18.06 Recitation 6

Isabel Vogt

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1 Problems

4. (essentially Strang, Section 4.4, Problem 33) Find all matrices that are both orthogonal and lower triangular in two parts:

   (a) What are all orthogonal and lower-triangular 2×2 matrices?

   (b) Can you generalize this to $m \times m$ matrices?

   Solution:

   (a) Any 2×2 lower-triangular matrix is of the form

   $$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

   In order to be orthogonal, we need:

   $$a^2 + c^2 = 1, \qquad cd = 0, \qquad d^2 = 1.$$

   Let's look at these backwards: first $d^2 = 1$ tells us that $d = \pm 1$. The second equation tells us that $c = 0$, since we can divide by the nonzero number d. Finally, the first equation simplifies to $a^2 = 1$ and so $a = \pm 1$. So all orthogonal, lower-triangular 2×2 matrices are of the form

   $$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

   (b) In general, we might surmise that "working backwards" is helpful. Let's write our lower-triangular $m \times m$ matrix as

   $$Q = \begin{pmatrix} q_1 & \cdots & q_m \end{pmatrix},$$

   where the ith component of the jth column vector satisfies

   $$(q_j)_i = 0 \qquad \text{for } i < j.$$

   In particular, for last column q_m, the only nonzero entry is in the last position. So first starting with the condition

   $$q_m \cdot q_m = 1,$$

   we see that $((q_m)_m)^2 = 1$ so $(q_m)_m = \pm 1$. Now for $i < m$, the ith column and the mth column are orthogonal. But the only nonzero entry of the mth column is the last one, so

   $$q_i \cdot q_m = (q_i)_m (q_m)_m = 0 \qquad \Rightarrow \qquad (q_i)_m = 0,$$

since we can divide by the nonzero $(q_m)_m$. Therefore the bottom row of Q is all zeros except for the final position.

Carrying on, let's look now at the condition $q_{m-1} \cdot q_{m-1} = 1$. Originally, when we only knew that Q was upper-triangular, q_{m-1} could only have two nonzero entries: $(q_{m-1})_m$ and $(q_{m-1})_{m-1}$. But from the last step, we learned that $(q_{m-1})_m = 0$. So q_{m-1} also has only 1 nonzero entry. Exactly as above we can therefore show that

$$(q_i)_{m-1} = \begin{cases} 0 & \text{if } i \neq m-1 \\ \pm 1 & \text{if } i = m-1. \end{cases}$$

And so we learn that the second to bottom row is also almost all 0, except in the $(m-1, m-1)$ position.

Carrying on like this, we see that as before, Q must be diagonal

$$Q = \begin{pmatrix} \pm 1 & & & & \\ & \pm 1 & & & \\ & & \ddots & & \\ & & & \pm 1 & \\ & & & & \pm 1 \end{pmatrix}.$$