Robert Gray

January 2007 Quals

Given

- Θ is a random variable described by a probability density function $f_\Theta(\theta) = 1/2\pi$ for $0 \le \theta < 2\pi$.
- $\{X_n;\ n = 0, 1, 2, \cdots\}$ is a discrete-time random process defined by $X_n = e^{jn\Theta}$.
- Fix a positive integer N and define

$$Y_k = \sum_{n=0}^{N-1} X_n e^{\frac{-j2\pi kn}{N}};\ k = 0, 1, \ldots, N-1.$$

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1. Evaluate the mean $E(X_n)$ and autocorrelation function $R_X(n,k) = E(X_n X_k^*)$.
2. Is X_n stationary?
3. Evaluate or approximate the following sums assuming that N is very large:

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n \qquad \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} X_n$$

$$\frac{1}{N}\sum_{k=0}^{N-1} Y_k \qquad \frac{1}{N}\sum_{k=0}^{N-1} |Y_k|^2 .$$

4. Suppose that we redefine X_n as $X_n = e^{jn\Theta_n}$ where now the Θ_n are independent identically distributed uniform random variables on $[0, 2\pi)$. Which of the above answers *change*?

Solutions

In all cases points are approximate, as more points could be awarded for clever solutions and fewer for meandering solutions or detours.

1. The first problem was intended to test basic probability skills. It counted for about 1.5 points out of the 10 as it is very elementary probability.

$$\begin{aligned} E(X_n) &= E(e^{jn\Theta}) \\ &= \int f_\Theta(\theta) e^{jn\theta}\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2\pi} e^{jn\theta}\, d\theta = \begin{cases} 1 & n = 0 \\ 0 & n = 1, 2, \cdots \end{cases} \\ R_X(n,k) &= E(X_n X_k^*) \\ &= E(e^{jn\Theta} e^{-jk\Theta}) = E(e^{j(n-k)\Theta}) \\ &= \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases} \end{aligned}$$

Ideally the first integral was done by inspection since the integral is obviously 1 for $n = 0$ and the integral of a period of a complex exponential (or, equivalently, of a sine and cosine) is 0. Ideally the student would realize the second integral was identical to the first with $n - k$ replacing n and not redo the entire calculation.

2. The goal of the second problem was to find out what the student knew about stationarity. The problem counted for about 1/2 point, mainly for seeing this process is not stationary and why. The process cannot be stationary because the mean at time 0 is different from the mean at all other times. If the process were begun at time $n = 1$, however, this problem goes away. In that case, the fact that the autocorrelation depends only on the time difference would mean the process is weakly or wide-sense stationary.

3. Most students spent most of their time on this problem. The intent was that most students would spend the remaining time on this problem. The first sum counted for about 2.5 points and the remainder for about 1.5 points each or about 7 total.

   - $\frac{1}{N}\sum_{n=0}^{N-1} X_n$ if the law of large numbers holds, this should converge to the mean of the process, which is 0. Except for $n = 0$, the process is weakly stationary and it is an uncorrelated process, this is enough to ensure that the weak las of large numbers and hence the sum from $n = 1$ will converge in both mean-square and in probability to the common mean, which is 0. The $n = 0$ term does not effect the limit, so the answer is 0. Some students remembered this fact and a few derived the result as follows:

$$
\begin{aligned}
E\left[\left(\frac{1}{N}\sum_{n=0}^{N-1} X_n\right)^2\right] &= (\frac{1}{N})^2 \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} R_X(n,k) \\
&= (\frac{1}{N})^2 \sum_{n=0}^{N-1} R_X(n,n) = \frac{1}{N} \to 0.
\end{aligned}
$$

     Convergence in probability follows from this result and the Tchebychev inequality. Alternately and equally good was to use the method described in the next item and simply directly prove that the sum goes to zero by evaluating it using the geometric progression.

   - The $\sqrt{N}$ term instead of the N term in the denominator was intended to make the student think of the central limit theorem, but that does not work here because this process does not meet any of the conditions required for the central limit theorem to hold. So here something different is needed, and the trick is to try to find the sum exactly. Here the geometric progression can be used to write

$$
\sum_{n=0}^{N-1} X_n = \sum_{n=0}^{N-1} e^{jn\Theta} = \frac{1 - e^{j\Theta(N+1)}}{1 - e^{j\Theta}}
$$

     With probability 1 Θ will take on a sample value that is not 0, so the denominator has some fixed value independent of N and the numerator is a well behaved function of N that can never have magnitude greater than 2. Thus dividing by $\sqrt{N}$ in the denominator will drive $(1/\sqrt{N})\sum_{n=0}^{N-1} X_n$ to zero.
     If this method were used for the first sum, then this part was an obvious variation.

   - $\frac{1}{N}\sum_{k=0}^{N-1} Y_k$ The fast way to do this one is to realize that the Y_k constitute the DFT of the X_n, so this is just the inverse DFT for $n = 0$, which is 1.
     If the inverse DFT was not remembered, it could be derived as

$$
\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1} Y_k e^{\frac{j2\pi kn}{N}} &= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} X_m e^{\frac{-j2\pi km}{N}}\right) e^{\frac{j2\pi kn}{N}} \\
&= \sum_{m=0}^{N-1} X_m \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m)}{N}} = X_n
\end{aligned}
$$

since

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m)}{N}} = \begin{cases} 1 & n = m \text{ (modulo } N) \\ 0 & \text{otherwise} \end{cases}$$

- $\frac{1}{N}\sum_{k=0}^{N-1} |Y_k|^2$ This sum was intended to recall Parceval's equation for the DFT. So either from that relation directly or from

$$\begin{aligned} \frac{1}{N}\sum_{k=0}^{N-1} |Y_k|^2 &= \frac{1}{N}\sum_{k=0}^{N-1} Y_k Y_k^* \\ &= \frac{1}{N}\sum_{k=0}^{N-1} Y_k \left(\sum_{n=0}^{N-1} X_n e^{\frac{-j2\pi kn}{N}}\right)^* \\ &= \sum_{n=0}^{N-1} X_n^* \frac{1}{N}\sum_{k=0}^{N-1} Y_k e^{\frac{j2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} X_n^* X_n = \sum_{n=0}^{N-1} |X_n|^2 \end{aligned}$$

Since $X_n = e^{jn\Theta}$ has magnitude 1 for all Θ, the answer is just N.

4. The last problem counted for about 1 point. If $X_n = e^{jn\Theta_n}$, then X_n becomes iid. The mean of X_0 changes to 0, but the remaining means and the autocorrelation are unchanged. The process is now strictly stationary since all iid processes are. All of the sums remain as before except the second sum, which now converges in distribution to a Gaussian random vector from the central limit theorem (both the real and imaginary parts converge to Gaussian random variables).

January 2008

The questions are colored red.
Solutions to R.M. Gray's 2008 qualifying exam problem.

An ideal band-pass filter (BPF) with passband $\mathcal{B} = \{f : |f - f_c| \leq W\}$ is a linear system for which an input of the form $e^{j2\pi ft}$ produces an output of $Ae^{j2\pi f(t-t_0))}$ for $f \in \mathcal{B}$ and an output of 0 otherwise. f_c, W, A, and t_0 are system parameters.

Write a formula for the transfer function $H_{\text{BP}}(f) = \int_{-\infty}^{\infty} h_{\text{BP}}(t)e^{-j2\pi ft}$ where $h_{\text{BP}}(t)$ is the impulse response of the ideal BPF filter.

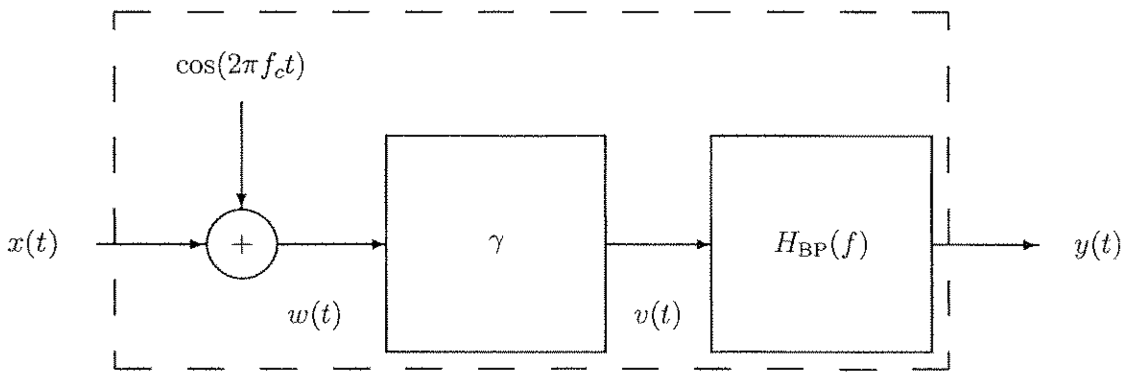
Is the filter causal?

