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Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 52 (2016) 213–220 www.elsevier.com/locate/endm

On the computational complexity of the virtual network embedding problem

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Abstract

Given a graph representing a substrate (or physical) network with node and edge capacities and a set of virtual networks with node capacity demands and node-to-node traffic demands, the Virtual Network Embedding problem (VNE) calls for an embedding of (a subset of) the virtual networks onto the substrate network which maximizes the total profit while respecting the physical node and edge capacities. In this work, we investigate the computational complexity of VNE. In particular, we present a polynomial-time reduction from the maximum stable set problem which implies strong $\mathcal{NP}$-hardness for VNE even for very special subclasses of graphs and yields a strong inapproximability result for general graphs. We also consider the special cases obtained when fixing one of the dimensions of the problem to one. We show that VNE is still strongly $\mathcal{NP}$-hard when a single virtual network request is present or when each virtual network request consists of a single virtual node and that it is weakly $\mathcal{NP}$-hard for the case with a single physical node.

Keywords: Virtual network embedding, combinatorial optimization, computational complexity, inapproximability

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http://dx.doi.org/10.1016/j.endm.2016.03.028

1 Introduction

Network virtualization techniques are a prominent topic in the recent networking literature [CB10]. The general idea of this paradigm is that of decoupling the high level role of service provisioning from the low level one of management and operation of the substrate physical network. This allows for a largely more flexible environment which, at least partially, helps preventing ossification phenomena due to the difficulty of upgrading the physical topology of large, preexisting networks.

We face a *Virtual Network Embedding* problem (VNE) when, given a substrate network and set of Virtual Network (VN) requests, we have to decide which VN requests to accept or reject and how to allocate the physical resources to the accepted VNs so as to maximize a profit function. Each VN is typically endowed with *node demands*, i.e., physical capacity requirements for its virtual nodes, and *traffic demands* between pairs of virtual nodes sharing a virtual edge. By performing a mapping, for the accepted VNs, of the virtual nodes onto the physical ones and of the virtual edges onto physical paths, a solution to VNE realizes an *embedding* of the virtual networks onto the substrate network.

As discussed in [CGK+15], see also the comprehensive survey in [CB10], VNE has a number of features. First, it can be either *online*, with VN requests arriving dynamically over time, or *offline*. Second, according to some authors, it may or may not involve *admission control*, thus either allowing or preventing the rejection of VN requests. The node mapping aspect can then be subject to *locality requirements* which, when present, restrict the set of physical nodes onto which a virtual node can be mapped. Also, different *routing schemes* can be considered, typically allowing for either a splittable or an unsplittable routing. Lastly, the substrate graph, as well as the different VNs, can be either directed or undirected graphs.

To the best of our knowledge, most of the papers where the complexity of VNE is mentioned claim that the problem is $\mathcal{NP}$-hard by reduction from the *k-multiway separator problem*, citing an (unpublished) technical report [And02] devoted to the (therein called) testbed allocation problem. Although the latter problem is connected to VNE, to establish its $\mathcal{NP}$-hardness the technical report sketches a reduction which lacks sufficient detail to verify its correctness. The only alternative $\mathcal{NP}$-hardness result which we are aware of (mentioned, among others, in [YYRC08]) relies on a straightforward reduction from the unsplittable multicommodity flow problem. However, this result only holds for the special case of VNE where the node mapping is already given and an unsplittable routing is employed.

In this work, we present a general polynomial-time reduction which implies the $\mathcal{NP}$-hardness of VNE even when restricting the problem to very specific subclasses of graphs and a strong inapproximability result for general graphs. We also investigate the complexity of VNE when one of its dimensions (i.e., the number of requests or that of virtual or physical nodes) is fixed to one. Throughout this paper, we focus on the offline version of the problem, assuming that both the physical and the virtual networks are undirected graphs and assuming an unsplittable single path routing (an option which is often favored in the applications as it avoids the issue of packet reordering). To be as general as possible, we will consider the case where locality constraints can be present.

