8 Exponentials and Laplace

8.1 The Exponential Function

module *DSLsofMath.W08* **where**
import *DSLsofMath.W05*
import *DSLsofMath.W06*

One of the classical analysis textbooks, Rudin's Rudin [1987] starts with a prologue on the exponential function. The first sentence is

> This is undoubtedly the most important function in mathematics.

Rudin goes on

> It is defined, for every complex number z, by the formula
>
> $$exp\ z = \Sigma\ (z\hat{\ }n)\ /\ n\,!$$

We have defined the exponential function as the function represented by the power series

$expx :: Fractional\ a \Rightarrow PowerSeries\ a$
$expx = integ\ expx\ 1$

and approximated by

$expf :: Fractional\ a \Rightarrow a \to a$
$expf = eval\ 100\ expx$

It is easy to see, using the definition of *integ* that the power series *expx* is, indeed

$$expx = [1, 1\ /\ 2, 1\ /\ (2 * 3), ..., 1\ /\ (2 * 3 * ... * n), ..]$$

We can compute the exponential for complex values if we can give an instance of *Fractional* for complex numbers. We could use the datatype *Data.Complex* from the Haskell standard library, but we prefer to roll our own in order to remind the basic operations on complex numbers.

As we saw in week 1, complex values can be represented as pairs of real values.

newtype $Complex\ r = C\ (r, r)$ **deriving** $(Eq, Show)$
$i :: Num\ a \Rightarrow Complex\ a$
$i = C\ (0, 1)$

Now, we have, for example

$ex1 :: Fractional\ a \Rightarrow Complex\ a$
$ex1 = expf\ i$

We have $ex1 = C\ (0.5403023058681398, 0.8414709848078965)$. Note that

$cosf\ 1 = 0.5403023058681398$
$sinf\ 1 = 0.8414709848078965$

and therefore $expf\ i == C\ (cosf\ 1, sinf\ 1)$. Coincidence?

Instead of evaluating the sum of the terms $a_n * z\hat{\ }n$, let us instead collect the terms in a series:

$$
\begin{aligned}
&terms\ as\ z = terms1\ as\ z\ 0\ \mathbf{where}\\
&\quad terms1\ (Cons\ a\ as)\ z\ n = Cons\ (a * z\hat{\ }n)\ (terms1\ as\ z\ (n+1))
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&ex2 :: Fractional\ a \Rightarrow PowerSeries\ (Complex\ a)\\
&ex2 = takePoly\ 10\ (terms\ expx\ i)
\end{aligned}
$$

$$
\begin{aligned}
ex2 = [\ &C\ (1.0, && 0.0), && C\ (0.0, 1.0 &&)\\
,\ &C\ (-0.5, && 0.0), && C\ (0.0, -0.16666666666666666 &&)\\
,\ &C\ (4.1666666666666664\text{e}{-2}, && 0.0), && C\ (0.0, 8.333333333333333\text{e}{-3} &&)\\
,\ &C\ (-1.3888888888888887\text{e}{-3}, && 0.0), && C\ (0.0, -1.9841269841269839\text{e}{-4} &&)\\
,\ &C\ (2.4801587301587298\text{e}{-5}, && 0.0), && C\ (0.0, 2.7557319223985884\text{e}{-6} &&)\\
]
\end{aligned}
$$

We can see that the real part of this series is the same as

$$ex2R = takePoly\ 10\ (terms\ cosx\ 1)$$

and the imaginary part is the same as

$$ex2I = takePoly\ 10\ (terms\ sinx\ 1)$$

(within approx 20 decimals). But the terms of a series evaluated at 1 are the coefficients of the series. Therefore, the coefficients of $cosx$ are

$$[1, 0, -1\ /\ 2!, 0, 1\ /\ 4!, 0, -1\ /\ 6!, ...]$$

i.e. The function representation of the coefficients for cos is

$$
\begin{aligned}
&cosa\ (2 * n) &&= (-1)\hat{\ }n\ /\ (2 * n)\ !\\
&cosa\ (2 * n + 1) &&= 0
\end{aligned}
$$

and the terms of $sinx$ are

$$[0, 1, 0, -1\ /\ 3!, 0, 1\ /\ 5!, 0, -1\ /\ 7!, ...]$$

i.e., the corresponding function for sin is

$$
\begin{aligned}
&sina\ (2 * n) &&= 0\\
&sina\ (2 * n + 1) &&= (-1)\hat{\ }n\ /\ (2 * n + 1)\ !
\end{aligned}
$$

