Tomographic reconstruction
Lecture 1: Radon transform, filtered backprojection and inverse problems

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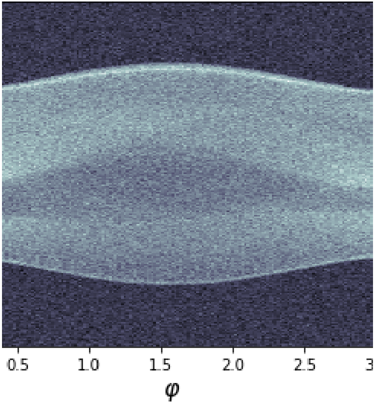
[1]based on a previous lecture by Sebastian Banert, Ozan Öktem

Overview of the tomographic reconstruction module (1)

Main goal: understanding how one gets an image out of a scanner.

Here is what you would (typically) get out of CT-scanner...



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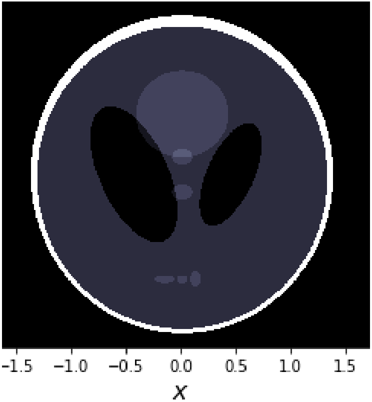
... would you have guessed it comes from the image below?



Overview of the tomographic reconstruction module (1)

Main goal: understanding how one gets an image out of a scanner.

- X-rays of incidental energy I_0 are attenuated when passing through the body/object, recovered energy is

$$I_1 = I_0 \ \exp(-\text{"matter encountered by the ray"}).$$

- Scanner measures I_1 for many different incident rays, so you have many values "matter encountered along some ray".

From all this information, want to recover the matter density everywhere: that's the image you are looking for.

Overview of the tomographic reconstruction module (2)

Main goal: understanding how one gets an image out of a scanner.

This requires

- defining what we mean by an image: mathematical viewpoint of an image as a function f,
- then "matter encountered by the ray" $= \int_L f(z)\, dz$ where L is some line,
- becomes the problem of *recovering a function knowing its integral on many lines*.

For this and because it covers most instances of medical imaging, we will build a general abstract setting, that of inverse problems, which covers CT-scan **but not only**.

Overview of the tomographic reconstruction module (3)

Main goal: understanding how one gets an image out of a scanner.

Problem: one would want to find the value of f at each point in space. This is in practice impossible and one only wants the averages of f over some small subsets of space (pixels/voxels). Requires the notion of discretisation of a space. In the end, the problem boils down to a set of (linear) equations.

You will

- get notions about why solving the equations is difficult, this is the notion of ill-posedness,
- learn some algorithms that tackle the problem, thanks to the theory of regularisation,
- implement some of them.

Formalising the notion of an image

What is an image?

- User's viewpoint: a quantity spatially distributed in 2D/3D.
- Mathematical viewpoint: an image is a function in 2D/3D:
  - Grey-scale images: real valued function in 2D/3D, *i.e.* $f : \mathbb{R}^n \to \mathbb{R}$ where $n = 2$ for 2D images and $n = 3$ for 3D images.
  - Colour images: vector valued function in 2D/3D, *i.e.* $f : \mathbb{R}^n \to \mathbb{R}^k$ where $n = 2$ or 3 and k is the number of colour channels (*e.g.* $k = 3$ for red, green & blue).

  We will only work with grey-scale images.
- Support of an image: the set $\Omega \subset \mathbb{R}^n$ where the image is defined, usually a rectangular region.
- Dynamic range of an image: the range of values in $\mathbb{R}^k$ that an image $f : \Omega \to \mathbb{R}^k$ can attain.

Image reconstruction

In many imaging applications, the image we seek is only indirectly observed. This is especially the case when we seek to observe a 3D image.

