Program Reasoning

5. First-order Logic

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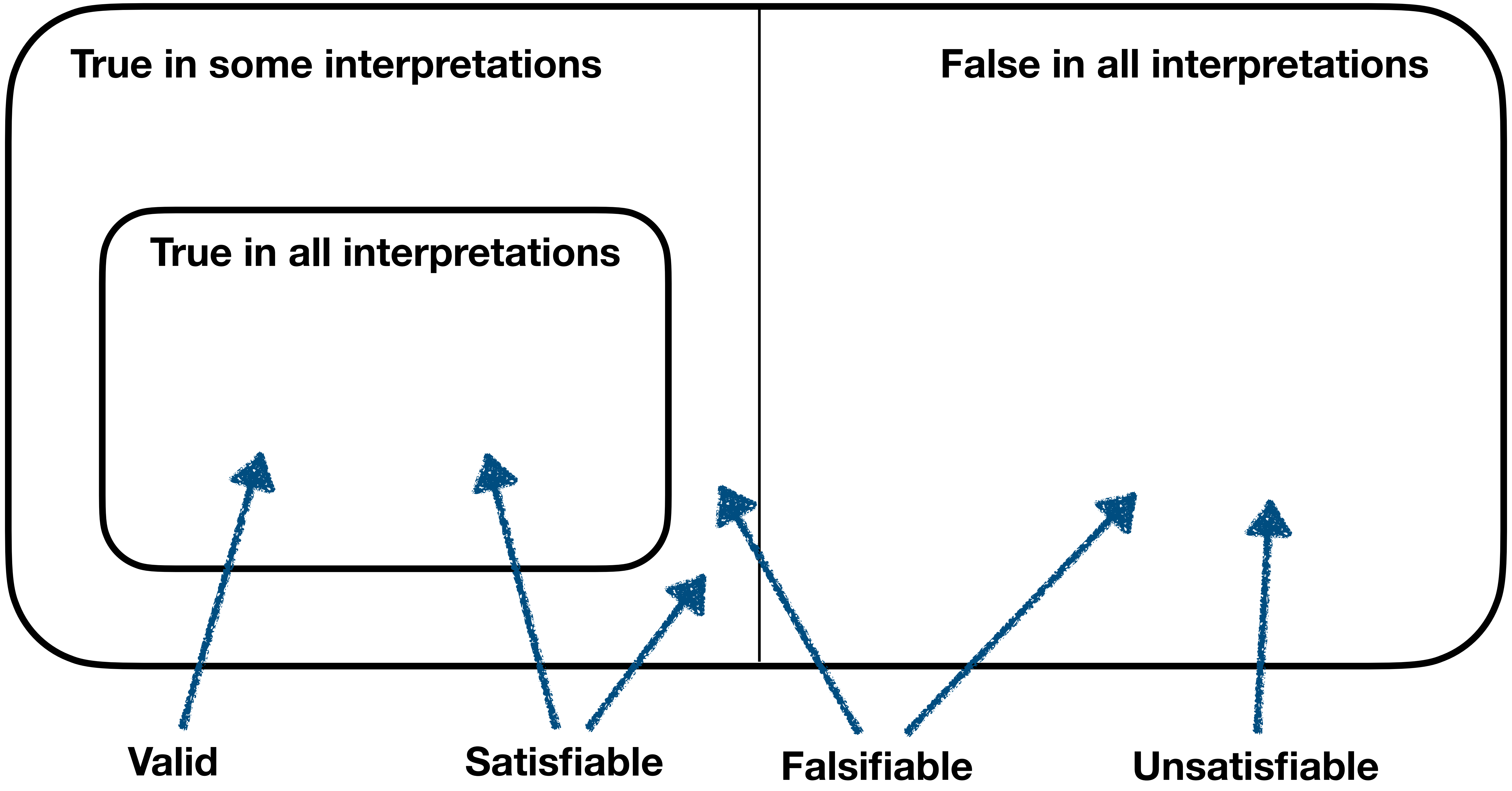
First-order Logic

