

Fourier Transform theorems and properties

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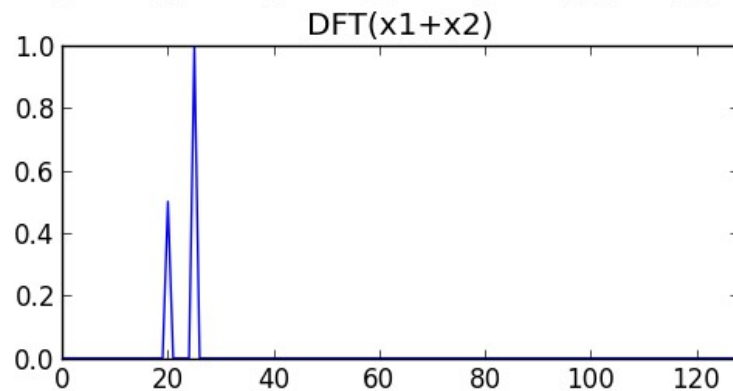
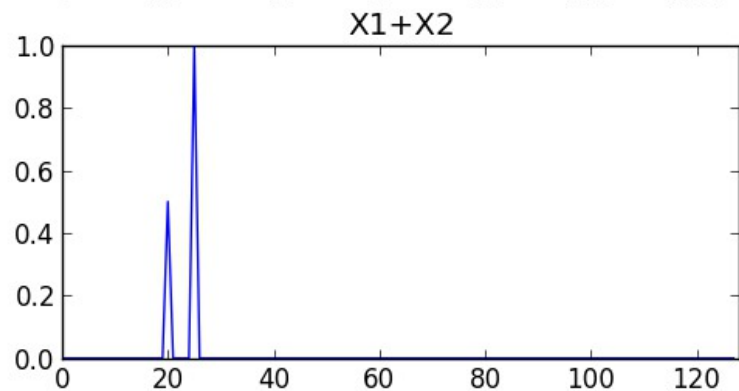
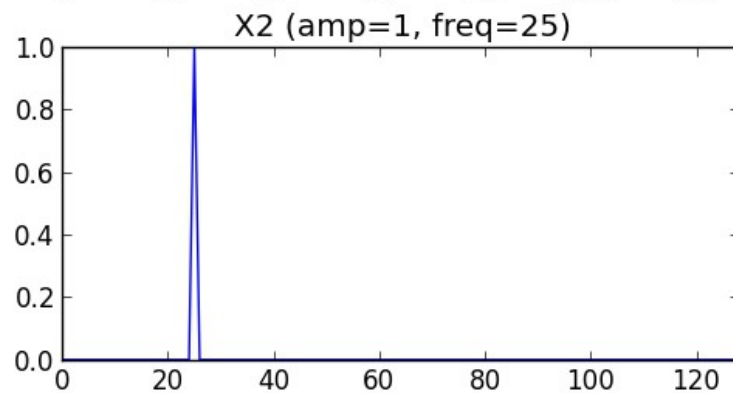
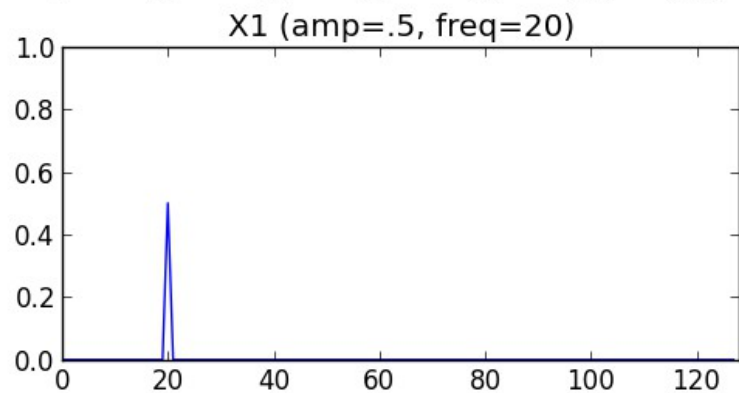
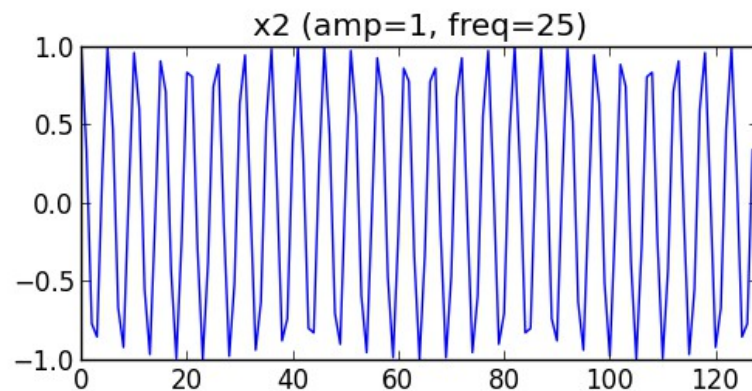
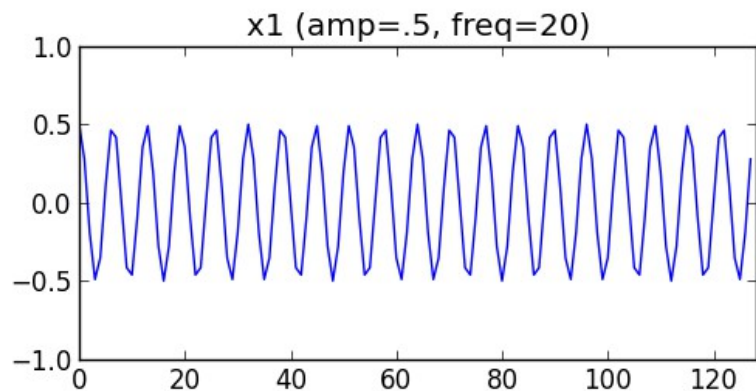
Index

- Linearity
- Shift
- Evenness
- Convolution
- Phase unwrapping
- Zero-padding
- Power & amplitude in dB
- Fast Fourier Transform (FFT)
- FFT and zero-phase

