Some basic Mathematics

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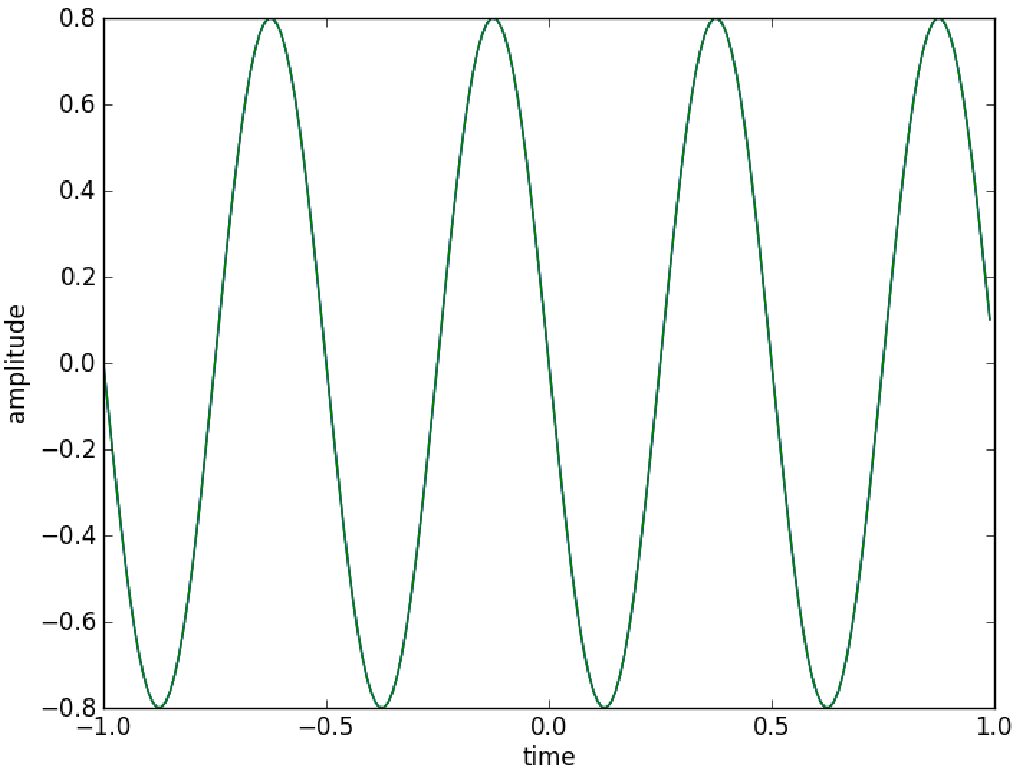
Sinusoidal functions (sine wave)

$$x(t) = A\cos(\omega_0 t + \varphi) = A\cos(2\pi f_0 t + \varphi)$$

$A =$ amplitude
$\omega_0 =$ frequency in radians
$f_0 =$ frequency in Hz
$\varphi =$ phase in radians
$t =$ time in seconds

Sinusoid plot



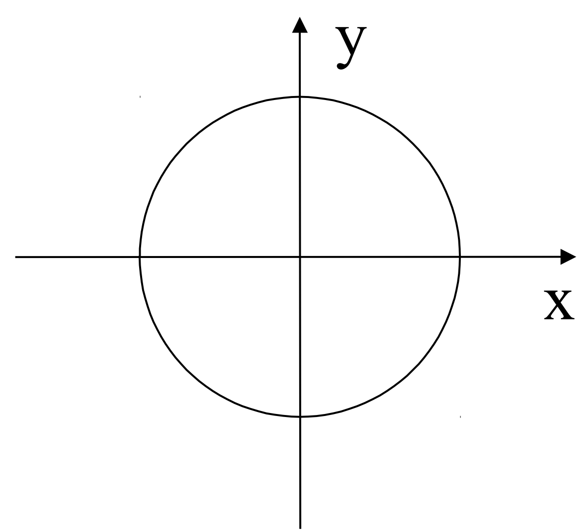

```
import matplotlib.pyplot as plt
import numpy as np

A0 = .8
f0 = 2
phi0 = np.pi/2
fs = 100
t = np.arange(-1, 1, 1.0/fs)
x = A0 * np.cos(2*np.pi*f0*t+phi0)
plt.plot(t, x)
```