- 3D/2D tomography in medical imaging
- Geophysical exploration
- 3D microscopy
- Radar imaging

Image reconstruction

Image reconstruction as an inverse problem

Recover the image $f_{\text{true}} \in X$ from measured data $g \in \mathbb{R}^m$ assuming

$$g = \mathcal{S}\big(\mathcal{A}(f_{\text{true}})\big) + g_{\text{noise}}.$$

- $f_{\text{true}} \colon \Omega \to \mathbb{R}$ is the image that is to be recovered and X (reconstruction space) is the set of feasible images.
- $g \in \mathbb{R}^m$ is the measured data, *i.e.*, the m numeric quantities recorded by the imaging device.
- $\mathcal{A} \colon X \to Y$ is the forward operator and Y (data space) is the set of possible continuum data. $\mathcal{A}$ models the imaging device for continuum data in absence of noise and measurement errors.
- $\mathcal{S} \colon Y \to \mathbb{R}^m$ models how continuum data is digitized (sampling operator) during the imaging.
- $g_{\text{noise}} \in \mathbb{R}^m$ is the noise component of measured data.

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Reconstruction space X

- Set of possible functions $f : \Omega \to \mathbb{R}$ (images)
- Usually some (infinite dimensional) vector space over $\mathbb{R}$, *i.e.*,

$$f + h \in X \text{ and } \alpha f \in X \quad \text{for any } f, h \in X \text{ and } \alpha \in \mathbb{R}.$$

Square integrable functions: $L^2(\Omega)$ is infinite dimensional and

$$\int_\Omega |f(x)|^2 \, dx < \infty \quad \text{whenever } f \in L^2(\Omega).$$

$L^2(\Omega)$ has inner-product and norm (Hilbert space):

$$\langle f, h \rangle_{L^2(\Omega)} := \int_\Omega f(x)h(x) \, dx \quad \text{for } f, h \in L^2(\Omega)$$

$$\|f\|_2 := \sqrt{\langle f, f \rangle^2_{L^2(\Omega)}} := \Big(\int_\Omega f(x)^2 \, dx\Big)^{1/2}.$$

Hilbert spaces: generalises the notion of an inner product to infinite dimensional vector spaces.

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Discretisation: computers cannot handle elements in infinite dimensional vector spaces. Need to replace the infinite dimensional vector space with a finite dimensional counterpart.

Basis expansion

Assume X is a Hilbert space with norm $\|\cdot\|_X$ and $\{\phi_j\}_j \subset X$ is a fixed set (dictionary/frame). Define $X_n \subset X$ as the linear span of $\{\phi_1, \ldots, \phi_n\}$, so for any element $f \in X_n$ then there exist real numbers $\alpha_j \in \mathbb{R}$ (that depend on f) such that

$$f(x) = \sum_{j=1}^{n} \alpha_j \phi_j(x) \quad \text{for all } x \in \Omega.$$

If $\{\phi_j\}_j$ are a basis (linearly independent), then X_n is an n-dimensional vector space over $\mathbb{R}$.

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How to construct α_j's from a given f

Given $f \in X$, it is natural to seek an element $\tilde{f} \in X_n$ that is the "best approximation" to f.

Stated equivalently, given f we seek real numbers $\alpha_j \in \mathbb{R}$ (that depend on f) that minimises the "difference" between f and

$$\tilde{f}(x) := \sum_{j=1}^{n} \alpha_j \phi_j(x) \quad \text{for } x \in \Omega.$$

Solution: choose $\tilde{f}$ that minimises $\|f - \tilde{f}\|_X$. If $\{\phi_j\}_j$ is an orthonormal basis ($\langle \phi_i, \phi_j \rangle_X = 0$ if $i \neq j$ and $= 1$ if $i = j$), then this corresponds to choosing $\tilde{f} \in X_n$ as the orthogonal projection of f onto X_n, so

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Sampling (uniform): sub-divide (the rectangular) region $\Omega \subset \mathbb{R}^3$ into n non-overlapping (rectangular) sub-regions Ω_j, voxels, of equal size that cover Ω.

