# Greedy Algorithms*

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Algorithm Course: Shanghai University

[*] Special thanks is given to Prof. Xiaofeng Gao for sharing her slides.

# Outline

1. Basic Methodology
    - Interval Scheduling
    - Interval Partitioning
    - Scheduling to Minimize Lateness
2. More Examples
    - Optimal Caching
    - Coin Changing

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# Interval Scheduling: An Introductory Example

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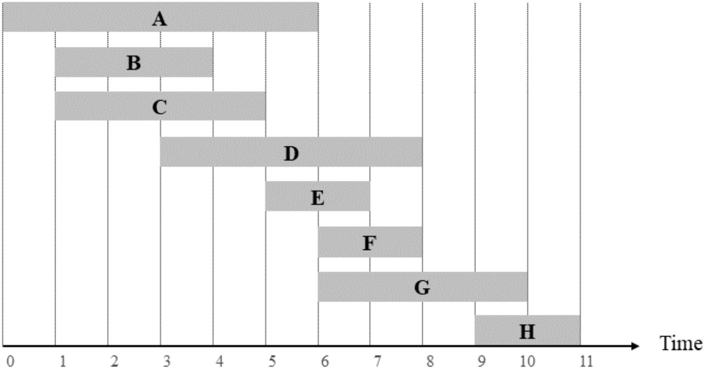

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**Interval Scheduling Problem:** Consider jobs in some natural order. Take each job provided to judge its compatibility with the ones already taken.

# Attempts

[Earliest start time] Consider jobs in ascending order of $s_j$.

[Earliest finish time] Consider jobs in ascending order of $f_j$.

[Shortest interval] Consider jobs in ascending order of $f_j - s_j$.

[Fewest conflicts] For each job $j$, count the number of conflicting jobs $c_j$. Schedule in ascending order of $c_j$.

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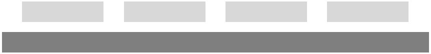
Counter Example:



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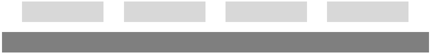

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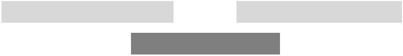

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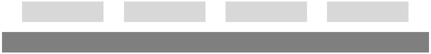

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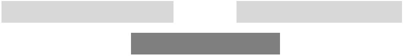

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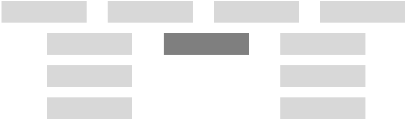

# Greedy Interval Scheduling Algorithm

**Algorithm 1:** Greedy Interval Scheduling

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Sort jobs by finish times so that $f_1 \leq f_2 \leq ... \leq f_n$;
$A \leftarrow \emptyset$ ;                    // set of jobs selected
for $j = 1$ to $n$ do
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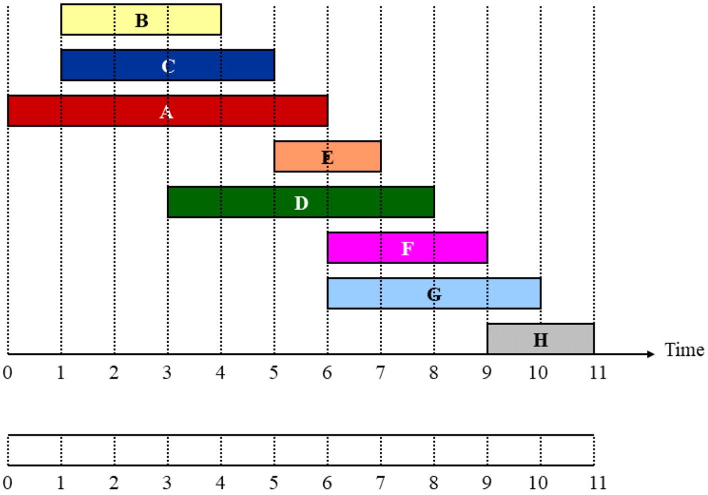
Implementation: $O(n \log n)$.

- After each iteration, set job $j^*$ that was added last to $A$.
- Job $j$ is compatible with $A$ if $s_j \geq f_{j^*}$.

