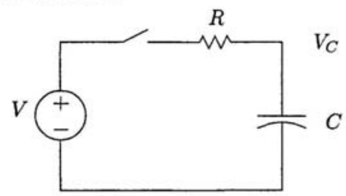
2003-2004 Electrical Engineering Qualifying Examination

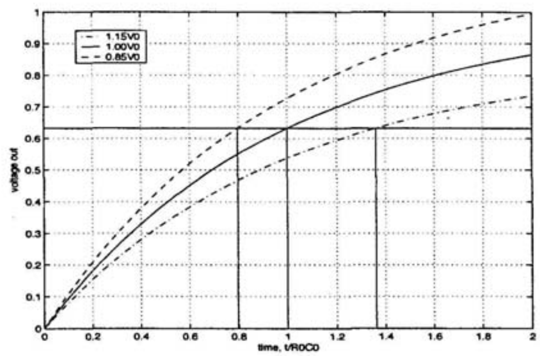
John Gill

The resistance R, capacitance C, and voltage V in the circuit shown below are independent random variables.



$R \sim \text{Uniform}[\,(1-\delta)\mathrm{R}_0,\ (1+\delta)\mathrm{R}_0\,]$
$C \sim \text{Uniform}[\,(1-\delta)\mathrm{C}_0,\ (1+\delta)\mathrm{C}_0\,]$
$V \sim \text{Uniform}[\,(1-\delta)\mathrm{V}_0,\ (1+\delta)\mathrm{V}_0\,]$
$0 < \delta < 1$
$V_C(t) = V(1 - e^{-t/RC}) \qquad (t > 0)$

The *time constant* of this random RC circuit is defined to be the time T that is needed for the capacitor to charge to $V_0(1-e^{-1})$. Examples of $V_C(t)$ and T are shown in the following figure.



Question 1 For this random circuit, the time constant is a random variable T. Find the conditional probability density of T given fixed values R and C.

Question 2 Find the expected value of the random time constant T.

Solution 1 Let $V_1 = V_0(1-e^{-1}) = 0.6321\,V_0$ denote the nominal threshold voltage, that is, the voltage on the capacitor after one nominal time constant R_0C_0 when $V = V_0$. The time constant random variable T is a function of V, R, and C; it is the solution of the equation

$$V_C(t) = V(1 - e^{-T/RC}) = V_1\,.$$

Solving the equation is straightforward:

$$V(1-e^{-T/RC}) = V_1 \;\Rightarrow\; 1 - e^{-T/RC} = \frac{V_1}{V} \;\Rightarrow$$
$$e^{-T/RC} = 1 - \frac{V_1}{V} \;\Rightarrow\; \frac{T}{RC} = -\ln\left(1 - \frac{V_1}{V}\right) \;\Rightarrow\; T = -RC\ln\left(1-\frac{V_1}{V}\right)$$