Solution The solution I was looking for was this: You are told the system is linear and what the input/output relation is for complex exponentials. You can either assume it is also time-invariant or argue it is time invariant since you are told the system has an impulse response that depends on only a single time argument. (Or you could ask.) Since a complex exponential is an eigenvalue of an LTI system, an input time signal $e^{j2\pi ft}$ yields an output $H(f)e^{j2\pi ft}$, which you are told is $Ae^{j2\pi f(t-t_0)}$ for f in the pass band and 0 outside. Hence

$$H_{\text{BP}}(f) = \begin{cases} \frac{Ae^{j2\pi f(t-t_0))}}{e^{j2\pi ft}} = Ae^{-j2\pi ft_0} & f \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

There were many more complicated ways to do this, but recognizing the input as a complex exponential and hence an eigenvalue of the system with $H(f)$ as the eigenvalue was the quickest. The inverse Fourier transform will give the impulse response, which is a modulated sinc function, which is not causal. (To be causal, the impulse response must be 0 for negative time.) I was not after detailed analysis here, rather I wanted to see what people could infer from the form of $H_{\text{BP}}(f)$ without grinding through the computation. An even more direct answer was to observe that since the spectrum is bandlimited and symmetric, its inverse Fourier transform could not be 0 for all $t \leq 0$.

Consider the following system:



where

$$w(t) = x(t) + \cos(2\pi f_c t)$$

$$v(t) = \gamma(w(t)) \quad , \quad \gamma(w) = a_0 + a_1 w + a_2 w^2$$

$H_{\mathrm{BP}}(f)$ as before
Require

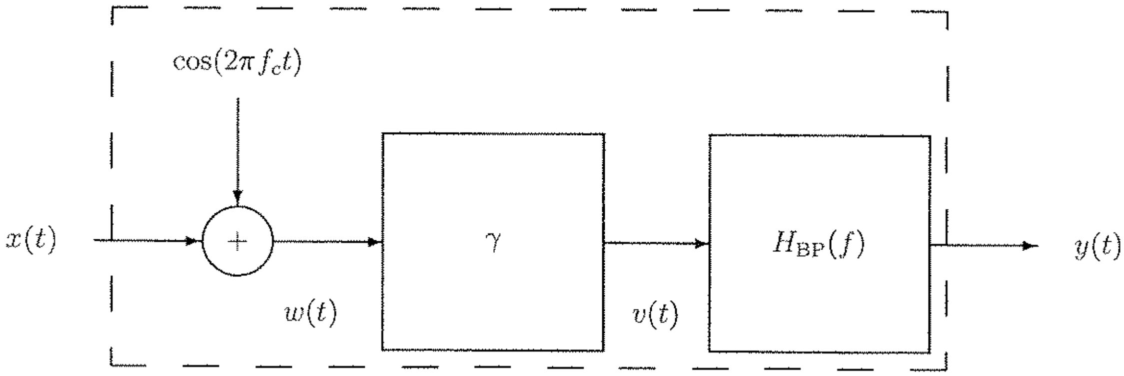
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt = 0 \text{ for } \begin{cases} |f| \geq W \\ f = 0 \end{cases}$$

(bandlimited to $(-W, W)$ and no DC)

- Is this system time-invariant? linear?

Solution
Here I looked for one of two approaches. Either a student started trying to find the output $y(t)$ to see if the system was linear or time-invariant. When this approach was clear, I moved ahead to the next question to focus on $y(t)$ first. The other approach was to look at the system components and either argue for a property or say it looked like a property held or not. In this case the answers I sought were (1) that the system is probably not time-invariant because of the cosine, and probably not linear because it has nonlinear components (and also because of the cosine term that will get through the nonlinearity and bandpass filter). To answer this question definitively, you really need to find $y(t)$. This question usually served as a warmup for the next.

Continue with the system



$$w(t) = x(t) + \cos(2\pi f_c t)$$

$$v(t) = \gamma(w(t)) \quad , \quad \gamma(w) = a_0 + a_1 w + a_2 w^2$$

$H_{\mathrm{BP}}(f)$ as before

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt = 0 \text{ for } \begin{cases} |f| \geq W \\ f = 0 \end{cases}$$

(bandlimited to $(-W, W)$ and no DC)

- Find a simple expression for $y(t)$.

Solution

$$\begin{aligned} v(t) &= a_0 + a_1\,(x(t) + \cos(2\pi f_c t)) \\ &\quad + a_2\,(x(t) + \cos(2\pi f_c t))^2 \\ &= \underbrace{a_0 + a_1 x(t) + a_2 x(t)^2}_{baseband} \\ &\quad + \underbrace{a_1 \cos(2\pi f_c t) + 2a_2 x(t)\cos(2\pi f_c t)}_{passband} \\ &\quad + a_2 \cos(2\pi f_c t)^2 \end{aligned}$$

Since $\cos(2\pi f_c t)^2 = (1 + \cos(4\pi f_c t))/2$, this is

$$\begin{aligned} v(t) &= \underbrace{a_0 + a_2/2 + a_1 x(t) + a_2 x(t)^2}_{baseband} \\ &\quad + \underbrace{a_1 \cos(2\pi f_c t) + 2a_2 x(t)\cos(2\pi f_c t)}_{passband} \\ &\quad + \underbrace{(a_2/2)\cos(4\pi f_c t))}_{highband} \end{aligned}$$

This can also be expressed in the Fourier domain. Note that $x(t)^2$ occupies a frequency band of $(-2W, 2W)$ so we need at least $f_c \geq 3W$.
Passing $v(t)$ through the bandpass filter will produce a delayed and scaled version of the passband signal,

$$y(t) = Aa_1 \cos(2\pi f_c(t - t_0)) + 2Aa_2 x(t - t_0) \cos(2\pi f_c(t - t_0))$$

a classical AM modulated signal. This is clearly not time invariant, but it is a linear system if $a_1 = 0$.
I included the complete expansions for completeness, but most people only wrote down the terms that survive the BPF.

Suppose now that a signal $y(t) = (c_0 + c_1 x(t)) \cos(2\pi f_c t + \theta)$ is received, but θ is unknown to the receiver.

- Can the signal $x(t)$ be recovered from $y(t)$ using only LTI filtering?

Solution

- No, to recover (or demodulate) $x(t)$ new frequencies must be introduced, which cannot occur with LTI systems. The system must be either nonlinear or time varying.

As before:
$y(t) = (c_0 + c_1 x(t))\cos(2\pi f_c t + \theta)$ is received, but θ is unknown to the receiver.

- Can the signal $x(t)$ be recovered from $\gamma(y(t))$, $\gamma(w) = a_0 + a_1 w + a_2 w^2$, and LTI filtering?

Solution

- $$\begin{aligned}
&\gamma(y(t)) \\
&\quad = a_0 + a_1\left[(c_0 + c_1 x(t))\cos(2\pi f_c t + \theta)\right] \\
&\qquad + a_2\left[(c_0 + c_1 x(t))\cos(2\pi f_c t + \theta)\right]^2 \\
&\quad = \underbrace{a_0 + \frac{a_2}{2}(c_0^2 + 2c_1 x(t) + x(t)^2)}_{baseband} \\
&\qquad + \underbrace{a_1\left[(c_0 + c_1 x(t))\cos(2\pi f_c t + \theta)\right]}_{passband} \\
&\qquad + \underbrace{\frac{a_2}{2}(c_0 + c_1 x(t))^2\cos(4\pi f_c t + \theta)}_{highpass}
\end{aligned}$$

  The bandpass and highpass information can be knocked out by a low pass filter, and they cannot be brought down to baseband by a LTI. So only the baseband terms can be used. There the $x(t)^2$ covers the $x(t)$, so there is no way in general to recover $x(t)$ alone from this signal using only LTI filtering.

$\boxed{\text{January 2009}}$

The questions are colored red.
Solutions to R.M. Gray's 2009 qualifying exam problem.

You are told that a discrete-time complex signal $x[n]$ for integer n has a Fourier series representation

$$x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi \frac{k}{K} n}$$

where $j = \sqrt{-1}$.
In particular, for this question an integer K and complex numbers $a_k, k = 0, 1, \ldots, K-1$ are given.

