Prof. Stephen Boyd

Ask me about anything that isn't clear.

Consider the autonomous linear dynamical system $\dot{x} = Ax$, with $x(t) \in \mathbf{R}^n$.

When is it true that $x_1(0) \geq 0$ implies $x_1(t) \geq 0$ for all $t \geq 0$?

Feel free to start with special cases.

Discussion/solution. The solution is very simple: This holds when (and only when)

$$A_{12} = \cdots = A_{1n} = 0,$$

$i.e.$, all entries in the first row of A, except the first one, are zero.

This condition implies that $\dot{x}_1 = A_{11}x_1$, so $x_1(t) = x_1(0)e^{tA_{11}}$, so we have $x_1(t) \geq 0$ for all t if $x_1(0) = 0$.

Conversely, suppose that $A_{1k} \neq 0$, where $k \geq 2$. Let's take $x(0) = -A_{1k}e_k$. Then $x_1(0) = 0$. We have

$$\frac{dx_1}{dt}(0) = e_1^T Ax(0) = -A_{1k}^2 < 0.$$

So for small $t > 0$, we have $x_1(t) < 0$. This shows the condition is also necessary.

A good warm-up would be to take $n = 1$. In this case you find that the condition *always* holds. The next thing to try is $n = 2$. And you'd find that the condition is $A_{12} = 0$.