# Demo



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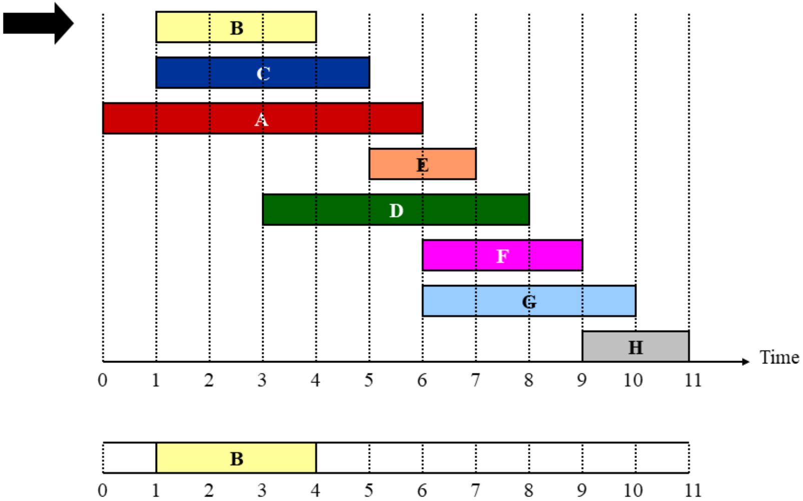

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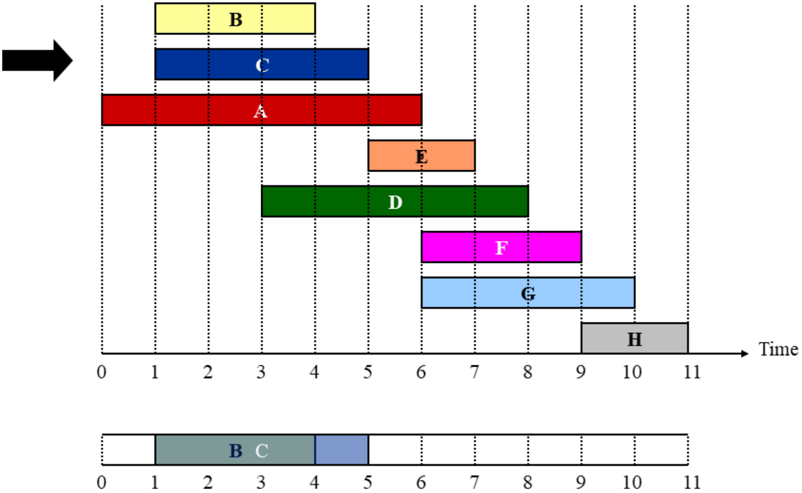

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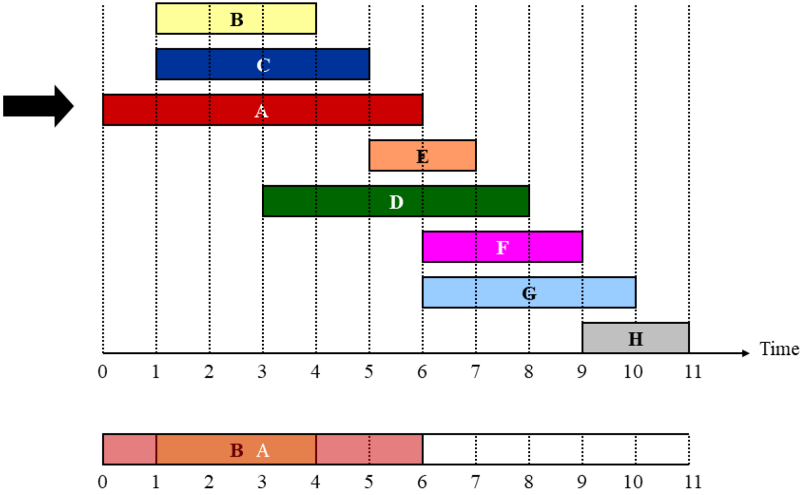

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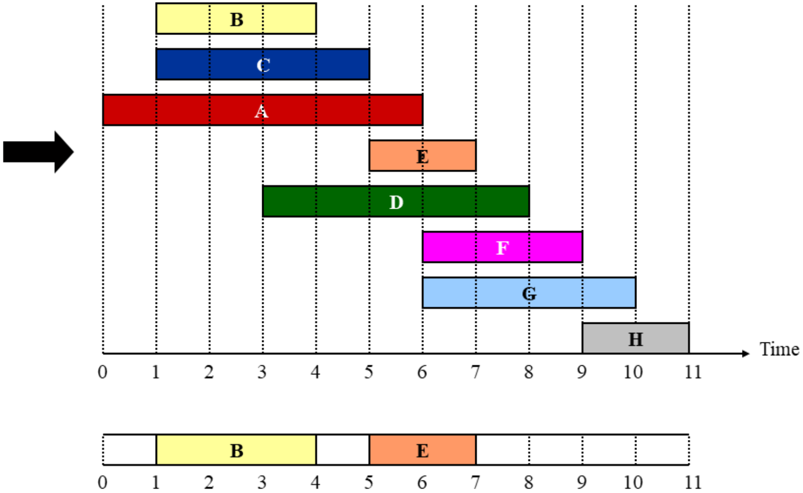

