

Program Reasoning

4. Propositional Logic

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Logic

- What is logic? A tool for reasoning about truths
- Why logic for computer scientists? Reasoning about computation
- For example,
 - “Does this program accept an array of integers and produce a sorted array?”
 - “Does this program access an unallocated memory?”
 - “Does this function always halt?”
- This course: propositional logic (PL) and first-order logic (FOL)

Syntax

- Atom: basic elements
 - Truth symbols: \top (“true”) and \perp (“false”)
 - Propositional variables: P, Q, R, \dots
- Literal: an atom α or its negation $\neg\alpha$
- Formula: a literal or the application of a logical connective to formulae

$$\begin{array}{c} F \rightarrow \begin{array}{l} \perp \\ \top \\ P, Q, R, \dots \\ \neg F \\ F_1 \wedge F_2 \\ F_1 \vee F_2 \\ F_1 \rightarrow F_2 \\ F_1 \leftrightarrow F_2 \end{array} \end{array}$$

Semantics

- Give meaning to formulae
 - In propositional logic, the truth values
- The semantics of a formula is defined with an interpretation I
 - An interpretation assigns to every propositional variable exactly one truth value
- For example, $F : P \wedge Q \rightarrow P \vee \neg Q$ and $I : \{P \mapsto \top, Q \mapsto \perp\}$

Inductive Definition of PL

- Notation:

- $I \models F$ if F evaluates to true under I
- $I \not\models F$ if F evaluates to false under I

$$I \models \top$$

$$I \not\models \perp$$

$$I \models P \quad \text{iff } I[P] = \text{true}$$

$$I \not\models P \quad \text{iff } I[P] = \text{false}$$

$$I \models \neg F \quad \text{iff } I \not\models F$$

$$I \models F_1 \wedge F_2 \quad \text{iff } I \models F_1 \text{ and } I \models F_2$$

$$I \models F_1 \vee F_2 \quad \text{iff } I \models F_1 \text{ or } I \models F_2$$

$$I \models F_1 \rightarrow F_2 \quad \text{iff, if } I \models F_1 \text{ then } I \models F_2$$

$$I \models F_1 \leftrightarrow F_2 \quad \text{iff, if } I \models F_1 \text{ and } I \models F_2, \text{ or if } I \not\models F_1 \text{ and } I \not\models F_2$$

Example

- $F : P \wedge Q \rightarrow P \vee \neg Q$ and $I : \{P \mapsto \top, Q \mapsto \perp\}$

Satisfiability and Validity

- Two important tasks in logic (why? when?)
- A formula F is **satisfiable** iff there exists an interpretation I such that $I \models F$
- A formula F is **valid** iff for all interpretations I , $I \models F$
- Satisfiability and validity are dual: F is valid iff $\neg F$ is unsatisfiable
- We are free to focus on either one; the other will follow

Determining Validity and Satisfiability (1)

- Truth table method
 - For example, $F : P \wedge Q \rightarrow P \vee \neg Q$
- Impractical: 2^n interpretations
- Impossible: for any other logic where the domain is not finite (e.g., first-order logic)

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Determining Validity and Satisfiability (2)

- Semantic argument method (proof by contradiction)
 - Assume F is invalid: $I \not\models F$
 - Apply proof rules to derive
 - Derive a contradiction in every branch of the proof
 - Then, F is valid

Proof Rules (1)

- According to semantics of negation,

$$\frac{I \models \neg F}{I \not\models F}$$

$$\frac{I \not\models \neg F}{I \models F}$$

- According to semantics of conjunction,

$$\frac{I \models F \wedge G}{I \models F, I \models G}$$

$$\frac{I \not\models F \wedge G}{I \not\models F \mid I \not\models G}$$

Proof Rules (2)

- According to semantics of disjunction,

$$\frac{I \models F \vee G}{I \models F \mid I \models G}$$

$$\frac{I \not\models F \vee G}{I \not\models F, I \not\models G}$$

- According to semantics of implication,

$$\frac{I \models F \rightarrow G}{I \not\models F \mid I \models G}$$

$$\frac{I \not\models F \rightarrow G}{I \models F, I \not\models G}$$

Proof Rules (3)

- According to semantics of iff,

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \mid I \models \neg F \wedge \neg G}$$

$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \mid I \models \neg F \wedge G}$$

- Contradiction

$$\frac{I \models F, I \not\models F}{I \models \perp}$$

Example

- Prove $F : P \wedge Q \rightarrow P \vee \neg Q$ is valid

Example

- Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid

Proof Tree

- Proof evolves as a **tree** rather than **linearly**
 - A branch of the tree is a sequence of lines descending from the root
 - A branch is **closed** if it contains a contradiction, otherwise open
 - A semantic argument is **finished** when no more proof rules are applicable
- Proof of the validity of F : if every branch is closed
 - Otherwise, each open branch describes a falsifying interpretation of F

Derived Rules

- The proof rules are theoretically sufficient
- However, derived proof rules can make proofs more concise (c.f., procedure, subroutine)
- Example: modus ponens

$$\frac{I \models F, \quad I \models F \rightarrow G}{I \models G}$$

Example

- Prove $F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is valid

Proof of Satisfiability

- Dual of the validity proof
- Truth-table or semantic argument methods
- Example: $\neg F : \neg(P \vee Q \rightarrow P \wedge Q)$

P	Q	$P \vee Q$	$P \wedge Q$	F	$\neg F$
0	0	0	0	1	0
0	1	1	0	0	1
1	0	1	0	0	1
1	1	1	1	1	0

Example

- Prove $G : \neg(P \vee Q \rightarrow P \wedge Q)$ is satisfiable

Equivalence and Implication

- Important properties of pairs of formulae
- Two formulae F_1 and F_2 are **equivalent** iff $F_1 \leftrightarrow F_2$ is valid: $F_1 \iff F_2$
 - $F_1 \iff F_2$ is not a formula but a statement
- Formula F_1 **implies** formula F_2 iff $F_1 \rightarrow F_2$ is valid: $F_1 \implies F_2$
 - $F_1 \implies F_2$ is not a formula but a statement

Examples

- Prove $P \iff \neg\neg P$
(using the truth table method)

We prove that $P \leftrightarrow \neg\neg P$ is valid:

P	$\neg P$	$\neg\neg P$	$P \leftrightarrow \neg\neg P$
0	1	0	1
1	0	1	1

- Prove $P \rightarrow Q \iff \neg P \vee Q$
(using the truth table method)

We prove that $F : P \rightarrow Q \leftrightarrow \neg P \vee Q$ is valid:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$	F
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

Example

- Prove that $R \wedge (\neg R \vee P) \implies P$

We prove that $R \wedge (\neg R \vee P) \rightarrow P$ is valid

Summary

- Propositional logic: the simplest form of logic
- Interpretation: decide the meaning of a formula (either true or false)
- Satisfiability: is there any interpretation that makes the formula be true?
- Validity: does the formula evaluate to be true for all interpretations?
- Duality of satisfiability and validity
 - E.g., “no input can trigger this bug” = “work well with all inputs”
- Equivalence and implication: properties of two formulae