- An extension of propositional logic with predicates, functions and quantifiers
- FOL is more expressive than propositional logic
  - Expressive enough to reason about programs
- Not admit completely automated reasoning (i.e., undecidable)
  - "Yes, F is valid" (so, $\neg F$ is unsatisfiable)
  - "Yes, $\neg F$ is valid" (so, F is unsatisfiable)
  - "…" (may not terminate if F is invalid)
  - Note: "F is invalid" $\neq$ "$\neg F$ is valid"

Valid, Satisfiable, Falsifiable and Unsatisfiable



Syntax (1): Terms

- Objects that we are reasoning about
- Terms evaluate to values in an underlying domain (e.g., integers, strings, lists, etc)
  - C.f., All formulae in PL evaluate to true or false
- Basic terms: variables (x, y, z, …) and constants (a, b, c, …)
- Composite terms: n-ary functions applied to n terms
  - A constant can be viewed as a 0-ary function
- Example:
  - $a,\ x,\ f(a),\ g(x, b),\ f(g(x, f(b)))$

Syntax (2): Predicates

- Generalization of propositional variables in PL (p, q, r, …)
- An n-ary predicate takes n terms as arguments
    - A FOL propositional variable is a 0-ary predicate (P, Q, R, …)
- Example:
    - P, $p(f(x), g(x, f(x)))$
    - *isHappy*(x), *love*(x, y), *betterThan*(x, y)

Predicates and Functions

- They look similar but different
- Function terms can be nested within each other and inside relation constants
  - E.g., $f(f(x))$, $p(f(x))$
- Predicates cannot be nested within function terms or other predicates
  - E.g., $f(p(x))$, $p(p(x))$

Syntax (3): Formula

- Atom: basic elements
  - truth symbols ($\bot$ and $\top$), n-ary predicates applied to n terms
- Literal: an atom α or its negation $\neg\alpha$
- Formula: literal, the app. of a logical conn. to formulae, or the app. of a quantifier to a formula

$$
\begin{array}{rcl}
F & \rightarrow & \bot \mid \top \mid p(t_1, \ldots, t_n) \\
 & \mid & \neg F \\
 & \mid & F_1 \wedge F_2 \\
 & \mid & F_1 \vee F_2 \\
 & \mid & F_1 \rightarrow F_2 \\
 & \mid & F_1 \leftrightarrow F_2 \\
 & \mid & \exists x. F[x] \\
 & \mid & \forall x. F[x]
\end{array}
$$

Quantification

quantified variable

$\exists x.F[x]$

$\forall x.F[x]$

scope of quantifier

"x is bound in $F[x]$"

$$\underbrace{\forall x.p(f(x),x) \rightarrow (\overbrace{\exists y.p(f(g(x,y)),g(x,y))}^{\text{scope of y}}) \wedge q(x,f(x))}_{\text{scope of x}}$$

- A variable is free in $F[x]$ if it is not bound
- free(F) and bound(F) denote the free and bound variables of F
- A formula F is closed if F has no free variables
- If free$(F) = \{x_1, \ldots, x_n\}$, the universal closure is $\forall x_1, \ldots, x_n . F$ (usually $\forall * .F$) and its existential closure is $\exists x_1, \ldots, x_n . F$ (usually $\exists * .F$)

Example

- Every dog has its day $\forall x.dog(x) \rightarrow \exists y.day(y) \wedge itsDay(x, y)$

- Some dogs have more days than others $\exists x, y.dog(x) \wedge dog(y) \wedge \#days(x) > \#days(y)$

- The length of one side of a triangle is less than the sum of the lengths of the other two sides

$$\forall x, y, z.triangle(x, y, z) \rightarrow length(x) < length(y) + length(z)$$

- Fermat's Last Theorem

$$\begin{aligned}
&\forall n.integer(n) \wedge n > 2 \\
&\quad \rightarrow \forall x, y, z. \\
&\qquad integer(x) \wedge integer(y) \wedge integer(z) \wedge x > 0 \wedge y > 0 \wedge z > 0 \\
&\qquad\quad \rightarrow x^n + y^n \neq z^n
\end{aligned}$$

Interpretation (1)

- A FOL interpretation $I : (D_I, \alpha_I)$ is a pair of a domain and an assignment
  - D_I : a nonempty set of values such as integers, real numbers, etc
  - α_I : a mapping from variables, constants, functions, and predicate symbols to elements, functions, and predicates over D_I
    - Each variable x is assigned to a value from D_I
    - Each n-ary function symbol f is assigned an n-ary function $f_I : D_I^n \rightarrow D_I$
    - Each n-ary predicate symbol p is assigned an n-ary predicate $p_I : D_I^n \rightarrow \{\text{true}, \text{false}\}$

