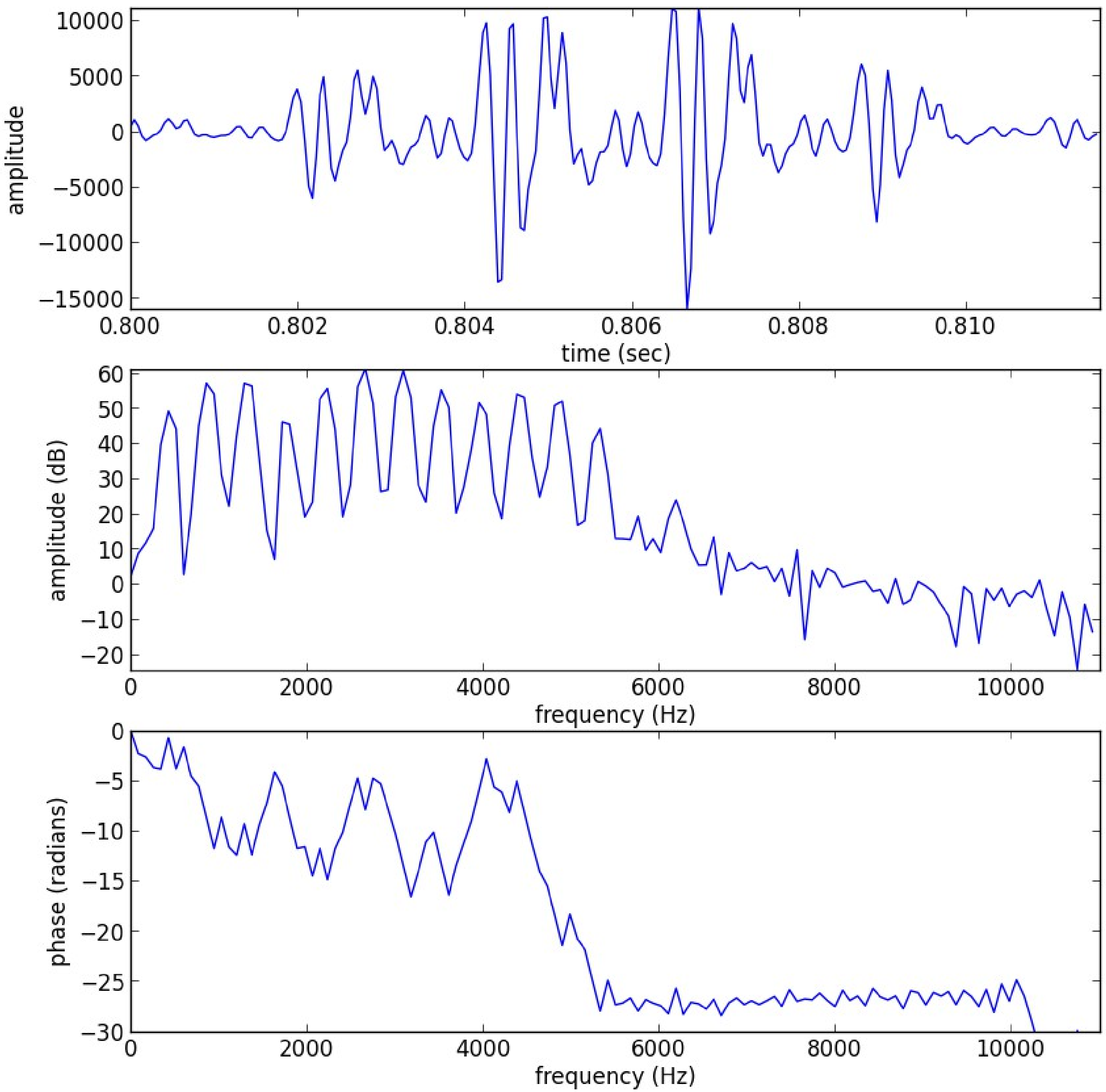
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \quad k = 0, 1, \ldots, N-1$$

n: discrete time index in samples (normalized time, T=1)

k: discrete frequency index in bins

$\omega_k = 2\pi k/N$ frequency in radians

$f_k = f_s k/N$ frequency in Hz, where fs is the sampling rate



fragment of a sound, x[n]