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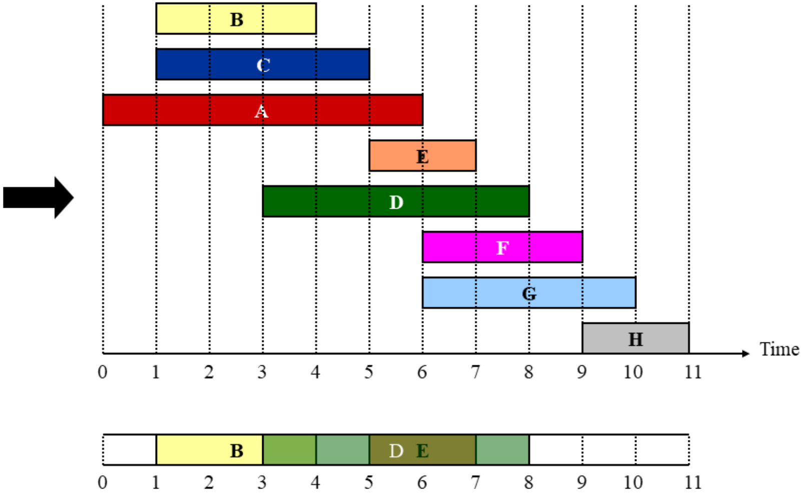

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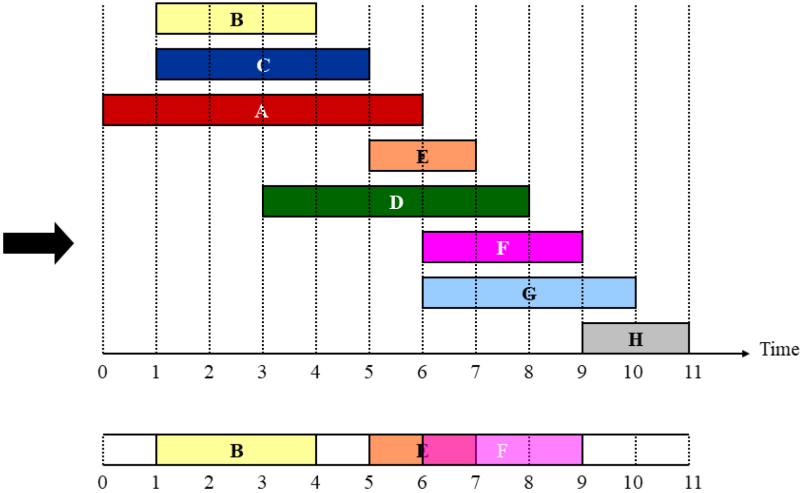

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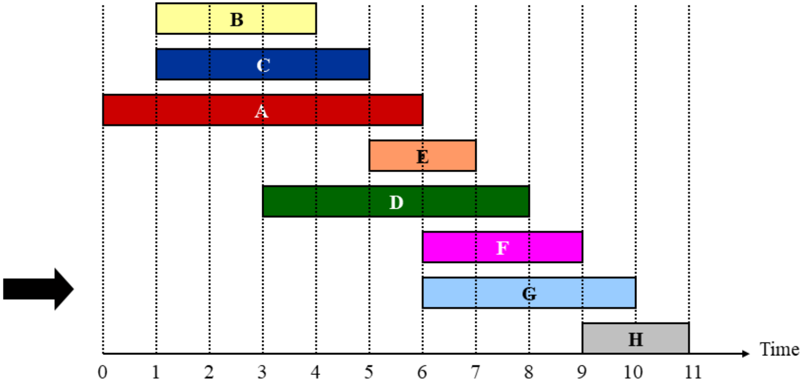

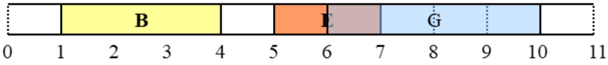

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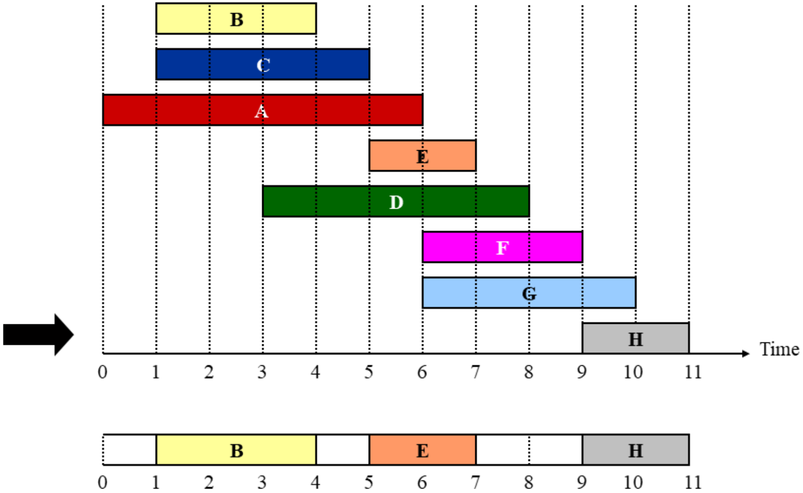