Voxel basis: given a sampling, define the voxel basis as

$$\phi_j(x) := \frac{1}{Vol(\Omega_j)} \begin{cases} 1 & \text{if } x \text{ is in the } j\text{:th voxel } \Omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\phi_j\}_j$ is an orthonormal basis and the orthogonal projection $\tilde{f}$ of f onto X_n is given by

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Basic assumptions

- $X \subset L^2(\Omega)$ is a sub-space
- A sampling of the domain Ω into voxels
- $\{\phi_j\}_j$ is a voxel basis based on the sampling

Discretisation: to each image $f \in X$ we can associate a unique vector (a finite dimensional representation) $\boldsymbol{\alpha} \in \mathbb{R}^n$ as follows:

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Digital image

- A vector representing grey-scale intensities corresponding to a voxel.
- Grey-scale values are usually mapped to integers $\{0, \ldots, 2^p - 1\}$ from white to black (p-bit image). We usually scale it back to, say, $[0, 1]$.
  - 2-bit image (binary images) corresponds to two values.
  - 8-bit corresponds to 256 different values.
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- Accuracy: for inverse problems, the discretisation must be accurate in both reconstruction and data spaces, *i.e.*,

$$f \approx \sum_{j=1}^{n} \alpha_j \phi_j \quad \text{and} \quad \mathcal{A}(f) \approx \mathcal{A}\Big(\sum_{j=1}^{n} \alpha_j \phi_j\Big)$$

  for any $f \in X$ where $\alpha_j := \langle f, \phi_j \rangle_X$.

- Discretised forward operator: fefine $A \colon \mathbb{R}^n \to \mathbb{R}^m$ as

$$A(\alpha) := \mathcal{S}\bigg(\mathcal{A}\Big(\sum_{j=1}^{n} \alpha_j \phi_j\Big)\bigg) \quad \text{for } \boldsymbol{\alpha} \in \mathbb{R}^n.$$

- Linearity & measurement matrix: assume $A \colon \mathbb{R}^n \to \mathbb{R}^m$ is linear (*e.g.*, both $\mathcal{A} \colon X \to Y$ and $\mathcal{S} \colon Y \to \mathbb{R}^m$ are linear). There exists an $(m \times n)$-matrix $\mathbf{A}$ (measurement matrix) so that $A(\alpha) = \mathbf{A} \cdot \alpha$.

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- Elements of the measurement matrix: when $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

$$A(\boldsymbol{\alpha}) = \sum_{j=1}^{n} \alpha_j \mathcal{S}\bigl(\mathcal{A}(\phi_j)\bigr).$$

Hence, the measurement matrix **A** is given as

$$\mathbf{A} := \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where

$$a_{i,j} := i\text{:th component of } \mathcal{S}\bigl(\mathcal{A}(\phi_j)\bigr) = \mathcal{S}\bigl(\mathcal{A}(\phi_j)\bigr)_i \in \mathbb{R}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$.

Remember: n is number of voxels and m is number of data points.

Original formulation (sampled noisy data)

Recover the image $f_{\text{true}} \in X$ from measured data $g \in \mathbb{R}^m$ assuming

$$g = \mathcal{S}\big(\mathcal{A}(f_{\text{true}})\big) + g_{\text{noise}}.$$

Here, $\mathcal{A}\colon X \to Y$ is the forward operator and $\mathcal{S}\colon Y \to \mathbb{R}^m$ is the sampling operator.

Fully discretised formulation

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Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.
Discretising has allowed us to go from trying to recover $f_{\text{true}} \in X$ (infinite-dimensional object) to trying to recover only its average value α_j in the voxel Ω_j, for $j = 1, \ldots, n$.

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The reconstruction problem
Tomography

- The data space: Y is an infinite dimensional vector space of real-valued functions defined on a manifold M of lines in $\mathbb{R}^3$, so continuum data $g \in Y$ is a function

$$g : M \to \mathbb{R}.$$

- The manifold of lines M: a line ℓ in $\mathbb{R}^3$ is uniquely determined by its direction ω (directional vector) and a point $x \in \omega^\perp$ that lies on ℓ ($\omega^\perp$ is the 2D plane orthogonal to ω):

$$\ell : t \mapsto x + t\omega.$$

  Hence, the pair (ω, x) corresponds to a unique line ℓ and vice versa, so M can be considered as a vector space of such pairs. Note: each $x \in \omega^\perp$ corresponds to a unique point on the detector surface.

- Data acquisition geometry: the arrangement of the m lines in M that correspond to the measurements, *i.e.*, the sampling of ω and $x \in \omega^\perp$.

