EE Ph.D. Qualifying Exam, January 2006 Question

David Miller

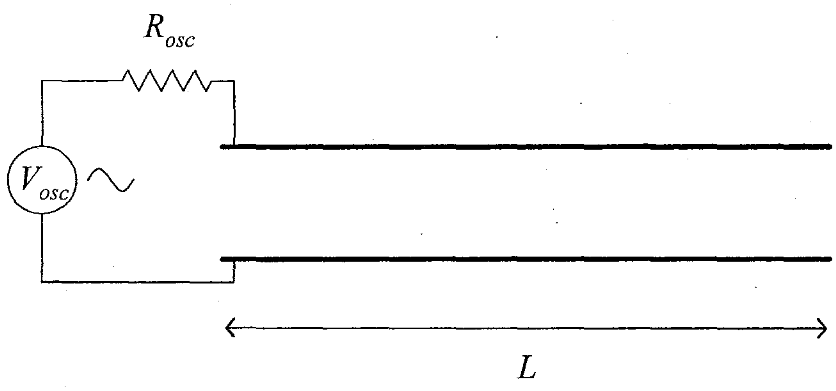
Transmission Line Resonances

Note: if you finish the questions on this sheet, subsequent questions will be asked.

Consider a uniform electrical transmission line of length L. An electrical oscillator produces a sinusoidally oscillating voltage of some fixed amplitude V_{osc} at a frequency f that can be varied. The oscillator is connected through a resistor to the transmission line at a point just inside one end of the transmission line as shown in the figures. The value of this resistor, R_{osc}, is very much greater than the characteristic impedance of the transmission line, Z. We will consider resonant frequencies in this line, i.e., frequencies at which the amplitude of the oscillating voltage on the line can build up to very large values at points or regions on the line.

For this problem, the line can be assumed to have very low loss, and to have a wave propagation velocity of c, the velocity of light in free space.

(a) For the case where the line has an open circuit at both ends,



(i) what is the lowest frequency at which there is a resonance?

(ii) what is the next frequency at which there is a resonance?

(iii) in general, at what frequencies are there resonances?

(b) For the case where the line is short-circuited at one end,

(i) what is the lowest frequency at which there is a resonance?

(ii) what is the next frequency at which there is a resonance?

(iii) in general, at what frequencies are there resonances?

(c) Returning again to case (a) (open circuit at both ends), what is the effect on the resonances of adding a resistor with value $R_{shunt} << Z$ across the line at the middle?

[Part (d) was on a separate sheet.]

(d) For the case (b) above of the line short circuited at one end,

(i) where would you place a resistor so as to suppress the lowest frequency resonance of the line, but leave the next higher frequency resonance approximately unchanged?

(ii) what fraction of all resonances would this resistor then suppress?

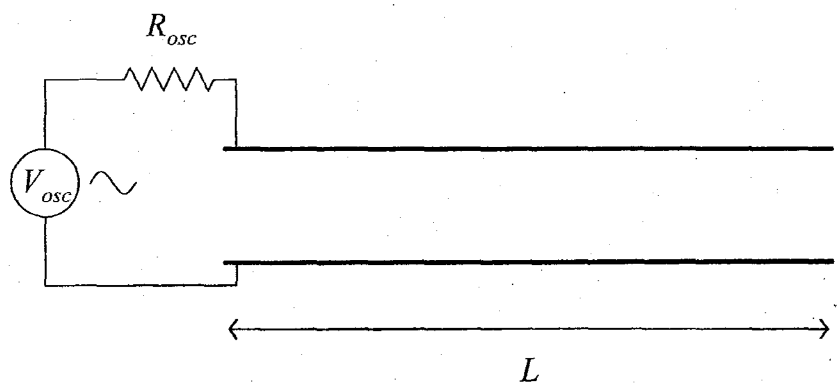
[Part (e) was on another separate sheet.]

(e) What would happen to the resonances on the line in case (a) above (both ends open circuit) if I cut a very small gap in the electrical conductors in the middle of the line? What would happen as I made the gap larger?

[Note: Most students got through parts (a) through (c) with some help. The more successful students got through most of part (d). Only a very few students got to start thinking about part (e).]

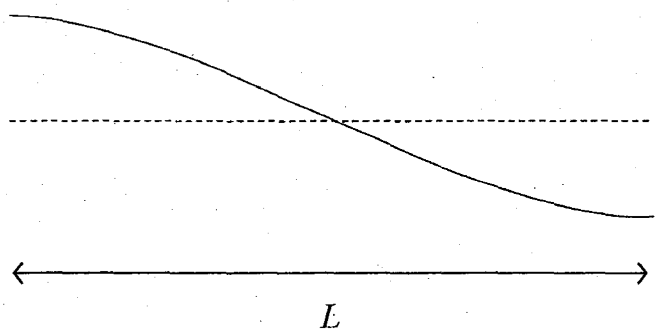
EE Ph.D. Qualifying Exam, January 2006 Solution

(a) For the case where the line has an open circuit at both ends,



(i) what is the lowest frequency at which there is a resonance?