Evaluate the long-term time averages

$$\begin{aligned} < x[n] > &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ < |x[n]|^2 > &= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 \end{aligned}$$

Solution
There are several ways to solve the problem.
First method
Those who remember their discrete time Fourier series will know that the $x[n]$ is periodic in n with period K and hence

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \frac{1}{K} \sum_{n=0}^{K-1} x[n]$$

and that the Fourier coefficients are given by

$$a_k = \frac{1}{K} \sum_{n=0}^{K-1} x[n] e^{-j2\pi \frac{k}{K} n} \tag{1}$$

and hence

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \frac{1}{K} \sum_{n=0}^{K-1} x[n] = a_0$$

If necessary, the formula for the coefficients could be derived e.g., as follows:

$$\begin{aligned} \sum_{n=0}^{K-1} x[n] e^{-j2\pi \frac{k}{K} n} &= \sum_{n=0}^{K-1} \left(\sum_{\ell=0}^{K-1} a_\ell e^{j2\pi \frac{\ell}{K} n} \right) e^{-j2\pi \frac{k}{K} n} \\ &= \sum_{\ell=0}^{K-1} a_\ell \sum_{n=0}^{K-1} e^{j2\pi \frac{\ell}{K} n} e^{-j2\pi \frac{k}{K} n} \\ &= \sum_{\ell=0}^{K-1} a_\ell \sum_{n=0}^{K-1} e^{j2\pi \frac{1}{K} n(\ell - k)} \end{aligned}$$

But

$$\sum_{n=0}^{K-1} e^{j2\pi \frac{n}{K} m} = \begin{cases} K & m = 0 \bmod K \\ \frac{1 - e^{j2\pi \frac{K}{K} m}}{1 - e^{j2\pi m/K}} = 0 & \text{otherwise} \end{cases} \tag{2}$$

and therefore

$$a_k = \frac{1}{K}\sum_{n=0}^{K-1} x[n]e^{-j2\pi\frac{k}{K}n}$$

Proving (1).

Second method Just plugging in and manipulating using the standard Fourier methods of interchanging summations with each other and limits works quite well here. This is what I looked for when students did not remember the basic properties of Fourier series.

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} x[n] &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{k=0}^{K-1} a_k e^{j2\pi\frac{k}{K}n}\right)\\ &= \sum_{k=0}^{K-1} a_k\left(\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{k}{K}n}\right)\end{aligned}$$

Here two ways are possible: you can recognize that since complex exponentials of this form are periodic with period K,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{k}{K}n} = \frac{1}{K}\sum_{n=0}^{K-1} e^{j2\pi\frac{k}{K}n} = \begin{cases}1 & k = 0 \bmod K\\ 0 & \text{otherwise}\end{cases}$$

using (2), which implies that the result is a_0, or you can use the geometric progression to evaluate the limit (which saves time on the second question):

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{k}{K}n} = \lim_{N\to\infty}\frac{1}{N}\left(\begin{cases}N & k = 0 \bmod K\\ \frac{1-e^{j2\pi\frac{k}{K}N}}{e^{j2\pi\frac{k}{K}N}} & \text{otherwise}\end{cases}\right) = \begin{cases}1 & k = 0 \bmod K\\ 0 & \text{otherwise}\end{cases}$$

which implies that the result is a_0.
Some students wrote down the geometric series formula to do the sum of the N exponential terms, but forgot that the formula requires a nonzero denominator (k cannot be 0). This led to the incorrect conclusion that the answer was 0 because the formula is bounded and the $1/N \to 0$. I gave a hint to the effect that the answer could not be correct and suggested the consider the $K = 1$ case, where $N^{-1}\sum_{n=0}^{N-1} x[n] = N^{-1}\sum_{n=0}^{N-1} a_0 = a_0$. Ideally people realized that by linearity this gave them the complete answer, their geometric progression formula forces the other terms to 0 and all that is left is a_0.
Some students tried to apply DTFTs here, which do not work since the signal does not have finite power. Others correctly recognized a connection with DFTs, but could not relate the DFTs to the Fourier coefficients correctly to get the answer.

The second sum is the time average power and it follows from Parseval's relation or by direct computation, again using (1): First, since the signal is periodic, the limit as $N \to \infty$ is given by the finite sum

$$\begin{aligned}
\frac{1}{K}\sum_{n=0}^{K-1}|x[n]|^2 &= \frac{1}{K}\sum_{n=0}^{K-1}x[n]\left(\sum_{k=0}^{K-1}a_k e^{j2\pi\frac{k}{K}n}\right)^* \\
&= \frac{1}{K}\sum_{k=0}^{K-1}a_k^*\sum_{n=0}^{K-1}x[n]e^{-j2\pi\frac{k}{K}n} \\
&= \frac{1}{K}\sum_{k=0}^{K-1}|a_k|^2
\end{aligned}$$

This can also be done with more work by plugging in the Fourier series representation for $x[n]$, taking the magnitude squared, and manipulating the sums and limits.

Before: $x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi \frac{k}{K} n}$

Next, you are told that $x[n]$ instead has the following form for integer n:

$$x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi k f_0 n}$$

where f_0 is a fixed real number.

Evaluate the time averages

$$\begin{aligned} < x[n] > &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x[n] \\ < |x[n]|^2 > &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 \end{aligned}$$

Solution

If f_0 is rational, the answer is the same as before. So the question is what happens when f_0 is irrational. This case occurs in modeling real-world systems, such as A/D converter error with sinusoidal inputs.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{k=0}^{K} a_k e^{j2\pi k f_0 n} \right) = \sum_{k=0}^{K} a_k \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi k f_0 n} \right)$$

As in (2)

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi k f_0 n} = \begin{cases} 1 & k f_0 = 0 (\bmod N) \\ \frac{1}{N} \frac{1-e^{j2\pi k f_0 N}}{1-e^{j2\pi k f_0}} & \text{otherwise} \end{cases}$$

If f_0 is irrational, then the first case only occurs for $k = 0$. The second case converges to 0 as $N \to \infty$. Therefore

$$< x[n] >= a_0$$

as in the periodic case.

Altenatively, by linearity you can separately consider what happens for each k in the sum. As in the periodic case, the $k = 0$ term immediately gives a_0 and the limit is trivial. The remaining terms are summing rotations around the circle and then dividing by N. The sum is bounded and the N blows up, so hey all go to zero, leaving only a_0.

The average power follows similarly:

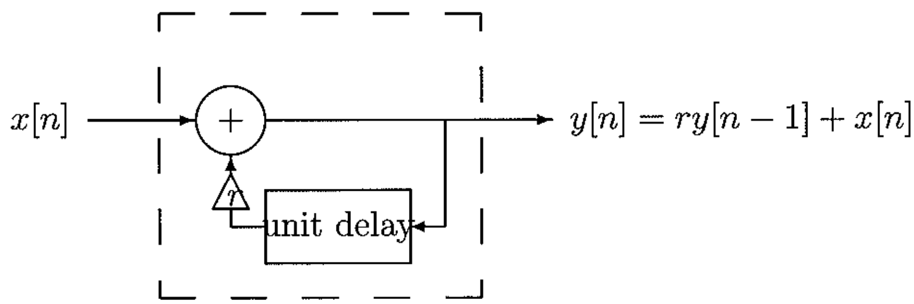
$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{\ell=0}^{K}a_\ell e^{j2\pi k\ell f_0 n}\right)\left(\sum_{k=0}^{K}a_k e^{j2\pi k f_0 n}\right)^* \\
&= \lim_{N\to\infty}\sum_{\ell=0}^{K}\sum_{k=0}^{K}a_\ell a_k^*\left(\frac{1}{N}\sum_{n=0}^{N-1}e^{j2\pi k\ell f_0 n}e^{-j2\pi k f_0 n}\right) \\
&= \lim_{N\to\infty}\sum_{\ell=0}^{K}\sum_{k=0}^{K}a_\ell a_k^*\left(\frac{1}{N}\sum_{n=0}^{N-1}e^{j2\pi k(\ell-k)f_0 n}\right) \\
&= \lim_{N\to\infty}\sum_{k=0}^{K}|a_k|^2
\end{aligned}$$

A signal representation of the type just considered,

$$x[n] = \sum_{k=0}^{K-1} a_k e^{j2\pi k f_0 n},$$

where f_0 is not a rational number, is an example of what is called a *generalized Fourier series* or an *almost periodic function.*

Suppose that such a signal is put into a discrete-time system as depicted below with $|r| < 1$:



Find a generalized Fourier series for $y[n]$.

Solution: The system is a linear system described by a linear difference equation with constant coefficients. There are many ways to find the output. The standard method from linear systems would be to recognize that (as in the case of Fourier series) the input signal is a linear combination of complex exponentials, and complex exponentials are eigenfunctions of linear systems. Together these facts lead to a solution for $y[n]$. Before going into details, the output must be in the form

$$y[n] = \sum_{k=0}^{K} a_k b_k e^{j2\pi k f_0 n}$$

and the problem is solved by finding b_k.

