Before: $x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi \frac{k}{K} n}$

Next, you are told that $x[n]$ instead has the following form for integer n:

$$x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi k f_0 n}$$

where f_0 is a fixed real number.

Evaluate the time averages

$$\begin{aligned} < x[n] > &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ < |x[n]|^2 > &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 \end{aligned}$$

Solution

If f_0 is rational, the answer is the same as before. So the question is what happens when f_0 is irrational. This case occurs in modeling real-world systems, such as A/D converter error with sinusoidal inputs.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{k=0}^{K} a_k e^{j2\pi k f_0 n} \right) = \sum_{k=0}^{K} a_k \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi k f_0 n} \right)$$

As in (2)

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi k f_0 n} = \begin{cases} 1 & k f_0 = 0 (\mathrm{mod} N) \\ \frac{1}{N} \frac{1 - e^{j2\pi k f_0 N}}{1 - e^{j2\pi k f_0}} & \text{otherwise} \end{cases}$$

If f_0 is irrational, then the first case only occurs for $k = 0$. The second case converges to 0 as $N \to \infty$. Therefore

$$< x[n] >= a_0$$

as in the periodic case.

Altenatively, by linearity you can separately consider what happens for each k in the sum. As in the periodic case, the $k = 0$ term immediately gives a_0 and the limit is trivial. The remaining terms are summing rotations around the circle and then dividing by N. The sum is bounded and the N blows up, so hey all go to zero, leaving only a_0.