2 Problem definition and previous work

Let R be the set of indices of the collection of VN requests. Let $G^0 = (V^0, E^0)$ denote the undirected graph representing the substrate network. Physical node and edge capacities are denoted by B_i, for $i \in V^0$, and K_{ij}, for $\{i, j\} \in E^0$. For each VN request of index $r \in R$, let $G^r = (V^r, E^r)$ be the corresponding virtual graph. For each virtual node $v \in V^r$, t_v^r denotes the corresponding node demand. Virtual links $\{v, w\} \in E^r$ are specified implicitly via the traffic demand d_{vw}^r between their end points $v, w \in V^r$. For each $r \in R$, the overall traffic matrix is denoted by $D^r \in \mathbb{R}_+^{|V^r| \times |V^r|}$. Profits for embedded requests are denoted by $p^r \geq 0$, for $r \in R$. For each $r \in R$ and $v \in V^r$, let $V^0(r, v)$ be the set of physical nodes on which the virtual node v can be mapped (due to locality restrictions). When $|V^0(r, v)| = 1$ for some $r \in R$ and $v \in V^r$, we refer to VNE with *extreme locality* constraints.

The (offline) version of VNE that we consider in this paper (see Figure 1 for the sketch of an instance) is defined as follows:

Given: i) an undirected substrate graph $G^0 = (V^0, E^0)$ with node and edge capacities B_i and K_{ij} and ii) for each $r \in R$, an undirected virtual network $G^r = (V^r, E^r)$ with a profit p^r, node demands t_v^r and traffic demands d_{vw}^r, select a subset $R' \subseteq R$ of requests and, for each $r \in R'$, determine an embedding (a mapping of the virtual nodes $v \in V^r$ onto eligible physical nodes $V^0(r, v)$ and of the virtual links $\{v, w\} \in E^r$ onto paths of G^0, for every $r \in R'$) which does not exceed physical node and link capacities and maximizes the sum of the profits of the accepted VN requests in R'.

In the literature, among the many previous and related works, most of

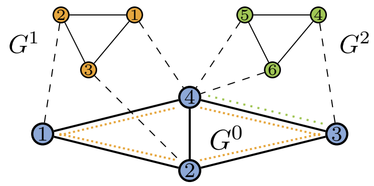

Fig. 1. A VNE instance with a mapping of two VN requests G^1 and G^2 onto the substrate G^0.

those which tackle variants of VNE as optimization problems adopt a two-phase heuristic approach, splitting the problem into two subproblems solved in sequence: a node mapping problem and a link mapping one, see [ZA06] and the references in [CB10]. Note that the second phase problem of these methods can be seen as a special case of VNE with extreme locality constraints for all the virtual nodes.

To the best of our knowledge, little attention has been devoted, so far, to exact approaches. Among the few cases, we mention [HLBAZ11,CGK$^+$15,CKT15]. In [HLBAZ11], a Mixed-Integer Linear Programming (MILP) formulation is proposed and used as an (optimal) oracle in an algorithm for the online version of VNE. In [CGK$^+$15], the authors propose an MILP formulation encompassing admission control, splittable or unsplittable routing, and locality restrictions. Their model also entails an extra network design aspect, according to which physical capacities must be rented, in bulks, before the corresponding physical nodes and links can be used. For a robust optimization approach where Γ-robustness is adopted to account for data uncertainties in the node and traffic demands, see [CKT15].

3 Strong $\mathcal{NP}$-hardness and inapproximability results

In this section, we propose a reduction from the Maximum Stable Set Problem (MSSP) which, as we shall see, implies a strong inapproximability result for VNE. In MSSP, given an undirected graph $G = (V, E)$, we look for a stable set (i.e., a subset of nodes that are not pairwise adjacent) that is as large as possible. We first propose the reduction for VNE with extreme locality constraints and then extend it to the less restrictive case where each virtual node can be mapped to any physical node.

