Divide and Conquer*

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* Special thanks is given to Prof. Xiaofeng Gao sharing her teaching materials.

Outline

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Divide-and-Conquer Strategy

The divide-and-conquer strategy solves a problem P by:

(1) Breaking P into smaller subproblems of the same type.

(2) Recursively solving these subproblems.

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The key works lay in three different places:

(1) How to partition problem into subproblems.
(2) At the very tail end of the recursion, how to solve the smallest subproblems outright.
(3) How to glue together the partial answers.

Outline

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1777 - 1855

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$$1 + 2 + \cdots + 100 = \frac{100 \cdot (1 + 100)}{2} = 5050.$$

Multiplication for Complex Numbers

Gauss once noticed that although the product of two complex numbers

$$(a + bi)(c + di) = ac - bd + (bc + ad)i$$

seems to involve four real-number multiplications, it can in fact be done with just three: ac, bd, and $(a + b)(c + d)$, since

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In our big-O way of thinking, reducing the number of multiplications from four to three seems wasted ingenuity. However, this modest improvement becomes *very significant when applied recursively*.

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x &= \boxed{\quad x_L \quad}\boxed{\quad x_R \quad} = 2^{n/2}x_L + x_R \\
y &= \boxed{\quad y_L \quad}\boxed{\quad y_R \quad} = 2^{n/2}y_L + y_R.
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The additions take linear time, as do the multiplications by powers of 2 (merely left-shifts). The significant operations are the four $n/2$**-bit multiplications**; these we can handle by *four recursive calls*.

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Optimization: By **Gauss**'s trick, three multiplications, $x_L y_L$, $x_R y_R$, and $(x_L + x_R)(y_L + y_R)$, suffice, as

$$x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R.$$

A Divide-and-Conquer Algorithm for Integer Multiplication

Algorithm 1: MULTIPLY(x,y)

Input: Positive integers x and y, in binary.
Output: Their product xy.

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if n = 1 then
    return xy;
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return P_1 × 2^n + (P_3 − P_1 − P_2) × 2^(n/2) + P_2
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New recurrence relation: $T(n) = 3T(n/2) + O(n)$ $\rightarrow$ How well?

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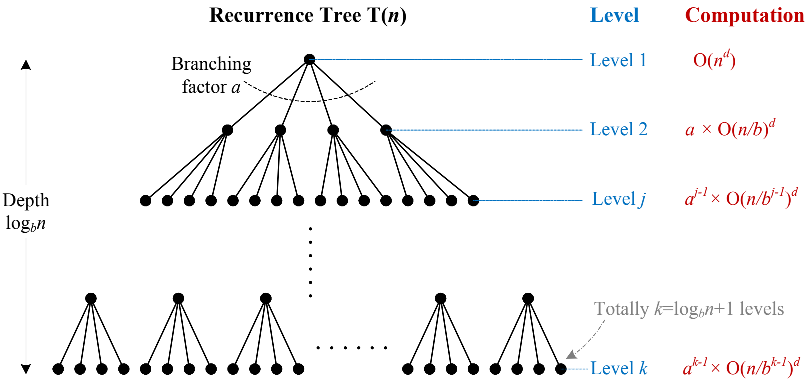
$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a. \end{cases}$$

Proof of Master Theorem: $T(n) = aT(\lceil n/b \rceil) + O(n^d)$



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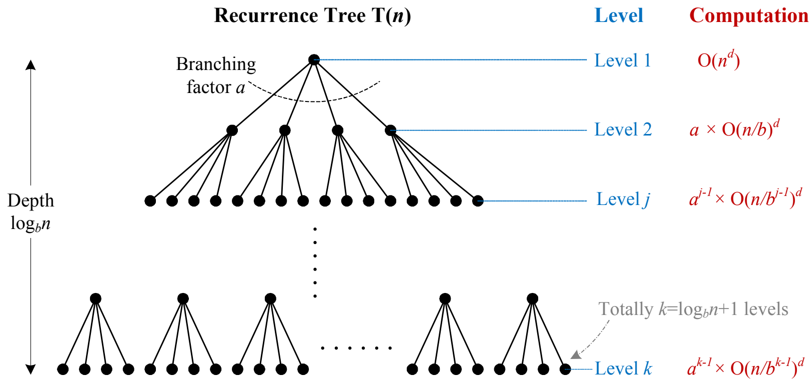

Complexity of $T(n)$ = Sum up all computations at each level.

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The size of the subproblems decreases by a factor of b with each level of recursion, and reaches the base case when

$$\frac{n}{b^{k-1}} = 1 \Rightarrow k = \log_b n + 1$$

(k is the level of the recursion tree, which equals to tree height + 1.)

