# Domain Specific Languages of Mathematics
# Course codes: DAT326 / DIT982

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2017-08-22

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**Results** Announced within 19 days

**Exam check** Fri. 2017-09-01 in EDIT 5468 at 12.30-12.55

**Aids** One textbook of your choice (e.g., Adams and Essex, or Rudin). No printouts, no lecture notes, no notebooks, etc.

**Grades** 3: 40p, 4: 60p, 5: 80p, max: 100p

Remember to write legibly. Good luck!

For reference: the DSLsofMath learning outcomes. Some are tested by the hand-ins, some by the written exam.

- Knowledge and understanding
  - design and implement a DSL (Domain Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [30pts] Algebraic structure: a DSL for semirings.

   A semiring is a set $R$ equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, such that:

   - $(R, +, 0)$ is a commutative monoid with identity element 0:
     $(a + b) + c = a + (b + c)$
     $0 + a = a + 0 = a$
     $a + b = b + a$
   - $(R, \cdot, 1)$ is a monoid with identity element 1:
     $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
     $1 \cdot a = a \cdot 1 = a$
   - Multiplication left and right distributes over $(R, +, 0)$:
     $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
     $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$
     $a \cdot 0 = 0 \cdot a = 0$

   (a) Define a type class *SemiRing* that corresponds to the semiring structure.

   (b) Define a datatype *SR v* for the language of semiring expressions (with variables of type *v*) and define a *SemiRing* instance for it. (These are expressions formed from applying the semiring operations to the appropriate number of arguments, e.g., all the left hand sides and right hand sides of the above equations.)

   (c) Find two other instances of the *SemiRing* class.

   (d) Gice a type signature for, and define, a general evaluator for *SR v* expressions on the basis of an assignment function.

   (e) Specialise the evaluator to the two *SemiRing* instances defined in (1c). Take three semiring expressions of type *SR String*, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 6pts.

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2. [20pts] Multiplication for matrices (from the matrix algebra DSL).

   Consider the following definition, from "Linear Algebra" by Donald H. Pelletier:

   **Definition:** If $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix, then the *product*, $AB$, is an $m \times p$ matrix; the $(i, j)^{th}$ entry of $AB$ is the sum of the products of the pairs that are obtained when the entries from the $i^{th}$ row of the left factor, $A$, are paired with those from the $j^{th}$ column of the right factor, $B$.

   (a) [7pts] Introduce precise types for the variables involved: $A$, $m$, $n$, $B$, $p$, $i$, $j$. You can write *Fin n* for the type of the values $\{0, 1, ..., n - 1\}$.

   (b) [6pts] Introduce types for the functions *mul* and *proj* where $AB = \mathit{mul}\ A\ B$ and $\mathit{proj}\ i\ j\ M =$ "take the $(i, j)^{th}$ entry of $M$". What class contraints (if any) are needed on the type of the matrix entries in the two cases?

   (c) [7pts] Implement *mul* in Haskell. You may use the functions *row* and *col* specified by $\mathit{row}\ i\ M =$ "the $i^{th}$ row of $M$" and $\mathit{col}\ j\ M =$ "the $j^{th}$ column of $M$". You don't need to implement them and here you can assume they return plain Haskell lists.

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3. [25pts] Consider the following differential equation:

$$f''\,t - 3\sqrt{2} * f'\,t + 4 * f\,t = 0, \quad f\,0 = 2, \quad f'\,0 = 3\sqrt{2}$$

(a) [10pts] Solve the equation assuming that $f$ can be expressed by a power series $fs$, that is, use *integ* and the differential equation to express the relation between $fs$, $fs'$, and $fs''$. What are the first three coefficients of $fs$?

(b) [15pts] Solve the equation using the Laplace transform. You should need this formula (and the rules for linearity + derivative):

$$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\, s = 1/(s - \alpha)$$

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4. [25pts] Adequate notation for mathematical concepts and proofs (or "50 shades of continuity").

A formal definition of "$f : X \to \mathbb{R}$ is continuous" and "$f$ is continuous at $c$" can be written as follows (using the helper predicate $Q$):

$$\begin{aligned} C\ (f) &= \forall\, c : X.\ Cat\ (f, c) \\ Cat\ (f, c) &= \forall\, \varepsilon > 0.\ \exists\, \delta > 0.\ Q\ (f, c, \varepsilon, \delta) \\ Q\ (f, c, \varepsilon, \delta) &= \forall\, x : X.\ |x - c| < \delta \Rightarrow |f\ x - f\ c| < \varepsilon \end{aligned}$$

By moving the existential quantifier outwards we can introduce the function $get\delta$ which computes the required $\delta$ from $c$ and $\varepsilon$:

$$C'\ (f) = \exists\, get\delta : X \to \mathbb{R}_{>0} \to \mathbb{R}_{>0}.\ \forall\, c : X.\ \forall\, \varepsilon > 0.\ Q\ (f, c, \varepsilon, get\delta\ c\ \varepsilon)$$

Now, consider this definition of *uniform continuity*:

**Definition:** Let $X \subseteq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is *uniformly continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x$ and $y$ in the domain of $f$, if $|x - y| < \delta$, then $|f\ x - f\ y| < \varepsilon$.

(a) [5pts] Write the definition of $UC\ (f)$ = "$f$ is uniformly continuous" formally, using logical connectives and quantifiers. Try to use $Q$.

(b) [10pts] Transform $UC\ (f)$ into a new definition $UC'\ (f)$ by a transformation similar to the one from $C\ (f)$ to $C'\ (f)$. Explain the new function $new\delta$ introduced.

(c) [10pts] Prove that $\forall f : X \to \mathbb{R}.\ UC'\ (f) \Rightarrow C'\ (f)$. Explain your reasoning in terms of $get\delta$ and $new\delta$.