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GMAT™

Official Guide Quantitative Review 2019

Book + Online

The ONLY source of real GMAT® questions from past exams

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GMAT® Official Guide Quantitative Review 2019

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Dear GMAT Test-Taker,

Thank you for your interest in graduate management education. Taking the GMAT® exam lets schools know that you're serious about your educational goals. By using the *Official Guide* to prepare for the GMAT exam, you're taking a very important step toward achieving your goals and pursuing admission to the MBA or business master's program that is the best fit for you.

This book, *GMAT® Official Guide Quantitative Review 2019*, is designed to help you prepare for and build confidence to do your best on the GMAT exam. It's the only guide of its kind on the market that includes real GMAT exam questions published by the Graduate Management Admission Council (GMAC), the makers of the exam.

In 1954, leading business schools joined together to launch a standardized way of assessing candidates for business school programs. For 65 years, the GMAT exam has helped people demonstrate their command of the skills needed for success in the classroom. Schools use and trust the GMAT exam as part of their admissions process because it's a proven predictor of classroom success and your ability to excel in your chosen program.

Today more than 7,000 graduate programs around the world use the GMAT exam to establish their MBA, graduate-level management degrees and specialized business master's programs as hallmarks of excellence. Nine out of 10 new MBA enrollments globally are made using a GMAT score.*

We are driven to keep improving the GMAT exam as well as to help you find and gain admission to the best school or program for you. We're committed to ensuring that no talent goes undiscovered and that more people around the world can pursue opportunities in graduate management education.

I applaud your commitment to educational success, and I know that this book and the other GMAT Official Prep materials available at mba.com will give you the confidence to achieve your personal best on the GMAT exam and launch or reinvigorate a rewarding career.

I wish you success on all your educational and professional endeavors in the future.

Sincerely,

Sangeet Chowfla
President & CEO of the Graduate Management Admission Council

*Top 100 *Financial Times* full-time MBA programs

GMAT® Official Guide 2019 Quantitative Review

1.0 What Is the GMAT® Exam?

1.0 What Is the GMAT® Exam?

The Graduate Management Admission Test® (GMAT®) exam is a standardized exam used in admissions decisions by more than 7,000 graduate management programs worldwide, at approximately 2,300 graduate business schools worldwide. It helps you gauge, and demonstrate to schools, your academic potential for success in graduate-level management studies.

The four-part exam measures your Analytical Writing, Integrated Reasoning, Verbal, and Quantitative Reasoning skills—higher-order reasoning skills that management faculty worldwide have identified as important for incoming students to have. "Higher-order" reasoning skills involve complex judgments, and include critical thinking, analysis, and problem solving. Unlike undergraduate grades and curricula, which vary in their meaning across regions and institutions, your GMAT scores provide a standardized, statistically valid, and reliable measure of how you are likely to perform academically in the core curriculum of a graduate management program. The GMAT exam's validity, fairness, and value in admissions have been well-established through numerous academic studies.

The GMAT exam is delivered entirely in English and solely on a computer. It is not a test of business knowledge, subject-matter mastery, English vocabulary, or advanced computational skills. The GMAT exam also does not measure other factors related to success in graduate management study, such as job experience, leadership ability, motivation, and interpersonal skills. Your GMAT score is intended to be used as one admissions criterion among other, more subjective, criteria, such as admissions essays and interviews.

1.1 Why Take the GMAT® Exam?

Launched in 1954 by a group of nine business schools to provide a uniform measure of the academic skills needed to succeed in their programs, the GMAT exam is now used by more than 7,000 graduate management programs at approximately 2,300 institutions worldwide.

Taking the GMAT exam helps you stand out in the admissions process and demonstrate your readiness and commitment to pursuing graduate management education. Schools use GMAT scores to help them select the most qualified applicants—because they know that candidates who take the GMAT exam are serious about earning a graduate business degree, and it's a proven predictor of a student's ability to succeed in his or her chosen program. When you consider which programs to apply to, you can look at a school's use of the GMAT exam as one indicator of quality. Schools that use the GMAT exam typically list score ranges or average scores in their class profiles, so you may also find these profiles helpful in gauging the academic competitiveness of a program you are considering and how well your performance on the exam compares with that of the students enrolled in the program.

Myth -vs- FACT

M – **If I don't achieve a high score on the GMAT, I won't get into my top choice schools.**

F – **There are great schools available for candidates at any GMAT score range.**

Fewer than 50 of the more than 250,000 people taking the GMAT exam each year get a perfect score of 800; and many more get into top business school programs around the world each year. Admissions Officers use GMAT scores as one component in their admissions decisions, in conjunction with undergraduate records, application essays, interviews, letters of recommendation, and other information when deciding whom to accept into their programs. Visit School Finder on mba.com to learn about schools that are the best fit for you.

