18.06 Recitation 2

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1 Pictures/Words Problems

1. If S_1 stands for the operation of putting on your socks, and S_2 stands for the operation of putting on your shoes (so $S_2 \circ S_1$ stands for first putting on your socks and then putting on your shoes), what is the inverse of $S_2 \circ S_1$?

2. The following are a bunch of True/False problems. If True, try to explain in words why; if False, try to give a counter-example!

   (a) **(T/F)** For any matrix A, if I use Gaussian Elimination to find the $A = LU$ factorization, then the diagonal entries of U are the same as the diagonal entries of A.

   (b) **(T/F)** The matrix $\begin{pmatrix} 0 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is upper-triangular.

   (c) **(T/F)** If Gaussian Elimination on A produces the identity matrix, then A is lower-triangular.

   (d) **(T/F)** The inverse of a permutation matrix P is always the same permutation matrix P.

   (e) **(T/F)** Fix a vector u in $\mathbb{R}^3$. Then all vectors v so that the dot product $u \cdot v = 0$ is a vector subspace of $\mathbb{R}^3$.

   (f) **(T/F)** Given a nonsingular matrix A and m vectors $b_1, \ldots, b_m$, it will necessarily take on the order of m^4 operations to solve all the equations

   $$Ax_1 = b_1, Ax_2 = b_2, \ldots, Ax_m = b_m.$$

   (g) **(T/F)** The set of all invertible 2×2 matrices (with usual addition and scaler multiplication) forms a vector subspace of $\mathbf{M}$ the space of all 2×2 matrices.

3. Which of the following are vector subspaces of $\mathbb{R}^2$:

   (a) The origin $(0, 0)$.

   (b) The first quadrant.

   (c) The vectors corresponding to points on the line $y = x + 1$.

   (d) The vectors corresponding to points on the line $y = 4x$.

2 Problems

1. The following problems concern the vector space **M** of all 2×2 matrices. What do we implicitly mean are the operations of addition and scaler multiplication. Why is this a vector space?

   (i) (Strang 3.1 Problem 4) The matrix $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ is a vector in the space **M**. Write sown the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

   (ii) (Strang 3.1 Problem 5)

   (a) Describe a subspace of **M** that contains $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ but not $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

   (b) If a subspace of **M** does contain A and B, must it contain I?

   (c) Describe a subspace of **M** that contains no nonzero diagonal matrices.

2. (Strang 3.1 Problem 15)

   (a) The intersection of two planes through $(0, 0, 0)$ in $\mathbb{R}^3$ is probably a ____________ in $\mathbb{R}^3$ but it could be a ____________. It can't be just $(0, 0, 0)$, why?

   (b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a ____________, but it could be a ____________ .

   (c) If **S** and **T** are subspaces of $\mathbb{R}^5$, prove that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbb{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of the vectors that lie in both spaces.

3. (Strang 3.1 Problem 23) If we add an extra column b to a matrix A, then the column space get larger unless ____________. Give an example where the column space gets larger and an example where it doesn't. Why is $Ax = b$ solvable exactly when the column space doesn't get larger – i.e. it is the same for A and $\begin{bmatrix} A & b \end{bmatrix}$.

4. Suppose that E, A, and R are $n \times n$ matrices. Further assume that E is invertible, and that we have a factorization $EA = R,$.

   (a) If y is an n component vector and $Ey = 0$, what can we say about y?

   (b) If x is an n component vector and $Rx = 0$, then what can we say about EAx and Ax?

   (c) If z is an n component vector and $Az = 0$, then what can we say about Rz?

   (d) What does this tell us about the null spaces of the matrices E, A and R?

5. Let A be an $n \times n$ matrix and let b, c, x, y, z be n component vectors. Suppose that $Ax = b$ and $Ay = c$ are both solvable.

   (a) Show that $Az = 2b + 3c$ is solvable: what is a possible solution z?

   (b) Can you rephrase this in terms of column spaces?

   (c) If $z + u + v$ is another solution to $A(z + u + b) = 2b + 3c$ for some vectors u and v, then ____________ is in ____________.

   (d) If $z + \alpha u + \beta v$ is also a solution for any α and β, then ____________ is in ____________.