magnitude spectrum

$20 * \log_{10}(X[k])$

phase spectrum

$\measuredangle X[k]$

```
import matplotlib.pyplot as plt
import numpy as np
from scipy.io.wavfile import read
from scipy.fftpack import fft

(fs, x) = read('oboe.wav')
size = 256
start = .8*fs
xw = x[start:start+size] * np.hamming(size)

plt.subplot(311)
plt.plot(np.arange(start, (start+size), 1.0)/fs, xw)

X = fft(xw)
mX = 20 * np.log10(abs(X)/size)
pX = np.unwrap(np.angle(X))

plt.subplot(312)
plt.plot(np.arange(0, fs/2.0, float(fs)/size), mX[0:size/2])

plt.subplot(313)
plt.plot(np.arange(0, fs/2.0, float(fs)/size), pX[:size/2])
```

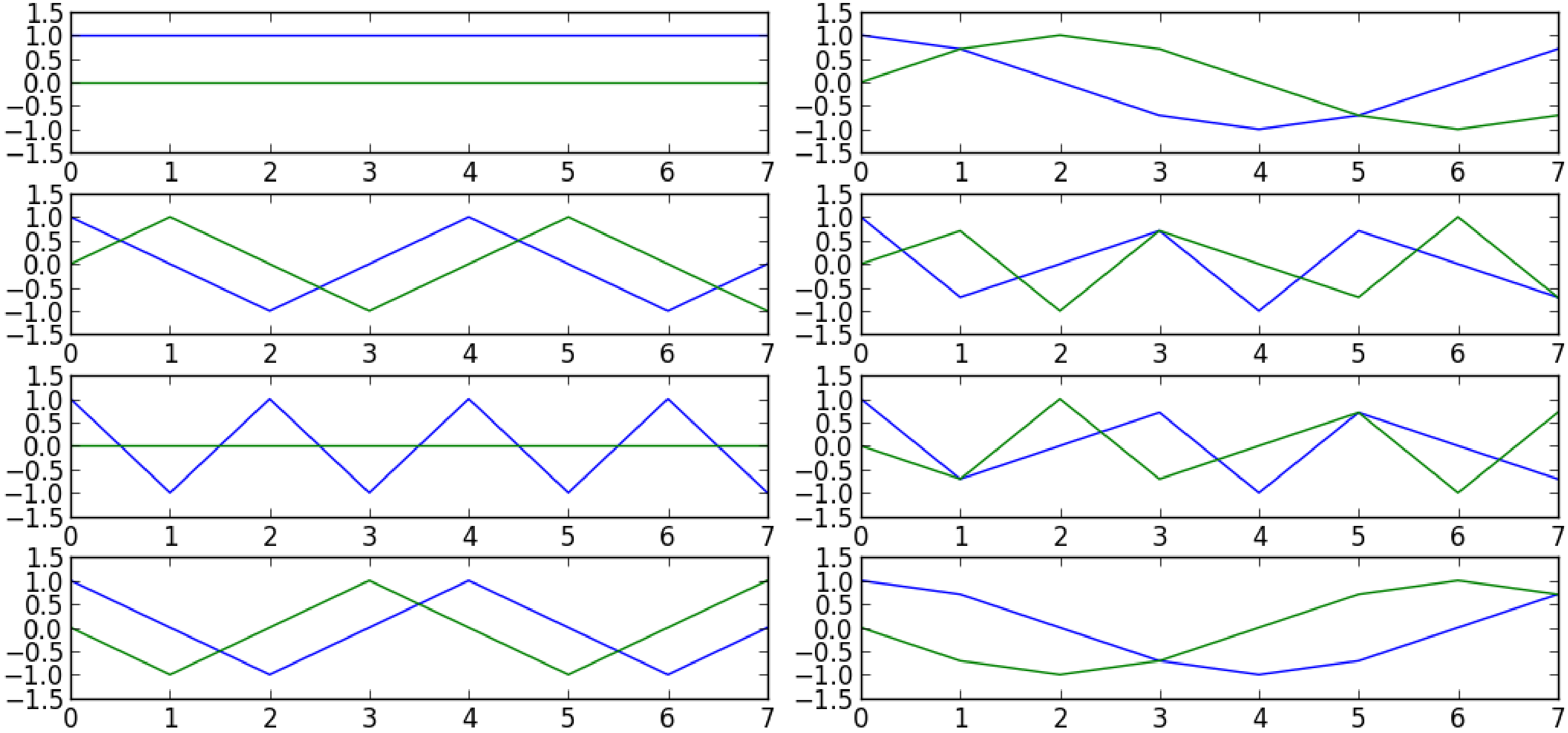
Complex exponentials

$$\bar{s}_k = e^{-j2\pi kn/N} = \cos(2\pi kn/N) - j\sin(2\pi kn/N)$$

for $N=4$, thus for $n=0,1,2,3\,;\,k=0,1,2,3$

$$\bar{s}_0 = \cos(2\pi*0*n/4) - j\sin(2\pi*0*n/4) = [1,1,1,1]$$
$$\bar{s}_1 = \cos(2\pi*1*n/4) - j\sin(2\pi*1*n/4) = [1,j,-1,-j]$$
$$\bar{s}_2 = \cos(2\pi*2*n/4) - j\sin(2\pi*2*n/4) = [1,-1,1,-1]$$
$$\bar{s}_3 = \cos(2\pi*3*n/4) - j\sin(2\pi*3*n/4) = [1,-j,-1,j]$$

Complex exponentials



```
import matplotlib.pyplot as plt
import numpy as np
N = 8
