Program Reasoning

7. Hoare Logic

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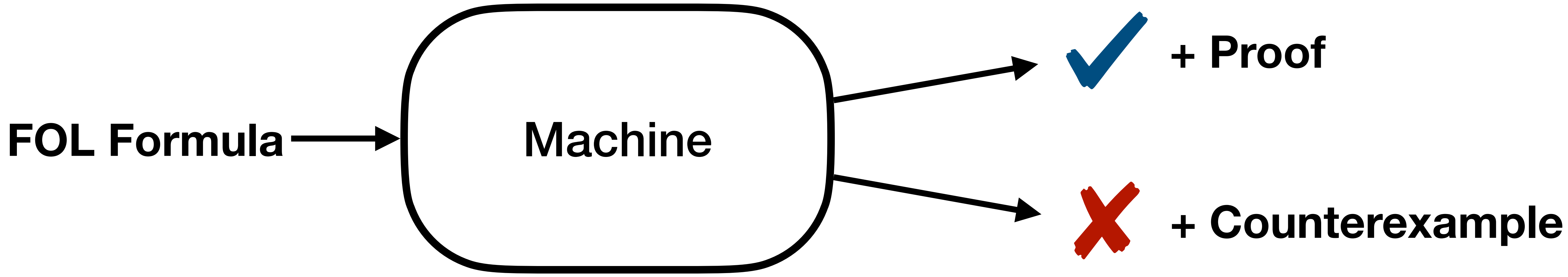
The Story So Far

