ECON312 Problem Set B3

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Contents


```
library(dplyr)
library(tidyr)
library(ggplot2)
library(purrr)
library(knitr)
library(latex2exp)
set.seed(112345)
```

2: Roy Model of College

$$Y_1 = \alpha + \bar{\beta} + U_1$$
$$Y_0 = \alpha + U_0$$

$$D = \mathbf{1}\{Y_1 - Y_0 - C \geq 0\}$$

$$\begin{pmatrix} U_0 \\ U_1 \\ V \end{pmatrix} \sim N(0, \begin{bmatrix} \sigma_0^2 & \sigma_{01} & \sigma_{0,V} \\ \sigma_{01} & \sigma_1^2 & \sigma_{1,V} \\ \sigma_{0,V} & \sigma_{1,V} & \sigma_V^2 \end{bmatrix})$$

Where $C = Z + V$, Z is tuition and V is psychic costs

$$Z \perp\!\!\!\perp (U_1, U_0, V)$$

Where $\alpha = 0.67$ and $\bar{\beta} = 0.2$

Assume $Z \sim U[-0.5, 0.5]$

Monte Carlo simulation of model

```
roy_MC <- function(config_num, sigma_11, sigma_00, sigma_V, sigma_01, sigma_0V, sigma_1V,
                   alpha = 0.67, beta = 0.2, N = 100000,
                   zmin = -0.5, zmax = 0.5
                   ){

  Z <- runif(N, zmin, zmax)

  if (sigma_V > 0){
      Sigma <- matrix(c(sigma_00, sigma_01, sigma_0V, sigma_01, sigma_11, sigma_1V, sigma_0V, sigma_1V,

        latent_vars <- MASS::mvrnorm(N, c(0,0,0), Sigma) %>%
           as_tibble()

        colnames(latent_vars) <- c("U_0", "U_1", "V")

  } else {
          Sigma <- matrix(c(sigma_00, sigma_01, sigma_01, sigma_11), nrow = 2)

        latent_vars <- MASS::mvrnorm(N, c(0,0), Sigma) %>%
           as_tibble()

        colnames(latent_vars) <- c("U_0", "U_1")

        latent_vars <- latent_vars %>%
          mutate(V = 0)

  }

  sigma_v_tilde <- sqrt(sigma_00 + sigma_11 + sigma_V -2*sigma_01 +2*sigma_0V - 2*sigma_1V)


  outcomes <- latent_vars %>%
    cbind(Z) %>%
    mutate(Y_1 = alpha + beta + U_1,
           Y_0 = alpha + U_0,
           beta_i = Y_1 - Y_0,
           C = Z + V,
           D = ifelse(Y_1 - Y_0 - C >= 0, 1, 0),
           V_tilde = V + U_0 - U_1,
           U_D = pnorm(V_tilde/sigma_v_tilde),
           MTE = beta + V_tilde*(-sigma_11 - sigma_00 +2*sigma_01 +sigma_1V - sigma_0V)/sigma_v_tilde^2
           Y = D*Y_1 + (1-D)*Y_0,
           config_num = config_num)

  outcomes

}
```

Simulations for Z $U(-0.5, 0.5)$

```
config_1 <- roy_MC("I", 1,1,1,0,0,0)
config_2 <- roy_MC("II",1,1,1,0.5,-0.5,0)
config_3 <- roy_MC("III",1,1,1,0,0,0.5)
config_4 <- roy_MC("IV",1,1,0, 0.5,0,0)
config_5 <- roy_MC("V", 1,0.25,1,0.1,-0.2,0.2)
config_6 <- roy_MC("VI",0.25,0.25,1,-0.2,0.1,-0.2)
config_7 <- roy_MC("VII",1,1,1,-0.2,0.6,0.5)
config_list <- list(config_1, config_2, config_3,
                    config_4, config_5, config_6, config_7)
```

A. Derive and graph the MTE

The propensity score $P(Z)$ is

$$P(D = 1|Z) = E[\mathbf{1}\{Y_1 - Y_0 - C \geq 0\}|Z]$$

plugging in for $Y_1 - Y_0 - C = \bar{\beta} + U_1 - U_0 - Z - V$, we have

$$P(Z) = P(\bar{\beta} + U_1 - U_0 - Z - V \geq 0)$$

re-arranging we have

$$P(Z) = P(\bar{\beta} - Z \geq V + U_0 - U_1)$$

Or

$$P(Z) = P(V + U_0 - U_1 \leq \bar{\beta} - Z)$$

Writing it this way, we can define a latent variable $\tilde{V} = -U_1 + U_0 + V$ that will contribute to treatment choice. The Computing the variance $Var(-U_1 + U_0 + V)$, we have

$$Var(-U_1 + U_0 + V) = \sigma^2_{\tilde{V}} = \sigma^2_v + \sigma^2_0 + \sigma^2_1 - 2\sigma_{01} + 2\sigma_{0V} - 2\sigma_{1V}$$