Linearity

$$ax_1 + bx_2 \leftrightarrow aX_1 + bX_2$$

$$\begin{aligned} & DFT(a x_1[n] + b x_2[n]) \\ &= \sum_{n=0}^{N-1} (a x_1[n] + b x_2[n]) e^{-j 2 \pi kn / N} \\ &= \sum_{n=0}^{N-1} a x_1[n] e^{-j 2 \pi kn / N} + \sum_{n=0}^{N-1} b x_2[n] e^{-j 2 \pi kn / N} \\ &= a \sum_{n=0}^{N-1} x_1[n] e^{-j 2 \pi kn / N} + b \sum_{n=0}^{N-1} x_2[n] e^{-j 2 \pi kn / N} \\ &= a X_1[k] + b X_2[k] \end{aligned}$$



```
import matplotlib.pyplot as plt
import numpy as np
from scipy.fftpack import fft, ifft

a = 0.5
b = 1.0
k1 = 20
k2 = 25
N = 128
x1 = a*np.exp(1j*2*np.pi*k1/N*np.arange(N))
x2 = b*np.exp(1j*2*np.pi*k2/N*np.arange(N))

plt.subplot(321)
plt.plot(np.arange(0, N, 1.0), np.real(x1))

plt.subplot(322)
plt.plot(np.arange(0, N, 1.0), np.real(x2))

X1 = fft(x1)
mX1 = abs(X1)/N
plt.subplot(323)
plt.plot(np.arange(0, N, 1.0), mX1)

X2 = fft(x2)
mX2 = abs(X2)/N
plt.subplot(324)
plt.plot(np.arange(0, N, 1.0), mX2)

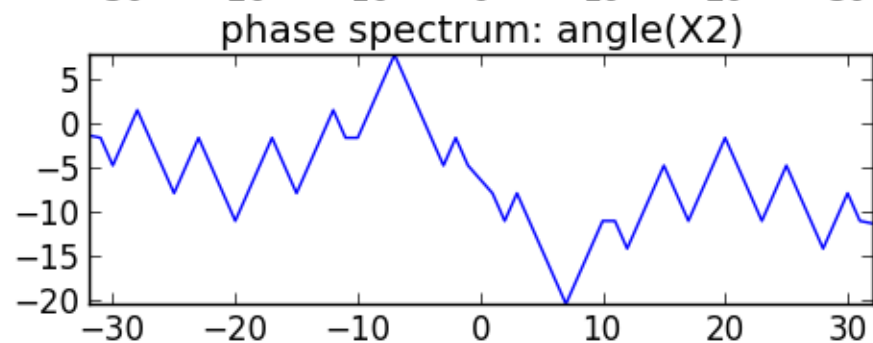
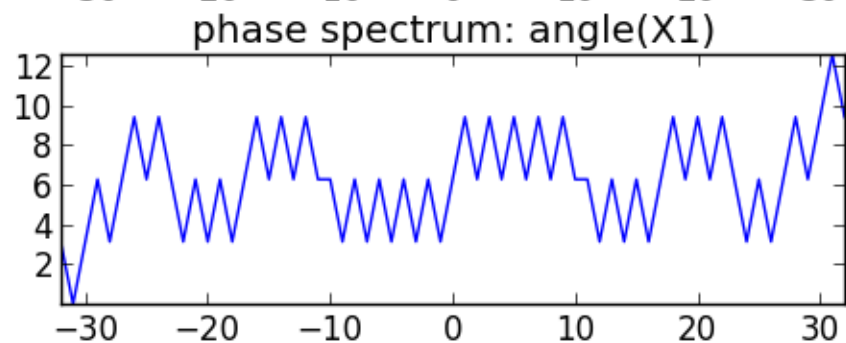
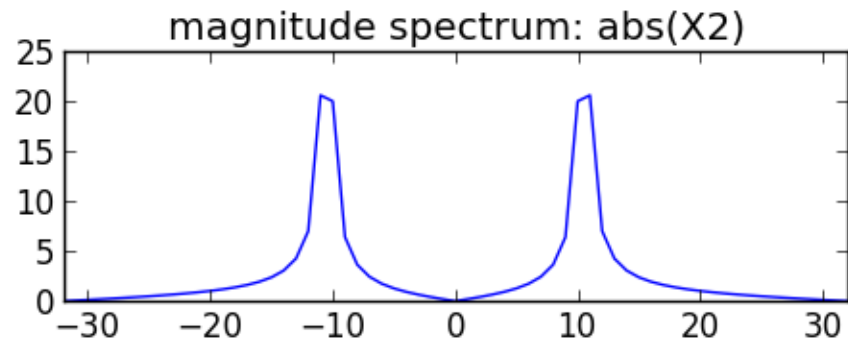
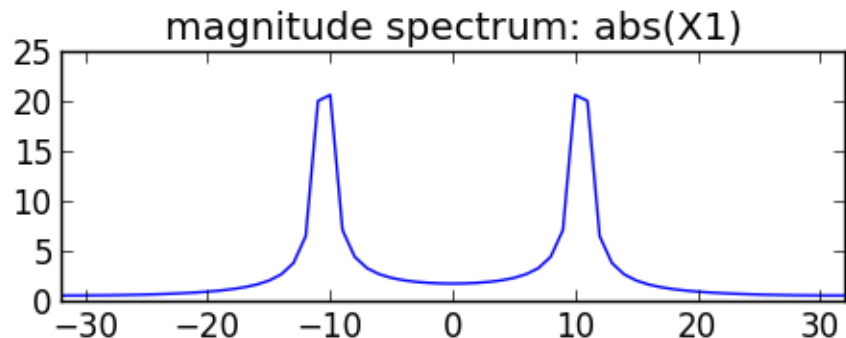
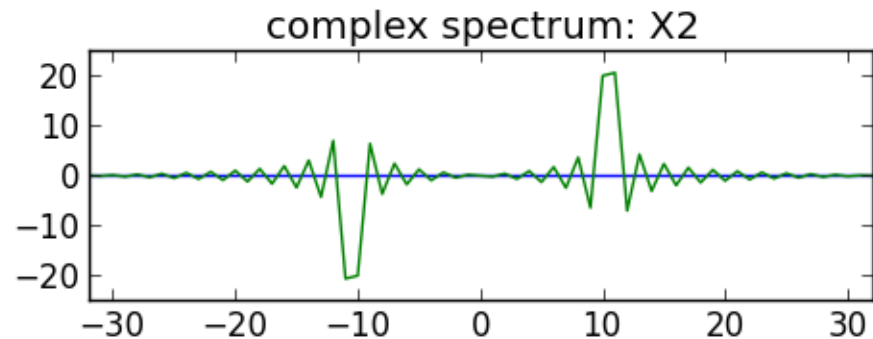
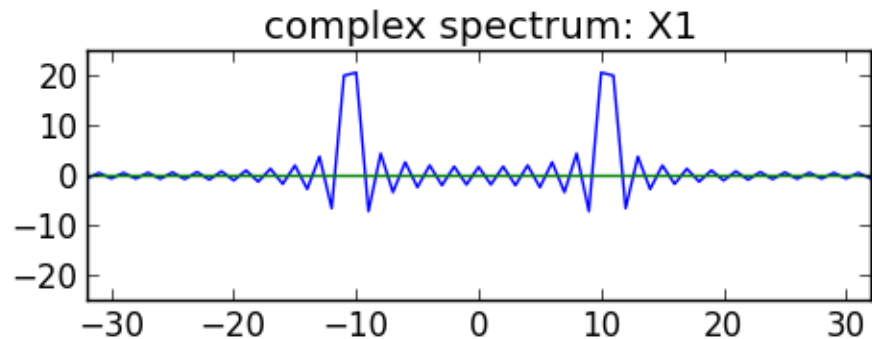
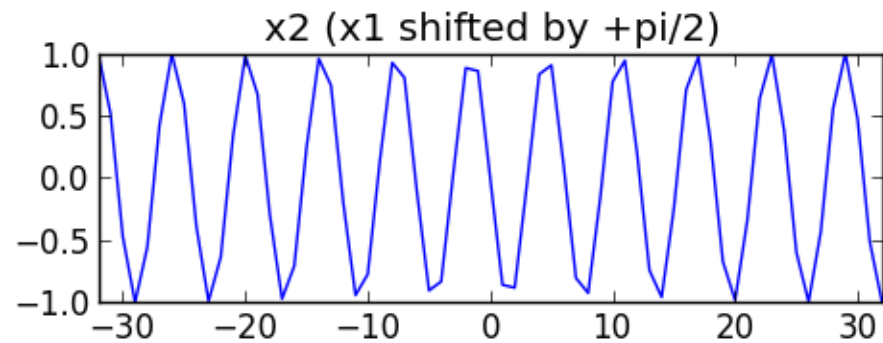
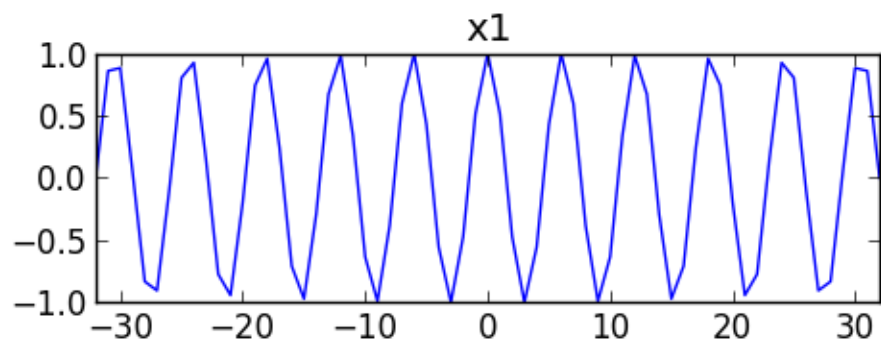
plt.subplot(325)
plt.plot(np.arange(0, N, 1.0), mX1+mX2)

X = fft(x1+x2)
mX= abs(X)/N
plt.subplot(326)
plt.plot(np.arange(0, N, 1.0), mX)
```