The line will have a standing wave resonance on it with voltage maxima at the ends. The voltage is a maximum at either end because the voltage wave reflects constructively on itself there. The voltage wave looks like



Equivalently, one can say there are zeros in the current at both ends. The lowest frequency for which this can happen is the one for which one half wavelength fits in between one end and the other, i.e.,

$$\frac{\lambda}{2} = L$$

The wavelength is given by

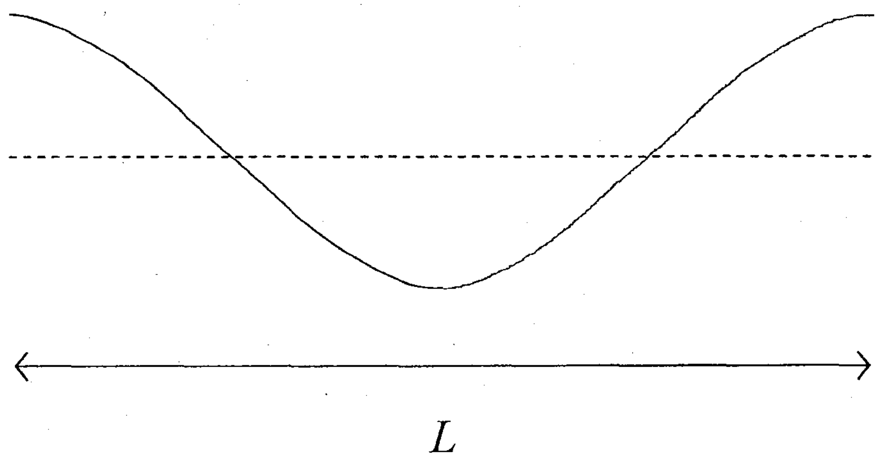
$$\lambda = c / f$$

So the frequency of this first mode is

$$f_1 = c / 2L$$

(ii) what is the next frequency at which there is a resonance?

The next possible resonance is when two half waves fit within the line, i.e., $L = \lambda$ and so the frequency is $2f_1$



Note this wave has a maximum in the middle.

(iii) in general, at what frequencies are there resonances?

The resonances are at frequencies mf_1, where m is a positive integer (starting at 1)

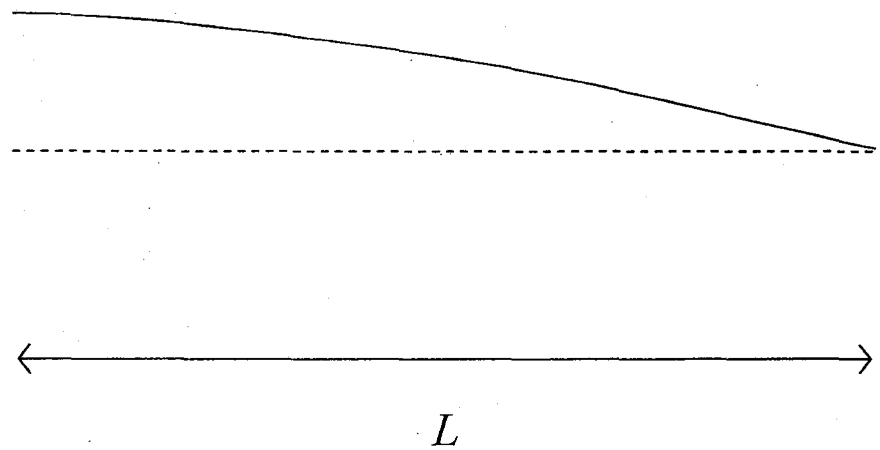
(b) For the case where the line is short-circuited at one end,

(i) what is the lowest frequency at which there is a resonance?

In this case, the voltage is zero at the short-circuited end, while still being maximum at the open end, so the standing wave for the lowest frequency is a quarter wave, and so the frequency of this lowest resonance is

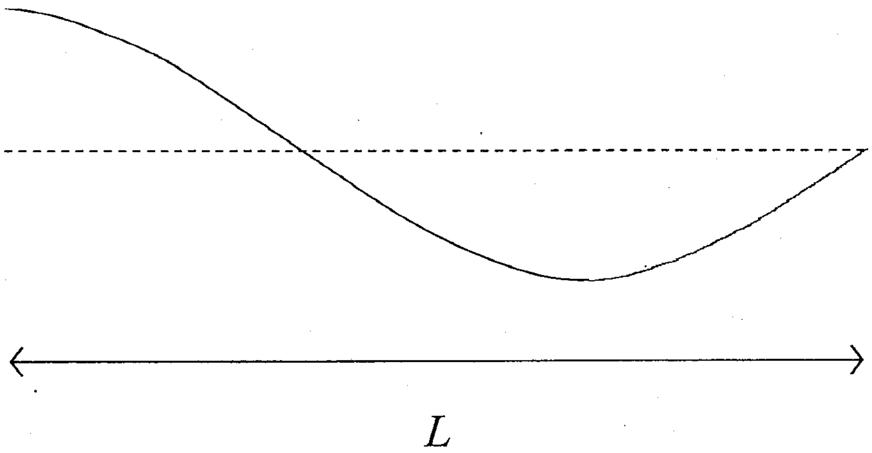
$$f_{1short} = c / 4L$$

i.e., for the case where the short circuit is on the right end of the line



(ii) what is the next frequency at which there is a resonance?

The next possibility is for three quarter waves to fit on the line, so the corresponding frequency is $3 f_{1short}$.



(iii) in general, at what frequencies are there resonances?

We have to fit an odd number of quarter waves in the length of the line in general, so the frequencies are $(2m-1)f_{1short}$ where m is a positive integer (starting at 1)

(c) Returning again to case (a) (open circuit at both ends), what is the effect on the resonances of adding a resistor with value $R_{shunt} << Z$ across the line at the middle?

It will have no effect on all of the resonances with m odd, because they all have a zero in the voltage at the middle, so there will be no current across the resistor in those cases, and the resistor does nothing. For all of the resonances with m even, there is a maximum in the voltage there (without the resistor). Adding the resistor in the middle therefore suppresses those resonances because they would now become very lossy.

(d) For the case (b) above of the line short circuited at one end,

(i) where would you place a resistor so as to suppress the lowest frequency resonance of the line, but leave the next higher frequency resonance approximately unchanged?

The resistor should be placed one third of the wave along the line from the open end, so as to coincide with the zero in the second resonance (hence not affecting it). Though not at a maximum of the first resonance, the amplitude of the first resonance is substantial, and so it would experience significant loss from this resistor, and is therefore suppressed.

