Stanford University, Department of Electrical Engineering
Qualifying Examination, Systems Area, Winter 2014-15
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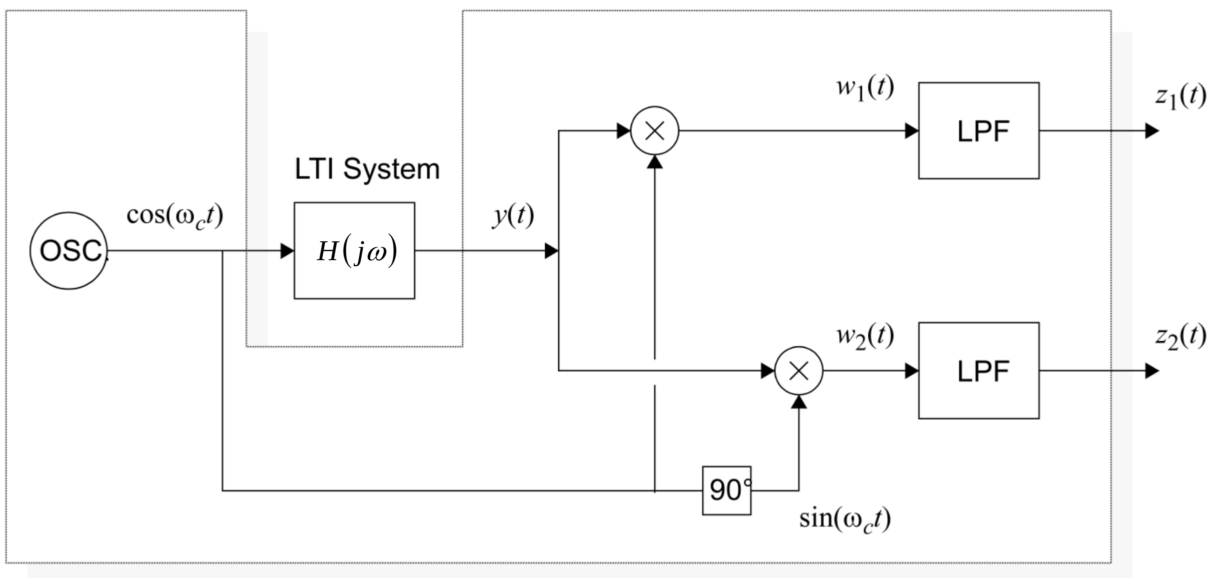
The instrument shown below (within the dashed box) is said to be useful for measuring the frequency response of a linear time-invariant (LTI) system, but someone misplaced the instruction manual. Evidently, the user connects an LTI system as shown. The oscillator frequency ω_c is swept over a frequency range of interest, and the two outputs $z_1(t)$ and $z_2(t)$ yield information about the frequency response $H(j\omega)$. Explain how the instrument works by deriving expressions for $y(t)$, $w_1(t)$ and $w_2(t)$ and $z_1(t)$ and $z_2(t)$. Assume the LTI system has a real impulse response $h(t)$. Also assume the lowpass filters (LPFs) are ideal and have cutoff frequencies much less than ω_c.



Solution

Since the LTI system's impulse response $h(t)$ is real, the frequency response $H(j\omega)$ has conjugate symmetry, $H(-j\omega) = H^*(j\omega)$. Equivalently, $|H(-j\omega_c)| = |H(j\omega_c)|$ and $\angle H(-j\omega_c) = -\angle H(j\omega_c)$.

We write the system input signal as

$$\cos\omega_c t = \frac{1}{2}e^{j\omega_c t} + \frac{1}{2}e^{-j\omega_c t}.$$

The system output signal is

$$\begin{aligned} y(t) &= \frac{1}{2}H(j\omega_c)e^{j\omega_c t} + \frac{1}{2}H(-j\omega_c)e^{-j\omega_c t} \\ &= \frac{1}{2}|H(j\omega_c)|\left[e^{j\angle H(j\omega_c)}e^{j\omega_c t} + e^{-j\angle H(j\omega_c)}e^{-j\omega_c t}\right]. \\ &= |H(j\omega_c)|\cos(\omega_c t + \angle H(j\omega_c)) \end{aligned}$$

Recall the identities

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)].$$

Using these identities, the multiplier outputs are

$$\begin{aligned} w_1(t) &= |H(j\omega_c)|\cos\omega_c t\cos(\omega_c t + \angle H(j\omega_c)) \\ &= \frac{1}{2}|H(j\omega_c)|[\cos(2\omega_c t + \angle H(j\omega_c)) + \cos\angle H(j\omega_c)] \end{aligned}$$

$$\begin{aligned} w_2(t) &= |H(j\omega_c)|\sin\omega_c t\cos(\omega_c t + \angle H(j\omega_c)) \\ &= \frac{1}{2}|H(j\omega_c)|[\sin(2\omega_c t + \angle H(j\omega_c)) - \sin\angle H(j\omega_c)] \end{aligned}$$

Each of the multiplier outputs contains a term at $\omega = 2\omega_c$ and a term at $\omega = 0$. The lowpass filters pass only the terms at $\omega = 0$, yielding

$$z_1(t) = \frac{1}{2}|H(j\omega_c)|\cos\angle H(j\omega_c)$$

$$z_2(t) = -\frac{1}{2}\left|H(j\omega_c)\right| \sin \angle H(j\omega_c).$$

Using $z_1(t)$ and $z_2(t)$, the magnitude and phase of $H(j\omega_c)$ can be computed as

$$\left|H(j\omega_c)\right|^2 = 4\left[z_1^2(t) + z_2^2(t)\right]$$

$$\angle H(j\omega_c) = -\tan^{-1}\left(\frac{z_2(t)}{z_1(t)}\right)$$