1994 Electrical Engineering Qualifying Examination Questions

John Gill

Signal

A *packet* or *frame* of data to be transmitted over a network consists of 8-bit bytes followed by a single end of frame (EOF) byte consisting of all ones ($11111111_2 = \text{FF}_{16}$).

byte #1	byte #2	⋯	byte #n	11111111
←		n data bytes	→	← EOF →

Question 1: Because arbitrary binary data can be transmitted over this network, there is a possibility that the EOF byte may occur within the data frame, thereby misleading the receiver into thinking that the frame has ended. Suppose that n data bytes are generated at random (all bits are independent and 0 or 1 occur with equal probability). Find the probability that the EOF byte occurs within the data.

Answers:

1. The probability that any particular byte equals the EOF byte is $p = 2^{-8} = 1/256$. The probability that any particular byte is *not* the EOF byte is $1 - p = 255/256$. The probability that all n bytes are *not* the EOF byte is $(1-p)^n$ because the bytes are independent. Finally, the probability that at least one of the n bytes equals the EOF byte is $1-(1-p)^n$.

2. Using the binomial probability distribution, the probability of at least one occurrence of the EOF byte is given by
$$\sum_{i=1}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \binom{n}{0} p^0 (1-p)^{n-0} = 1-(1-p)^n .$$

3. If there is at least one occurrence of EOF, then the first occurrence is in either byte 1 or byte 2 ... or byte n. The probability that the *first* occurrence is in byte i is $(1-p)^{i-1}p$, which is the probability that the first $i-1$ bytes are not EOF but byte i does match. Therefore the probability of at least one occurrence of EOF is
$$\sum_{i=1}^{n} p(1-p)^{i-1} = p\sum_{i=0}^{n-1}(1-p)^i = p\cdot\frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n .$$

Question 2: What happens to the above probability as n ranges from 1 to ∞?

Answers: The probability that the EOF byte occurs approaches 1 as n gets large.

Question 3: Suppose that data bytes are generated at random until the EOF byte is produced. What is the average number of *bytes* generated?

Answers: Let X be the random variable that counts the number of bytes generated, up to and including the EOF byte. Then X has possible values $1, 2, 3, \ldots$, and the probability that $X = i$ is given by

$$\Pr(X = i) = q^{i-1}p \qquad \text{where } p = \frac{1}{256},\ q = \frac{255}{256}.$$

The expected value of X is defined by

$$E[X] = \sum_{i=1}^{\infty} i \Pr(X = i) = \sum_{i=1}^{\infty} iq^{i-1}p.$$

The expected value can be determined by several methods:

1. Memory. The expected value of a geometric random variable with success probability p is $1/p$. In this case, $1/p = 256$.

2. Calculus. The series for the expected value is the derivative of a simpler series:

$$\sum_{i=1}^{\infty} iq^{i-1}p = p\sum_{i=1}^{\infty} iq^{i-1} = p\frac{d}{dq}\sum_{i=1}^{\infty} q^i = p\frac{d}{dq}\frac{q}{1-q} = p\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

3. Recursion. With probabiltity p, the value of X is 1. Otherwise, with probability q, the first trial is wasted, the conditional expectation is now 1 plus the original expectation. In other words, $E[X]$ satisfies the equation

$$E[X] = p \cdot 1 + (1-p)(1 + E[X]) = 1 + (1-p)E[X],$$

which is easily solved to obtain $E[X] = 1/p = 256$.

Question 4: Suppose now that we relax the requirement that the EOF byte must be byte-aligned. In other words, we generate data one bit at a time until the EOF pattern (8 consecutive ones) appears. What is the average number of *bits* generated up to and including the last EOF bit?

Answers: Let X be the random variable that counts the number of bits generated. The probability distribution for X is easy to determine for small values. For example,

$$\begin{aligned}
\Pr(X = 8) &= \Pr 11111111 = 2^{-8} \\
\Pr(X = 9) &= \Pr 011111111 = 2^{-9} \\
\Pr(X = 10) &= \Pr x011111111 = 2^{-9}
\end{aligned}$$

But the general formula is rather complex:

$$\begin{aligned}
\Pr(X = i) &= \Pr(\text{first } i-9 \text{ bits do } \textit{not} \text{ contain 8 consecutive ones}) \times \\
&\quad\ \Pr(\text{last 9 bits are } 01111111)
\end{aligned}$$

The expected value will have to be determined by some other method. Here are several solutions.

John Gill

1. Let X_1 be the expected number of bits until the first 1 is generated. Obviously, $E[X_1] = 2$, since X_1 has a geometric probability distribution. Let X_2 be the expected number bits until two consecutive ones are generated. Once the first one occurs, the next bit is a one with probability 1/2; otherwise the next bit is zero, which causes the search for two consecutive ones to start over. This leads to the following formula for $E[X_2]$ in terms of $E[X_1]$:

$$E[X_2] = E[X_1] + \frac{1}{2}\Big(1 + (1 + E[X_2])\Big) \quad \Rightarrow \quad E[X_2] = 2E[X_1] + 2 = 6\,.$$

Similarly, $E[X_3] = 2E[X_2] + 2 = 14$. The sequence $\{E[X_i]\}$ for $i = 1, \ldots, 8$ is $\{2, 6, 14, 30, 62, 126, 254, 510\}$. (Obviously, $E[X_i] = 2^i - 2$.) Thus $E[X_8] = 510$.