[Note for interest: This is the reason why the register key in a clarinet is placed 1/3 of the way along the pipe from the mouthpiece end of the instrument. The mouthpiece behaves as a stopped end on the pipe; the other end of the pipe is open. In this acoustic case, though, one should consider the pressure, in which case the mouthpiece end is a pressure maximum, so that stopped end is analogous to the open end of this transmission line if we are drawing an analogy between pressure and voltage, and the open end of the pipe is a pressure minimum, analogous to the closed end of the transmission line. The register key hole is therefore placed at a pressure node for the next resonance of the entire pipe. Opening this hole makes the clarinet play in its second, "clarion" register, a frequency three times higher than that of the lower, "chalumeau" register.]

(ii) what fraction of all resonances would this resistor then suppress?

It will suppress 2/3 of all of the resonances of the initial line. Since we must now have a zero at $L/3$ from the open end of the line, it is as if the line is only one third as long, so the (low-loss) resonances are three times as far apart as those on the original line. All the other resonances of the original line are suppressed.

(e) What would happen to the resonances on the line in case (a) above (both ends open circuit) if I cut a very small gap in the electrical conductors in the middle of the line?

For a very small gap, it is as if we put large capacitors connecting two pieces of transmission line, and such capacitors behave as short-circuits for oscillating voltages, so there is no effect on the resonaces of the line for a very small gap. It is also true that there will simply be wave coupling between the different parts of the line (which is not really a separate statement from the capacitive coupling).

As the gap size is increased, the break does become a significant, and in the limit of a large gap, the two parts of the line will behave as if they are separate lines of half the length.

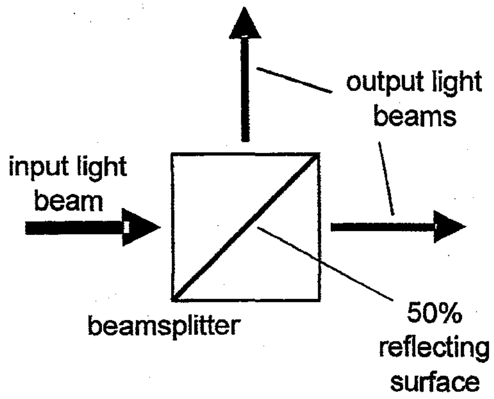
EE Ph.D. Qualifying Exam, January 2007 Question

David Miller

Backwards Beamsplitters

Note: if you finish the questions on this sheet, subsequent questions will be asked.

A mirror with 50% reflectivity can be used as a beamsplitter, turning one light beam into two, as shown in the figure. (A beamsplitter is often made in the shape of a cube of glass, with the reflecting surface in the middle of the cube, as sketched in the figure.)



Starter question (not part of the graded exam):

What happens if we shine the input light beam onto the bottom face of the beamsplitter, instead of the left face?

Question:

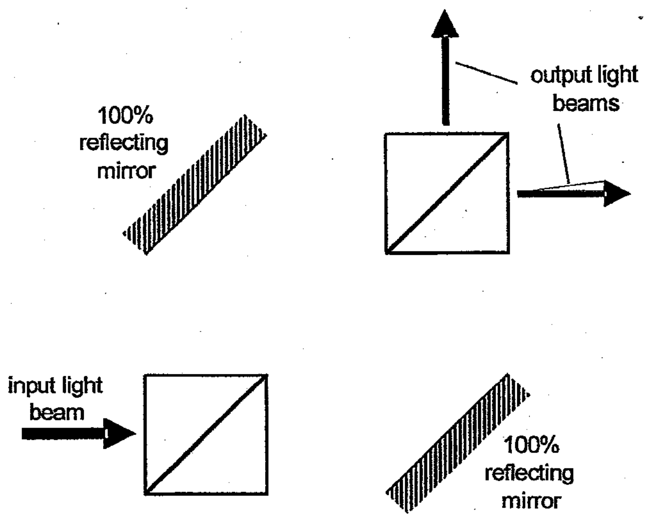
Can we run this beamsplitter backwards, using it to *combine* the power of two input light beams into one? (For example, we could have input light beams incident on the top and right faces, and want to combine them to give an output light beam only from the left face.) You may presume that the two light beams to be combined are monochromatic light beams (i.e., one color or frequency) of exactly the same frequency.

If we can combine them, what determines whether the combined beam comes out of the left face or out of the bottom face?

Supplementary questions (all on separate sheets)

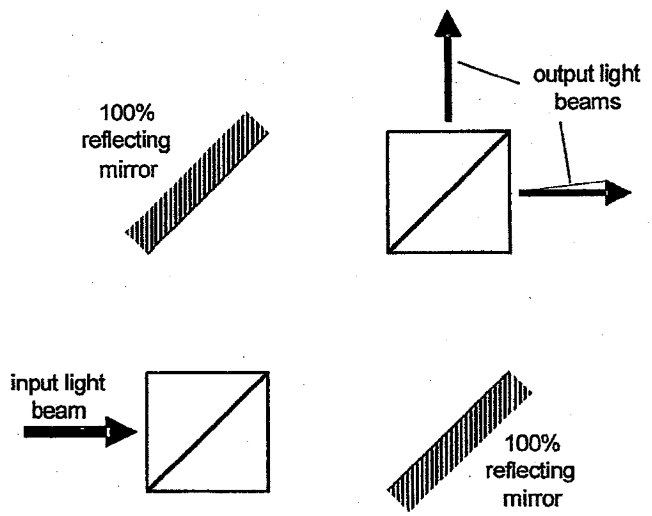
Supplementary question 1

In the apparatus below, in which there are two beamsplitters (each 50% reflecting) and two mirrors (each 100% reflecting), what would you do to the apparatus, or what would you add, that would allow you to control how much of the power came out in each of the output beams?



Supplementary question 2

In the apparatus below, what do you think will happen if we turn down the power in the input light beam so much that there is only ever one photon in the apparatus at a time? In that case, will we still be able to control the relative average powers in the two output light beams just as we did before?



Supplementary question 3

Suppose we had two hot bodies (e.g., like light-bulb filaments) emitting light. Do you think you could use a beamsplitter to combine the emitted light power from these two so that you could heat up another object (e.g., another light bulb filament) to a hotter temperature? If so, how would you do it? If not, why could you not do it?

