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since $\underline{\mathcal{F}\,\mathrm{f}}[n] = 0$ if $n \in I'$. This will be 0 *for all* $\underline{\mathrm{f}} \in \mathbb{B}^I$ if and only if $\underline{\mathcal{F}\mathrm{g}}[n] = 0$ for all $n \in I$. This says that $\underline{\mathrm{g}}$ must be in $\mathbb{B}^{I'}$. Sybolically,

$$(\mathbb{B}^I)^\perp = \mathbb{B}^{I'}.$$

For the second question, to show that $K\underline{\mathrm{f}} = \underline{\mathrm{h}} * \underline{\mathrm{f}}$ defines the orthogonal projection onto $\mathbb{B}^I$ we have to do several things. First, if $\underline{\mathrm{f}}$ is any signal we have to show that $\underline{\mathrm{h}} * \underline{\mathrm{f}} \in \mathbb{B}^I$. For this, for any m the convolution theorem and the definition of $\underline{\mathrm{h}}$ gives

$$\begin{aligned} \underline{\mathcal{F}}(\underline{\mathrm{h}} * \underline{\mathrm{f}})[m] &= (\underline{\mathcal{F}\,\mathrm{h}}[m])(\underline{\mathcal{F}\,\mathrm{f}}[m]) \\ &= \begin{cases} \underline{\mathcal{F}\,\mathrm{f}}[m], & m \in I \\ 0, & m \in I' \end{cases} \end{aligned}$$

Thus $\underline{\mathrm{f}} * \underline{\mathrm{h}}$ is supported on I, i.e., $\underline{\mathrm{h}} * \underline{\mathrm{f}} \in \mathbb{B}^I$.

Second, if $\underline{\mathrm{f}}$ is already in $\mathbb{B}^I$ we should have $\underline{\mathrm{h}} * \underline{\mathrm{f}} = \underline{\mathrm{f}}$. But if $\underline{\mathrm{f}} \in \mathbb{B}^I$ then already $\underline{\mathcal{F}\,\mathrm{f}}[m] = 0$ for $m \in I$ and so by the definition of $\underline{\mathrm{h}}$,

$$(\underline{\mathcal{F}\,\mathrm{h}})(\underline{\mathcal{F}\,\mathrm{f}}) = \underline{\mathrm{f}}.$$

Taking the inverse DFT gives

$$\underline{\mathrm{h}} * \underline{\mathrm{f}} = \underline{\mathrm{f}}.$$

As an aside, another way to do this part of the problem is to observe that

$$(\underline{\mathcal{F}\,\mathrm{h}})(\underline{\mathcal{F}\,\mathrm{h}}) = \underline{\mathcal{F}\,\mathrm{h}},$$

hence

$$\underline{\mathrm{h}} * \underline{\mathrm{h}} = \underline{\mathrm{h}}.$$

Thus for any signal $\underline{\mathrm{f}}$,

$$K^2\underline{\mathrm{f}} = K(K(\underline{\mathrm{f}})) = \underline{\mathrm{h}} * (\underline{\mathrm{h}} * \underline{\mathrm{f}}) = (\underline{\mathrm{h}} * \underline{\mathrm{h}}) * \underline{\mathrm{f}} = \underline{\mathrm{h}} * \underline{\mathrm{f}} = K\underline{\mathrm{f}},$$

that is

$$K^2 = K,$$

which is the definition of a projection.

Why is this an orthogonal projection? If $\underline{\mathrm{f}}$ is in $\mathcal{B}^{I'}$, the orthogonal complement of $\mathbb{B}^I$, then $\underline{\mathcal{F}\,\mathrm{f}}[m] = 0$ for $m \in I$, hence, by definition of $\underline{\mathrm{h}}$,

$$(\underline{\mathcal{F}\,\mathrm{h}})(\underline{\mathcal{F}\,\mathrm{f}}) = 0,$$

whence

$$K\underline{\mathrm{f}} = \underline{\mathrm{h}} * \underline{\mathrm{f}} = 0.$$