

INFDEV026A - Algoritmiek

Week 6

G. Costantini, F. Di Giacomo, G. Maggiore

costg@hr.nl, giacf@hr.nl, maggg@hr.nl - Office H4.204

Today

- ▶ ~~Why is my code slow?~~
 - ▶ ~~Empirical and complexity analysis~~
- ▶ ~~How do I order my data?~~
 - ▶ ~~Sorting algorithms~~
- ▶ ~~How do I structure my data?~~
 - ▶ ~~Linear, tabular, recursive data structures~~
- ▶ How do I represent relationship networks?
 - ▶ Graphs

More detailed agenda

- ▶ What is **dynamic programming**?
 - ▶ Fibonacci example; memoization and bottom-up approaches
- ▶ How can we find the shortest paths between **all pairs** of nodes in a graph?
 - ▶ Floyd Warshall algorithm

Dynamic Programming

Fibonacci sequence, Memoization, Bottom up, General idea

Fibonacci Sequence

Definition

- ▶ Sequence of integer numbers, built according to the following rules
 - ▶ Elements 0 and 1 are 1
 - ▶ Any other element is the result of the sum of the two preceding elements in the sequence
- ▶ Recursive formulation

$$F(0) = 1$$

$$F(1) = 1$$

$$F(n) = F(n - 1) + F(n - 2)$$

Fibonacci Sequence

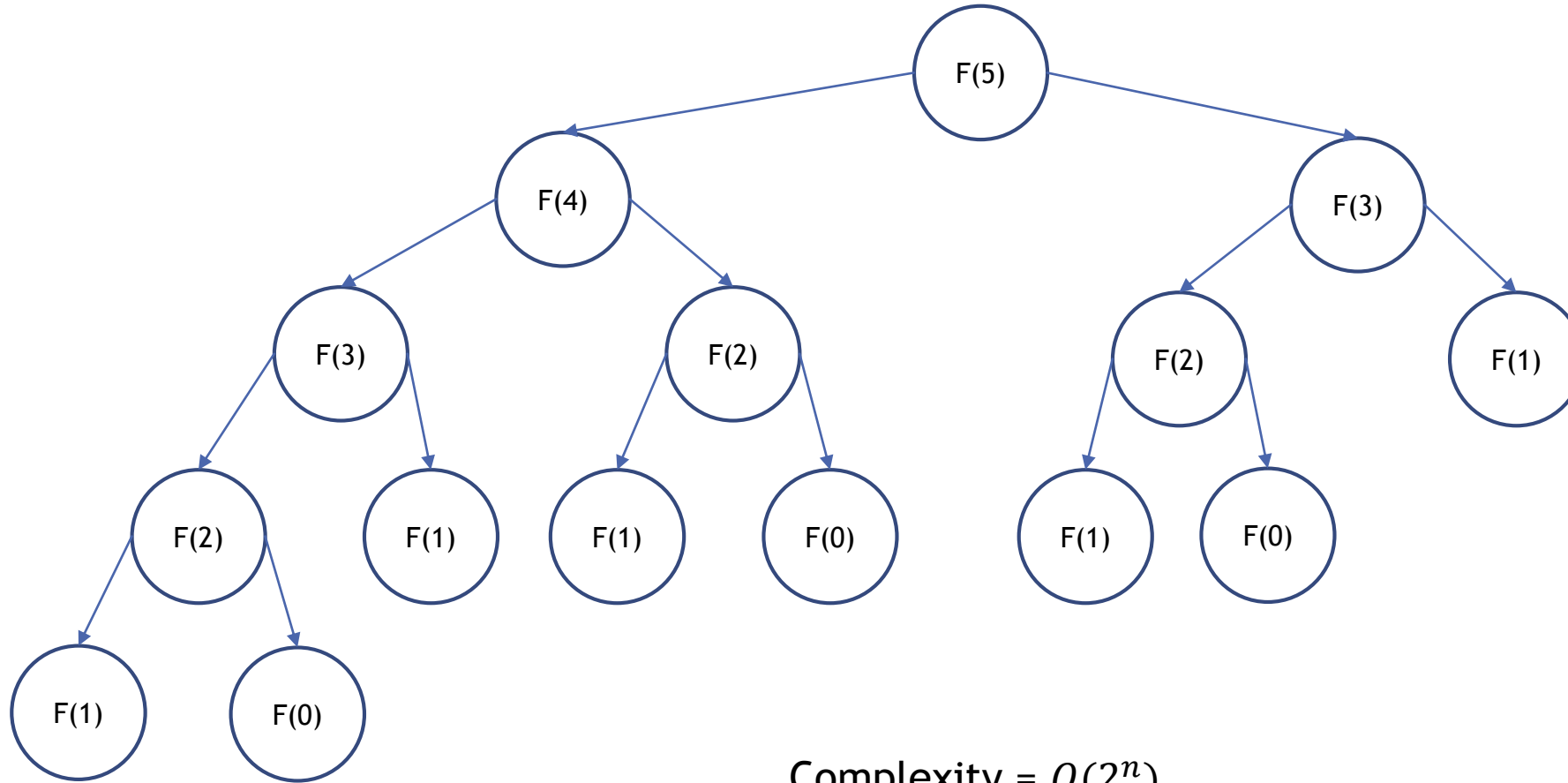
Recursive function

```
function fib(n)  
    if n <= 1 return 1  
    return fib(n - 1) + fib(n - 2)
```