Solution

Starter question:

If we shine the light beam onto the bottom of the beamsplitter, it will still come out of the top and right sides with equal powers.

(The reasons for asking this question are to make sure the examinee understands beamsplitters (so the exam is not a test of basic optics knowledge), and to set up the paradox in the examinee's mind that underlies the actual question – namely, if we are to run this beamsplitter backwards, what is it about the light beams that will make sure the light comes out of the left rather than out of the bottom of the beamsplitter.)

Question:

The answer is that we can run the beamsplitter backwards to get the light to come out of only one port.

Often students starting to answer this question conclude we cannot run the beamsplitter backwards because they think in powers or intensities, which leads them to conclude that the power will always be split between two outputs when we attempt to run the beam splitter backwards. To see that it can be run backwards, one needs to think in terms of waves, and wave equations can typically be time reversed, rather like taking a movie of the wave splitting and then running the movie backwards. That idea of reversibility is sufficient to allow the student to conclude that, at least for some conditions, the beamsplitter can be run backwards.

The key concept is to understand that the property that determines whether the beams come out of the bottom or the left of the beamsplitter is the relative phase of the two beams. If they have one particular relative phase, they will both come out of the left face, and add perfectlyFor that one particular phase, they add constructively out of the left face, and destructively out of the bottom face, and that is the relative phase required to run the beamsplitter backwards to get an output from the left. If the relative phase is changed by 180 degrees, the beam will come out of the bottom face.

It is also true that the two light beams have to be in the same spatial mode (beam shape and direction) on reflection off of, or transmission through, the mirror, and of the same polarization, though to keep the problem simple, I would not introduce these attributes unless the examinee brought them up. Without those two attributes being the same, the perfect cancellation of the waves in one output direction would not be possible.

Supplementary question 1:

The answer is that we need to do something to change the relative phase of one path light beam path within the apparatus compared to the other. We could do this by very slightly displacing one of the mirrors, or we could add something into one of the paths, such as a thin piece of glass, to otherwise change the relative phase of the two beams going into the upper beamsplitter. In this way, we can actually arbitrarily change the relative power in the two output beams.

This structure, by the way, is known as a Mach-Zehnder interferometer.

Supplementary question 2:

The answer is that it makes no difference to the relative average powers – we can still control the average power to come out in one or the other of the output beams, or any relative ratio between them that we wish, just as we could with higher powers. In this question, I am just interested to see how the examinee reasons on this one, and I do not expect that they know the quantum mechanics that might otherwise help with the answer.

Supplementary question 3:

The answer is that you could not combine the powers in such a way as to heat up another body to a hotter temperature. There are several ways of looking at the answer to this. One is to rely on the Second Law of Thermodynamics, which forbids such processes. A second way is to look at the microscopic physics. One argument would be that the phase of the light from the two different bodies is completely independent, and so on the average, one could not add the powers reliably into one output port or another. A third way is to rely on the Constant Brightness Theorem (or Constant Radiance Theorem) of optics, which also forbids such processes, though it is unlikely the examinee would have heard of this, and the most convincing way to prove that theorem is anyway by starting from the Second Law of Thermodynamics.

EE Ph.D. Qualifying Exam, January 2009 Question

David Miller

Computing without dissipation?

Notes:

There may not be a single "correct" answer to this question. The goal of this question is to see how you think about it.

If you finish the question on this sheet, subsequent questions will be asked.

[Examinees were also told verbally that the examiner was most interested in the way they thought rather than being concerned with right or wrong answers or statements, and that the exam would be conducted primarily by talking between the examinee and the examiner.]

Question:

Is it fundamentally possible to make a computer that dissipates no power while still performing a useful calculation?

If so, explain how to make the computer.

If not, explain why this cannot be done.

Answer

On this question as stated, essentially all the examinees stated that they believed (useful) computing without any dissipated power or energy was impossible (and, indeed, it is not possible to do useful computing, as defined later to the examinees, without dissipation of energy).

Most examinees were not initially clear exactly why a computer had to dissipate energy for a useful calculation, other than that they could not imagine computing parts that did not dissipate. Some students knew that the action of a logic gate, such as an AND gate, necessarily involves some dissipation because it is not reversible; knowing the output of an AND gate is not sufficient to tell you what the inputs had been.

There are, at least in principle, ways of performing computations without dissipating any energy. Some students knew that there have been proposals for reversible logic that does not throw away any information. (The Controlled-NOT gate of quantum computing is one such gate. Classical reversible gates were considered some years ago (e.g., by Fredkin and Toffoli) that also do not throw away information.) It is also true that ordinary analog physical systems, such as two balls bouncing elastically off one another, could be viewed as performing calculations of results without any particular dissipation; as an analog computer, such a system does obviously compute the result of two balls bouncing off of one another. But there is a catch in all such reversible systems, which relates to the idea of a useful calculation. At this point in the exam, after some discussion on points like these, the examinees were verbally asked the following question.

Question (second part)

A computation here will not be considered useful unless the result of the calculation is written down somewhere, such as in a memory register of some kind (e.g., in a USB memory stick). Given that requirement, and given a result that will be of some specific length (e.g., a 10 bit binary number), is there some lower bound to the amount of energy that must be dissipated for the computation to give a useful answer?

Answer (second part)

The key to the answer here is to consider entropy. The idea of entropy occurs both in discussing information and in thermodynamics, especially the Second Law of thermodynamics. Before the computation, I do not know the answer. Equivalently, the memory at the start contains an arbitrary 10 bit number. There are 2^{10} possible initial states of the memory. Hence it has an entropy

$$S = k_B \log\left(2^{10}\right) = 10k_B \log 2$$

Here, k_B is Boltzmann's constant. This result follows from the more general statistical mechanics formula that the entropy $S = k_B \log \Omega$ where Ω is the "multiplicity" or the number of accessible states, each of which is presumed to be equally likely. (Many examinees knew this formula, either from a physics discussion of entropy or from the discussion of entropy in the context of information (in which case they would probably not have the Boltzmann's constant in the formula). If the examinee got the general point that the concept of entropy was the key to the answer and did not know this formula, some help was given to work towards an expression like this.)

