18.06 Recitation 2

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1 Pictures/Words Problems

3. Which of the following are vector subspaces of $\mathbb{R}^2$:

   (a) The origin $(0, 0)$.

   (b) The first quadrant.

   (c) The vectors corresponding to points on the line $y = x + 1$.

   (d) The vectors corresponding to points on the line $y = 4x$.

   Solution:

   (a) The origin is the zero vector, which is always a subspace.

   (b) The first quadrant is *not* a subspace of $\mathbb{R}^2$. For any vector $v \in \mathbb{R}^2$ in the first quadrant, it's negative is in the third quadrant. But if the first quadrant were a subspace, it would have to include all scalar multiples of vectors in the space.

   (c) We think of solutions to $y = x + 1$ as vectors $\begin{bmatrix} x \\ y \end{bmatrix}$. Then this is not a subspace, since a subspace of $\mathbb{R}^2$ must contain all scalar multiples of vectors in the space. But if $\begin{bmatrix} x \\ y \end{bmatrix}$ is in the space, then the zero vector $0 \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a scalar multiple that is not in the space, since $1 \neq 0$.

   (d) This is a subspace! It is simply the span of the vector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

2 Problems

1. The following problems concern the vector space $\mathbf{M}$ of all 2×2 matrices. What do we implicitly mean are the operations of addition and scaler multiplication. Why is this a vector space?

   (i) (Strang 3.1 Problem 4) The matrix $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ is a vector in the space $\mathbf{M}$. Write sown the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

   (ii) (Strang 3.1 Problem 5)

(a) Describe a subspace of $\mathbf{M}$ that contains $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ but not $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) If a subspace of $\mathbf{M}$ does contain A and B, must it contain I?

(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

Solution:

The space $\mathbf{M}$ stands for the set of all 2×2 matrices with the operation of $+$ for matrix addition (in the usual sense) and scalar multiplication. To check that this is a vector space, the main two things we need are

(S) If $A \in \mathbf{M}$ (meaning: A is a 2×2 matrix), then for any scalar c, the scaled matrix cA is also in $\mathbf{M}$ (meaning: cA is also a 2×2 matrix). This is clearly true!

(A) If A and B are in $\mathbf{M}$ (meaning: A and B are both 2×2 matrices), then their sum $A + B$ is als in $\mathbf{M}$ (meaning: their sum is again a 2×2 matrix). This is also clearly true!

So there is nothing *deep* going on with the fact that $\mathbf{M}$ is a vector space – we are just keeping track of the fact that $\mathbf{M}$ behaves in the ways that we expect.

(i) The zero vector is $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It is $0 \cdot A$ (or 0 times any matrix in $\mathbf{M}$). It is also the matrix that can be added to any other matrix without changing the matrix: $0 + A = A$. The example scalar multiples of A are:

$$\frac{1}{2}A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad -A = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}.$$

The smallest subspace containing A is also known as the span of A. Intuitively we should understand what to do to find this: a subspace must be closed under taking arbitrary linear combinations (scalar multiples as in condition (S) above and additions as in condition (A) above). So to find the span of A, we must take all multiples of A. This subspace is

$$\langle A \rangle = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}, \qquad a \in \mathbb{R}.$$

Note that this does contain the zero matrix ($a = 0$) as it should!

(ii) (a) An example subspace that contains A but not B is just the span of A:

$$\langle A \rangle = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

A *non-example* is the subspace of diagonal matrices:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

This space has another name: the span of A and B.

(b) Any subspace containing A and B must contain all linear combinations of A and B. So in particular it must contain

$$A - B = \begin{pmatrix} 1 & 0 \\ 0 & -(-1) \end{pmatrix} = I.$$

(c) We could take the subspace of matrices

$$V := \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

Let's check that this is a subspace. We need to test two conditions:

(S) If $A \in V$, then for all scalars c, we need $cA \in V$. Well, if A is in V, then A is of the form $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ for some a. And so

$$cA = c \cdot \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ca \\ 0 & 0 \end{pmatrix},$$

which is again of the form of a matrix in V, with $x = ca$ this time.

(A) If A and B are both in V, then we need that $A + B$ is in V. Well, we have

$$A + B = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a+b \\ 0 & 0 \end{pmatrix}.$$

So this is again of the form of matrices in V, but with $x = a + b$.

