EE Qualifying Exam 2016 – John Duchi

Question 1: You have two coins, both of which are fair (i.e. the probability of heads is $\frac{1}{2}$ for each of them), and they are independent.

(a) Describe how to use the two coins to generate an event that occurs with probability exactly 2/3.

(b) How many coin flips, on average, does your solution to part (a) require?

(c) Now you are given a value $p \in [0, 1]$, where p may be irrational. Generalize your solution to part (a) to use the coins to generate an event that occurs with probability exactly p. How many flips are required to generate your event (on average)?

(d) Now you wish to generate a series of independent random variables, $X_1, X_2, \ldots$, each with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$. Can you devise a procedure, similar to part (c), that uses fewer flips on average? What is this procedure, and how many coin flips does it require on average?

Answer:

(a) Let $E \in \{0, 1\}$ be the event. We use rejection sampling. Let HH denote the event that both coins flip to heads; HT that the first coin is heads and the second is tails; TH the first being tails, second heads; and TT the event that both coins are tails. We flip both coins. If we achieve HH, we set $E = 0$. If we have HT or TH, we set $E = 1$. If we see TT, we simply re-flip the coins and ignore the result. It is clear that the probability our procedure stops at HH, yielding $E = 0$, is 1/3.

(b) The probability of failing to stop in any round of the procedure is $\mathbb{P}(\mathsf{TT}) = \frac{1}{4}$, and on each round we perform two coin flips. Let R be the number of rounds. Then for $k \in \{1, 2, \ldots\}$, we have $\mathbb{P}(R \ge k) = (\frac{1}{4})^{k-1}$, because $R \ge k$ if and only if the first $k - 1$ rounds are failures. Thus

$$\begin{aligned}\mathbb{E}[R] = \sum_{k=1}^{\infty} k\mathbb{P}(R = k) &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mathbf{1}\{l \le k\}\, \mathbb{P}(R = k) \\ &= \sum_{l=1}^{\infty}\sum_{k=l}^{\infty} \mathbb{P}(R = k) = \sum_{l=1}^{\infty} \mathbb{P}(R \ge l) = \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.\end{aligned}$$

We expect to perform 8/3 flips. (One could also note simply that this is the expectation of a geometric random variable.)

(c) Let $p = 0.b_1 b_2 b_3 \ldots$, where $b_j \in \{0, 1\}$, be the binary decimal expansion of p (so that $p = 1$ has expansion $1 = b_1 = b_2 = \cdots$, and $p = \frac{1}{2}$ has expansion $p = 0.10000\ldots$). The following procedure suffices. Let $C_1, C_2, \ldots$ be an infinite sequence of coin flips, where

$C_i = 1$ if the coin flip is heads and $C_i = 0$ if it is tails. Let $\widehat{p} = 0.C_1C_2C_3\ldots$. Then return

$$E = \begin{cases} 1 & \text{if } \widehat{p} \le p, \quad \text{i.e. } 0.C_1C_2C_3\ldots \le 0.b_1b_2b_3\ldots \\ 0 & \text{otherwise.} \end{cases}$$

As $\widehat{p}$ is uniform in $[0,1]$, we see that $\mathbb{P}(E = 1) = p$ exactly.

An implementable version of this abstract procedure is as follows: repeat the following for iterations $k = 1, 2, \ldots$.

i. At iteration k, flip coin and get value C_k.
ii. If $C_k = 0$ and $b_k = 1$, then we know that $\widehat{p} \le p$, so return $E = 1$.
iii. If $C_k = 1$ and $b_k = 0$, we know that $\widehat{p} > p$, so return $E = 0$.
iv. If $C_k = b_k$, continue.

If as above we let R denote the number of rounds of this procedure, we see that $\mathbb{P}(R \ge k) = (\frac{1}{2})^{k-1}$, so that $\mathbb{E}[R] = \sum_{k=1}^{\infty} \mathbb{P}(R \ge k) = \sum_{k=0}^{\infty} 2^{-k} = 2$.

(d) We provide a more general solution to the problem, which allows us to generate events $E = 1, \ldots, E = m$ with probabilities $p_1 \ge \ldots \ge p_m$, and $\sum_{i=1}^m p_i = 1$. (In the case above, we fix some n, set $m = 2^n$, and let $E = 1$ correspond to $X_1 = 0, X_2 = 0, \ldots, X_n = 0$, $E = 2$ correspond to $X_1 = 0, \ldots, X_{n-1} = 0, X_n = 0$, and so on until $E = 2^n$ corresponds to $X_i = 1$ for all i.) Let $s_0 = 0, s_1 = p_1, s_2 = p_1 + p_2, \ldots, s_k = 1$, and give them binary expansions $s_i = 0.b_1^i b_2^i b_3^i \cdots$. Let $\widehat{p} = 0.C_1C_2C_3\ldots$ be the binary expansion of an i.i.d. sequence of coin flips. Then return

$$E = \{i \text{ such that } s_{i-1} \le \widehat{p} < s_i\}.$$

We have $\mathbb{P}(E = i) = p_i$, and we can use the ideas from part (c) to terminate the procedure.

In particular, it is possible to show that $\mathbb{E}[R] \le -\sum_{i=1}^m p_i \log_2 p_i + 2$, though no one made it this far.

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