Shift

shift \leftrightarrow multiplication by a complex exponential

$$\begin{aligned} & DFT(x[n - n_0]) \\ &= \sum_{n=0}^{N-1} x[n - n_0] e^{-j2\pi kn/N} \\ &= \sum_{m=-n_0}^{N-1-n_0} x[m] e^{-j2\pi k(m+n_0)/N} \quad (m = n - n_0) \\ &= \sum_{m=0}^{N-1} x[m] e^{-j2\pi km/N} e^{-j2\pi kn_0/N} \\ &= e^{-j2\pi kn_0/N} \sum_{m=0}^{N-1} x[m] e^{-j2\pi km/N} \\ &= e^{-j2\pi kn_0/N} X[k] \end{aligned}$$



```

import numpy as np
import matplotlib.pyplot as plt
N = 64
k0 = 10.5

x = np.cos(2*np.pi*k0/N*np.arange(-N/2, N/2+1))
plt.subplot(4,2,1)
plt.plot(np.arange(-N/2, N/2+1), x)

for k in range(-N/2, N/2+1):
    s = np.exp(1j*2*np.pi*k/N*np.arange(-N/2, N/2+1))
    X = np.append(X, sum(x*np.conjugate(s)))
plt.subplot(4,2,3)
plt.plot(np.arange(-N/2, N/2+1), np.real(X))
plt.plot(np.arange(-N/2, N/2+1), np.imag(X))

plt.subplot(4,2,5)
plt.plot(np.arange(-N/2, N/2+1), abs(X))

plt.subplot(4,2,7)
plt.plot(np.arange(-N/2, N/2+1), np.angle(X))

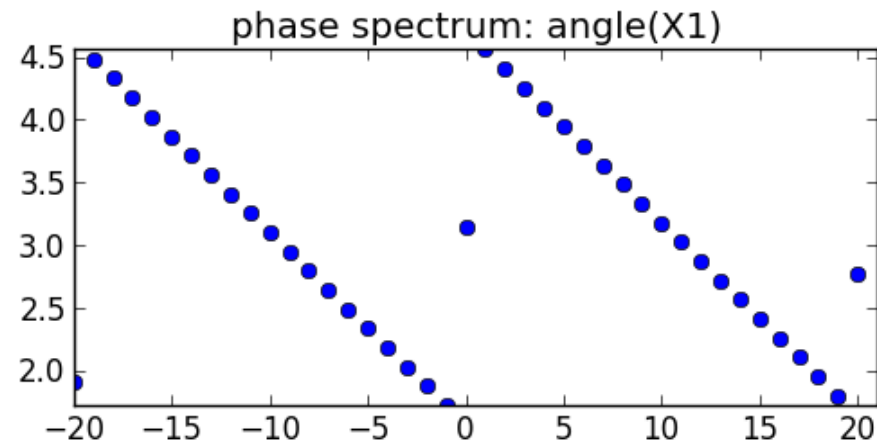
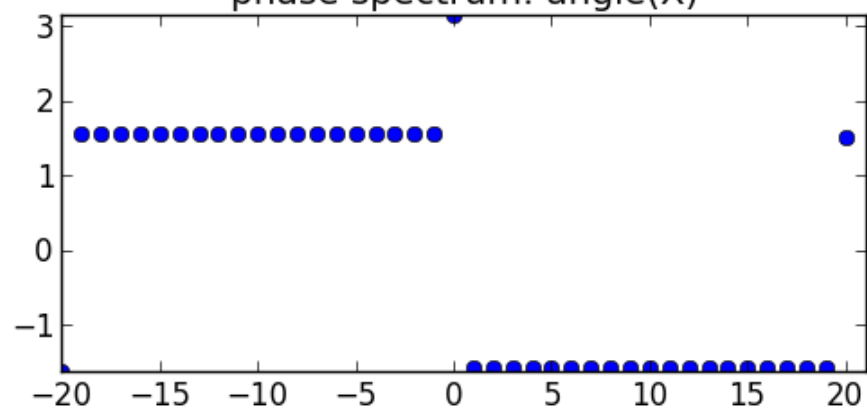
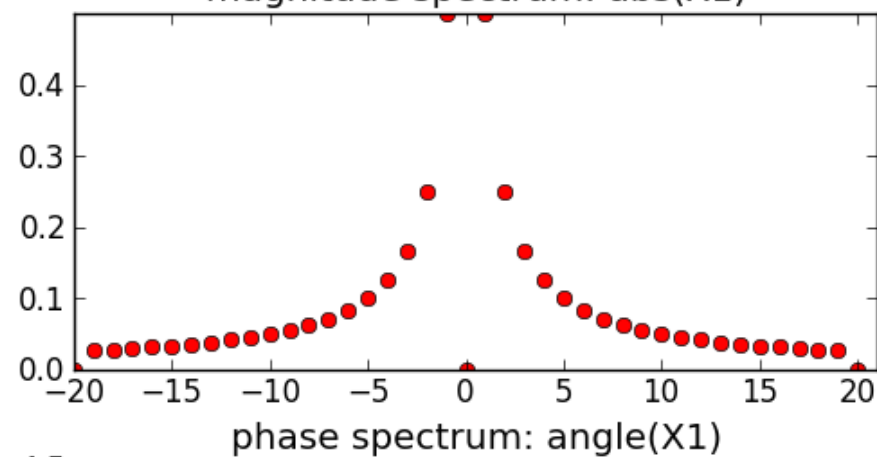
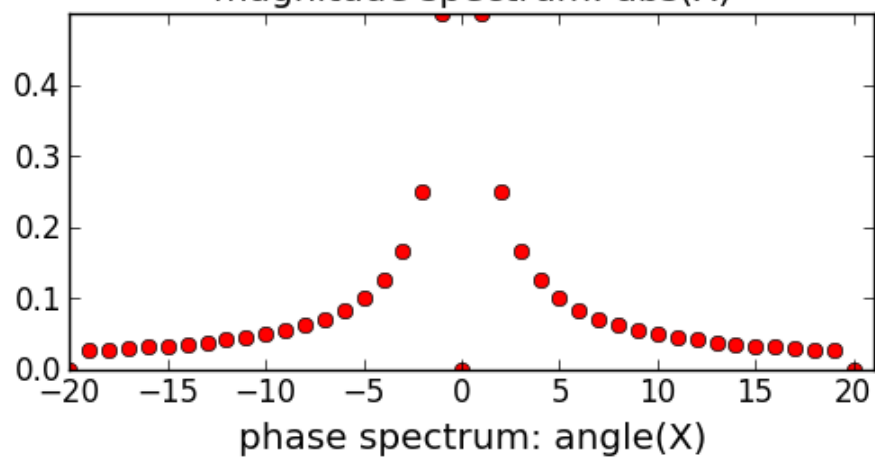
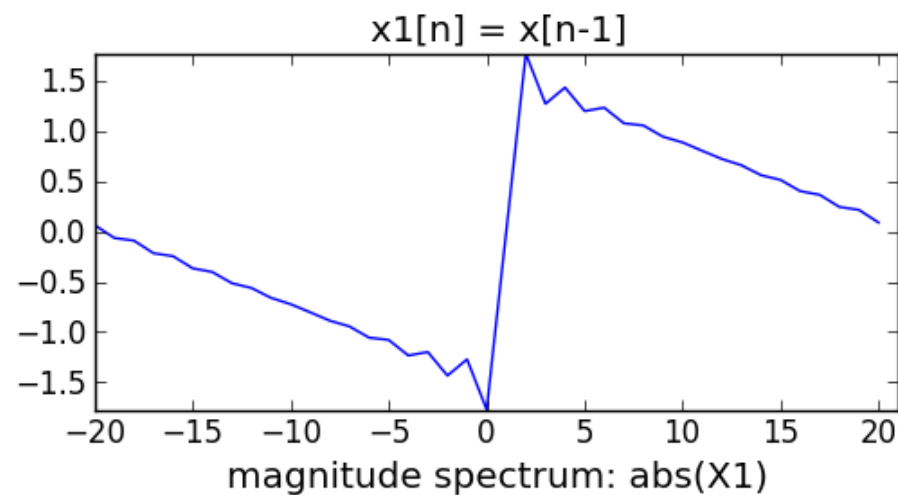
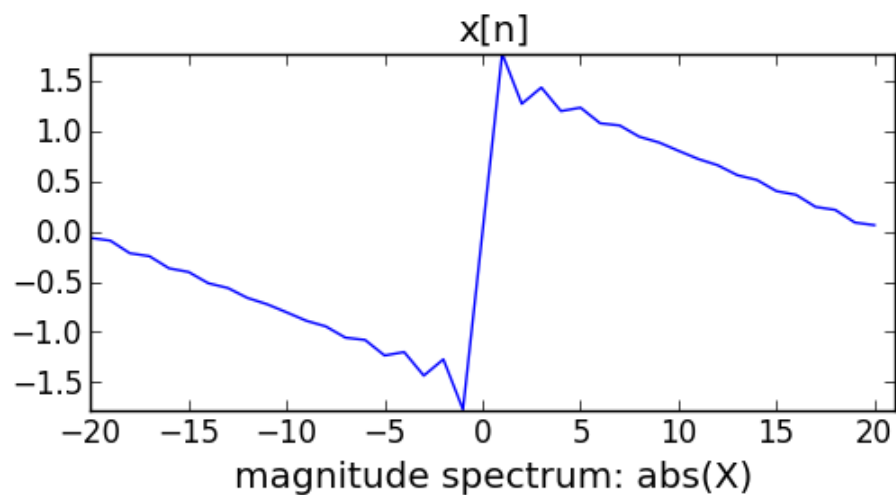
X = np.array([])
x = np.cos(2*np.pi*k0/N*np.arange(-N/2, N/2+1)+np.pi/2)
plt.subplot(4,2,2)
plt.plot(np.arange(-N/2, N/2+1), x)

for k in range(-N/2, N/2+1):
    s = np.exp(1j*2*np.pi*k/N*np.arange(-N/2, N/2+1))
    X = np.append(X, sum(x*np.conjugate(s)))
plt.subplot(4,2,4)
plt.plot(np.arange(-N/2, N/2+1), np.real(X))
plt.plot(np.arange(-N/2, N/2+1), np.imag(X))

plt.subplot(4,2,6)
plt.plot(np.arange(-N/2, N/2+1), abs(X))

plt.subplot(4,2,8)
plt.plot(np.arange(-N/2, N/2+1), np.angle(X))

```

```

import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft, fftshift

N = 41
nsines = 20
for k in range(1,nsines):
    x = x + (1.0/k)*np.sin(2*np.pi*k/N*np.arange(-(N-1)/2.0, (N+1)/2.0))
plt.subplot(3,2,1)
plt.plot(np.arange(-(N-1)/2.0,(N+1)/2), x)

for k in range(N):
    s = np.exp(1j*2*np.pi*k/N*np.arange(-(N-1)/2.0, (N+1)/2.0))
    X = np.append(X, sum(x*np.conjugate(s)))

plt.subplot(3,2,3)
plt.plot(np.arange(-(N-1)/2.0,(N+1)/2), abs(fftshift(X))/N, 'ro')

plt.subplot(3,2,5)
plt.plot(np.arange(-(N-1)/2.0,(N+1)/2), np.angle(fftshift(X)), 'bo')

for k in range(1,nsines):
    x = x + (1.0/k)*np.sin(2*np.pi*k/N*np.arange(-(N-1)/2.0-1, (N+1)/2.0-1))
plt.subplot(3,2,2)
plt.plot(np.arange(-(N-1)/2.0,(N+1)/2), x)

for k in range(N):
    s = np.exp(1j*2*np.pi*k/N*np.arange(-(N-1)/2.0, (N+1)/2.0))
    X = np.append(X, sum(x*np.conjugate(s)))

plt.subplot(3,2,4)
plt.plot(np.arange(-(N-1)/2.0,(N+1)/2), abs(fftshift(X))/N,'ro')

plt.subplot(3,2,6)
plt.plot(np.arange(-(N-1)/2.0,(N+1)/2), np.unwrap(np.angle(fftshift(X))), 'bo')

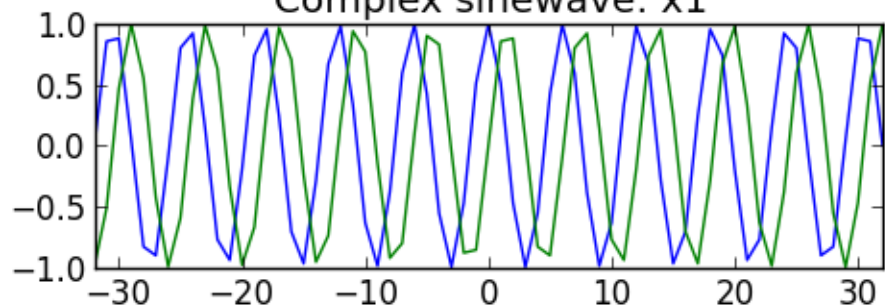
```