After the calculation, the memory is in one specific state, so the multiplicity has been reduced to 1, leading to $S = k_B \log 1 = 0$. Hence, the change in entropy as a result of writing the answer into the memory is $\Delta S = 10k_B \log 2$.

If the entropy of the memory has been reduced by an amount ΔS, then, by the Second Law of Thermodynamics, the entropy somewhere else or in some other aspect of the memory must have increased by an equal amount. If that is done by dissipation of energy, then we can use the thermodynamic formula

$$\Delta S = \frac{\Delta Q}{T}$$

where ΔQ is the dissipated energy (heat) and T is the temperature. (Not many examinees remembered a formula like this, though at this point any way of getting to an expression like this, including intelligent guesswork or even dimensional analysis based on the units of Boltzmann's constant being Joules/Kelvin, would have been sufficient.) Hence, finally, the dissipated energy is

$$\Delta Q = 10k_B T \log 2$$

It is just possible that this increase in entropy does not appear as heat but instead as the disordering of some other previously ordered system, and if any examinee had given that answer it would certainly have been acceptable. With that minor caveat, we cannot make a computing system that gives us a useful result (i.e., one that is written down somewhere) without dissipating an energy ΔQ of this form.

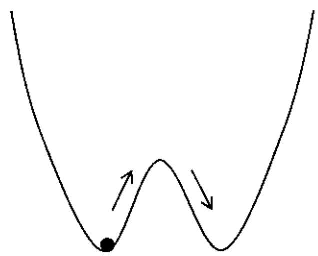
Question (third part)

If any examinee got this far in the question, then they were asked to consider mechanistically exactly how the dissipation of energy arises in setting the state of a memory in some toy example of a binary memory.

Answer (third part)

The key difficulty here is that, in writing the answer into a memory, we have to be able to write our answer (e.g., a logic one) into the memory *regardless of the current state of the memory*, and we have to make a mechanism that does that. (One might think that one could just measure the current state of the memory, and hence make a contingent mechanism that does different operations depending on the current state of the memory, but that merely postpones the problem. Our contingent mechanism would have to have a one bit memory in it to write down the answer to the measurement, and we would have to write down that answer independent of the current state of that memory, and so on.)

So, imagine that our memory element is in the form of a potential with two minima in it, and we have a ball that sits in one minimum or the other to represent either a logic zero (left minimum) or a logic one (right minimum).



Our proposed "set to logic one" mechanism is then one that simply pushes the ball to the right with sufficient pushing that it will push the ball over the top of the local maximum (the "hill") if it happens to be on the left. (Equivalently, we could temporarily lift the potential of the left well sufficiently that the ball will then roll gently into the right hand well, and then we reset the potential to its original form.) But this will not actually work yet, because the ball will simply then oscillate backwards and forwards between the wells because it now has enough energy to do so. To stop it oscillating, we need to introduce friction, such as immersing the entire apparatus in a viscous fluid, and it is in that addition that we introduce the dissipation to the system. To have the system settle into a specific state independent of its starting state, we need to introduce something irreversible, in this case the damping fluid.

Similarly, in a memory based on voltage stored on a capacitor, applying a voltage pulse through a wire will lead to oscillation from the combination of the capacitance together with the inductance of the wire unless there is resistance in the wire to damp out the oscillation.

A classic reference on the topics discussed in this question is

R. Landauer, "Dissipation and heat generation in the computing process," IBM J. Research and Development **5**, 183 – 191 (1961)

Subsequent discussions include

C. H. Bennett, "The thermodynamics of computation – a review," International Journal of Theoretical Physics **21**, 905 – 940 (1982)

C. H. Bennett, "Notes on Landauer's principle, reversible computation, and Maxwell's Demon," Studies in History and Philosophy of Modern Physics **34**, 501-510 (2003)

EE Ph.D. Qualifying Exam, January 2012 Question

David Miller

Energy in charging and discharging capacitors

Notes:

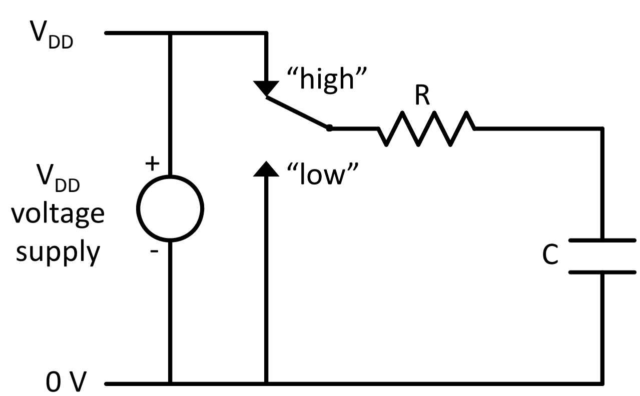
There may not be single "correct" answer to parts of this question. The goal of this question is to see how you think about it.

If you finish the question on this sheet, subsequent questions will be asked.

[In the exam, most students got through the main question, possibly with some help. Most of those then got through the first supplementary question. A fair fraction got to the second supplementary question, though only a few had time to finish that. Those that did finish mostly also got through the third supplementary question.]

Question:

In the circuit below, the switch has initially been connected in the "low" position for a long time; at the start of our experiment, therefore, the capacitor C is completely discharged. Then we move the switch to the "high" position and leave it there for a long time, charging the capacitor up to a voltage V_{DD}.



(i) What electrostatic energy is now stored in the capacitor C?

(ii) What energy has been dissipated in the resistor R during the charging process?

(iii) What total energy has been provided by the V_{DD} power supply?