This can be proven from the definitions of $cosx$ and $sinx$. From this we obtain *Euler's formula*:

$$exp\ (i * x) = cos\ x + i * sin\ x$$

One thing which comes out of Euler's formula is the fact that the exponential is a *periodic function*. A function $f : A \to B$ is said to be periodic if there exists $T \in A$ such that

$$f\ x = f\ (x + T) \quad \text{-- } \forall x \in A$$

(therefore, for this definition to make sense, we need addition on A; in fact we normally assume at least group structure, i.e., addition and subtraction).

Since *sin* and *cos* are periodic, with period $2 * \pi$, we have, using the standard notation $a + i * b$ for some $z = C\ (a, b)$:

$$
\begin{aligned}
& e\hat{}(z + 2 * \pi * i) && = \{\text{- Def. of } z \text{ -}\} \\
& e\hat{}((a + i * b) + 2 * \pi * i) && = \{\text{- Rearranging -}\} \\
& e\hat{}(a + i * (b + 2 * \pi)) && = \{\text{- } \mathit{exp} \text{ is a homomorphism from } (+) \text{ to } (*) \text{ -}\} \\
& e\hat{}a * e\hat{}(i * (b + 2 * \pi)) && = \{\text{- Euler's formula -}\} \\
& e\hat{}a * (\mathit{cos}\ (b + 2 * \pi) + i * \mathit{sin}\ (b + 2 * \pi)) && = \{\text{- } \mathit{cos} \text{ and } \mathit{sin} \text{ are } 2 * \pi\text{-periodic -}\} \\
& e\hat{}a * (\mathit{cos}\ b + i * \mathit{sin}\ b) && = \{\text{- Euler's formula -}\} \\
& e\hat{}a * e\hat{}(i * b) && = \{\text{- } \mathit{exp} \text{ is a homomorphism -}\} \\
& e\hat{}(a + i * b) && = \{\text{- Def. of } z \text{ -}\} \\
& e\hat{}z
\end{aligned}
$$

Thus, we see that *exp* is periodic, because $\mathit{exp}\ z = \mathit{exp}\ (z + T)$ with $T = 2 * \pi * i$, for all z.

8.1.1 Exponential function: Associated code

```
instance Num r ⇒ Num (Complex r) where
  (+) = addC
  (∗) = mulC
  fromInteger = toC ∘ fromInteger
  -- abs = absC – requires Floating r as context
toC :: Num r ⇒ r → Complex r
toC x = C (x, 0)
addC :: Num r ⇒ Complex r → Complex r → Complex r
addC (C (a, b)) (C (x, y)) = C ((a + x), (b + y))
mulC :: Num r ⇒ Complex r → Complex r → Complex r
mulC (C (ar, ai)) (C (br, bi)) = C (ar ∗ br − ai ∗ bi, ar ∗ bi + ai ∗ br)
modulusSquaredC :: Num r ⇒ Complex r → r
modulusSquaredC (C (x, y)) = xˆ2 + yˆ2
absC :: Floating r ⇒ Complex r → Complex r
absC = toC ∘ √· ∘ modulusSquaredC
scaleC :: Num r ⇒ r → Complex r → Complex r
scaleC a (C (x, y)) = C (a ∗ x, a ∗ y)
conj :: Num r ⇒ Complex r → Complex r
conj (C (x, y))     = C (x, −y)
instance Fractional r ⇒ Fractional (Complex r) where
  (/) = divC
  fromRational = toC ∘ fromRational
divC :: Fractional a ⇒ Complex a → Complex a → Complex a
divC x y = scaleC (1 / modSq) (x ∗ conj y)
  where modSq = modulusSquaredC y
```

8.2 The Laplace transform

This material was inspired by Quinn and Rai [2008], which is highly recommended reading.

