USgood

# EE Qualifying Exam
# January 2006

Let $\underline{\mathrm{f}}$ and $\underline{\mathrm{g}}$ be discrete, real signals, each periodic of period $N$. Define their correlation by the formula

$$(\underline{\mathrm{f}} \star \underline{\mathrm{g}})[m] = \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} \underline{\mathrm{f}}[n]\underline{\mathrm{g}}[n+m]$$

(a) Show that

$$\underline{\mathcal{F}}\,(\underline{\mathrm{f}} \star \underline{\mathrm{g}}) = \overline{\underline{\mathcal{F}}\,\underline{\mathrm{f}}}\,\underline{\mathcal{F}}\,\underline{\mathrm{g}}\,,$$

where $\underline{\mathcal{F}}$ is the discrete Fourier transform.

(b) Give an upper bound for $(\underline{\mathrm{f}} \star \underline{\mathrm{g}})[0]$.

Brad Osgood

# EE Qualifying Exam
## January 2007

An important property of linear, time-invariant systems is that complex exponentials are eigenfunctions.

What about the converse? That is, suppose $L$ is a linear system and the complex exponentials $e^{2\pi i\nu x}$ are eigenfunctions of $L$ for all $\nu \in \mathbf{R}$. Is $L$ necessarily a time-invariant system?

O5400D

# EE QUALIFYING EXAM
# JANUARY 2010

This is a problem about discrete signals (vectors) of length $N$.

Let $I$ be a subset of $\{0, 1, \ldots, N-1\}$ and let $I'$ be the complementary subset. (For example, $I$ could be the set of even numbers in $\{0, 1, \ldots, N-1\}$ and $I'$ would then be the set of odd numbers.)

Let $\mathbb{B}^I$ be the set of signals whose spectrum is supported on $I$, i.e.,

$$\underline{f} \in \mathbb{B}^I \iff \underline{\mathcal{F}}\,\underline{f}[m] = 0 \quad \text{if } m \in I'.$$

Here $\underline{\mathcal{F}}$ is the discrete Fourier transform.

- What is the set of signals that are *orthogonal* to $\mathbb{B}^I$, i.e., what is the orthogonal complement to $\mathbb{B}^I$?

Let $\underline{h}$ be the signal defined by

$$\underline{\mathcal{F}}\,\underline{h}[m] = \begin{cases} 1, & m \in I \\ 0, & m \in I' \end{cases}$$

- Show that the orthogonal projection onto $\mathbb{B}^I$ is given by

$$K\underline{f} = \underline{h} * \underline{f}.$$

- What is the orthogonal projection onto the orthogonal complement of $\mathbb{B}^I$?

*Solutions*

For the first question, two signals $\underline{f}$ and $\underline{g}$ are orthogonal if their inner product, $\underline{f} \cdot \underline{g}$ is 0. By Parseval's theorem

$$\underline{f} \cdot \underline{g} = \frac{1}{N}(\underline{\mathcal{F}}\,\underline{f} \cdot \underline{\mathcal{F}}\,\underline{g}).$$

If $\underline{f} \in \mathbb{B}^I$ then

$$\begin{aligned} \underline{\mathcal{F}}\,\underline{f} \cdot \underline{\mathcal{F}}\,\underline{g} &= \sum_{n=0}^{N-1} \underline{\mathcal{F}}\,\underline{f}[n]\overline{\underline{\mathcal{F}}\,\underline{g}[n]} \\ &= \sum_{n \in I} \underline{\mathcal{F}}\,\underline{f}[n]\overline{\underline{\mathcal{F}}\,\underline{g}[n]} \end{aligned}$$

since $\mathcal{F}\underline{\mathrm{f}}[n] = 0$ if $n \in I'$. This will be 0 *for all* $\underline{\mathrm{f}} \in \mathbb{B}^I$ if and only if $\mathcal{F}\underline{\mathrm{g}}[n] = 0$ for all $n \in I$. This says that $\underline{\mathrm{g}}$ must be in $\mathbb{B}^{I'}$. Sybolically,

$$(\mathbb{B}^I)^\perp = \mathbb{B}^{I'}.$$

For the second question, to show that $K\underline{\mathrm{f}} = \underline{\mathrm{h}} * \underline{\mathrm{f}}$ defines the orthogonal projection onto $\mathbb{B}^I$ we have to do several things. First, if $\underline{\mathrm{f}}$ is any signal we have to show that $\underline{\mathrm{h}} * \underline{\mathrm{f}} \in \mathbb{B}^I$. For this, for any $m$ the convolution theorem and the definition of $\underline{\mathrm{h}}$ gives

$$\begin{aligned}\mathcal{F}(\underline{\mathrm{h}} * \underline{\mathrm{f}})[m] &= (\mathcal{F}\underline{\mathrm{h}}[m])(\mathcal{F}\underline{\mathrm{f}}[m]) \\ &= \begin{cases} \mathcal{F}\underline{\mathrm{f}}[m], & m \in I \\ 0, & m \in I' \end{cases}\end{aligned}$$

Thus $\underline{\mathrm{f}} * \underline{\mathrm{h}}$ is supported on $I$, i.e., $\underline{\mathrm{h}} * \underline{\mathrm{f}} \in \mathbb{B}^I$.