Theorem 3.1 *VNE with extreme locality is strongly $\mathcal{NP}$-hard.*

Proof. Consider the decision version of MSSP which, given a graph $G = (V, E)$ and a positive integer k, asks whether G contains a stable set of cardi-

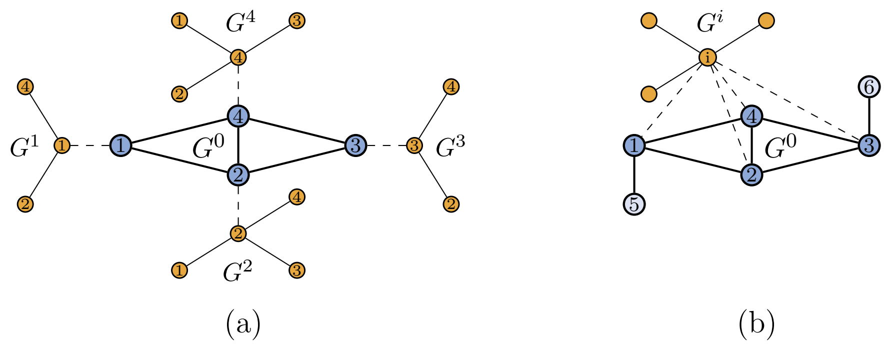

Fig. 2. Illustrations of the reductions in Theorem 3.1 (a) and Corollary 3.2 (b).

nality at least k. We describe a polynomial time reduction from this problem to VNE with extreme locality constraints. For any instance of the decision version of MSSP, we construct (in linear time) the special instance of VNE consisting of the substrate network $G^0 = G$ with unit node and edge capacities, and $|V^0|$ VN requests with unit profits, one for each node $i \in V^0$. The VN request G^r with $r \in R$ corresponding to physical node $i \in V^0$ is isomorphic to the closed neighborhood of node i, i.e., is a star graph with $1 + |\delta(i)|$ nodes (a central virtual node corresponding to $i \in V^0$ and $|\delta(i)|$ virtual leaf nodes, one for each neighbor $j \in \delta(i)$). The central virtual node has a unit demand, while all the virtual leaf nodes have a demand of $\frac{1}{\Delta}$, where $\Delta = \max_{i \in V^0} |\delta(i)|$. The extreme locality constraints are as follows: for each $r \in R$, the central node of the VN G^r corresponding to node $i \in V^0$ can only be mapped to node i, while each virtual leaf node only to the corresponding neighbor $j \in \delta(i)$ in G^0. See Figure 2 (a) for an illustration, where the numbers in the nodes of G^0 and of the VNs indicate the extreme node mapping constraints.

Due to the unit node and edge capacities, if a VN of index $r \in R$ (whose central virtual node corresponds to the physical node $i \in V^0$) is accepted, then no other VN of index $r' \in R$ with a center corresponding to some $j \in \delta(i)$ can be accepted. Therefore, the VNE instance admits a feasible solution of total profit k (where k VNs are simultaneously embedded) if and only if the graph G of the MSSP instance contains a stable set of cardinality at least k. □

It is worth pointing out that, since MSSP is $\mathcal{NP}$-hard even for planar graphs of maximum degree 3 [Pol74] and for triangle-free graphs (i.e., graphs with chordless cycles of size 4 or more) [GJS76], this also holds for VNE (with extreme locality constraints).

The above complexity result can be extended as follows:

Corollary 3.2 *VNE is strongly $\mathcal{NP}$-hard even without locality constraints.*

Proof. We modify the reduction in the proof of Theorem 3.1 as follows. In the substrate network, for each node $i \in V^0$, we add $\Delta - |\delta(i)|$ nodes (leaves) connected solely to i. All the VNs are now identical stars with exactly Δ leaves, with unit traffic demands, a demand of 1 for the central node, and one of $\frac{1}{\Delta}$ for all the leaf nodes. See Figure 2 (b) for an illustration. Because of the traffic requirements, no two virtual nodes of a VN can be mapped onto the same physical node. As before, due to the node capacity constraints, no central node of a VN corresponding to a request r can be mapped next (adjacent) to a node where the central node of another request r' has been mapped. $\square$