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The total work done at the j-th level is

$$a^{j-1} \times O\left(\frac{n}{b^{j-1}}\right)^d = O(n^d) \times \left(\frac{a}{b^d}\right)^{j-1}.$$

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$$\sum_{j=1}^{\log_b n+1}\left(a^{j-1}\times O\left(\frac{n}{b^{j-1}}\right)^d\right)=\sum_{j=0}^{\log_b n}\left(O(n^d)\times\left(\frac{a}{b^d}\right)^j\right)=O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j.$$

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(1) $\frac{a}{b^d}<1\Rightarrow d>\log_b a$:

$$O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j\le O(n^d)\frac{1}{1-\frac{a}{b^d}}=O(n^d).$$

(Sum of GS: $S_n=\sum_{j=1}^{n}a_1q^{j-1}=a_1\frac{1-q^n}{1-q}\le a_1\frac{1}{1-q}$ if $q<1$)

Proof of Master Theorem

(2) $\frac{a}{b^d} = 1 \Rightarrow d = \log_b a$:

$$O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j = O(n^d)(\log_b n + 1) = O(n^d \log_b n) = O(n^d \log n).$$

($\log_b n = \frac{\log n}{\log b} = \frac{1}{\log b}\log n = O(\log n)$ by changing the base)

Proof of Master Theorem

(3) $\dfrac{a}{b^d} > 1 \Rightarrow d < \log_b a$: (reverse the GS in decreasing order)

$$
\begin{aligned}
O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j &= O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^{\log_b n}\cdot\left(\frac{b^d}{a}\right)^j \\
&= O(n^d)\sum_{j=0}^{\log_b n}\frac{a^{\log_b n}}{(b^{\log_b n})^d}\cdot\left(\frac{b^d}{a}\right)^j \\
&\leq O(n^d)\frac{n^{\log_b a}}{n^d}\cdot\frac{1}{1-\frac{b^d}{a}} \\
&= O(n^{\log_b a})
\end{aligned}
$$

$\left(a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a}\right)$

Time Complexity of Multiplication

Original recurrence relation: $T(n) = 4T(n/2) + O(n)$

$a = 4, b = 2, d = 1, d < \log_b a.$

$\Rightarrow$ **Time complexity:** $O(n^{\log_b a}) = O(n^2).$

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Optimized recurrence relation: $T(n) = 3T(n/2) + O(n)$

$a = 3$, $b = 2$, $d = 1$, $d < \log_b a$.

$\Rightarrow$ **Time complexity:** $O(n^{\log_2 3}) \approx O(n^{1.59})$.

Outline

Binary Search

Algorithm 2: BinarySearch

Input: An array $A[1..n]$ of n elements sorted in nondecreasing order and an element x.

Output: j if $x = A[j]$, $1 \le j \le n$, and 0 otherwise.

$low \leftarrow 1$; $high \leftarrow n$; $j \leftarrow 0$;
while $low \le high$ **and** $j = 0$ **do**
    $mid \leftarrow \lfloor (low + high)/2 \rfloor$;
    if $x = A[mid]$ **then**
        $j \leftarrow mid$ **break**;
    else if $x < A[mid]$ **then**
        $high \leftarrow mid - 1$;
    else
        $low \leftarrow mid + 1$;
return j;

Time Complexity

To find a key x in $A[1, \cdots, n]$ in sorted order, we first compare x with $A[n/2]$, and depending on the result we recurse either on the first half $A[1, \cdots, n/2-1]$, or on the second half $A[n/2+1, \cdots, n]$.

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$$T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(1),$$

By Master Theorem, $a = 1$, $b = 2$, $d = 0$, and thus the running time should be $O(\log n)$.

Outline

1. Divide-and-Conquer
   - Basic Technique
   - An Introductory Example: Multiplication
   - Recurrence Relations
2. Applications
   - Binary Search
   - Merge Sort
   - Matrix Multiplication

Merging Two Sorted Lists

Algorithm 3: Merge

Input: $A[1..m]$, p, q and r with $1 \leq p \leq q < r \leq m$.
Output: $A[p..r]$ (merging two sorted subarrays $A[p..q]$, $A[q+1..r]$).

$s \leftarrow p$; $t \leftarrow q+1$; $k \leftarrow p$;
while $s \leq q$ **and** $t \leq r$ **do**
    if $A[s] \leq A[t]$ **then**
        $B[k] \leftarrow A[s]$; $s \leftarrow s+1$; ($B[p..r]$ is an auxiliary array)
    else $B[k] \leftarrow A[t]$; $t \leftarrow t+1$;
    $k \leftarrow k+1$;
if $s = q+1$ **then**
    $B[k..r] \leftarrow A[t..r]$;
else $B[k..r] \leftarrow A[s..q]$;
return $A[p..r] \leftarrow B[p..r]$;

Analysis of Merge

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and

7	11	13	45	57

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If the two array sizes are $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, the number of comparisons is between $\lfloor n/2 \rfloor$ and $n - 1$.

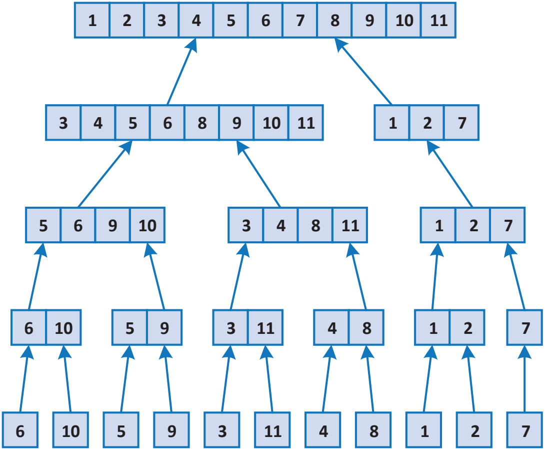
Bottom-Up MergeSort Algorithm

Algorithm 4: MergeSort

Input: An array $A[1..n]$ of n elements.
Output: $A[1..n]$ sorted in nondecreasing order.

