18.06 Recitation 3

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1 Problems

1. (Strang 3.2, Problem 13) The plane $x - 3y - z = 12$ is parallel to $x - 3y - z = 0$. One particular point on this plane is $(12, 0, 0)$. All points on this plane have the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Solution: The general form of points will be a particular point ($(12, 0, 0)$) plus all linear combinations of point on the parallel plane $x - 3y - z = 0$. So it suffices to find a description of these points (x, y, z) so that $x - 3y - z = 0$. That is, we want to find a basis of the null space of the matrix $\begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$. This has two free columns, with corresponding special solutions

$$s_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Using this, we fill in the above blanks to get:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. (Strang 3.2, Problem 16) Construct A so that $N(A)$ is the span of the vector $(4, 3, 2, 1)$. Its rank is ________________.

Solution: One possible A is

$$A = \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the vector $(4, 3, 2, 1)$ is the special solution for the x_4 variable.

Note: there is some ambiguity in choosing A, we could multiply on the left by any invertible matrix, for example.

Since we know that x_4 is the only free variable (the null space is 1-dimensional (the span of this one vector)), the rank of A must be $4 - 1 = 3$.

3. **(Strang 3.3, Problem 30)** Find the complete solution to

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Solution: First we put the augmented matrix $[A|b]$ into rref.

$$[A|b] = \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = [R|d]$$

From this we see that A has rank 3: columns 1, 2 and 4 (or variables x_1, x_2, x_4) are pivots. Column 3 (or variable x_3) is free. Since # rows $= 3 =$ rank, we are in a **full row rank** situation. Therefore solutions are guaranteed to exist, but they may not be unique (and since we are *not* in a full column rank situation, they will not be unique).

However, the form of all solutions will be $x = x_p + x_n$: the sum of a particular solution x_p and elements of the null space x_n ($Ax_n = 0$).

It is easy to find a particular solution from the augmented matrix $[R|d]$: set the free variables to 0 and back-substitute. This will give

$$x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}.$$

It is also easy from R to find a basis for the null space; there is a special solution corresponding to each free variable and these form a basis. In this case, there is just one free variable, so the null space is 1-dimensional spanned by the special solution

$$x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We can read this off from R: it has a 1 in the position of the free variable x_3, and the entry in position i is $-R_{i3}$: the negative of the corresponding entry of R.

So the complete solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

4. **(Strang 3.3, Problem 6)** What conditions on b_1, b_2, b_3, b_4 make the system

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

solvable? Find x in that case.

Solution: Again, the best first thing to do is get everything in rref:

$$\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & b_1 - 2(b_3 - 2b_1) \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_1 - 3(b_3 - 2b_1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4b_1 - 2b_3 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 + 3b_1 - 3b_3 \end{bmatrix}$$

Now we see immediately that A has rank 2: the columns 1 and 2 are pivots. So this is a **full column rank** situation. Therefore if a solution exists, it is unique, but the column space is not all of $\mathbb{R}^4$, so a solution may not exist.

The constraints on b_1, b_2, b_3, b_4 for a solution to exist are given by the **zero rows** of R:

$$\begin{aligned} &\text{(row 2):} && b_2 - 2b_1 = 0, \\ &\text{(row 4):} && b_4 + 3b_1 - 3b_3 = 0. \end{aligned}$$

Under those conditions, we find a solution by back substitution from the **nonzero rows** of R:

$$\begin{aligned} &\text{(row 1):} && x_1 = 4b_1 - 2b_3, \\ &\text{(row 3):} && x_2 = b_3 - 2b_1. \end{aligned}$$

So the complete solution is given by

$$\begin{cases} \text{no solution} & \text{if } b_2 - 2b_1 \neq 0 \textbf{ or } b_4 + 3b_1 - 3b_3 \neq 0, \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} & \text{if } b_2 - 2b_1 = 0 \textbf{ and } b_4 + 3b_1 - 3b_3 = 0. \end{cases}$$

5. (Strang 3.2, Problem 24) Give examples of matrices A for which the number of solutions to $Ax = b$ is

   (a) 0 or 1 depending on b

   (b) ∞, regardless of b

   (c) 0 or ∞, depending on b

   (d) 1 regardless of b

   Solution:

   (a) This is the case when A has **full column rank** but **not full row rank**. So an example is

   $$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix}$$

   from problem 4 above.

(b) This is the case when A has **full row rank** but **not full column rank**. So an example is

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix}$$

from problem 3 above.

(c) This is the case when A has **not full row rank** and **not full column rank** (also called the **rank deficit** case). An example is

$$A = \begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix}.$$

(d) This is the case when A has both **full row rank** and **full column rank** (also called the **invertible** case). An example is

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

6. (Strang 3.2, Problem 34) Suppose you know that a 3×4 matrix A has the vector $s = (2, 3, 1, 0)$ as the only special solution to $Ax = 0$.

   (a) What is the rank of A?

   (b) What is the exact reduced row echlon form R of A?

   (c) How do you know that $Ax = b$ can be solved for all b?

   Solution:

   (a) If s is the only special solution, then the null space of A is dimension 1. Since A has 4 columns by assumptions, we must have that the other three columns/variables are pivots. The rank of A is the number of pivot columns, which is therefore 3.

   (b) From the form of s, we know that variable x_3 is the free variable. Therefore variables x_1, x_2 and x_4 are pivots. So in the rref, the columns 1, 2, and 4 are identity matrix columns, so R must be of the form

$$R = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 1 \end{bmatrix}.$$

   To fill in column 3, we know that it is free, so it must have a 0 in component 3. And components 1 and 2 must be such that the vector $(2, 3, 1, 0)$ is in the null space. That is, we must solve the equation

$$\begin{bmatrix} 1 & 0 & r_1 & 0 \\ 0 & 1 & r_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   Because R is in rref, this is easy to solve and works out precisely to $r_1 = -2$ and $r_2 = -3$:

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(c) This is a case of full row rank, so every equation is solvable. Alternatively, to see directly, the column space of R contains the column space of the smaller matrix where I delete the free column 3:

$$C(R) \supseteq C\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right).$$

But this columns space is all of $\mathbb{R}^3$, since it is the span of the three standard basis vectors. So every b gives rise to a solvable equation.