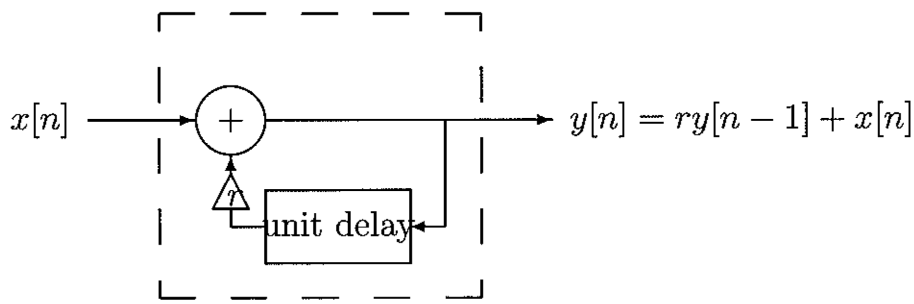
A signal representation of the type just considered,

$$x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi k f_0 n},$$

where f_0 is not a rational number, is an example of what is called a *generalized Fourier series* or an *almost periodic function.*
Suppose that such a signal is put into a discrete-time system as depicted below with $|r| < 1$:



Find a generalized Fourier series for $y[n]$.

Solution: The system is a linear system described by a linear difference equation with constant coefficients. There are many ways to find the output. The standard method from linear systems would be to recognize that (as in the case of Fourier series) the input signal is a linear combination of complex exponentials, and complex exponentials are eigenfunctions of linear systems. Together these facts lead to a solution for $y[n]$. Before going into details, the output must be in the form

$$y[n] = \sum_{k=0}^{K} a_k b_k e^{j2\pi k f_0 n}$$

and the problem is solved by finding b_k.
Observe that $y[n]$ can be found directly:

$$\begin{aligned}
y[n] &= ry[n-1] + x[n] \\
&= r(ry[n-2] + x[n-1]) + x[n] = r^2 y[n-2] + rx[n-1] + x[n] \\
&= r^3 y[n-3] + r^2 x[n-2] + rx[n-1] \\
&\vdots \\
&= \sum_{m=0}^{\infty} r^m x[n-m]
\end{aligned}$$

Therefore

$$\begin{aligned}
y[n] &= \sum_{m=0}^{\infty} r^m \left(\sum_{k=0}^{K} a_k e^{j2\pi k f_0 (n-m)} \right) \\
&= \sum_{k=0}^{K} a_k e^{j2\pi k f_0 n} \sum_{m=0}^{\infty} r^m e^{-j2\pi k f_0 m} \\
&= \sum_{k=0}^{K} a_k e^{j2\pi k f_0 n} \frac{1}{1 - re^{-j2\pi f_0}}
\end{aligned}$$