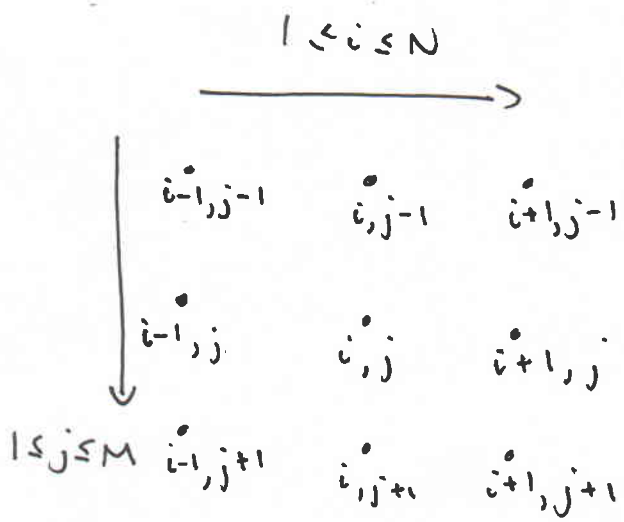
The Discrete Laplacian



Any derivative can be represented as a weighted average of these 9 points

h = Step size (1 in our case - we renormalize Poisson's equation)

$$h^2 \nabla^2 u\Big|_{(i,j)} = \frac{1}{6}\left(U_{i-1,j-1} + U_{i+1,j-1} + U_{i+1,j+1} + U_{i-1,j+1}\right) + \frac{4}{6}\left(U_{i,j-1} + U_{i-1,j} + U_{i+1,j} + U_{i,j+1}\right) - \frac{20}{6} U_{i,j}$$

This is the optimized 9-point stencil

If you need improved accuracy, move to a 16-point stencil

We also have

$$\frac{du}{dx}\Big|_{(i,j)} = \frac{1}{2h}\left[\frac{2}{3}\left(U_{i+1,j} - U_{i-1,j}\right) + \frac{1}{12}\left(U_{i+1,j+1} + U_{i+1,j-1} - U_{i-1,j+1} - U_{i-1,j-1}\right)\right]$$

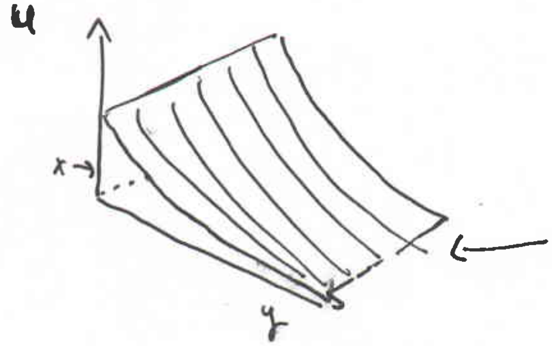
$$\frac{du}{dy}\Big|_{(i,j)} = \frac{1}{2h}\left[\frac{2}{3}\left(U_{i,j+1} - U_{i,j-1}\right) + \frac{1}{12}\left(U_{i+1,j+1} + U_{i-1,j+1} - U_{i+1,j-1} - U_{i-1,j-1}\right)\right]$$

The Poisson Solver

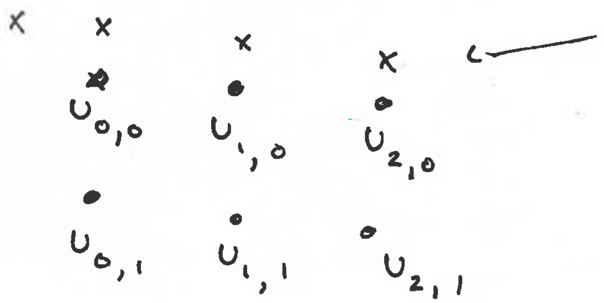
By representing the Laplacian by a discretized version, it is straightforward to represent this as a matrix equation

The 9-point stencil requires well defined boundary conditions

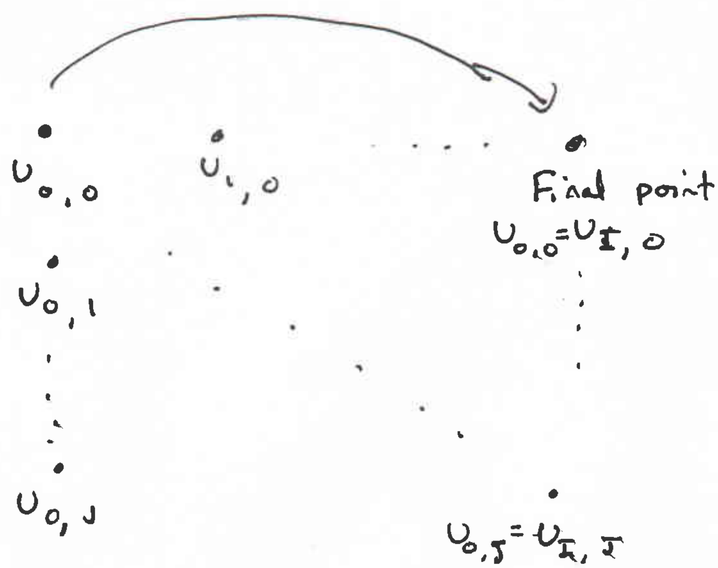
My notes on how to construct the matrix are attached



we fix this bottom edge to 0



We add a row of 'ghost points' above the top edge, which are used in the 9-point stencils in the first row of $U_{i,0}$, but we do not construct a stencil centered on any of these points



