15/9/21 Solution of

NON LINEAR EQUATIONS

$$\left.\begin{array}{l} 3x+2y=1 \\ 4x+6y=-2 \end{array}\right\} \text{linear}$$

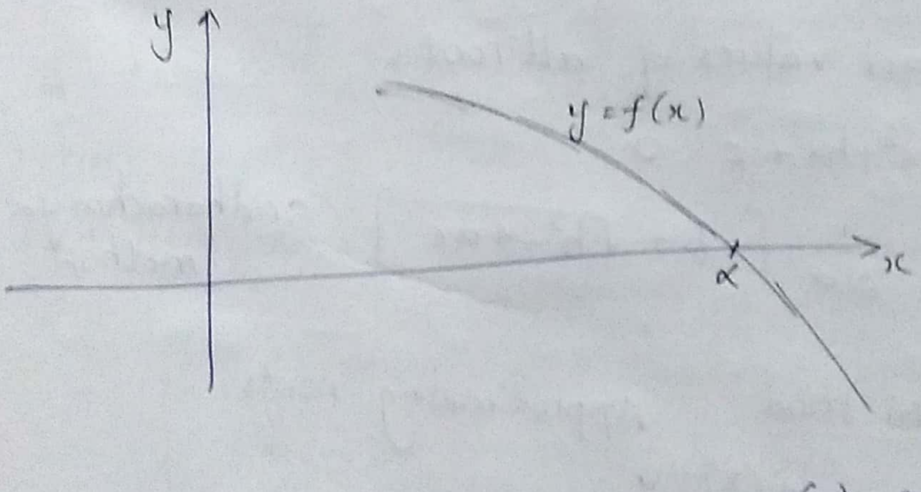
Algebraic/ Polynomial Equation $f(x)=0$

$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$

$3x^3 - 2x + 1 = 0$

Transcendental eq^n $\cos x - xe^x = 0$

$xe^{2x} - 1 = 0$



$f(x)=0$
$x=\alpha$
$f(\alpha)=0$
α is a root/zero

Simple root: α is simple root of $f(x)=0$

if $f(\alpha)=0$ and $f'(\alpha)\neq 0$

or $f(x) = (x-\alpha)\,g(x)$, $g(\alpha)\neq 0$

$f(x) = x^3 + x - 2 = 0$

$f(x) = (x-1)(x^2+x+2) = (x-1)\,g(x)$, $g(1)\neq 0$

$x=1$ $f(1)=0$

$\downarrow$

simple root $f'(x) = 3x^2+1$, $f'(1) = 4 \neq 0$

Multiple Root: α is a multiple root of multiplicity m, of $f(x)=0$ if

$$f(\alpha)=0,\ f'(\alpha)=0,\ \ldots\ f^{(m-1)}(\alpha)=0$$

and $f^{m}(\alpha)\neq 0$

$$f(x) = (x-\alpha)^m\, g(x),\quad g(\alpha)\neq 0$$

$$f(x) = x^3-3x^2+4 = 0$$

$$f(x) = (x-2)^2 \underbrace{(x+1)}_{g(x)}$$

$g(2) \neq 0$

g

$f(x)$ has multiple root 2 with multiplicity 2

Direct Method:

Exact values of all roots

$$ax^2+bx+c=0$$

$$x = \frac{1}{2a}\left[-b \pm \sqrt{b^2-4ac}\right]$$ (Sridharacharya method)

Iterative Method: Approximating roots.

$$f(x)=0$$

$$x_{k+1} = \phi(x_k)$$

$$\phi(x_k, x_{k-1})$$

Criterion to terminate iterative procedure

i) $|f(x_k)| \leq \epsilon$

ii) $|x_{k+1} - x_k| \leq \epsilon$

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A polynomial eq^n of degree 'n' has exactly 'n' roots, real or complex, simple or multiple.

Iterative Method:

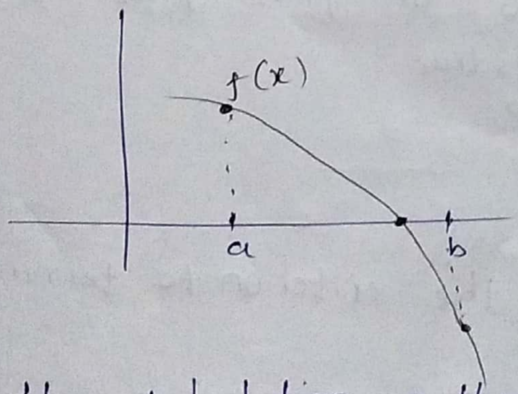
Initial Approximation for an iterative method:

- → Descarte's rule.
- → Bolzano's theorem.

Bolzano's theorem (Intermediate value theorem).

on continuity

Th If the function $f(x)$ is continous in $[a,b]$ and if $f(a)$ and $f(b)$ are of opposite signs, then there exists at least 1 real root of $f(x) = 0$ between a & b.