2. (Strang 3.1 Problem 15)

(a) The intersection of two planes through $(0,0,0)$ in $\mathbb{R}^3$ is probably a ____________ in $\mathbb{R}^3$ but it could be a ____________. It can't be just $(0,0,0)$, why?

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ____________, but it could be a ____________ .

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbb{R}^5$, prove that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbb{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of the vectors that lie in both spaces.

Solution:

(a) The intersection of two planes through the origin is usually a **line**, but if the two planes are equal, then it is a **plane**. It is never just $(0,0,0)$ – a 2×3 matrix always has a nontrivial nullspace. Do you see why this is what the problem is asking?

(b) The intersection of a plane through the origin with a line through the origin is usually a **point** – namely the origin $(0,0,0)$ – but it could be a **line**, if the original line was contained in the plane.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are both subspaces, let's check that $\mathbf{S} \cap \mathbf{T}$ is also a subspace. As a set, $\mathbf{S} \cap \mathbf{T}$ is the set of vectors v so that $v \in \mathbf{S}$ and $v \in \mathbf{T}$. We need to check two conditions

(S) If $v \in \mathbf{S} \cap \mathbf{T}$, then for any scalar c, we need that $cv \in \mathbf{S} \cap \mathbf{T}$. Because $v \in \mathbf{S} \cap \mathbf{T}$, we know that $v \in \mathbf{S}$ and $v \in \mathbf{T}$. In order for cv to be in $\mathbf{S} \cap \mathbf{T}$, we need that both $cv \in \mathbf{S}$ and $cv \in \mathbf{T}$. But $\mathbf{S}$ and $\mathbf{T}$ are both subspaces, so this follows from condition (S) for $\mathbf{S}$ and $\mathbf{T}$.

(A) If v and v' are in $\mathbf{S} \cap \mathbf{T}$, we need that $v + v'$ is in $\mathbf{S} \cap \mathbf{T}$. The assumption that both v and v' are in $\mathbf{S} \cap \mathbf{T}$ means that v and v' are in $\mathbf{S}$ and v and v' are in $\mathbf{T}$ as well. So by (A) for the subspace $\mathbf{S}$, we have

$$v + v' \in \mathbf{S}.$$

And by (A) for the subspace $\mathbf{T}$, we have

$$v + v' \in \mathbf{T}.$$

Since $v + v'$ is in both $\mathbf{S}$ and $\mathbf{T}$, it is in $\mathbf{S} \cap \mathbf{T}$ as desired.

5. Let A be an $n \times n$ matrix and let b, c, x, y, z be n component vectors. Suppose that $Ax = b$ and $Ay = c$ are both solvable.

   (a) Show that $Az = 2b + 3c$ is solvable: what is a possible solution z?

   (b) Can you rephrase this in terms of column spaces?

   (c) If $z + u + v$ is another solution to $A(z + u + b) = 2b + 3c$ for some vectors u and v, then ____________ is in ____________.

   (d) If $z + \alpha u + \beta v$ is also a solution for any α and β, then ____________ is in ____________.

   Solution:

   (a) We can solve this by
   $$z = 2x + 3y.$$
   To check:
   $$Az = A(2x + 3y) = 2(Ax) + 3(Ay) = 2b + 3c.$$

   (b) The column space of A is the set of all vectors $\mathbf{b}$ so that $Ax = \mathbf{b}$ is solvable. So part (a) is asking if $2b + 3c$ is in the column space of A. But the columns space is a subspace, so it is closed under linear combinations. We know by assumption that b and c are in the column space of A, so the linear combination $2b + 3c$ must also be in the column space of A.

   (c) If we have another solution "$u + v$" (the fact that this happens to be decomposable into a sum is a red herring – we can always write a vector like this, if we, for example, take $v = 0$) to the equation
   $$A\mathbf{z} = 2b + 3c$$
   besides the original solution z we already knew about, then expanding out, we have
   $$A(z + u + v) = Az + A(u + v) = (2b + 3c) + A(u + v) = 2b + 3c,$$
   so $A(u + v) = 0$. Therefore $\mathbf{u} + \mathbf{v}$ is in the **null space** of A. (Note that this doesn't necessarily mean that u and v are individually in the null space of A, only that $Au = -Av$).

   (d) If for all scalars α and β, we know that $A(z + \alpha u + \beta v) = 2b + 3c$, then $\alpha u + \beta v$ is in the null space of A for any α and β. Therefore **the span of u and v** is in the **null space** of A. In particular, in this case, both u and v are independently in the null space of A.