Consider the differential equation

$$f''\ x - 3 * f'\ x + 2 * f\ x = exp\ (3 * x), f\ 0 = 1, f'\ 0 = 0$$

We can solve such equations with the machinery of power series:

$$\begin{aligned} &fs = integ\ fs'\ 1 \\ &\quad \textbf{where}\ fs' = integ\ (exp\ (3 * x) + 3 * fs' - 2 * fs)\ 0 \end{aligned}$$

We have done this by "zooming in" on the function f and representing it by a power series, $f\ x = \Sigma\ a_n * x\hat{}n$. This allows us to reduce the problem of finding a function $f : \mathbb{R} \to \mathbb{R}$ to that of finding a function $a : \mathbb{N} \to \mathbb{R}$ (or finding a list of sufficiently many a-values for a good approximation).

Still, recursive equations are not always easy to solve (especially without a computer), so it's worth looking for alternatives.

When "zooming in" we go from f to a, but we can also look at it in the other direction: we have "zoomed out" from a to f via an infinite series:

$$a : \mathbb{N} \to \mathbb{R} \xrightarrow{\sum a_n * x^n} f : \mathbb{R} \to \mathbb{R}$$

We would like to go one step further

$$a : \mathbb{N} \to \mathbb{R} \xrightarrow{\sum a_n * x^n} f : \mathbb{R} \to \mathbb{R} \xrightarrow{??} F{:}?$$

That is, we are looking for a transformation of f to some F in a way which resembles the transformation from a to f. The analogue of "sum of an infinite series" for a continuous function is an integral:

$$a : \mathbb{N} \to \mathbb{R} \xrightarrow{\sum a_n * x^n} f : \mathbb{R} \to \mathbb{R} \xrightarrow{\int (f t) * x^t dt} F{:}?$$

We note that, for the integral $\int_0^\infty (f\ t) * x\hat{}t\ dt$ to converge for a larger class of functions (say, bounded functions), we have to limit ourselves to $|x| < 1$. Both this condition and the integral make sense for $x \in \mathbb{C}$, so we could take

$$a : \mathbb{N} \to \mathbb{R} \xrightarrow{\sum a_n * x^n} f : \mathbb{R} \to \mathbb{R} \xrightarrow{\int (f t) * x^t dt} F : \{ z \mid |z| < 1 \} \to \mathbb{C}$$

but let us stick to $\mathbb{R}$ for now.

Writing, somewhat optimistically

$$\mathscr{L}\ f\ x = \int_0^\infty (f\ t) * x\hat{}t\ dt$$

we can ask ourselves what $\mathscr{L}\ f'$ looks like. After all, we want to solve *differential* equations by "zooming out". We have

$$\mathscr{L}\ f'\ x = \int_0^\infty (f'\ t) * x\hat{}t\ dt$$

Remember that $D\ (f * g) = D\ f * g + f * D\ g$, therefore

$$\begin{aligned} &\mathscr{L}\ f'\ x && = \{\text{-}\ g\ t = x\hat{}t;\ g'\ t = log\ x * x\hat{}t\ \text{-}\} \\ &\int_0^\infty (D\ (f\ t * x\hat{}t)) - f\ t * log\ x * x\hat{}t\ dt && = \\ &\int_0^\infty (D\ (f\ t * x\hat{}t))\ dt - \int_0^\infty f\ t * log\ x * x\hat{}t\ dt = \\ &lim_{t \to \infty}(f\ t * x\hat{}t) - (f\ 0 * x\hat{}0) - log\ x * \int_0^\infty f\ t * x\hat{}t\ dt = \end{aligned}$$

$$
\begin{aligned}
&-f\ 0 - \mathit{log}\ x * \textstyle\int_0^\infty\ f\ t * x\hat{\ }t\ dt = \\
&-f\ 0 - \mathit{log}\ x * \mathscr{L}\ f\ x
\end{aligned}
$$

