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January 2007 Quals

Given

- Θ is a random variable described by a probability density function $f_\Theta(\theta) = 1/2\pi$ for $0 \le \theta < 2\pi$.
- $\{X_n;\ n = 0, 1, 2, \cdots\}$ is a discrete-time random process defined by $X_n = e^{jn\Theta}$.
- Fix a positive integer N and define

$$Y_k = \sum_{n=0}^{N-1} X_n e^{\frac{-j2\pi kn}{N}};\ k = 0, 1, \ldots, N-1.$$

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1. Evaluate the mean $E(X_n)$ and autocorrelation function $R_X(n,k) = E(X_n X_k^*)$.
2. Is X_n stationary?
3. Evaluate or approximate the following sums assuming that N is very large:

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n \qquad \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X_n$$

$$\frac{1}{N}\sum_{k=0}^{N-1} Y_k \qquad \frac{1}{N}\sum_{k=0}^{N-1} |Y_k|^2 .$$

4. Suppose that we redefine X_n as $X_n = e^{jn\Theta_n}$ where now the Θ_n are independent identically distributed uniform random variables on $[0, 2\pi)$. Which of the above answers *change*?

Solutions

In all cases points are approximate, as more points could be awarded for clever solutions and fewer for meandering solutions or detours.

1. The first problem was intended to test basic probability skills. It counted for about 1.5 points out of the 10 as it is very elementary probability.

$$\begin{aligned}
E(X_n) &= E(e^{jn\Theta}) \\
&= \int f_\Theta(\theta) e^{jn\theta}\, d\theta \\
&= \int_0^{2\pi} \frac{1}{2\pi} e^{jn\theta}\, d\theta = \begin{cases} 1 & n = 0 \\ 0 & n = 1, 2, \cdots \end{cases} \\
R_X(n,k) &= E(X_n X_k^*) \\
&= E(e^{jn\Theta} e^{-jk\Theta}) = E(e^{j(n-k)\Theta}) \\
&= \begin{cases} 1 & n = k \\ 0 & n \ne k \end{cases}
\end{aligned}$$

Ideally the first integral was done by inspection since the integral is obviously 1 for $n = 0$ and the integral of a period of a complex exponential (or, equivalently, of a sine and cosine) is 0. Ideally the student would realize the second integral was identical to the first with $n - k$ replacing n and not redo the entire calculation.