- Mechanization of logic and mathematics

"Is the following machine possible?"
- D. Hilbert, 1930

"NO!"
-K. Gödel, 1931

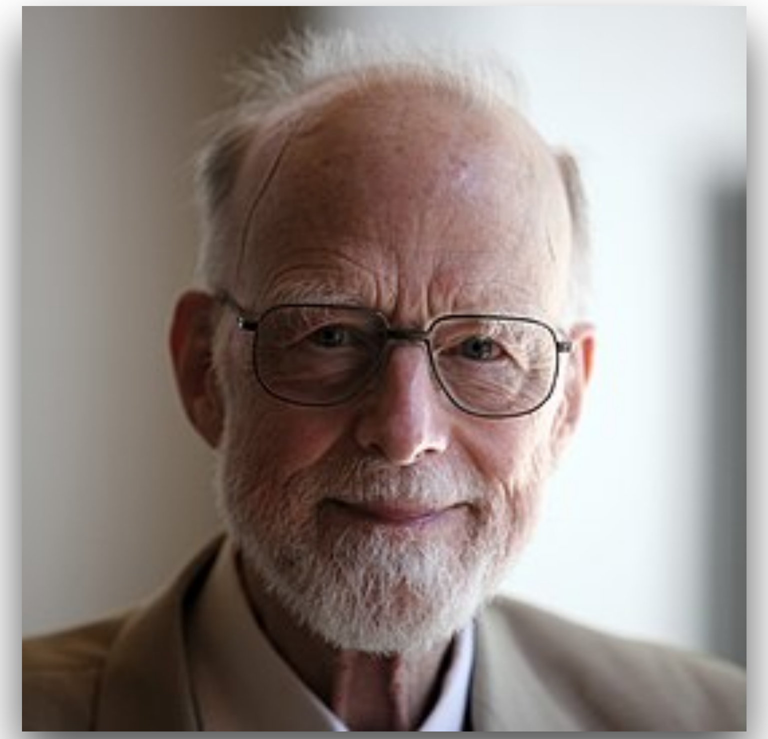
"NO!"
- A. Turing, 1936



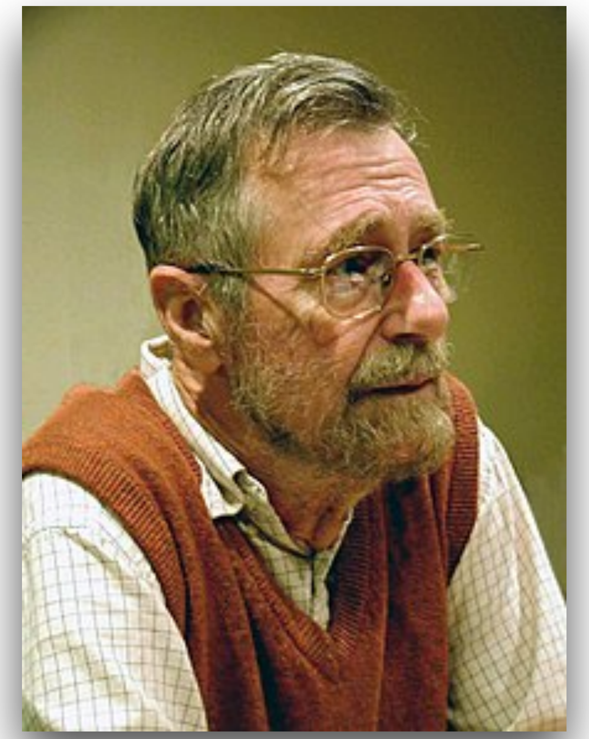
Never Ending Story

- Reasoning about programs

“How to check a program is correct?”
- A. Turing, 1949



“Hoare Logic”
- T. Hoare, 1969

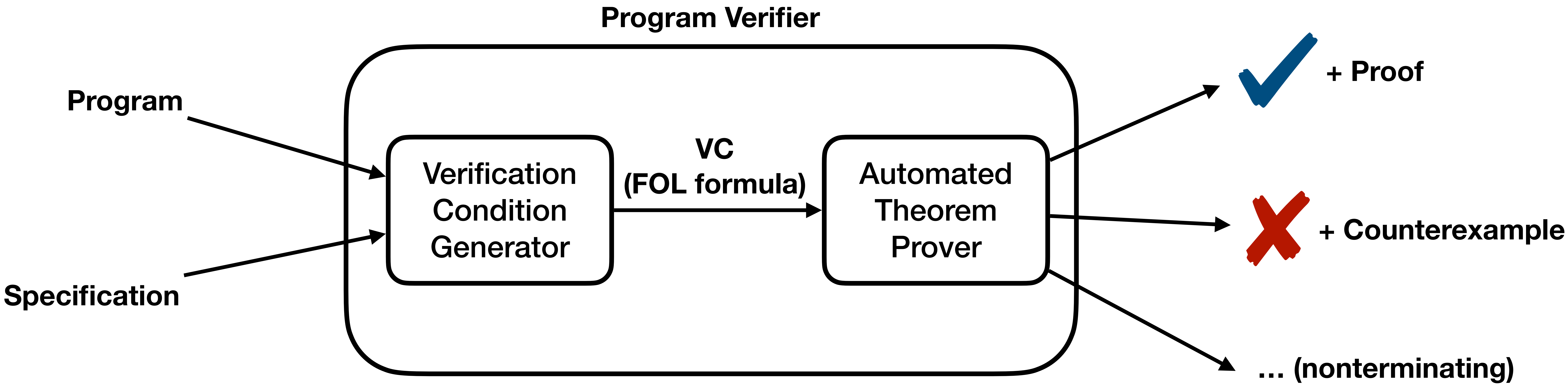


“Weakest Precondition Calculus”
- E. Dijkstra, 1975

Program Verification

- Specifying and proving properties of programs
- Specification: precise statement of properties that a program should exhibit in FOL
- Partial correctness properties: certain states cannot ever occur during the execution
  - “Bad things never happen” (e.g., integer overflow, buffer overflow, deadlock, etc)
  - Proof by inductive assertion method
- Total correctness properties: certain states are eventually reached during the execution
  - “Good things will eventually happen” (e.g., termination, fairness)
  - Proof by ranking function method

Overview



Program Verifier
Program
Specification
Verification Condition Generator
VC (FOL formula)
Automated Theorem Prover
+ Proof
+ Counterexample
… (nonterminating)

Specification

- Typically embedded into program text as program annotations
- An annotation is a FOL formula F
- An annotation F at location L asserts that F is true whenever program control reaches L
- Three types of annotations:
  - Function specification
  - Loop invariant
  - Assertion

Function Specification

- A pair of annotations: precondition and postcondition
- Precondition: a formula whose free variables include only the formal parameters
  - “What should be true upon entering the function?”
- Postcondition: a formula whose free variables include only the formal parameters and the return value
  - “What is the relationship between the input and output?”