Use tabulation method to find a & b

-5
-4
-3
-2 $\left.\begin{matrix}+\\-\end{matrix}\right\}$ change of sign.
-1
0
1

Method of Bisection (similar to Binary search) based on Bolzano's theorem
(Iterative Method)

i) Initial Approximation

ii) ϕ

iii) Criteria for termination

Let $f(x)=0$ has a root in $[a,b]$ f is continous
in $[a,b]$ Then $f(a)\cdot f(b)<0$
A real root is in b/w a & b.

(a) (b) $f(a)\cdot f(b)<0$

i) $x_1 = \frac{a+b}{2}$ If $f(x_1)=0$, x_1 is the root of $f(x)=0$

else either $f(x_1)\cdot f(a)<0$ or $f(x_1)\cdot f(b)<0$

$a_1 = a, b_1 = x_1$ if this is true

ii) $x_2 = \frac{a_1+b_1}{2}$, If $f(x_2)=0$ x_2 is the root of $f(x)=0$

else either $f(x_2)f(a_1)<0$ or $f(x_2)\cdot f(b_1)<0$

if this is true
$a_2 = x_2$, $b_2 = b_1$

⋮

Do till we reach the criterion to terminate

Ex: Find by bisection method, a real root of

$$2x - \log_{10} x = 7$$

$$f(x) = 2x - \log_{10} x - 7 = 0$$

$f(1) = -5$

$f(2) = 2(3) - \log_{10} 2 - 7 = -3.3$

$f(3) = -1.477$

$f(4) = 0.3979$

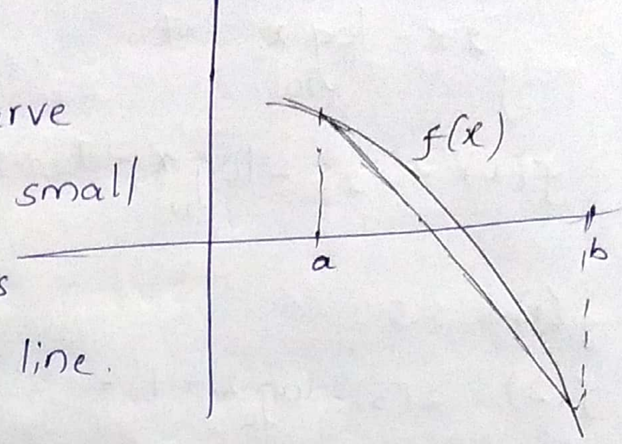
$f(3) \cdot f(4) < 0$

n	a_n	b_n	$x_{n+1} = \frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	3	4	3.5	-0.5441
1	3.5	4	3.75	-0.0740
2	3.75	4	3.875	0.1617
3	3.75	3.875	3.8125	0.0438
4	3.75	3.8125	3.7813	-0.0151
5	3.7813	3.8125	3.7969	0.0143
6	3.7813	3.7969	3.7891	-0.0004
7	3.7891	3.7969	3.7930	0.0070
8	3.7891	3.7930	3.7910	0.0033
9	3.7891	3.7910	3.79	

$\alpha = 3.79$

Chord Methods:

We approximate the curve $f(x)=0$ in a sufficiently small interval which contains the root, by a straight line.



Method of False Position (Regula - Falsi)

Start: Find the interval in which the root lies.
(Bolzano's theorem)

Let root of $f(x)=0$ lies in (x_{k-1}, x_k) i.e., $f_{k-1} f_k < 0$

This $P(x_{k-1}, f_{k-1})$ & $Q(x_k, f_k)$ are points on the curve $f(x)$.

Draw a st. line joining the points P & Q.
(Approximation of the curve in the interval)

Eq^n of PQ $\quad \dfrac{y-f_k}{f_{k-1}-f_k} = \dfrac{x-x_k}{x_{k-1}-x_k}$

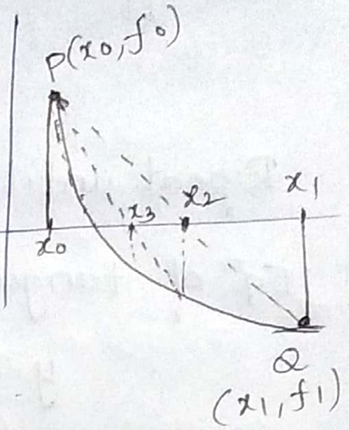
Find x for which $y=0$

Next approximation: where PQ and X-axis intersect

set $y=0$ & solve for x.

$$x = x_k - \left(\frac{x_{k-1}-x_k}{f_{k-1}-f_k}\right) f_k$$

$$x = x_K - \left(\frac{x_K - x_{K-1}}{f_K - f_{K-1}}\right) f_K$$



$$x_{K+1} = x_K - \left(\frac{x_K - x_{K-1}}{f_K - f_{K-1}}\right) f_K$$

$$x_{K+1} = \frac{x_{K-1} f_K - x_K f_{K-1}}{f_K - f_{K-1}}, K = 1, 2, \ldots$$

Starting with initial interval (x_0, x_1) in which root lies $x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$

Now if $f(x_0) \cdot f(x_2) < 0$ then root lies b/w (x_0, x_2) otherwise in (x_2, x_1). The iteration is continued using the interval in which the root lies, until the required accuracy criterion is satisfied.