Complex numbers

$(x + jy)$ where x: real part
y: imaginary part
$j = \sqrt{-1}$

Complex plane
x-axis (real part)
y-axis (imaginary part)

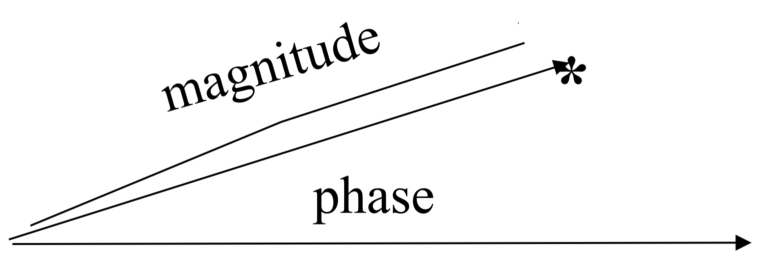


- Rectangular co-ordinates

- Polar co-ordinates

magnitude : $\sqrt{x^2 + y^2}$

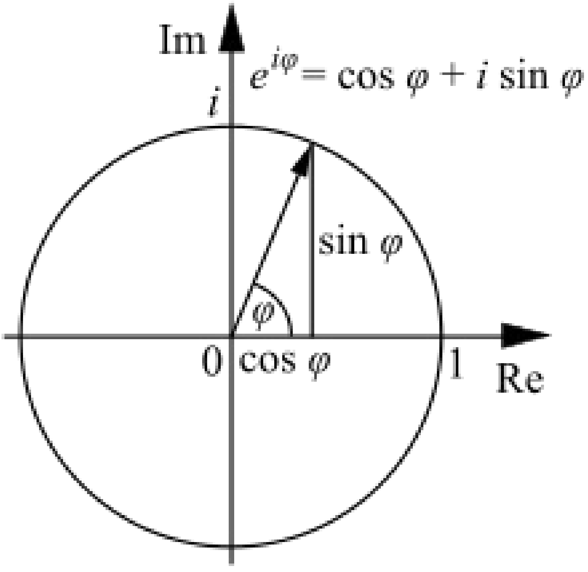
phase : $tan^{-1}(y/x)$



Euler's formula

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

$$\cos(\varphi) = \frac{e^{j\varphi} + e^{-j\varphi}}{2} \quad \sin(\varphi) = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$



Complex sinusoids

$$\bar{x}(t) = Ae^{j(\omega_0 t+\phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