As far as the approximability of VNE is concerned, the reduction from MSSP in the proof of Theorem 3.1 and the inapproximability result for MSSP in [Has99] imply that:

Corollary 3.3 *Unless $\mathcal{P} = \mathcal{NP}$, VNE with extreme locality constraints cannot be approximated in polynomial time within a factor of $n^{1/2-\epsilon}$ for any $\epsilon > 0$.*

4 Special cases of VNE with a constant dimension

We consider three cases where one of the dimensions of VNE, namely, the number of requests $|R|$, the size of the VNs $|V^r|$ for $r \in R$, and the size of the substrate $|V^0|$, is equal to 1.

The first result is obtained when fixing the number of requests to one. It relies on minor containment and, specifically, on the Maximum Clique Minor Problem (MCMP). Given a graph G and an integer k, MCMP calls for a subgraph G' which, after an edge contraction operation, is isomorphic to a clique of cardinality k. Since MCMP is strongly $\mathcal{NP}$-hard [Epp09], we have the following:

Proposition 4.1 *VNE is strongly $\mathcal{NP}$-hard even when $|R| = 1$.*

Proof. For any MCMP instance with a graph G asking for a clique minor of at least size k, it suffices to construct a VNE instance with $G^0 = G$ (with unit node and edge capacities) and with a single VN request (with k nodes, a complete demand matrix with unit entries, and unit node demands). $\square$

The second result concerns the case of VNs with a single node and relies on the strong $\mathcal{NP}$-hardness of the Multi-Knapsack Problem (MKP) [GZ86]:

Proposition 4.2 *VNE is strongly $\mathcal{NP}$-hard even when $|V^r| = 1$ for all $r \in R$.*

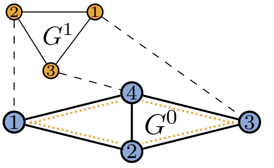

Fig. 3. Illustration of the reduction used in Proposition 4.1.

Proof. Consider any MKP instance with p knapsacks and q items, with knapsack capacities $b_j \in \mathbb{Z}_+$, for all $j = 1, \ldots, p$, item weights $w_i \in \mathbb{Z}_+$ and profits $p_i \in \mathbb{Z}_+$, for all $i = 1, \ldots, q$, and a nonnegative integer k. Construct a VNE instance with a physical node for each knapsack of index j in MKP, also having the same capacity b_j, and as many VN requests as the MKP items. Construct each VN request as a single virtual node with demand t_i^r equal to the weight w_i of the corresponding MKP item, also having the same profit p_i. Then, the MKP instance has a solution of value at least k if and only if a subset of VN requests with a total profit of at least k can be embedded. □

It is well known that the standard dynamic programming algorithm for the (single) Knapsack Problem (KP) can be adapted for the solution of MKP. Although the algorithm becomes exponential in the number of knapsacks, it is pseudo-polynomial whenever the number of knapsacks is a constant. This immediately implies that the special case of VNE with a single physical node can be solved in pseudo-polynomial time. Note that this holds regardless of the size or topology of each VN which, due to the presence of a single physical node, can be transformed w.l.o.g. into a single virtual node. By a straightforward reduction from KP, we also have the following:

Corollary 4.3 *VNE with $|V^0| = 1$ is weakly $\mathcal{NP}$-hard.*

5 Concluding remarks

We have addressed the computational complexity of VNE, showing its strong $\mathcal{NP}$-hardness even for a restricted subclass of graphs and derived a strong inapproximability result for general graphs. We have also shown $\mathcal{NP}$-hardness for the special cases of VNE where a dimension of the problem is fixed to one.

Acknowledgment

This work was partially supported by the German Federal Ministry of Education and Research (BMBF grant 05M13PAA, joint project 05M2013 - VINO: Virtual Network Optimization).

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