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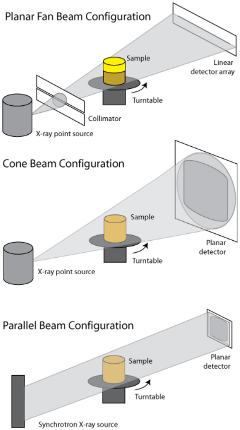
Common data acquisition geometries

- Parallel beam tomography
- Fan-beam
- 3D cone-beam tomography
- 3D helical/spiral tomography



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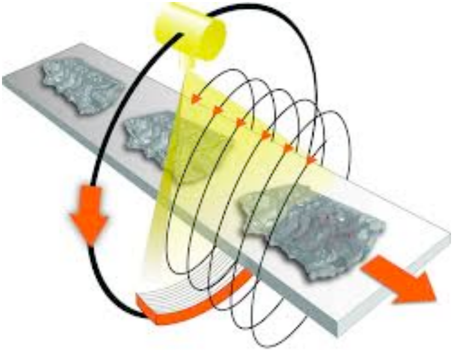


- Transmission tomography: forward operator given by the ray transform:

$$\mathcal{A}(f)(\omega, x) = \int_{-\infty}^{\infty} f(x + t\omega)\,\mathrm{d}t \quad \text{for a line } (\omega, x) \in M.$$

  Note that $g := \mathcal{A}(f)$ is a real-valued function defined on M a set of lines in $\mathbb{R}^3$.

- Exact reconstruction in 2D: write x and ω in polar coordinates:

$$x = \begin{pmatrix} t\cos\theta \\ t\sin\theta \end{pmatrix}$$

$$\omega = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

- Forward operator in polar coordinates:

$$\mathcal{A}(f)(t,\theta) = \int_{-\infty}^{\infty} f\left[t\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + s\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right] \mathrm{d}s$$

  for $t \in \mathbb{R}$ and $0 < \theta < \pi$

- Backprojection $=$ average over all lines through one point:

$$\mathcal{B}(g)(x,y) = \frac{1}{\pi}\int_0^{\pi} g(x\cos\theta + y\sin\theta, \theta)\,\mathrm{d}\theta.$$

  The backprojection takes a sinogram (=image on M) and maps it back to an image in $\mathbb{R}^2$. But it is **not** the inverse mapping of $\mathcal{A}$.

- Fourier transform: we transform only with respect to the first variable

$$\mathcal{F}(g)(t,\theta) = \int_{-\infty}^{\infty} g(s,\theta)\exp(-ist)\,\mathrm{d}s$$
$$\mathcal{F}^{-1}(g)(t,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s,\theta)\exp(+ist)\,\mathrm{d}s$$

Each of these transformations maps a function on M to another function on M, and $\mathcal{F}^{-1}$ is the inverse of $\mathcal{F}$. There exist fast implementations for both CPU (e.g. (py)FFTW) and GPU (cuFFT).

Filtered backprojection
The formula

$$f(x,y) = \frac{1}{2}\mathcal{B}(\mathcal{F}^{-1}(|t|\mathcal{F}(\mathcal{A}f)(t,\theta)))(x,y)$$

Step by step

1. Start with the sinogram $\mathcal{A}f$, which you obtain from the scanner.
2. Calculate the Fourier transform $\mathcal{F}(\mathcal{A}f)$.
3. Multiply by the absolute value of the first (radial) component. (This is the filtering).
4. Calculate the inverse Fourier transform. You still have a sinogram.
5. Apply the (nonfiltered) backprojection. Here you obtain an image.
6. Divide by 2.

Filtered backprojection
Challenges

- Requires full knowledge of all line integrals to give an exact reconstruction.
- The multiplication with the abolute value amplifies errors, in particular high frequencies. Solution: use a different filter, which does not grow so rapidly.
- But it is fast in comparison with iterative and variational methods.