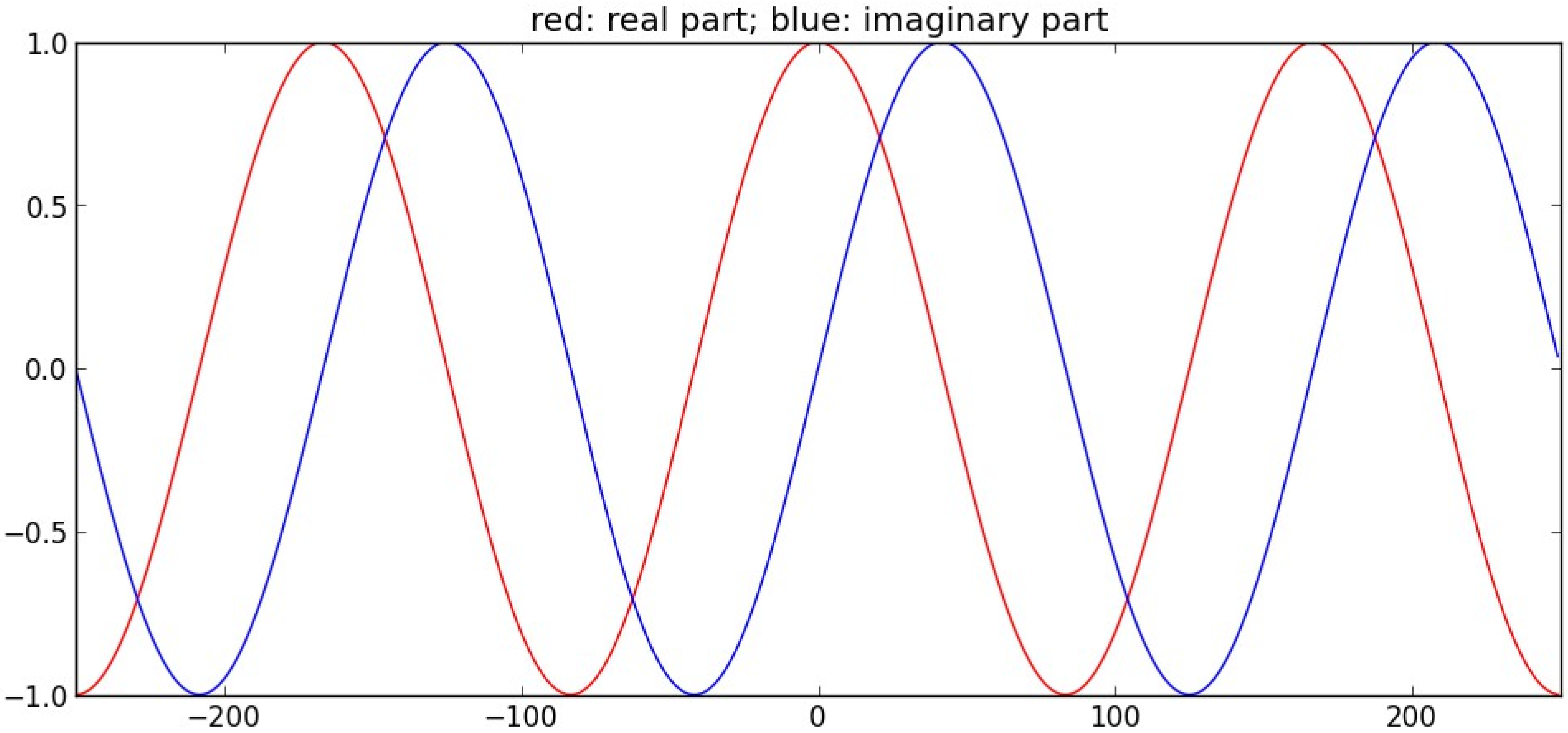
Real sinusoid

$$x(t) = A\cos(\omega_0 t + \phi) = A\left(\frac{e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}}{2}\right)$$

$$= \frac{1}{2}Xe^{j\omega_0 t} + \frac{1}{2}X^* e^{-j\omega_0 t} = \frac{1}{2}\bar{x}(t) + \frac{1}{2}\bar{x}^*(t)$$

$$= \mathrm{Re}\{\bar{x}(t)\}$$

Complex sinewave



Inner product of signals

Operation of two signals (vectors) which produces a scalar.

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n] * \bar{y}[n]$$

Example:

$$x[n] = [0, j, 1]; \; y[n] = [1, j, j]$$
$$\langle x, y \rangle = 0*1 + j*(-j) + 1*(-j) = 0 + 1 + (-j) = 1 - j$$

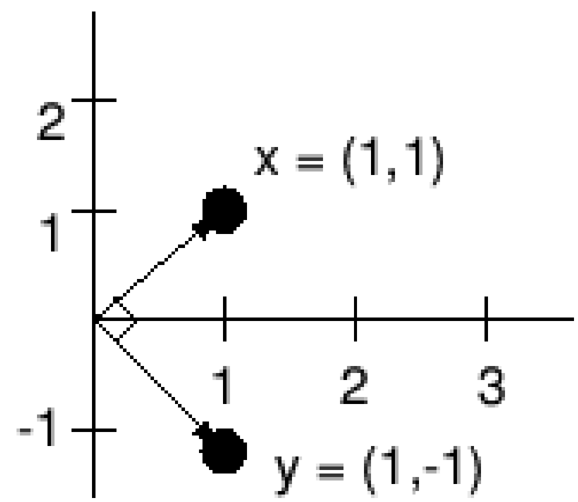
Orthogonality of vectors (signals)

Two vectors (signals) are orthogonal if their inner product is equal to zero

$$x \perp y \Leftrightarrow \langle x, y \rangle = 0$$

Example:

$$x[n] = [1,1];\ y[n] = [1,-1]$$
$$\langle x, y \rangle = 1 * \bar{1} + 1 * \left(\overline{-1}\right) = 1 - 1 = 0$$

