

You are told that a discrete-time complex signal $x[n]$ for integer n has a Fourier series representation

$$x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi \frac{k}{K}n}$$

where $j = \sqrt{-1}$.

In particular, for this question an integer K and complex numbers $a_k, k = 0, 1, \dots, K-1$ are given.

Evaluate the long-term time averages

$$\begin{aligned} \langle x[n] \rangle &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ \langle |x[n]|^2 \rangle &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 \end{aligned}$$

Solution

There are several ways to solve the problem.

First method

Those who remember their discrete time Fourier series will know that the $x[n]$ is periodic in n with period K and hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \frac{1}{K} \sum_{n=0}^{K-1} x[n]$$

and that the Fourier coefficients are given by

$$a_k = \frac{1}{K} \sum_{n=0}^{K-1} x[n] e^{-j2\pi \frac{k}{K}n} \quad (1)$$

and hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \frac{1}{K} \sum_{n=0}^{K-1} x[n] = a_0$$

If necessary, the formula for the coefficients could be derived e.g., as follows:

$$\begin{aligned} \sum_{n=0}^{K-1} x[n] e^{-j2\pi \frac{k}{K}n} &= \sum_{n=0}^{K-1} \left(\sum_{\ell=0}^{K-1} a_\ell e^{j2\pi \frac{\ell}{K}n} \right) e^{-j2\pi \frac{k}{K}n} \\ &= \sum_{\ell=0}^{K-1} a_\ell \sum_{n=0}^{K-1} e^{j2\pi \frac{\ell}{K}n} e^{-j2\pi \frac{k}{K}n} \\ &= \sum_{\ell=0}^{K-1} a_\ell \sum_{n=0}^{K-1} e^{j2\pi \frac{1}{K}n(\ell-k)} \end{aligned}$$

But

$$\sum_{n=0}^{K-1} e^{j2\pi \frac{n}{K}m} = \begin{cases} K & m = 0 \bmod K \\ \frac{1-e^{j2\pi \frac{K}{K}m}}{1-e^{j2\pi m/K}} = 0 & \text{otherwise} \end{cases} \quad (2)$$