Since we know that (U_1, U_0, V) is distributed as multivariate normal, $\tilde{V} \sim N(0, \sqrt{\sigma^2_v + \sigma^2_0 + \sigma^2_1 - 2\sigma_{01} + 2\sigma_{0V} - 2\sigma_{1V}})$. So we can re-write the propensity score as

$$P(D = 1|Z) = P(Z = z) = P(\frac{\tilde{V}}{\sigma_{\tilde{V}}} \leq \frac{\bar{\beta} - Z}{\sigma_{\tilde{V}}})$$

and since $\frac{\tilde{V}}{\sigma_{\tilde{V}}} \sim N(0, 1)$, finally we can write

$$P(Z) = U_D = \Phi(\frac{\bar{\beta} - Z}{\sigma_{\tilde{V}}})$$

Therefore we define the MTE as

$$MTE(U_D) = E[Y_1 - Y_0|U_D = u_D] = E[Y_1 - Y_0|\tilde{V} = \tilde{v}]$$

Because of **index sufficiency**.

$$E[Y_1 - Y_0|\tilde{V} = \tilde{v}] = \bar{\beta} + E[U_1 - U_0|\tilde{V} = \tilde{v}]$$

Because of the given multivariate normal relationship, we can expand the second term

$$E[Y_1 - Y_0|Z = z] = \bar{\beta} + Cov(U_1 - U_0, \frac{-U_1 + U_0 + V}{\sigma_{\tilde{V}}}) * (-U_1 + U_0 + V)$$

Note

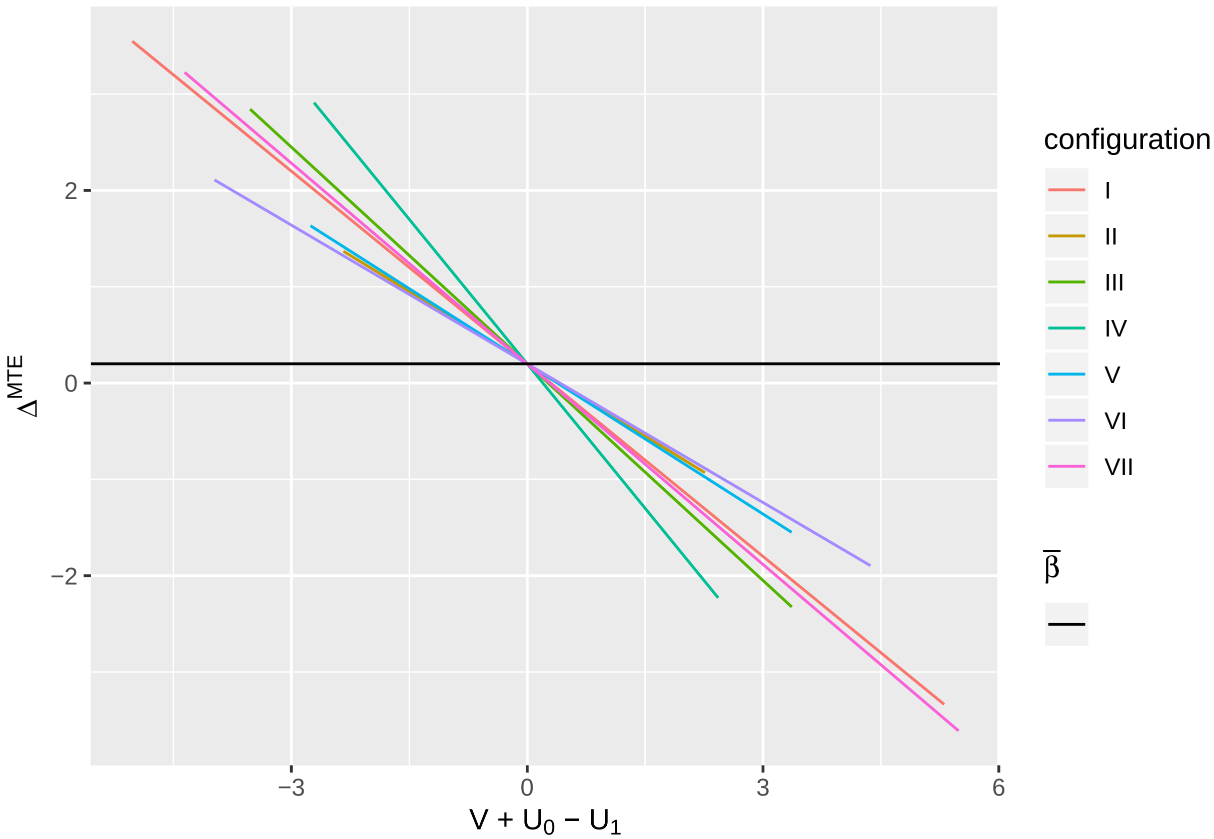
$$Cov(U_1 - U_0, \frac{-U_1 + U_0 + V}{\sigma_{\tilde{V}}}) = \frac{1}{\sigma_{\tilde{V}}}(-Var(U_1 - U_0) + Cov(U_1 - U_0, V)) = \frac{1}{\sigma_{\tilde{V}}}(-\sigma_1^2 - \sigma_0^2 + 2\sigma_{01} + \sigma_{V1} - \sigma_{V0})$$

