The Four Fundamental Subspaces

A is $m \times n$ matrix of rank r

$R = \text{rref}(A)$, $M = \text{rref}(A^T)$, The SVD of A is $U\Sigma V^T$

Nullspace: $N(A)$

- definition:
- subspace of _________ of dimension _____
- orthogonal complement:
- relation to $N(R)$ or $N(M^T)$:
- if $N(A) = 0$ then we say that A has

  ____
- an orthonormal basis given by:

Row space: $C(A^T)$

- definition:
- subspace of _________ of dimension _____
- orthogonal complement:
- relation to $C(R^T)$ or $C(M)$:
- if $C(A^T) = \mathbb{R}^n$ then we say that A has

  ____
- an orthonormal basis given by:

Column space: $C(A)$

- definition:
- subspace of _________ of dimension _____
- orthogonal complement:
- relation to $C(R)$ or $C(M^T)$:
- if $C(A) = \mathbb{R}^m$ then we say that A has

  ____
- an orthonormal basis given by:

Left nullspace: $C = N(A^T)$

- definition:
- subspace of _________ of dimension _____
- orthogonal complement:
- relation to $N(R^T)$ or $N(M)$:
- if $N(A^T) = 0$ then we say that A has

  ____
- an orthonormal basis given by:

Number of Operations

1. Multiplying an $m \times n$ matrix A by an n component vector v:

2. Multiplying an $m \times n$ matrix A by an $n \times r$ matrix B:

3. Computing the dot product of two m component vectors v and w:

4. Finding the inverse of a matrix A using the Gauss-Jordan method:

5. Finding the LU factorization of an $m \times m$ matrix A using Gaussian elimination:

6. Finding the LU factorization of an $m \times m$ upper-triangular matrix A using Gaussian elimination:

7. Finding the LU factorization of an $m \times m$ matrix A such that $a_{ij} = 0$ if $i > j + 1$ using Gaussian elimination:

8. Finding the LU factorization of an $m \times m$ matrix A such that $a_{ij} = 0$ if $i > j + 1$ or $j > i + 1$ using Gaussian elimination:

9. Solving an upper-triangular system $Ux = b$, where U is an $m \times m$ upper-triangular matrix and b is an m component vector:

10. Solving an upper-triangular system $Ux = b$, where U is an $m \times m$ upper-bidiagonal matrix (so $u_{ij} = 0$ if $i > j$ or $j > i + 1$) and b is an m component vector:

Linear Algebraic Algorithms

Gaussian Elimination:

- Purpose:
- Input:
- Any assumptions on the input?
- Output:
- Approximate # of operations (based on size of input):
- Factorizations?
- Pattern of nonzero entries if input is:
  - tridiagonal
  - lower triangular
  - upper triangular

Gram-Schmidt:

- Purpose:
- Input:
- Any assumptions on the input?
- Output:
- Approximate # of operations (based on size of input):
- Factorizations?
- Pattern of nonzero entries if input is:
  - tridiagonal
  - lower triangular
  - upper triangular

The matrix is...	Definition	Special about **eigenvalues**	Special about **eigenvectors**	Special about **det**	Factorizations	Consequences
invertible						
projection						
Markov						
positive Markov						
(real orthogonal) unitary						
(real symmetric) Hermitian						
(real anti-symmetric) anti-Hermitian						
positive/negatie (semi)definite Hermitian						

Suppose that A is a matrix with eigenvectors $v_1, \cdots, v_n$ and eigenvalues $\lambda_1, \cdots, \lambda_n$. Write as much as you know about the eigenvectors and eigenvalues of the following matrices:

	$-A$	A^{2017}	$A^4 + 3A + I$	A^T	e^{A^2}	$(A+I)^{-1}$	SAS^{-1}
eigenvalues							
eigenvectors							

Suppose instead that A is 3×3 and not diagonalizable, with a basis of generalized eigenvectors v_1, j_1, v_2, so that $Aj_1 = \lambda_1 j_1 + v_1$. Now write down as much as you know about the eigenvalues, (generalized) eigenvectors, and diagonalizability of the following matrices:

	$-A$	A^2	$A + I$	A^T	e^A	A^{-1}	SAS^{-1}
eigenvalues							
(generalized) eigenvectors							
diagonalizable?							

Suppose that A is an $(m+1)\times(m+1)$ nondiagonalizable matrix with eigenvalues λ_1 (with multiplicity 2), $\lambda_2,\ldots,\lambda_m$, and a corresponding basis of (generalized) eigenvectors $x_1, j_1, x_2, \ldots, x_m$ (so $Aj_1 = \lambda_1 j_1 + x_1$ and $Ax_i = \lambda_i x_i$).

1. What is the solution to the differential equation $\frac{dx}{dt} = Ax$ with initial condition $x(0)$?

2. For *generic* $x(0)$, under what conditions does:

   (a) $|x(t)| \to \infty$, as $t \to \infty$?

   (b) $|x(t)| \to 0$, as $t \to \infty$?

   (c) $|x(t)| \to$ a nonzero constant, as $t \to \infty$?

   (d) $x(t) \to$ a nonzero constant, as $t \to \infty$?

   (e) $x(t)$ oscillates, as $t \to \infty$?

3. For *some choice of* $x(0)$, under what conditions does:

   (a) $|x(t)| \to \infty$, as $t \to \infty$?

   (b) $|x(t)| \to 0$, as $t \to \infty$?

   (c) $|x(t)| \to$ a nonzero constant, as $t \to \infty$?

   (d) $x(t) \to$ a nonzero constant, as $t \to \infty$?

   (e) $x(t)$ oscillate, as $t \to \infty$?

4. What is the solution to the recurrence $x_k = Ax_{k-1}$ with initial condition x_0?

5. For *generic* x_0, under what conditions does:

   (a) $|x_n| \to \infty$, as $n \to \infty$?

   (b) $|x_n| \to 0$, as $n \to \infty$?

   (c) $|x_n| \to$ a nonzero constant, as $n \to \infty$?

   (d) $x_n \to$ a nonzero constant, as $n \to \infty$?

   (e) x_n oscillates, as $n \to \infty$?

6. For *some choice of* x_0, under what conditions does:

   (a) $|x_n| \to \infty$, as $n \to \infty$?

   (b) $|x_n| \to 0$, as $n \to \infty$?

   (c) $|x_n| \to$ a nonzero constant, as $n \to \infty$?

   (d) $x_n \to$ a nonzero constant, as $n \to \infty$?

   (e) x_n oscillates, as $n \to \infty$?