Example: Linear Search

- What would be the pre and post conditions?

@pre: $0 \leq l \wedge u < |a|$

@post: $rv \leftrightarrow \exists i \, . \, l \leq i \leq u \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {
  for (int i := l; i <= u; i := i + 1) {
    if (a[i] = e) return true;
  }
  return false;
}
```

- BTW, is this nontrivial precondition (a formula other than $\top$) is always acceptable?
  - In terms of the software engineering practice (e.g., public API)

Example: More Robust Linear Search

- What would be the pre and post conditions?

@pre: $\top$

@post: $rv \leftrightarrow \exists i \,.\, l \leq i \leq u < |a| \land a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {
  if (l < 0 \/ u >= |a|) return false;
  for (int i := l; i <= u; i := i + 1) {
    if (a[i] = e) return true;
  }
  return false;
}
```

Example: Binary Search

- What would be the pre and post conditions?

@pre: $0 \leq l \land u < |a| \land \text{sorted}(a, l, u)$

@post: $rv \leftrightarrow \exists i \,.\, l \leq i \leq u \land a[i] = e$

```
bool BinarySearch(int[] a, int l, int u, int e) {
  if (l > u) return false;
  int m := (l + u) / 2;
  if (a[m] = e) return true;
  else if (a[m] < e) return BinarySearch(a, m + 1, u, e)
  else return BinarySearch(a, l, m - 1, e)
}
```

- The sorted predicate is defined in the combined theory of integers and arrays:

$$\text{sorted}(a, l, u) \iff \forall i, j \,.\, l \leq i \leq j \leq u \rightarrow a[i] \leq a[j]$$

Example: Bubble Sort

- What would be the pre and post conditions?

@pre: $\top$

@post: sorted$(rv, 0, |rv| - 1)$

```
bool BubbleSort(int[] a0) {
  int a[] := a0;
  for (int i := |a| - 1; i > 0; i := i - 1) {
    for (int j := 0; j < i; j := j + 1) {
      if (a[j] > a[j + 1]) {
        int t := a[j];
        a[j] := a[j + 1];
        a[j + 1] := t;
      }
    }
  }
  return a;
}
```

Enough?

Loop Invariants

- Each loop has an annotation called loop invariant

```
while
      @F
      (⟨condition⟩) {
      ⟨body⟩
}
```

- The assertion F must hold at the beginning of every iteration
  - $F \wedge \langle condition \rangle$ holds on entering the body
  - $F \wedge \neg \langle condition \rangle$ holds when exiting the loop
- Why are loop invariants needed?

Example: Linear Search

- What would be the loop invariant?

@pre: $0 \leq l \wedge u < |a|$

@post: $rv \leftrightarrow \exists i \,.\, l \leq i \leq u \wedge a[i] = e$

```
bool LinearSearch(int[] a, int l, int u, int e) {
  int i := l;
  while
    @L : l ≤ i ∧ (∀j . l ≤ j < i → a[j] ≠ e)
    (i <= u) {
    if (a[i] = e) return true;
    i := i + 1;
  }
  return false;
}
```