So plugging in we have

$$\Delta^{MTE} = \bar{\beta} + \frac{-\sigma_1^2 - \sigma_0^2 + 2\sigma_{01} + \sigma_{V1} - \sigma_{V0}}{\sigma_v^2 + \sigma_0^2 + \sigma_1^2 - 2\sigma_{01} + 2\sigma_{0V} - 2\sigma_{1V}} * (V + U_0 - U_1)$$

```
MTE <- function(config_num,
                     sigma_11, sigma_00, sigma_V, sigma_01, sigma_0V, sigma_1V,
                     beta = 0.2){

  sigma_v_tilde <- sqrt(sigma_00 + sigma_11 + sigma_V -2*sigma_01 +2*sigma_0V - 2*sigma_1V)

  V_tilde <- rnorm(n = 200, mean = 0, sd = sigma_v_tilde)

  tibble(V_tilde) %>%
    mutate(configuration = config_num,
           U_d = pnorm(V_tilde/sigma_v_tilde),
           MTE = beta + V_tilde*(-sigma_11 - sigma_00 +2*sigma_01 +sigma_1V - sigma_0V)/sigma_v_tilde^2)

}
all_MTEs <-
  MTE("I", 1,1,1,0,0,0) %>%
  rbind(MTE("II", 1,1,1,0.5,-0.5,0),
        MTE("III", 1,1,1,0,0,0.5),
        MTE("IV", 1,1,0, 0.5, 0, 0),
        MTE("V", 1, 0.25, 1, 0.1, -0.2, 0.2),
        MTE("VI", 0.25, 0.25, 1, -0.2, 0.1, -0.2),
        MTE("VII", 1, 1, 1, -0.2, 0.6, 0.5))
```

```
all_MTEs %>%
  ggplot(aes(x = V_tilde, y = MTE, color = configuration)) +
  geom_line() + labs(x = TeX("V + U_0 - U_1"), y = TeX("$\\Delta^{MTE}$"), linetype = TeX("$\\bar{\\beta
```



Alternatively with the latent variable normalized

Compute and plot LIV estimates for the configurations of parameters and data from different societies listed below.

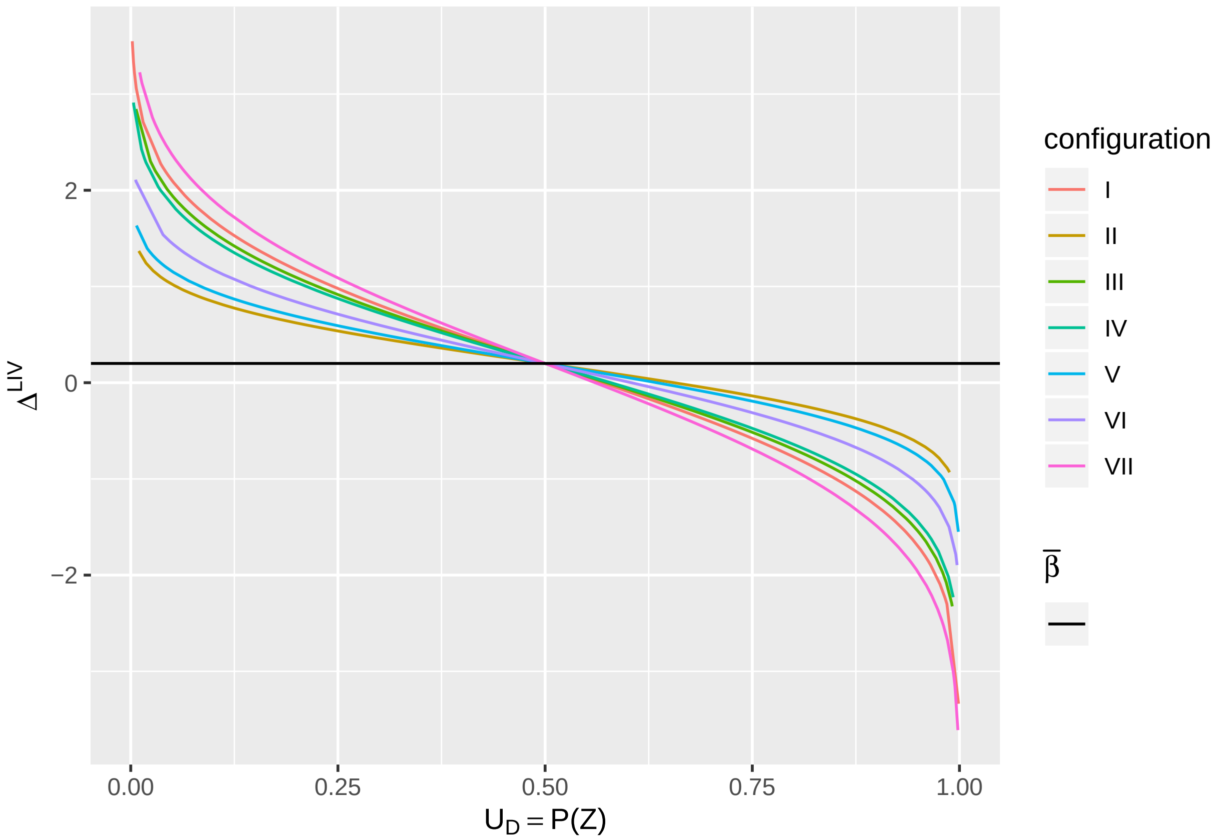
The Local IV is defined to be

$$\Delta^{LIV}(P(Z)) = \frac{\partial E[Y|P(Z) = P(z)]}{\partial P(z)} = E[Y_1 - Y_o|P(Z) = P(z)]$$