n	Time (ms)
30	4
35	50
40	551
45	3792
>50	Happy watching!

Fibonacci Sequence

Why so slow?



Complexity = $O(2^n)$

Fibonacci Sequence

Why so slow?

“*Did I ever tell you what the definition of insanity is? Insanity is doing the exact... same *** thing... over and over again expecting... things to change...*”

- ▶ We are doing the same thing over and over again
 - ▶ $F(3)$ computed twice, $F(2)$ computed three times...
- ▶ Ideas on how to speed up the process?
 - ▶ We could save the result of sub-problems (= *memoization*)!

Fibonacci Sequence

Memoization

- ▶ Save the result of sub-problems into a data structure (called *lookup table*)
- ▶ Before making a recursive call, we check the lookup table...
 - ▶ Sub-problem never solved yet? → Make the recursive call
 - ▶ Sub-problem already solved? → Read the result stored in the table
- ▶ Add the result of the current recursive call to the lookup table

Fibonacci Sequence

Memoized algorithm (recursive)

- Suppose m is a map object which maps each value of `fib` that has already been calculated to its result

```
var m := map(0 → 1, 1 → 1)
function fib(n)
  if key n is not in map m
    m[n] := fib(n - 1) + fib(n - 2)
  return m[n]
```

Fibonacci Sequence

Memoization

```
var m := map(0 → 0, 1 → 1)
function fib(n)
  if key n is not in map m
    m[n] := fib(n - 1) + fib(n - 2)
  return m[n]
```

- ▶ Time complexity?
 - ▶ Accessing the lookup table requires $O(1)$
 - ▶ Sub-problems are n
 - ▶ Time complexity then is $O(n)$: we compute n times sub-problems that require $O(1)$ time
- ▶ Space complexity?
 - ▶ $O(n)$ memory space to save the sub-problems results

Fibonacci Sequence

Bottom up algorithm (iterative)

- ▶ Other possible approach
 - ▶ Build the result of the computation starting from the base case of the recursion
 - ▶ At each iteration save the intermediate results to use at the next step
- ▶ Same time complexity $O(n)$ as the recursive version, but only $O(1)$ of space complexity

```
function fib(n)
  if n = 0
    return 1
  else
    var previousFib := 1, currentFib := 1
    repeat n - 1 times // loop is skipped if n = 1
      var newFib := previousFib + currentFib
      previousFib := currentFib
      currentFib := newFib
    return currentFib
```

Floyd Warshall algorithm

All pairs shortest paths problem

Shortest path problem

- ▶ Remember Dijkstra?
 - ▶ Shortest paths starting from ONE node to ALL other nodes
- ▶ What if we need shortest paths from multiple source nodes?
 - ▶ Floyd-Warshall: shortest paths starting from ALL nodes to ALL nodes
 - ▶ Example of dynamic programming

Floyd-Warshall algorithm

- ▶ Compares all possible paths through the graph between each pair of vertices
- ▶ Every combination of edges is tested
- ▶ It works by incrementally improving an estimate on the shortest path between two vertices, until the estimate is optimal
- ▶ Based on a recursive idea

Floyd-Warshall algorithm

- ▶ Consider
 - ▶ A graph G with vertices V numbered from 1 to N
 - ▶ A function $shortestPath(i, j, k)$ which returns the shortest possible path from i to j using vertices only from the set $\{1, 2, \dots, k\}$ as intermediate points along the way
- ▶ Goal
 - ▶ Find the shortest path from each i to each j using only vertices from 1 to $k + 1$
 - ▶ What could this shortest path be?