Interpretation (2)

- Interpretation of complicated atoms: recursively defined
- Evaluate arbitrary terms recursively:
  - $\alpha_I[f(t_1, \ldots, t_n)] = \alpha_I[f](\alpha_I[t_1], \ldots, \alpha_I[t_n])$
- Evaluate arbitrary terms recursively:
  - $\alpha_I[p(t_1, \ldots, t_n)] = \alpha_I[p](\alpha_I[t_1], \ldots, \alpha_I[t_n])$

Example

$$F : x + y > z \rightarrow y > z - x$$

- Note: +, -, > are just symbols and no meaning is given without an interpretation
  - Alternative form: $p(f(x, y), z) \rightarrow p(y, g(z, x))$
- The standard interpretation
  - Domain $D_I = \mathbb{Z}$
  - Assignment $\alpha_I = \{+ \mapsto +_{\mathbb{Z}}, - \mapsto -_{\mathbb{Z}}, > \mapsto >_{\mathbb{Z}}, x \mapsto 13, y \mapsto 42, z \mapsto 1, \ldots\}$

Semantics

- Given an interpretation $I : (D_I, \alpha_I)$, $I \vDash F$ or $I \nvDash F$

$$
\begin{array}{ll}
I \models \top & \\
I \not\models \bot & \\
I \models p(t_1, \ldots, t_n) & \text{iff } \alpha_I[p(t_1, \ldots, t_n)] = \text{true} \\
I \models \neg F & \text{iff } I \not\models F \\
I \models F_1 \wedge F_2 & \text{iff } I \models F_1 \text{ and } I \models F_2 \\
I \models F_1 \vee F_2 & \text{iff } I \models F_1 \text{ or } I \models F_2 \\
I \models F_1 \rightarrow F_2 & \text{iff, if } I \models F_1 \text{ then } I \models F_2 \\
I \models F_1 \leftrightarrow F_2 & \text{iff, if } I \models F_1 \text{ and } I \models F_2, \text{ or if } I \not\models F_1 \text{ and } I \not\models F_2 \\
I \models \forall x.F & \text{iff for all } v \in D_I, I \triangleleft \{x \mapsto v\} \models F \\
I \models \exists x.F & \text{iff there exists } v \in D_I, I \triangleleft \{x \mapsto v\} \models F
\end{array}
$$

where $J : I \triangleleft \{x \mapsto v\}$ denotes an $\boldsymbol{x}$-variant of $\boldsymbol{I}$

- $D_J = D_I$
- $\alpha_J[y] = \alpha_I[y]$ for all constant, free variable, function, and predicate symbols y except that $\alpha_J(x) = v$

Example

$$F : \exists x. f(x) = g(x)$$

- Consider the interpretation $I : (D_I, \alpha_I)$
  - $D_I = \{0,1\}$
  - $\alpha_I = \{f(0) \mapsto 0,\ f(1) \mapsto 1,\ g(0) \mapsto 1,\ g(1) \mapsto 0\}$
- Compute the truth value of F under I
  - $I \triangleleft \{x \mapsto v\} \not\models f(x) = g(x)$ for $v \in D_I$
  - $I \not\models \exists x . f(x) = g(x)$ since $v \in D_I$ is arbitrary

Satisfiability and Validity

- A formula F is satisfiable iff there exists an interpretation I such that $I \vDash F$
- A formula F is valid iff for all interpretations I, $I \vDash F$
- Satisfiability and validity are dual: F is valid iff $\neg F$ is unsatisfiable
- Satisfiability and validity are defined for closed FOL, but conventionally
    - A formula with free variables is valid : $\forall * .F$ is valid
    - A formula with free variables is satisfiable : $\exists * .F$ is satisfiable

Proof Rules (1)

- According to the semantics of negation,

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

- According to the semantics of conjunction,

$$\frac{I \models F \land G}{I \models F, I \models G} \qquad \frac{I \not\models F \land G}{I \not\models F \mid I \not\models G}$$

Proof Rules (2)

