Electrostatics_3

Continued from Electrostatics_2

Electric Quadrupole:

An electric quadrupole is a system of two dipoles aligned in such a way that the net electric field or effect don't cancel each other.

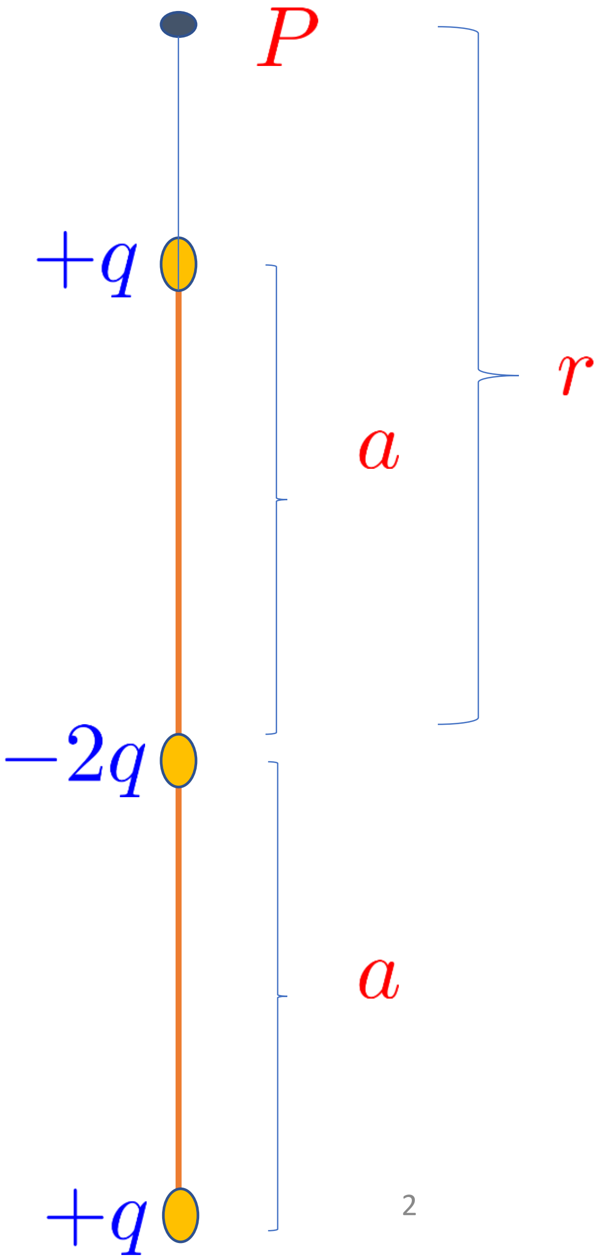
Consider an electric quadrupole which has two dipoles each of length 'a'. We have to determine the field and potential at a point P.

Here, we assume OP = r.

Then the electric potential is given by

$$V = \sum V_i$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0(r-a)} - \frac{2q}{4\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0(r+a)}$$



$$\Rightarrow \quad V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r-a} - \frac{2}{r} + \frac{1}{r+a}\right)$$

$$\Rightarrow \quad V = \frac{q}{4\pi\epsilon_0}\left[\frac{r(r+a) - 2r^2 + 2a^2 + r(r-a)}{r(r^2-a^2)}\right]$$

$$\Rightarrow \quad V = \frac{q.2a^2}{4\pi\epsilon_0 r(r^2-a^2)}$$

$$\Rightarrow \quad \boxed{V = \frac{Q}{4\pi\epsilon_0 r(r^2-a^2)}}$$

Cases: For a short quadrupole, $r \gg a$

$$So,\ r^2 - a^2 \approx r^2$$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r^3}}$$

Here, $Q = q.2a^2, \; is \; the \; quadrupole \; moment.$

This gives the electric potential due to a short quadrupole.