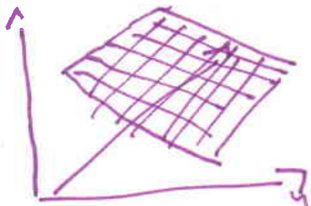
Periodic, so that the first and last columns are equal and the derivates at those points agree

③ CASES TO TEST

① Plane, $u_x + u_y = \text{Const}$ at both boundaries

(capacitor)

$\cancel{\nabla^2 \phi = 0}$



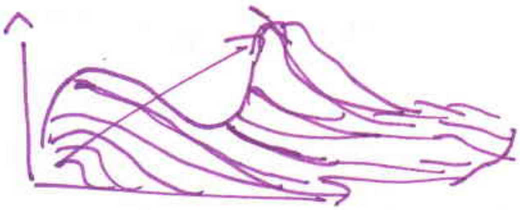
$\nabla^2 u = 0$

$y=0: u_x + u_y = -1$

$y = L_m: u_x + u_y = -1$

② Sinusoidal w/ decaying exponential
Obtained by separation of variables $\nabla^2 u = 0$

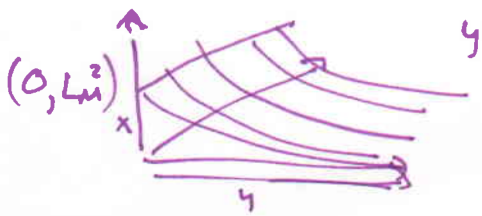
Laplace's



$u = e^{\omega x}\cos(\phi + \omega y)$

③ Construct arbitrary solutions to Poisson's

~~Capac.~~ $\cancel{u_x + u_y = -2L,\ u_x + u_y}$



$y=0: u_x + u_y = -2L_m$

$y = L_m: u_x + u_y = 0$

$u = (y - L_m)^2$

$\nabla^2 u = 2$

$f(x,y) = -2$

④ Being periodic - practically only trig functions or constants $\therefore\ u(x,y) = f(x),\ f(x)\sin(\phi + \omega y)$

$u = (x - L_m)^2 \sin(\omega y)$

$u_x = 2(x - L_m)\sin(\omega y)$

$u_y = (x - L_m)^2\, \omega \cos(\omega y)$

$u_{xx} = 2\sin(\omega y)$

$u_{yy} = -(x - L_m)^2\, \omega^2 \sin(\omega y)$

$N \longrightarrow$ $(2l+1)$ t_z

$N=3,\ M=2$

$$
\begin{array}{c|c|c|c|c}
H(1,1) & t_x & \cdot & & \\
\hline
t_x & H(1,2) & t_x \cdot & & \\
\hline
\cdot & t_x & H(1,3) & \cdot t_x \cdot & \\
\hline
\cdot & & \cdot & H(1,4) & \\
\hline
\cdot & & & \ddots & \\
1M & & & & \cdot \\
\hline
21 & t_z & & & \\
22 & & t_z & &
\end{array}
\qquad
\begin{pmatrix} 11 \\ 12 \\ 13 \\ \vdots \\ 1N \\ 21 \\ \vdots \\ 2N \\ \vdots \\ M1 \\ \vdots \\ MN \end{pmatrix}
$$

$H(i,j)$ couples to
$H(i-1,j)$
$H(i+1,j)$
$H(i,j-1)$
$H(i,j+1)$

$$
\begin{tikzcd}
1,1 \arrow[r, "t_x"] \arrow[d, "t_z"] & 1,2 \arrow[r, "t_x"] \arrow[d, "t_z"] & 1,3 \arrow[d, "t_z"] \\
2,1 \arrow[r, "t_x"] \arrow[d, "t_z"] & (2,2) \arrow[r, "t_x"] \arrow[d, "t_x"] & \cdot \\
3,1 & &
\end{tikzcd}
$$

$$t_x = \begin{pmatrix} t' & & & & & \\ & t & & & \emptyset & \\ & & t & & & \\ & & & t' & & \\ & \emptyset & & & t & \\ & & & & & t \end{pmatrix}$$

$$= \begin{pmatrix} t' & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} \otimes \mathbb{1}_{2\times 2}$$

$$= \begin{pmatrix} yz & & \\ & zx & \\ & & xy \end{pmatrix} \otimes \begin{pmatrix} \uparrow & 0 \\ 0 & \downarrow \end{pmatrix}$$

$$t_z = \begin{pmatrix} t & & & & & \\ & t & & & \emptyset & \\ & & t' & & & \\ & & & t & & \\ & \emptyset & & & t & \\ & & & & & t' \end{pmatrix}$$

$$= \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t' \end{pmatrix} \otimes \begin{pmatrix} \uparrow & 0 \\ 0 & \downarrow \end{pmatrix}$$

