Discussion/solution.

As always, the point is not the solution; the point is the clarity of the arguments used.

The identity matrix is an example showing it's possible for all entries of A and A^{-1} to be integers. Another more interesting example is an upper or lower triangular matrix, with its diagonal entries all 1 or -1.

Let's start with 1×1 matrices, $i.e.$, scalars. Here the inverse is an integer only if $A = 1$ or $A = -1$.

Now let's look at 2×2 matrices. We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

so if $\det A = 1$ or -1, then all entries of A^{-1} are integers. The converse is also true: if all entries of the inverse are integers, then $\det A = 1$ or -1. To see this, we note that

$$1 = \det I = \det(AA^{-1}) = (\det A)(\det A^{-1}).$$

If A and A^{-1} have all integer entries, then $\det A$ and $\det A^{-1}$ are both integers (since they are sums of products of entries). These two integers have a product equal to 1, so they can only be both 1, or both -1.

Now we can guess the general case: A^{-1} has integer entries if and only if $\det A$ is 1 or -1. To show one way, assume that $\det A = 1$ or -1. Cramer's formula for the inverse is

$$(A^{-1})_{ij} = \frac{(-1)^{i+j} \det \tilde{A}}{\det A},$$

where $\tilde{A}$ is formed from A by removing a column and a row. The numerator is an integer, and the denominator is 1 or -1, so $(A^{-1})_{ij}$ is an integer. To prove the converse, the argument above works: if A and A^{-1} both have integer entries, then $(\det A)(\det A^{-1})1$, and we conclude that $\det A = 1$ or -1.