So we ccan write the LIV as a function of the MTE

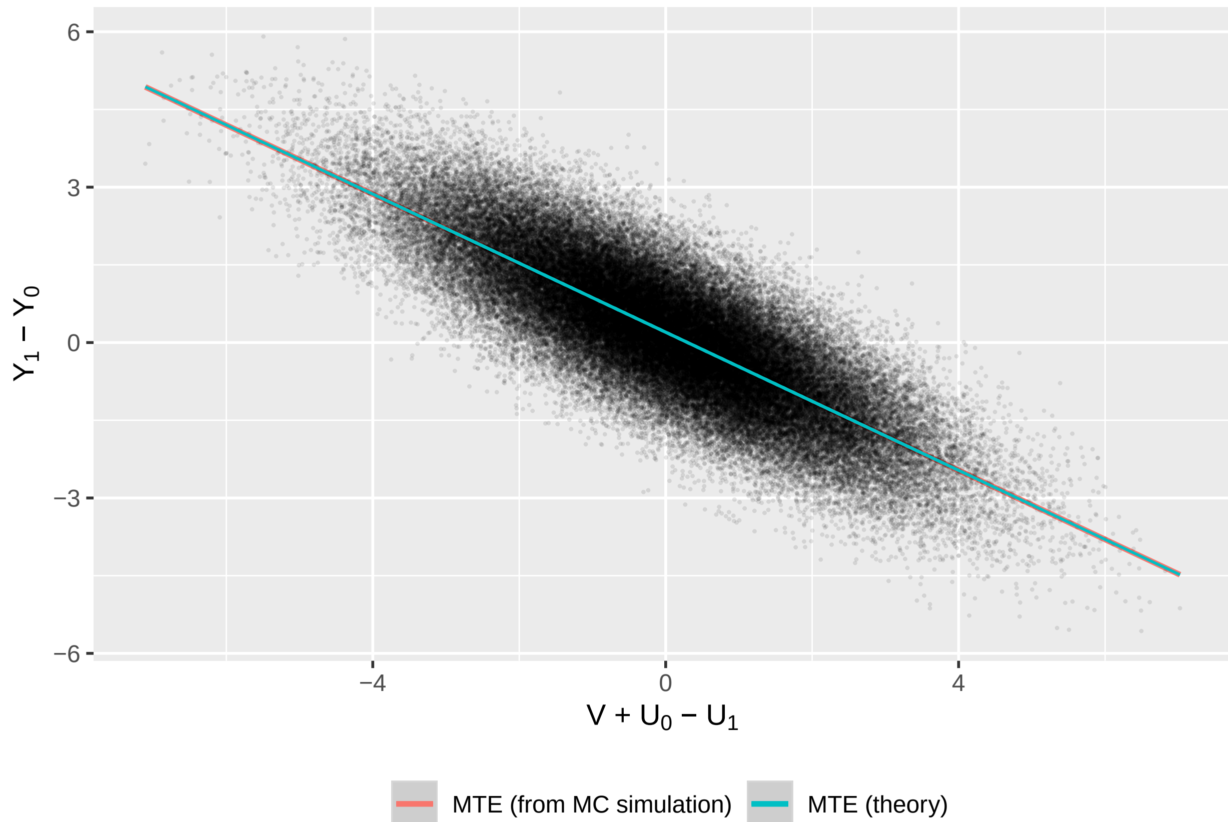
$$\Delta^{LIV}(P(Z)) = MTE(U_D = P(z))$$

```
all_MTEs %>%
  ggplot(aes(x = U_d, y = MTE, color = configuration)) +
  geom_line() + labs(x = TeX("$U_D = P(Z)$"), y = TeX("$\\Delta^{LIV}$"), linetype = TeX("$\\bar{\\beta
```