2. If the first bit is 0, then the conditional expected value of X_8 is $E[X_8] + 1$, since the first bit is wasted and the experiment has returned to its initial state. Similarly, if the first two bits are 10, then conditional expected value of X_8 is $E[X_8] + 2$. Continuing in this way we obtain the following table:

Initial bits	Probability	Conditional expectation
0	2^{-1}	$E[X_8] + 1$
10	2^{-2}	$E[X_8] + 2$
110	2^{-3}	$E[X_8] + 3$
1110	2^{-4}	$E[X_8] + 4$
11110	2^{-5}	$E[X_8] + 5$
111110	2^{-6}	$E[X_8] + 6$
1111110	2^{-7}	$E[X_8] + 7$
11111110	2^{-8}	$E[X_8] + 8$
11111111	2^{-8}	8

The unconditional expected value of X_8 is obtained by averaging the conditional expectations in the above table. This leads to a formula for $E[X_8]$:

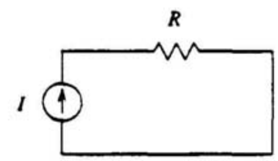
$$E[X_8] = 8 \cdot 2^{-8} + \sum_{i=1}^{8} 2^{-i}(E[X_8] + i) = \frac{255}{256}E[X_8] + \frac{510}{256}\,.$$

Therefore $E[X_8] = 256 \cdot \dfrac{510}{256} = 510$ bits.

3. Each time a zero is generated, the experiment ends if the next 8 bits are all ones. This occurs with probability $p = 2^{-8}$. Since the trials separated by zeroes are independent, the expected number of trials is $1/p = 2^8$. That is, the expected number of zeroes seen before 8 consecutive ones is 256. In fact, the first trial does not require an initial zero, so the average number of zeroes is 255. Since zeroes and ones are equally likely, the average number of ones is also 255. Therefore the expected number of bits generated is 510.

1995 Electrical Engineering Qualifying Examination Questions
John Gill *Signal*

Consider the following electrical circuit.



1. What is the voltage drop across the resistor?

   ANSWER: $V = IR$

2. Suppose that the resistor is a random variable $\tilde{R}$ with uniformly distributed in the range $R \pm \Delta R$. What is the expected value of the voltage drop?

   ANSWER: $E[V] = E[I\tilde{R}] = IE[\tilde{R}] = IR$

3. Suppose that the current source is also a random variable $\tilde{I}$. What is the expected value of the voltage drop.

   EXPECTED QUESTIONS: Are $\tilde{I}$ and $\tilde{R}$ independent? What is the joint probability distribution of $\tilde{I}$ and $\tilde{R}$.

   ANSWER: If $\tilde{I}$ and $\tilde{R}$ are independent, then $E[V] = E[\tilde{I}\tilde{R}] = E[\tilde{I}]E[\tilde{R}] = IR$, where $I = E[\tilde{I}]$.

4. What is a weaker condition than independence that guarantees that $E[\tilde{I}\tilde{R}] = IR$?

   ANSWER: Uncorrelated.

5. Suppose that two random resistors $\tilde{R}_1$ and $\tilde{R}_2$ are connected in series. What is the average resistance?

   ANSWER: $E[\tilde{R}_1 + \tilde{R}_2] = E[\tilde{R}_1 + \tilde{R}_2] = 2R$

6. What if the resistors values are not statistically independent?

   ANSWER: The expected value of a sum is always the sum of the expected values.

7. Suppose that two random resistors $\tilde{R}_1$ and $\tilde{R}_2$ are connected in parallel. What is the average resistance?

   EXPECTED QUESTION: What is the joint probability distribution of $\tilde{R}_1$ and $\tilde{R}_2$? Suppose the resistors are independent.

ANSWER: For any two values of $\tilde{R}_1$ and $\tilde{R}_2$, the parallel resistance is

$$\frac{1}{1/R_1 + 1/R_2} = \frac{R_1 R_2}{R_1 + R_2}.$$

Therefore the expected value of the resistance is

$$E\Big[\frac{R_1 R_2}{R_1 + R_2}\Big] = \frac{1}{(2\Delta R)^2} \int_{R-\Delta R}^{R+\Delta R} \int_{R-\Delta R}^{R+\Delta R} \frac{r_1 r_2}{r_1 + r_2}\, dr_1 dr_2$$

The details of the integration are messy and not very interesting.