Solution

(i) The electrostatic energy now stored on the capacitor is $\frac{1}{2}CV_{DD}^2$

Essentially all the students knew this answer, and they were not asked to prove or derive it.

(ii) A few students already knew or intelligently guessed that the answer to this part is also $(1/2)CV_{DD}^2$, though I would require them to justify this result. There are two main ways of answering this part. The first is to attempt some integration of the power or energy dissipated in the resistor. The most common approach students would take here was to integrate the known exponential behavior of the current or voltage in time, using either I^2R or V^2/R formulae for the power being dissipated in the resistor at a current I or voltage V. A more compact version is to integrate over voltage directly. The energy dissipated in flowing a charge δQ through a resistor at a voltage V is $\delta E = V\delta Q$. When flowing a charge δQ onto a capacitor C, the resulting change in voltage δV on the capacitor is such that $\delta Q = C\delta V$. In the circuit, the voltage V across the resistor R is $V = V_{DD} - V_{OUT}$. Hence the total energy dissipated in the resistor in charging the capacitor C from 0V to V_{DD} is

$$\Delta E_R = C\int_0^{V_{DD}} \left(V_{DD} - V_{OUT}\right) dV_{OUT} = \frac{1}{2}CV_{DD}^2$$

No student actually took this approach to start with. Anyway, once I could see that the student would be able to work out this energy by some integral approach, I generally stopped them, taking it for granted that eventually they would get to the right answer here.

The easy way to solve this problem is to solve part (iii) first, which avoids all this integration. Students were told to look at parts (ii) and (iii) together, though very few actually did that!

(iii) Some students would try to approach this problem (correctly, though not optimally) by figuring out an answer to part (ii) and adding it to the answer to part (i). The easy way to solve this is simply to calculate the energy that must be supplied by the power supply because it provides a charge $Q = CV_{DD}$ through the circuit to charge the capacitor. That energy is $E = QV_{DD} = CV_{DD}^2$. Few students got this without help. The temptation to integrate the power over time was generally too strong. Most students also had temporarily forgotten that Volts are Joules/Coulomb and that by definition in moving a 1 Coulomb charge through 1 Volt I must do 1 Joule of work. In general, students tended to be too frozen in to thinking in powers rather than energies in this whole problem, which made it harder for them than ideally it needed to be.

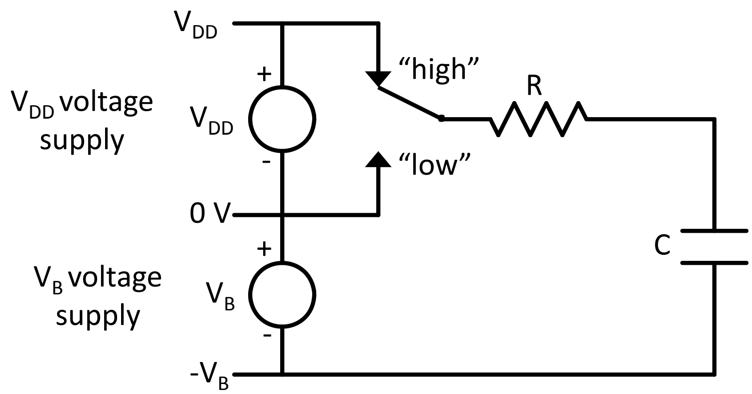
Armed with the result CV_{DD}^2 from this part, the answer to part (ii) is easy to deduce by conservation of energy.

Supplementary question 1

How would the above answers change if the resistor was nonlinear – i.e., the current through it was not proportional to voltage (as could be the case in a transistor or a diode, for example)?

Supplementary question 2

Now consider the circuit below. Here, initially, the switch has been connected in the "low" position for a long time; at the start of our experiment, therefore, the capacitor C has a voltage V_B across it. Then we move the switch to the "high" position and leave it there for a long time, charging the capacitor up to a voltage $V_{TOT} = V_{DD} + V_B$.



(a) What has been the change in the electrostatic energy stored in the capacitor C as a result of moving the switch to the "high" position?

(b) What energy has been dissipated in the resistor R during the charging process?

(c) What energies have been provided by each power supply (i.e., by the V_{DD} power supply and by the V_B power supply)?

(d) What happens to the energy provided by the capacitor C to the circuit when we now connect the switch back to the "low" position?

Supplementary question 3

Can you think of a way in which we can avoid dissipating the energy that the capacitor tries to put back into the V_B power supply?

Supplementary question 1 solution

Note that the energy calculated in part (ii) abo ve is quite independent of the value of the resistance R, and does not require any particular relation between current and voltage for that resistor; hence the resistor may be nonlinear and there will be no difference in the energy results here. This answer is particularly easy to see if the dissipation in the resistor is deduced by subtracting the energy stored in the capacitor from the total energy provided by the power supply; again, no particular relation between voltage and current is required in the resistor (at least as long as finite current flows at all voltages).

Supplementary question 2 solution

(a) The change in energy stored in the capacitor is the difference in energy between having a voltage V_B over the capitor and having a voltage $V_B + V_{DD}$ over it, which is

$$\Delta E = \frac{1}{2}C\left[\left(V_B + V_{DD}\right)^2 - V_B^2\right] = \frac{1}{2}CV_{DD}^2 + CV_{DD}V_B$$

(b) The answer to this is $(1/2)CV_{DD}^2$, exactly as in the main question, because the situation as seen by the resistor is exactly the same as it was in the first part, charging up a capacitor from 0V to V_{DD}, passing a charge $Q = CV_{DD}$ onto the capacitor just as before.