Observe that $y[n]$ can be found directly:

$$\begin{aligned}
y[n] &= ry[n-1] + x[n] \\
&= r(ry[n-2] + x[n-1]) + x[n] = r^2 y[n-2] + rx[n-1] + x[n] \\
&= r^3 y[n-3] + r^2 x[n-2] + rx[n-1] \\
&\vdots \\
&= \sum_{m=0}^{\infty} r^m x[n-m]
\end{aligned}$$

Therefore

$$\begin{aligned}
y[n] &= \sum_{m=0}^{\infty} r^m \left(\sum_{k=0}^{K} a_k e^{j2\pi k f_0 (n-m)} \right) \\
&= \sum_{k=0}^{K} a_k e^{j2\pi k f_0 n} \sum_{m=0}^{\infty} r^m e^{-j2\pi k f_0 m} \\
&= \sum_{k=0}^{K} a_k e^{j2\pi k f_0 n} \frac{1}{1 - re^{-j2\pi f_0}}
\end{aligned}$$

Alternatively, if a signal $e^{j2\pi k f_0 n}; n = 0, 1, 2, \ldots$ is put into the system, the output must be $H(kf_0)e^{j2\pi k f_0 n}$, where $H(kf_0)$ is the discrete-time Fourier transform of the Kronecker delta (discrete-time impulse) response $h_k = r^k$, $k = 0, 1, 2, \ldots$:

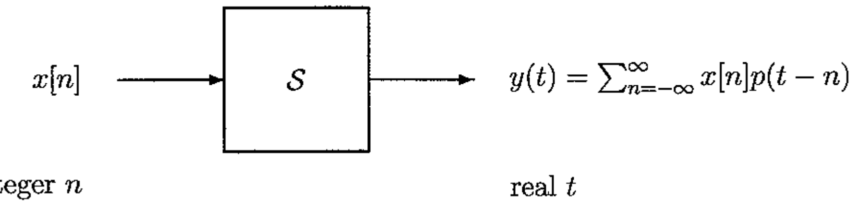
$$H(kf) = \sum_{n=0}^{\infty} r^k e^{-j2\pi k f_0} = \frac{1}{1 - re^{-j2\pi f_0}},$$

which yields the same answer.

$\boxed{\text{January 2010}}$

The questions are colored red.
Solutions to R.M. Gray's 2010 qualifying exam problem.

The following system is useful as a model in pulse amplitude modulation (PAM) systems and digital-to-analog (D/A) converters:



where $p(t)$ is a real-valued continuous-time signal satisfying

$$\int_{-\infty}^{\infty} p(t)p(t-n)dt = \delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{all nonzero integers} \end{cases} \tag{1}$$

Define the discrete-time Fourier transform (DTFT) of a signal $x[n]$ by

$$X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}; -\frac{1}{2} \leq f \leq \frac{1}{2}$$

and the continuous-time Fourier transform (CTFT) of a signal $y(t)$

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt; -\infty < f < \infty,$$

where $j = \sqrt{-1}$.

First Question:
Find a *simple* relationship between $Y(f)$ and $X(f)$.

Solution This was intended as a straightforward start using standard Fourier proof techniques — substitute (plug in) the definition of the signal to the definition of the transform $Y(f)$ asked for, interchange the order of summation and integral, and then simplify.

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x[n]p(t-n)\right] e^{-j2\pi ft}dt \\ &= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} p(t-n)e^{-j2\pi ft}dt = \sum_{n=-\infty}^{\infty} x[n]P(f)e^{-j2\pi fn} \end{aligned}$$

where the last step is the usual Fourier shift theorem for CT signals (or just change variables in the integral). Thus

$$Y(f) = X(f)P(f).$$

A tricky point here is that $Y(f)$ should be defined for all real f, but $X(f)$ was defined only for f in $[-1/2, 1/2]$. The formula makes sense, however, if we take $X(f)$ to be the periodic extension, that is, just use the sum in the DTFT definition for all real f.

Many people complicated the problem by trying to convert the DT signal into a CT signal using impulse trains. This way leads to the answer, but it makes things much more complicated. Some people observed correctly that the left hand side resembles a convolution and tried to quote the convolution theorem, but here the "convolution" is discrete time while the output signal is continuous time, so the usual convolution theorems (for DT or CT) do not directly apply.

Recall that $p(t)$ is real and

$$\begin{aligned} \int_{-\infty}^{\infty} p(t)p(t-n)dt &= \delta_n && (1) \\ P(f) &= \int_{-\infty}^{\infty} p(t)e^{-j2\pi ft}dt \end{aligned}$$

Second Question:
Find a *simple* expression for

$$\int_{-\infty}^{\infty} |P(f)|^2 e^{j2\pi fn} df$$

Solution There are *many* ways to do this problem.
The most straightforward approach is the standard Fourier proof method of substitution and interchanging order of integration.

$$\begin{aligned} \int_{-\infty}^{\infty} |P(f)|^2 e^{j2\pi fn} df &= \int_{-\infty}^{\infty} P(f) \left[\int_{-\infty}^{\infty} p(t)e^{-j2\pi ft}dt \right]^* e^{j2\pi fn} df \\ &= \int_{-\infty}^{\infty} p(t) \left[\int_{-\infty}^{\infty} P(f)e^{j2\pi f(t+n)}df \right] dt \\ &= \int_{-\infty}^{\infty} p(t)p(t+n)dt \end{aligned}$$

where we have used the Fourier inversion formula to recover p from P. From (1) the answer is

$$\int_{-\infty}^{\infty} |P(f)|^2 e^{j2\pi fn} df = \delta_n$$

Some people got bogged down by substituting the time domain intergral for both occurrences of $P(f)$, which is messier because of the triple integration. I tried to warn people who took a path that was likely to get tangled in details.
A shortcut to the answer is to recognize the integral as the continuous-time inverse Fourier transform of $|P(f)|^2$ evaluated at time n and that $|P(f)|^2$ is the transform of the CT autocorrelation of p,

$$r_p(\tau) = \int_{-\infty}^{\infty} p(t)p(t-\tau)dt$$

(from the correlation theorem for continuous time Fourier transforms), which when evaluated at an integer time yields the Kronecker delta δ_n from (1).
Equivalently, the integral asked for is the integral of the product of $P(f)$ and $P^*(f)e^{j2\pi fn} = \left(P(f)e^{-j2\pi fn}\right)^*$. From the generalized Parseval's theorem this is the integral in the time domain of the product of the inverse Fourier transforms of these signals, which are $p(t)$ and $p^*(t-n) = p(t-n)$, which from (1) is the Kronecker delta δ_n.
Several people tried another short cut that does not work. They correctly recognized (1) as an autocorrelation and reasoned that therefore if they transformed both sides the left hand side should be $|P(f)|^2$ (the transform of a correlation) and the right hand side should be 1 (the transform of a delta function), thus $|P(f)|^2 = 1$ for "all" f. But the correlation is a continuous time correlation, while the equation is a discrete time relation – the delta function is a Kronecker

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Recall that

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Third Question:
Find a waveform $p(t)$ which has the desired property (1) and has the additional property that $P(t)$ *also* satisfies property (1); that is,

$$\int_{-\infty}^{\infty} P(t)P(t-n)dt = \delta_n \qquad (2)$$

Solution Here are two possible approaches:

1. Guess and show it works. There are not many signals p which satisfy (1), so it is easy to see if they also satisfy (2) if you either know or can find the CTFT.
2. Look at the formulas the signals must satisfy and find a solution.

First Approach: A simple signal which satisfies (1) is the box or rect function or square pulse

$$p_0(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Its Fourier transform is easily found (or recalled from memory) as

$$\begin{aligned}P_0(f) &= \int_{-1/2}^{1/2} e^{-j2\pi ft}dt = \left.\frac{e^{-j2\pi ft}}{-j2\pi ft}\right|_{-1/2}^{1/2}\\ &= \frac{e^{-j\pi f}}{-j\pi f} - \frac{e^{j\pi f}}{-j\pi f} = \frac{\sin(\pi f)}{\pi f}\end{aligned}$$

which is the sinc function, $P_0(f) = \text{sinc}(f)$. But these are also orthogonal as in (1) from Parseval's theorem and the modulation theorem:

$$\int_{-\infty}^{\infty} \text{sinc}(f-n)\,\text{sinc}(f)df = \int_{-\infty}^{\infty} p_0(t)e^{-j2\pi tn}p_0(t)dt = \int_{-1/2}^{1/2} e^{-j2\pi tn}dt = \delta_n.$$

Second Approach: Combining the Second Question with (2) yields

$$\int_{-\infty}^{\infty} |P(f)|^2 e^{j2\pi fn}df = \int_{-\infty}^{\infty} P(f)P(f-n)df = \delta_n$$

so $P(f)$ must satisfy

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$$P(f) = P_1(f) = \begin{cases} 1 & |f| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}.$$

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$$\boxed{\text{January 2010}}$$

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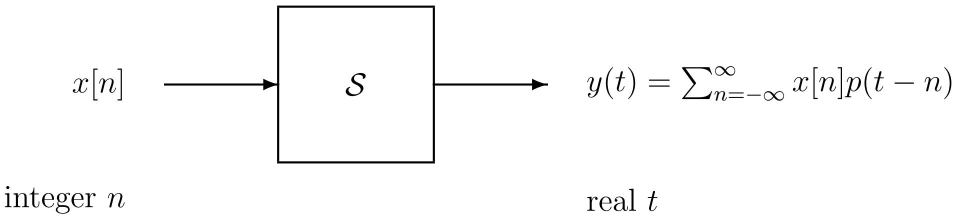

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January 2011

Solutions to R.M. Gray's 2011 qualifying exam problem.

