1 Problems

1. (Spring 2009, Exam 1, Problem 3) A is a matrix with nullspace $N(A)$ spanned by the following 3 vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ 1 \end{pmatrix}$$

   (a) Give a matrix B so that its column space $C(B)$ is the same as $N(A)$.

   (b) Give a different possible answer to (a): another B with $C(B) = N(A)$.

   (c) For some vector $\mathbf{b}$, you are told that a particular solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x}_p = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

   Now your classmate Zarkon tells you that a second solution is

$$\mathbf{x}_Z = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix},$$

   while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct."

$$\mathbf{x}_H = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

   Is Zarkon's solution correct, or Hastur's solution, or are both correct?

2. (essentially Fall 2005, Exam 2, #9 from 2009 practice problems) We take measurements $1, 4, b_3$ at times $t = 1, 2, 3$. We want to find the nearest line $C + Dt$ using least squares.

   (a) Which value of b_3 will put all three points on the same line. Give C and D for this line.

   (b) Will least squares choose this line if $b_3 = 9$?

   (c) What is the linear system $Ax = b$ that would be solved exactly for $x = (C, D)$ if all three points lie on a line. Write down a formula for the projection matrix onto the column space of A. What are the dimensions of the matrix? What is its rank? What is its column space?

   (d) More generally, what is the linear system $Gx = f$ that is solved exactly by the least squares solution $\hat{x} = (\hat{C}, \hat{D})$.

3. (Fall 2012, Exam 2, Problem 1) P is any $n \times n$ projection matrix. Compute the ranks of A, B, and C below. Your method must be visibly correct for every such P, not just one example.

   (a) $A = (I - P)P$.

(b) $B = (I - P) - P$. (Hint: squaring B might be helpful.)

(c) $C = (I - P)^{2017} + P^{2017}$.

4. (Fall 2012, Exam 2, Problem 3) The 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has QR decomposition

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = Q \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$$

(a) What is r_{11} in terms of $a, b, c, d, e, f, g, h, i$? (but not any of the elements of Q!)

(b) Solve for x in the equation

$$Q^T x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

expressing your solution possibly in terms of r_{11}, r_{22}, r_{33} and $a, b, c, d, e, f, g, h, i$ (but again not any of the elements of Q.)