Electric Field at P:

The electric field at P is given by

$$E = \frac{q}{4\pi\epsilon_0(r-a)^2} - \frac{2q}{4\pi\epsilon_0 r^2} + \frac{q}{4\pi\epsilon_0(r+a)^2}$$

H.W.: Simplify above expression carefully (school level simplification) to arrive at the following expression

$$\Rightarrow \quad E = \frac{q}{4\pi\epsilon_0}\left[\frac{6a^2r^2 - 2a^4}{r^2(r-a)^2)(r+a)^2}\right]$$

$$\Rightarrow \quad E = \frac{q.2a^2(3r^2 - a^2)}{4\pi\epsilon_0 r^2(r^2 - a^2)^2}$$

$$\Rightarrow \quad E = \frac{Q(3r^2 - a^2)}{4\pi\epsilon_0 r^2(r^2 - a^2)^2}$$

where $Q = q.2a^2,$
is the quadrupole moment.

Special case:

For a short quadrupole, $r \gg a$

$$So, 3r^2 - a^2 \approx 3r^2 \;\; and \;\; r^2 - a^2 \approx r^2$$

$$E = \frac{Q.3\cancel{r^2}}{4\pi\epsilon_0 \cancel{r^2} r^4} \Longrightarrow \boxed{E = \frac{3Q}{4\pi\epsilon_0 r^4}}$$

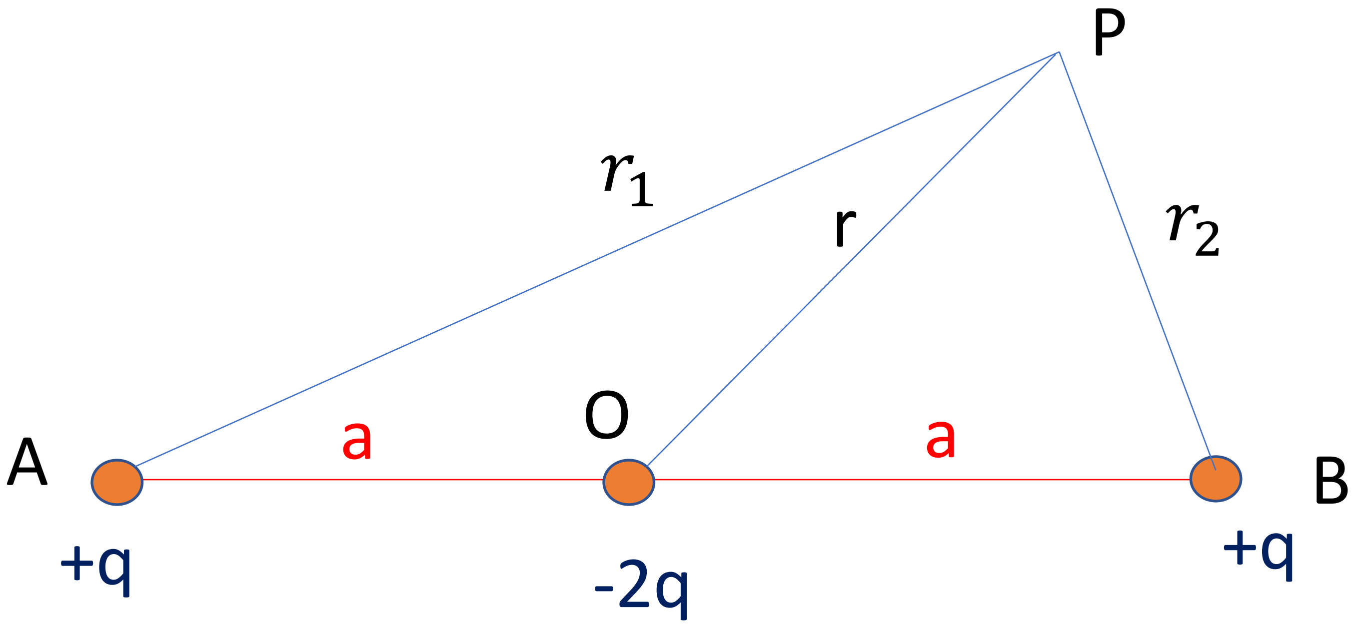
This is the required electric field due to a short dipole at a point on the axis of the quadrupole.