Solutions include more information than was expected from the student during the exam, specific exams could emphasize one thread or another depending on the student's performance on the earlier material.

Several systems are described below by their input/output relations.
For each system:

1. is the system linear?
2. is the system time-invariant?
3. Is the system stable?
4. is the system invertible?

Systems:

- *Input:* $x(t)$; all real t

  Output: $y(t) = \int_{-\infty}^{t} x(\tau)e^{-\alpha(t-\tau)}d\tau$; all real t

- *Input:* $x(t)$; all real t

  Output: $y(t) = [a + mx(t)]\cos(2\pi f_0 t + \theta)$; all real t

- *Input:* $x(t)$; all real t

  Output: $y[n] = x(n)$; all integer n

- *Input:* $x[n]$; all integer n

  Output: satisfies difference equation $y[n] = ay[n-1] + x[n]$; all integer n. The system is assumed to be causal.

Solution

- The system is defined by a convolution, so it is linear and time invariant. The impulse response can be recognized as $h(t) = e^{-\alpha t}u(t)$, where $u(t)$ is the unit step function. The system is stable provided $\alpha > 0$, otherwise it is an integrator (and has a Fourier transform only in the generalized, limiting, sense). Assuming that $\alpha > 0$, The Fourier transform of the impulse response is
$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt = \frac{1}{\alpha + j2\pi f}.$$
The inverse filter will have transform $\alpha + j2\pi f$, which has inverse Fourier transform $\alpha\delta(t) + \delta'(t)$, a scaled identity plus differentiation.

- The system is ordinary amplitude modulation. It is linear if $a = 0$, but only affine otherwise. It is time varying because of the cosine. It is invertible if θ is known provided the positive bandwidth of $x(t)$, say W Hz, satisfies $W < f_0$. E.g., multiply by $\cos(2\pi f_0 t + \theta)$ and low pass filter and DC block. If θ is not known, then demodulation is still possible but takes more work.

- This is a sampling system. It is linear. The system is not time invariant because an input shift of an arbitrary amount does not correspond to an output shift by the same amount (unless the shift is by an integer). The system will be invertible if the sampling theorem holds, which means that the sampling frequency f_s (here 1) satisfies $f_s > 2W$, where W is the maximum frequency in the signal. Thus provided the Fourier transform of the signal is nonzero only for $f \in (-1/2, 1/2)$, the signal can be reconstructed from its samples using the sampling expansion or by low pass filtering the a signal with the samples imbedded on impulses.

- A difference equation defines a linear system, and the system is time invariant since the coefficients are. The system can be characterized as convolving the input with the response to a Kronecker delta $x[n] = \delta(n) = 1$ for $n = 0$, and 0 otherwise. Since the system is assumed causal, $y[n] = 0$ for $n < 0$ and hence $y[0] = 1$, $y[1] = a$, $y[2] = a^2$, etc. so that the response to a Kronecker delta is $h[n] = a^n$ for $n = 0, 1, 2, \ldots$ and 0 otherwise. The inverse filter has Kronecker delta response $g[n] = \delta[n] - a\delta[n-1]$, the discrete time convolution of h and g is δ. Alternatively, from the geometric series (assuming $|a| < 1$)

$$H(f) = \sum_{n=0}^{\infty} a^n e^{-j2\pi fn} = \frac{1}{1 - ae^{-j2\pi f}}$$

and the inverse filter is $1/H(f)$.

Alternatively, taking the transform of the difference equation and changing variables (or using the delay theorem)

$$\begin{aligned} Y(f) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j2\pi fn} \\ &= \sum_{n=-\infty}^{\infty} ay[n-1]e^{-j2\pi fn} + \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn} \\ &= \sum_{n'=-\infty}^{\infty} ay[n']e^{-j2\pi f(n'+1)} + X(f) \\ &= e^{-j2\pi f}Y(f) + X(f) \end{aligned}$$

so that

$$Y(f) = \frac{X(f)}{1 - e^{-j2\pi f}}$$

If $a = 1$, then the first argument still works and the system is still invertible, but the transform arguments get more complicated.

The continuous time Fourier transform (CTFT) of a continuous time signal $x(t)$ is
$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

The discrete time Fourier transform (DTFT) of a discrete time signal $x[n]$ is
$X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$
For the same set of systems, describe how the Fourier transform of the output can be determined from that of the input (using the appropriate type of Fourier transform)
Systems:

- *Input:* $x(t)$; all real t

  Output: $y(t) = \int_{-\infty}^{t} x(\tau)e^{-\alpha(t-\tau)}d\tau$; all real t

- *Input:* $x(t)$; all real t

  Output: $y(t) = [a + mx(t)]\cos(2\pi f_0 t + \theta)$; all real t

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- *Input:* $x[n]$; all integer n

  Output: satisfies difference equation $y[n] = ay[n-1] + x[n]$; all integer n. The system is assumed to be causal.

Solution: Some of these may have been derived in answering the first question, in which case they were skipped or just rephrased.

- $Y(f) = X(f)H(f)$ as before.

- $Y(f) = \frac{a}{2}(\delta(f-f_0) + \delta(f+f_0))e^{j\theta} + \frac{m}{2}[X(f-f_0) + X(f+f_0)]$

- If the signal is bandlimited to $[-1/2, 1/2]$, then DTFT $Y(f)$ of the sampled waveform is the same as that of the original waveform for $f \in [-1/2, 1/2]$, the only range of f needed for inversion. If f is allowed to range over the entire real line, then the DTFT has periodic replicas of $X(f)$ with period 1.

  A careful proof was not expected, I was more interested in either memory or intuition. A short proof is the following: If $X(f)$ is nonzero only in $[-1/2, 1/2]$, then in that region it can be expanded as a Fourier series in f as

  $$X(f) = \sum_{k=-\infty}^{\infty} a_k e^{-j2\pi kf};\, f \in [-1/2, 1/2]$$

  with

  $$a_k = \int_{-1/2}^{1/2} X(f)e^{j2\pi kf}df$$

where the signs are reversed from the usual convention because this is in the frequency domain. But from the bandlimited assumption,

$$a_k = \int_{-\infty}^{\infty} X(f) e^{j2\pi k f} df = x(k) = y[k]$$

so that

$$X(f) = \sum_{k=-\infty}^{\infty} y[k] e^{-j2\pi k f} = Y(f)$$

- As earlier,

$$Y(f) = \frac{X(f)}{1 - e^{-j2\pi f}}$$

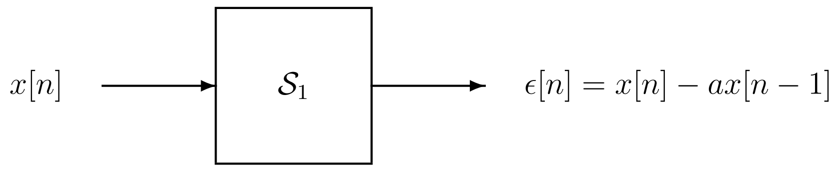
January 2012

$\boxed{\text{The questions are boxed for emphasis.}}$

Reminders of useful facts from previous slides are in blue.

Solutions to R.M. Gray's 2012 qualifying exam problem.
Solutions include much more information than was expected from the student during the exam, specific exams could emphasize one thread or another depending on the student's performance on the earlier material.

Consider the following discrete-time system with input $x[n]$ and output $\epsilon[n]$ defined by the input/output relation



Only causal signals are allowed, i.e., $x[n] = 0$ for all $n < 0$.