8. What about the average conductance?

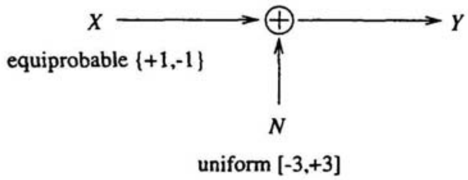
ANSWER: The average conductance is the sum of the average conductance for each resistor (whether or not the resistors are independent). Since $G = 1/R$,

$$E[G] = E\Big[\frac{1}{R}\Big] = \frac{1}{2\Delta R} \int_{R-\Delta R}^{R+\Delta R} \frac{1}{r}\, dr = \frac{1}{2\Delta R}(\ln(R + \Delta R) - \ln(R - \Delta R))$$

1996 Qualifying Exam Questions

John Gill

A communications system transmits binary data by sending one bit per unit time, representing binary values by the analog values $X = +1$ and $X = -1$.



The received signal is corrupted by additive noise; that is, $Y = X + N$. The noise N is uniformly distributed for $-3 \leq N \leq +3$. The two input values are equally probable.

1. Sketch the pdf (probability density function) of N.

2. What is the variance of N?

3. The noise power is defined to be its variance. What is the signal-to-noise ratio (SNR)?

4. Sketch the pdf of the received signal Y.

5. What is a good decision rule for estimating X given Y?

6. Suppose we use the simple decision rule:

$$\hat{X} = \begin{cases} +1 & \text{if } Y > 0 \\ -1 & \text{if } Y < 0 \end{cases}$$

   What is $\Pr(X = -1 \mid Y = +1)$, that is, the conditional error probability given $Y = +1$?

7. Find the overall error probability, $\Pr(\hat{X} \neq X)$.

8. To reduce the probability of error, we increase signal power by sending the same signal twice. The received signal Y is the sum of the two transmission:

$$Y = 2X + N_1 + N_2\,,$$

   where N_1 and N_2 are independent. What is the pdf of the received signal Y?

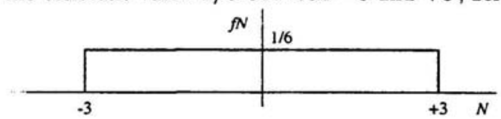
9. What is the optimum decision rule for estimating X given Y?

1996 Qualifying Exam Answers

John Gill

The first three questions are warmup questions. The last two questions are bonus questions; about 15% of the examinees reached the last two questions.

1. The pdf of N has the constant value 1/6 between -3 and $+3$, zero elsewhere.



2. The variance of a random variable uniformly distributed on the interval $[a, b]$ is $(b-a)^2/12$. So the variance of the noise N is $6^2/12 = 3$. The variance can also be calculated from the definition:

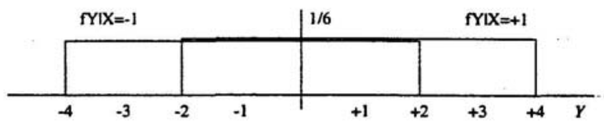
$$\sigma_N^2 = E[N^2] - E[N]^2 = \int_{-3}^{+3} \frac{1}{6} x^2\, dx = \frac{x^3}{18}\Big[_{-3}^{+3} = \frac{(+3)^3 - (-3)^3}{18} = 3\,.$$

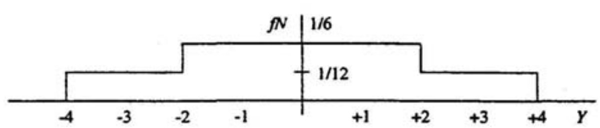
3. The power of the signal X is 1 since the magnitude of X is always 1. The variance of X is also 1 because

$$\sigma_X^2 = E[X^2] - E[X]^2 = \frac{1}{2}(+1)^2 + \frac{1}{2}(-1)^2 = 1\,.$$

Therefore the signal-to-noise ratio is 1/3.

4. For each value of X, the conditional probability density of Y is uniformly distributed about that value of X. The conditional densities $f_Y(y \mid X = \pm 1)$ and the unconditional density $f_Y(y)$ are shown below.





Combining the two conditional densities is the same as convolving the pdf of X, which is two impulses of height 1/2 located at ± 1 with the pdf of N.

5. The obvious decision rule is to estimate that $X = +1$ when $Y > 0$ and that $X = -1$ when $Y < 0$. This decision rule is optimal, as shown below.

6. The conditional probability of error given any particular received value y is

$$\Pr(X \neq \hat{X} \mid Y = y) = \frac{\Pr(X \neq \hat{X} \text{ and } Y = y)}{\Pr(Y = y)}.$$

When $Y = +1$ the estimate of X is $\hat{X} = +1$, so the conditional error probability is

$$\frac{\Pr(X \neq -1 \text{ and } Y = +1)}{\Pr(Y = +1)} = \frac{\Pr(X = -1)\Pr(Y = +1 \mid X = -1)}{\Pr(Y = +1)} = \frac{(1/2)\cdot(1/6)}{(1/6)} = \frac{1}{2}.$$

Both $\Pr(Y = +1)$ and $\Pr(Y = +1 \mid X = -1)$ are values of probability density functions, so their quotient is meaningful.