Evenness even \leftrightarrow real - valued

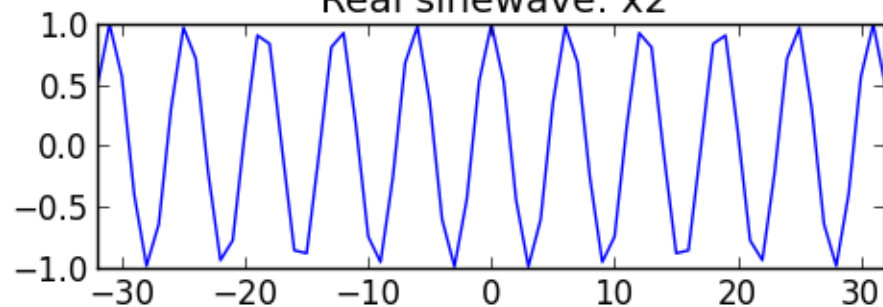
$$\begin{aligned}x_0[n] &= A_0 \cos(2\pi k_0 n/N + \varphi_0) \\&= \frac{1}{2} A_0 e^{j\varphi_0} e^{j2\pi k_0 n/N} + \frac{1}{2} A_0 e^{-j\varphi_0} e^{-j2\pi k_0 n/N}\end{aligned}$$

$$\begin{aligned}DFT(x[n]) &= \sum_{k=0}^{N-1} \left(\frac{1}{2} A_0 e^{j\varphi_0} e^{j2\pi k_0 n/N} + \frac{1}{2} A_0 e^{-j\varphi_0} e^{-j2\pi k_0 n/N} \right) e^{-j2\pi k n/N} \\&= \sum_{k=0}^{N-1} \frac{1}{2} A_0 e^{j\varphi_0} e^{j2\pi k_0 n/N} e^{-j2\pi k n/N} \\&\quad + \sum_{k=0}^{N-1} \frac{1}{2} A_0 e^{-j\varphi_0} e^{-j2\pi k_0 n/N} e^{-j2\pi k n/N} \\&= \frac{1}{2} A_0 e^{j\varphi_0} X_0[k] + \frac{1}{2} A_0 e^{-j\varphi_0} \bar{X}_0[k]\end{aligned}$$

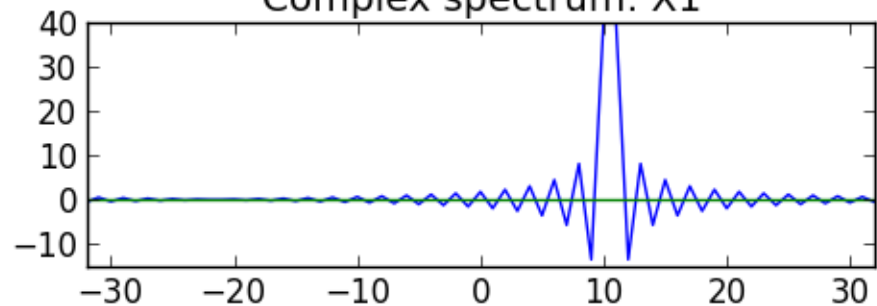
Complex sinewave: x1



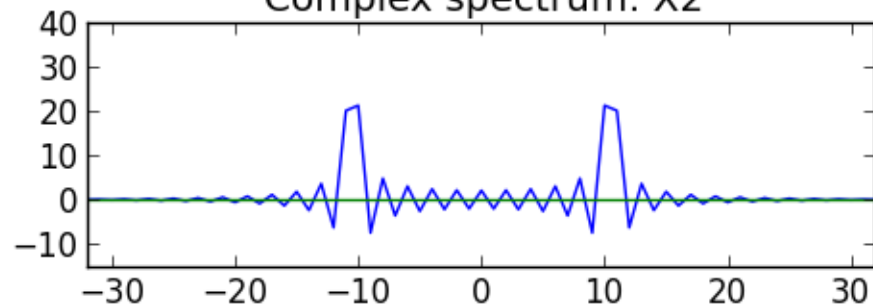
Real sinewave: x2



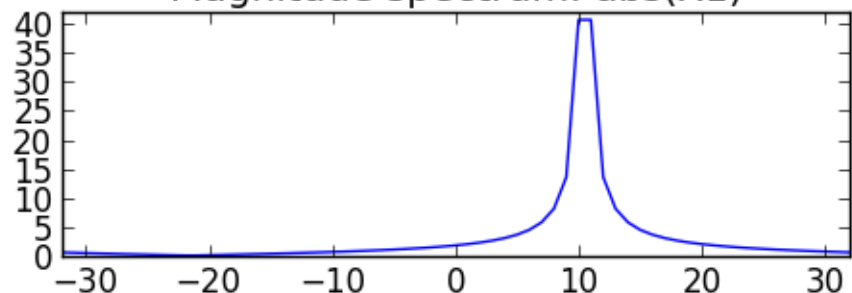
Complex spectrum: X1



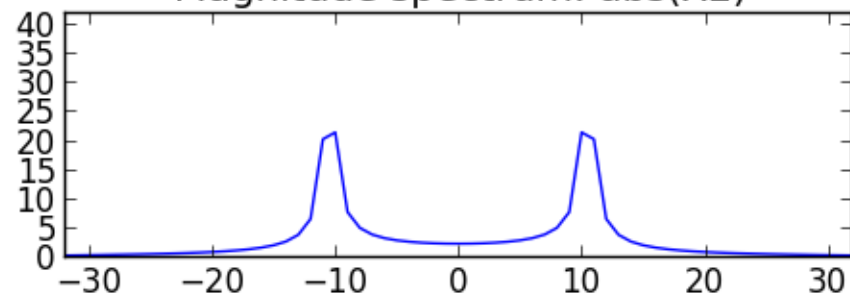
Complex spectrum: X2



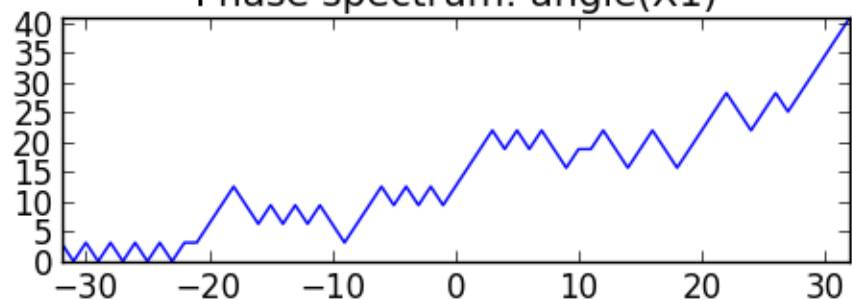
Magnitude spectrum: abs(X1)



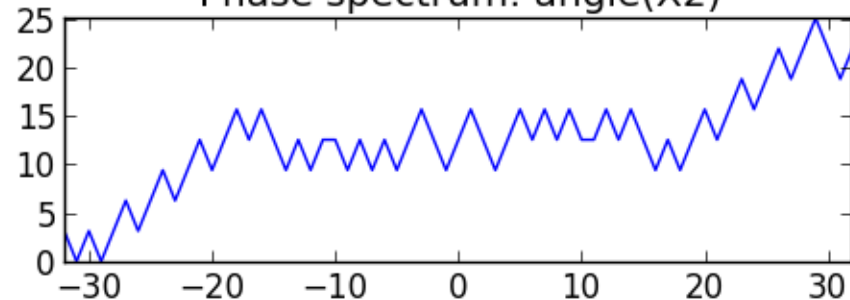
Magnitude spectrum: abs(X2)



Phase spectrum: angle(X1)



