18.06 Recitation 8 Solutions

Isabel Vogt

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1 Problems

1. The Fibonacci sequence is defined recursively by specifying initial values $F_0 = 0, F_1 = 1$, and the relation
$$F_{n+1} = F_n + F_{n-1}.$$

   (a) Given the input vector $v_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$, what is the matrix A, so that $Av_n = v_{n+1}$?
   Solution: By the recursive definition, we know that
$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_n + F_{n-1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}.$$
   So we take
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

   (b) Give an expression for v_{n+1} in terms of A and $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   Solution: We know from above that $v_{n+1} = Av_n$, so applying this recursively, we see:
$$v_{n+1} = A(Av_{n-1}) = A(A(Av_{n-2})) = \cdots = A^n v_1.$$

   (c) What are the eigenvectors and eigenvalues of A?
   Solution: The eigenvalues of A are roots of the characteristic polynomial
$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda(\lambda - 1) - 1 = \lambda^2 - \lambda - 1.$$
   By the quadratic formula, these roots are
$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$
   To find the corresponding eigenvectors, we can find the nullspaces of the matrices
$$A - \lambda_1 I = \begin{pmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{pmatrix}, \qquad A - \lambda_2 I = \begin{pmatrix} 1-\lambda_2 & 1 \\ 1 & -\lambda_2 \end{pmatrix}.$$
   Using row reduction, we can see
$$\begin{pmatrix} 1-\lambda_i & 1 \\ 1 & -\lambda_i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1-\lambda_i & 1 \\ 0 & -\lambda_i - \frac{1}{1-\lambda_i} \end{pmatrix} = \begin{pmatrix} 1-\lambda_i & 1 \\ 0 & \frac{-\lambda_i(1-\lambda_i)-1}{1-\lambda_i} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda_i & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -\lambda_i \\ 0 & 0 \end{pmatrix}$$

Therefore we see that the nullspace is spanned by the vector

$$x_i = \begin{pmatrix} \lambda_i \\ 1 \end{pmatrix}.$$

An eigenbasis is therefore given by

$$\lambda_1 = \frac{1+\sqrt{5}}{2},\ x_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2},\ x_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}.$$

(d) Using the eigenbasis, give an exact formula for F_n. What do you know about $|F_n|$ as $n \to \infty$?

Solution:

We know from part (b) that

$$v_{n+1} = A^n v_1 = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

First we should express the vector v_1 in the eigenbasis above. We can do this by solving the linear system

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(We did this using elimination in the last recitation). Alternatively, let's use change of basis matrices. The matrix

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$$

takes vectors expressed in terms of the x_1, x_2 eigenbasis to vectors expressed in terms of the standard basis. The inverse matrix

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

expresses vectors in the standard basis in terms of vectors in the eigenbasis. So we want

$$S^{-1} v_1 = \begin{pmatrix} \frac{1}{\lambda_1 - \lambda_2} \\ \frac{-1}{\lambda_1 - \lambda_2} \end{pmatrix},$$

therefore

$$a = \frac{1}{\lambda_1 - \lambda_2}, \qquad b = \frac{-1}{\lambda_1 - \lambda_2} = \frac{1}{\lambda_2 - \lambda_1}.$$

This gives

$$v_{n+1} = A^n v_1 = A^n (a x_1 + b x_2) = a\lambda_1^n x_1 + b\lambda_2^n x_2.$$

Extracting the second coefficient, which is F_n, we see

$$F_n = a\lambda_1^n + b\lambda_2^n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

We expect that after multiplying by A many times, the output should be approximately a multiple of x_1, the eigenvector corresponding to the largest eigenvalue. A consequence is that

$$\frac{F_{n+1}}{F_n} \rightsquigarrow \lambda_1.$$

Let's check this:

$$\begin{aligned}
\frac{F_{n+1}}{F_n} &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} \\
&= \frac{\lambda_1^{n+1} - \lambda_1\lambda_2^n + \lambda_1\lambda_2^n - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} \\
&= \frac{\lambda_1(\lambda_1^n - \lambda_2^n) + \lambda_2^n(\lambda_1 - \lambda_2)}{\lambda_1^n - \lambda_2^n} \\
&= \lambda_1 + \frac{\lambda_2^n(\lambda_1 - \lambda_2)}{\lambda_1^n - \lambda_2^n} \\
&= \lambda_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n \left(\frac{(\lambda_1 - \lambda_2)}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n}\right),
\end{aligned}$$

as n gets large, since

$$\left|\frac{\lambda_2}{\lambda_1}\right| < 1,$$

the right-hand term goes to 0.