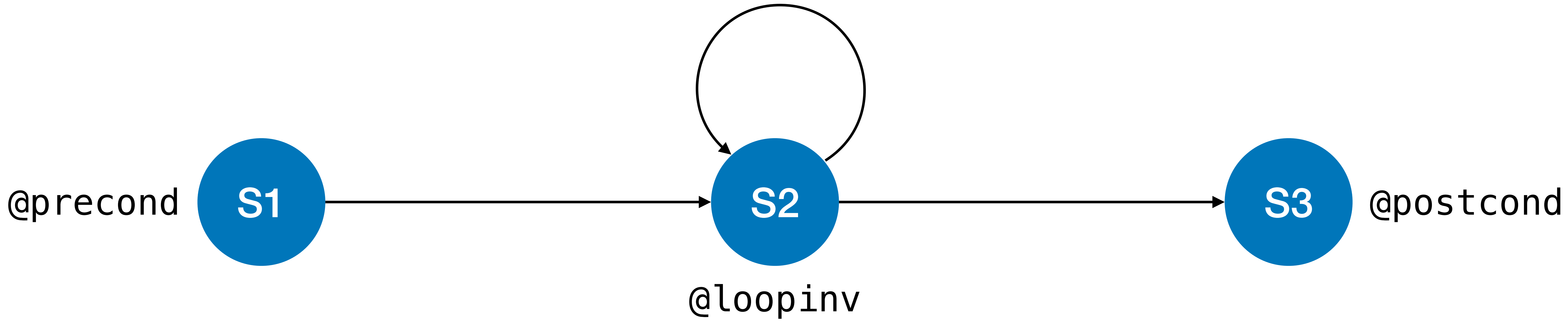
Assertions

- Allows programmers to provide a formal comment
- Runtime assertions: a special class of assertions
  - E.g., division by 0, null dereference, etc
- Example: linear search with runtime assertions

```
bool LinearSearch(int[] a, int l, int u, int e) {
  int i := l;
  while (i <= u) {
    @ 0 ≤ i < |a|
    if (a[i] = e) return true;
    i := i + 1;
  }
  return false;
}
```

Inductive Assertion Method

- Proof technique for partial correctness of programs
- Idea: derive verification conditions from a function given annotations
- Example:



Hoare Triple

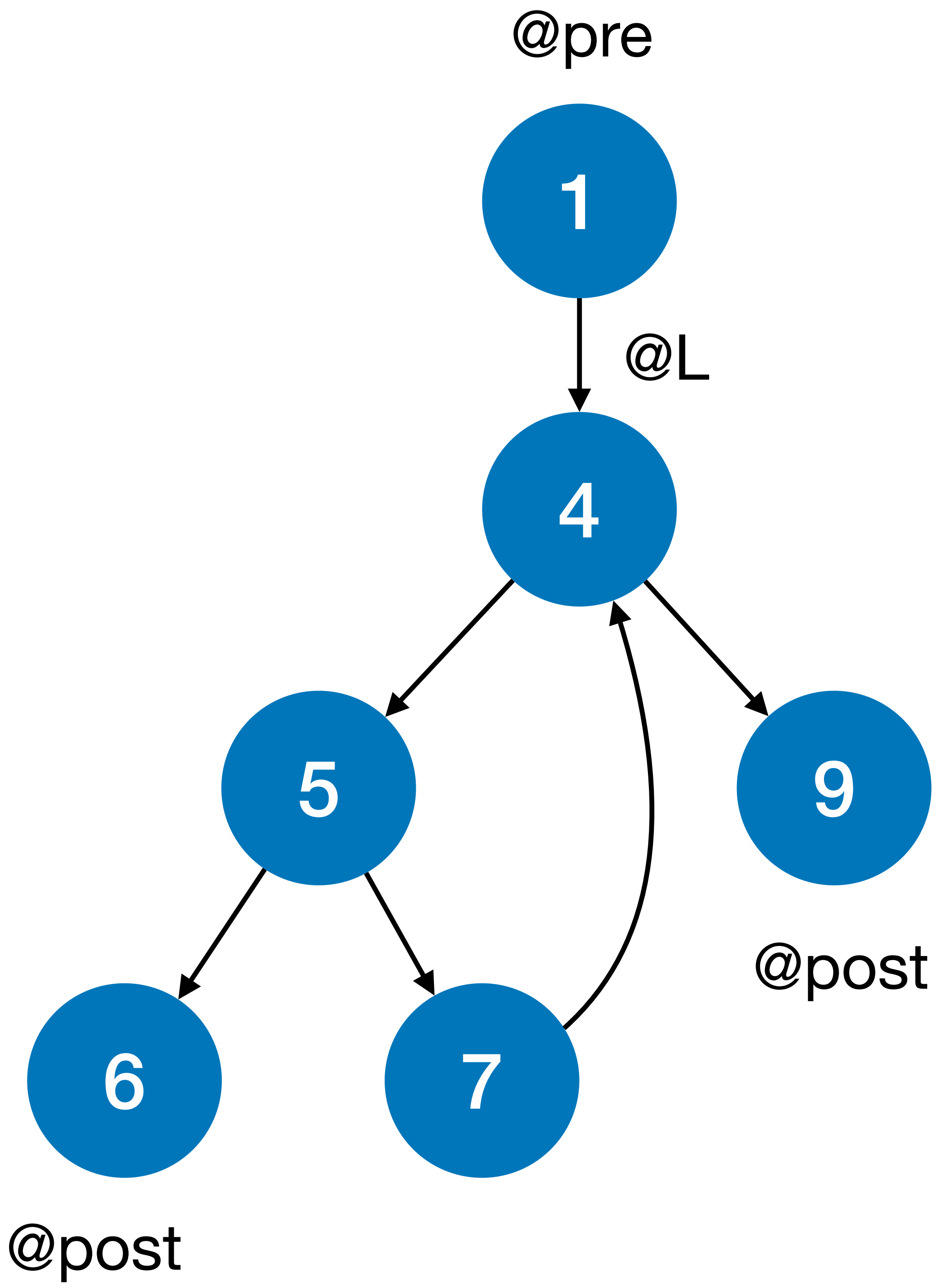
- Partial correctness specified using Hoare triple: $\{P\}\ S\ \{Q\}$
    - S : program fragment
    - P : precondition
    - Q : postcondition
- Meaning of Hoare triple:
    - If S is executed in a state satisfying P and if the execution of S terminates
    - Then, the program state after S terminates satisfies Q

