Domain Specific Languages of Mathematics
Practice Exam

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Results Announced within ?

Aids One textbook of your choice (e.g., Adams and Essex, or Rudin). No printouts, no lecture notes, no notebooks, etc.

Grades 3: 40p, 4: 60p, 5: 80p, max: 100p

Remember to write legibly. Good luck!

1. [30pts]

   A vector space over $\mathbb{R}$ is a set `V` together with a constant (or nullary) operation `0 : V`, an operation `+ : V -> V -> V`, and an *external* operation $\cdot$ `:` $\mathbb{R}$ `-> V -> V`, such that

   - `0` is the unit of `+`:

     $$\forall\ v \in V \quad v + 0 = 0 + v = v$$

   - `+` is associative:

     $$\forall\ v_1,\ v_2,\ v_3 \in V \quad (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$$

   - `+` is invertible:

     $$\forall\ v \in V\ \ \exists\ (-\ v) \in V \quad v + (-v) = (-v) + v = 0$$

   - `+` is commutative:

     $$\forall\ v_1,\ v_2 \in V \quad v_1 + v_2 \ = \ v_2 + v_1$$

     Remarks:

     – we usually denote $v_1 + (-\ v_2) = v_1 - v_2$
     – the first two conditions say that `(V, +, 0)` is a *monoid*
     – the first three conditions say that `(V, +, 0)` is a *group*
     – the four conditions say that `(V, +, 0)` is a *commutative group*

   - $\cdot$ is associative

     $$\forall\ x_1,\ x_2 \in \mathbb{R},\ v \in V \quad x_1 \cdot (x_2 \cdot v) = (x_1 * x_2) \cdot v$$

     Remark: `*` denotes the standard multiplication in $\mathbb{R}$

   - 1 is a unit of $\cdot$:

     $$\forall\ v \in V \qquad 1 \cdot v \ = \ v$$

   - $\cdot$ distributes over `+`:

     $$\forall\ x \in \mathbb{R},\ v_1,\ v_2 \in V \quad x \cdot (v_1 + v_2) = x \cdot v_1 + x \cdot v_2$$

   - $\cdot$ distributes over `+`

     $$\forall\ x_1,\ x_2 \in \mathbb{R},\ v \in V \quad (x_1 + x_2) \cdot v = x_1 \cdot v + x_2 \cdot v$$

   i. Define a type class `Vector` that corresponds to the vector space over $\mathbb{R}$ structure.

   ii. Define a datatype for the language of vector space expressions and define a `Vector` instance for it.

   iii. Find two other instances of the `Vector` class.

   iv. Define a general evaluator for `Vector` expressions on the basis of *two* given assignment functions.

   v. Specialise the evaluator to the two `Vector` instances defined at point iii. Take three vector expressions, give the appropriate assignments and compute the results of evaluating, in each case, the three expressions.

   Each question carries 6pts.

2. [25pts]

   Consider the following differential equation:

   ```
   f'' t - 2 · f' t + f t - 2 = 3 * e^{2 * t}, f 0 = 5, f' 0 = 6
   ```

   i. [10pts] Solve the equation assuming that `f` can be expressed by a power series `fs`, that is, use `deriv` and `integ` to compute `fs`. What are the first three coefficients of `fs`?

   ii. [15pts] Solve the equation using the Laplace transform. You should need only one formula:

   ```
   ℒ (e^(α*t)) s  = 1 / (s - α)
   ```

3. [25pts]

   Consider the following definition for the limit of a sequence, adapted from Adams and Essex 2010:

   > We say that sequence a_n converges to the limit L, and we write $lim_{n \to \infty} a_n = L$, if for every positive real number ε there exists an integer N (which may depend on ε) such that if $n > N$, then $|a_n - L| < \varepsilon$.

   i. [5pts] Write the definition formally, using logical connectives and quantifiers.

   ii. [10pts] Introduce functions and types to simplify the definition.

   iii. [10pts] Prove the following proposition: If `lim a = L`$_1$ and `lim b = L`$_2$, then `lim (a + b) = L`$_1$ `+ L`$_2$.

4. [20pts]

   Consider the following text from Mac Lane's *Mathematics: Form and Function* (page 168):

   > If $z = g(y)$ and $y = h(x)$ are two functions with continuous derivatives, then in the relevant range $z = g(h(x))$ is a function of x and has derivative
   >
   > $$z'(x) = g'(y) * h'(x)$$

   Give the types of the elements involved (`x`, `y`, `z`, `g`, `h`, `z'`, `g'`, `h'`, `*` and `'`).