Phase spectrum: angle(X2)



```

import numpy as np
import matplotlib.pyplot as plt

N = 64
k0 = 10.5
x = np.exp(1j*2*np.pi*k0/N*np.arange(-N/2, N/2+1))

plt.subplot(4,2,1)
plt.plot(np.arange(-N/2, N/2+1), np.real(x))
plt.plot(np.arange(-N/2, N/2+1), np.imag(x))

for k in range(-N/2, N/2+1):
    s = np.exp(1j*2*np.pi*k/N*np.arange(-N/2, N/2+1))
    X = np.append(X, sum(x*np.conjugate(s)))
plt.subplot(4,2,3)
plt.plot(np.arange(-N/2, N/2+1), np.real(X))
plt.plot(np.arange(-N/2, N/2+1), np.imag(X))

plt.subplot(4,2,5)
plt.plot(np.arange(-N/2, N/2+1), abs(X))

plt.subplot(4,2,7)
plt.plot(np.arange(-N/2, N/2+1), np.angle(X))

x = np.cos(2*np.pi*k0/N*np.arange(-N/2, N/2+1))
plt.subplot(4,2,2)
plt.plot(np.arange(-N/2, N/2+1), x)

for k in range(-N/2, N/2+1):
    s = np.exp(1j*2*np.pi*k/N*np.arange(-N/2, N/2+1))
    X = np.append(X, sum(x*np.conjugate(s)))
plt.subplot(4,2,4)
plt.plot(np.arange(-N/2, N/2+1), np.real(X))
plt.plot(np.arange(-N/2, N/2+1), np.imag(X))

plt.subplot(4,2,6)
plt.plot(np.arange(-N/2, N/2+1), abs(X))

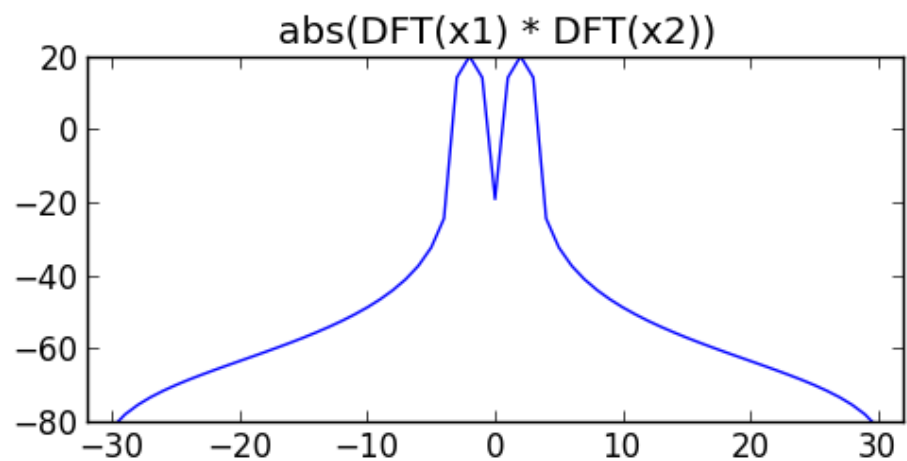
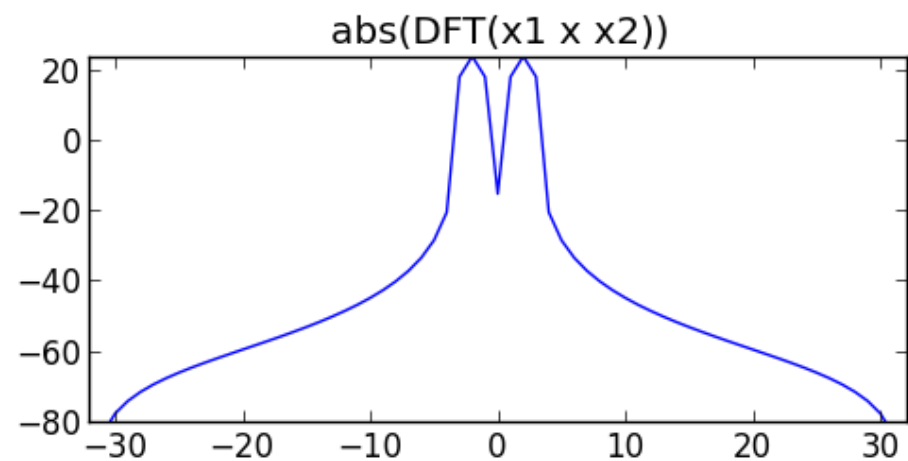
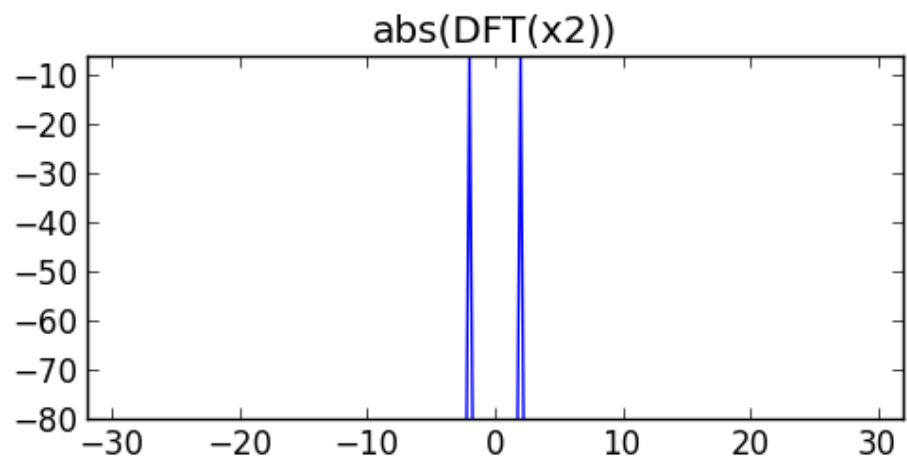
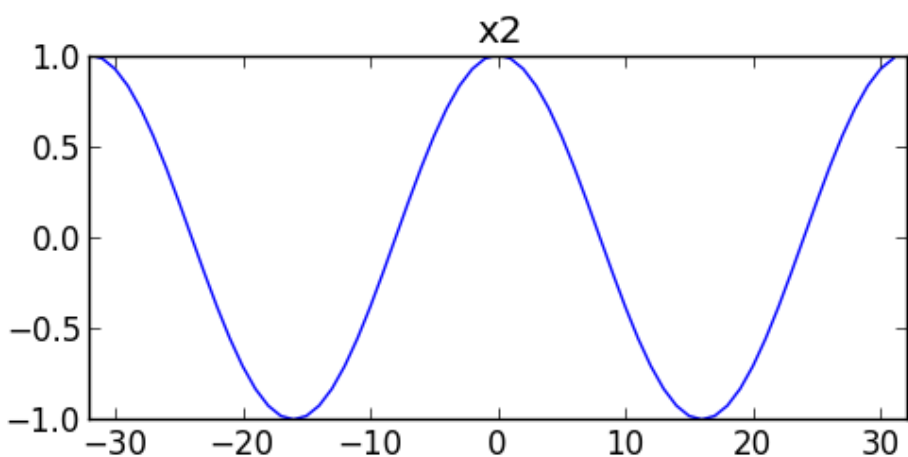
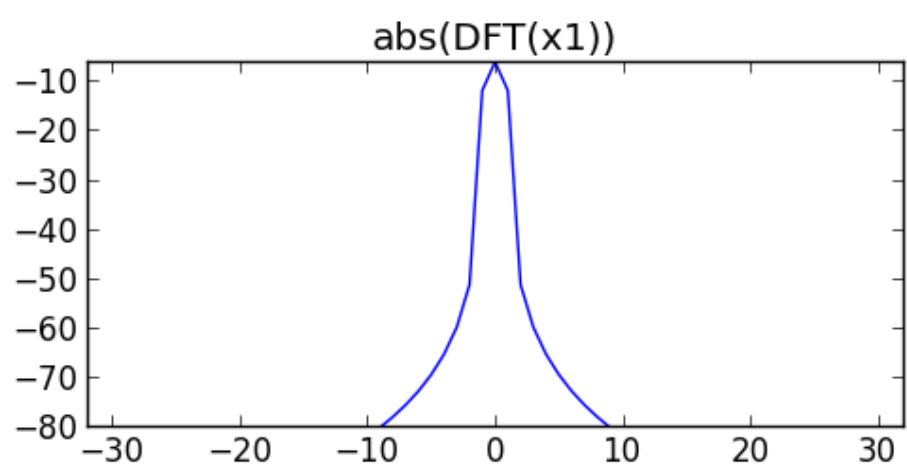
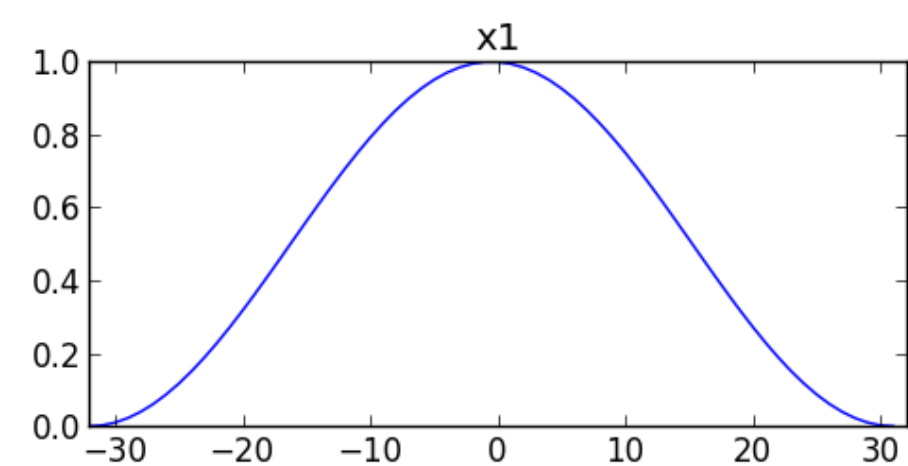
plt.subplot(4,2,8)
plt.plot(np.arange(-N/2, N/2+1), np.angle(X))