$t \leftarrow 1$;
while $t < n$ **do**
    $s \leftarrow t$; $t \leftarrow 2s$; $i \leftarrow 0$;
    while $i + t \leq n$ **do**
        $Merge(A, i+1, i+s, i+t)$;
        $i \leftarrow i + t$;
    if $i + s < n$ **then**
        $Merge(A, i+1, i+s, n)$;
return $A[1..n]$;

An Example


1 2 3 4 5 6 7 8 9 10 11
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7

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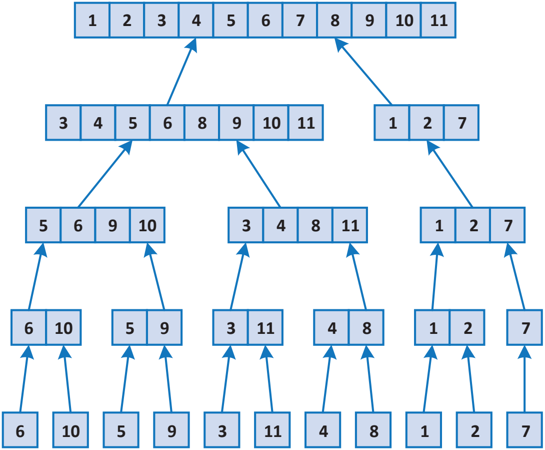


Time Complexity:

Recurrence:

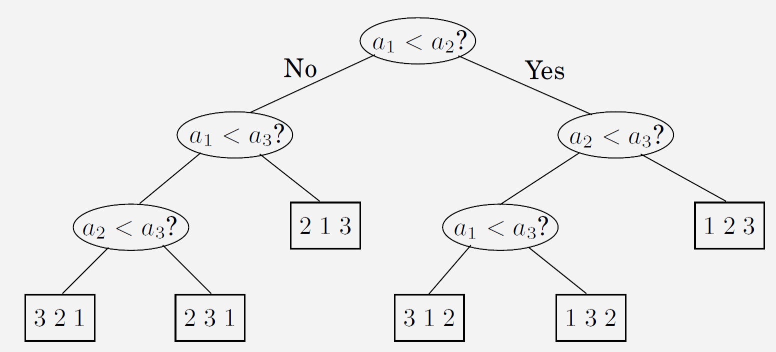
$$T(n) = 2T(n/2) + O(n);$$

By Master Theorem

$$T(n) = O(n \log n).$$

An $n \log n$ Lower Bound for Sorting

An example sorting permutation tree for $\{a_1, a_2, a_3\}$:



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This is a binary tree with $n!$ leaves. Thus, the depth of our tree – and the complexity of our algorithm – must be at least

$$\log(n!) \approx \log\left(\sqrt{\pi\,(2n + 1/3)} \cdot n^n \cdot e^{-n}\right) \overset{?}{=} \Omega(n \log n),$$

where we use Stirling's formula.

Outline

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$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}$$

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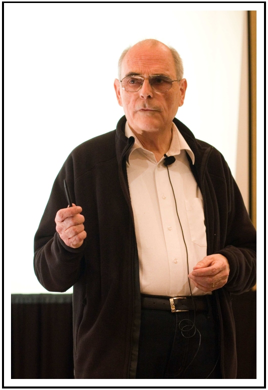
The preceding formula implies an $O(n^3)$ algorithm for matrix multiplication.

Volker Strassen



Volker Strassen (1936 –)

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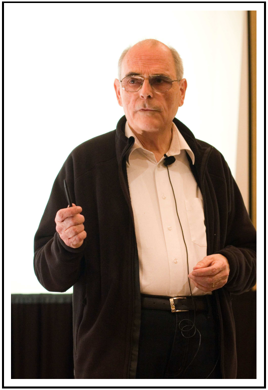

Volker Strassen (1936 –)

In 1969, the German mathematician **Volker Strassen** announced a surprising $O(n^{2.81})$ algorithm.

Divide and conquer

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$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

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The recurrence is

$$T(n) = 8T(n/2) + O(n^2)$$

with solution $O(n^3)$.

Strassen’s trick

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$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

where

$$\begin{aligned} P_1 &= A(F - H) & P_5 &= (A + D)(E + H) \\ P_2 &= (A + B)H & P_6 &= (B - D)(G + H) \\ P_3 &= (C + D)E & P_7 &= (A - C)(E + F) \\ P_4 &= D(G - E) \end{aligned}$$

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The recurrence is

$$T(n) = 7T(n/2) + O(n^2)$$

with solution $O(n^{\log_2 7}) \approx O(n^{2.81})$.