The continuous time Fourier transform (CTFT) of a continuous time signal $x(t)$ is
$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

The discrete time Fourier transform (DTFT) of a discrete time signal $x[n]$ is
$X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$
For the same set of systems, describe how the Fourier transform of the output can be determined from that of the input (using the appropriate type of Fourier transform)
Systems:

- *Input:* $x(t)$; all real t

  Output: $y(t) = \int_{-\infty}^{t} x(\tau)e^{-\alpha(t-\tau)}d\tau$; all real t

- *Input:* $x(t)$; all real t

  Output: $y(t) = [a + mx(t)]\cos(2\pi f_0 t + \theta)$; all real t

- *Input:* $x(t)$; all real t

  Output: $y[n] = x(n)$; all integer n

- *Input:* $x[n]$; all integer n

  Output: satisfies difference equation $y[n] = ay[n-1] + x[n]$; all integer n. The system is assumed to be causal.

Solution: Some of these may have been derived in answering the first question, in which case they were skipped or just rephrased.

- $Y(f) = X(f)H(f)$ as before.

- $Y(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))e^{j\theta} + \frac{m}{2}[X(f-f_0) + X(f+f_0)]$

- If the signal is bandlimited to $[-1/2, 1/2]$, then DTFT $Y(f)$ of the sampled waveform is the same as that of the original waveform for $f \in [-1/2, 1/2]$, the only range of f needed for inversion. If f is allowed to range over the entire real line, then the DTFT has periodic replicas of $X(f)$ with period 1.

  A careful proof was not expected, I was more interested in either memory or intuition. A short proof is the following: If $X(f)$ is nonzero only in $[-1/2, 1/2]$, then in that region it can be expanded as a Fourier series in f as

  $$X(f) = \sum_{k=-\infty}^{\infty} a_k e^{-j2\pi kf}; f \in [-1/2, 1/2]$$

  with

  $$a_k = \int_{-1/2}^{1/2} X(f)e^{j2\pi kf}df$$