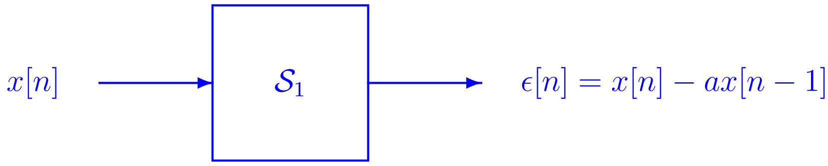
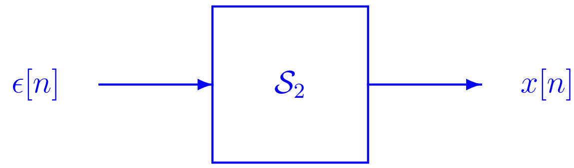
As before:





Next Question:

$$\boxed{\text{What are the eigenvalues and eigenfunctions of the system } \mathcal{S}_1\text{?}}$$

$$\boxed{\text{of the system } \mathcal{S}_2\text{?}}$$

Solution: An eigenfunction $u[n]$ of a discrete-time system is a nonzero signal with the property that an input of $u[n]$ yields an output of $\lambda u[n]$, where λ is the associated eigenvalue. From linear systems theory, discrete-time time-invariant systems have complex exponentials as eigenvalues, that is, signals of the form

$$u[n] = e^{j2\pi fn}$$

with a corresponding eigenvalue of $H(j2\pi f)$, where H is the DTFT of the Kronecker delta response (the *system function*) and where f is any real number in $[-1/2, 1/2]$. (Note that the eigenfunctions are not in the family of inputs considered up until now since they do not have finite energy.) Both $\mathcal{S}_1$ and $\mathcal{S}_2$ are LTI systems and hence have such eigenfunctions.
For $\mathcal{S}_1$, using the previously found expression for $E(f)$ yields $H(f) = E(f)/X(f) = 1 - e^{-j2\pi f}$. Alternatively, take the DTFT of the Kronecker delta response $h[n] = \delta[n] - a\delta[n-1]$ to get the same result.
For $\mathcal{S}_2$ you can either observe from the properties of linear systems that

$$G(f) = \frac{1}{H(f)} = \frac{1}{1 - e^{-j2\pi f}}$$

or you can compute $G(f)$ directly using the geometric progression and the geometric Kronecker delta response found earlier.
It is straightforward to verify directly that these eigenfunctions and eigenvalues work by plugging them into the specific linear difference equations describing the two systems.