# Notation

**Greedy Solution**: $\{B, E, H\}$

**Optimal Solutions**: (not necessarily unique)

$\{A, F, H\}, \{B, E, H\}, \{B, F, H\}, \{C, E, H\}, \{C, F, H\}$

**Feasible Solutions**: (can work, but may not be the best)

$\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \{G\}, \{H\};$
$\{A, F\}, \{A, G\}, \{A, H\}, \{B, E\}, \{B, F\}, \{B, G\}, \{B, H\},$
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Let $j_1, j_2, \cdots, j_m$ denote set of jobs in an optimal solution with $i_1 = j_1$, $i_2 = j_2, \cdots, i_r = j_r$ for the largest possible value of $r$.

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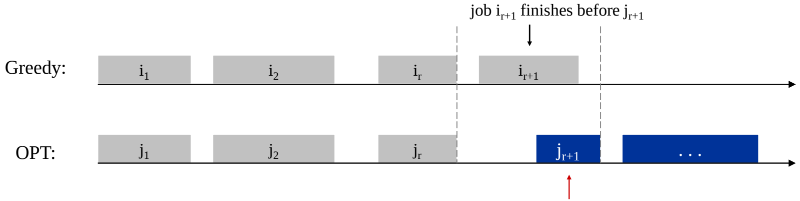

why not replace job $j_{r+1}$
with job $i_{r+1}$?

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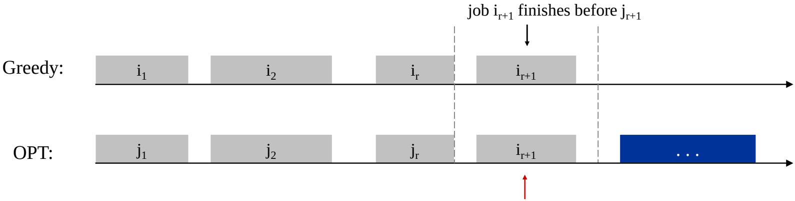

solution still feasible and optimal,
but contradicts the maximality of r.

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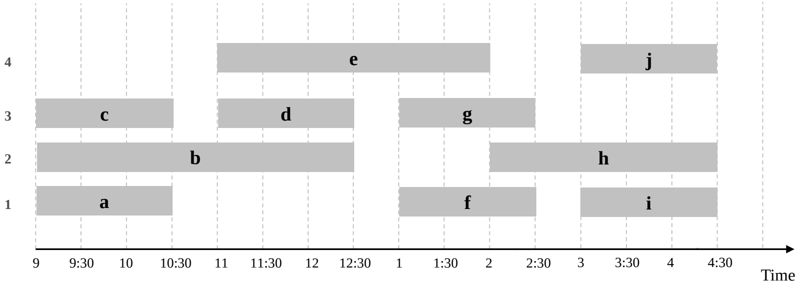
**Example:** This schedule uses 4 classrooms to schedule 10 lectures.



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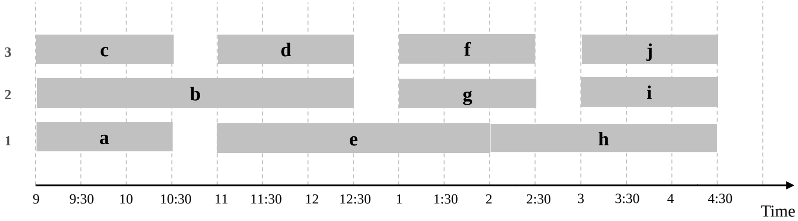

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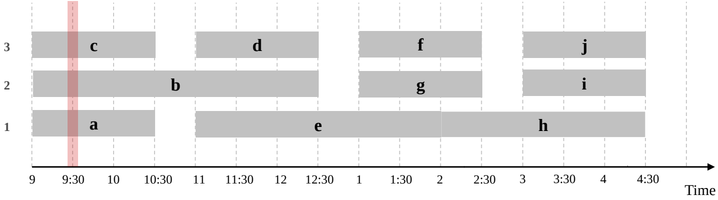

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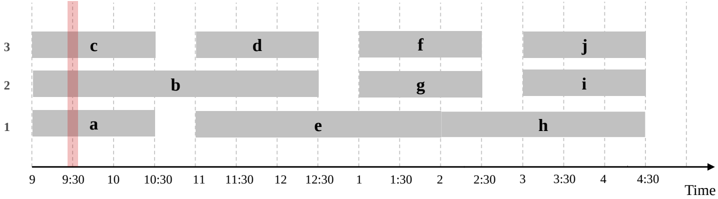

Question. Does there always exist a schedule equal to depth of intervals?

