2016 EE Qualifying Examination
Electromagnetics
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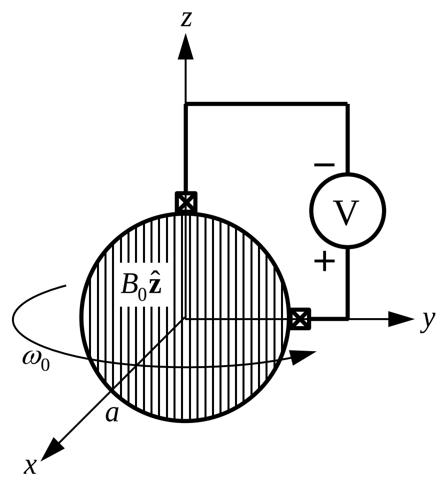
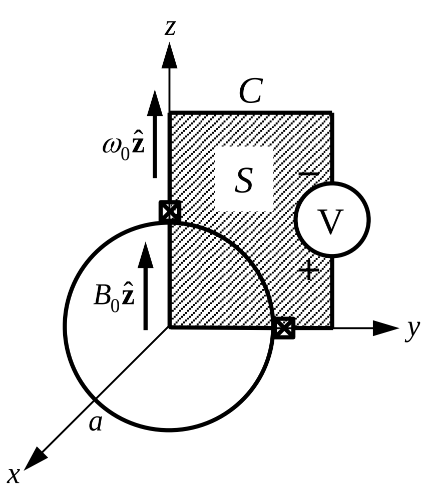
Question

A conducting sphere of radius a is uniformly magnetized along the z axis so that, inside the sphere, there is a uniform magnetic induction $\mathbf{B}_{in}(\mathbf{r}) = B_0\hat{\mathbf{z}}$. The sphere is rotated about its central axis at a constant angular velocity $\boldsymbol{\omega} = \omega_0\hat{\mathbf{z}}$. One sliding electrical contact ("brush") is touched against the top of the sphere (along the positive z axis), while another is touched to the sphere along the equator (in the x-y plane). Find an expression for the voltage measured between these two contacts. You may refer to the figure below.



Answer

Consider a contour C enclosing a surface S.



We apply Faraday's Law of Induction. The EMF around the contour C is

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} + \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \,.$$

On the right-hand side, the first term is a “transformer EMF” that is present when there is a time-varying magnetic induction. The second terms is a reflection of the Lorentz force exerted on a unit test charge moving at velocity $\mathbf{v}$ in a magnetic induction. In this problem, there is no time-varying magnetic induction, $\partial \mathbf{B} / \partial t = 0$ everywhere, so the first term vanishes. Considering the second term, $\mathbf{v} \times \mathbf{B}$ is nonzero only along the *y* axis inside the sphere, where it has the value

$$\mathbf{v} \times \mathbf{B} = (\omega_0 r \hat{\boldsymbol{\varphi}}) \times (B_0 \hat{\mathbf{z}}) = \omega_0 r B_0 \hat{\mathbf{r}} \,,$$

and where $d\mathbf{l} = dr\hat{\mathbf{r}}$ .The voltage measured is given by the integral

$$V = \int_0^a \omega_0 r B_0 \, dr = \frac{\omega_0 a^2 B_0}{2} \,.$$