Problem 1. In order to get a feeling for the difficulties with inverse problems, we consider the problem of numerical differentiation.

Let $f : [0,1] \to \mathbb{R}$ be a differentiable function. You have some knowledge of the function at equidistant points $\frac{k}{n}$, but up to some measurement errors. In other words, we have $n+1$ values $f_0, \ldots, f_n \in \mathbb{R}$ such that

$$|f_k - f(k/n)| \leq \varepsilon \qquad \text{for } k = 0, \ldots, n$$

(i.e., that the f_k's approximate the function values with precision ε) for some $\varepsilon > 0$. For some k, approximate the derivative with the difference quotient

$$f'(k/n) \approx \frac{f((k+1)/n) - f(k/n)}{1/n}.$$

Show that the error by replacing $f(k/n)$ with f_k is $2n\varepsilon$ and this estimation is sharp. What happens for $n \to \infty$?

Problem 2. Compute the forward operator $\mathcal{A}$ (in polar coordinates)

$$\mathcal{A}(f)(t,\theta) = \int_{-\infty}^{\infty} f \begin{pmatrix} t\cos\theta - s\sin\theta \\ t\sin\theta + s\cos\theta \end{pmatrix} \mathrm{d}s$$

of the image $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} 1 - (x^2 + y^2) & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: $\cos^2\theta + \sin^2\theta = 1$.

Problem 3. Compute the matrix $\boldsymbol{A}$ of the discretized forward operator for a (2×2) image with horizontal and vertical rays. What do the rows and columns of the matrix correspond to? Invert this matrix if you can.

Problem 4. Consider a $(2 \times 2 \times 2)$ image on $[0,2]^3$ and the ray given by

$$\begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, t \in \mathbb{R}.$$

Compute the intersections of this ray with the planes given by each of the planes separating the voxels of the images.