- Sampling operator: given m data sampling points $(\boldsymbol{\omega}_1, x_1), \ldots, (\boldsymbol{\omega}_m, x_m)$ (which are lines) that correspond to actual measured data, define

$$\mathcal{S}(g) = \big(g(\boldsymbol{\omega}_1, x_1), \ldots, g(\boldsymbol{\omega}_m, x_m)\big) \in \mathbb{R}^m \quad \text{for } g \in Y.$$

  In particular, the i:th measurement corresponding to the i:th line $(\boldsymbol{\omega}_i, x_i)$ is modeled by

$$\mathcal{S}\big(\mathcal{A}(f)\big)_i = \mathcal{A}(f)(\boldsymbol{\omega}_i, x_i) = \int_{-\infty}^{\infty} f(x_i + t\boldsymbol{\omega}_i)\,\mathrm{d}t.$$

- Measurement matrix: fix a basis $\{\phi_i\}_i$ of X and a data acquisition geometry $(\boldsymbol{\omega}_1, x_1), \dots, (\boldsymbol{\omega}_m, x_m)$. Then, the elements in the $(m \times n)$-measurement matrix

$$\mathbf{A} := \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

are given as

$$a_{i,j} = \mathcal{A}(\phi_i)(\boldsymbol{\omega}_j, x_j) = \int_{-\infty}^{\infty} \phi_i(x_j + t\boldsymbol{\omega}_j)\,\mathrm{d}t$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$.

- Voxel basis: the ray transform assumes the x-ray beam is one-dimensional (pencil beam), so

$$a_{i,j} = \text{length of part of } i\text{:th line in } j\text{:th voxel.}$$

- Voxel basis and x-ray with 2D-dimensional cross section: the lines are replaced by 3D strips, so

$$a_{i,j} = \text{volume of part of } i\text{:th strip in } j\text{:th voxel.}$$

More complicated beam profiles: can be included by weighting different parts of the strips differently. The calculation of $a_{i,j}$ requires a lot of work and much of the literature on iterative methods discuss effects of using various schemes to approximate the measurement matrix.

Historical note: in the early approaches for 2D tomography by Hounsfield, one used

$$a_{i,j} = \begin{cases} 1 & \text{if center of } j\text{:th pixel is in } i\text{:th strip,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, values of $a_{i,j}$ can be computed at run time even with slow computers and do not have to be stored.

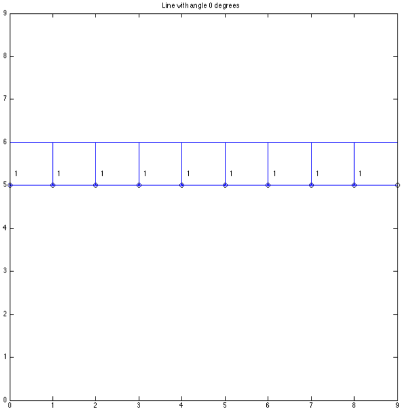
The reconstruction problem

Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 0^\circ$ with horizontal axis



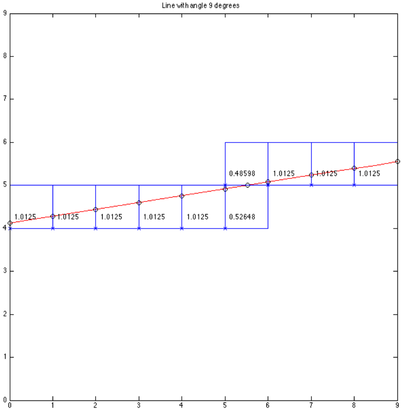
The reconstruction problem

Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 9^\circ$ with horizontal axis



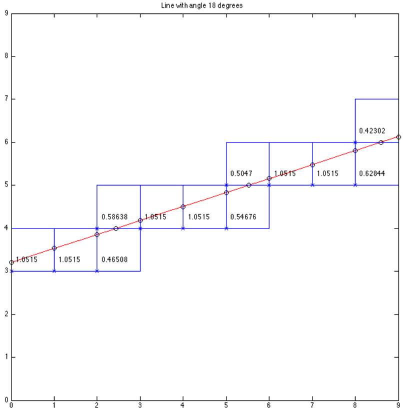
The reconstruction problem

Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 18^\circ$ with horizontal axis

