18.06 Recitation 9

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1. Suppose that A is a 4×4 square matrix with eigenvalues $-0.7 \pm 0.1i, -0.3, 0.01$ and corresponding eigenvectors v_1, v_2, v_3, v_4.

   (a) If
   $$\frac{dx}{dt} = Ax,$$
   for some initial $x(0)$, what does the solution $x(t)$ probably look like after a long time?

   (b) For large t, the matrix e^{At} is approximately a rank ______ matrix because the columns are roughly spanned by ______.

   (c) What must be true of $x(0)$ for the solution to be decaying, and what does the normalized solution
   $$\frac{x(t)}{||x(t)||}$$
   then probably look like after a long time?

   (d) Suppose that instead we compute the recurrence $x_{n+1} = Ax_n$ for some initial vector x_0. What can you say about the solution x_n for large n for a typical x_0?

2. (Strang 6.3, Problem 18) By explicitly differentiating the infinite series definition for e^{At}, show that $e^{At}x(0)$ solves $dx/dt = Ax$ with initial condition $x(0)$.

3. (Strang 6.3, Problem 22) If $A^2 = A$ show that $e^{At} = I + (e^t - 1)A$. For the 2×2 $A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$ this gives $e^{At} =$ ______________.

4. (Strang 6.3, Problem 3)

   (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

   (b) With negative diagonal and positive off-diagonal adding to 0, $u' = Au$ will be a "continuous" Markov equation. For example for
   $$\frac{du}{dt} = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} u, \qquad u(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$
   This has an eigenbasis
   $$\lambda_1 = 0, v_1 = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}, \qquad \lambda_2 = -5, v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
   What is the steady state $u(\infty)$ as $t \to \infty$?

5. (Strang 6.3, Problem 26) Give two reasons why the matrix e^{At} is never singular.

   (a) Write down its inverse.

   (b) Why are these eigenvalues nonzero? If $Ax = \lambda x$ then

$$e^{At}x = \underline{\hspace{3cm}}.$$