Second, if $\underline{\mathrm{f}}$ is already in $\mathbb{B}^I$ we should have $\underline{\mathrm{h}} * \underline{\mathrm{f}} = \underline{\mathrm{f}}$. But if $\underline{\mathrm{f}} \in \mathbb{B}^I$ then already $\mathcal{F}\underline{\mathrm{f}}[m] = 0$ for $m \in I$ and so by the definition of $\underline{\mathrm{h}}$,

$$(\mathcal{F}\underline{\mathrm{h}})(\mathcal{F}\underline{\mathrm{f}}) = \underline{\mathrm{f}}.$$

Taking the inverse DFT gives

$$\underline{\mathrm{h}} * \underline{\mathrm{f}} = \underline{\mathrm{f}}.$$

As an aside, another way to do this part of the problem is to observe that

$$(\mathcal{F}\underline{\mathrm{h}})(\mathcal{F}\underline{\mathrm{h}}) = \mathcal{F}\underline{\mathrm{h}},$$

hence

$$\underline{\mathrm{h}} * \underline{\mathrm{h}} = \underline{\mathrm{h}}.$$

Thus for any signal $\underline{\mathrm{f}}$,

$$K^2\underline{\mathrm{f}} = K(K(\underline{\mathrm{f}})) = \underline{\mathrm{h}} * (\underline{\mathrm{h}} * \underline{\mathrm{f}}) = (\underline{\mathrm{h}} * \underline{\mathrm{h}}) * \underline{\mathrm{f}} = \underline{\mathrm{h}} * \underline{\mathrm{f}} = K\underline{\mathrm{f}},$$

that is

$$K^2 = K,$$

which is the definition of a projection.

Why is this an orthogonal projection? If $\underline{\mathrm{f}}$ is in $\mathcal{B}^{I'}$, the orthogonal complement of $\mathbb{B}^I$, then $\mathcal{F}\underline{\mathrm{f}}[m] = 0$ for $m \in I$, hence, by definition of $\underline{\mathrm{h}}$,

$$(\mathcal{F}\underline{\mathrm{h}})(\mathcal{F}\underline{\mathrm{f}}) = 0,$$

whence

$$K\underline{\mathrm{f}} = \underline{\mathrm{h}} * \underline{\mathrm{f}} = 0.$$

Finally, for the last question, the orthogonal projection onto $\mathbb{B}^{I'}$ is given by

$$K' = I - K.$$

With

$$(I - K)\underline{\mathrm{f}} = \underline{\mathrm{f}} - K\underline{\mathrm{f}} = \underline{\mathrm{f}} - \underline{\mathrm{h}} * \underline{\mathrm{f}}$$

we can also write

$$K'\underline{\mathrm{f}} = \underline{\mathrm{f}} - \underline{\mathrm{h}} * \underline{\mathrm{f}}.$$

or as a convloution

$$K\underline{\mathrm{f}} = \underline{\mathrm{h}}' * \underline{\mathrm{f}},$$

where $\underline{\mathrm{h}}'$ is given by

$$\underline{\mathcal{F}\mathrm{h}}'[m] = \begin{cases} 1, & m \in I' \\ 0, & m \in I \end{cases}$$

EE Qualifying Exam
January 2012

1. Define
$$\mathrm{III}_p(t) = \sum_{k=-\infty}^{\infty} \delta(t - pk), \quad p > 0.$$
What is $\mathcal{F}\mathrm{III}_p$, the Fourier transform of $\mathrm{III}_p$? Deduce from your answer that $\mathrm{III}_p$ is even.

2. Suppose we input $\mathrm{III}_p$ into a linear time-invariant system $L$ and measure the output, $w = L\mathrm{III}_p$. Is it possible to recover the impulse response $h$ of the system from this information?

Prof. Brad Osgood

# EE Qualifying Exam
January 2013

From Wikipedia:

> Pink noise or 1/f noise (sometimes also called flicker noise) is a signal or process with a frequency spectrum such that the power spectral density is inversely proportional to the frequency. In pink noise, each octave carries an equal amount of noise power. (The name arises from the pink appearance of visible light with this power spectrum.)

- Please explain the terms in the description above. How would you go about measuring a signal to see if it's pink noise? How would you generate a signal that has pink noise? How does pink noise differ from white noise?

1/f noise occurs in many physical, biological and economic systems. Some researchers describe it as being ubiquitous. In physical systems, it is present in some meteorological data series, the electromagnetic radiation output of some astronomical bodies, and in almost all electronic devices (referred to as flicker noise). In biological systems, it is present in, for example, heart beat rhythms, neural activity, and the statistics of DNA sequences. . . . Also, it describes the statistical structure of many natural images (images from the natural environment).

Richard F. Voss and J. Clarke claim that almost all musical melodies, when each successive note is plotted on a scale of pitches, will tend towards a pink noise spectrum. There are many theories of the origin of 1/f noise. Some theories attempt to be universal, while others are applicable to only a certain type of material, such as semiconductors. Universal theories of 1/f noise remain a matter of current research interest.