Discussion/solution. We can write $\mathbf{avg}(x) = (1/n)\mathbf{1}^T x$, so

$$\mathbf{avg}(Ax) = \mathbf{avg}(x) \iff (1/m)\mathbf{1}^T Ax = (1/n)\mathbf{1}^T x. \iff \mathbf{1}^T Ax = (m/n)\mathbf{1}^T x.$$

This holds for all x if and only if $\mathbf{1}^T A = (m/n)\mathbf{1}^T$, which can be expressed as $A^T\mathbf{1} = (m/n)\mathbf{1}$. This means that all columns of A must sum to m/n.

Another way to say it is: If you add up the rows of A, you get a row vector all of whose entries as m/n.

If A is square (which it need not be), the condition also means that A has $\mathbf{1}$ as a left eigenvector, with associated eigenvalue m/n.

The second question is a bit trickier. The solution is: $|\mathbf{avg}(Ax)| \leq |\mathbf{avg}(x)|$ for all x if and only if $A^T\mathbf{1} = \alpha(m/n)\mathbf{1}$ for some α with $|\alpha| \leq 1$. In other words, all columns of A must sum to m/n, times a constant (which is the same for all columns) less than or equal to one in magnitude. In terms of eigenvectors, the condition can be expressed as: A has $\mathbf{1}$ as a left eigenvector, with associated eigenvalue λ, with $|\lambda| \leq m/n$.

The "if" direction is clear: If $A^T\mathbf{1} = \alpha(m/n)\mathbf{1}$, where $|\alpha| \leq 1$, then for any x we have

$$\mathbf{avg}(Ax) = (1/m)|(A^T\mathbf{1})^T x| = (|\alpha|/n)|\mathbf{1}^T x| \leq (1/n)|\mathbf{1}^T x| = \mathbf{avg}(x).$$

Now we'll show the opposite direction. Let $a = A^T\mathbf{1}$ and $b = (m/n)\mathbf{1}$. Then $|\mathbf{avg}(Ax)| \leq |\mathbf{avg}(x)|$ can be written as $|a^T x| \leq |b^T x|$. Suppose that $|a^T x| \leq |b^T x|$ for all x. We'll show that $a = \alpha b$, for some $\alpha \in [-1, 1]$. Note that this holds if $a = 0$, with $\alpha = 0$, so we will assume that $a \neq 0$.

Clearly if $b^T x = 0$, then $a^T x = 0$. Thus $\mathcal{N}(b^T) \subseteq \mathcal{N}(a^T)$. Taking orthogonal complements we get $\mathcal{R}(b) \supseteq \mathcal{R}(a)$. In particular $b \in \mathcal{R}(a)$, which means that $b = \alpha a$ for some $\alpha \in \mathbf{R}$. Taking $x = b$ in $|a^T x| \leq |b^T x|$ yields

$$|a^T b| = |\alpha| a^T a \leq |b^T b| = \alpha^2 a^T a,$$

so $|\alpha| \leq \alpha^2$. From this we conclude $|\alpha| \leq 1$.

Another way to come to the conclusion that the sums of the columns of A must be equal is to consider the particular values of x given by $x = e_i - e_j$, with $i \neq j$. Then $\mathbf{avg}(x) = 0$, so we have to have $\mathbf{avg}(Ax) = 0$. But $\mathbf{avg}(Ax)$ is exactly half the difference of the sum of column i and column j. We conclude that these column sums must be equal; since i and j were arbitrary, we see that all columns of A must have the same sum.