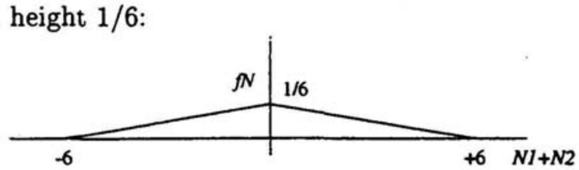
The same conditional error probability of 1/2 is obtained for all values of Y between -2 and $+2$. For these values of Y, the receiver knows no more about X after receiving Y than before X was transmitted. Therefore $\hat{X} = +1$ and $\hat{X} = -1$ are equally good estimates, so the simple decision rule based on the sign of Y is optimal. (Or $\hat{X}$ could be decided by tossing a coin when $-2 < Y < +2$.)

7. When $Y < -2$ it is certain that $X = -1$, and when $Y > +2$ it is certain that $X = +1$. For Y in these two intervals, the conditional error probability is 0. When $-2 < Y < +2$, the conditional error probability is 1/2. The overall error probability is

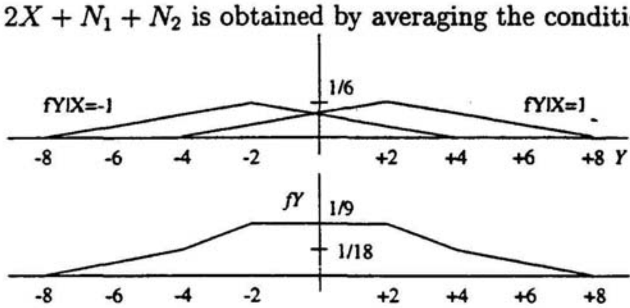
$$\Pr(|Y| > 2)\cdot 0 \;+\; \Pr(|Y| < 2)\cdot \tfrac{1}{2} \;=\; \tfrac{2}{3}\cdot\tfrac{1}{2} \;=\; \tfrac{1}{3}.$$

Another way to evaluate the error probability is to condition on X. When $X = -1$, an error occurs when $Y = X + N > 0$, that is, when $N > 1$. From the pdf for N, we see that $\Pr(N > 1) = 1/3$, so $\Pr(\hat{X} \neq X \mid X = -1) = 1/3$. By a similar calculation, $\Pr(\hat{X} \neq X \mid X = +1) = 1/3$. The overall error probability, which is the average of these two conditional probabilities, is 1/3.

8. The combined noise is the sum of two independent uniformly distributed random variables. The pdf of the sum is the convolution of two rectangle functions, which is a triangle with range $[-6, +6]$ and height 1/6:



The pdf for $Y = 2X + N_1 + N_2$ is obtained by averaging the conditional pdfs:



9. The optimum decision rule is the obvious decision rule:

$$\hat{X} = \begin{cases} +1 & \text{if } Y > 0 \\ -1 & \text{if } Y < 0 \end{cases}$$

Since the values of X are equiprobable, minimum error decoding is the same as maximum-likelihood decoding. Because the pdfs of Y given X are triangular, $\Pr(Y = y \,|\, X = -1) > \Pr(Y = y \mid X = +1)$ when $y < 0$, whereas $\Pr(Y = y \mid X = -1) < \Pr(Y = y \mid X = +1)$ when $y > 0$.

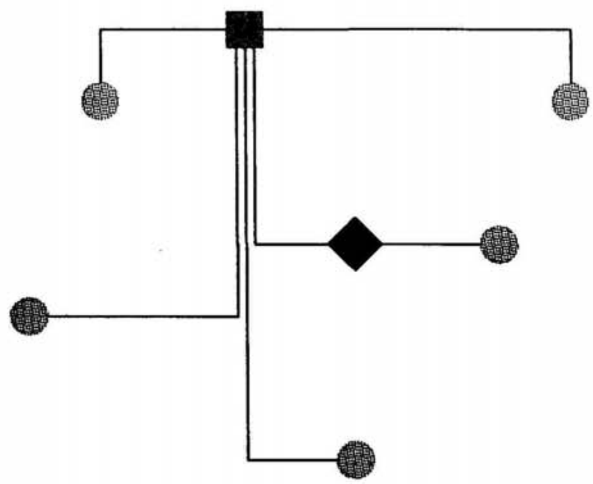
The overall error probability is the conditional error probability given either value of X. For $X = -1$ this conditional probability is the area under the left triangle to the right of the vertical axis. Thus $\Pr(\hat{X} \neq X) = \Pr(N_1 + N_2 > 2) = 2/9$. As expected, this error probability is smaller than $1/3$, the error probability for a single transmission of X.

Signal

1998-1999 Qualifying Examination Question

JOHN GILL

A set of nodes (blue circles in the figure below) must be connected to a power node (red square) using separate wires that run horizontally or vertically.



Where in general should the power node be placed to minimize the *total* wire length? (The location shown in the figure above is *not* optimal.)

