18.06 Recitation 4

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1 Pictures/Words Problems

1. (Strang, 3.4 Problem 24 + 3.5 Problem 25 + ϵ) True or False (give a good reason if True/example if False)

   (a) If the zero vector is in the column space of a matrix A, then the columns of A are linearly dependent.

   (b) If the columns of a matrix are dependent, so are the rows.

   (c) The column space of a 2×2 matrix is the same as its row space.

   (d) The column space of a 2×2 matrix has the same dimension as its row space.

   (e) The columns of a matrix are a basis for the column space.

   (f) A and A^T have the same number of pivots.

   (g) A and A^T have the same left nullspace.

   (h) If the row space equals the column space then $A^T = A$.

   (i) If $A^T = -A$, then the row space of A equals the column space.

   Solution:

   (a) **False:** every subspace contains the zero vector. Take for instance $A = I$.

   (b) **False:** take a matrix with more columns than rows but full row rank, for example

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

   (c) **False:** take a matrix such as

   $$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

   Then $C(A) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$, but $R(A) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, which is not the same.

   (d) **True:** this is always true: the dimensions of both spaces are the rank of A.

   (e) **False:** the columns of a matrix span the columns space, but they might be linearly dependent. Take for example a matrix with more columns than rows:

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(f) **True:** the number of (nonzero) pivots is the rank of A, which is equal to the rank of A^T.

(g) **False:** the left nullspace of A is $N(A^T)$ and the left nullspace of A^T is $N(A)$ and in general these don't even lie in the same space if A is not square!

(h) **False:** take any invertible but nonsymmetric matrix, like

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

In this case, the row space and the column space are both all of $\mathbb{R}^2$, but $A \neq A^T$.

(i) **True:** here is another way to make a counter-example to the previous part! Note that $C(A) = C(-A) = C(A^T) = R(A)$.

2 Problems

1. (Strang 3.4, Problem 7) If w_1, w_2, w_3 are independent vectors in $\mathbb{R}^3$, show that the differences

$$\begin{aligned} v_1 &= w_2 - w_3 \\ v_2 &= w_1 - w_3 \\ v_3 &= w_1 - w_2. \end{aligned}$$

are *dependent*. Find the matrix A so that

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}.$$

Which matrices above are singular?

Solution:

We can show that the differences are dependent by finding a relation:

$$v_1 - v_2 + v_3 = (w_2 - w_3) - (w_1 - w_3) + (w_1 - w_2) = 0.$$

The matrix A is

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

As has to be the case, this matrix is singular. So is the matrix $(v_1\ v_2\ v_3)$.

3. **(Strang 3.4, Problem 22)** Construct $A = uv^T + wz^T$ whose column space has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and whose row space has basis $(1,0),(1,1)$. Write A as a 3×2 matrix times a 2×2 matrix.

Solution:

Let's try to find A such that the columns are an invertible linear combination of $u = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, and $w = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, and the rows contain $(1,0)$ and $(1,1)$ as the first two rows.

Inspection (or other techniques for solving linear equations) will show you that this is possible by

$$\left(0 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + (1/2) \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} , \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} - (1/2) \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ (1/2) & (7/2) \end{pmatrix} = A.$$

And so we see that

$$\begin{aligned} A &= \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & -1/2 \end{pmatrix}. \end{aligned}$$

So we can take

$$v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad z = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$$

5. **(Strang 3.5, Problem 23)** If a subspace S is contained in a subspace V, prove that $S^{\perp}$ contains $V^{\perp}$.

   Solution:

   Suppose that $w \in V^{\perp}$. Then for all $v \in V$, we have that $w \cdot v = 0$. Since S is contained in V, for all $s \in S$, we also have $s \in V$, and therefore $w \cdot s = 0$. So $w \in S^{\perp}$. This shows every vector in $V^{\perp}$ is also in $S^{\perp}$ and so we have $V^{\perp} \subset S^{\perp}$.