for k in range(N):
 s = np.exp(1j*2*np.pi*k/N*np.arange(N))
 plt.subplot(N/2, 2, k+1)
 plt.plot(np.real(s))
 plt.subplot(N/2, 2, k+1)
 plt.plot(np.imag(s))
```

Inner product - DFT

$$\langle x, s_k \rangle = \sum_{n=0}^{N-1} x[n] * \bar{s}_k[n] = \sum_{n=0}^{N-1} x[n] * e^{-j2\pi kn/N}$$

Example:

$$x[n] = [1, -1, 1, -1];\ N = 4$$

$$\langle x, s_0 \rangle = 1*1 + (-1)*1 + 1*1 + (-1)*1 = 0$$
$$\langle x, s_1 \rangle = 1*1 + (-1)*(-j) + 1*(-1) + (-1)*j = 0$$
$$\langle x, s_2 \rangle = 1*1 + (-1)*(-1) + 1*1 + (-1)*(-1) = 4$$
$$\langle x, s_3 \rangle = 1*1 + (-1)*(-j) + 1*(-1) + (-1)*j = 0$$

Inner product - DFT

```
import numpy as np

x = np.array([1,-1,1,-1])
print 'x = {}'.format(x)
N = 4
for k in range(N):
    s = np.exp(1j*2*np.pi*k/N*np.arange(N))
    print 's{0} = {1}'.format(k, s)
    X = sum(x*np.conjugate(s))
    print '<x,s{0}> = {1}'.format(k,X)
```

Output:

```
x = [ 1 -1  1 -1]
s0 = [ 1.+0.j  1.+0.j  1.+0.j  1.+0.j]
<x,s0> = 0.0
s1 = [ 1.+0.j  0.+1.j -1.+0.j -0.-1.j]
<x,s1> = 1.32693504719e-16
s2 = [ 1.+0.j -1.+0.j  1.-0.j -1.+0.j]
<x,s2> = 4.0
s3 = [ 1.+0.j -0.-1.j -1.+0.j  0.+1.j]
<x,s3> = 5.52708599219e-16
```