So this tells us that $|F_n|$ as $n \to \infty$ is getting very large: at every stage, we're roughly multiplying the previous number by $\lambda_1 > 1$.

2. Let A be an $m \times m$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix},$$

and let A' be the matrix obtained from A be reversing the rows and columns:

$$A' = \begin{pmatrix} a_{mm} & a_{m(m-1)} & \dots & a_{m1} \\ a_{(m-1)m} & a_{(m-1)(m-1)} & \dots & a_{(m-1)1} \\ & & \ddots & \\ a_{1m} & a_{1(m-1)} & \dots & a_{11} \end{pmatrix}.$$

(a) When $m = 2$, what do you notices about the eigenvalues of A'?

Solution: We're dealing with the special case

$$A' = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial $\det(A - \lambda I)$ and the eigenvalues of A' are the roots of the characteristic polynomial $\det(A' - \lambda I)$. Let's compute this:

$$\det(A'-\lambda I) = \det\begin{pmatrix} a_{22}-\lambda & a_{21} \\ a_{12} & a_{11}-\lambda \end{pmatrix} = (a_{22}-\lambda)(a_{11}-\lambda)-a_{12}a_{21} = \det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}.$$

So we notice that *the eigenvalues of A and A' are the same.*

(b) What is true in general and why?

Solution: For the general $m \times m$ case, we can't compute simply by hand. We'll have to realize the operation done to take A to A' as matrix operations. Similar to a problem in an earlier recitation, if we let P be the permutation matrix

$$P = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \iddots & 1 & 0 \\ 0 & \iddots & 0 & 0 \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Then multiplication by P on the left swaps the order of the rows. Good! This is one of the things we need to do to A to get A'.

Multiplication by P on the right also swaps the order of the columns. Double great! This is the second thing we have to do to A to get A'.

So we see that

$$A' = PAP.$$

There is one more important thing, which is that $P = P^{-1}$, because, as you can check, $P^2 = I$ (swapping the order of the rows and then swapping again is the same as doing nothing). So really

$$A' = PAP^{-1},$$

so A and A' are similar, and so have the same eigenvalues.

3. (Strang, Section 10.3, Problem 6)

(a) For a Markov matrix, show that the sum of the components of x equals the sum of the components of Ax.

Solution:

If A is a Markov matrix then for $o = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, we know that

$$o^T A = o^T.$$

The sum of the components of a vector y is given by $o^T y$. Using the above we have

$$o^T Ax = (o^T A)x = o^T x,$$

and so Ax and x have the same sum of coefficients.

(b) If $Ax = \lambda x$ with $\lambda \neq 1$, prove that the components of this non-steady eigenvector x add to zero.

Solution: If $\lambda \neq 1$, then from above we have

$$o^T x = o^T Ax = o^T(\lambda x) = \lambda o^T x,$$

and so

$$o^T x(1 - \lambda) = 0.$$

Since we assumed that $\lambda \neq 1$, the first term $o^T x$ must be 0.

4. (a) Show that every square matrix is similar to its transpose. That is,

$$A^T = SAS^{-1},$$

for some invertible matrix S.

Solution:

(b) Assuming A is diagonalizable, give a formula for S in terms of the matrix X of eigenvectors of A and the matrix Y of eigenvectors of A^T (which is also diagonalizable).

Solution: If Λ is the matrix of eigenvalues of A and A^T, then we have that

$$A = X\Lambda X^{-1}, \qquad Y^{-1}A^TY = \Lambda.$$

So putting these together,

$$A = XY^{-1}A^TYX^{-1},$$

and so $S = XY^{-1}$.