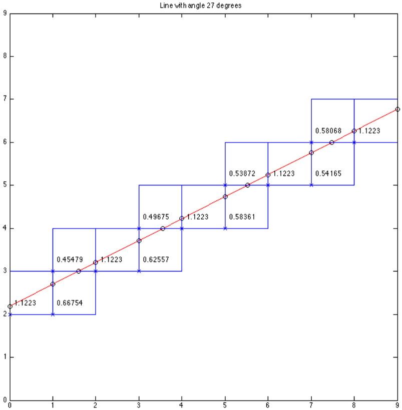


The reconstruction problem
Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 27^\circ$ with horizontal axis

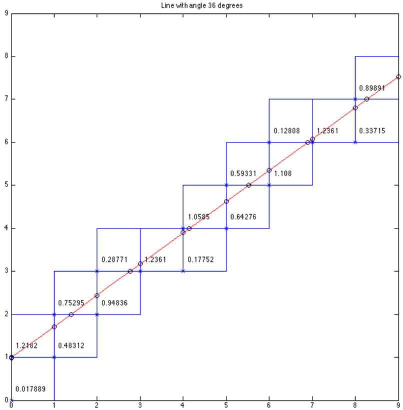


The reconstruction problem
Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 36^\circ$ with horizontal axis



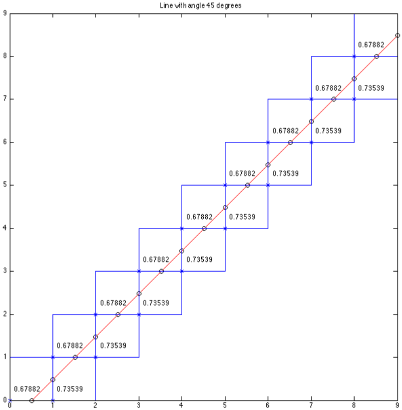
The reconstruction problem

Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 45^\circ$ with horizontal axis

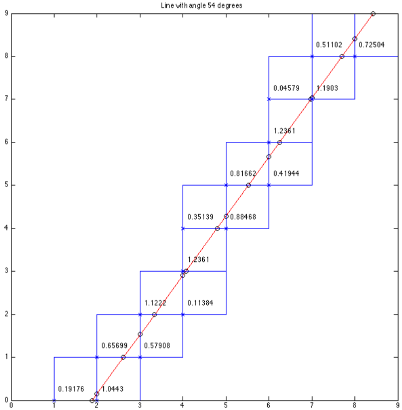


The reconstruction problem
Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 54^\circ$ with horizontal axis

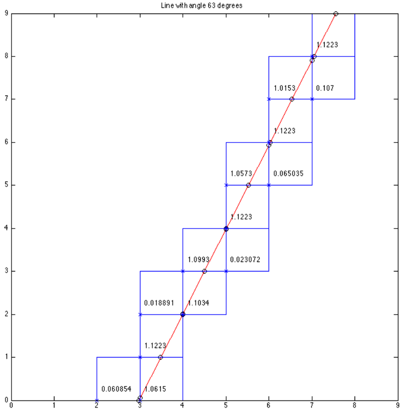


The reconstruction problem
Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 63^\circ$ with horizontal axis



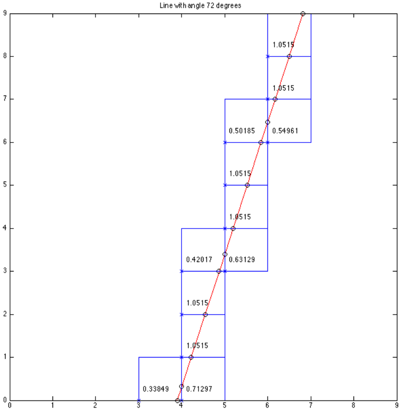
The reconstruction problem

Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 72^\circ$ with horizontal axis

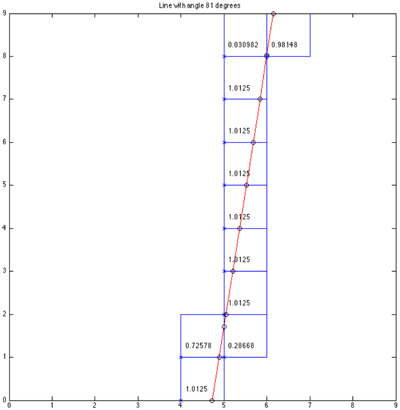


The reconstruction problem
Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 81^\circ$ with horizontal axis

