Domain Specific Languages of Mathematics
Exam 2016–03–15

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Results Announced within 19 days (by Monday 2016-04-04)

Exam check Mo 2016-04-12 and Tu 13. Both at 12.30-12.55 in EDIT 5468.

Aids One textbook of your choice (e.g., Adams and Essex, or Rudin). No printouts, no lecture notes, no notebooks, etc.

Grades 3: 40p, 4: 60p, 5: 80p, max: 100p

Remember to write legibly. Good luck!

1. [30pts] A *lattice* is a set `L` together with two operations $\vee$ and $\wedge$ (usually pronounced "sup" and "inf") such that

   - $\vee$ and $\wedge$ are associative:

     ```
     ∀ x, y, z ∈ L        (x ∨ y) ∨ z = x ∨ (y ∨ z)
     ∀ x, y, z ∈ L        (x ∧ y) ∧ z = x ∧ (y ∧ z)
     ```

   - $\vee$ and $\wedge$ are commutative:

     ```
     ∀ x, y ∈ L            x ∨ y = y ∨ x
     ∀ x, y ∈ L            x ∧ y = y ∧ x
     ```

   - $\vee$ and $\wedge$ satisfy the *absorption laws*:

     ```
     ∀ x, y ∈ L            x ∨ (x ∧ y) = x
     ∀ x, y ∈ L            x ∧ (x ∨ y) = x
     ```

   i. Define a type class `Lattice` that corresponds to the lattice structure.

   ii. Define a datatype for the language of lattice expressions and define a `Lattice` instance for it.

   iii. Find two other instances of the `Lattice` class.

   iv. Define a general evaluator for `Lattice` expressions on the basis of an assignment function.

   v. Specialise the evaluator to the two `Lattice` instances defined at point iii. Take three lattice expressions, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 6pts.

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2. [20pts] Consider the following text from Mac Lane's *Mathematics: Form and Function* (page 182):

   > In these cases one tries to find not the values of `x` which make a given function `y = f(x)` a minimum, but the values of a given function `f(x)` which make a given quantity a minimum. Typically, that quantity is usually measured by an integral whose integrand is some expression `F` involving both `x`, values of the function `y = f(x)` at interest and the values of its derivatives - say an integral
   >
   > $$\int_a^b F(y, y', x)dx, \quad y = f(x).$$

   Give the types of the variables involved (`x`, `y`, `y'`, `f`, `F`, `a`, `b`) and the type of the four-argument integration operator:

   $$\int_\cdot^\cdot \cdot d\cdot$$

3. [25pts] Consider the following differential equation:

$$f''\,t - 2 * f'\,t + f\,t = e^{2*t}, \quad f\,0 = 2, \quad f'\,0 = 3$$

i. [10pts] Solve the equation assuming that `f` can be expressed by a power series `fs`, that is, use `deriv` and `integ` to compute `fs`. What are the first three coefficients of `fs`?

ii. [15pts] Solve the equation using the Laplace transform. You should need only one formula (and linearity):

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

4. [25pts] Consider the classical definition of continuity:

> *Definition:* Let $X \subseteq \mathbb{R}$, and $c \in X$. A function $f : X \to \mathbb{R}$ is *continuous at* c if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every x in the domain of f, if $|x - c| < \delta$, then $|f x - f c| < \varepsilon$.

i. [5pts] Write the definition formally, using logical connectives and quantifiers.

ii. [10pts] Introduce functions and types to simplify the definition.

iii. [10pts] Prove the following proposition: If `f` is continuous at `c`, and `g` is continuous at `f c`, then `g ∘ f` is continuous at `c`.