The factor $\mathit{log}\ x$ is somewhat awkward. Let us therefore return to the definition of $\mathscr{L}$ and operate a change of variables:

$$
\begin{aligned}
&\mathscr{L}\ f\ x = \textstyle\int_0^\infty\ (f\ t) * x\hat{\ }t\ dt && \Leftrightarrow \{\text{- } x = \mathit{exp}\ (\mathit{log}\ x) \text{ -}\} \\
&\mathscr{L}\ f\ x = \textstyle\int_0^\infty\ (f\ t) * (\mathit{exp}\ (\mathit{log}\ x))\hat{\ }t\ dt && \Leftrightarrow \{\text{- } (a\hat{\ }b)\hat{\ }c = a\hat{\ }(b * c) \text{ -}\} \\
&\mathscr{L}\ f\ x = \textstyle\int_0^\infty\ (f\ t) * \mathit{exp}\ (\mathit{log}\ x * t)\ dt
\end{aligned}
$$

Since $\mathit{log}\ x < 0$ for $|x| < 1$, we make the substitution $-s = \mathit{log}\ x$. The condition $|x| < 1$ becomes $s > 0$ (or, in $\mathbb{C}$, $\mathit{real}\ s > 0$), and we have

$$
\mathscr{L}\ f\ s = \textstyle\int_0^\infty\ (f\ t) * \mathit{exp}\ (-s * t)\ dt
$$

This is the definition of the Laplace transform of the function f. Going back to the problem of computing $\mathscr{L}\ f'$, we now have

$$
\begin{aligned}
&\mathscr{L}\ f'\ s && = \{\text{- The computation above with } s = -\mathit{log}\ x. \text{ -}\} \\
&-f\ 0 + s * \mathscr{L}\ f\ s
\end{aligned}
$$

We have obtained

$$
\mathscr{L}\ f'\ s = s * \mathscr{L}\ f\ s - f\ 0 \quad \texttt{-- The "Laplace-D" law}
$$

From this, we can deduce

$$
\begin{aligned}
&\mathscr{L}\ f''\ s && = \{\text{- Laplace-D for } f' \text{ -}\} \\
&s * \mathscr{L}\ f'\ s - f'\ 0 && = \{\text{- Laplace-D for } f \text{ -}\} \\
&s * (s * \mathscr{L}\ f\ s - f\ 0) - f'\ 0 && = \{\text{- Simplification -}\} \\
&s\hat{\ }2 * \mathscr{L}\ f\ s - s * f\ 0 - f'\ 0
\end{aligned}
$$

Exercise 8.1: what is the general formula for $\mathscr{L}\ f^{(k)} s$?

Returning to our differential equation, we have

$$
\begin{aligned}
&\ \ f''\ x - 3 * f'\ x + 2 * f\ x = \mathit{exp}\ (3 * x), f\ 0 = 1, f'\ 0 = 0 \\
&\Leftrightarrow \{\text{- point-free form -}\} \\
&\ \ f'' - 3 * f' + 2 * f = \mathit{exp} \circ (3*), f\ 0 = 1, f'\ 0 = 0 \\
&\Rightarrow \{\text{- applying } \mathscr{L} \text{ to both sides -}\} \\
&\ \ \mathscr{L}\ (f'' - 3 * f' + 2 * f) = \mathscr{L}\ (\mathit{exp} \circ (3*)), f\ 0 = 1, f'\ 0 = 0 \quad \texttt{-- Eq. (1)}
\end{aligned}
$$

Remark: Note that this is a necessary condition, but not a sufficient one. The Laplace transform is not injective. For one thing, it does not take into account the behaviour of f for negative arguments. Because of this, we often assume that the domain of definition for functions to which we apply the Laplace transform is $\mathbb{R}_{\geqslant 0}$. For another, it is known that changing the values of f for a countable number of its arguments does not change the value of the integral.

For the definition of $\mathscr{L}$ and the linearity of the integral, we have that, for any f and g for which the transformation is defined, and for any constants α and β

$$
\mathscr{L}\ (\alpha * f + \beta * g) = \alpha * \mathscr{L}\ f + \beta * \mathscr{L}\ g
$$

Note that this is an equality between functions. (Comparing to last week we can also see f and g as vectors and $\mathscr{L}$ as a linear transformation.)