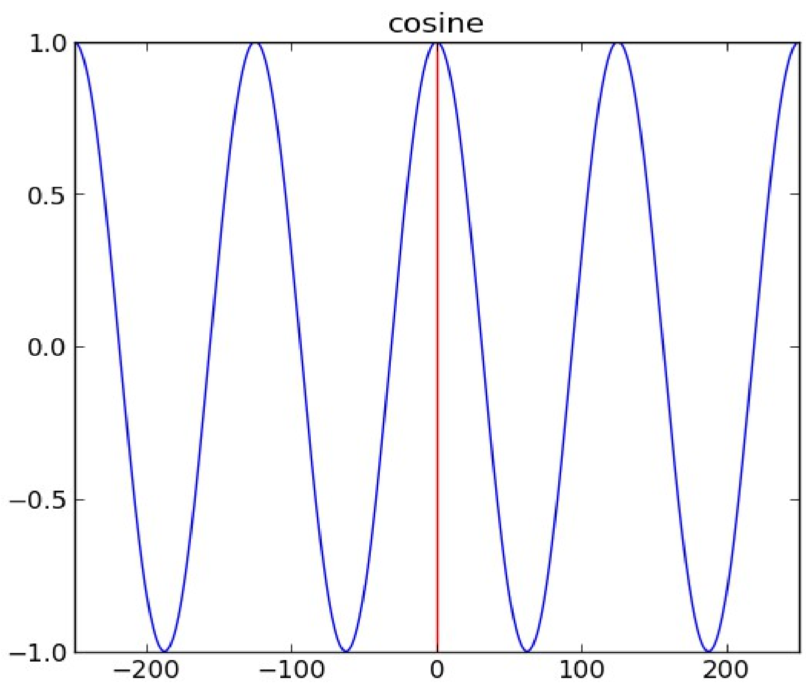


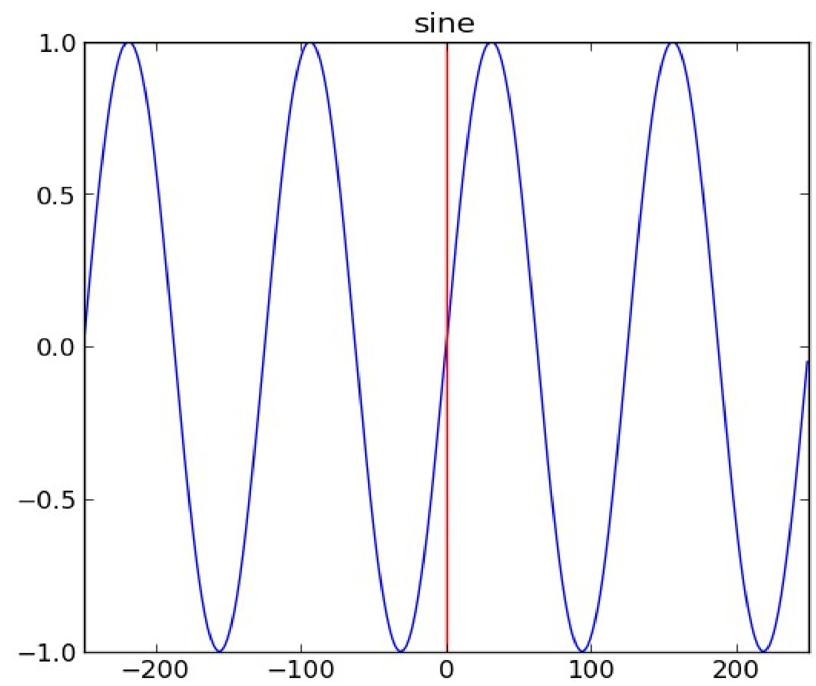
Even and odd functions

f(a) is *even* if f(-a) = f(a). An *even* function is symmetric.

f(a) is *odd* if f(-a) = -f(a). An *odd* function is antisymmetric.