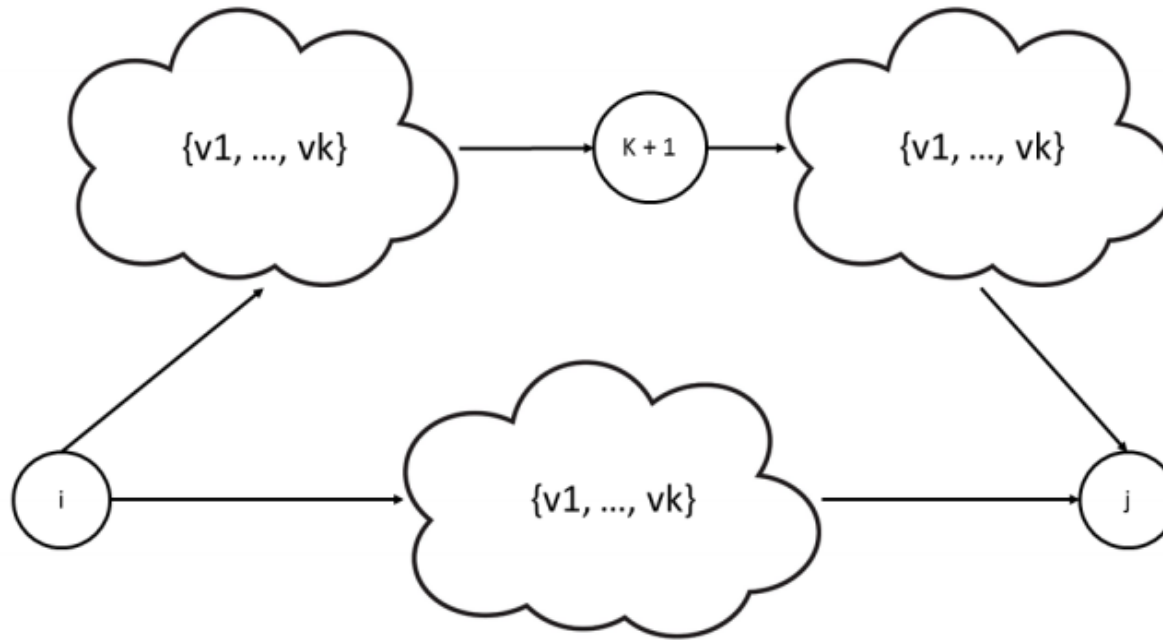
Floyd-Warshall algorithm

► Goal

- Find the shortest path from each i to each j using only vertices from 1 to $k + 1$
- What could this shortest path be?
 1. a path that only uses vertices in the set $\{1, \dots, k\}$
 2. a path that goes from i to $k + 1$ and then from $k + 1$ to j
- In other words... can we improve the shortest path between i and j if we pass through the vertex $k + 1$?

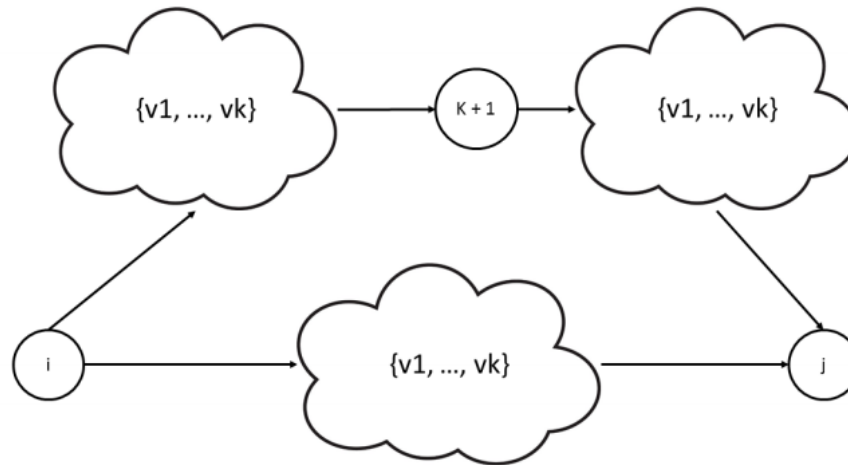
Floyd-Warshall algorithm

- Can we improve the shortest path between i and j if we pass through the vertex $k + 1$?