For quadrupole, we have found the electric potential and electric field (for the short quadrupole)at a point on the axis.

Now, we wish to find the potential due to a short quadrupole at any point.

Electric Potential due to a (short) quadrupole at any point):

Consider an electric quadrupole of length 2a. We have to find the electric potential at a point P such that OP = r, AP = r_1 and BP = r_2.



Now, the potential at the point P is given by the algebraic sum of the individual potentials.

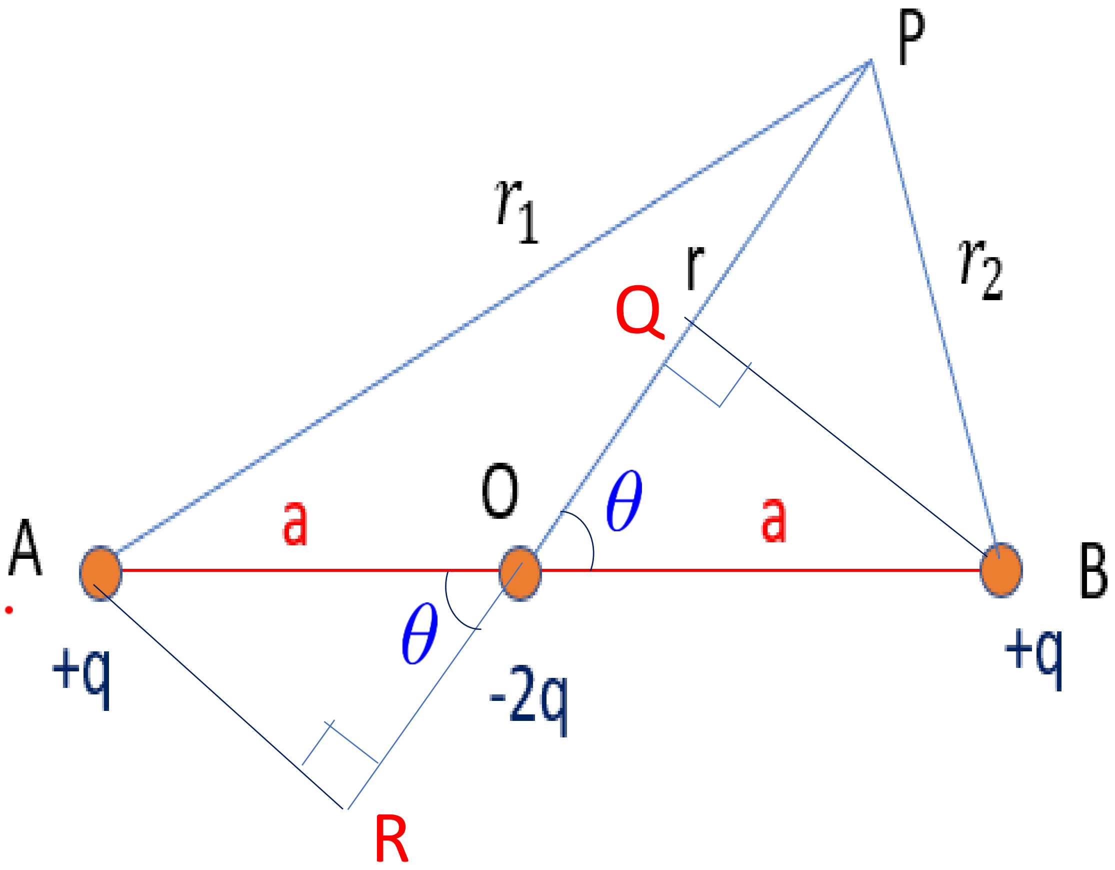
i.e. $$V = \frac{q}{4\pi\epsilon_0 AP} - \frac{2q}{4\pi\epsilon_0 OP} + \frac{q}{4\pi\epsilon_0 BP}$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r}\right]$$

