Domain Specific Languages of Mathematics

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Results Announced within 19 days (by Monday 2017-04-03)

Exam check Wed. 2017-04-05 and Fri. -07. Both at 12.30-12.55 in EDIT 5468.

Aids One textbook of your choice (e.g., Adams and Essex, or Rudin). No printouts, no lecture notes, no notebooks, etc.

Grades 3: 40p, 4: 60p, 5: 80p, max: 100p

Remember to write legibly. Good luck!

For reference: the DSLsofMath learning outcomes:

- Knowledge and understanding
  - design and implement a DSL (Domain Specific Language) for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of elementary real and complex analysis, algebra, and linear algebra
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

1. [20pts] Consider the following text from page 169 of Mac Lane [1968]:

> [...] a function $z = f\ (x, y)$ for all points (x, y) in some open set U of the cartesian (x, y)-plane. [...] If one holds y fixed, the quantity z remains just a function of x; its derivative, when it exists, is called the *partial derivative* with respect to x. Thus at a point (x, y) in U this derivative for $h \neq 0$ is
>
> $$\partial z/\partial x = f'_x(x, y) = \lim_{h\to 0}(f(x + h, y) - f(x, y))/h$$

What are the types of the elements involved in the equation on the last line? You are welcome to introduce functions and names to explain your reasoning.

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2. [25pts] Consider the following differential equation:

$$f''\,t + 4 * f\,t = 6 * \cos\,t, \quad f\,0 = 0, \quad f'\,0 = 0$$

   (a) [10pts] Solve the equation assuming that f can be expressed by a power series *fs*, that is, use *integ* and the differential equation to express the relation between *fs*, *fs*′, *fs*″, and *rhs* where *rhs* is the power series representation of $(6*) \circ \mathit{cos}$. What are the first four coefficients of *fs*?

   (b) [15pts] Solve the equation using the Laplace transform. You should need only two formulas (and linearity):

$$\mathscr{L}\,(\lambda t.\, e^{\alpha * t})\,s = 1/(s - \alpha)$$

$$2 * \cos\,t = e^{i*t} + e^{-i*t}$$

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3. [20pts] One definition of *derivative* is (inspired by [Rudin, 1964], p. 89):

> *Definition:* Let $f : [a, b] \to \mathbb{R}$. For an $x \in [a, b]$, consider the function $\phi_f(x) : [a, b] \to \mathbb{R}$ by
>
> $$\phi_f(x)(t) = (f(t) - f(x))/(t - x), \qquad \text{for } t \neq x$$
>
> and define
>
> $$f'(x) = \lim_{t\to x} \phi_f(x)(t)$$
>
> provided that this limit exists. We thus associate with f a function f' whose domain of definition is the set of points x at which the limit (2) exists; f' is called the *derivative* of f.

   (a) [5pts] Let $r : [1, 2] \to \mathbb{R}$ with $r(x) = 1/x$. Compute r' using this definition.

   (b) [5pts] Let $h = g \circ f$ for $f, g : [a, b] \to [a, b]$. Formulate the chain rule (the derivative of h in terms of operations on f and g).

   (c) [10pts] Prove your formulation of the chain rule using the definition above.

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4. [15pts] Recall the type of expressions

```
data FunExp = Const Rational
            | Id
            | FunExp :+: FunExp
            | FunExp :*: FunExp
            | FunExp :/: FunExp
            | Exp FunExp
            | Sin FunExp
            | Cos FunExp
            -- and so on
          deriving Show
```

and consider the function

```
f :: Double -> Double
f x = exp (sin x) + x
```

(a) Find an expression e such that $eval\ e == f$ and show this using equational reasoning.

(b) Implement a function *deriv2* such that, for any $f : Fractional\ a \Rightarrow a \to a$ constructed with the grammar of *FunExp* and any x in the domain of f, we have that $deriv2\ f\ x$ computes the second derivative of f at x. Use the function $derive :: FunExp \to FunExp$ from the lectures ($eval\ (derive\ e)$ is the derivative of $eval\ e$). What instance declarations do you need?

The type of $deriv2\ f$ should be $Fractional\ a \Rightarrow a \to a$.

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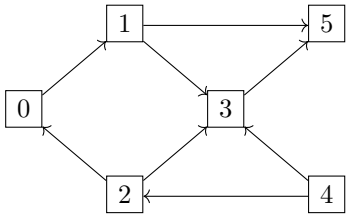
5. [20pts] Consider a non-deterministic system with a transition function $f : G \to [G]$ (for $G = \{0\,..\,5\}$) represented in the following graph



The transition matrix can be given the type $m :: G \to (G \to Bool)$ and the canonical vectors have type $e\ i :: G \to Bool$ for i in G.

(a) (General questions.) What do the canonical vectors represent? What about non-canonical ones? What are the operations on *Bool* used in the matrix-vector multiplication?

(b) (Specific questions.) Write the transition matrix m of the system. Compute, using matrix-vector multiplication, the result of three steps of the system starting in state 2.