# Greedy Interval Partitioning Algorithm

**Algorithm 2:** Interval Partitioning Greedy Algorithm

```
Sort intervals by starting time so that $s_1 \leq s_2 \leq \ldots \leq s_n$;
$d \leftarrow 0$ ;                    // number of allocated classrooms
for $j = 1$ to $n$ do
    if lecture $j$ is compatible with some classroom $k$ then
        schedule lecture $j$ in classroom $k$;
    else
        allocate a new classroom $d + 1$;
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```

Implementation: $O(n \log n)$. 最小堆

- For classroom $k$, maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.

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Key observation ⇒ all schedules use $\geq d$ classrooms.

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# Scheduling to Minimize Lateness

- Single resource processes one job at a time.
- Job $j$ requires $t_j$ units of processing time and is due at time $d_j$.
- If $j$ starts at time $s_j$, it finishes at time $f_j = s_j + t_j$.
- Lateness: $l_j = \max\{0, f_j - d_j\}$.

**Goal**: schedule all jobs to minimize maximum lateness $L = \max l_j$.

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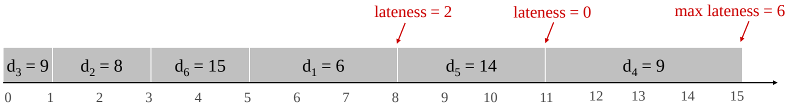
| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |



# Attempt: Consider jobs in ascending order by some strategy

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[Earliest deadline first] Sort by deadline $d_j$.

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counterexample

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# A Greedy Algorithm: Earliest Deadline First

**Algorithm 3:** Greedy Minimizing Lateness

```
Sort $n$ jobs by deadline so that $d_1 \leq d_2 \leq ... \leq d_n$;
$t \leftarrow 0$;
for $j = 1$ to $n$ do
    Assign job $j$ to interval $[t, t + t_j]$;
    $s_j \leftarrow t, f_j \leftarrow t + t_j$;
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return intervals $[s_j, f_j]$;
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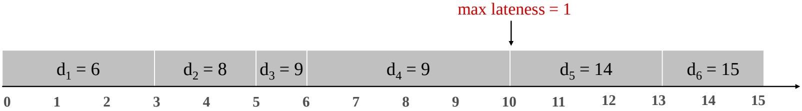
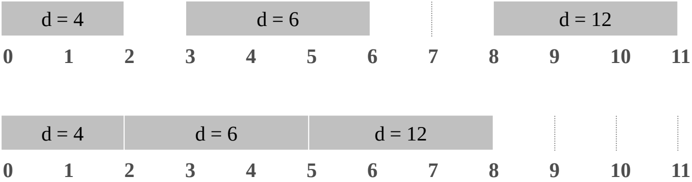
Implementation: $O(n \log n)$.



# Correctness Proof: Reduce Optimal Solution

Observation. There exists an optimal schedule with no idle time.



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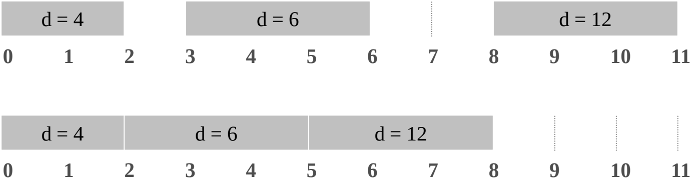
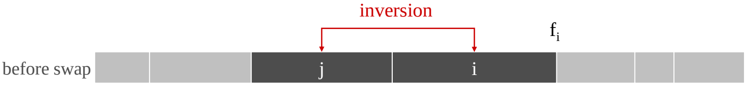


Observation. The greedy schedule has no idle time.

# Correctness Proof: Optimal Solution vs Algorithm Solution

**Definition**. Given a schedule $S$, an inversion is a pair of jobs $i$ and $j$ such that: $i < j$ but $j$ scheduled before $i$.



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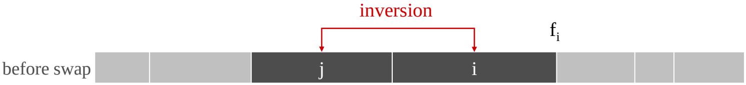

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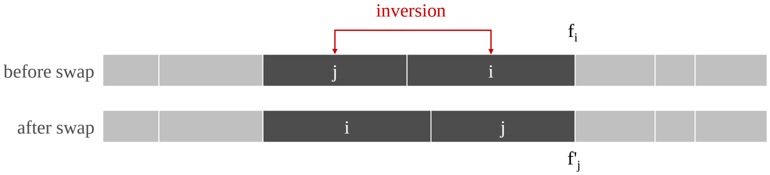
Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

