18.06 Recitation 7 Solutions

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1 Problems

1. Suppose that A is the matrix
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   (a) What is the pattern when you multiply A repeatedly by some vector? After _______ multiplications, you get back the same vector, so
$$A^{\text{---}} = \underline{\qquad\qquad}.$$

   Solution: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. We can calculate
$$\begin{aligned} Av &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}, \\ A^2v = A(Av) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}, \\ A^3v = A(A(Av)) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}, \\ A^4v = A(A(A(Av))) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \end{aligned}$$

   Therefore we see that after **four** multiplications, you get back the same vector, so $\mathbf{A^4} = \mathbf{I}$. (Notice also that after 2 multiplications we get back the negative of the original vector, so $A^2 = -I$).

   (b) What are eigenvalues and eigenvectors of A? Is this consistent with the previous part?
   Solution:
   The eigenvalues are the roots of the characteristic polynomial
$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1,$$
   which are $\lambda = \pm i$.
   To find eigenvectors we can do several things:
   <u>Approach 1:</u> The eigenvector associated to eigenvalue $\lambda_1 = i$ is the basis vector of the nullspace of the matrix
$$A - iI = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}.$$

To find a basis for the nullspace, we can first put this into its rref:

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \rightsquigarrow \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore the nullspace is spanned by the vector

$$x_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \lambda_1 = i.$$

We can do a similar process to find the second eigenvector

$$x_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \lambda_1 = -i.$$

<u>Approach 2:</u> We want to solve the equation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Since the eigenspace associated to an eigenvalue λ is a subspace of $\mathbb{R}^2$ (or $\mathbb{R}^m$ more generally), as long as there is an eigenvector in this subspace with $v_1 \neq 0$, we can assume that $v_1 = 1$, by scaling appropriately. Then we want to solve the equation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} = i \begin{pmatrix} 1 \\ v_2 \end{pmatrix},$$

or more simply

$$v_2 = i, \qquad -1 = iv_2,$$

which is consistent, since $i^2 = -1$. This gives the eigenvector $x_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ associated to the eigenvalue $\lambda_1 = i$, as before. We could do the same thing to find the second one.

We could also note that since A has real entries, $\overline{A}$ (meaning the matrix composed of the complex conjugates of the entires of A) is equal to A, so taking complex conjugates we have

$$\overline{Ax_1} = \overline{A}\,\overline{x_1} = A\overline{x_1}.$$

But on the other hand, as x_1 is an eigenvector

$$\overline{Ax_1} = \overline{\lambda_1 x_1} = \overline{\lambda_1}\,\overline{x_1}.$$

Putting these two together, we see

$$A\overline{x_1} = \overline{\lambda_1}\,\overline{x_1},$$

or in other words: if A is real, then the eigenvalues and eigenvectors come in complex conjugate pairs. Therefore

$$x_2 = \overline{x_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad \lambda_2 = \overline{\lambda_1} = -i.$$

(c) Write the vector $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the basis of the eigenvectors and give a formula for $A^n x$.

Solution: We want to find $a, b \in \mathbb{C}$ so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ i \end{pmatrix} + b \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

This is pretty easy to see by inspection, but for the sake of demonstrating that nothing magical is happening, let's do this systematically. The a and b above are the solution to the linear system

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We can solve this by row reduction on the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ i & -i & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2i & -i \end{pmatrix}.$$

So we see we should take

$$b = 1/2, \qquad a = 1/2.$$

Therefore,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1/2) \begin{pmatrix} 1 \\ i \end{pmatrix} + (1/2) \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

So

$$\begin{aligned} A^n e_1 = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= A^n \left((1/2) \begin{pmatrix} 1 \\ i \end{pmatrix} \right) + A^n \left((1/2) \begin{pmatrix} 1 \\ -i \end{pmatrix} \right), \\ &= (1/2) A^n \begin{pmatrix} 1 \\ i \end{pmatrix} + (1/2) A^n \begin{pmatrix} 1 \\ -i \end{pmatrix}, \\ &= (1/2)(i)^n \begin{pmatrix} 1 \\ i \end{pmatrix} + (1/2)(-i)^n \begin{pmatrix} 1 \\ -i \end{pmatrix}, \\ &= (1/2) \begin{pmatrix} i^n(1 + (-1)^n) \\ i^{n+1}(1 + (-1)^{n+1}) \end{pmatrix}. \end{aligned}$$