Floyd-Warshall algorithm

- Can we improve the shortest path between i and j if we pass through the vertex $k + 1$?



$$\begin{aligned} & \textit{shortestPath}(i, j, k + 1) \\ &= \min(\textit{shortestPath}(i, j, k), \textit{shortestPath}(i, k + 1, k) + \textit{shortestPath}(k + 1, j, k)) \end{aligned}$$

Floyd-Warshall algorithm

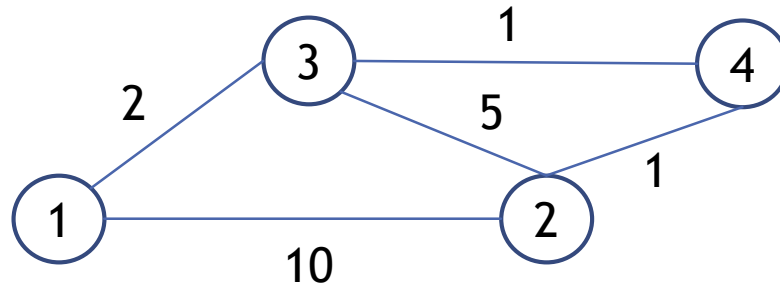
- ▶ What is the base (easiest) case of $shortestPath(i, j, k)$?
 - ▶ $k = 0 \rightarrow$ the path between i and j does not involve any other vertex
 - ▶ Length of the path = weight of the edge $w(i, j)$
 - ▶ Remember that if two vertices are not directly connected, then $w(i, j) = \infty$

Floyd-Warshall algorithm

- ▶ What is the base (easiest) case of $shortestPath(i, j, k)$?
 - ▶ $k = 0 \rightarrow$ the path between i and j does not involve any other vertex
 - ▶ Length of the path = weight of the edge $w(i, j)$
 - ▶ Remember that if two vertices are not directly connected, then $w(i, j) = \infty$
- ▶ Combination of the two formulas \rightarrow core of the Floyd Warshall algorithm

$$\begin{aligned} & \textcolor{red}{shortestPath(i, j, 0) = w(i, j)} \\ & \textcolor{red}{shortestPath(i, j, k + 1)} \\ & \textcolor{red}{= \min(shortestPath(i, j, k), shortestPath(i, k + 1, k) + shortestPath(k + 1, j, k))} \end{aligned}$$

Floyd-Warshall and dynamic programming



We are recomputing the same paths multiple times, for example:

- ▶ $sp(1,2,4) = \min(sp(1,2,3), sp(1,4,3) + sp(4,2,3))$
- ▶ $sp(1,2,3) = \min(sp(1,2,2), sp(1,3,2) + sp(3,2,2))$
- ▶ $sp(1,4,3) = \min(sp(1,4,2), sp(1,3,2) + sp(3,4,2))$

Floyd-Warshall memoization

- ▶ How do we avoid to compute the same thing multiple times?
 - ▶ Saving the intermediate results into a matrix!
 - ▶ This matrix contains the *length of the path from vertex i to vertex j*
- ▶ The algorithm uses an iterative bottom up approach
 - ▶ Start by: filling the matrix with the *base case solutions*
 - ▶ 0 is the distance between a vertex and itself
 - ▶ $w(i,j)$ is the distance between adjacent vertices
 - ▶ ∞ is the distance between non-adjacent vertices
 - ▶ Iterative part: filling in the matrix by iteratively expanding the set of intermediate vertices to use in the path

Floyd-Warshall algorithm

- ▶ Initialization ($k = 0$)
 - ▶ $shortestPath(i, j, 0) = w(i, j)$
- ▶ Loop
 - ▶ compute $shortestPath(i, j, k)$ for all (i, j) pairs when $k = 1$
 - ▶ See if passing for vertex 1 improves some shortest paths
 - ▶ compute $shortestPath(i, j, k)$ for all (i, j) pairs when $k = 2$
 - ▶ See if passing for vertex 2 improves some shortest paths
 - ▶ ...
 - ▶ compute $shortestPath(i, j, k)$ for all (i, j) pairs when $k = N$
 - ▶ See if passing for vertex N improves some shortest paths
 - ▶ After this, we have found the shortest path for all (i, j) pairs using **any** intermediate vertices