$\varepsilon_0^{ij} = 0$

$\varepsilon_{yz} = \cancel{2}t' \cos k_x + t \cos k_y + \cancel{2}t \cos k_z + \varepsilon_0^{yz}$

$\varepsilon_{zx} = t \cos k_x + t' \cos k_y + \cancel{2}t \cos k_x + \varepsilon_0^{zx}$

$\varepsilon_{xy} = t \cos k_x + \cancel{2}t \cos k_y + t' \cos k_z + \varepsilon_0^{xy}$

(No approximations)

$$\cancel{H(i,j)} = \quad H(i,j) = \begin{pmatrix} \varepsilon_{yz} & 0 & 0 \\ 0 & \varepsilon_{zx} & 0 \\ 0 & 0 & \varepsilon_{xy} \end{pmatrix} \otimes \mathbb{1}_{2\times 2} + SO$$

(2)

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \otimes \mathcal{H}(i,j)$$

(1)

$$\mathcal{H}(i,j) = \begin{pmatrix} \varepsilon_{yz} & 0 & 0 \\ 0 & \varepsilon_{zx} & 0 \\ 0 & 0 & \varepsilon_{xy} \end{pmatrix} \otimes \mathbb{1}_{2\times 2}$$

$$= \mathbb{1}_{NM\times NM} \otimes \mathcal{H}(i,j)$$

(4)

$$\begin{pmatrix} 0 & & & \\ t_x & 0 & & \emptyset \\ & t_x & 0 & \\ & & \ddots & \end{pmatrix} + \begin{pmatrix} 0 & t_x & & \\ & 0 & t_x & \\ \emptyset & & 0 & \ddots \\ & & & 0 \end{pmatrix}$$

= Diagonal matrix $(1, t_x)$

$+$ Diag $(-1, t_x)$

(3)

$$t_x = \begin{pmatrix} t' & \\ & t_t \end{pmatrix} \otimes \mathbb{1}_{2\times 2}$$

$$t_z = \begin{pmatrix} t_t & \\ & t' \end{pmatrix} \otimes \mathbb{1}_{2\times 2}$$

(5)

$$\begin{pmatrix} 0 & & & \emptyset \\ \vdots & 0 & & \\ & & 0 & \\ t_z & & & 0 \\ & \ddots & & \end{pmatrix} + \begin{pmatrix} 0 & \cdots & t_z \\ & 0 & \vdots \\ \emptyset & & 0 \end{pmatrix} = \text{Diag}(N, t_z) + \text{Diag}(-N, t_z)$$

21, 11

Right so + did I mix up t_x, t_z?

∇^2:

U_{11} $\frac{10}{3}$ U_{21} $-\frac{2}{3}$ $-\frac{1}{6}$ $U_{N-1,1}$ $U_{N,1}$

$\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$, …, $\vec{V}_N$

V_{11}, V_{12}, …, V_{1M}, V_{21}, V_{22}, …, V_{2M}

U_{21}

$U_{N-1,1}$ $-\frac{2}{3}$ $-\frac{1}{6}$ ϕ

Periodic

Ghost row 1

Ghost row 2

Dimensions $[(N-1)(M-1)]^2$

$M=N$; $(N-1)^2 \times (N-1)^2$

U_N Append

Exterior point $i \to i$

Top edge

4 Periodic

U_2 U_3 U_4 U_5 U_6 $U_7 = U_1$

N odd $N = 5$

Bottom edge

U_{11} U_{21} U_{31} U_{41} $U_{51} = U_{11}$

Don't solve for

Last P_{ij}

Append Flux

Mixing w/ other ghosts

U_{12}, U_{13} connect to U_{12}

$U_{N-1,1}$ U_{11} U_{21} U_{31} U_{41} … $U_{N-1,1}$

$U_{N-1,2}$ U_{12}

$U_{N-1,3}$ U_{13}

$U_{N-1,4}$

U_{1M}

$U_{1M} = U_N$

N even

$N=4$ U_{11} U_{21} U_{31} $U_{41} = U_{11}$

New dimensions: $[(N-1)(N+1)]^2$

$(N^2-1) \times (N^2-1)$