18.06 Recitation 4

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1 Pictures/Words Problems

1. Describe geometrically in pictures/words these subspaces:

   (a) The span of two vectors $v_1, v_2 \in \mathbb{R}^n$ if v_1 and v_2 are linearly dependent.

   (b) The span of three linearly independent vectors $v_1, v_2, v_3 \in \mathbb{R}^n$.

   (c) Let $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The space of vectors $w \in \mathbb{R}^3$ so that $u \cdot w = 0$. Can you describe the geometry of this situation in terms of fundamental subspaces?

2. (Strang, 3.4 Problem 24 + 3.5 Problem 25 + ϵ) True or False (give a good reason if True/example if False)

   (a) If the zero vector is in the column space of a matrix A, then the columns of A are linearly dependent.

   (b) If the columns of a matrix are dependent, so are the rows.

   (c) The column space of a 2×2 matrix is the same as its row space.

   (d) The column space of a 2×2 matrix has the same dimension as its row space.

   (e) The columns of a matrix are a basis for the column space.

   (f) A and A^T have the same number of pivots.

   (g) A and A^T have the same left nullspace.

   (h) If the row space equals the column space then $A^T = A$.

   (i) If $A^T = -A$, then the row space of A equals the column space.

2 Problems

1. (Strang 3.4, Problem 7) If w_1, w_2, w_3 are independent vectors in $\mathbb{R}^3$, show that the differences

$$\begin{aligned} v_1 &= w_2 - w_3 \\ v_2 &= w_1 - w_3 \\ v_3 &= w_1 - w_2. \end{aligned}$$

are *dependent*. Find the matrix A so that

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}.$$

Which matrices above are singular?

2. **(Strang 3.4, Problem 11)** A is an $m \times n$ matrix of rank r. Suppose there are right sides b for which $Ax = b$ has no solution. (What does this mean in terms of fundamental subspaces?)

   (a) What are all of the inequalities ($<$ or $\leq$) that must be true between m and n and r?

   (b) How do you know that $A^T y = 0$ has solutions other than $y = 0$?

3. **(Strang 3.4, Problem 22)** Construct $A = uv^T + wz^T$ whose column space has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and whose row space has basis $(1,0),(1,1)$. Write A as a 3×2 matrix times a 2×2 matrix.

4. (Strang 3.5, Problem 5)

   (a) If $Ax = b$ has a solution and $A^T y = 0$, is

   $$y^T x = 0, \qquad \text{or} \qquad y^T b = 0?$$

   (b) If $A^T y = (1,1,1)$ has a solution and $Ax = 0$, then ______________.

5. **(Strang 3.5, Problem 23)** If a subspace S is contained in a subspace V, prove that $S^\perp$ contains $V^\perp$.

6. **(Strang 3.5, Problem 27)** The lines $3x + y = b_1$ and $6x + 2y = b_2$ are ______________. They are the same line if ______________. In that case (b_1, b_2) is perpendicular to the vector ______________. The nullspace of the matrix is the line $3x + y =$ ________. One particular vector in the nullspace is ______________.

7. (Strang 3.5, Problem 33)** Suppose I give you 8 vectors $r_1, r_2, n_1, n_2, c_1, c_2, l_1, l_2$ in $\mathbb{R}^4$.

   (a) What are the conditions for those pairs to be bases for the our fundamental subspaces of a 4×4 matrix?

   (b) What is one possible matrix A?