$$Let\ \angle BOP = \theta$$

Then, we have to express r_1 and r_2 in terms of r, a and θ

$$Now,\ draw\ AR \perp PR\ and\ BQ \perp\ PO.$$



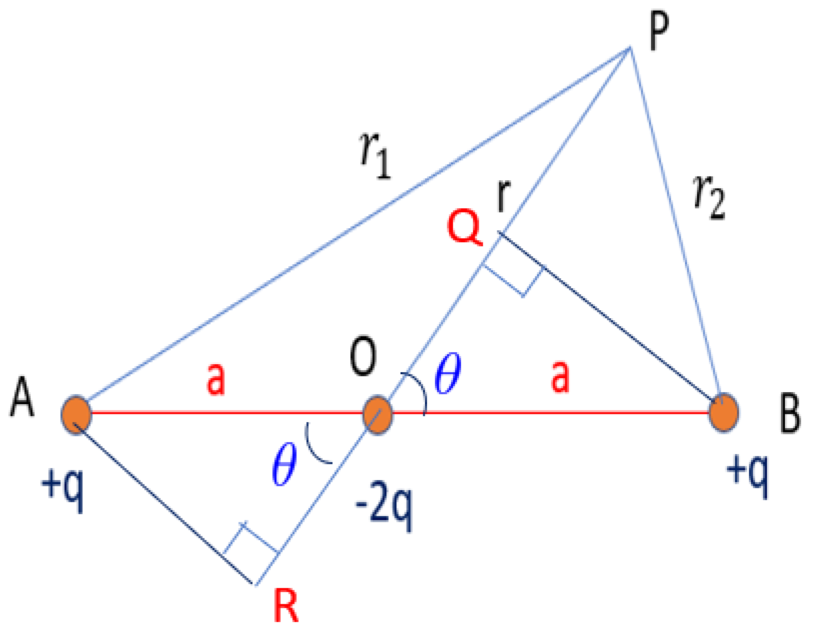
$$In\ \triangle APR,\ AP^2 = PR^2 + AR^2$$

$$or,\ AP^2 = (PO + OR)^2 + AR^2$$

$$or,\ AP^2 = (PO + a\cos\theta)^2 + (a\sin\theta)^2$$

$$or,\ r_1 = AP = [(r + a\cos\theta)^2 + (a\sin\theta)^2]^{\frac{1}{2}}$$

$$or,\ r_1 = AP = (r^2 + 2ar\cos\theta + a^2)^{\frac{1}{2}}$$



$$Similarly,\ in\ \triangle BOQ,\ BP^2 = PQ^2 + BQ^2$$

$$or,\ BP^2 = (PO - OQ)^2 + BQ^2$$

$$or,\ r_2 = BP = (r^2 - 2ar\cos\theta + a^2)^{\frac{1}{2}}$$

Using the values of r_1 and r_2, we get

$$V = \frac{q}{4\pi\epsilon_0}[\frac{1}{(r^2 + a^2 + 2ar\cos\theta)^{\frac{1}{2}}} + \frac{1}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{1}{2}}} - \frac{2}{r}]$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0}[\frac{1}{r(1 + \frac{a^2}{r^2} + \frac{2a\cos\theta}{r})^{\frac{1}{2}}} + \frac{1}{r(1 + \frac{a^2}{r^2} - \frac{2a\cos\theta}{r})^{\frac{1}{2}}} - \frac{2}{r}]$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0 r}[\frac{1}{(1 + \frac{a^2}{r^2} + \frac{2a\cos\theta}{r})^{\frac{1}{2}}} + \frac{1}{(1 + \frac{a^2}{r^2} - \frac{2a\cos\theta}{r})^{\frac{1}{2}}} - 2]$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0 r}[(1 + \frac{a^2}{r^2} + \frac{2a\cos\theta}{r})^{-\frac{1}{2}} + (1 + \frac{a^2}{r^2} - \frac{2a\cos\theta}{r})^{-\frac{1}{2}} - 2]$$