No matter how you perform on the GMAT exam, you should contact the schools that interest you to learn more and to ask how they use GMAT scores and other criteria (such as your undergraduate grades, essays, and letters of recommendation) in their admissions processes. School admissions offices, web sites, and materials published by schools are the key sources of information when you are doing research about where you might want to go to business school.

For more information on the GMAT, test preparation materials, registration, how to use and send your GMAT scores to schools, and applying to business school, please visit mba.com.

1.2 GMAT® Exam Format

The GMAT exam consists of four separately timed sections (see the table on the next page). The Analytical Writing Assessment (AWA) section consists of one essay. The Integrated Reasoning section consists of graphical and data analysis questions in multiple response formats. The Quantitative and Verbal Reasoning sections consist of multiple-choice questions.

The Verbal and Quantitative sections of the GMAT exam are computer adaptive, which means that the test draws from a large bank of questions to tailor itself to your ability level, and you won't get many questions that are too hard or too easy for you. The first question will be of medium difficulty. As you answer each question, the computer scores your answer and uses it—as well as your responses to all preceding questions—to select the next question.

Computer-adaptive tests become more difficult the more questions you answer correctly, but if you get a question that seems easier than the last one, it does not necessarily mean you answered the last question incorrectly. The test has to cover a range of content, both in the type of question asked and the subject matter presented.

Myth -vs- FACT

M – **Getting an easier question means I answered the last one wrong.**

F – **You should not become distracted by the difficulty level of a question.**

Most people are not skilled at estimating question difficulty, so don't worry when taking the test or waste valuable time trying to determine the difficulty of the question you are answering.

To ensure that everyone receives the same content, the test selects a specific number of questions of each type. The test may call for your next problem to be a relatively hard data sufficiency question involving arithmetic operations. But, if there are no more relatively difficult data sufficiency questions involving arithmetic, you might be given an easier question.

Because the computer uses your answers to select your next questions, you may not skip questions or go back and change your answer to a previous question. If you don't know the answer to a question, try to eliminate as many choices as possible, then select the answer you think is best.

Though the individual questions are different, the mix of question types is the same for every GMAT exam. Your score is determined by the difficulty and statistical characteristics of the questions you answer as well as the number of questions you answer correctly. By adapting to each test-taker, the GMAT exam is able to accurately and efficiently gauge skill levels over a full range of abilities, from very high to very low.

The test includes the types of questions found in this book and online at gmat.wiley.com, but the format and presentation of the questions are different on the computer. When you take the test:

- Only one question or question prompt at a time is presented on the computer screen.
- The answer choices for the multiple-choice questions will be preceded by circles, rather than by letters.

- Different question types appear in random order in the multiple-choice and Integrated Reasoning sections.
- You must select your answer using the computer.
- You must choose an answer and confirm your choice before moving on to the next question.
- You may not go back to previous screens to change answers to previous questions.

Format of the GMAT® Exam

	Questions	Timing
Analytical Writing Assessment	1	30 min.
Integrated Reasoning Multi-Source Reasoning Table Analysis Graphics Interpretation Two-Part Analysis	12	30 min.
Quantitative Reasoning Problem Solving Data Sufficiency	31	62 min.
Verbal Reasoning Reading Comprehension Critical Reasoning Sentence Correction	36	65 min.
	Total Time:	187 min.

You will now have the flexibility to select the order for the section of the GMAT exam from three options.

<table>
<tr><th>Order #1</th><th>Order #2</th><th>Order #3</th></tr>
<tr><td>Analytical Writing Assessment</td><td rowspan="2">Verbal</td><td rowspan="2">Quantitative</td></tr>
<tr><td>Integrated Reasoning</td></tr>
<tr><td colspan="3">Optional 8-minute break</td></tr>
<tr><td>Quantitative</td><td>Quantitative</td><td>Verbal</td></tr>
<tr><td colspan="3">Optional 8-minute break</td></tr>
<tr><td rowspan="2">Verbal</td><td>Integrated Reasoning</td><td>Integrated Reasoning</td></tr>
<tr><td>Analytical Writing Assessment</td><td>Analytical Writing Assessment</td></tr>
</table>

The section order selection will take place at the test center on exam date, immediately prior to the start of the GMAT exam.

1.3 What Is the Content of the Exam Like?

The GMAT exam measures higher-order analytical skills encompassing several types of reasoning. The Analytical Writing Assessment asks you to analyze the reasoning behind an argument and respond in writing; the Integrated Reasoning section asks you to interpret and synthesize information from multiple sources and in different formats to make reasoned conclusions; the Quantitative section asks you to reason quantitatively using basic arithmetic, algebra, and geometry; and the Verbal section asks you to read and comprehend written material and to reason and evaluate arguments.

