18.06 Recitation 12

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October 1, 2018

1. Exploring defective matrices
   (a) What are the eigenvalues and eigenvectors of the 2×2 matrix
   $$A = \begin{pmatrix} 0 & 1 \\ 0 & \epsilon \end{pmatrix}?$$
   (b) What happens to the two eigenvectors as $\epsilon \to 0$. What does this tell you about diagonalizability of A?
   (c) For $\epsilon = 0$, $N(A)$ is dimension ____________. What is $N(A^2)$?
   (d) For $\epsilon = 0$, we know that $A^2 =$ ____________. Therefore
   $$e^{At} = \underline{\hspace{3cm}}.$$
   (e) Given input vector $x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$, give the solution $x(t)$ to
   $$\frac{dx}{dt} = Ax$$
   with this input. What is the behavior as $t \to \infty$?
   (f) How does e^{At} act on eigenvectors?
   (g) What is a basis of eigenvectors and "Jordan vectors" or "generalized eigenvectors"? Does this explain the above behavior?

2. (a) Which of the following matrices have the same Jordan forms, and what are they?
   $$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
   (b) Consider a 6×6 matrix whose bottom triangle is
   $$\begin{pmatrix} 0 & & & & & \\ 0 & 0 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
   and whose upper triangle is all 0 except for a single 1 in one of the 15 possible positions. What positions give rise to what possible Jordan forms?

3. Suppose that A is a 3×3 matrix with roots $\lambda = 1, -0.3$ of its characteristic polynomial $\det(A - \lambda I)$.

   (a) Suppose that for some vector x_0, the powers $A^n x_0$ grow in magnitude as $n \to \infty$. What do you know about A?

   (b) What is $||A^{500} x_0|| / ||A^{100} x_0||$ approximately?