Applying this to the left-hand side of (1), we have for any s

$$
\begin{aligned}
&\ \mathscr{L}\ (f'' - 3 * f' + 2 * f)\ s \\
= &\ \{\text{- } \mathscr{L} \text{ is linear -}\} \\
&\ \mathscr{L}\ f''\ s - 3 * \mathscr{L}\ f'\ s + 2 * \mathscr{L}\ f\ s \\
= &\ \{\text{- re-writing } \mathscr{L}\ f'' \text{ and } \mathscr{L}\ f' \text{ in terms of } \mathscr{L}\ f \text{ -}\} \\
&\ s\hat{\ }2 * \mathscr{L}\ f\ s - s * f\ 0 - f'\ 0 - 3 * (s * \mathscr{L}\ f\ s - f\ 0) + 2 * \mathscr{L}\ f\ s \\
= &\ \{\text{- } f\ 0 = 1,\ f'\ 0 = 0 \text{ -}\} \\
&\ (s\hat{\ }2 - 3 * s + 2) * \mathscr{L}\ f\ s - s + 3
\end{aligned}
$$

For the right-hand side, we apply the definition:

$$
\begin{aligned}
&\mathscr{L}\ (exp \circ (3*))\ s && = \{\text{- Def. of } \mathscr{L} \text{ -}\} \\
&\int_0^\infty\ exp\ (3 * t) * exp\ (-s * t)\ dt && = \\
&\int_0^\infty\ exp\ ((3 - s) * t))\ dt && = \\
&lim_{t \to \infty} \tfrac{exp\ ((3-s)*t)}{3-s} - \tfrac{exp\ ((3-s)*0)}{3-s} && = \{\text{- for } s > 3 \text{ -}\} \\
&\tfrac{1}{s-3}
\end{aligned}
$$

Therefore, we have, writing F for $\mathscr{L}\ f$

$$(s\hat{\ }2 - 3 * s + 2) * F\ s - s + 3 = \tfrac{1}{s-3}$$

and therefore

$$
\begin{aligned}
&F\ s && = \{\text{- Solve for } F\ s \text{ -}\} \\
&\frac{\frac{1}{s-3} + s - 3}{s\hat{\ }2 - 3 * s + 2} && = \{\text{- } s\hat{\ }2 - 3 * s + 2 = (s - 1) * (s - 2) \text{ -}\} \\
&\frac{10 - 6 * s + s\hat{\ }2}{(s-1)*(s-2)*(s-3)}
\end{aligned}
$$

We now have the problem of "recovering" the function f from its Laplace transform. The standard approach is to use the linearity of $\mathscr{L}$ to write F as a sum of functions with known inverse transforms. We know one such function:

$$exp\ (\alpha * t)\ \{\text{- is the inverse Laplace transform of -}\}\ 1\ /\ (s - \alpha)$$

In fact, in our case, this is all we need.

The idea is to write $F\ s$ as a sum of three fractions with denominators $s - 1$, $s - 2$, and $s - 3$ respectively, i.e., to find A, B, and C such that

$$
\begin{aligned}
&A\ /\ (s - 1) + B\ /\ (s - 2) + C\ /\ (s - 3) = (10 - 6 * s + s\hat{\ }2)\ /\ ((s - 1) * (s - 2) * (s - 3)) \\
&\Rightarrow \\
&A * (s - 2) * (s - 3) + B * (s - 1) * (s - 3) + C * (s - 1) * (s - 2) = 10 - 6 * s + s\hat{\ }2 \quad \text{-- (2)}
\end{aligned}
$$

We need this equality (2) to hold for values $s > 3$. A *sufficient* condition for this is for (2) to hold for *all* s. A *necessary* condition for this is for (2) to hold for the specific values 1, 2, and 3.

$$
\begin{aligned}
&\text{For } s = 1 : A * (-1) * (-2) && = 10 - 6 + 1 && \Rightarrow A = 2.5 \\
&\text{For } s = 2 : B * 1 * (-1) && = 10 - 12 + 4 && \Rightarrow B = -2 \\
&\text{For } s = 3 : C * 2 * 1 && = 10 - 18 + 9 && \Rightarrow C = 0.5
\end{aligned}
$$

It is now easy to check that, with these values, (2) does indeed hold, and therefore that we have

$$F\ s = 2.5 * (1 \,/\, (s-1)) - 2 * (1 \,/\, (s-2)) + 0.5 * (1 \,/\, (s-3))$$

The inverse transform is now easy:

$$f\ t = 2.5 * \mathit{exp}\ t - 2 * \mathit{exp}\ (2 * t) + 0.5 * \mathit{exp}\ (3 * t)$$

Our mix of necessary and sufficient conditions makes it necessary to check that we have, indeed, a solution for the differential equation. The verification is in this case trivial.

