18.06 Recitation 10

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1. Consider unitary matrices $Q^H Q = I$. If Q is real, then we are saying that Q is orthogonal (e.g., $Q^T Q = I$).

   (a) Find the flaw in this false proof: **False Claim: all eigenvalues of a real orthogonal matrix are ± 1.** Indeed, if $Qx = \lambda x$,
   $$\lambda^2 x^T x = (Qx)^T (Qx) = x^T (Q^T Q) x = x^T x,$$
   therefore $\lambda^2 = 1$, so $\lambda = \pm 1$. If you want, you can think about what happens for the rotation matrices
   $$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

   (b) Correct the proof to show **True Claim: all eigenvalues of a unitary matrix have magnitude 1 (e.g. $\lambda = e^{i\phi}$ for some ϕ).**

   (c) Show that the eigenvectors for different eigenvalues of a unitary matrix are orthogonal.

   (d) Show that the determinant of any real unitary matrix (e.g., an orthogonal matrix) is ± 1 using eigenvalues. (Note: you already proved this on a previous pset in a different way.)

2. Let A be a Hermitian matrix and B be a positive definite Hermitian matrix. Consider the generalized eigenproblem
   $$Ax = \lambda Bx.$$

   (a) Show that these "generalized" eigenvalues λ above are real.

   (b) Show that eigenvectors for different "generalized eigenvalues" are orthogonal with respect to the "B-dot-product":
   $$x_1 \cdot_B x_2 = x_1^H B x_2.$$

3. (Strang 6.4, Problem 12) Here is a quick "proof" that the eigenvalues of **every** real matrix A are real:
   $$\textbf{False Proof: } Ax = \lambda x \text{ gives } x^T A x = \lambda x^T x, \quad \text{so } \lambda = \frac{x^T A x}{x^T x} = \frac{\text{real}}{\text{real}}.$$
   Find the flaw in this reasoning – a hidden assumption that is not justified. You can test those steps on the 90° rotation matrix
   $$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda = i, \; x = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

4. (Strang 9.2, Problem 23) How are the eigenvalues of A^H related to the eigenvalues of the square matrix A?

5. (a) If S is a positive definite matrix, show that S^{-1} is also positive definite.

   (b) If S and T are positive definite, their sum $S+T$ is also positive definite. If $S = A^H A$ and $T = B^H B$ for full-column-rank matrices A and B, then can you write down a full-column-rank matrix C so that $S + T = C^T C$?

6. We saw on homework 2 that if we represent complex numbers $x + iy$ by the vector of real and imaginary parts $\begin{pmatrix} x \\ y \end{pmatrix}$, then multiplication by the fixed complex number $a + bi$ is given by matrix multiplication by the real matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

What real matrix represents multiplication by $\overline{a + bi}$?