First Question: Given the following definitions and assumptions:

$$
\begin{aligned}
< x > &\triangleq \sum_{n=0}^{\infty} x[n] = 0 \quad \text{mean} \\
< x^2 > &\triangleq \sum_{n=0}^{\infty} x[n]^2 = \mathcal{E}_x < \infty \quad \text{energy} \\
r_x(k) &\triangleq \sum_{n=k}^{\infty} x[n]x[n-k] \quad \text{autocorrelation } (\mathcal{E}_x = r_x(0)) \\
X(f) &\triangleq \sum_{n=0}^{\infty} x[n]e^{-j2\pi fn}; -\frac{1}{2} \leq f \leq \frac{1}{2} \qquad \text{DTFT,}
\end{aligned}
$$

$$\boxed{\text{Find simple expressions for } < \epsilon >, \mathcal{E}_\epsilon, \text{ and the DTFT } E(f) \text{ of } \epsilon[n].}$$

Solution: Using the fact that $x[-1] = 0$ since inputs must be causal,

$$
\begin{aligned}
< \epsilon > &= \sum_{n=0}^{\infty} \epsilon[n] = \sum_{n=0}^{\infty} (x[n] - ax[n-1]) = \sum_{n=0}^{\infty} x[n] - a \sum_{n=0}^{\infty} x[\underbrace{n-1}_{n'}] \\
&= < x > -a \sum_{n'=0}^{\infty} x[n'] =< x > -a < x >= 0 \\
\mathcal{E}_\epsilon &= \sum_{n=0}^{\infty} \epsilon[n]^2 = \sum_{n=0}^{\infty} (x[n] - ax[n-1])^2 \\
&= \sum_{n=0}^{\infty} \left(x[n]^2 - 2ax[n]x[n-1] + a^2 x[n-1]^2\right) \\
&= r_x(0) - 2ar_x(1) + a^2 r_x(0) = (1 + a^2) r_x(0) - 2ar_x(1) \\
E(f) &= \sum_{n=0}^{\infty} \epsilon[n] e^{-j2\pi fn} = \sum_{n=0}^{\infty} (x[n] - ax[n-1]) e^{-j2\pi fn} \\
&= \sum_{n=0}^{\infty} x[n]e^{-j2\pi fn} - a \sum_{n=0}^{\infty} x[\underbrace{n-1}_{n'}] e^{-j2\pi fn} = X(f) - a \sum_{n'=0}^{\infty} x[n'] e^{-j2\pi f(n+1)} = X(f)(1 - ae^{-j2\pi f})
\end{aligned}
$$

or just quote linearity and the shift property of DTFTs

As before:

$$x[n] \longrightarrow \boxed{\mathcal{S}_1} \longrightarrow \epsilon[n] = x[n] - ax[n-1]$$

Autocorrelation $r_x(k) = \sum_{n=k}^{\infty} x[n]x[n-k]$

Error energy $\mathcal{E}_\epsilon = \sum_{n=0}^{\infty} \epsilon[n]^2 = \sum_{n=0}^{\infty} (x[n] - ax[n-1])^2$

Next Question:
Suppose $ax[n-1]$ is interpreted as a linear prediction of $x[n]$ based on a single past sample so that $\epsilon[n]$ is the linear prediction error sequence.

$$\boxed{\text{What value of } a \text{ minimizes } \mathcal{E}_\epsilon\text{?}}$$

Solution: Use calculus: Solve for

$$0 = \frac{d}{da}\mathcal{E}_\epsilon = \frac{d}{da}\big((1+a^2)r_x(0) - 2ar_x(1)\big) = 2ar_x(0) - 2r_x(1)$$

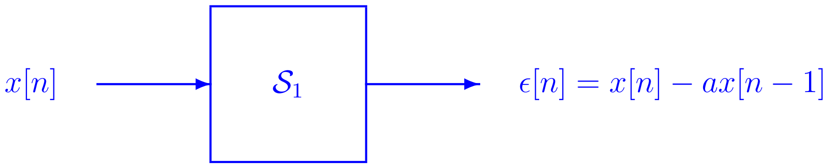
or $a = r_x(1)/r_x(0)$. Note that since $r_x(0) = \mathcal{E}_x \geq 0$, the second derivative satisfies

$$\frac{d^2}{da^2}\mathcal{E}_\epsilon \geq 0$$

so that the $a = r_x(1)/r_x(0)$ indeed minimizes $\mathcal{E}_\epsilon$. The resulting minimum is

$$\begin{aligned}
\mathcal{E}_\epsilon &= (1+a^2)r_x(0) - 2ar_x(1) = \left[1 + \left(\frac{r_x(1)}{r_x(0)}\right)^2\right] r_x(0) - 2\left(\frac{r_x(1)}{r_x(0)}\right) r_x(1) \\
&= r_x(0) + \frac{r_x(1)^2}{r_x(0)} - 2\frac{r_x(1)^2}{r_x(0)} = r_x(0) - \frac{r_x(1)^2}{r_x(0)}
\end{aligned}$$

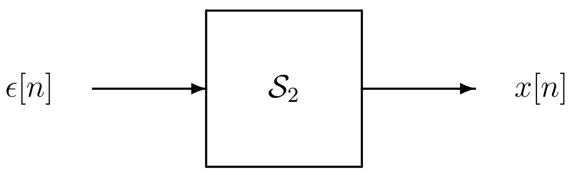
As before:



Notation: Kronecker delta function $\delta[n]$ for integer n defined by

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Next Question:
A system $\mathcal{S}_2$ is implied by the block diagram below.



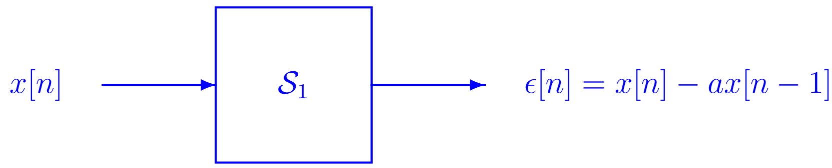
$$\boxed{\text{Find the response } g[n] \text{ of } \mathcal{S}_2 \text{ to an input } \delta[n].}$$

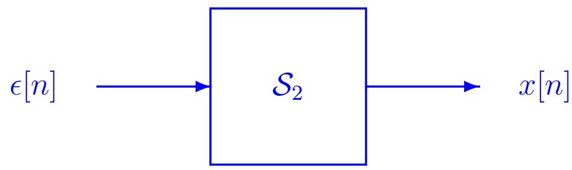
Solution: The question effectively asks for the inverse filter. There are many ways to approach this, some of which can get quite complicated. The simplest way to solve the problem is to directly find linear difference equations that produce $x[n]$ from $\epsilon[n]$. To do this just rewrite the equation for the output to put $x[n]$ alone on the left and iterate back to time 0:

$$\begin{aligned} x[n] &= \epsilon[n] + a \underbrace{x[n-1]}_{\epsilon[n-1]+ax[n-2]} \\ &= \epsilon[n] + a\epsilon[n-1] + a^2 \underbrace{x[n-2]}_{\epsilon[n-2]+ax[n-3]} \\ &\vdots \\ &= \epsilon[n] + a\epsilon[n-1] + a^2\epsilon[n-2] + \cdots + a^n\epsilon[0] \\ &= \sum_{k=0}^{n} \epsilon[k]a^{n-k} \end{aligned}$$

If the input is $\epsilon[k] = \delta[k]$, then the output at time n is $g[n] = a^n$. The first line of the above equations can also be used to identify the filter as a first-order *autoregressive* or *all-pole* filter defined by the linear difference equation $x[n] = \epsilon[n] + ax[n-1]$

As before:





Next Question:

$$\boxed{\text{What are the eigenvalues and eigenfunctions of the system } \mathcal{S}_1?}$$

$$\boxed{\text{of the system } \mathcal{S}_2?}$$

Solution: An eigenfunction $u[n]$ of a discrete-time system is a nonzero signal with the property that an input of $u[n]$ yields an output of $\lambda u[n]$, where λ is the associated eigenvalue. From linear systems theory, discrete-time time-invariant systems have complex exponentials as eigenvalues, that is, signals of the form

$$u[n] = e^{j2\pi fn}$$

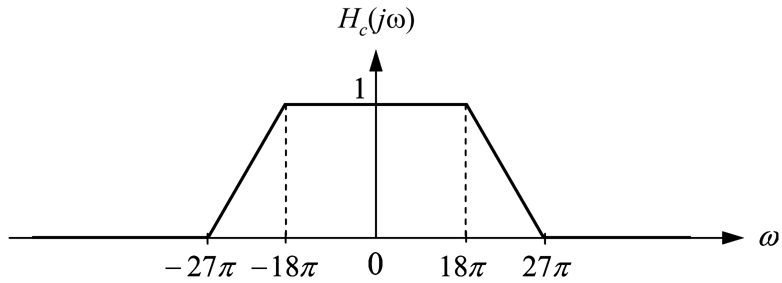
with a corresponding eigenvalue of $H(j2\pi f)$, where H is the DTFT of the Kronecker delta response (the *system function*) and where f is any real number in $[-1/2, 1/2]$. (Note that the eigenfunctions are not in the family of inputs considered up until now since they do not have finite energy.) Both $\mathcal{S}_1$ and $\mathcal{S}_2$ are LTI systems and hence have such eigenfunctions.
For $\mathcal{S}_1$, using the previously found expression for $E(f)$ yields $H(f) = E(f)/X(f) = 1 - e^{-j2\pi f}$. Alternatively, take the DTFT of the Kronecker delta response $h[n] = \delta[n] - a\delta[n-1]$ to get the same result.
For $\mathcal{S}_2$ you can either observe from the properties of linear systems that

$$G(f) = \frac{1}{H(f)} = \frac{1}{1 - e^{-j2\pi f}}$$

or you can compute $G(f)$ directly using the geometric progression and the geometric Kronecker delta response found earlier.
It is straightforward to verify directly that these eigenfunctions and eigenvalues work by plugging them into the specific linear difference equations describing the two systems.