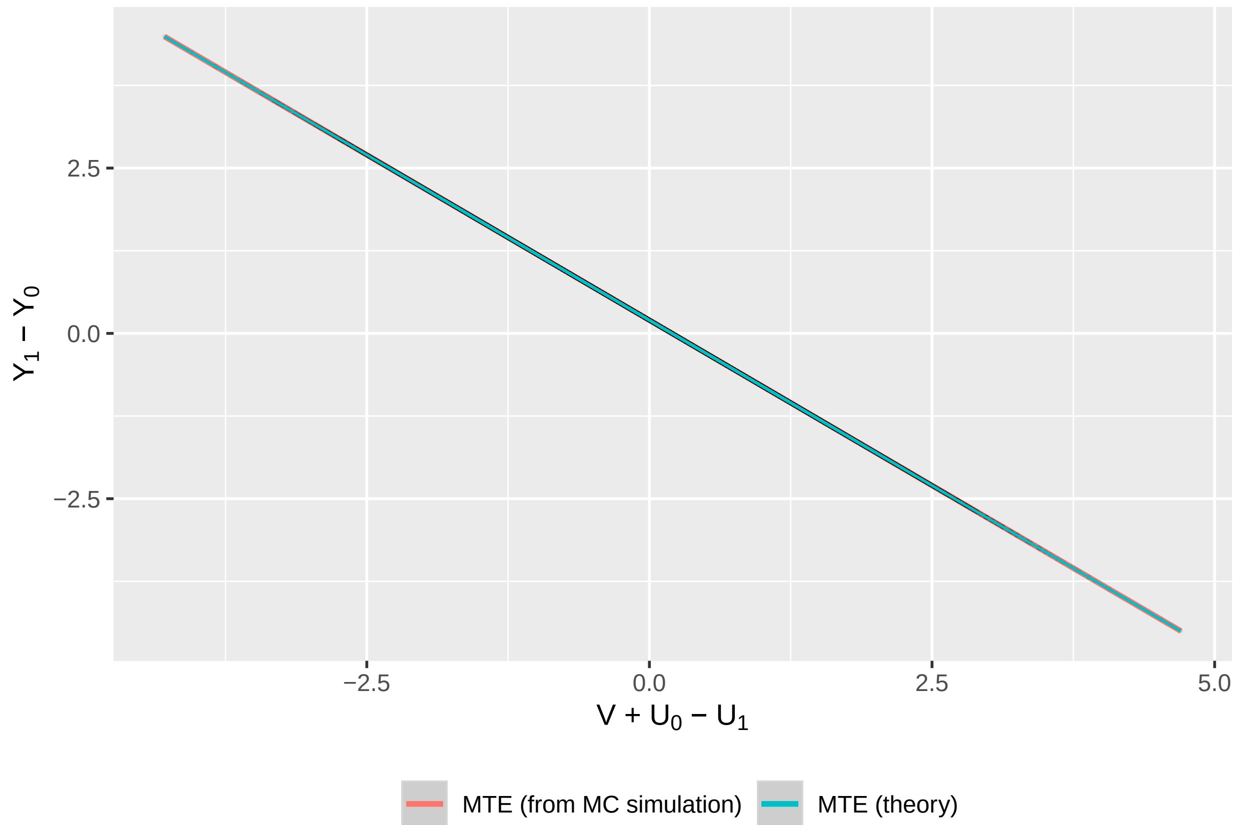


Verifying result with MC simulations

```
config_1 %>%
  ggplot(aes(x = V_tilde))+
  geom_smooth(aes(y = beta_i, color = "MTE (from MC simulation)"), method = "lm") +
  geom_point(aes(y = beta_i), alpha = 0.1, size = 0.1) +
  geom_line(aes(y = MTE, color = "MTE (theory)")) + labs(color = "") +
  theme(legend.position = "bottom") + labs(x = TeX("V + U_0 - U_1"), y = TeX("Y_1 - Y_0"))
```




```
config_4 %>%
  ggplot(aes(x = V_tilde))+
  geom_smooth(aes(y = beta_i, color = "MTE (from MC simulation)"), method = "lm") +
  geom_point(aes(y = beta_i), alpha = 0.1, size = 0.1) +
  geom_line(aes(y = MTE, color = "MTE (theory)")) + labs(color = "") +
  theme(legend.position = "bottom") + labs(x = TeX("V + U_0 - U_1"), y = TeX("Y_1 - Y_0"))
```



Compute both objective and subjective treatment effects

???

I am assuming the objective effect is just

$$Y_1 - Y_0$$

D: Derive weights and estimate treatment parameters

values estimated from monte-carlo simulations

```
treat_effects <- function(config){

  config_num <- filter(config, row_number() ==1)$config_num


  Prob_treated <- config$D %>% mean()

  ATE <- config$beta_i %>% mean()

  TT <- filter(config, D == 1)$beta_i %>% mean()

  TUT  <- filter(config, D == 0)$beta_i %>% mean()
```

```r
  ols_model <- lm(Y ~ D, data = config)

  OLS <- as.numeric(ols_model$coefficients['D'])

  #linear IV
  first_stage <- lm(D ~ Z, data = config)

  D_hat <- predict(first_stage)

  second_stage <- lm(config$Y ~ D_hat)

  linear_IV <- as.numeric(second_stage$coefficients['D_hat'])


  #PRTE?

  tibble(config_num,
         Prob_treated,
         ATE,
         TT,
         TUT,
         OLS,
         linear_IV)


}
map_df(config_list, treat_effects) %>%
  kable(digits = 3,
        col.names = c("Configuration", "$Pr(D =1)$",
                      "ATE", "TOT", "TUT", "OLS", "Linear IV"))
```

Configuration	$Pr(D = 1)$	ATE	TOT	TUT	OLS	Linear IV
I	0.545	0.200	1.030	-0.796	0.122	0.122
II	0.575	0.200	0.526	-0.241	0.528	0.099
III	0.555	0.204	0.943	-0.717	-0.165	0.121
IV	0.574	0.198	0.853	-0.686	0.092	0.015
V	0.570	0.202	0.592	-0.315	0.660	0.068
VI	0.550	0.199	0.735	-0.457	0.189	0.082
VII	0.542	0.203	1.151	-0.919	-0.338	0.068

Taking our expression for

$$\Delta^{MTE} = \bar{\beta} + \frac{-\sigma_1^2 - \sigma_0^2 + 2\sigma_{01} + \sigma_{V1} - \sigma_{V0}}{\sigma_v^2 + \sigma_0^2 + \sigma_1^2 - 2\sigma_{01} + 2\sigma_{0V} - 2\sigma_{1V}} * (V + U_0 - U_1)$$