Test questions may address a variety of subjects, but all of the information you need to answer the questions will be included on the exam, with no outside knowledge of the subject matter necessary. The GMAT exam is not a test of business knowledge, English vocabulary, or advanced computational skills. You will need to read and write in English and have basic math and English skills to perform well on the test, but its difficulty comes from analytical and critical thinking abilities.

The questions in this book are organized by question type and from easiest to most difficult, but keep in mind that when you take the test, you may see different types of questions in any order within each section.

1.4 Analytical Writing Assessment

The Analytical Writing Assessment (AWA) consists of one 30-minute writing task: Analysis of an Argument. The AWA measures your ability to think critically, communicate your ideas, and formulate an appropriate and constructive critique. You will type your essay on a computer keyboard.

1.5 Integrated Reasoning Section

The Integrated Reasoning section highlights the relevant skills that business managers in today's data-driven world need in order to analyze sophisticated streams of data and solve complex problems. It measures your ability to understand and evaluate multiple sources and types of information—graphic, numeric, and verbal—as they relate to one another. This section will require you to use both quantitative and verbal reasoning to solve complex problems and solve multiple problems in relation to one another.

Four types of questions are used in the Integrated Reasoning section:

- Multi-Source Reasoning
- Table Analysis
- Graphics Interpretation
- Two-Part Analysis

Integrated Reasoning questions may be quantitative, verbal, or a combination of both. You will have to interpret graphics and sort tables to extract meaning from data, but advanced statistical knowledge and spreadsheet manipulation skills are not necessary. You will have access to an on-screen calculator with basic functions for the Integrated Reasoning section, but note that the calculator is *not* available on the Quantitative section.

1.6 Quantitative Section

The GMAT Quantitative section measures your ability to reason quantitatively, solve quantitative problems, and interpret graphic data.

Two types of multiple-choice questions are used in the Quantitative section:

- Problem Solving
- Data Sufficiency

Both are intermingled throughout the Quantitative section, and require basic knowledge of arithmetic, elementary algebra, and commonly known concepts of geometry.

To review the basic mathematical concepts that you will need to answer Quantitative questions, see the math review in chapter 3. For test-taking tips specific to the question types in the Quantitative section, practice questions, and answer explanations, see chapters 4 and 5.

1.7 Verbal Section

The GMAT Verbal section measures your ability to read and comprehend written material and to reason and evaluate arguments. The Verbal section includes reading sections from several different content areas. Although you may be generally familiar with some of the material, neither the reading passages nor the questions assume detailed knowledge of the topics discussed.

Three types of multiple-choice questions are intermingled throughout the Verbal section:

- Reading Comprehension
- Critical Reasoning
- Sentence Correction

All three require basic knowledge of the English language, but the Verbal section is not a test of advanced vocabulary.

For test-taking tips specific to each question type in the Verbal section, practice questions, and answer explanations, see *GMAT® Official Guide 2019*, or *GMAT® Official Guide Verbal Review 2019*; both are available for purchase at mba.com

1.8 What Computer Skills Will I Need?

The GMAT exam requires only basic computer skills. You will type your AWA essay on the computer keyboard using standard word-processing keystrokes. In the Integrated Reasoning and multiple-choice sections, you will select your responses using either your computer mouse or the keyboard. The Integrated Reasoning section includes basic computer navigation and functions, such as clicking on tabs and using drop-down menus to sort tables and select answers. You will also have access to an on-screen calculator in the Integrated Reasoning section (calculator is not available in any other section of the exam).

1.9 What Are the Test Centers Like?

The GMAT exam is administered under standardized conditions at test centers worldwide. Each test center has a proctored testing room with individual computer workstations that allow you to sit for the exam under quiet conditions and with some privacy. You will be able to take two optional 8-minute breaks during the course of the exam. You may not take notes or scratch paper with you into the testing room, but an erasable notepad and marker will be provided for you to use during the test. For more information about exam day visit mba.com.

1.10 How Are Scores Calculated?

Verbal and Quantitative sections are scored on a scale of 6 to 51, in one-point increments. The Total GMAT score ranges from 200 to 800 and is based on your performance in these two sections. Your score is determined by:

- The number of questions you answer
- The number of questions you answer correctly or incorrectly
- The level of difficulty and other statistical characteristics of each question

Your Verbal, Quantitative, and Total GMAT scores are determined by an algorithm that takes into account the difficulty of the questions that were presented to you and how you answered them. When you answer the easier questions correctly, you get a chance to answer harder questions, making it possible to earn a higher score. After you have completed all the questions on the test, or when your time is expired, the computer will calculate your scores. Your scores on the Verbal and Quantitative sections are combined to produce your Total score which ranges from 200 to 800 in10-point increments.

