18.06 Recitation 13

Isabel Vogt

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1. Let
$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$
be a tridiagonal matrix. You can imagine that A is the 5×5 version of a family of tridiagonal matrices of size $n \times n$.

   (a) How can you use Gaussian elimination to solve $Ax = b$ in linear time?

   (b) What is special about the LU factorization of A? What is the pattern of nonzero entries.

   (c) What is the pattern of nonzero entries of A^{-1}?

2. Suppose that A is a 4×4 matrix with singular value decomposition $AV = U\Sigma$ given by
$$A \begin{pmatrix} v_1 & v_2 & v_3 & \begin{matrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{matrix} \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{matrix} & u_2 & u_3 & \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0.2 & & \\ & & 0.00001 & \\ & & & 0 \end{pmatrix}.$$

   (a) What is the SVD of A^T?

   (b) What is the rank of A? Give an expression for a smaller rank matrix that approximates A.

   (c) Is $Ax = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$ solvable?

   (d) What is the matrix P that projects $\mathbb{R}^4$ onto $N(A)$?

   (e) Let $B = AA^T$ and let $y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. As $n \to \infty$, what do you know about $B^n y$?

   (f) Suppose that you wanted to use the power method to compute v_1. How would you set this up? How does the error behave after k iterates?

3. Let

$$A = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}.$$

   (a) What are the eigenvectors of A? What special properties do they have?

   (b) What are eigenvalues corresponding to these eigenvectors?

4. Assume that A is diagonalizable with eigenvalues

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|.$$

   We may find λ_1 and a corresponding eigenvector v_1 using the power method. Let P be the projection matrix onto the orthogonal complement to the span of v_1.

   Now suppose that we want to find λ_2 and a corresponding eigenvector v_2. Consider the following algorithm

   1. Start with a random vector x_0.
   2. Iteratively set

$$x_k = \frac{PAx_{k-1}}{||PAx_{k-1}||}.$$

   (a) Assuming that A is normal (the eigenvectors for distinct eigenvalues are orthogonal) what does this converge to?

   (b) If A is not normal what can you say about the vector $y = \lim_{k\to\infty} x_k$ that this converges to? Let

$$Q = \begin{pmatrix} v_1 & y \end{pmatrix}$$

   What do you know about the matrix Q?

   (c) Write $AQ = QS$ for some matrix S. Do you know anything special about S (i.e. does it have a particular pattern of zero entries)?

   (d) What is λ_2 and a choice of v_2? (hint: use S!)

5. Suppose that A is a 3×3 matrix that is not diagonalizable, but has a basis of generalized eigenvectors $\{x_1, j_1, x_2\}$ where

$$Ax_1 = \lambda_1 x_1, \qquad Aj_1 = \lambda_1 j_1 + x_1, \qquad Ax_2 = \lambda_2 x_2.$$

   (a) Is the matrix $A + I$ diagonalizable? Give the eigenvalues and a basis of (generalized) eigenvectors.

   (b) What are the eigenvalues of the matrix A^n? What is a basis of (generalized) eigenvectors of A^n in terms of x_1, j_1, x_2?

   (c) What are the eigenvalues of the matrix e^{At}? What is a basis of (generalized) eigenvectors of e^{At} in terms of x_1, j_1, x_2?