Example

@pre: $0 \le l \wedge u < |a|$

@post: $rv \leftrightarrow \exists i \,.\, l \le i \le u \wedge a[i] = e$

```
   bool LinearSearch(int[] a, int l, int u, int e) {
1:   int i := l;
2:   while
3:     @L : l ≤ i ∧ (∀j . l ≤ j < i → a[j] ≠ e)
4:     (i <= u) {
5:     if (a[i] = e)
6:        return true;
7:     i := i + 1
8:   }
9:   return false;
   }
```



- What conditions should be proven?

  {@pre} S1 {@L} {@L} S4; S5; S6 {@post} {@L} S4; S5; S7 {@L} {@L} S4; S9 {@post}

- What about {@L} S4; S5; S7; S4; S5; S6 {@post} ? Why?

Hoare Logic

- A logic to prove the validity of Hoare triple
- A set of logical rules for reasoning about the partial correctness of programs
- In this lecture, we assume the following simple imperative language

$$
\begin{aligned}
S \rightarrow\ & \mathsf{skip} \\
\mid\ & x := E \\
\mid\ & S; S \\
\mid\ & \texttt{if } E \texttt{ then } S \texttt{ else } S \\
\mid\ & \texttt{while } E \texttt{ do } S
\end{aligned}
$$

Example

- Which one is valid?
  - $\{x = 0\}$ `x := x + 1` $\{x = 1\}$
  - $\{x = 0 \land y = 1\}$ `x := x + 1` $\{x = 1 \land y = 2\}$
  - $\{x = 0\}$ `x := x + 1` $\{x = 1 \lor y = 2\}$
  - $\{x = 0\}$ `while true do x := 0` $\{x = 1\}$

Hoare Rules (1)

- Rule for skip $\dfrac{}{\{P\}\ \texttt{skip}\ \{P\}}$

- Rule for assignment $\dfrac{}{\{Q[E/x]\}\ x := E\ \{Q\}}$

  - Intuition: revert to the state before the assignment

  - Example:

$$\{\mathsf{true}\}\ x := 1\ \{x = 1\} \qquad \{x + 1 > 0\}\ x := x + 1\ \{x > 0\}$$

$$\{y = 1\}\ x := y\ \{x = 1\} \qquad \{\mathsf{false}\}\ x := y + 3\ \{y = 0 \land x = 12\}$$

Hoare Rules (2)

- Rule for precondition strengthening $$\frac{\{P'\}\ S\ \{Q\} \quad P \implies P'}{\{P\}\ S\ \{Q\}}$$
  - Example:

$$\dfrac{\dfrac{\{y > 0[x/y]\}\ y := x\ \{y > 0\}}{\{x > 0\}\ y := x\ \{y > 0\}} \qquad x = 2 \implies x > 0}{\{x = 2\}\ y := x\ \{y > 0\}}$$