(c) The same charge $Q = CV_{DD}$ has to be passed through both power supplies (they are in series), so the energies they provide are, for the V_{DD} supply, CV_{DD}^2, and for the V_B supply $QV_B = CV_{DD}V_B$ (which we recognize as the second term on the far right on the above equation). Note that the energies do all add up correctly here, with the energy provided by the power supplies equaling ΔE plus the energy dissipated in the resistor.

$$CV_{DD}^2 + CV_B V_{DD} = \Delta E + \frac{1}{2}CV_{DD}^2$$

(d) When we discharge the system back to zero volts, an energy $(1/2)CV_{DD}^2$ is dissipated in the resistor, because as far as the resistor is concerned, we are discharging a capacitor from V_{DD} to 0V so all the currents and dissipations look the same as for such a problem. We are therefore left with an energy of magnitude $QV_B = CV_{DD}V_B$. This is energy that the circuit now tries to put back into the V_B power supply. Whether it can do that successfully depends on the nature of the power supply. If it is just an ordinary non-rechargeable battery, then the energy will end up dissipated as heat instead.

Supplementary question 3 solution

The answer is to make the V_B power supply effectively rechargeable, and the easiest way to do that is to put a large capacitor across it (sometimes called a bypass capacitor). A large capacitor effectively functions as a perfectly rechargeable power supply. In an actual circuit we might formally decouple the V_B power supply from the circuit with series resistance and/or inductance, leaving this bypass capacitor as the effective local, perfectly rechargeable “battery”.

EE Ph.D. Qualifying Exam, January 2013 Question

David Miller

Waves and transmission lines

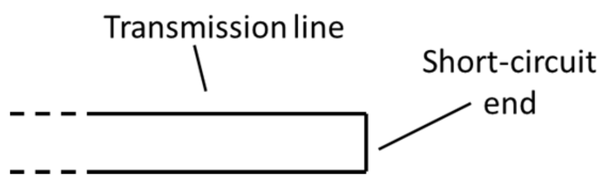
Notes: There may not be single "correct" answer to parts of this question. The goal of this question is to see how you think about it. The answers are mostly qualitative, and little or no algebra should be required for them. If you finish the question on this sheet, subsequent questions will be asked.

Question:

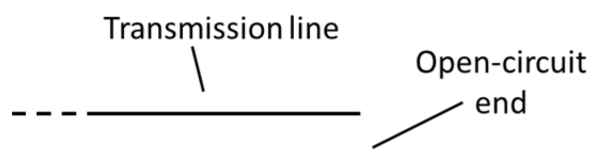
Consider a transmission line – that is, a two-conductor electrical line with some well-defined impedance (for example, 50 Ω). Presume for simplicity that the line has no loss (e.g., perfectly conducting wires) and that the wave propagation velocity on the line is c, the velocity of light in free space. A monochromatic (i.e., single-frequency) wave of frequency f has been launched onto the line from some source on the far left.

a) Sketch the form of the voltage on the line near the right end at some specific time for the two cases below, in both cases indicating any characteristic length involved and giving its magnitude.

(i) a line that is short-circuited at its right end, as shown in the diagram below

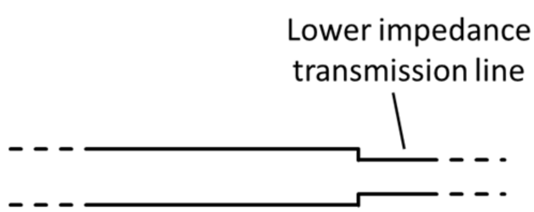


(ii) a line that is open-circuited at its right end, as shown in the diagram below



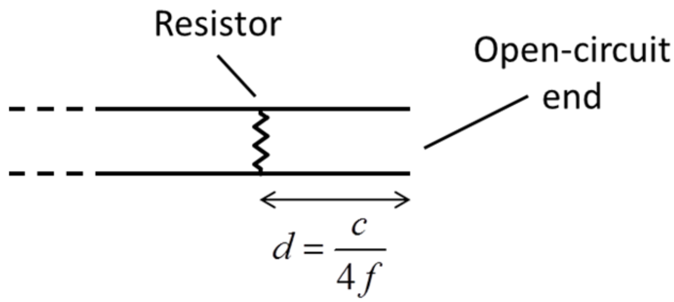
b) Sketch the form of the time-averaged electrostatic energy density in the line (which you can take to be proportional to the square of the voltage from your answers above) for both cases above.

c) Suppose that, instead of an open circuit at the right, we connect to another, lower impedance (for example, 25 Ω) line. Sketch the form of the energy density (i.e., the square of the voltage) in the line to the left of this connection (you need not calculate any numbers, but you should show the qualitative behavior).



Supplementary question 1

a) What happens if we add a resistor between the two conductors on the open-circuited line, at a distance $d = c/(4f)$ from the right (open-circuit) end?



b) Where else could you position the resistor to have the same effect?

Supplementary question 2

I want to have some signal that varies with frequency so that I can monitor possible frequency changes in a single-frequency electrical voltage, but I want to avoid constructing any resonant filter or frequency counter. I don’t need an accurate measurement of the frequency – just some signal that varies predictably as the frequency changes. I do, however, want to be able to distinguish between the frequency increasing or decreasing relative to some central frequency.

Can you think of a way of doing this, exploiting some of the concepts explored here with transmission lines?

(Note: there is certainly no single correct answer to this question!)