Stanford University, Department of Electrical Engineering
Qualifying Examination, Winter 2011-12
Professor Joseph M. Kahn

Consider a continuous-time filter $h_c(t) \overset{FT}{\leftrightarrow} H_c(j\omega)$ having the frequency response shown below.



A sampling frequency $\omega_s = 2\pi/T = 48\pi$ rad/s is assumed. Using three different approaches, a discrete-time filter $h[n] \overset{DTFT}{\leftrightarrow} H(e^{j\Omega})$ is derived from the continuous-time filter. In each case, you are asked to sketch the magnitude response $|H(e^{j\Omega})|$ and answer a few questions. The discrete-time and continuous-time frequencies are related by $\Omega = \omega T$.

(a) An infinite impulse response filter $h_1[n] \overset{Z}{\leftrightarrow} H_1(z)$ is designed using the impulse invariance criterion:

$$h_1[n] = T \cdot h_c(t)\big|_{t=nT} \ .$$

Sketch the magnitude response $|H_1(e^{j\Omega})|$. Does aliasing occur?

(b) An infinite impulse response filter $h_2[n] \overset{Z}{\leftrightarrow} H_2(z)$ is designed using the bilinear transformation:

$$H_2(z) = H_c(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \ .$$

Sketch the magnitude response $|H_2(e^{j\Omega})|$. Does aliasing occur? The continuous-time frequencies $\omega_c = 18\pi$ and $\omega_s = 27\pi$ map to discrete-time frequencies Ω_c and Ω_s. Can you obtain expressions for Ω_c and Ω_s?

(c) A finite impulse response filter $h_3[n] \overset{Z}{\leftrightarrow} H_3(z)$ is designed by performing a Fourier series expansion of:

$$H_c\left(j\tfrac{\Omega}{T}\right)$$

over the frequency range $-\pi < \Omega < \pi$. (This is equivalent to performing a Fourier series expansion of $H_c(j\omega)$ over the range $-24\pi < \omega < 24\pi$.) Sketch the magnitude response $|H_3(e^{j\Omega})|$. Does aliasing occur? Will the Gibbs phenomenon be observed if $h_3[n]$ is not multiplied by a window function?

Solution

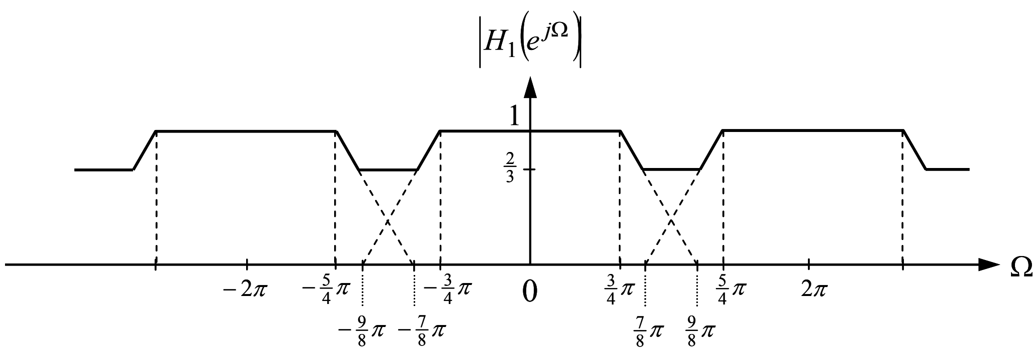
(a) Since $h_1[n] \overset{Z}{\leftrightarrow} H_1(z)$ is designed using the impulse invariance criterion, we have:

$$H_1\left(e^{j\omega T}\right) = \sum_{l=-\infty}^{\infty} H_c\left(j\left(\omega - \frac{l2\pi}{T}\right)\right),$$

In terms of discrete-time frequency Ω, we have:

$$H_1\left(e^{j\Omega}\right) = \sum_{l=-\infty}^{\infty} H_c\left(j\left(\frac{\Omega - l2\pi}{T}\right)\right).$$

A plot of $\left|H_1\left(e^{j\Omega}\right)\right|$ is shown below. Aliasing occurs because the sampling rate is less than twice the highest frequency in $H_c(j\omega)$.



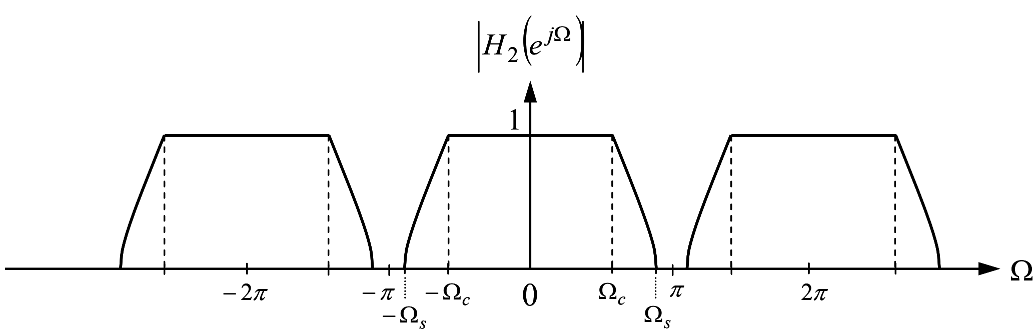
(b) Since $h_2[n] \overset{Z}{\leftrightarrow} H_2(z)$ is designed using bilinear transformation:

$$H_2\left(e^{j\Omega}\right) = H_c(j\omega),$$

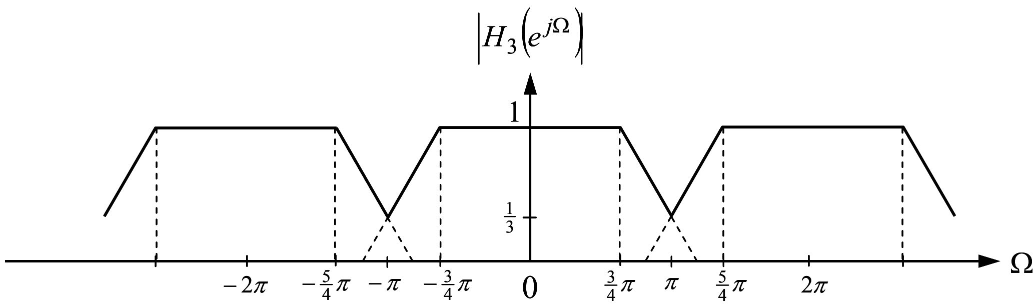
where the discrete-time frequency ω and continuous-time frequency Ω are related by:

$$\Omega = 2\tan^{-1}(\omega T/2).$$

Thus, the passband edge frequency $\omega_c = 18\pi$ maps to $\Omega_c = 2\tan^{-1}(3\pi/8)$, while the stopband edge frequency $\omega_s = 27\pi$ maps to $\Omega_s = 2\tan^{-1}(9\pi/16)$. A plot of $\left|H_2\left(e^{j\Omega}\right)\right|$ is shown below.



(c) Since $h_3[n] \overset{Z}{\leftrightarrow} H_3(z)$ is designed by frequency matching over the Nyquist bandwidth $H_3\left(e^{j\Omega}\right) = H_c\left(j\frac{\Omega}{T}\right)$, the magnitude response $\left|H_3\left(e^{j\Omega}\right)\right|$ is shown below. As in part (b), there is no aliasing. Since $\left|H_3\left(e^{j\Omega}\right)\right|$ has no discontinuities, there is no Gibbs phenomenon.



January 2013

Solutions to R.M. Gray's 2013 qualifying exam problem.

The goal of the problem was to test understanding and familiarity with basic probability and expectation in an unfamiliar context.

The problem treats a notion of "distance" between two distributions. The quotes reflect the fact that this is not a distance or metric in the mathematical sense since it does not satisfy the triangle inequality. The square root of the quantity is a distance. This "distance" is very old and goes by many names, including Monge-Kantorovich, transportation, Gini, Wasserstein, and Mallow distance. Most recently it was rediscovered in 1998 in the CS literature and renamed the "earth mover's distance," but its primary origins were in work by Monge in 1781 and Kantorovich in 1942. Kantorovich shared the Nobel prize in economics for the development of linear programming, which is intimately connected with a general version of this distance. It is useful in signal processing and communications as a measure of the mismatch resulting when designing a system for one random variable, but then applying it to another. The distance extends naturally to random vectors and random processes. Here, however, only elementary probability is needed.

Let X be a (real-valued) random variable described by a cumulative distribution function (cdf) $F_X(x) = \Pr(X \leq x)$, which in turn is described either by a probability density function (pdf) $f_X(x) = dF_X(x)/dx$ if X is continuous, or a probability mass function (pmf) $p_X(x)$ if X is discrete. Let Y be another random variable with cdf F_Y etc. A joint cdf for both X and Y is denoted by $F_{XY}(x,y) = \Pr(X \leq x, Y \leq y)$.