Using the binomial expansion, $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots..$

$$Here,\ n = -\frac{1}{2},\ so\ \frac{n(n-1)}{2!} = \frac{3}{8}$$

$$V = \frac{q}{4\pi\epsilon_0}[\frac{1}{(r^2+a^2+2ar\cos\theta)^{\frac{1}{2}}} + \frac{1}{(r^2+a^2-2ar\cos\theta)^{\frac{1}{2}}} - \frac{2}{r}]$$

$$V = \frac{q}{4\pi\epsilon_0}[\frac{1}{r(1+\frac{a^2}{r^2}+\frac{2a\cos\theta}{r})^{\frac{1}{2}}} + \frac{1}{r(1+\frac{a^2}{r^2}-\frac{2a\cos\theta}{r})^{\frac{1}{2}}} - \frac{2}{r}]$$

$$V = \frac{q}{4\pi\epsilon_0 r}[\frac{1}{(1+\frac{a^2}{r^2}+\frac{2a\cos\theta}{r})^{\frac{1}{2}}} + \frac{1}{(1+\frac{a^2}{r^2}-\frac{2a\cos\theta}{r})^{\frac{1}{2}}} - 2]$$

$$V = \frac{q}{4\pi\epsilon_0 r}[(1+\frac{a^2}{r^2}+\frac{2a\cos\theta}{r})^{-\frac{1}{2}} + (1+\frac{a^2}{r^2}-\frac{2a\cos\theta}{r})^{-\frac{1}{2}} - 2]$$

Using the binomial expansion, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 \ldots .$

$Here,\ n = -\frac{1}{2},\ so\ \frac{n(n-1)}{2!} = \frac{3}{8}$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0 r}[1 - \frac{1}{2}(\frac{a^2}{r^2}+\frac{2a\cos\theta}{r}) + \frac{3}{8}(\frac{a^2}{r^2}+\frac{2a\cos\theta}{r})^2 + \ldots 1 - \frac{1}{2}(\frac{a^2}{r^2}-\frac{2a\cos\theta}{r}) + \frac{3}{8}(\frac{a^2}{r^2}-\frac{2a\cos\theta}{r})^2 + \ldots - 2]$$

$$\Rightarrow V = \frac{q}{4\pi\epsilon_0 r}[2 - \frac{1}{2}(\frac{a^2}{r^2}+\frac{2a\cos\theta}{r}) - \frac{1}{2}(\frac{a^2}{r^2}-\frac{2a\cos\theta}{r}) + \frac{3}{8}(\frac{2a\cos\theta}{r})^2 + \frac{3}{8}(\frac{2a\cos\theta}{r})^2 - 2]$$

Here, we have assumed that, for a short quadrupole, $r \gg a$

So, the term $\frac{3}{8}(\frac{a^2}{r^2} + \frac{2a\cos\theta}{r})^2$ when expanded has, the significant contribution of only the square of second term.

$$V = \frac{q}{4\pi\epsilon_0 r}[-\frac{a^2}{r^2} - \cancel{\frac{a\cos\theta}{r}} + \cancel{\frac{a\cos\theta}{r}} + \frac{6}{8}(\frac{2a\cos\theta}{r})^2]$$

$$\Rightarrow \quad V = \frac{q}{4\pi\epsilon_0 r}[\frac{3a^2\cos^2\theta}{r^2} - \frac{a^2}{r^2}]$$

$$\Rightarrow \quad V = \frac{q}{4\pi\epsilon_0 r}\frac{a^2}{r^2}(3\cos^2\theta - 1)$$

$$\Rightarrow \quad \boxed{V = \frac{qa^2}{4\pi\epsilon_0 r^3}(3\cos^2\theta - 1)}$$

Cases: $(i) At\ the\ axial\ line,\ \theta = 0^\circ\ or\ 180^\circ,$

$$V = \frac{qa^2}{4\pi\epsilon_0 r^3}.2 = \frac{Q}{4\pi\epsilon_0 r^3}$$

$$(ii) At\ the\ equatorial\ line,\ \theta = 90^\circ$$

$$V = \frac{qa^2}{4\pi\epsilon_0 r^3}(-1) = -\frac{Q}{8\pi\epsilon_0 r^3}$$

$$So,\ \frac{V_{axial}}{V_{equatorial}} = 2:1$$