Qualifying Examinations

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1. **Expectation.**

Explain intuitively why expectation is linear.

2. Randomized Quicksort.

Consider a set of n numbers, S, that we wish to sort using the following randomized algorithm.

> **Input: The set of numbers, S.**
> **Output: Elements of S in increasing order.**
>
> **Steps:**
>
> **1. Choose y uniformly and at random from S.**
> **2. Compare every element with y to find: S_1, the set of elements smaller than y, and S_2, the set of elements larger than y.**
> **3. Recursively sort S_1 and S_2. Output S_1, y, S_2.**

In this question, we're going to find out how long the algorithm takes to run. We'll define the expected running time to be the *expected number of comparisons* needed to sort the set S. Let $S_{(1)}$ be the smallest element of S, $S_{(n)}$ the largest, and $S_{(i)}$ the ith smallest.

(a) Consider two elements $S_{(i)}$ and $S_{(j)}$. How many times can they be compared to each other by the algorithm?

(b) Define X_{ij} to be equal to 1 if elements $S_{(i)}$ and $S_{(j)}$ are compared by the algorithm, and 0 otherwise. Write down an expression for the total number of comparisons in terms of X_{ij}.

(c) Find an expression for the expected number of comparisons as an expression of $E[X_{ij}] \equiv p_{ij}$. .

In what follows, we'll find an expression for p_{ij}.

(d) Draw a binary decision tree, rooted on y, that shows the sets evolving at each stage of the recursion. Show the set with smaller elements to the left. What order do we need to read out the elements in the tree to output them in sorted order?

(e) Pick two elements from S at random. Under what conditions will they be compared during the execution of the algorithm?