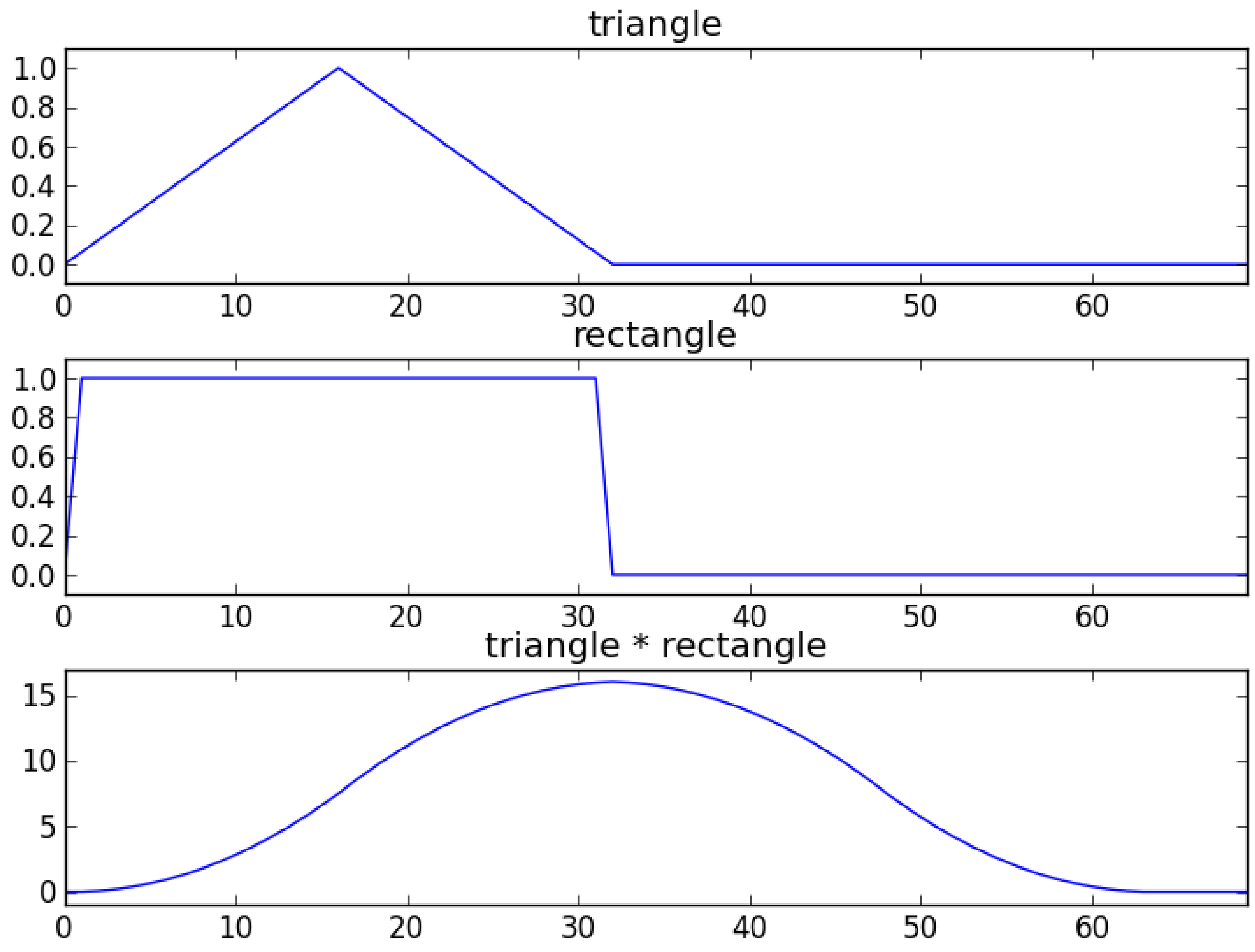
The *cosine* function is *even* since cos (-a) = cos(a). The *sine* function is *odd* since sin(-a) = - sin(a)





Convolution

$$\left(x_1[n] * x_2[n]\right)_n = \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]$$



References

- https://en.wikipedia.org/wiki/Sinusoid
- https://en.wikipedia.org/wiki/Complex_numbers
- https://en.wikipedia.org/wiki/Euler_formula
- https://en.wikipedia.org/wiki/Negative_frequency
- https://en.wikipedia.org/wiki/Inner_product
- https://en.wikipedia.org/wiki/Convolution
- Full code of plots available at: https://github.com/MTG/sms-tools
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Credits