This might look terrible, since there are all of these i terms floating around, but we are just multiplying a real matrix by a real vector! Never fear... we can simplify more! If n is odd, then $(-1)^n = -1$ and the top coordinate is 0. And if n is odd, then $n + 1$ is even, and so the bottom coordinate simplifies too to

$$n \text{ odd}, \qquad A^n e_1 = \begin{pmatrix} 0 \\ i^{n+1} \end{pmatrix}.$$

And if $n = 1+4*m$ then $i^{n+1} = i^{2+4m} = i^2 = -1$. If $n = 3+4m$ then $i^{n+1} = i^{4(m+1)} = 1$ and so we cna further simplify this to

$$A^n e_1 = \begin{cases} \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{if } n = 1 + 4m \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } n = 3 + 4m. \end{cases}.$$

Similarly, if n is even, then the bottom entry is 0 and we can similarly show that

$$A^n e_1 = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } n = 4m \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{if } n = 2 + 4m. \end{cases}.$$

This should not be surprising, since $A^n e_1$ is the first column of the matrix A^n. And as we saw (basically in part (a)):

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and then it repeats! This confirms our answers computed above using the eigenbasis.

(d) What are the eigenvectors and eigenvalues of $B = 2A + I$?

Solution: Suppose that x is an eigenvector of A with eigenvalue λ, so that

$$Ax = \lambda x.$$

Then

$$Bx = (2A + I)x = 2Ax + x = 2\lambda x + x = (2\lambda + 1)x.$$

Therefore x is also an eigenvector of B with eigenvalue $2\lambda + 1$. Going in the other way, using $A = (1/2)(B - I)$, we see than any eigenvector of B is also an eigenvector of A.

Alternatively, we can see the eigenvalues from the definition:

$$\det(B - \lambda I) = \det \begin{pmatrix} (1-\lambda) & 2 \\ -2 & (1-\lambda) \end{pmatrix} = (1-\lambda)^2 + 4.$$

This has roots

$$(1 - \lambda) = \pm 2i, \qquad \lambda = 1 \pm 2i,$$

as expected.

(e) What do you know about $B^n x$ as $n \to \infty$ and $n \to -\infty$?

Solution: B now has eigenvalues with magnitude

$$|\lambda_1| = |\lambda_2| = (1 + 2i)(1 - 2i) = 1 + 4 = 5 > 1.$$

Therefore positive powers of B times the eigenvectors x_1 and x_2 will become larger and larger multiples of x_1 or x_2. Negative powers will become smaller and smaller. You would then expect that B^n times a random vector will blow up if $n \to \infty$ and go to 0 if $n \to -\infty$. Let's see this in action in the case as above.

This is the level you're probably expected to understand, but to go a bit deeper:

Using our formula $x = (1/2)x_1 + (1/2)x_2$, we see

$$\begin{aligned} B^n x &= (1/2)B^n x_1 + (1/2)B^n x_2 \\ &= (1/2)(1+2i)^n x_1 + (1/2)(1-2i)^n x_2 \\ &= (1/2) \begin{pmatrix} (1+2i)^n + (1-2i)^n \\ i\Big((1+2i)^n - (1-2i)^n\Big) \end{pmatrix}. \end{aligned}$$

Note that these entries are <u>real</u> numbers! We can check as follows:

$$\begin{aligned}
\overline{(1+2i)^n + (1-2i)^n} &= \overline{(1+2i)^n} + \overline{(1-2i)^n} \\
&= (\overline{1+2i})^n + (\overline{1-2i})^n \\
&= (1-2i)^n + (1+2i)^n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{i(1+2i)^n - i(1-2i)^n} &= \overline{i(1+2i)^n} - \overline{i(1-2i)^n} \\
&= \bar{i}(\overline{1+2i})^n - \bar{i}(\overline{1-2i})^n \\
&= -i(1-2i)^n - (-i)(1+2i)^n \\
&= -i(1-2i)^n + i(1+2i)^n.
\end{aligned}$$

Let's see that these "blow up" as $n \to \infty$. Let's look just at the first entry. We can expand

$$\begin{aligned}
(1+2i)^n + (1-2i)^n &= \sum_{j=0}^{n} \binom{n}{j} \left((2i)^j + (-2i)^j \right) \\
&= \sum_{j=0}^{n} 2^j \binom{n}{j} (i^j + (-i)^j) \\
&= \sum_{j=0}^{n} 2^j \binom{n}{j} i^j (1 + (-1)^j).
\end{aligned}$$

As we saw before, $(1+(-1)^j)$ is only nonzero if j is even. So we can rewrite as

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{2k+1} \binom{n}{2k} (-1)^k.$$

The numbers are all huge, and you can see that they won't cancel (the binomial coefficients are palindromic, but the powers of 2 are increasing, giving different weights to the two palindromic terms. The largest one will be in the middle/just above the middle.) The sign of this expression will be positive or negative depending on whether the central/just above central k is even or odd.