- Rule for postcondition weakening $$\frac{\{P\}\ S\ \{Q'\} \quad Q' \implies Q}{\{P\}\ S\ \{Q\}}$$
  - Example:

$$\dfrac{\dfrac{\dots}{\{\text{true}\}\ S\ \{x = y \land z = 2\}} \qquad x = y \land z = 2 \implies x = y}{\{\text{true}\}\ S\ \{x = y\}}$$

Hoare Rules (3)

- Rule for composition

$$\frac{\{P\}\ S_1\ \{Q\} \quad \{Q\}\ S_2\ \{R\}}{\{P\}\ S_1; S_2\ \{R\}}$$

$$\dfrac{\dfrac{\{x = 2[2/x]\}\ x := 2\ \{x = 2\}}{\{\mathsf{true}\}\ x := 2\ \{x = 2\}} \quad \dfrac{\{x = 2 \land y = 2[x/y]\}\ y := x\ \{x = 2 \land y = 2\}}{\{x = 2\}\ y := x\ \{x = 2 \land y = 2\}}}{\{\mathsf{true}\}\ x := 2;\ y := x\ \{x = 2 \land y = 2\}}$$

- Rule for if statement

$$\frac{\{P \land E\}\ S_1\ \{Q\} \quad \{P \land \neg E\}\ S_2\ \{Q\}}{\{P\}\ \texttt{if}\ E\ \texttt{then}\ S_1\ \texttt{else}\ S_2\ \{Q\}}$$

$$\dfrac{\dfrac{\{y \geq 0[x/y]\}\ y := x\ \{y \geq 0\} \quad x \geq 0 \implies x > 0}{\{x > 0\}\ y := x\ \{y \geq 0\}} \quad \dfrac{\{y \geq 0[-x/y]\}\ y := -x\ \{y \geq 0\}}{\{x \leq 0\}\ y := -x\ \{y \geq 0\}}}{\{\texttt{true}\}\ \texttt{if}\ x > 0\ \texttt{then}\ y := x\ \texttt{else}\ y := -x\ \{y \geq 0\}}$$

Hoare Rules (4)

- Rule for loop

$$\frac{\{P \land E\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ E\ \texttt{do}\ S\ \{P \land \neg E\}}$$

$$\frac{\dfrac{\cdots}{\{x \le n \land x < n\}\ x := x + 1\ \{x \le n\}}}{\{x \le n\}\ \texttt{while}\ x < n\ \texttt{do}\ x := x + 1\ \{x \ge n\}}$$

Loop Invariant

- Challenge: impossible to know how many times a given loop iterates
- How to prove the partial correctness of a loop within finite time?
- Analogy: mathematical induction
- Loop invariant I satisfies the following properties:
  - I holds initially before the loop
  - I holds after each iteration of the loop
- Example

```
i := 0; j := 0; n := 10;
while (i < n) { // loop invariants?
  i := i + 1;
  j := i + j;
}
```

Inductive Invariant

- Not all invariants are provable
- Example:

```
i := 5;
while (i > 1) { // invariant: i > 0
  i := i - 2;
}
assert(i = 1);
```

$$\frac{\{P \land E\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ E\ \texttt{do}\ S\ \{P \land \neg E\}}$$

- Inductive invariant: invariant we can prove using induction
- Challenge: finding inductive loop invariants
  - Practice: human, static analysis, machine learning, etc

Automatically Proving Partial Correctness

- $\{P\}\ S\ \{Q\}$: Given the precondition satisfied, the postcondition is satisfied after the execution (if it terminates)
- Assumption: loop invariants are given by an oracle
  - Oracle: human, static analysis, machine learning, etc
- How to automatically prove correctness?
- Idea: deriving verification conditions (VCs) and check the validity

Verification Condition

- A FOL formula F such that the program is correct iff F is valid
- Automatically proving partial correctness
  - Generating VCs from a program + checking the validity of VCs by a theorem prover
- Two ways to generate verification conditions
  - Forward: starting from prediction, generate formulas to prove postcondition (strongest postconditions)
  - Backward: starting from postcondition, generate formulas to prove precondition (weakest preconditions)