Answer

Suppose that the i-th blue node is at location (x_i, y_i). If (x, y) is the location of the red node, then the total wire length is

$$\sum_{i=1}^{n}(|x - x_i| + |y - y_i|) = \sum_{i=1}^{n} |x - x_i| + \sum_{i=1}^{n} |y - y_i| .$$

The two sums can be independently minimized; that is, the x-coordinate of the best location for the power node depends only on the x-coordinates of the blue nodes, and similarly for the y-coordinates. We can assume that $x_1 \le x_2 \le \cdots \le x_n$. If $x_k \le x \le x_{k+1}$ then the total x cost is

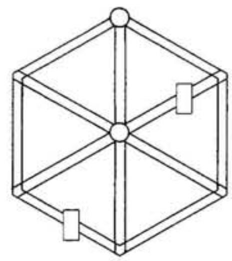
$$\sum_{i=1}^{k}(x - x_i) + \sum_{i=k+1}^{n} (x_i - x) .$$

The derivative of this piecewise-linear function is $k-(n-k) = 2k-n$. The derivative is negative if $k < n/2$ and is positive if $k > n/2$. Thus the x cost is minimized by locating x at the *median* of the x-coordinates, where an equal number of x_i are less than x and greater than x. Similarly, the best y location is the median of the y-coordinates. The optimum power node location for the figure above is the green diamond. (If n is even then all values of x between $x_{n/2}$ and $x_{n/2+1}$ have the same cost.)

1999-2000 Qualifying Examination Questions

John Gill

The Hexagon has two sets of corridors as shown in the figure below. Each corridor segment is 1 km long. These corridors contain a very large number of light bulbs, which burn out at random independent times. Two battery-powered robots (represented by rectangles) are available to replace the light bulbs. There are two recharging stations (circles), one at the top of the Hexagon and one in the center.



One robot is dedicated to the outside (circumferential) corridors and the other robot serves the inside (radial) corridors.

Primary questions

1. Suppose that the outside robot returns to the charging station at the top of the Hexagon after replacing a light bulb. On average how far does it travel to change the light bulb and return to the charging station?

2. Suppose that the outside robot waits until two light bulbs have burned out. It then travels by the most efficient overall path to replace the two bulbs, finally returning to the charging station. On average how far does it travel?

Secondary and bonus questions

3. What is the average distance traveled by the outside robot to change a single light bulb if it remains at the location of the last replaced light bulb?

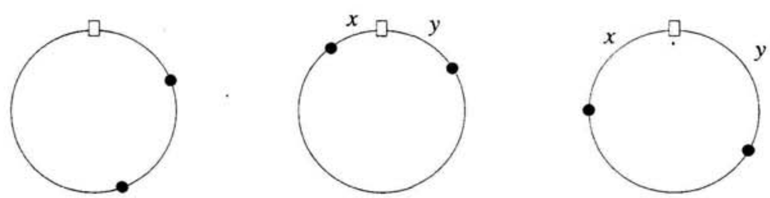
4. What is the average distance traveled by the inside robot to change a single light bulb and return to the charging station at the center?

5. What is the average distance traveled by the inside robot to change a single light bulb if it remains at the location of the last replaced light bulb?

Answers to primary questions

1. The maximum round trip is 6 km, since the robot travels by the shortest path. The distance is uniform from 0 to 6, so the average round trip is 3 km.

2. With probability $\frac{2}{4} = \frac{1}{2}$, the two light bulbs are on the same side of the Hexagon In this case, the robot travels to the closer light bulb, then to the more distant one, then returns by the reverse path. The distance to the farther light bulb is a random variable that is the maximum of two independent random variables that are uniformly distributed from 0 to 3. The average value of the distance is $\frac{2}{3}$ of the maximum, so the average round trip in this case is 4 km.



When the light bulbs are on opposite sides of the Hexagon, the robot has two possible routes. If the bulbs are close together, the robot travels first to one, then returns to the charging station, then travels to the other and returns. The total distance is $2x + 2y$. If the light bulbs are far apart, it is more efficient to travel to one, then continue in the same direction to the second, then continue in the same direction to return to the charging station, for a total distance of 6. The robot chooses the shorter of these two paths, depending on whether $x + y < 3$. Since x and y are independent and uniformly distributed from 0 to 3, their sum $x+y$ ranges from 0 to 6 and $\Pr(x+y < 3) = \frac{1}{2}$. The conditional probability density for $z = x + y$ given $x + y < 3$ is the triangle $f(z) = 2z/9$, so the conditional expectation is $x + y$ is 2 and the average round trip in this case is 4. Obviously, when $x + y > 3$ the average distance is 6. The overall average round trip distance is

$$\tfrac{1}{2} \cdot 4 + \tfrac{1}{4} \cdot 4 + \tfrac{1}{4} \cdot 6 = 4\tfrac{1}{2},$$

which is obtained by combining the conditional expectations for the three cases.

