18.06 Recitation 1

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5. What is the LU factorization of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}?$$

Solution: This matrix doesn't have an LU factorization! Here's why: suppose that there were a lower triangular matrix

$$L = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}$$

and an upper triangular matrix

$$U = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = LU.$$

In particular, this means that the upper left entry of the matrix A would satisfy

$$0 = x \cdot a + 0 \cdot 0 = xa.$$

But this means that one of x or a is 0, as the product of two nonzero numbers is always nonzero.

So either x or a is 0. Let's suppose that it's x. The the matrix L is really

$$L = \begin{bmatrix} 0 & 0 \\ y & z \end{bmatrix},$$

and we still have

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ y & z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

But now look at the upper right entry of the matrix A. The equation for this is

$$1 = 0 \cdot b + 0 \cdot c = 0.$$

But this is absurd! So no matrices L and U can exist. We'd get the same contradiction (but for $2 \neq 0$) if we had $a = 0$.

The problem was the the first column pivot was not in row 1, we had to do a row swap. If we were finding the LU factorization in order to solve some equations, doing a row swap just corresponds to switching the order of equations, which we can clearly do without changing the solutions. But in terms of the ordering we choose, an LU factorization does not exist!

5. Consider the company Widgets-R-Us (WRU). WRU makes 100 kinds of widgets and sells them in all 50 US states. Consider the 100 by 50 matrix A with i, j-entry recording the number of widgets of type i sold in state j in the month of August 2017. What kind of 50 by 1 column vectors x could you cook up so that the 100 by 1 column vector Ax records

   (a) the number of each kind of widgets sold in Kansas,

   (b) the number of each kind of widget sold in all western states combined, and

   (c) the gross revenue obtained from each kind of widget (suppose all widgets have the same price in each state but vary state to state)?

   What kind of 1 by 100 row vectors y could you cook up so that the 1 by 50 row vector yA records

   (a) total widgets sold by state,

   (b) total revenue by state (suppose each kind of widget has the same price in all states, but that different widgets can be priced differently)?

   Solution: The matrix A has rows indexed by widgets $w_1, \ldots, w_{100}$ and columns indexed by the states $s_1, \ldots, s_{50}$.

   (a) Say that Kansas is the state with index s_k. The we want to extract the s_kth column of the matrix A. This is achieved by the elementary vector with 0 all components $j \neq k$ and a 1 in the kth entry. So if $k = 3$, this would be

   $$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

   for example.

   (b) If the western states have indices $s_{r_1}, \ldots, s_{r_n}$, then the vector should have a 1 in components $r_1, \ldots, r_n$ and a 0 elsewhere.

   (c) Suppose that all widgets in state s_j have price p_j. Then this is achieved by the vector

   $$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \\ p_{50} \end{bmatrix}.$$

   (a) To find total widgets by state, we want to sum each column of the matrix. This is achieved by the matrix

   $$\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}.$$

   (d) Suppose that widget i has price q_i in all states. Then this is achieved by the matrix

   $$\begin{bmatrix} q_1 & q_2 & \cdots & q_{100} \end{bmatrix}.$$

6. (Strang 2.4.31, essentially) Consider a fixed complex number $A + Bi$, where A, B are real. View complex numbers $z = x + yi$ as column vectors

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

Can you think of a 2 by 2 matrix M such that left multiplication by M is the same as complex multiplication by $A + Bi$? When is this matrix singular?

Solution: The product

$$(A + Bi)(x + yi) = (Ax + i^2 By) + (Bx + Ay)i = (Ax - By) + (Bx + Ay)i.$$

Putting this back in vector form, this is represented by

$$\begin{bmatrix} Ax - By \\ Bx + Ay \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

So the matrix representing "multiplication by the complex number $A + Bi$" is

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

This is singular if and only if it has fewer than 2 pivots. This will happen if (1) $A = 0$ and $B = 0$, or if (2) $A \neq 0$ but $A + B^2/A = 0$ (the second pivot is 0 after clearing the first column). These two conditions can be combined into one by saying $A^2 + B^2 = 0$.