And subsituting in for $\tilde{V} = \sigma_{\tilde{V}} * \Phi^{-1}(U_D)$, we have

$$\Delta^{MTE} = \bar{\beta} + \frac{-\sigma_1^2 - \sigma_0^2 + 2\sigma_{01} + \sigma_{V1} - \sigma_{V0}}{\sqrt{\sigma_v^2 + \sigma_0^2 + \sigma_1^2 - 2\sigma_{01} + 2\sigma_{0V} - 2\sigma_{1V}}} * \Phi^{-1}(U_D)$$

We will label the ratio of the variances

$$c = \frac{-\sigma_1^2 - \sigma_0^2 + 2\sigma_{01} + \sigma_{V1} - \sigma_{V0}}{\sqrt{\sigma_v^2 + \sigma_0^2 + \sigma_1^2 - 2\sigma_{01} + 2\sigma_{0V} - 2\sigma_{1V}}}$$

ATE

$$ATE = \int_0^1 (1) * \Delta^{MTE} du_D$$

So plugging in our expression above we have

$$ATE = \bar{\beta} + c * \int_0^1 \Phi^{-1}(u_d) du_D$$

And note with a change of variables

$$\int_0^1 \Phi^{-1}(u_d) du_D = \int_{-\infty}^{\infty} \tilde{v} * f(\tilde{v}) d\tilde{v}$$

Where $f(\tilde{v})$ is the pdf of $\tilde{V} \sim N(0, \sigma_{\tilde{v}})$ as derived in part A. The RHS is just $E[\tilde{V}] = 0$, so we have

$$ATE = \bar{\beta} = 0.2$$

So the ATE should be the same for all the configurations and constant across u_d ## TT

$$\Delta^{TT} = \int_0^1 \omega_{TT} * \Delta^{MTE} du_D$$

Where:

