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2016 Ph.D. Qualifying Examination

(There were three problems that I posed one after another. I did not expect anyone to do the last problem but one student did everything.)

1. Given a non-negative integer k, and a positive integer $n \geq k$, let $P = \{(x_1, \ldots, x_n) : 0 \leq x_i \leq 1, \sum_i x_i = k\}$. ($P$ is the subset of the n-dimensional unit cube on which the coordinates sum to k.) Suppose $c_1 \geq \cdots \geq c_n$ are real numbers. Find

$$\max_{x \in P} \sum_{i=1}^{n} c_i x_i.$$

2. Suppose $X \in \mathbb{C}^{n \times k}$, $k \leq n$, and $X^\dagger X = I_k$, that is, the columns of X are orthonormal in $\mathbb{C}^n$. Show that

$$\|X^\dagger y\|^2 \leq \|y\|^2 \quad \text{for all } y \in \mathbb{C}^n.$$

Conclude that $0 \leq (XX^\dagger)_{ii} \leq 1$ for all $1 \leq i \leq n$.

3. Suppose A is a Hermitian matrix. We know that such a matrix can be written as $A = U\Lambda U^\dagger$ where U is unitary and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, with $\lambda_i \in \mathbb{R}$. Without loss of generality, assume that we have permuted the rows and columns of A so that $a_{11} \geq a_{22} \geq \cdots \geq a_{nn}$, and we have indexed the eigenvalues so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) Show that $\max\limits_{x \in \mathbb{C}^n : x^\dagger x = 1} x^\dagger A x = \lambda_1$.

(b) Show that $a_{11} \leq \lambda_1$.

(c) Show that for any $k = 1, \ldots, n$,

$$\max_{X \in \mathbb{C}^{n \times k} : X^\dagger X = I_k} \mathrm{tr}(X^\dagger A X) = \max_{X \in \mathbb{C}^{n \times k} : X^\dagger X = I_k} \mathrm{tr}(X^\dagger \Lambda X) = \sum_{i=1}^{k} \lambda_i.$$

[Hint: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and you may find the previous two problems to be useful.]

(d) Show that for any $k = 1, \ldots, n$, $\sum_{i=1}^k a_{ii} \leq \sum_{i=1}^k \lambda_i$, and that equality holds when $k = n$.