Floyd-Warshall algorithm

```
dist  $\leftarrow$   $|V| \times |V|$  matrix of minimum distances initialized to  $\infty$ 
for each vertex  $v$ 
    dist[v][v]  $\leftarrow$  0
for each edge  $(u,v)$ 
    dist[u][v]  $\leftarrow$   $w(u,v)$ 
for  $k$  from 1 to  $|V|$ 
    for  $i$  from 1 to  $|V|$ 
        for  $j$  from 1 to  $|V|$ 
            if dist[i][j] > dist[i][k] + dist[k][j]
                dist[i][j]  $\leftarrow$  dist[i][k] + dist[k][j]
            end if
```

Floyd-Warshall algorithm

- ▶ Previous code only computed the minimum distance between pairs of vertices, but not the actual minimum path
- ▶ Small changes to the algorithm make it possible to save also the information on the actual path

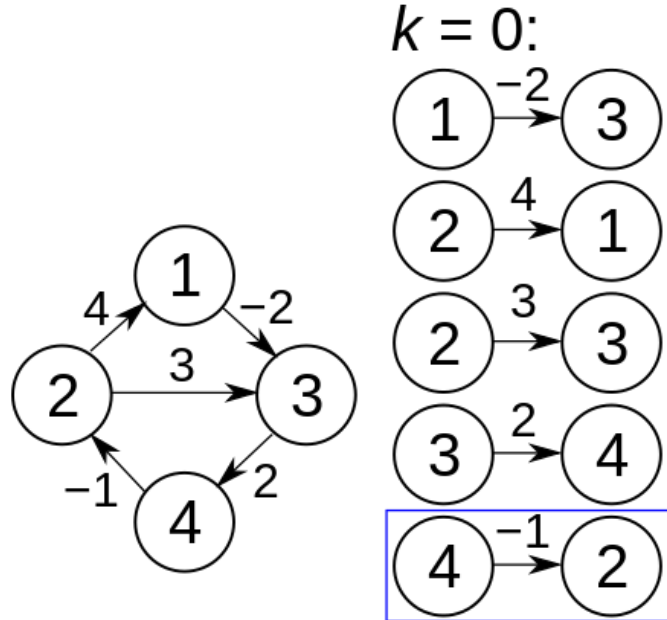
```
dist ← |V| × |V| matrix of minimum distances initialized to ∞
next ← |V| × |V| matrix of vertex indices initialized to null
for each edge (u,v)
    dist[u][v] ← w(u,v)
    next[u][v] ← v
for k from 1 to |V|
    for i from 1 to |V|
        for j from 1 to |V|
            if dist[i][k] + dist[k][j] < dist[i][j] then
                dist[i][j] ← dist[i][k] + dist[k][j]
                next[i][j] ← next[i][k]
```

Floyd-Warshall algorithm

- ▶ Complexity?
 - ▶ Three nested loops
 - ▶ Each loop performs $|V|$ iterations
 - ▶ $|V| \times |V| \times |V|$ iterations in total $\rightarrow O(|V|^3)$
 - ▶ Or, if we call n the number of nodes ($|V|$), $O(n^3)$