# Correctness Proof: Inversions

**Claim.** Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.



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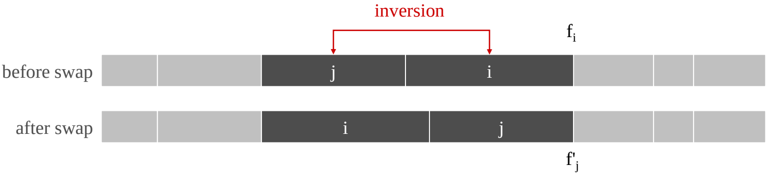

**Proof.** Let $l$ be the lateness before the swap, and let $l'$ be it afterwards.

- $l'_k = l_k$ for all $k \neq i, j$
- $l'_i \leq l_i$
- If job $j$ is late:

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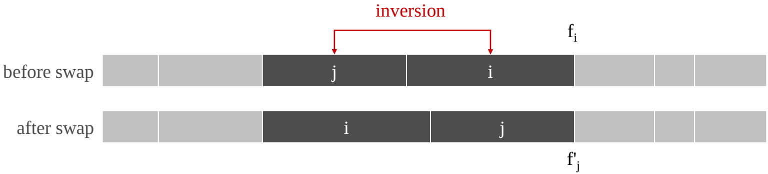

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- If job $j$ is late:

$$\begin{aligned} l'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq l_i && \text{(definition)} \end{aligned}$$

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- If $S^*$ has no inversions, then $S = S^*$.
- If $S^*$ has an inversion, let $i - j$ be an adjacent inversion.

  Swapping $i$ and $j$ does not increase the maximum lateness and strictly decreases the number of inversions. This contradicts definition of $S^*$. □

# Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

Other greedy algorithms. Kruskal, Prim, Dijkstra, Huffman, ...

# Outline

1. Basic Methodology
   - Interval Scheduling
   - Interval Partitioning
   - Scheduling to Minimize Lateness

2. More Examples
   - Optimal Caching
   - Coin Changing

# Optimal Offline Caching

- ○ Cache with capacity to store $k$ items.
- ○ Sequence of $m$ item requests $d_1, d_2, ..., d_m$.
- ○ Cache hit: item already in cache when requested.
- ○ Cache miss: item not in cache when requested: must bring requested item into cache, and evict some existing item, if full.

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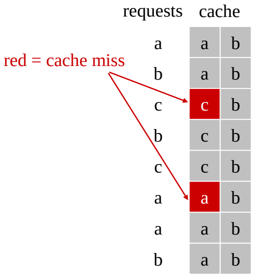
**Example**. $k = 2$, initial cache = $ab$,

requests: $a, b, c, b, c, a, a, b$.



red = cache miss

| requests | cache | |
|---|---|---|
| a | a | b |
| b | a | b |
| c | c | b |
| b | c | b |
| c | c | b |
| a | a | b |
| a | a | b |
| b | a | b |

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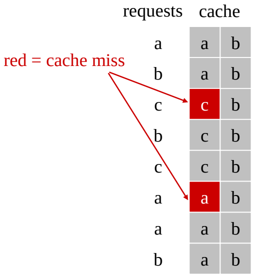
Optimal eviction schedule: 2 cache misses.



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| requests | cache | |
|---|---|---|
| a | a | b |
| b | a | b |
| c | c | b |
| b | c | b |
| c | c | b |
| a | a | b |
| a | a | b |
| b | a | b |

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current cache:

| a | b | c | d | e | f |
|---|---|---|---|---|---|

future queries: **g a b c e d a b b a c d e a f a d e f g h . . .**

↑ cache miss

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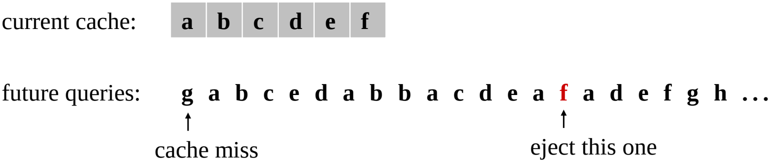

current cache: a b c d e f

future queries: g a b c e d a b b a c d e a f a d e f g h ...

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eject this one

**Theorem**. [Bellady, 1960s] FF is an optimal eviction schedule.

**Proof.** Algorithm and theorem are intuitive; proof is subtle.

# Reduced Eviction Schedules

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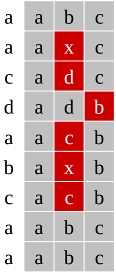
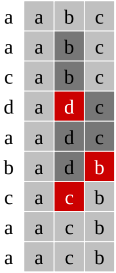
**Intuition.** Can transform an unreduced schedule into a reduced one with no more cache misses (means no more cache insertion/replacement here).



| | | | |
|---|---|---|---|
| a | a | b | c |
| a | a | x | c |
| c | a | d | c |
| d | a | d | b |
| a | a | c | b |
| b | a | x | b |
| c | a | c | b |
| a | a | b | c |
| a | a | b | c |

an unreduced schedule



| | | | |
|---|---|---|---|
| a | a | b | c |
| a | a | b | c |
| c | a | b | c |
| d | a | d | c |
| a | a | d | c |
| b | a | d | b |
| c | a | c | b |
| a | a | c | b |
| a | a | c | b |

a reduced schedule

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**Claim.** Given any unreduced schedule $S$, we can transform it into a reduced schedule $S'$ with no more cache replacement (insertion).