$$\omega_{TT} = \frac{Pr(P(z) \geq u_D)}{Pr(D = 1)}$$

Evaluating the denominator

$$E(P(Z)) = E[\Phi(\frac{\bar{\beta} - Z}{\sigma_{\tilde{V}}})]$$

Taking the expection w.r.t Z, since $Z\ U(-0.5, 0.5)$

$$E(P(Z)) = \Phi(\frac{\bar{\beta}}{\sigma_{\tilde{V}}})$$

Note that because $P(z) \sim U(0,1)$, we know

$$P(P(z) \geq u_D) = 1 - u_D$$

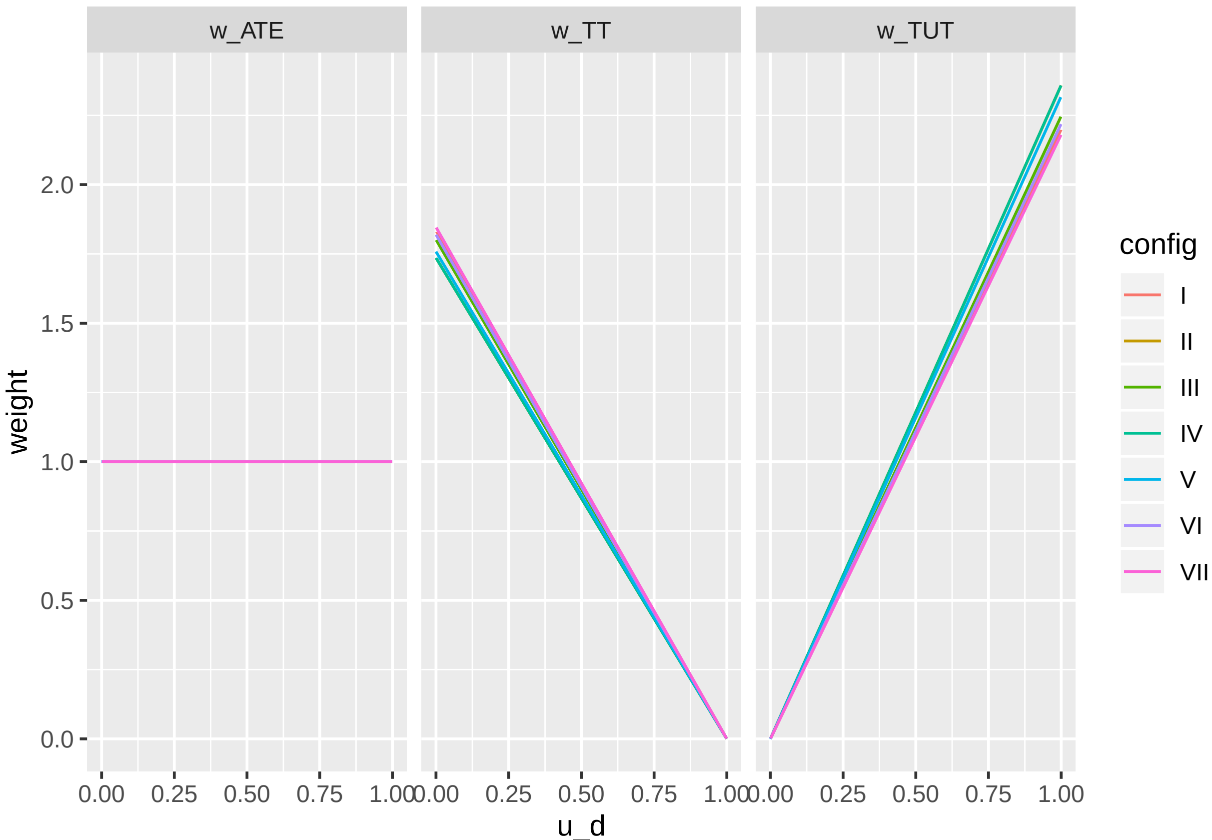
We know the

$$\omega_{TT} = \frac{1 - u_D}{\Phi(\frac{\bar{\beta}}{\sigma_{\tilde{V}}})}$$

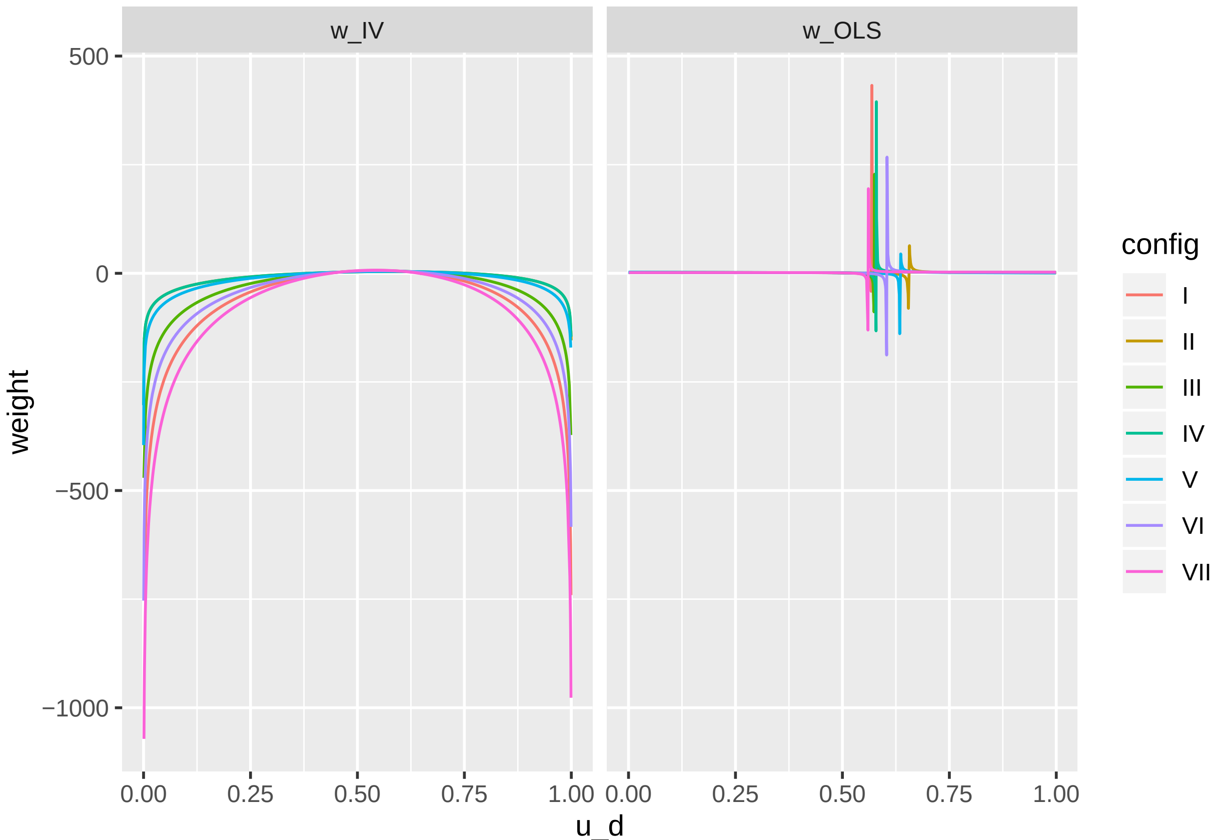
So pluggin in our equation for MTE

$$\Delta^{TT}(u_D) = \bar{\beta} * \int_0^1 \omega_{TT} du_D + c * \int_0^1 \Phi^{-1}(u_d) * \omega_{TT} du_D$$

```r
weights <- function(config_num,
                    sigma_11, sigma_00, sigma_V, sigma_01, sigma_0V, sigma_1V,
                    beta = 0.2){

  sigma_v_tilde <- sqrt(sigma_00 + sigma_11 + sigma_V -2*sigma_01 +2*sigma_0V - 2*sigma_1V)
  V_tilde <- rnorm(n = 1000, mean = 0, sd = sigma_v_tilde)
  u_d <- pnorm(V_tilde/sigma_v_tilde)

  u <- -0.3/sigma_v_tilde
  l <- 0.7/sigma_v_tilde
  k_TT <- 0.3*pnorm(u)+0.7*pnorm(l)-sigma_v_tilde*(dnorm(u)-dnorm(l))

  MTE <- beta + V_tilde*(-sigma_11 - sigma_00 +2*sigma_01 +sigma_1V - sigma_0V)/sigma_v_tilde^2
  EU1_UD <- (-sigma_11+sigma_01+sigma_1V)*V_tilde/sigma_v_tilde^2
  EU0_UD <- (sigma_00-sigma_01+sigma_0V)*V_tilde/sigma_v_tilde^2
  w_1 <- (1-u_d)/k_TT
  w_0 <- u_d/(1-k_TT)

  tibble(u_d) %>%
    mutate(config = config_num,
           w_ATE = 1,
           w_TT = (1-u_d)/k_TT,
           w_TUT = u_d/(1-k_TT),
           w_IV = (0.5*(beta-V_tilde)^2-0.125)/((0.105-0.5*sigma_v_tilde^2)*(pnorm(u)-pnorm(l))-0.5*sig
           w_OLS = 1+(EU1_UD*w_1-EU0_UD*w_0)/MTE
           )
}
all_weights <-
    weights("I", 1,1,1,0,0,0) %>%
  rbind(weights("II", 1,1,1,0.5,-0.5,0),
        weights("III", 1,1,1,0,0,0.5),
        weights("IV", 1,1,0, 0.5, 0, 0),
        weights("V", 1, 0.25, 1, 0.1, -0.2, 0.2),
        weights("VI", 0.25, 0.25, 1, -0.2, 0.1, -0.2),
        weights("VII", 1, 1, 1, -0.2, 0.6, 0.5)) %>%
  pivot_longer(cols = -one_of(c("u_d", "config")), names_to = "parameter", values_to = "weight")

all_weights %>% filter (parameter %in% c('w_ATE', 'w_TT', 'w_TUT')) %>%
  ggplot(aes(x = u_d, y = weight, color = config)) +
