Answers: Let X be the random variable that counts the number of bytes generated, up to and including the EOF byte. Then X has possible values $1, 2, 3, \ldots$, and the probability that $X = i$ is given by

$$\Pr(X = i) = q^{i-1}p \qquad \text{where } p = \frac{1}{256},\ q = \frac{255}{256}.$$

The expected value of X is defined by

$$E[X] = \sum_{i=1}^{\infty} i \Pr(X = i) = \sum_{i=1}^{\infty} iq^{i-1}p.$$

The expected value can be determined by several methods:

1. Memory. The expected value of a geometric random variable with success probability p is $1/p$. In this case, $1/p = 256$.

2. Calculus. The series for the expected value is the derivative of a simpler series:
$$\sum_{i=1}^{\infty} iq^{i-1}p = p\sum_{i=1}^{\infty} iq^{i-1} = p\frac{d}{dq}\sum_{i=1}^{\infty} q^i = p\frac{d}{dq}\frac{q}{1-q} = p\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

3. Recursion. With probabiltity p, the value of X is 1. Otherwise, with probability q, the first trial is wasted, the conditional expectation is now 1 plus the original expectation. In other words, $E[X]$ satisfies the equation
$$E[X] = p \cdot 1 + (1-p)(1 + E[X]) = 1 + (1-p)E[X],$$
which is easily solved to obtain $E[X] = 1/p = 256$.

Question 4: Suppose now that we relax the requirement that the EOF byte must be byte-aligned. In other words, we generate data one bit at a time until the EOF pattern (8 consecutive ones) appears. What is the average number of *bits* generated up to and including the last EOF bit?

Answers: Let X be the random variable that counts the number of bits generated. The probability distribution for X is easy to determine for small values. For example,

$$\begin{aligned}
\Pr(X = 8) &= \Pr 11111111 = 2^{-8} \\
\Pr(X = 9) &= \Pr 011111111 = 2^{-9} \\
\Pr(X = 10) &= \Pr x011111111 = 2^{-9}
\end{aligned}$$

But the general formula is rather complex:

$$\begin{aligned}
\Pr(X = i) = {} & \Pr(\text{first } i-9 \text{ bits do } \textit{not} \text{ contain 8 consecutive ones}) \times \\
& \Pr(\text{last 9 bits are } 01111111)
\end{aligned}$$

The expected value will have to be determined by some other method. Here are several solutions.