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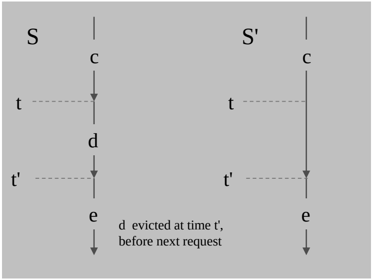
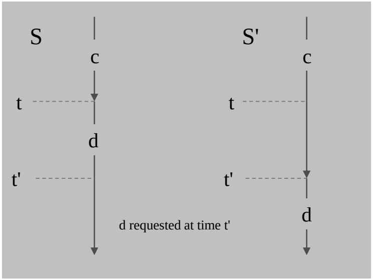
**Claim.** Given any unreduced schedule $S$, we can transform it into a reduced schedule $S'$ with no more cache replacement (insertion).

**Proof.** (by induction on number of unreduced[†] items)
Suppose $S$ brings $d$ into the cache at time $t$, without a request. Let $c$ be the item $S$ evicts when it brings $d$ into the cache.



Case 1



Case 2

[†] doesn't enter cache at requested time

# Theorem. FF is an optimal eviction algorithm

**Proof.** (by induction on number of requests $j$)

**Invariant:** There exists an optimal reduced schedule $S$ that makes the same eviction schedule as $S_{FF}$ through the first $j+1$ requests.

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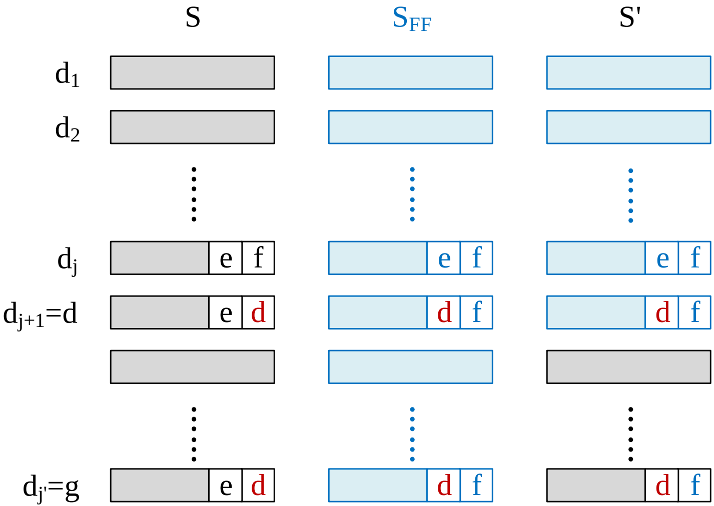
Let $S$ be reduced schedule that satisfies invariant through $j$ requests. We produce $S'$ that satisfies invariant after $j+1$ requests.

Consider $(j+1)^{th}$ request $d = d_{j+1}$. Since $S$ and $S_{FF}$ have agreed up until now, they have the same cache contents before request $j+1$.

- Case 1: ($d$ is already in the cache) $S' = S$ satisfies invariant.
- Case 2: ($d$ is not in the cache; $S$, $S_{FF}$ evict same element)
  $S' = S$ satisfies invariant.
- Case 3: ($d$ is not in the cache; $S_{FF}$ evicts $e$; $S$ evicts $f \neq e$)
  Let $S'$ agree with $S_{FF}$ at the $j+1$ requests; we show that having element $f$ in cache is no worse than having element $e$.

# An Illustration of Case 3



# Correctness Proof (Continued)

Let $j'$ be the first time after $j+1$ that $S$ and $S'$ take a different action (must involve $e$ or $f$ or both), and let $g$ be item requested at time $j'$.

