

## DSLsofMath 2017: Assignment 2

### Optimisation using Newton's method

1. The evaluation of the second derivative is given by

$$eval'' = eval' \circ derive = eval \circ derive \circ derive$$

- (a) Show that  $eval''$  is not a homomorphism.
  - (b) What datatype is needed to have a homomorphism from  $Expr$  in this case? Show this for the case of multiplication.
  - (c) What datatype is needed for the homomorphism that is analogous to apply? Give instances of the numerical classes ( $Num$ ,  $Fractional$ ,  $Floating$ ) for this datatype, as well as an embedding of  $Const$  and  $Id$ .
2. Newton's method allows us to find zeros of a large class of functions in a given interval. The following description of Newton's method follows Bird and Wadler [1988], page 23:

```
newton :: (Double → Double) → Double → Double → Double
newton f ε x = if |fx| < ε
               then x
               else if fx' ≠ 0 then newton f ε next
                       else newton f ε (x + ε)
  where fx    = f x
        fx'   = undefined -- f' x (derivative of f at x)
        next  = x - (fx / fx')
```

Implement Newton's method, using the datatype you introduced above for computing the derivatives. In other words, use the code above to implement

```
newton :: (FDD Double → FDD Double) → Double → Double → Double
```

where  $FDD\ a$  is the datatype from above, in order to obtain the appropriate value for  $f' x$ . Test your implementation on the following functions:

```
test0 = x^2           -- one (double) zero, in 0
test1 = x^2 - 1       -- two zeros, in -1 and 1
test2 = sin           -- many, many zeros ( $n * \pi$ )
test3 n x y = y^n - fddId x -- nth root of x
-- where fddId is the embedding of Id
```

For each of these functions, apply Newton's method to a number of starting points from a sensible interval. For example:

```
fmap (newton test1 0.001) [-2.0, -1.5 .. 2.0]
```

but be aware that the method might not always converge!

3. We can find the optima of a twice-differentiable function on an interval by finding the zeros of its derivative on that interval, and checking the second derivative. If  $x_0$  is a zero of  $f'$ , then
  - if  $f'' x_0 < 0$ , then  $x_0$  is a maximum
  - if  $f'' x_0 > 0$ , then  $x_0$  is a minimum
  - if  $f'' x_0 = 0$ , then, if  $f'' (x_0 - \epsilon) * f'' (x_0 + \epsilon) < 0$  (i.e.,  $f''$  changes its sign in the neighbourhood of  $x_0$ ),  $x_0$  is an inflection point (not an optimum)
  - otherwise, we don't know

Use Newton's method to find the optima of the test functions from point 2. That is, implement a function

*optim :: (FDD Double → FDD Double) → Double → Double → Result Double*

so that *optim f*  $\epsilon$   $x$  uses Newton's method to find a zero of  $f'$  starting from  $x$ . If  $y$  is the result (i.e.  $f' y$  is within  $\epsilon$  of 0), then check the second derivative, returning *Maximum y* if  $f'' y < 0$ , *Minimum y* if  $f'' y > 0$ , and *Dunno y* if  $f'' = 0$ .

As before, use several starting points.

Hint: you might want to modify the code you've written for Newton's method at point 2.

## Formalities

**Submission:** Assignments are to be submitted via Fire:

<https://ds1s-lp3-17.frs.cse.chalmers.se/login>

**Deadline:** Tuesday, 2017-02-28, 23:59.

**Grading:** Discussions with each of the teams during the exercises session of Friday, 2017-03-03.

## References

R. Bird and P. Wadler. *Introduction to Functional Programming, 1988*. Prentice-Hall, Englewood Cliffs, NJ, 1988.