Answers to secondary and bonus questions

3. The locations of successive light bulbs are independent, so distance from one to the next by the shortest path is a maximum of 3 km and an average of 1.5 km.

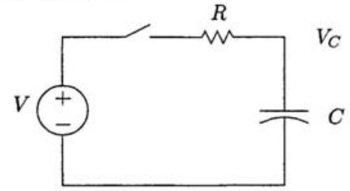
4. The maximum round trip for the inside robot is 2 km, so the average is 1 km.

5. With probability $\frac{5}{6}$ the next light bulb is in another radial corridor, in which case the average distance is $\frac{1}{2} \cdot 2 = 1$. With probability $\frac{1}{6}$, the next light bulb is in the same corridor, in which case the average distance is $\frac{1}{3}$. The overall average round trip for the inside robot is $\frac{5}{6} + \frac{1}{18} = \frac{8}{9}$.

2003-2004 Electrical Engineering Qualifying Examination

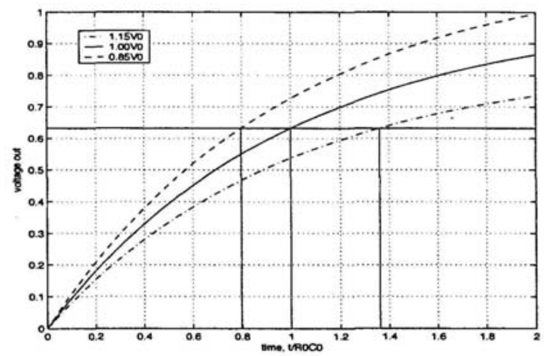
John Gill

The resistance R, capacitance C, and voltage V in the circuit shown below are independent random variables.



$$R \sim \text{Uniform}[\,(1-\delta)\mathrm{R_0},\ (1+\delta)\mathrm{R_0}\,]$$
$$C \sim \text{Uniform}[\,(1-\delta)\mathrm{C_0},\ (1+\delta)\mathrm{C_0}\,]$$
$$V \sim \text{Uniform}[\,(1-\delta)\mathrm{V_0},\ (1+\delta)\mathrm{V_0}\,]$$
$$0 < \delta < 1$$
$$V_C(t) = V(1-e^{-t/RC}) \qquad (t>0)$$

The *time constant* of this random RC circuit is defined to be the time T that is needed for the capacitor to charge to $V_0(1-e^{-1})$. Examples of $V_C(t)$ and T are shown in the following figure.



Question 1 For this random circuit, the time constant is a random variable T. Find the conditional probability density of T given fixed values R and C.

Question 2 Find the expected value of the random time constant T.

Solution 1 Let $V_1 = V_0(1-e^{-1}) = 0.6321\,V_0$ denote the nominal threshold voltage, that is, the voltage on the capacitor after one nominal time constant R_0C_0 when $V = V_0$. The time constant random variable T is a function of V, R, and C; it is the solution of the equation

$$V_C(t) = V(1-e^{-T/RC}) = V_1\,.$$

Solving the equation is straightforward:

$$V(1-e^{-T/RC}) = V_1 \;\Rightarrow\; 1-e^{-T/RC} = \frac{V_1}{V} \;\Rightarrow$$
$$e^{-T/RC} = 1-\frac{V_1}{V} \;\Rightarrow\; \frac{T}{RC} = -\ln\left(1-\frac{V_1}{V}\right) \;\Rightarrow\; T = -RC\ln\left(1-\frac{V_1}{V}\right)$$

Since T is a monotonically decreasing function of V, the range of T is

$$-RC\ln\left(1-\frac{V_1}{V_0(1+\delta)}\right) \le T \le -RC\ln\left(1-\frac{V_1}{V_0(1-\delta)}\right).$$

If the random voltage V is too small—namely, $V < V_1$—then the capacitor voltage $V_C(t)$ never reaches V_1. In this case $T = +\infty$. (The upper bound in the above equation is meaningless.) We assume from now on that $\delta < e^{-1}$.

Given the formula for T, there are two standard ways to find the pdf of T: find the cdf $F_T(t)$ and differentiate, or express the pdf $f_T(t)$ in terms of the pdf $f_V(v)$ of V.

We can find the cdf $F_T(t)$ by using its definition.

$$\begin{aligned}
\mathbf{P}\{T \le t\} &= \mathbf{P}\left\{-RC\ln\left(1-\tfrac{V_1}{V}\right) \le t\right\} = \mathbf{P}\left\{\ln\left(1-\tfrac{V_1}{V}\right) \ge -\frac{t}{RC}\right\} \\
&= \mathbf{P}\left\{1-\tfrac{V_1}{V} \ge e^{-t/RC}\right\} = \mathbf{P}\left\{\tfrac{V_1}{V} \le 1-e^{-t/RC}\right\} = \mathbf{P}\left\{V \ge \frac{V_1}{1-e^{-t/RC}}\right\} \\
&= 1-\mathbf{P}\left\{V \le \frac{V_1}{1-e^{-t/RC}}\right\} = F_V\left(\frac{V_1}{1-e^{-t/RC}}\right) = \frac{1}{2\delta V_0}\left(\frac{V_1}{1-e^{-t/RC}} - V_0(1-\delta)\right)
\end{aligned}$$