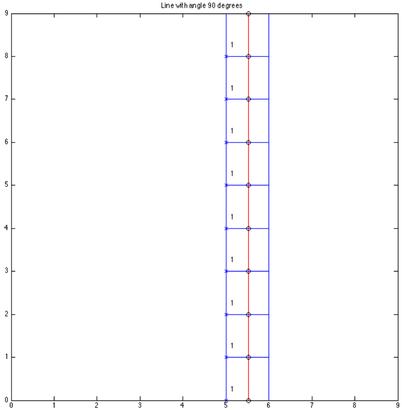


The reconstruction problem
Tomography: computing the measurement matrix for lines in 2D

Matrix element $a_{i,j}$ is the length of i:th line in j:th pixel.

Example: calculating i:th row a_i for a line through a 9×9 pixel 2D grid where pixel size $=1$.

Angle $= 90^\circ$ with horizontal axis

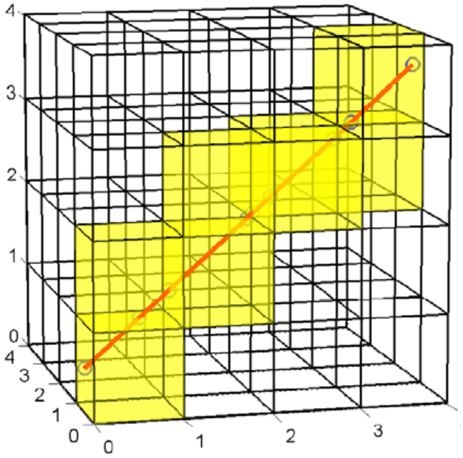


The reconstruction problem

Tomography: computing the measurement matrix for lines in 3D

Matrix element $a_{i,j}$ is the length of i:th line in j:th voxel.

Calculate the intersection of i:th line ℓ_i with j:th voxel.



The reconstruction problem

Tomography: computing the measurement matrix for lines in 3D

Matrix element $a_{i,j}$ is the length of i:th line in j:th voxel.

Calculate the intersection of i:th line ℓ_i with j:th voxel.

- Intersection with x_1-planes: consider the intersection point in $\mathbb{R}^3$ between a line ℓ and a hyperplane parallel to the x_1-axis:

$$(x_1, x_2, x_3) \mapsto (p, x_2, x_3) \quad \text{where } p \text{ is fixed.}$$

Assume ℓ is given by $(\boldsymbol{\omega}, x^0)$, *i.e.*,

$$\ell_i : t \mapsto x^0 + t\boldsymbol{\omega}.$$

Let $x^* = (x_1^*, x_2^*, x_3^*)$ denote this intersection, so $x_1^* = p$ by definition and

$$x_2^* = x_2^0 + \frac{p - x_1^0}{\omega_1}\omega_2 \quad \text{and} \quad x_3^* = x_3^0 + \frac{p - x_1^0}{\omega_1}\omega_3.$$

Matrix element $a_{i,j}$ is the length of i:th line in j:th voxel.

Calculate the intersection of i:th line ℓ_i with j:th voxel.

- Intersection with x_2- and x_3-planes: similar equations for the planes $x_2 = q$ and $x_3 = r$.
- From these intersections it is easy to compute the ray length in voxel j and there are fast methods for these computations:
    - R. L. Siddon. *Fast calculation of the exact radiological path for a 3-dimensional CT array*. Medical Physics, vol. 12, no. 2, pages 252–255, 1985.
    - H. Gao. *Fast parallel algorithms for the X-ray transform and its adjoint*. Medical Physics, vol. 39, no. 11, pages 7110–7120, 2012.

- For most tomography problems, the measurement matrix $\mathbf{A}$ is very large and it cannot be stored in computer memory.
  Problem size in a modern 64-slice helical CT:
  - a single 2D cross-section
    - about 1 000 detector elements arranged along 64 (detector) slices, each roughly corresponding to a 2D slice through the object. Radiation source attains 800 different positions each second (two rotations/second)
      $\Longrightarrow 1000 \cdot 800 \approx 800\,000$ data points.
    - 2D cross-section is 512×512 pixels in size $\Longrightarrow 260\,000$ pixels.