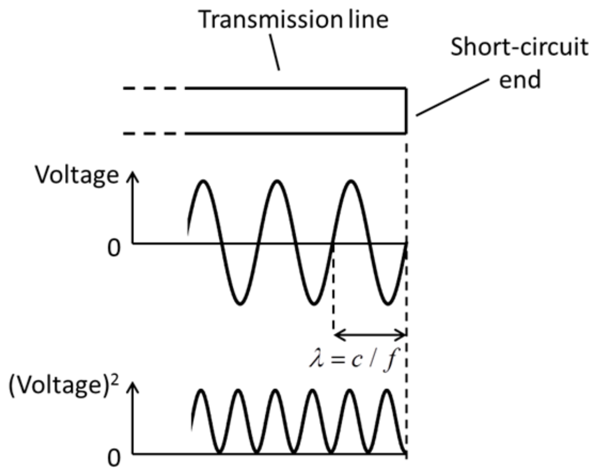
Answers

Main question

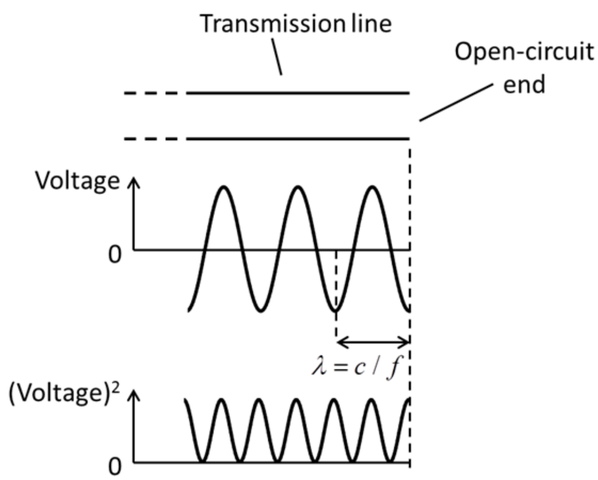
In both cases, of shorted or open ends on the line, we will form standing waves on the line.

a)

For case (i), the short circuit, there can be no voltage at the right end of the line, so that point will be a node (i.e., a zero) in the standing wave pattern. At any specific time, the voltage on the line will be a sine wave of wavelength $\lambda = c/f$ with a zero at the right end. So, the voltage will look as sketched below, or possibly minus this.



For case (ii), the open circuit, the voltage is maximum at the right end of the line, so that point will be an antinode (a maximum) in the standing wave pattern. The result is similar to the above, but shifted to have the maximum at the end.

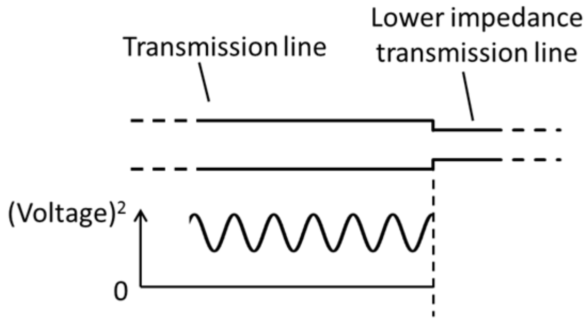


b)

The squares of the voltages are also sketched on the figures above for part (a).

c)

This situation is intermediate between an open-circuit and a short-circuit end. The phase of the reflection is the same as that for a short-circuit end, but the amplitude is smaller (the reflection cannot be total because some power is transmitted on into the lower-impedance line). Consequently, we have a partial standing wave, not going to zero at the minima, but with the same phase as that of the short-circuited line.



Notes on how students answered this

Most students had some notions of transmission lines. For those that did not, it was possible to recast this question in terms of waves on a string, waves in other kinds of waveguides, or waves propagating up to dielectric interfaces.

The most common minor error was that, instead of drawing a $\sin^2$ function for the standing wave, students drew a rectified sine wave. (In fact, nearly all students did this!) I took little or nothing off for that error in itself because it is not such big problem for parts (a) and (b), but it made it conceptually more difficult to answer part (c).

In part (c), some students correctly reasoned that the answer should be between that of parts (a) and (b) in some sense. This could cause them to guess (incorrectly) that the phase on reflection was somewhere between 0 and 180 degrees. I took no marks off for that, though, because it is not easy to get the phase here based on purely logical (rather than algebraic) reasoning. The main difficulty most students had was in realizing that the partial reflection would lead to a standing wave pattern that does not reach zeros. If so, I tried to help them reason towards the right answer by asking them what the pattern would look like if both lines had the same impedance (the answer is a completely flat line, of course).

Supplementary question 1

Since this position (one quarter wavelength from the open end) is a node, this resistor has no effect; there is no voltage across it at any time.

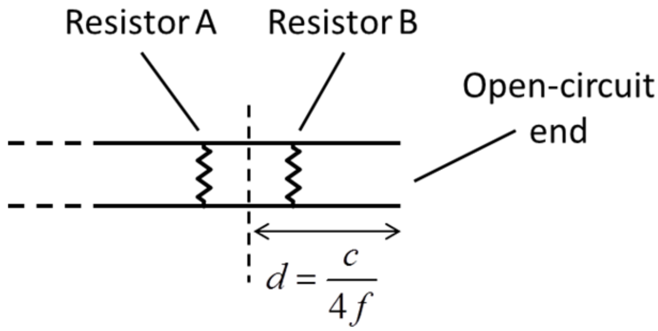
The resistor could be positioned at any node in the line for the same effect. The nodes are a one-quarter wavelength, 3/4 wavelength, 5/4 wavelength, … and so on, from the right end of the line.

Notes

The students who got to this point generally got this answer correct.

Supplementary question 2

There are many possible ways of doing this. One possible solution here is shown below.



Here, two resistors are placed on either side of the position of the node in the open-circuit line case. We presume we make these resistors rather large values compared to the line impedance so that, though they are not sitting at the node position, there is relatively little current flowing through them and so they do not greatly perturb the standing wave on the line. If the frequency rises higher, the node moves to the right, so there is less power dissipated in Resistor B and more in Resistor A. Hence, if we measured the temperature difference between these resistors, higher temperature in Resistor B compared to Resistor A would indicate somewhat higher frequency in the line. Conversely, moving to lower frequency would move the node towards Resistor A giving the opposite temperature difference. Hence measuring the temperature difference between these resistors could gives a measure, with an appropriate sign, of changes in the signal frequency in the line.

Notes

It is not hard to come up with an approach that tells you that the frequency is off – just looking for a finite voltage across the resistor in the circuit of Supplementary Question 1 will tell you that. The tricky part is to come up with something that also tells you the sign – i.e., whether your frequency is above or below. That requires that you compare to something. The solution shown above does that, in this case in a symmetric way that makes it easy to get the sign without any calculations.