Finding the pdf is now an exercise in using the Chain Rule to differentiate the cdf.

$$\begin{aligned}
f_T(t) &= \frac{d}{dt}\frac{1}{2\delta V_0}\left(\frac{V_1}{1-e^{-t/RC}} - V_0(1-\delta)\right) = \frac{V_1}{2\delta V_0}\frac{d}{dt}\frac{1}{1-e^{-t/RC}} \\
&= \frac{V_1}{2\delta V_0}\left(-\frac{1}{(1-e^{-t/RC})^2}\right)\left(-\frac{e^{-t/RC}}{RC}\right) = \frac{V_1 e^{-t/RC}}{2RC\delta V_0(1-e^{-t/RC})^2}
\end{aligned}$$

The pdf of $T = g(V)$ can be obtained directly from the pdf of V using a formula familiar to EE 278 students. Let $v_1, v_2, \ldots$ be the solutions of the equation $t = g(v)$ and let $g'(v_i)$ be the derivative of g evaluated at v_i. Then

$$f_T(t) = \sum_i \frac{f_V(v_i)}{|g'(v_i)|},$$

Since $g(v) = -RC\ln(1 - V_1/V)$ is monotonically decreasing, there is at most one solution to the equation. By definition of T, the value of v corresponding to t satisfies $v(1-e^{-t/RC}) = V_1$, hence $v = V_1/(1-e^{-t/RC})$. Therefore

$$\begin{aligned}
\frac{d}{dv}\left(\ln\left(1-\frac{V_1}{v}\right)\right) &= \frac{d}{dv}\left(\ln\left(\frac{V_1-v}{v}\right)\right) = \frac{d}{dv}\Big(\ln(v-V_1) - \ln v\Big) = \frac{1}{v-V_1} - \frac{1}{v} \\
&= \frac{1}{v - v(1-e^{-t/RC})} - \frac{1-e^{-t/RC}}{V_1} = \frac{1}{ve^{-t/RC}} - \frac{1-e^{-t/RC}}{V_1} \\
&= \frac{1-e^{-t/RC}}{V_1e^{-t/RC}} - \frac{1-e^{-t/RC}}{V_1} = \frac{1-e^{-t/RC} - e^{-t/RC}(1-e^{-t/RC})}{V_1e^{-t/RC}} = \frac{(1-e^{-t/RC})^2}{V_1e^{-t/RC}}
\end{aligned}$$

Putting it all together, we find the pdf of T:

$$f_T(t) = \frac{f_V(v)}{|g'(v)|} = \frac{1}{2\delta V_0}\left(RC\frac{(1-e^{-t/RC})^2}{V_1e^{-t/RC}}\right)^{-1} = \frac{V_1e^{-t/RC}}{2RC\delta V_0(1-e^{-t/RC})^2}$$

We note with satisfaction and relief that both methods yield the same answer.

Solution 2 In part 1 we found the formula for T as a function of R, C, and V. Because R, C, and V are independent,

$$E[T] = E\left[-RC\ln\left(1-\frac{V_1}{V}\right)\right] = E[R]\,E[C]\,E\left[-\ln\left(1-\frac{V_1}{V}\right)\right] = R_0C_0E\left[-\ln\left(1-\frac{V_1}{V}\right)\right].$$

The uniform pdf for V leads to the following integral.

$$E[T] = R_0C_0E\left[-\ln\left(1-\frac{V_1}{V}\right)\right] = \frac{R_0C_0}{2\delta V_0}\int_{V_0(1-\delta)}^{V_0(1+\delta)} -\ln\left(1-\frac{V_1}{v}\right)dv$$

The integrand is not defined when $V < V_1$, so $E[T]$ is undefined (or infinite) when $\delta \geq e^{-1}$.

For completeness, we perform the integration. This was *not* a requirement for the problem. From integration by parts, tables of integrals, or long term memory we obtain

$$\int \ln x \;=\; x\ln x - x\,.$$

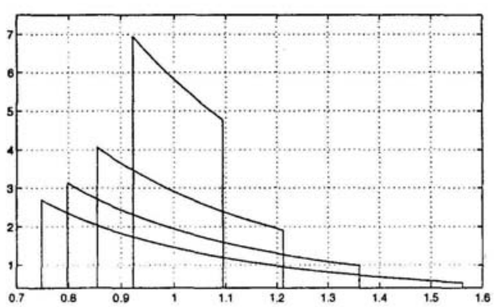
Next we find the *indefinite* integral needed for $E[T]$ (additive constants can be ignored).

