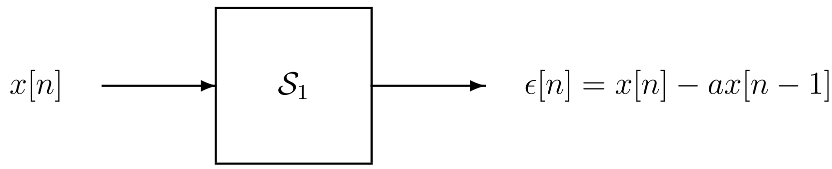
Consider the following discrete-time system with input $x[n]$ and output $\epsilon[n]$ defined by the input/output relation




**Only causal signals are allowed, i.e.,** $x[n] = 0$ **for all** $n < 0$**.**

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**First Question:** Given the following definitions and assumptions:

$$
\begin{aligned}
< x > &\triangleq \sum_{n=0}^{\infty} x[n] = 0 \quad \text{mean} \\
< x^2 > &\triangleq \sum_{n=0}^{\infty} x[n]^2 = \mathcal{E}_x < \infty \quad \text{energy} \\
r_x(k) &\triangleq \sum_{n=k}^{\infty} x[n]x[n-k] \quad \text{autocorrelation } (\mathcal{E}_x = r_x(0)) \\
X(f) &\triangleq \sum_{n=0}^{\infty} x[n]e^{-j2\pi fn}; -\frac{1}{2} \leq f \leq \frac{1}{2} \qquad \text{DTFT,}
\end{aligned}
$$

Find simple expressions for $< \epsilon >, \mathcal{E}_\epsilon$, and the DTFT $E(f)$ of $\epsilon[n]$.

**Solution:** Using the fact that $x[-1] = 0$ since inputs must be causal,

$$
\begin{aligned}
< \epsilon > &= \sum_{n=0}^{\infty} \epsilon[n] = \sum_{n=0}^{\infty}(x[n] - ax[n-1]) = \sum_{n=0}^{\infty} x[n] - a\sum_{n=0}^{\infty} x[\underbrace{n-1}_{n'}] \\
&= < x > -a\sum_{n'=0}^{\infty} x[n'] =< x > -a < x >= 0 \\
\mathcal{E}_\epsilon &= \sum_{n=0}^{\infty} \epsilon[n]^2 = \sum_{n=0}^{\infty} (x[n] - ax[n-1])^2 \\
&= \sum_{n=0}^{\infty} \left(x[n]^2 - 2ax[n]x[n-1] + a^2x[n-1]^2\right) \\
&= r_x(0) - 2ar_x(1) + a^2r_x(0) = (1 + a^2)r_x(0) - 2ar_x(1) \\
E(f) &= \sum_{n=0}^{\infty} \epsilon[n]e^{-j2\pi fn} = \sum_{n=0}^{\infty} (x[n] - ax[n-1])\, e^{-j2\pi fn} \\
&= \sum_{n=0}^{\infty} x[n]e^{-j2\pi fn} - a\sum_{n=0}^{\infty} x[\underbrace{n-1}_{n'}]e^{-j2\pi fn} = X(f) - a\sum_{n'=0}^{\infty} x[n']e^{-j2\pi f(n+1)} = X(f)(1 - ae^{-j2\pi f})
\end{aligned}
$$

or just quote linearity and the shift property of DTFTs