EE Qualifying Exam
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Lost in Translation?

For your research you'll need to develop the ability to translate scientific and mathematical statements from the language used in books and articles from another field to that of your own. Consider this theorem, copied verbatim from a mathematics text:

Theorem *If* (ζ_ν) *are the finite Fourier coefficients of the sequence* (z_ν) *and if we subject the* (z_ν) *to the cyclic transformation*

$$\begin{aligned} z_0' &= a_0 z_0 + a_1 z_1 + a_2 z_2 + \cdots + a_{k-1} z_{k-1} \\ z_1' &= a_{k-1} z_0 + a_0 z_1 + a_1 z_2 + \cdots + a_{k-2} z_{k-1} \\ &\vdots \\ &\vdots \\ z_{k-1}' &= a_1 z_0 + a_2 z_1 + a_3 z_2 + \cdots + a_0 z_{k-1}, \end{aligned}$$

then the finite Fourier coefficients of the new sequence (z_ν') *are*

$$\zeta_\nu' = \zeta_\nu f(e^{2\pi i \nu / k}),$$

where

$$f(z) = a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}.$$

This is actually a standard result in digital signal processing. Make the translation. Why is the result true?

Solution: Let's see, "finite Fourier coefficients ..." They must be talking about the DFT, and the ζ's are the "coefficients." The indexing goes from 0 to $k-1$ so it looks like the DFT of order k. We must have $\underline{z} = (z_0, z_1, \ldots, z_{k-1})$ as the input and $\underline{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_{k-1})$ as the output on applying the DFT, i.e.,

$$\underline{\zeta} = \mathcal{F}\underline{z}, \quad \zeta_\nu = \sum_{n=0}^{k-1} z_n e^{-2\pi i n \nu / k}.$$

We want to find $\underline{\zeta}' = \mathcal{F}\underline{z}'$ when $\underline{z}'$ is obtained from $\underline{z}$ by the indicated "cyclic transformation." We can write that relationship between the z_ν' and the z_ν as a matrix product:

$$\begin{pmatrix} z_0' \\ z_1' \\ z_2' \\ \vdots \\ z_{k-1}' \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-1} \\ a_{k-1} & a_0 & a_1 & \cdots & a_{k-2} \\ a_{k-2} & a_{k-1} & a_0 & \cdots & a_{k-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{k-1} \end{pmatrix}$$

The matrix, call it A, is *circulant*, and we know that the system

$$\underline{z}' = A\underline{z}$$

(being tine invariant) can be written as a convolution

$$\underline{z'} = \underline{h} * \underline{z}$$

where $\underline{h}$ is the first column of the matrix (convolution with the impulse response),

$$\underline{h} = \begin{pmatrix} a_0 \\ a_{k-1} \\ a_{k-2} \\ \vdots \\ a_1 \end{pmatrix}.$$

Then by the convolution theorem

$$\underline{\zeta'} = \mathcal{F}\underline{z'} = (\mathcal{F}\underline{h})(\mathcal{F}\underline{z}) = (\mathcal{F}\underline{h})\,\underline{\zeta},$$

so we have to see if the product on the right-hand side is the translation of what the author wrote, i.e.,

$$\zeta'_\nu = \zeta_\nu f(e^{2\pi i\nu/k}).$$

The DFT of $\underline{h}$ is

$$\mathcal{F}\underline{h}[\nu] = \sum_{n=0}^{k-1} \underline{h}_n e^{-2\pi i n\nu/k} = \sum_{n=0}^{k-1} a_{k-n} e^{-2\pi i n\nu/k}.$$

Note here that we have to regard $\underline{h}$ (as well as the other inputs and outputs) as being periodic of period k, so $a_k = a_0$, etc. According to this we have

$$\zeta'_\nu = \zeta_\nu \sum_{n=0}^{k-1} a_{k-n} e^{-2\pi i n\nu/k} = \zeta_\nu (a_0 + a_{k-1}e^{-2\pi i\nu/k} + a_{k-2}e^{-2\pi i 2\nu/k} + \cdots + a_1 e^{-2\pi i(k-1)\nu/k}).$$

Doesn't look quite like what the author has. But we can write

$$e^{-2\pi i n\nu/k} = e^{-2\pi i n\nu/k} e^{2\pi i k\nu/k} = e^{2\pi i(k-n)\nu/k}$$

and that brings the sum above into the form

$$\begin{aligned}\zeta'_\nu = \zeta_\nu \sum_{n=0}^{k-1} a_{k-n} e^{-2\pi i n\nu/k} &= \zeta_\nu (a_0 + a_{k-1}e^{-2\pi i\nu/k} + a_{k-2}e^{-2\pi i 2\nu/k} + \cdots + a_1 e^{-2\pi i(k-1)\nu/k}) \\ &= \zeta_\nu (a_0 + a_{k-1}e^{2\pi i\nu(k-1)/k} + a_{k-2}e^{2\pi i\nu(k-2)/k} + \cdots + a_1 e^{2\pi i\nu/k}) \\ &= \zeta_\nu f(e^{2\pi i\nu/k}),\end{aligned}$$

with

$$f(z) = a_0 + a_1 z + \cdots + a_{k-1}z^{k-1}.$$

Just as the author wrote!