O5400D

EE QUALIFYING EXAM
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This is a problem about discrete signals (vectors) of length N.

Let I be a subset of $\{0, 1, \ldots, N-1\}$ and let I' be the complementary subset. (For example, I could be the set of even numbers in $\{0, 1, \ldots, N-1\}$ and I' would then be the set of odd numbers.)

Let $\mathbb{B}^I$ be the set of signals whose spectrum is supported on I, i.e.,

$$\underline{f} \in \mathbb{B}^I \iff \underline{\mathcal{F}}\,\underline{f}[m] = 0 \quad \text{if } m \in I'.$$

Here $\underline{\mathcal{F}}$ is the discrete Fourier transform.

- What is the set of signals that are *orthogonal* to $\mathbb{B}^I$, i.e., what is the orthogonal complement to $\mathbb{B}^I$?

Let $\underline{h}$ be the signal defined by

$$\underline{\mathcal{F}}\,\underline{h}[m] = \begin{cases} 1, & m \in I \\ 0, & m \in I' \end{cases}$$

- Show that the orthogonal projection onto $\mathbb{B}^I$ is given by

$$K\underline{f} = \underline{h} * \underline{f}.$$

- What is the orthogonal projection onto the orthogonal complement of $\mathbb{B}^I$?

Solutions

For the first question, two signals $\underline{f}$ and $\underline{g}$ are orthogonal if their inner product, $\underline{f} \cdot \underline{g}$ is 0. By Parseval's theorem

$$\underline{f} \cdot \underline{g} = \frac{1}{N}(\underline{\mathcal{F}}\,\underline{f} \cdot \underline{\mathcal{F}}\,\underline{g}).$$

If $\underline{f} \in \mathbb{B}^I$ then

$$\begin{aligned} \underline{\mathcal{F}}\,\underline{f} \cdot \underline{\mathcal{F}}\,\underline{g} &= \sum_{n=0}^{N-1} \underline{\mathcal{F}}\,\underline{f}[n]\overline{\underline{\mathcal{F}}\,\underline{g}[n]} \\ &= \sum_{n \in I} \underline{\mathcal{F}}\,\underline{f}[n]\overline{\underline{\mathcal{F}}\,\underline{g}[n]} \end{aligned}$$