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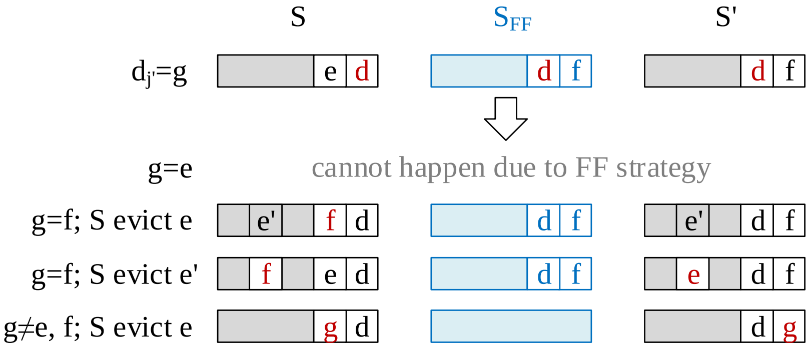

|  | S | $S_{FF}$ | S' |
|---|---|---|---|
| $d_{j'}$=g | e d | d f | d f |
| g=e | cannot happen due to FF strategy |  |  |
| g=f; S evict e | e' f d | d f | e' d f |
| g=f; S evict e' | f e d | d f | e d f |
| g≠e, f; S evict e | g d |  | d g |

# Correctness Proof (Continued)

**Case** 3$a$: $g = e$. Can't happen with Farthest-In-Future since there must be a request for $f$ before $e$.

**Case** 3$b$: $g = f$. Element $f$ can't be in cache of $S$, so let $e'$ be the element that $S$ evicts.

- if $e' = e$, $S'$ accesses $f$ from cache; now $S$ and $S'$ have same cache
- if $e' \neq e$, $S'$ evicts $e'$ and brings $e$ into the cache; now $S$ and $S'$ have the same cache.

**Case** 3$c$: $g \neq e, f$. $S$ must evict $e$ (otherwise $S'$ would take the same action). Make $S'$ evict $f$; now $S$ and $S'$ have the same cache. □

# Caching Perspective

**Online vs. Offline algorithms.**

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

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**Theorem.** FF is optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is $k$-competitive. [Section 13.8 in Cornell Book]
- LIFO is arbitrarily bad.

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**Goal**. Given US currency denominations:

1 (cent), 5 (nickel), 10 (dime), 25 (quarter), 100 (dollar),

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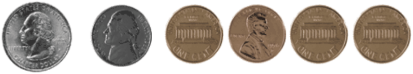

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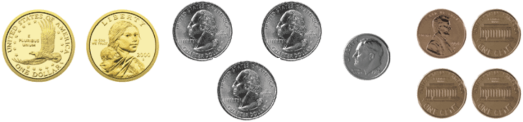
**Example**. $2.89.



# Cashier's Algorithm

**Algorithm 4:** Cashier's Algorithm

```
Sort coins denominations by value: $c_1 < c_2 < ... < c_n$;
$S \leftarrow \emptyset$ ;                                   // coins selected
while $x \neq 0$ do
    let $k$ be largest integer such that $c_k < x$;
    if $k = 0$ then
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    $S \leftarrow S \cup \{k\}$;
return $S$;
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**Question.** Is cashier's algorithm optimal?

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**Property**. Number of quarters $\leq 3$.

**Property**.Number of nickels + Number of dimes $\leq 2$.
**Proof**.

- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.
- Recall: at most 1 nickel.

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If not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to $x$.

| k | $c_k$ | All optimal solutions must satisfy | Max value of coins 1, 2, …, k-1 in any OPT |
|---|---|---|---|
| 1 | 1 | P ≤ 4 | - |
| 2 | 5 | N ≤ 1 | 4 |
| 3 | 10 | N + D ≤ 2 | 4 + 5 = 9 |
| 4 | 25 | Q ≤ 3 | 20 + 4 = 24 |
| 5 | 100 | no limit | 75 + 24 = 99 |

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Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by greedy algorithm.

# Is Cashier's Algorithm Work for Any Denominations?

**Observation 1.** Greedy is sub-optimal for US postal denominations:

$$1, 10, 21, 34, 70, 100, 350, 1225, 1500.$$

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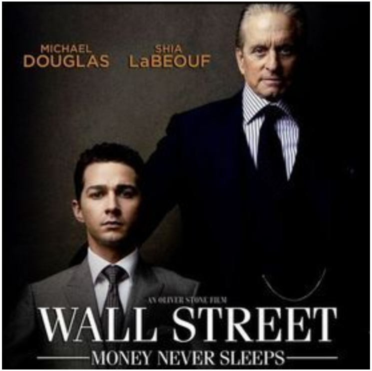
USA 1¢
American Kestrel
10
USA 21
34
70
USA
EXPLORING THE SOLAR SYSTEM
USA
$3⁵⁰
USA
$12²⁵
U.S. DEPARTMENT OF THE INTERIOR
$15
Mottled Duck
MIGRATORY BIRD HUNTING AND CONSERVATION STAMP

**Observation 2.** Even no feasible solution with system ¢={7, 8, 9}.

- Cashier's algorithm: 15¢= 9+???
- Optimal: 15¢= 7 + 8.

# Movie: Wall Street (1987)



*Greed is good.*

*Greed is right.*

*Greed works.*

*Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.*

— Gordon Gecko
(Michael Douglas)