DFT of complex sinewave

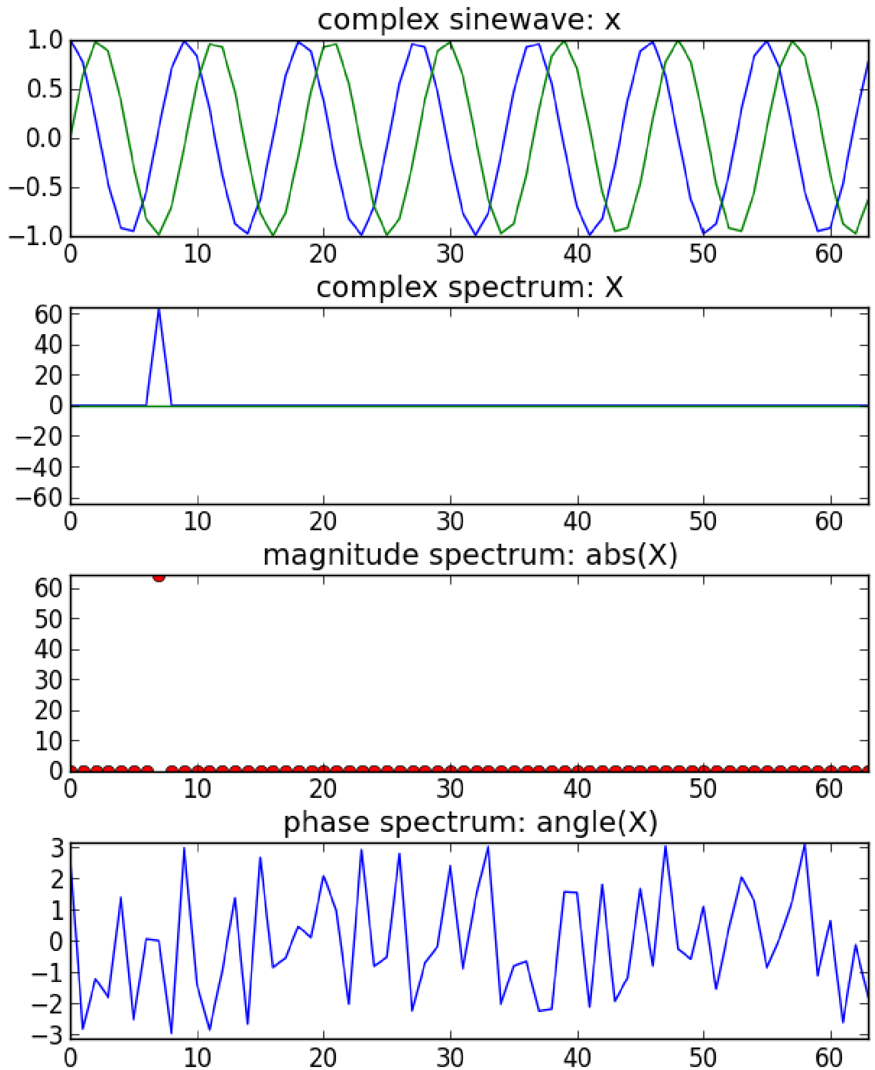
$$x_1[n] = e^{j(2\pi k_0/N)n} \qquad n = 0,1,\ldots N-1$$

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{N-1} e^{j(2\pi k_0/N)n} e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{N-1} e^{-j(2\pi/N)(k-k_0)n}$$

$$= \begin{cases} N & k_0 = k \\ 0 & k_0 \neq k \end{cases}$$




```
import numpy as np
import matplotlib.pyplot as plt

N = 64
k0 = 7
X = np.array([])
x = np.exp(1j*2*np.pi*k0/N*np.arange(N))

plt.subplot(411)
plt.plot(np.arange(N), np.real(x))
plt.plot(np.arange(N), np.imag(x))

for k in range(N):
 s = np.exp(1j*2*np.pi*k/N*np.arange(N))
 X = np.append(X, sum(x*np.conjugate(s)))

plt.subplot(412)
plt.plot(np.arange(N), np.real(X))
plt.plot(np.arange(N), np.imag(X))

plt.subplot(413)
plt.plot(np.arange(N), abs(X), 'ro')

plt.subplot(414)
plt.plot(np.arange(N), np.angle(X))
```

