This can also be expressed in the Fourier domain. Note that $x(t)^2$ occupies a frequency band of $(-2W, 2W)$ so we need at least $f_c \geq 3W$.
Passing $v(t)$ through the bandpass filter will produce a delayed and scaled version of the passband signal,

$$y(t) = Aa_1 \cos(2\pi f_c(t - t_0)) + 2Aa_2 x(t - t_0) \cos(2\pi f_c(t - t_0))$$

a classical AM modulated signal. This is clearly not time invariant, but it is a linear system if $a_1 = 0$.
I included the complete expansions for completeness, but most people only wrote down the terms that survive the BPF.