

Domain Specific Languages of Mathematics

Practice Exam

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Results Announced within ?

Aids One textbook of your choice (e.g., Adams and Essex, or Rudin). No printouts, no lecture notes, no notebooks, etc.

Grades 3: 40p, 4: 60p, 5: 80p, max: 100p

Remember to write legibly. Good luck!

1. [30pts]

A vector space over \mathbb{R} is a set V together with a constant (or nullary) operation $0 : V$, an operation $+$: $V \rightarrow V \rightarrow V$, and an *external* operation \cdot : $\mathbb{R} \rightarrow V \rightarrow V$, such that

- 0 is the unit of +:

$$\forall v \in V \quad v + 0 = 0 + v = v$$

- + is associative:

$$\forall v_1, v_2, v_3 \in V \quad (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

- + is invertible:

$$\forall v \in V \quad \exists (-v) \in V \quad v + (-v) = (-v) + v = 0$$

- + is commutative:

$$\forall v_1, v_2 \in V \quad v_1 + v_2 = v_2 + v_1$$

Remarks:

- we usually denote $v_1 + (-v_2) = v_1 - v_2$
- the first two conditions say that $(V, +, 0)$ is a *monoid*
- the first three conditions say that $(V, +, 0)$ is a *group*
- the four conditions say that $(V, +, 0)$ is a *commutative group*

- \cdot is associative

$$\forall x_1, x_2 \in \mathbb{R}, v \in V \quad x_1 \cdot (x_2 \cdot v) = (x_1 * x_2) \cdot v$$

Remark: $*$ denotes the standard multiplication in \mathbb{R}

- 1 is a unit of \cdot :

$$\forall v \in V \quad 1 \cdot v = v$$

- \cdot distributes over +:

$$\forall x \in \mathbb{R}, v_1, v_2 \in V \quad x \cdot (v_1 + v_2) = x \cdot v_1 + x \cdot v_2$$

- \cdot distributes over +

$$\forall x_1, x_2 \in \mathbb{R}, v \in V \quad (x_1 + x_2) \cdot v = x_1 \cdot v + x_2 \cdot v$$

- Define a type class **Vector** that corresponds to the structure “vector space over \mathbb{R} ”.
- Define a datatype for the language of vector space expressions and define a **Vector** instance for it.
- Find two other instances of the **Vector** class.
- Define a general evaluator for **Vector** expressions on the basis of *two* given assignment functions.
- Specialise the evaluator to the two **Vector** instances defined at point iii. Take three vector expressions, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

Each question carries 6pts.

2. [25pts]

Consider the following differential equation:

$$f'' t - 2 * f' t + f t - 2 = 3 * e^{2*t}, \quad f 0 = 5, \quad f' 0 = 6$$

- i. [10pts] Solve the equation assuming that f can be expressed by a power series fs , that is, use `deriv` and `integ` to compute fs . What are the first three coefficients of fs ?
- ii. [15pts] Solve the equation using the Laplace transform. You should need only one formula (and linearity):

$$\mathcal{L}(\lambda t. e^{\alpha * t}) s = 1 / (s - \alpha)$$

3. [25pts]

Consider the following definition for the limit of a sequence, adapted from Adams and Essex 2010:

We say that sequence a_n converges to the limit L , and we write $\lim_{n \rightarrow \infty} a_n = L$, if for every positive real number ε there exists an integer N (which may depend on ε) such that if $n > N$, then $|a_n - L| < \varepsilon$.

- i. [5pts] Write the definition formally, using logical connectives and quantifiers.
- ii. [10pts] Introduce functions and types to simplify the definition.
- iii. [10pts] Prove the following proposition: If $\lim a = L_1$ and $\lim b = L_2$, then $\lim (a + b) = L_1 + L_2$.

4. [20pts]

Consider the following text from Mac Lane's *Mathematics: Form and Function* (page 168):

If $z = g(y)$ and $y = h(x)$ are two functions with continuous derivatives, then in the relevant range $z = g(h(x))$ is a function of x and has derivative

$$z'(x) = g'(y) * h'(x)$$

Give the types of the elements involved ($x, y, z, g, h, z', g', h', *$ and $'$).