DFT of complex sinewave

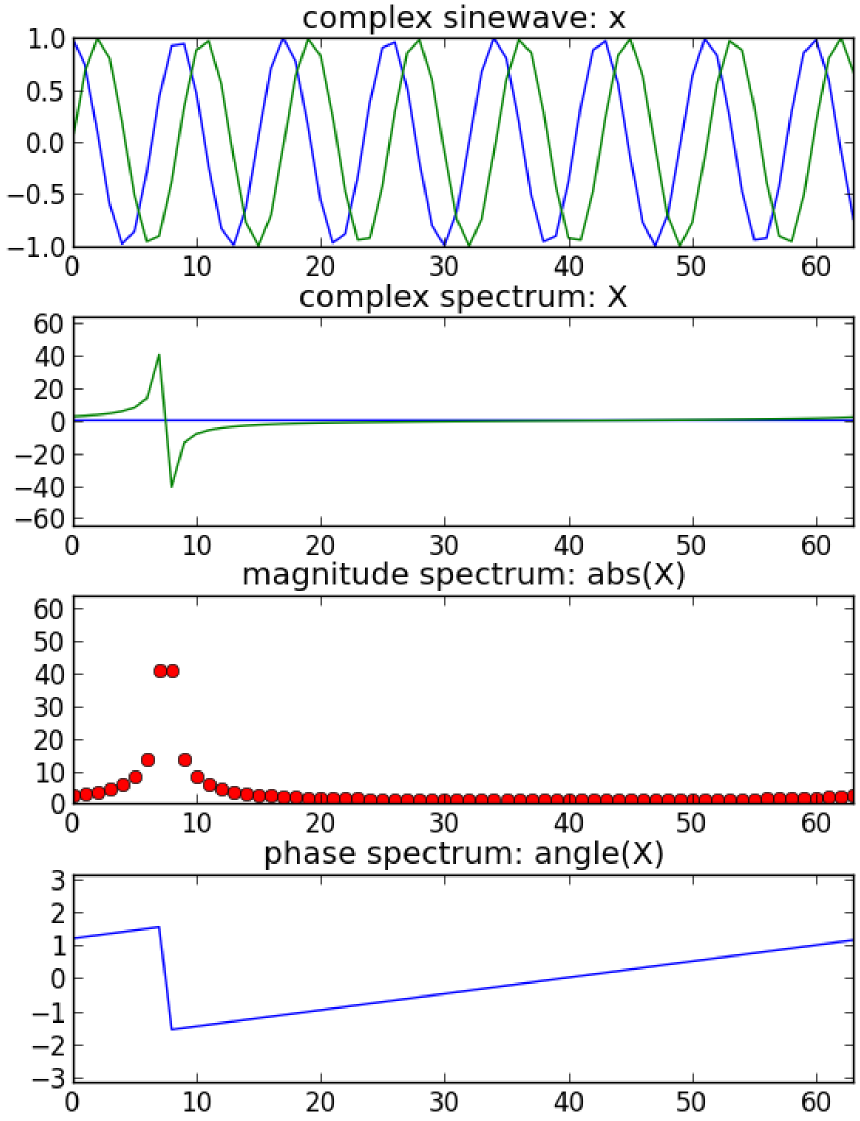
$$x_2[n] = e^{j(\hat{\omega}_0 n + \phi)} \qquad n = 0,1,\ldots N-1$$

$$X_3[k] = \sum_{n=0}^{N-1} e^{j(\hat{\omega}_0 n + \phi)} e^{-j(2\pi/N)kn}$$

$$= e^{j\phi} \sum_{n=0}^{N-1} e^{-j(2\pi k/N - \hat{\omega}_0)n}$$

$$= e^{j\phi} \left(e^{-j(0)} + e^{-j(2\pi k/N - \hat{\omega}_0)} + \ldots + e^{-j(2\pi k/N - \hat{\omega}_0)(N-1)} \right)$$

$$= e^{j\phi} \frac{1 - e^{-j(2\pi k/N - \hat{\omega}_0)N}}{1 - e^{-j(2\pi k/N - \hat{\omega}_0)}}$$




```
N = 64
k0 = 7.5
X = np.array([])
x = np.exp(1j*2*np.pi*k0/N*np.arange(N))

plt.subplot(411)
plt.plot(np.arange(N), np.real(x))
plt.plot(np.arange(N), np.imag(x))

for k in range(N):
 s = np.exp(1j*2*np.pi*k/N*np.arange(N))
 X = np.append(X, sum(x*np.conjugate(s)))

plt.subplot(412)
plt.plot(np.arange(N), np.real(X))
plt.plot(np.arange(N), np.imag(X))

plt.subplot(413)
plt.plot(np.arange(N), abs(X), 'ro')

plt.subplot(414)
plt.plot(np.arange(N), np.angle(X))
```

