18.06 Recitation 11

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1. **(Guided: What is SVD? From the perspective of AA^T and A^TA)** Let A be an $m \times n$ real matrix of rank r.

   (a) The matrix AA^T:
   - ...is dimensions _____ $\times$ _____
   - ...and is ____________________ and ____________________.
   - Therefore the eigenvalues λ_i of AA^T satisfy ____________________.
   - I can choose an eigenbasis $\{u_i\}$ for AA^T such that ____________________.
   - $N(AA^T) =$ ________________.
   - The rank of AA^T is _______________.
   - $C(AA^T) =$ ________________.

   (b) The matrix A^TA:
   - ...is dimensions _____ $\times$ _____
   - ...and is ____________________ and ____________________.
   - Therefore the eigenvalues ω_i of A^TA satisfy ____________________.
   - I can choose an eigenbasis $\{v_i\}$ for A^TA such that ____________________.
   - $N(A^TA) =$ ________________.
   - The rank of A^TA is _______________.
   - $C(A^TA) =$ ________________.

   (c) By considering $A(A^TA)v_i$ and $A^T(AA^T)u_i$, the eigenvalues (without multiplicity) of AA^T and A^TA are ____________________. Up to reordering we can therefore assume that
   $$A^TAv_i = \sigma_i^2 v_i \text{ and } AA^Tu_i = \sigma_i^2 u_i \qquad 1 \leq i \leq r,$$
   with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$; and that $A^TAv_i = AA^Tu_i = 0$ for $i > r$.

   (d) Using $(A^TA)v_i$, we know that $||Av_i|| =$ ________________.

   (e) Again from considering $A(A^TA)v_i$, we know that
   $$Av_i = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

   (f) Using the above, write $AV = U\Sigma$, where V is the matrix of vectors v_i and U is the matrix of vectors v_i, and Σ is the ______ $\times$ ______ diagonal matrix of the singular values

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

(g) Using this, write A as a sum of ____________ rank 1 matrices

$$A = \underline{\hspace{6cm}}.$$

2. Suppose that B is a real-symmetric matrix.

   (a) What do you know about the eigenvalues of B? What nice properties can we arrange for an eigenbasis?

   (b) Using the eigenbasis, what is the maximum possible value of the quoteint $\frac{x^TBx}{x^Tx}$?

   (c) What is the minimum possible value of the quoteint $\frac{x^TBx}{x^Tx}$?

3. Given the SVD of a real matrix A and singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, what is the maximum value of $\frac{||Av||}{||v||}$?

4. **(Beginning defective matrices!)**

   (a) What are the eigenvalues and eigenvectors of the 2×2 matrix

   $$A = \begin{pmatrix} 0 & 1 \\ 0 & \epsilon \end{pmatrix}?$$

   (b) What happens to the two eigenvectors as $\epsilon \to 0$. What does this tell you about diagonalizability of A?

   (c) For $\epsilon = 0$, $N(A)$ is dimension ____________. What is $N(A^2)$?

   (d) For $\epsilon = 0$, we know that $A^2 =$ ____________. Therefore

   $$e^{At} = \underline{\hspace{3cm}}.$$

   (e) Given input vector $x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$, give the solution $x(t)$ to

   $$\frac{dx}{dt} = Ax$$

   with this input. What is the behavior as $t \to \infty$?

   (f) How does e^{At} act on eigenvectors?

   (g) Similarly, what is $(I + A)^n$? How does this act on eigenvectors?