8.3 Laplace and other transforms

To sum up, we have defined the Laplace transform and shown that it can be used to solve differential equations. It can be seen as a continuous version of the transform between the infinite sequence of coeeficients $a : \mathbb{N} \to \mathbb{R}$ and the functions behind formal power series.

Laplace is also closely related to Fourier series, which is a way of expressing functions on a closed interval as a linear combination of dicrete frequency components rather than as a function of time. Finally, Laplace is also a close relative of the Fourier transform. Both transforms are used to express functions as a sum of "complex frequencies", but Laplace allows a wider range of functions to be transformed. A nice local overview and comparison is B. Berndtsson's "Fourier and Laplace Transforms"[6] Fourier analysis is a common tool in courses on Transforms, Signals and Systems.

8.4 Exercises

Exercise 8.1. Starting from the "Laplace-D" law

$$\mathscr{L}\ f'\ s = s * \mathscr{L}\ f\ s - f\ 0$$

Derive a general formula for $\mathscr{L}\ f^{(k)} s$.

Exercise 8.2. Find the Laplace transforms of the following functions:

a. $\lambda t.\, 3 * e^{5*t}$

b. $\lambda t.\, e^{\alpha * t} - \beta$

c. $\lambda t.\, e^{(t + \frac{\pi}{6})}$

Exercise 8.3.

a. Show that:

   (a) $\sin t = \frac{1}{2*i}\left(e^{i*t} - e^{-i*t}\right)$

   (b) $\cos t = \frac{1}{2}\left(e^{i*t} + e^{-i*t}\right)$

b. Find the Laplace transforms $\mathcal{L}\,(\lambda t.\ \sin t)$ and $\mathcal{L}\,(\lambda t.\ \cos t)$

[6] Available from `http://www.math.chalmers.se/Math/Grundutb/CTH/mve025/1516/Dokument/F-analys.pdf`.

8.4.1 Exercises from old exams

Exercise 8.4. *From exam 2016-03-15*

Consider the following differential equation:

$$f''\,t - 2 * f'\,t + f\,t = e^{2*t}, \quad f\,0 = 2, \quad f'\,0 = 3$$

Solve the equation using the Laplace transform. You should need only one formula (and linearity):

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\,s = 1/(s - \alpha)$$

Exercise 8.5. *From exam 2016-08-23*

Consider the following differential equation:

$$f''\,t - 5 * f'\,t + 6 * f\,t = e^{t}, \quad f\,0 = 1, \quad f'\,0 = 4$$

Solve the equation using the Laplace transform. You should need only one formula (and linearity):

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\,s = 1/(s - \alpha)$$

Exercise 8.6. *From exam 2016-Practice*

Consider the following differential equation:

$$f''\,t - 2 * f'\,t + f\,t - 2 = 3 * e^{2*t}, \quad f\,0 = 5, \quad f'\,0 = 6$$

Solve the equation using the Laplace transform. You should need only one formula (and linearity):

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\,s = 1/(s - \alpha)$$

Exercise 8.7. *From exam 2017-03-14*

Consider the following differential equation:

$$f''\,t + 4 * f\,t = 6 * \cos\,t, \quad f\,0 = 0, \quad f'\,0 = 0$$

Solve the equation using the Laplace transform. You should need only two formulas (and linearity):

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\,s = 1/(s - \alpha)$$

$$2 * \cos\,t = e^{i*t} + e^{-i*t}$$

Exercise 8.8. *From exam 2017-08-22*

Consider the following differential equation:

$$f''\,t - 3\sqrt{2} * f'\,t + 4 * f\,t = 0, \quad f\,0 = 2, \quad f'\,0 = 3\sqrt{2}$$

Solve the equation using the Laplace transform. You should need only one formula (and linearity):

$$\mathcal{L}\,(\lambda t.\, e^{\alpha * t})\,s = 1/(s - \alpha)$$