    Total: 260 000 unknowns and 800 000 equations.
  - a single 2D cross-section corresponds to a $1\ \mathrm{mm}$ thick slice in 3D, so 1 000 cross sections needed to cover $1\ \mathrm{m}$:
    $\Longrightarrow n \sim 260 \cdot 10^6 \quad \text{and} \quad m \sim 800 \cdot 10^6$.
- $\mathbf{A}$ is sparse since each ray only intersects a few voxels.

The reconstruction problem

Tomography: properties of the measurement matrix

- For most tomography problems, the measurement matrix **A** is very large and it cannot be stored in computer memory.
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The reconstruction problem

Naive inversion and concept of ill-posedness

Recover the digital image $\boldsymbol{\alpha}_{\text{true}} \in \mathbb{R}^n$ from measured data $g \in \mathbb{R}^m$ assuming

$$g = \mathbf{A} \cdot \boldsymbol{\alpha}_{\text{true}} + g_{\text{noise}}. \tag{1}$$

Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

Existence: naive inversion $\boldsymbol{\alpha}_{\text{naive}} := \mathbf{A}^{-1} \cdot g$ is not possible when $\mathbf{A}$ is not invertible (possibly no solutions to (1)).

Generalised solution: relax the notion of a solution to (1) to enforce existence. Define a data error function $E \colon \mathbb{R}^m \to \mathbb{R}_+$ and look for a vector $\boldsymbol{\alpha}^* \in \mathbb{R}^n$ that solves

$$\min_{\boldsymbol{\alpha}} E(\mathbf{A} \cdot \boldsymbol{\alpha} - g).$$

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Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

- Least-squares solutions: choose E as the 2-norm of the residual, *i.e.*, solve

$$\min_{\boldsymbol{\alpha}} \|\mathbf{A} \cdot \boldsymbol{\alpha} - g\|_2^2.$$

- Maximum likelihood solution: choosing E as the neglog of the data likelihood gives maximum likelihood solutions to (1).

Recover the digital image $\alpha_{\text{true}} \in \mathbb{R}^n$ from measured data $g \in \mathbb{R}^m$ assuming

$$g = \mathbf{A} \cdot \alpha_{\text{true}} + g_{\text{noise}}. \tag{1}$$

Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

Ill-posedness: generalised solutions handle the issue of existence (there is always a generalised solution to (1)). Two serious problems remain:

- Non-uniqueness: infinitely many generalised solutions.
- Instability: generalised solutions do not depend continuously on the data g.

The inverse problem (1) is ill-posed if any of these issues occurs.

The reconstruction problem
Naive inversion and concept of ill-posedness

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Here, $\mathbf{A}$ is the $(m \times n)$-measurement matrix.

Condition number and ill-posedness:

assuming that A is invertible (n = m), what happens if we try $\boldsymbol{\alpha}_{\text{naive}} := \mathbf{A}^{-1} \cdot g$.

Classical perturbation theory leads to the bound

$$\frac{\|\boldsymbol{\alpha}_{\text{true}} - \boldsymbol{\alpha}_{\text{naive}}\|_2}{\|\boldsymbol{\alpha}_{\text{true}}\|_2} \leq \text{cond}(\mathbf{A}) \frac{\|g - g_{\text{exact}}\|_2}{\|g_{\text{exact}}\|_2}$$

where $g_{\text{exact}} := \mathbf{A} \cdot \boldsymbol{\alpha}_{\text{true}}$ and cond($\mathbf{A}$) is the condition number (ratio of the maximal and minimal singular values).

cond($\mathbf{A}$) large $\Longrightarrow$ naive inversion not useful since $\boldsymbol{\alpha}_{\text{naive}}$ can be very far from $\boldsymbol{\alpha}_{\text{true}}$.

Regularisation: mathematical theory for handling Ill-posed problems. Stability and uniqueness is enforced by accounting for a priori knowledge about the true (unknown) image.

The main idea of regularisation

replace the original ill-posed reconstruction problem by a well-posed reconstruction problem (*i.e.*, it has a unique solution that depends continuously on the data) that is convergent as the data error goes to zero.

Well-posedness: guarantees stability

Convergence: the reconstructions obtained converge to a least squares solution when the data error approaches zero and the parameters in the reconstruction method (regularisation parameters) are chosen appropriately.

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