Signal

1998-1999 Qualifying Examination Question

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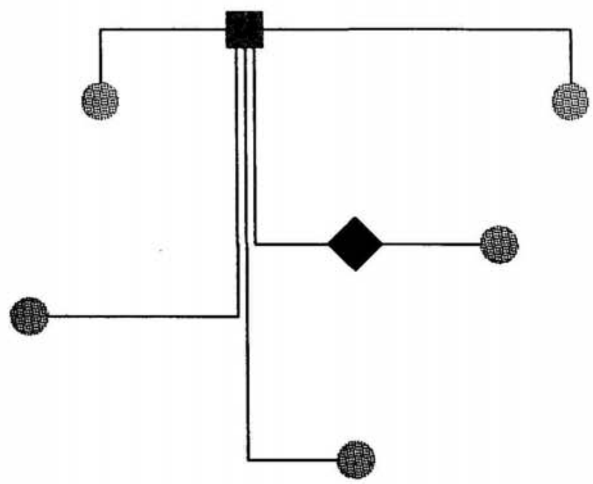
A set of nodes (blue circles in the figure below) must be connected to a power node (red square) using separate wires that run horizontally or vertically.



Where in general should the power node be placed to minimize the *total* wire length? (The location shown in the figure above is *not* optimal.)

Answer

Suppose that the i-th blue node is at location (x_i, y_i). If (x, y) is the location of the red node, then the total wire length is

$$\sum_{i=1}^{n}(|x - x_i| + |y - y_i|) = \sum_{i=1}^{n} |x - x_i| + \sum_{i=1}^{n} |y - y_i| \, .$$

The two sums can be independently minimized; that is, the x-coordinate of the best location for the power node depends only on the x-coordinates of the blue nodes, and similarly for the y-coordinates. We can assume that $x_1 \le x_2 \le \cdots \le x_n$. If $x_k \le x \le x_{k+1}$ then the total x cost is

$$\sum_{i=1}^{k}(x - x_i) + \sum_{i=k+1}^{n} (x_i - x) \, .$$

The derivative of this piecewise-linear function is $k - (n - k) = 2k - n$. The derivative is negative if $k < n/2$ and is positive if $k > n/2$. Thus the x cost is minimized by locating x at the *median* of the x-coordinates, where an equal number of x_i are less than x and greater than x. Similarly, the best y location is the median of the y-coordinates. The optimum power node location for the figure above is the green diamond. (If n is even then all values of x between $x_{n/2}$ and $x_{n/2+1}$ have the same cost.)