Assume throughout that $E(X) = E(Y) = 0$,
$E(X^2) = \sigma_X^2$, $E(Y^2) = \sigma_Y^2$.
Both σ_X^2 and σ_Y^2 are assumed to be nonzero and finite.

A very old and very useful measure of "distance" between two given cdfs F_X and F_Y is defined by

$$\overline{d}(F_X, F_Y) = \min_{F_{XY}} E[(X-Y)^2],$$

where the expectation is with respect to the joint cdf F_{XY} and the minimum is over all joint cdfs F_{XY} having the given F_X and F_Y as marginals.

First Question: Given arbitrary cdfs F_X and F_Y describing random variables X and Y, give a *simple* example of a joint cdf F_{XY} with the prescribed marginals and use it to find an upper bound to $\overline{d}(F_X, F_Y)$ which depends only on σ_X^2 and σ_Y^2.

Solution: Assume that X and Y are independent random variables, in which case $F_{XY}(x,y) = F_X(x)F_Y(y)$ and $E(XY) = E(X)E(Y) = 0$ and hence

$$E[(X-Y)^2] = E(X^2) + E(Y^2) - 2E(XY) = \sigma_X^2 + \sigma_Y^2.$$

Second Question: Again assume you are given arbitrary cdfs F_X and F_Y describing random variables X and Y. Find a nontrivial *lower* bound to $\overline{d}(F_X, F_Y)$ which depends only on σ_X^2 and σ_Y^2.

Solution: The trivial lower bound is 0, since the expected value of the square of a real random variable is nonnegative. As before we know that

$$E[(X-Y)^2] = E(X^2) + E(Y^2) - 2E(XY) = \sigma_X^2 + \sigma_Y^2 - 2E(XY)$$

so to get a lower bound to $\overline{d}(F_X, F_Y)$ we want an upper bound to the correlation $E(XY)$. One of the most important bounds in probability has exactly this form. The Cauchy-Schwartz inequality states that

$$\mid E(XY) \mid \leq \sqrt{E(X^2)E(Y^2)} = \sigma_X \sigma_Y$$

so that for any joint cdf with the required marginals,

$$E[(X-Y)^2] \geq \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2$$

and hence

$$\overline{d}(F_X, F_Y) \geq (\sigma_X - \sigma_Y)^2$$

is the desired lower bound.

A few people used the equivalent fact that the correlation coefficient $E(XY)/\sigma_X\sigma_Y$ has magnitude less than 1.

Third Question: Suppose that both X and Y are Gaussian (0 means, variances σ_X^2 and σ_Y^2 respectively).

Find $\overline{d}(F_X, F_Y)$.

Solution: In a short oral exam there is no time for formal optimization. The idea here is to realize (possibly with a hint) that if you can think of a joint distribution with the given marginals which hits the previous lower bound, then that must be the minimum since no joint distribution can do any better. Here there is a natural guess — you can turn a 0 mean Gaussian X with variance σ_X^2 into a 0 mean Gaussian Y with variance σ_Y^2 by a simple scaling $Y = \sigma_Y X/\sigma_X$, which yields

$$\begin{aligned} E[(X-Y)^2] &= \sigma_X^2 + \sigma_Y^2 - 2E(XY) \\ &= \sigma_X^2 + \sigma_Y^2 - 2\frac{\sigma_Y}{\sigma_X}E(X^2) \\ &= \sigma_X^2 + \sigma_Y^2 - 2\sigma_X\sigma_Y = (\sigma_X - \sigma_Y)^2. \end{aligned}$$

Since this is an unbeatable lower bound from the previous part, it must solve the minimization. Note that here one of the two random variables is defined as a deterministic random variable, which is how Monge defined his distance over two centuries ago. The definition in terms of fixed marginals and a best joint was Kantorovich's. A great deal of research has been done to determine when the two definitions are equivalent.

Fourth Question: Suppose that X is Gaussian with 0 mean and variance σ_X^2 as before, but now Y is discrete with pmf

$$p_Y(y) = \frac{1}{2}; y = \pm\sigma_Y.$$

Find an upper bound for $\overline{d}(F_X, F_Y)$ that is strictly better than the bound of the First Question.

Useful fact: X Gaussian, 0 mean, variance σ_X^2

$$E[|X|] = E[X \mid X \geq 0] = \sqrt{\frac{2}{\pi}}\sigma_X$$

Solution: To get an upper bound, a joint distribution is needed on X, Y with the correct marginals. The first question used a simple product distribution, making the random variables independent. If we want a close mean squared error match, however, we want the two random variables to be correlated, in fact we want them to be maximally correlated if we want to try to achieve the lower bound of Question 2. There are several ways to create a joint distribution with the desired properties. One way is to define Y as a deterministic function of X with

$$Y = \begin{cases} \sigma_Y & \text{if } X \geq 0 \\ -\sigma_Y & \text{otherwise} \end{cases}$$

the resulting joint distribution yields the desired marginals and using conditional expectation results in

$$\begin{aligned} E[(X-Y)^2] &= \sigma_X^2 + \sigma_Y^2 - 2E(XY) \\ &= \sigma_X^2 + \sigma_Y^2 - 2\left[E(XY \mid X \geq 0)\frac{1}{2} + E(XY \mid X < 0)\frac{1}{2}\right] \\ &= \sigma_X^2 + \sigma_Y^2 - 2\sqrt{\frac{2}{\pi}}\sigma_X\sigma_Y \end{aligned}$$

This is not as good as the lower bound of Question 2, but nonetheless it turns out to actually solve the minimization. Here the joint distribution results from one random variable being a deterministic function of the other, and the operation used here is simply a binary quantizer.
Another way to get the same joint distribution is to use a classic result often derived in elementary probability classes. If U is a uniform distribution on $[0, 1]$, then the random variable $F_X^{-1}(U)$ will have cdc F_X, where F_X^{-1} denotes the inverse cdf. Thus given a single uniform U, one can generate random variables with the correct marginals via $(X, Y) = F_X^{-1}(U), F_Y^{-1}(U)$. This results in the same joint distribution as using the quantizer, and hence yields the same bound. But everyone who made a guess chose the previous approach, which amounts to a binary quantizer.

Last Question: I did not expect anyone to get this far, but two people did.
Does the lower bound you found in the previous part actually solve the minimization? That is, does the lower bound equal the distance?

Solution: You might think the bound does not yield the maximum correlation and hence the minimum "distance" since it does not achieve the bound given by Cauchy-Schwartz in Question 2, but it turns out that it is the minimum and the lower bound of Question 2 is not achievable in this nonGaussian example. Intuitively, you can not make X and Y with the given distributions any more correlated then matching their signs (in this example).
To prove that our new bound actually yields the distance, we need to show that for *any* joint distribution *with the given marginals*, it must be true that

$$E(XY) \leq \sqrt{\frac{2}{\pi}}\sigma_X\sigma_Y,$$

the value we actually achieved by a specific joint distribution. While Cauchy-Schwartz is still true here, it is too optimistic, it is not achievable. We need a better bound to the maximum correlation. This question was intended to elicit thoughts on how such an inequality might be proved for this case. One way is to use the method of indicators.
Define the indicator function

$$1(X \geq 0) = \begin{cases} 1 & X \geq 0 \\ 0 & X < 0 \end{cases}$$

and define the other indicator $1(X < 0)$ similarly. Since $1 = 1(X \geq 0) + 1(X < 0)$, we have that

$$\begin{aligned} E(XY) &= E[XY(1(X \geq 0) + 1(X < 0))(1(Y \geq 0) + 1(Y < 0))] \\ &= E[XY1(X \geq 0)1(Y \geq 0)] + E[XY1(X \geq 0)1(Y < 0)] \\ &\quad + E[XY1(X < 0)1(Y \geq 0)] + E[XY1(X < 0)1(Y < 0)] \\ &\leq E[XY1(X \geq 0)1(Y \geq 0)] + E[XY1(X < 0)1(Y < 0)] \end{aligned}$$

since the removed terms are negative. Because Y is binary, the right hand side is

$$\begin{aligned} &\sigma_Y E[X1(X \geq 0)1(Y \geq 0)] - \sigma_Y E[X1(X < 0)1(Y < 0)] = \\ &\qquad \sigma_Y E[X1(X \geq 0)1(Y \geq 0)] + \sigma_Y E[-X1(X < 0)1(Y < 0)]. \end{aligned}$$

Again the terms in the brackets are nonnegative and indicator functions are bound above by 1, so we have

$$E(XY) \leq \sigma_Y E[X1(X \geq 0)] + \sigma_Y E[-X1(X < 0)] = \sigma_Y E(|X|) = \sqrt{\frac{2}{\pi}}\sigma_X\sigma_Y$$

as needed. This proves the bound is actually achieved and hence yields the transportation distance. I did not expect anyone to actually go through this (and there is probably a shorter proof), I was only looking for ideas on how to decompose the expectation using the structure of the given distributions.