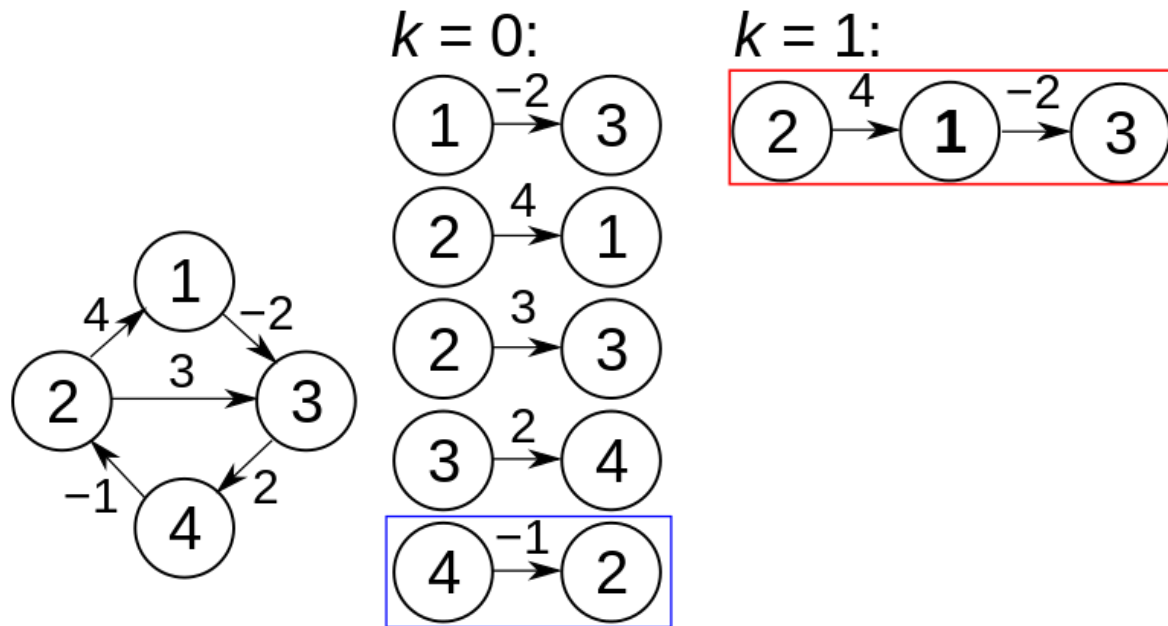
Floyd-Warshall algorithm

► Example



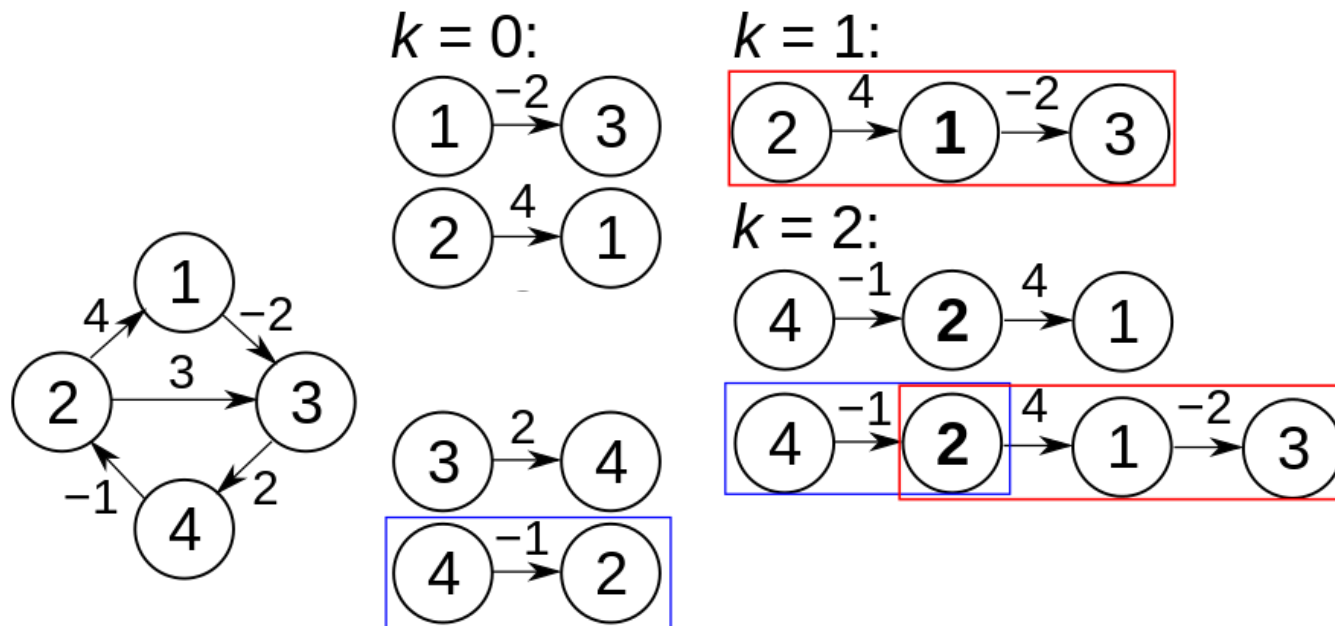
Floyd-Warshall algorithm

► Example



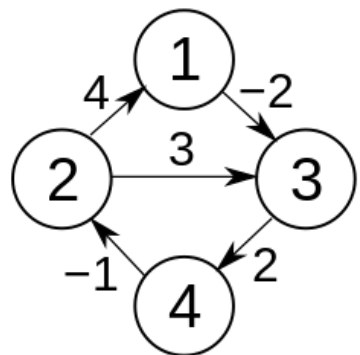
Floyd-Warshall algorithm

► Example

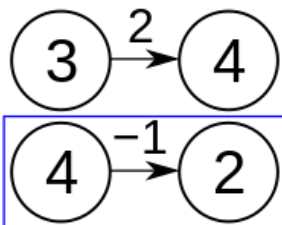
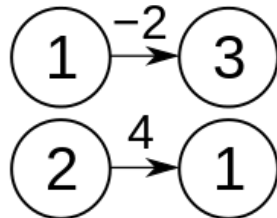


Floyd-Warshall algorithm

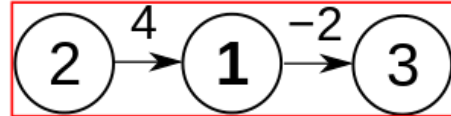
► Example



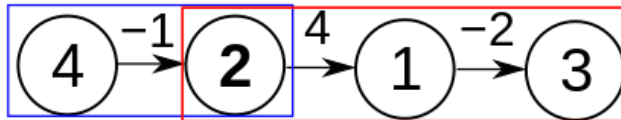
$k = 0$:



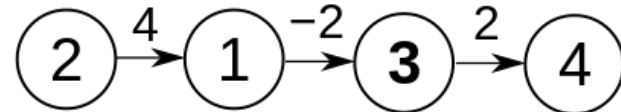
$k = 1$:



$k = 2$:

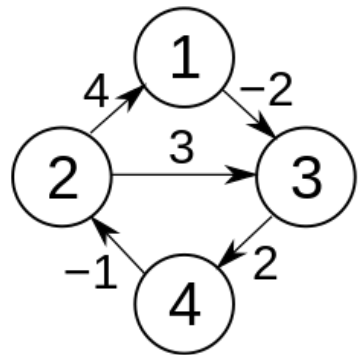


$k = 3$:

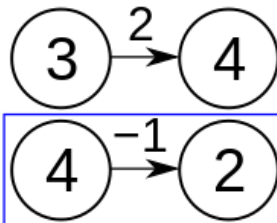
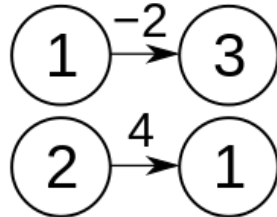


Floyd-Warshall algorithm