  geom_line() + facet_wrap(~parameter)
```




```r
all_weights %>% filter (parameter %in% c('w_IV', 'w_OLS')) %>%
  ggplot(aes(x = u_d, y = weight, color = config)) +
  geom_line() + facet_wrap(~parameter)
```




```
c <- NA; k_TT <- NA
treat_effects_theory <- function(config_num,
                        sigma_11, sigma_00, sigma_V, sigma_01, sigma_0V, sigma_1V,
                        beta = 0.2){

  sigma_v_tilde <- sqrt(sigma_00 + sigma_11 + sigma_V -2*sigma_01 +2*sigma_0V - 2*sigma_1V)

  c <<- (-sigma_11 - sigma_00 +2*sigma_01 +sigma_1V - sigma_0V)/sigma_v_tilde

  k_TT <<- 0.3*pnorm(-0.3/sigma_v_tilde)+0.7*pnorm(0.7/sigma_v_tilde)-sigma_v_tilde*(dnorm(-0.3/sigma_v
  integrand_TT <- function(u_D, beta.=beta,c.=c) {u_D*(beta.+c.*qnorm(u_D))}
  int_TT <- integrate(integrand_TT, lower = 0, upper = 1)$value
  TT <- (beta-int_TT)/k_TT
  TUT <- int_TT/(1-k_TT)

  integrand_OLS <- function (u_D,c.=c,k_TT.=k_TT) {
    ((-sigma_11+sigma_01+sigma_1V)*sigma_v_tilde*qnorm(u_D)/sigma_v_tilde^2)*(1-u_D)/k_TT.
    -((sigma_00-sigma_01+sigma_0V)*sigma_v_tilde*qnorm(u_D)/sigma_v_tilde^2)*(u_D)/k_TT.*c.*qnorm(u_D)
    }
  int_OLS <- integrate(integrand_OLS, lower = 0, upper = 1)$value
  beta_OLS <- beta+int_OLS

  tibble(config_num,
         P_D1 = k_TT,
         ATE = beta,
         TT = TT,
         TUT = TUT,
```

```r
         beta_OLS = beta_OLS,
         c = c)
}
all_tx_effects <-
  treat_effects_theory("I", 1,1,1,0,0,0) %>%
  rbind(treat_effects_theory("II", 1,1,1,0.5,-0.5,0),
        treat_effects_theory("III", 1,1,1,0,0,0.5),
        treat_effects_theory("IV", 1,1,0, 0.5, 0, 0),
        treat_effects_theory("V", 1, 0.25, 1, 0.1, -0.2, 0.2),
        treat_effects_theory("VI", 0.25, 0.25, 1, -0.2, 0.1, -0.2),
        treat_effects_theory("VII", 1, 1, 1, -0.2, 0.6, 0.5))
all_tx_effects %>%
  kable(digits = 3,
        col.names = c("Configuration","$Pr(D=1)$", "ATE", "TOT", "TUT", "Beta_{OLS}", "c"))
```

Configuration	$Pr(D = 1)$	ATE	TOT	TUT	Beta_{OLS}	c
I	0.545	0.2	0.781	-0.496	0.811	-1.155
II	0.576	0.2	0.418	-0.097	0.200	-0.500
III	0.555	0.2	0.719	-0.448	0.876	-1.061
IV	0.576	0.2	0.663	-0.430	0.634	-1.000
V	0.569	0.2	0.464	-0.148	0.177	-0.581
VI	0.550	0.2	0.572	-0.253	0.440	-0.759
VII	0.541	0.2	0.871	-0.593	1.354	-1.318