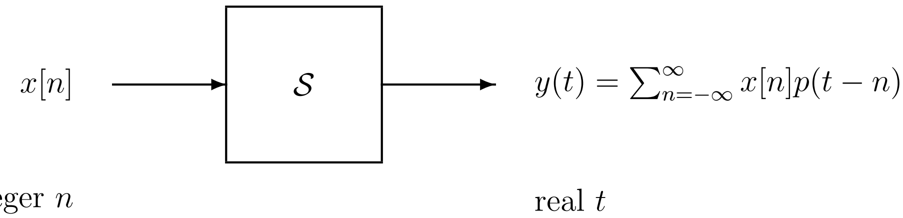
The following system is useful as a model in pulse amplitude modulation (PAM) systems and digital-to-analog (D/A) converters:



where $p(t)$ is a real-valued continuous-time signal satisfying

$$\int_{-\infty}^{\infty} p(t)p(t-n)dt = \delta_n = \begin{cases} 1 & n=0 \\ 0 & \text{all nonzero integers} \end{cases} \tag{1}$$

Define the discrete-time Fourier transform (DTFT) of a signal $x[n]$ by

$$X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}; -\frac{1}{2} \leq f \leq \frac{1}{2}$$

and the continuous-time Fourier transform (CTFT) of a signal $y(t)$

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt; -\infty < f < \infty,$$

where $j = \sqrt{-1}$.

First Question:

Find a *simple* relationship between $Y(f)$ and $X(f)$.

Solution This was intended as a straightforward start using standard Fourier proof techniques — substitute (plug in) the definition of the signal to the definition of the transform $Y(f)$ asked for, interchange the order of summation and integral, and then simplify.

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x[n]p(t-n) \right] e^{-j2\pi ft}dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} p(t-n)e^{-j2\pi ft}dt = \sum_{n=-\infty}^{\infty} x[n]P(f)e^{-j2\pi fn} \end{aligned}$$

where the last step is the usual Fourier shift theorem for CT signals (or just change variables in the integral). Thus

$$Y(f) = X(f)P(f).$$

A tricky point here is that $Y(f)$ should be defined for all real f, but $X(f)$ was defined only for f in $[-1/2, 1/2]$. The formula makes sense, however, if we take $X(f)$ to be the periodic extension, that is, just use the sum in the DTFT definition for all real f.

Many people complicated the problem by trying to convert the DT signal into a CT signal using impulse trains. This way leads to the answer, but it makes things much more complicated. Some people observed correctly that the left hand side resembles a convolution and tried to quote the convolution theorem, but here the "convolution" is discrete time while the output signal is continuous time, so the usual convolution theorems (for DT or CT) do not directly apply.