Weakest Liberal Preconditions

- Goal: verify Hoar triple $\{P\}\ S\ \{Q\}$
- Weakest liberal precondition $wlp(S, Q)$ [Dijkstra75]
  - Weakest: most general condition that guarantees Q will hold after S in any execution
  - Liberal: we do not care about termination
- Proof of the Hoar triple $\{P\}\ S\ \{Q\}$: $P \rightarrow wlp(S, Q)$
- Example: $\{y \geq 10\}\ x := y + 1\ \{x \geq 0\}$

Weakest Precondition Calculus

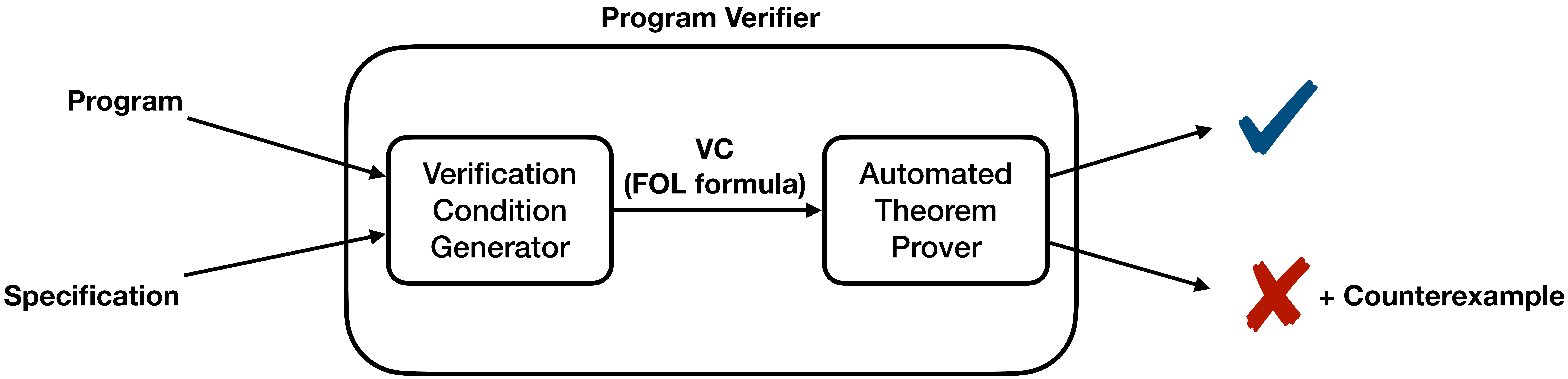
- Inductively define wlp following Hoare rules
- $wlp(x := E, Q) = Q[E/x]$
- $wlp(s_1; s_2, Q) = wlp(s_1, wlp(s_2, Q))$
- $wlp(\text{if } E \text{ then } s_1 \text{ else } s_2, Q) = E \rightarrow wlp(s_1, Q) \wedge \neg E \rightarrow wlp(s_2, Q)$
- $wlp(\text{while } E \text{ do } S, Q) = I \wedge (E \wedge I \rightarrow wlp(S, I)) \wedge (\neg E \wedge I \rightarrow Q)$
  - Assumption: an inductive invariant I is provided

Example

- $S : x := y + 1;$ if $x > 0$ then $z := 1$ else $z := -1$
- $wlp(S, z > 0)$?
- $wlp(S, z \leq 0)$?
- $\{y \geq -1\}\ S\ \{z = 1\}$?
- $\{y > -1\}\ S\ \{z = 1\}$?

Verification of Hoare Triple

- Validity of $\{P\}\ S\ \{Q\}$
- Verification condition: $P \rightarrow wlp(S, Q)$



Summary

- Hoare triple: specifications for partial correctness $\{P\}\ S\ \{Q\}$
- Hoare logic: a logic to prove the validity of Hoare triple
  - Proof rules for each program command
- Verification condition is valid iff the Hoare triple is valid
- Automated program verification: check whether the VC is valid using theorem provers