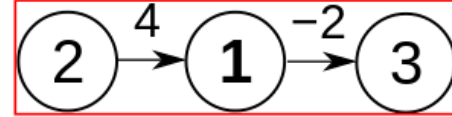
► Example



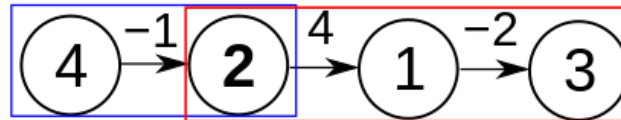
$k = 0$:



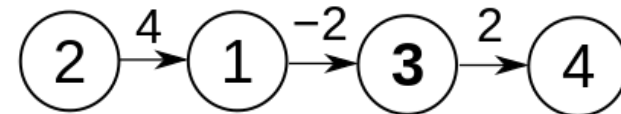
$k = 1$:



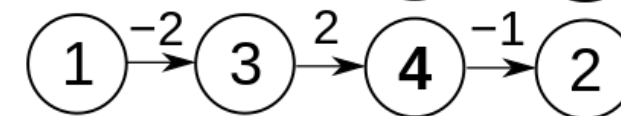
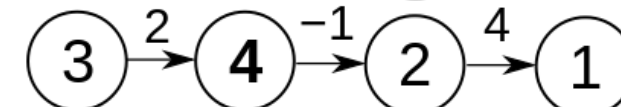
$k = 2$:



$k = 3$:



$k = 4$:



Homework

- ▶ **GO ON WITH THE ASSIGNMENT!!!**

- ▶ *Exercise 1 should be completed*
- ▶ *Exercise 2 should be completed (or at a good point)*
- ▶ *Exercise 3 can be started*

- ▶ **DO THE PROEFTENTAMEN!!!**

- ▶ *Can be found on N@tschool*
- ▶ *Next week we discuss together your solutions*
- ▶ *Note: the general structure of the exam is in the modulewijzer... take a look at it!*