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1. Let X_1 be the expected number of bits until the first 1 is generated. Obviously, $E[X_1] = 2$, since X_1 has a geometric probability distribution. Let X_2 be the expected number bits until two consecutive ones are generated. Once the first one occurs, the next bit is a one with probability 1/2; otherwise the next bit is zero, which causes the search for two consecutive ones to start over. This leads to the following formula for $E[X_2]$ in terms of $E[X_1]$:

$$E[X_2] = E[X_1] + \frac{1}{2}\Big(1 + (1 + E[X_2])\Big) \quad \Rightarrow \quad E[X_2] = 2E[X_1] + 2 = 6\,.$$

Similarly, $E[X_3] = 2E[X_2] + 2 = 14$. The sequence $\{E[X_i]\}$ for $i = 1, \ldots, 8$ is $\{2, 6, 14, 30, 62, 126, 254, 510\}$. (Obviously, $E[X_i] = 2^i - 2$.) Thus $E[X_8] = 510$.

2. If the first bit is 0, then the conditional expected value of X_8 is $E[X_8] + 1$, since the first bit is wasted and the experiment has returned to its initial state. Similarly, if the first two bits are 10, then conditional expected value of X_8 is $E[X_8] + 2$. Continuing in this way we obtain the following table:

Initial bits	Probability	Conditional expectation
0	2^{-1}	$E[X_8] + 1$
10	2^{-2}	$E[X_8] + 2$
110	2^{-3}	$E[X_8] + 3$
1110	2^{-4}	$E[X_8] + 4$
11110	2^{-5}	$E[X_8] + 5$
111110	2^{-6}	$E[X_8] + 6$
1111110	2^{-7}	$E[X_8] + 7$
11111110	2^{-8}	$E[X_8] + 8$
11111111	2^{-8}	8

The unconditional expected value of X_8 is obtained by averaging the conditional expectations in the above table. This leads to a formula for $E[X_8]$:

$$E[X_8] = 8 \cdot 2^{-8} + \sum_{i=1}^{8} 2^{-i}(E[X_8] + i) = \frac{255}{256}E[X_8] + \frac{510}{256}\,.$$

Therefore $E[X_8] = 256 \cdot \dfrac{510}{256} = 510$ bits.

3. Each time a zero is generated, the experiment ends if the next 8 bits are all ones. This occurs with probability $p = 2^{-8}$. Since the trials separated by zeroes are independent, the expected number of trials is $1/p = 2^8$. That is, the expected number of zeroes seen before 8 consecutive ones is 256. In fact, the first trial does not require an initial zero, so the average number of zeroes is 255. Since zeroes and ones are equally likely, the average number of ones is also 255. Therefore the expected number of bits generated is 510.