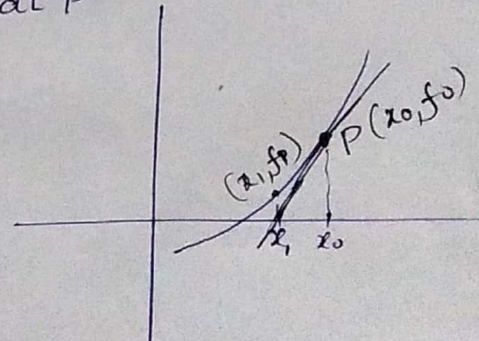
<u>Newton - Raphson Method</u>: (Tangent Method)

Let $x_0 \rightarrow$ initial approx to the root of $f(x) = 0$

$P(x_0, f_0)$ is a pt on the curve

Draw the tangent to the curve at P

Approximate the curve in the neighbourhood of the root by the tangent to the curve at P.



Next approximation: intersection of the tangent to x-axis

Repeat until required accuracy is obtained

Eq^n of tangent to the curve

$y = f(x)$ at P is given by

$y - f_0 = (x - x_0) f'(x_0)$ $\quad$ $f'(x_0) \rightarrow$ slope of tangent at $P(x_0, f_0)$

Set $y = 0$, Solve for x.

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \quad , f'(x_0) \neq 0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad , f'(x_k) \neq 0$$

Ex: $x^3 - 3x + 1 = 0$ Solve for root correct to 3 decimal places using Regula-Falsi method.

x	0	1	2	3
$f(x)$	1	-1	3	19

$x_0 = 0 \quad f_0 = f(x_0) = f(0) = 1$

$x_1 = 1 \quad f_1 = f(x_1) = f(1) = -1$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{0-1}{-1-1} = 0.5$$

$f(x_2) = f(0.5) = -0.375$

iii) $f(0) \cdot f(0.5) < 0 \Rightarrow (0, 0.5)$

$$x_3 = \frac{x_0 f_2 - x_2 f_0}{f_2 - f_0} = 0.36364$$

$f(x_3) = -0.04283$

iv) $(0, 0.36364)$

$$x_4 = \frac{x_0 f_3 - x_3 f_0}{f_3 - f_0} = 0.34870$$

$f(x_4) = -0.00370$

v) $(0, x_4)$

$$x_5 = \frac{x_0 f_4 - x_4 f_0}{f_4 - f_0} = 0.34741$$

$f_5 = -0.0030$

vi) $x_6 = 0.347306$

$|x_6 - x_5| = 0.0001 < 0.0005$

correct to 3 decimal places

$\alpha = 0.347$

Ex: Perform 4 iterations of Newton-Raphson method to find a root of $f(x) = x^3 - 5x + 1 = 0$

sol

$f(0) = 1 \qquad f(1) = -3$

$(0, 1) \Rightarrow x_0 = 0.5$

$$x_{k+1} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5}$$

$$x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 5} = 0.176471$$

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 5} = 0.201568$$

$x_3 = 0.201640$

$x_4 = 0.201640$

$$\boxed{\alpha = 0.201640}$$

Error of Approximation: (Rate of Approximation)

$$\epsilon_k = x_k - \alpha \quad , k = 0, 1, 2, \ldots$$

$$|\epsilon_k| \to 0 \quad \text{as } k \to \infty$$

A method has
Order: p or the rate of convergence p if
p is the largest +ve real number for which
$\exists$ a finite constante c such that

$$|\epsilon_{k+1}| \le c \cdot |\epsilon_k|^p$$

Linear rate at which the convergence is happening
If p is more Rate of Approximation is fast.

Bisection: $|\epsilon_{k+1}| = |x_{k+1} - \alpha| \le \frac{1}{2} |x_k - \alpha|$

$$|\epsilon_{k+1}| \le \frac{1}{2} |\epsilon_k|$$

$p = 1$ convergence rat is linear.

Regular Falsi: $p = 1$ linear
But Regular Falsi is faster than Bisection.

Newton Raphson: $p = 2$

$$|\epsilon_{k+1}| \le c \cdot |\epsilon_k|^2$$

Convergence rate is quadratic
Much faster than Regular Falsi.
But Newton Raphson has limititations.

i) May diverge if initial value x_0 is far away from α.

ii) $f'(x) = 0$ then will get struck.