Inverse DFT

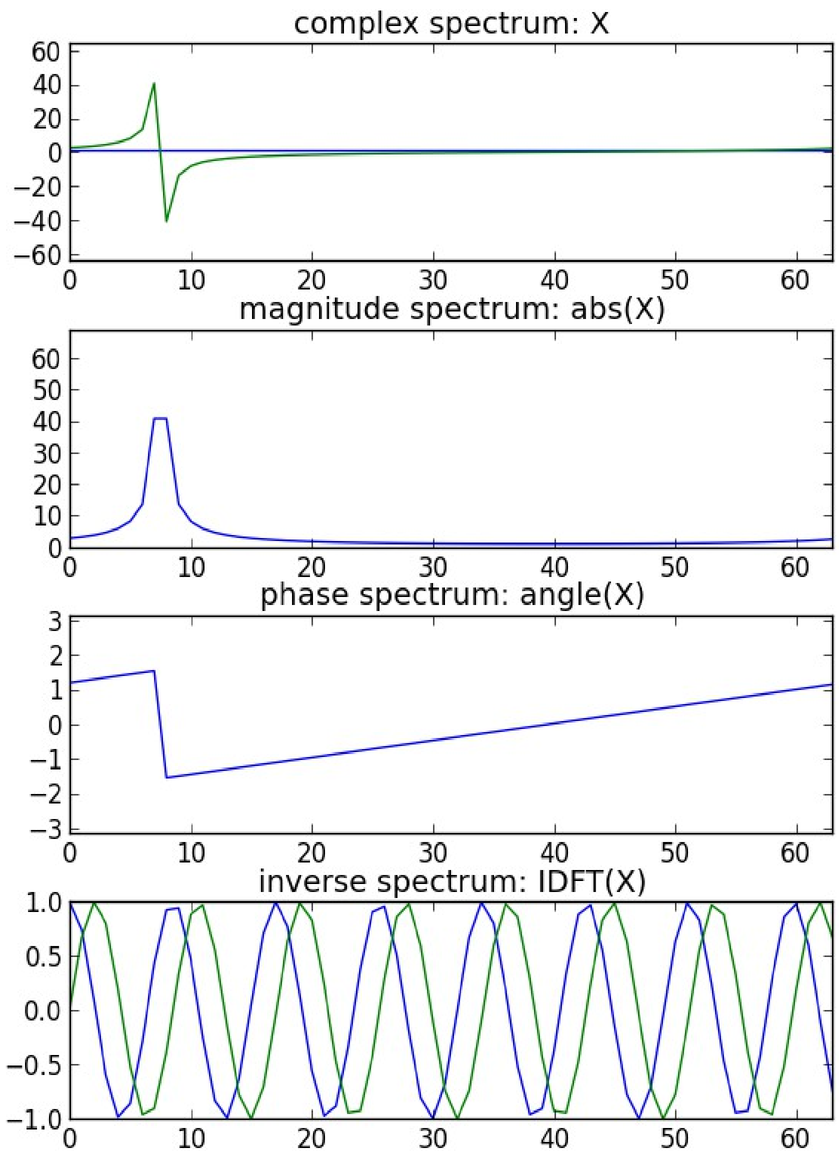
$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]*s_k[n]=\frac{1}{N}\sum_{k=0}^{N-1} X[k]*e^{j2\pi kn/N} \qquad n=0,1,\ldots,N-1$$

Example:

$$X[k]=[0,4,0,0];\, N=4$$

$$X*s_0=0*1+4*1+0*1+0*1=4$$
$$X*s_1=0*1+4*j+0*(-1)+0*(-j)=4\mathrm{j}$$
$$X*s_2=0*1+4*(-1)+0*1+0*(-1)=-4$$
$$X*s_3=0*1+4*(-j)+0*(-1)+0*j=-4\mathrm{j}$$

Inverse DFT example




```
import numpy as np
import matplotlib.pyplot as plt

N = 64
k0 = 7.5

x = np.exp(1j*2*np.pi*k0/N*np.arange(N))
for k in range(N):
 s = np.exp((1j*2*np.pi*k/N)*np.arange(N))
 X = np.append(X, sum(x*np.conjugate(s)))

plt.subplot(411)
plt.plot(np.arange(N), np.real(X))
plt.plot(np.arange(N), np.imag(X))

plt.subplot(412)
plt.plot(np.arange(N), abs(X))

plt.subplot(413)
plt.plot(np.arange(N), np.angle(X))

for n in range(N):
 s = np.exp((1j*2*np.pi*n/N)*np.arange(N))
 y = np.append(y, sum(X*s)/N)

plt.subplot(414)
plt.plot(np.arange(N), np.real(y))
plt.plot(np.arange(N), np.imag(y))
```

References

- https://ccrma.stanford.edu/~jos/mdft/
- Full code of plots and accompanying labs available at:https://github.com/MTG/sms-tools
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Credits