```

Convolution

convolution \leftrightarrow point - by - point multiplication

$$\begin{aligned} & DFT(x_1[n] * x_2[n]) \\ &= \sum_{n=0}^{N-1} (x_1[n] * x_2[n]) e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] x_2[n-m] e^{-j2\pi kn/N} \\ &= \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x_2[n-m] e^{-j2\pi kn/N} \\ &= \left(\sum_{m=0}^{N-1} x_1[m] e^{-j2\pi km/N} \right) X_2[k] \\ &= X_1[k] X_2[k] \end{aligned}$$



```

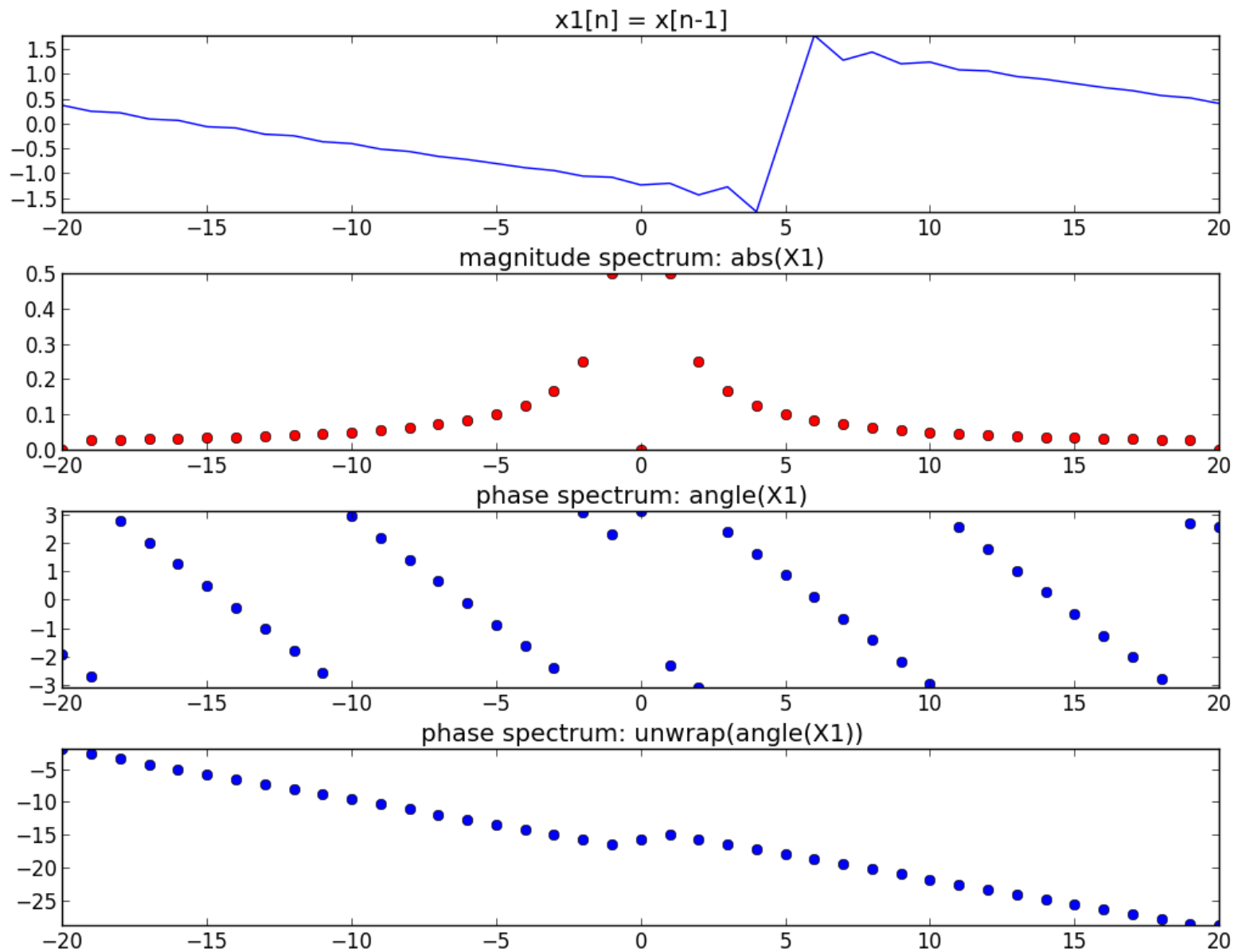
import matplotlib.pyplot as plt
import numpy as np
from scipy.fftpack import fft, fftshift

M = 64
N = 512
x1 = np.hanning(M)
x2 = np.cos(2*np.pi*2/M*np.arange(M))
y1 = x1*x2
mY1 = 20 * np.log10(np.abs(fftshift(fft(y1, M))))
plt.subplot(3,2,1)
plt.plot(x1)
plt.subplot(3,2,3)
plt.plot(x2)
plt.subplot(3,2,5)
plt.plot(mY1)

Y2 = np.convolve(fftshift(fft(x1, M)), fftshift(fft(x2, M)))
mY2 = 20 * np.log10(np.abs(Y2))
mX1 = 20 * np.log10(np.abs(fftshift(fft(x1, M)))/M)
mX2 = 20 * np.log10(np.abs(fftshift(fft(x2, M)))/M)
plt.subplot(3,2,2)
plt.plot(mX1)
plt.subplot(3,2,4)
plt.plot(mX2)
plt.subplot(3,2,6)
plt.plot(mY2[M/2:M+M/2])

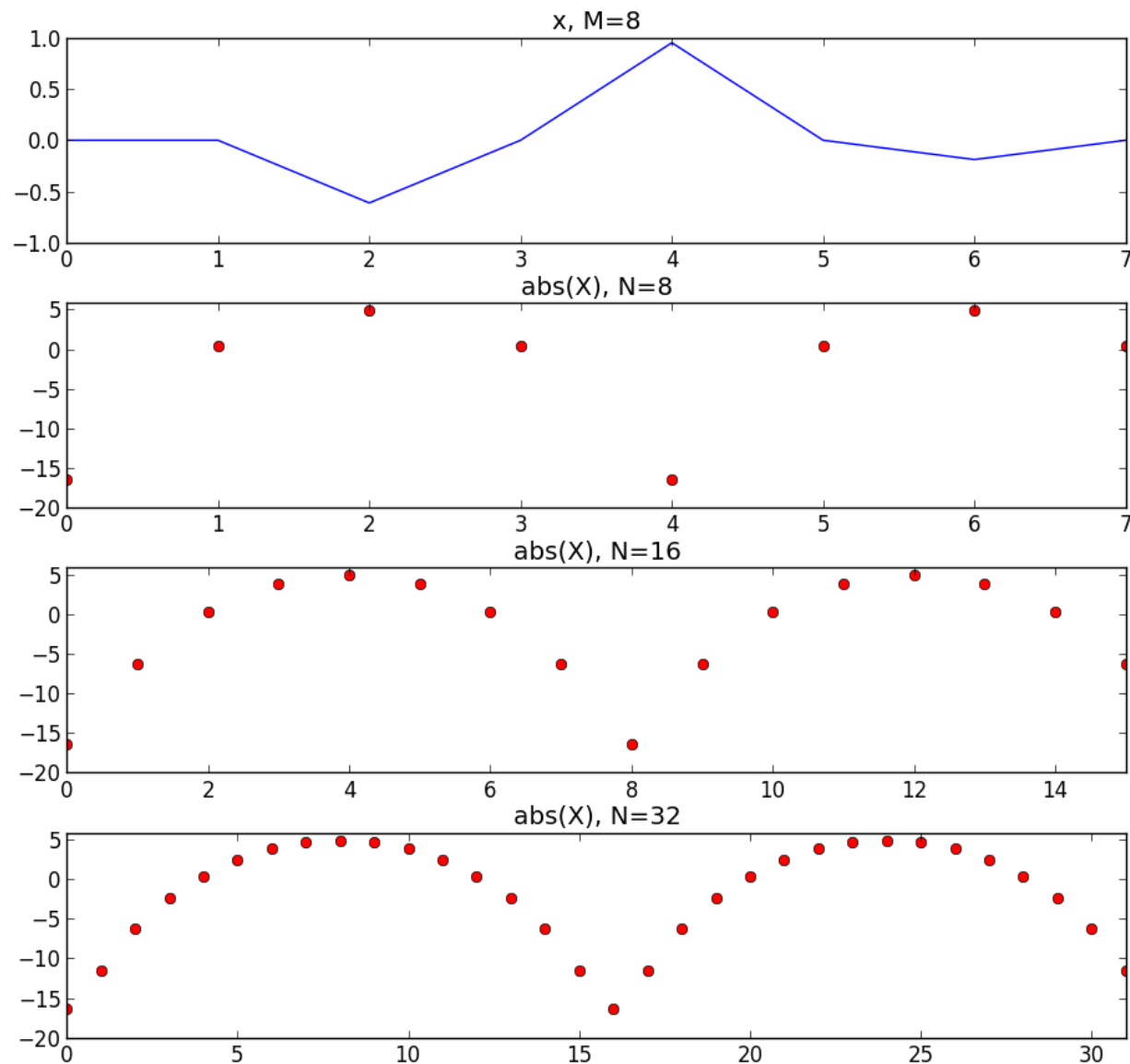
```


Phase unwrapping



Zero-padding

zero padding \leftrightarrow interpolation



```
import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft, fftshift

M = 8
N1 = 8
N2 = 16
N3 = 32
x = np.cos(2*np.pi*2/M*np.arange(M)) * np.hanning(M)

plt.subplot(4,1,1)
plt.plot(x, 'b')

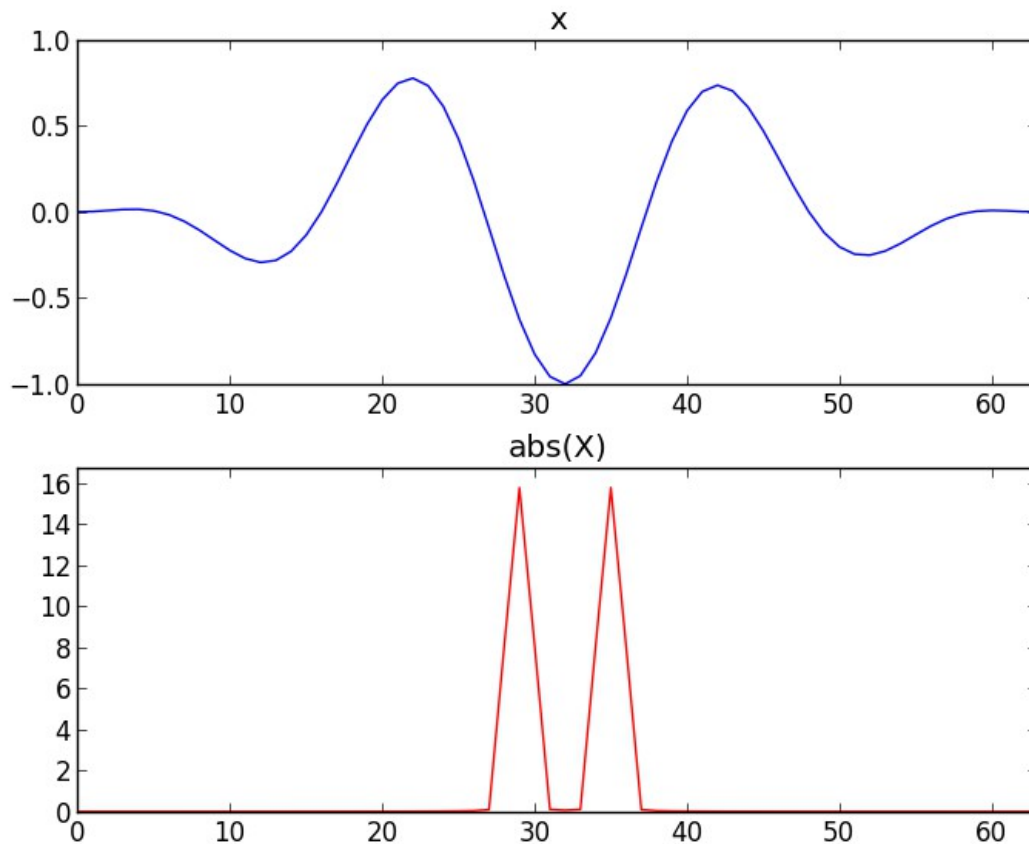
mX = 20 * np.log10(np.abs(fftshift(fft(x, N1))))
plt.subplot(4,1,2)
plt.plot(mX, 'ro')

mX = 20 * np.log10(np.abs(fftshift(fft(x, N2))))
plt.subplot(4,1,3)
plt.plot(mX, 'ro')

mX = 20 * np.log10(np.abs(fftshift(fft(x, N3))))
plt.subplot(4,1,4)
plt.plot(mX, 'ro')
```