- According to the semantics of disjunction,

$$\frac{I \models F \vee G}{I \models F \quad | \quad I \models G} \qquad\qquad \frac{I \not\models F \vee G}{I \not\models F,\ I \not\models G}$$

- According to the semantics of implication,

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G} \qquad\qquad \frac{I \not\models F \rightarrow G}{I \models F,\ I \not\models G}$$

Proof Rules (3)

- According to the semantics of iff,

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \ \mid \ I \models \neg F \wedge \neg G} \qquad \frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \ \mid \ I \models \neg F \wedge G}$$

Proof Rules (4)

- A contradiction exists if two variants of the original interpretation I disagree

$$\frac{\begin{array}{c} J : I \triangleleft \cdots \models p(s_1, \ldots, s_n) \\ K : I \triangleleft \cdots \not\models p(t_1, \ldots, t_n) \end{array}}{I \models \bot} \text{ for } i \in \{1, \ldots, n\}, \alpha_J[s_i] = \alpha_K[t_i]$$

- Example

$$\frac{I \triangleleft \{x \mapsto a\} \models p(x),\ I \triangleleft \{y \mapsto a\} \not\models p(y)}{I \models \bot}$$

Proof Rules (5)

- According to the semantics of universal quantification

$$\frac{I \models \forall x.F}{I \triangleleft \{x \mapsto v\} \models F} \text{ for any } v \in D_I$$

- According to the semantics of existential quantification

$$\frac{I \not\models \exists x.F}{I \triangleleft \{x \mapsto v\} \not\models F} \text{ for any } v \in D_I$$

(Usually applied using a domain element v that was introduced earlier in the proof)

Example

- Prove $F : (\forall x \,.\, p(x)) \rightarrow (\exists y \,.\, p(y))$ is valid

Proof Rules (6)

- According to the semantics of universal quantification

$$\frac{I \not\models \forall x.F}{I \triangleleft \{x \mapsto v\} \not\models F} \text{ for a fresh } v \in D_I$$

(v must not have been previously used in the proof)

- Example: prove $F : p(a) \rightarrow \forall x \,.\, p(x)$ is valid

Proof Rules (7)

- According to the semantics of existential quantification

$$\frac{I \models \exists x.F}{I \triangleleft \{x \mapsto v\} \models F} \text{ for a fresh } v \in D_I$$

(v must not have been previously used in the proof)

- Example: prove $F : \exists x \,.\, p(x) \rightarrow p(a)$ is valid

Example (1)

- Prove $F : (\forall x \,.\, p(x)) \rightarrow (\forall y \,.\, p(y))$ is valid

Example (2)

- Prove $F : (\forall x . p(x)) \to (\neg \exists y . \neg p(y))$ is valid

Example (3)

- Prove $F : (\forall x \,.\, p(x)) \rightarrow (\neg \exists y \,.\, \neg p(y))$ is valid

$$F : p(a) \rightarrow (\exists x . p(x))$$

Example (4)

$$F : (\forall x . p(x, x)) \rightarrow (\exists x . \forall y . p(x, y))$$

Soundness and Completeness

- Soundness: if every branch of a semantic argument proof reach $I \models \bot$ then F is valid
- Completeness: each valid formula F has a semantic argument proof in which every branch reaches $I \models \bot$
  - Gödel’s completeness theorem
  - “Anything universally true is provable”
- Note: DO NOT get confused with Gödel's **incompleteness** theorem
  - First-order logic: complete (completeness theorem)
  - First-order logic of (Peano) arithmetic: incomplete (incompleteness theorem)

Decidability

- Does there exist an algorithm to solve a problem?
  - Solve: eventually halt and return a correct answer
  - E.g., Halting problem
- Our problem: satisfiability (or dually, validity) of FOL
- Satisfiability of PL: decidable
- Satisfiability of FOL: semi-undecidable (by Church and Turing)
  - If F is valid, the algorithm says "Yes"
  - If F is invalid, the algorithm may not terminate

Summary

- FOL: an extension of PL with predicates, functions and quantifiers
  - Powerful enough to reason about properties of software
- Proof system (semantic argument method) for validity
  - Sound and complete
  - Undecidable
- How to use FOL for program reasoning, mathematical reasoning, etc?
  - Next topic: First-order theories