DSLsofMath 2017: Assignment 2

Optimisation using Newton's method

1. The evaluation of the second derivative is given by

$$eval'' = eval' \circ derive = eval \circ derive \circ derive$$

   (a) Show that $eval''$ is not a homomorphism.

   (b) What datatype is needed to have a homomorphism from $Expr$ in this case? Show this for the case of multiplication.

   (c) What datatype is needed for the homomorphism that is analogous to apply? Give instances of the numerical classes (Num, $Fractional$, $Floating$) for this datatype, as well as an embedding of $Const$ and Id.

2. Newton's method allows us to find zeros of a large class of functions in a given interval. The following description of Newton's method follows Bird and Wadler [1988], page 23:

$$
\begin{array}{l}
newton :: (Double \to Double) \to Double \to Double \to Double \\
newton\ f\ \epsilon\ x = \mathbf{if}\ |fx| < \epsilon \\
\qquad\qquad\qquad \mathbf{then}\ x \\
\qquad\qquad\qquad \mathbf{else\ if}\ fx' \not\equiv 0\ \mathbf{then}\ newton\ f\ \epsilon\ next \\
\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{else}\ \ newton\ f\ \epsilon\ (x + \epsilon) \\
\quad \mathbf{where}\ fx \quad = f\ x \\
\qquad\qquad fx' \quad = undefined \quad \text{-- } f'\ x \text{ (derivative of } f \text{ at } x\text{)} \\
\qquad\qquad next = x - (fx \ / \ fx')
\end{array}
$$

   Implement Newton's method, using the datatype you introduced above for computing the derivatives. In other words, use the code above to implement

$$newton :: (FDD\ Double \to FDD\ Double) \to Double \to Double \to Double$$

   where $FDD\ a$ is the datatype from above, in order to obtain the appropriate value for $f'\ x$.

   Test your implementation on the following functions:

$$
\begin{array}{ll}
test0 = x\ \hat{}\ 2 & \text{-- one (double) zero, in } 0 \\
test1 = x\ \hat{}\ 2 - 1 & \text{-- two zeros, in } -1 \text{ and } 1 \\
test2 = sin & \text{-- many, many zeros } (n * \pi) \\
test3\ n\ x\ y = y\ \hat{}\ n - fddId\ x & \text{-- nth root of } x \\
\quad \text{-- where } fddId \text{ is the embedding of } Id &
\end{array}
$$

   For each of these functions, apply Newton's method to a number of starting points from a sensible interval. For example:

$$fmap\ (newton\ test1\ 0.001)\ [-2.0, -1.5 \mathinner{..} 2.0]$$

   but be aware that the method might not always converge!

3. We can find the optima of a twice-differentiable function on an interval by finding the zeros of its derivative on that interval, and checking the second derivative. If x_0 is a zero of f', then

   - if $f''\ x_0 < 0$, then x_0 is a maximum
   - if $f''\ x_0 > 0$, then x_0 is a minimum
   - if $f''\ x_0 = 0$, then, if $f''\ (x_0 - \epsilon) * f''\ (x_0 + \epsilon) < 0$ (i.e., f'' changes its sign in the neighbourhood of x_0), x_0 is an inflection point (not an optimum)
   - otherwise, we don't know

   Use Newton's method to find the optima of the test functions from point 2. That is, implement a function

   $$optim :: (FDD\ Double \rightarrow FDD\ Double) \rightarrow Double \rightarrow Double \rightarrow Result\ Double$$

   so that $optim\ f\ \epsilon\ x$ uses Newton's method to find a zero of f' starting from x. If y is the result (i.e. $f'\ y$ is within ϵ of 0), then check the second derivative, returning $Maximum\ y$ if $f''\ y < 0$, $Minimum\ y$ if $f'' > y$, and $Dunno\ y$ if $f'' = 0$.

   As before, use several starting points.

   Hint: you might want to modify the code you've written for Newton's method at point 2.

Formalities

Submission: Assignments are to be submitted via Fire:

`https://dsls-lp3-17.frs.cse.chalmers.se/login`

Deadline: Tuesday, 2016-02-28, 23:59.

Grading: Discussions with each of the teams during the exercises session of Friday, 2016-03-03.

References

R. Bird and P. Wadler. *Introduction to Functional Programming, 1988.* Prentice-Hall, Englewood Cliffs, NJ, 1988.