Power

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$



$$\sum_{n=0}^{N-1} |x(n)|^2 = 11.81182$$

$$\frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 = 11.81182$$

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft, fftshift

M= 64
N = 64
x = np.cos(2*np.pi*3/M*np.arange(M)) * np.hanning(M)

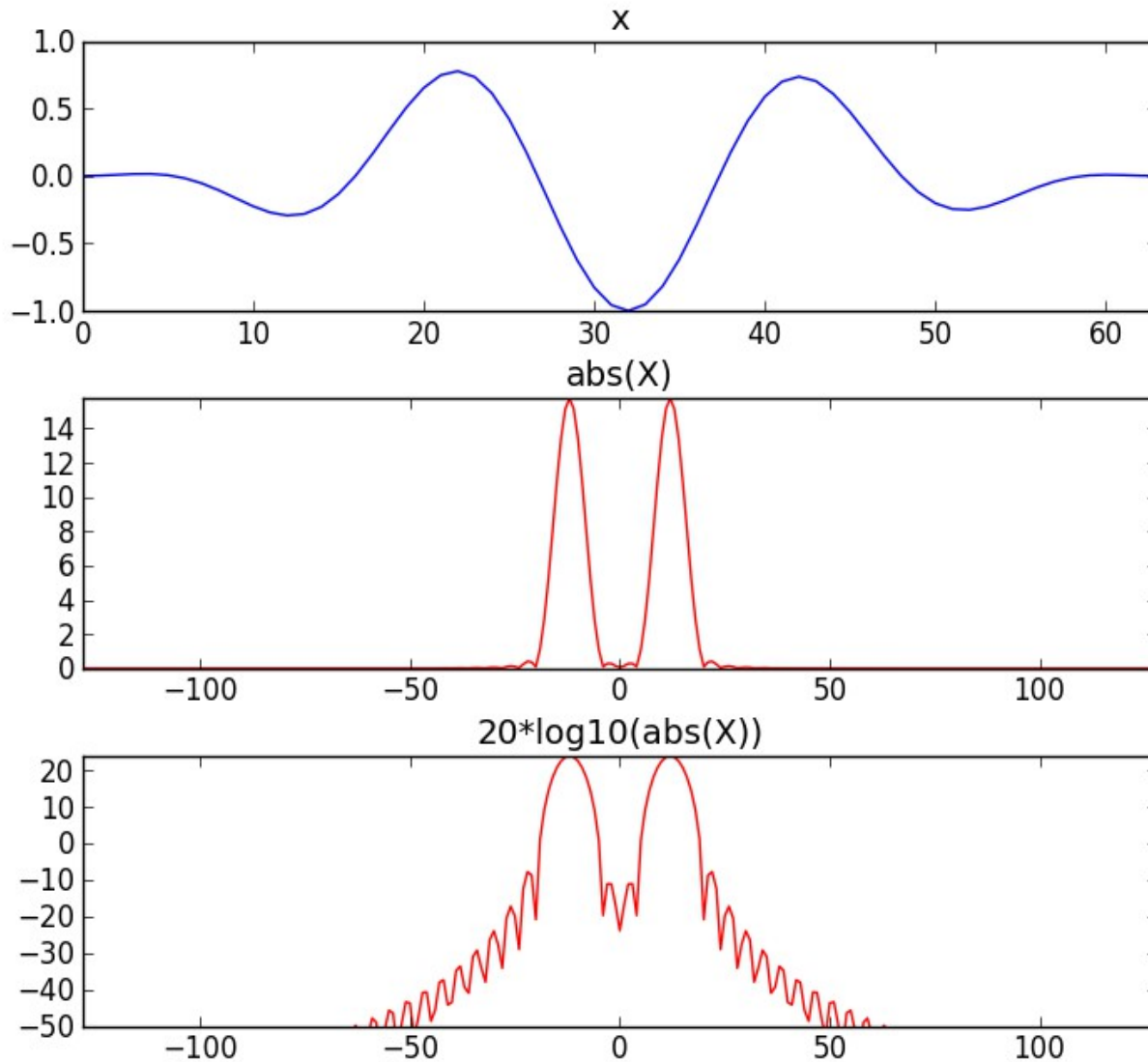
plt.subplot(2,1,1)
plt.plot(x, 'b')

powerx = sum(np.abs(x)**2)
print powerx

mX = np.abs(fftshift(fft(x, N)))
plt.subplot(2,1,2)
plt.plot(mX, 'r')

powerX = sum(mX**2) / N
print powerX
```

Amplitude in Decibels



```
import numpy as np
import matplotlib.pyplot as plt
from scipy.fftpack import fft, fftshift

M= 64
N = 256
x = np.cos(2*np.pi*3/M*np.arange(M)) * np.hanning(M)

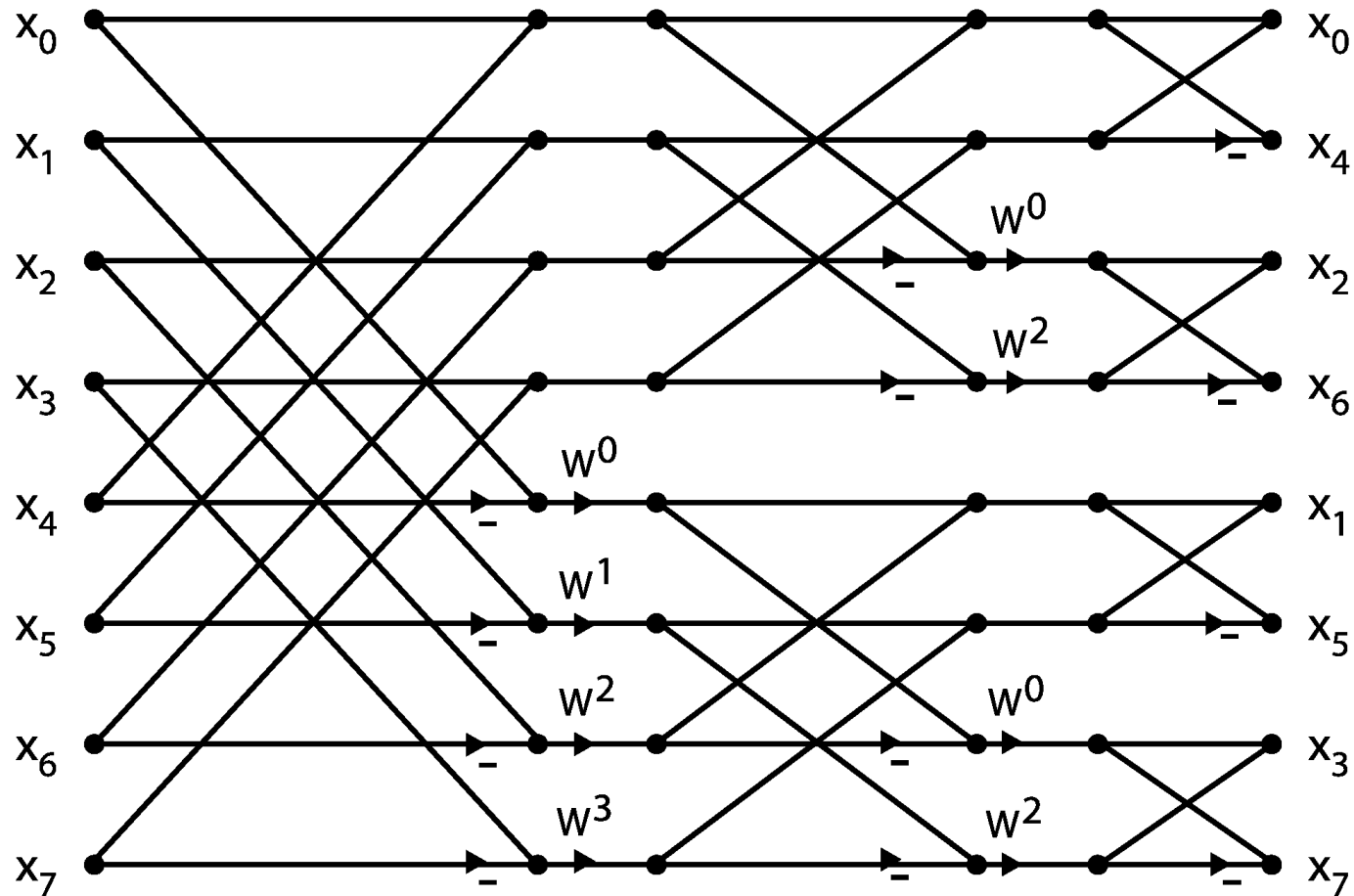
plt.subplot(3,1,1)
plt.plot(x, 'b')

