2006-2007 Electrical Enginering Qualifying Examination

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A *Bernoulli horse race* has m horses competing on a race course of length n steps.

At discrete times $k = 1, 2, 3, \ldots$ each horse flips an unbiased coin and advances one step if the coin comes up heads.

Consider just one horse. Let T be the number of coin flips that the horse takes to finish.

1. Find $\mathrm{E}(T)$, the average value of T.
2. Find $\mathrm{P}\{T \le 2n-1\}$, the probability that the horse finishes within $2n-1$ coin flips.
3. Find $p_T(r) = \mathrm{P}\{T = r\}$, the probability mass function.
4. Find the most probable value of T, that is, the largest value of $p_T(r)$.

SOLUTIONS

1. The random variable T can be written as the sum $T_1 + T_2 + \cdots + T_n$ where T_i is the number of coin flips needed to move from step $i-1$ to step i. Each T_i is an geometric random variable with success probability $p = 1/2$ and expected value $1/p = 2$. Therefore $\mathrm{E}(T) = \sum_{i=1}^{n} T_i = 2n$.

2. Consider all 2^{2n-1} sequences of $2n-1$ coin flips. A horse finishes in $2n-1$ flips if and only if at least n of the flips are heads. (If a horse finishes in less than $2n-1$ coin flips, then the remaining flips need not be looked at.) The probability of n heads in $2n-1$ flips of an unbiased coin is the probability that the majority of an odd number of flips is heads, namely, $1/2$.

3. If a horse finishes on the r-th coin flip, then the last flip is heads, and the first $r-1$ coin flips contain exactly $n-1$ heads. There are $\binom{r-1}{n-1}$ sequences satisfying these conditions. Each such sequence has probability 2^{-r}. Therefore $\mathrm{P}\{T = r\} = \binom{r-1}{n-1} 2^{-r}$.

   (The sum of n geometric random variables has a *negative binomial probability distribution*.)

4. Because the pmf values are products, the easiest way to determine the maximum is to look at ratios of successive probabilities.

$$\frac{p_T(r)}{p_T(r+1)} = \frac{\binom{r-1}{n-1} 2^{-r}}{\binom{r}{n-1} 2^{-r-1}} = \frac{r-n+1}{\frac{1}{2} r} \lesseqgtr 1 \;\Leftrightarrow\; r-n+1 \lesseqgtr \tfrac{1}{2} r \;\Leftrightarrow\; r < 2n-2\,.$$

   This shows that if $r < 2n-2$ then $p_T(r) < p_T(r+1)$ and if $r > 2n-2$ then $p_T(r) > p_T(r+1)$. The maximum value occurs at $r = 2n-2$ and $r = 2n-1$.