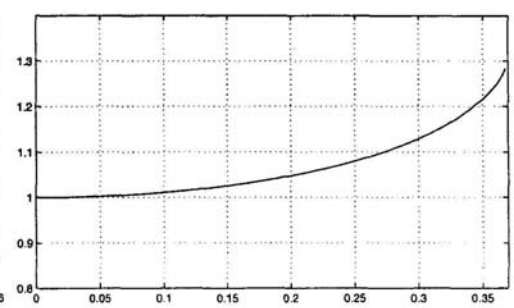
$$\begin{aligned}\int \ln\left(1-\frac{V_1}{v}\right)dv \;&=\; \int \ln\frac{v-V_1}{v}\,dv \;=\; \int\Big(\ln(v-V_1) \;-\; \ln v\Big)\,dv\\ &=\; (v-V_1)\ln(v-V_1) - (v-V_1) - v\ln v + v \;=\; (v-V_1)\ln(v-V_1) - v\ln v\,.\end{aligned}$$

The final answer has a closed form but no obvious simplications.

$$\begin{aligned}E[T] \;&=\; \frac{R_0C_0}{2\delta V_0}\int_{V_0(1-\delta)}^{V_0(1+\delta)} -\ln\left(1-\frac{V_1}{v}\right)dv = \frac{R_0C_0}{2\delta V_0}\Big(v\ln v \;-\; (v-V_1)\ln(v-V_1)\Big)\Big[_{V_0(1-\delta)}^{V_0(1+\delta)}\\ &=\; \frac{R_0C_0}{2\delta V_0}\Big(V_0(1+\delta)\ln(V_0(1+\delta)) \;-\; V_0(1-\delta)\ln(V_0(1-\delta)) \;-\\ &\qquad (V_0(1+\delta) - V_1)\ln(V_0(1+\delta) - V_1) \;+\; (V_0(1-\delta) - V_1)\ln(V_0(1-\delta) - V_1)\Big)\end{aligned}$$

The left graph shows the conditional pdfs of T given $R=1$, $C=1$ for $\delta = 0.20, 0.15, 0.10, 0.05$ (left to right). The right graph plots the expected value of T as a function of δ for $0 < \delta < e^{-1}$.





The pdf and mean of T are not defined for $\delta \geq e^{-1}$. It is somewhat surprising that as $\delta \to e^{-1}$ the mean of T converges to a finite number $\frac{1}{2}(e+1)\ln(e+1) - \frac{1}{2}(e-1)\ln(e-1) - \ln 2 = 1.2833$

2006-2007 Electrical Enginering Qualifying Examination

JOHN GILL

A *Bernoulli horse race* has m horses competing on a race course of length n steps.

At discrete times $k = 1, 2, 3, \ldots$ each horse flips an unbiased coin and advances one step if the coin comes up heads.

Consider just one horse. Let T be the number of coin flips that the horse takes to finish.

1. Find $\mathrm{E}(T)$, the average value of T.
2. Find $\mathrm{P}\{T \leq 2n - 1\}$, the probability that the horse finishes within $2n - 1$ coin flips.
3. Find $p_T(r) = \mathrm{P}\{T = r\}$, the probability mass function.
4. Find the most probable value of T, that is, the largest value of $p_T(r)$.

SOLUTIONS

1. The random variable T can be written as the sum $T_1 + T_2 + \cdots + T_n$ where T_i is the number of coin flips needed to move from step $i-1$ to step i. Each T_i is an geometric random variable with success probability $p = 1/2$ and expected value $1/p = 2$. Therefore $\mathrm{E}(T) = \sum_{i=1}^{n} T_i = 2n$.
2. Consider all 2^{2n-1} sequences of $2n - 1$ coin flips. A horse finishes in $2n - 1$ flips if and only if at least n of the flips are heads. (If a horse finishes in less than $2n - 1$ coin flips, then the remaining flips need not be looked at.) The probability of n heads in $2n-1$ flips of an unbiased coin is the probability that the majority of an odd number of flips is heads, namely, $1/2$.
3. If a horse finishes on the r-th coin flip, then the last flip is heads, and the first $r - 1$ coin flips contain exactly $n-1$ heads. There are $\binom{r-1}{n-1}$ sequences satisfying these conditions. Each such sequence has probability 2^{-r}. Therefore $\mathrm{P}\{T = r\} = \binom{r-1}{n-1}2^{-r}$.

   (The sum of n geometric random variables has a *negative binomial probability distribution*.)
4. Because the pmf values are products, the easiest way to determine the maximum is to look at ratios of successive probabilities.
$$\frac{p_T(r)}{p_T(r+1)} = \frac{\binom{r-1}{n-1}2^{-r}}{\binom{r}{n-1}2^{-r-1}} = \frac{r-n+1}{\frac{1}{2}r} \lesseqgtr 1 \iff r-n+1 \lesseqgtr \tfrac{1}{2}r \iff r < 2n-2\,.$$
   This shows that if $r < 2n - 2$ then $p_T(r) < p_T(r+1)$ and if $r > 2n - 2$ then $p_T(r) > p_T(r+1)$. The maximum value occurs at $r = 2n - 2$ and $r = 2n - 1$.