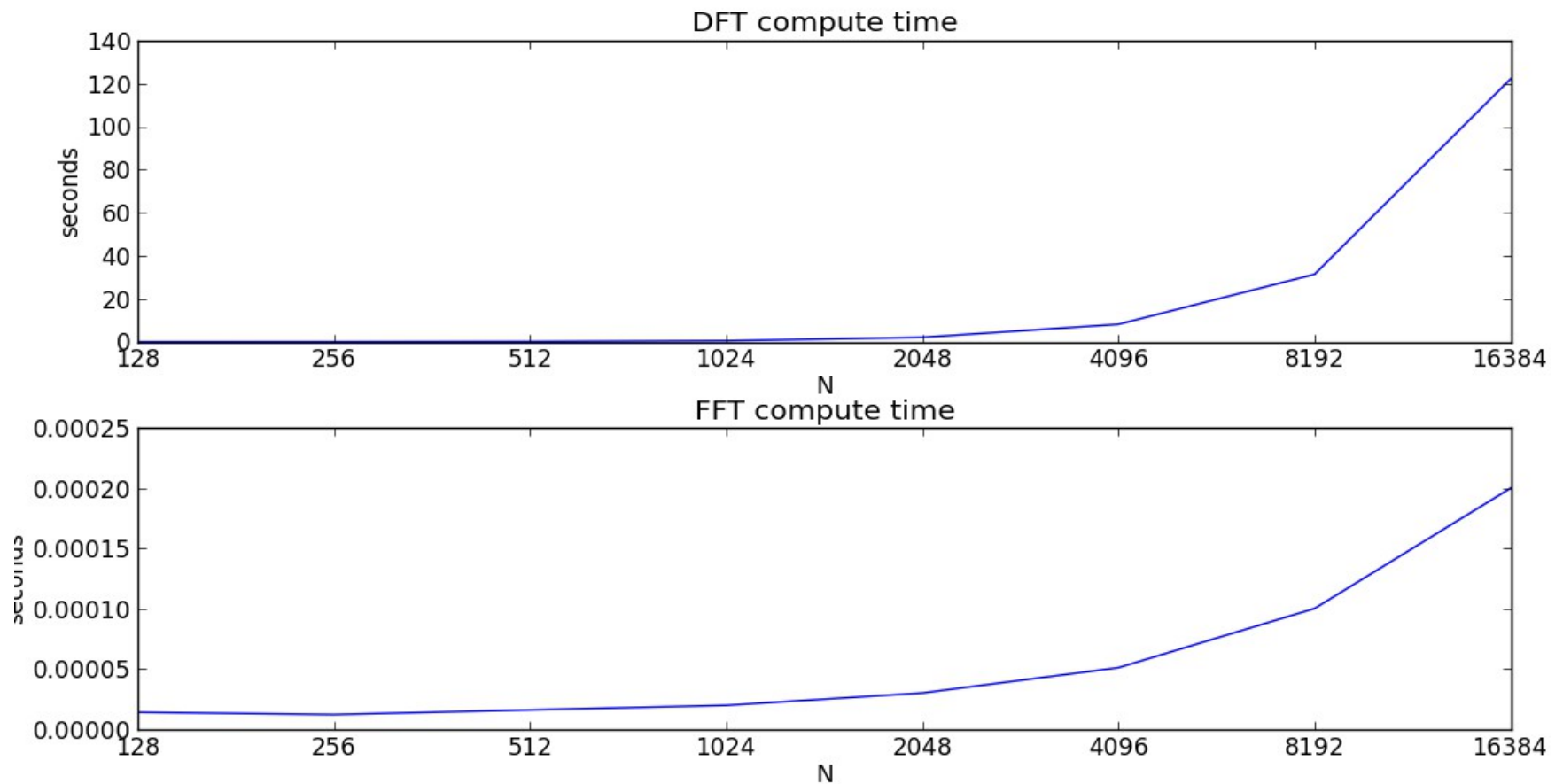
mX = np.abs(fftshift(fft(x, N)))
plt.subplot(3,1,2)
plt.plot(np.arange(-N/2,N/2,1), mX, 'r')

mX = 20 * np.log10(mX)
plt.subplot(3,1,3)
plt.plot(np.arange(-N/2,N/2,1), mX, 'r')
```

Fast Fourier Transform

The most common version uses the Cooley-Tukey algorithm. It breaks down recursively the DFT of a power of 2 size into two pieces of size $N/2$.

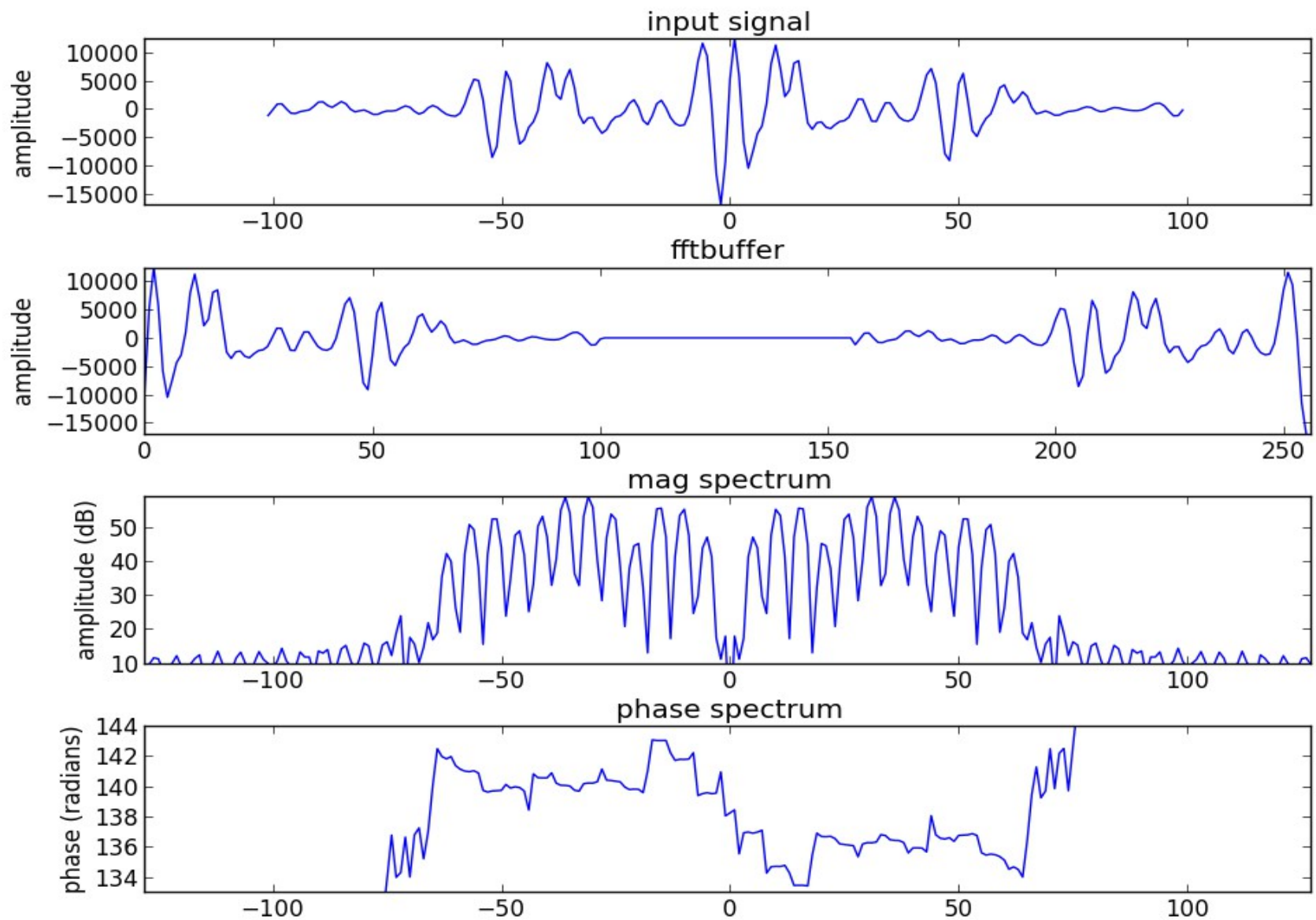




```
Ns = 2*np.arange(7,15)
for N in Ns:
    str_time = time.time()
    for k in range(N):
        s = np.exp(1j*2*np.pi*k/N*np.arange(N))
        X = np.append(X, sum(x*np.conjugate(s)))
        timeDFT = np.append(timeDFT, time.time()-str_time)

for N in Ns:
    str_time = time.time()
    X = fft(x)
    timeFFT = np.append(timeFFT, time.time()-str_time)
```

FFT and zero-phase



```
import matplotlib.pyplot as plt
import numpy as np
from scipy.io.wavfile import read
from scipy.fftpack import fft, fftshift

(fs, x) = read('oboe.wav')
N = 256
M = 201
hN = N/2
hM = (M+1)/2
start = .8*fs
xw = x[start-hM:start+hM-1] * np.hamming(M)
plt.subplot(411)
plt.plot(np.arange(-hM, hM-1), xw)

fftbuffer = np.zeros(N)
fftbuffer[:hM] = xw[hM-1:]
fftbuffer[N-hM+1:] = xw[:hM-1]
plt.subplot(412)
plt.plot(np.arange(0, N), fftbuffer)

X = fftshift(fft(fftbuffer))
mX = 20 * np.log10(abs(X)/N)
pX = np.unwrap(np.angle(X))
plt.subplot(413)
plt.plot(np.arange(-hN, hN), mX)

plt.subplot(414)
plt.plot(np.arange(-hN, hN), pX)
```

References

- <https://ccrma.stanford.edu/~jos/mdft/>
- https://en.wikipedia.org/wiki/Fast_Fourier_transform
- Full code of plots and accompanying labs available at: <https://github.com/MTG/sms-tools>

Credits

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