The Analytical Writing Assessment consists of one writing task. Your essay will be scored two times independently. Essays are evaluated by college and university faculty members from a variety of disciplines, including management education, who rate the overall quality of your critical thinking and writing. (For details on how readers are qualified, visit mba.com.) In addition, your response is also scored by an automated scoring program designed to reflect the judgment of expert readers.

Your essay is scored on a scale of 0 to 6, in half-point increments, with 6 being the highest score and 0 the lowest. A score of zero is given for responses that are off topic, are in a foreign language, merely attempt to copy the topic, consist only of keystroke characters, or are blank. Your AWA score is typically the average of two independent ratings. If the independent scores vary by more than a point, a third reader adjudicates, but because of ongoing training and monitoring, discrepancies are rare.

Your Integrated Reasoning section is scored on a scale of 1 to 8, in one-point increments. Many questions have multiple parts, and you must answer all parts of a question correctly to receive credit; partial credit will not be given.

Your Analytical Writing Assessment and Integrated Reasoning scores are computed and reported separately from the other sections of the test and have no effect on your Verbal, Quantitative, or Total scores. The schools that you have designated to receive your scores may receive a copy of your Analytical Writing Assessment essay with your score report. Your own copy of your score report will not include your essay.

Your GMAT score includes a percentile ranking that compares your skill level with other test-takers from the past three years. The percentile rank of your score shows the percentage of tests taken with scores lower than your score. Every July, percentile ranking tables are updated. Visit mba.com to view the most recent percentile rankings tables.

1.11 Test Development Process

The GMAT exam is developed by experts who use standardized procedures to ensure high-quality, widely-appropriate test material. All questions are subjected to independent reviews and are revised or discarded as necessary. Multiple-choice questions are tested during GMAT exam administrations. Analytical Writing Assessment tasks are tested on mba.com registrants and then assessed for their fairness and reliability. For more information on test development, visit mba.com.

2.0 How to Prepare

2.0 How to Prepare

2.1 How Should I Prepare to Take the Test?

The GMAT® exam is designed specifically to measure reasoning skills needed for management education, and the test contains several question formats unique to the GMAT exam. At a minimum, you should be familiar with the test format and the question formats before you sit for the test. Because the GMAT exam is a timed exam, you should practice answering test questions, not only to better understand the question formats and the skills they require, but also to help you learn to pace yourself so you can finish each section when you sit for the exam.

Because the exam measures reasoning rather than subject-matter knowledge, you most likely will not find it helpful to memorize facts. You do not need to study advanced mathematical concepts, but you should be sure your grasp of basic arithmetic, algebra, and geometry is sound enough that you can use these skills in quantitative problem solving. Likewise, you do not need to study advanced vocabulary words, but you should have a firm understanding of basic English vocabulary and grammar for reading, writing, and reasoning.

This book and other study materials released by the Graduate Management Admission Council (GMAC) are the ONLY source of questions that have been retired from the GMAT exam. All questions that appear or have appeared on the GMAT exam are copyrighted and owned by GMAC, which does not license them to be reprinted elsewhere. Accessing live Integrated Reasoning, Quantitative, or Verbal test questions in advance or sharing test content during or after you take the test is a serious violation, which could cause your scores to be canceled and schools to be notified. In cases of a serious violation, you may be banned from future testing and other legal remedies may be pursued.

Myth -vs- FACT

M – **You need very advanced math skills to get a high GMAT score.**

F – **The GMAT measures your reasoning and critical thinking abilities, rather than your advanced math skills.**

The GMAT exam only requires basic quantitative skills. You should review the math skills (algebra, geometry, basic arithmetic) presented in this guide (chapter 3). The difficulty of GMAT Quantitative questions stems from the logic and analysis used to solve the problems and not the underlying math skills.

2.2 What About Practice Tests?

The Quantitative and Verbal sections of the GMAT exam are computer adaptive, and the Integrated Reasoning section includes questions that require you to use the computer to sort tables and navigate to different sources of information. Our official practice materials will help you get comfortable with the format of the test and better prepare for exam day. Two full-length GMAT practice exams are available at no charge for those who have created an account on mba.com. The practice exams include computer-adaptive Quantitative and Verbal sections, plus additional practice questions, information about the test, and tutorials to help you become familiar with how the GMAT exam will appear on the computer screen at the test center.

To maximize your studying efforts with the free practice exams, you should leverage official practice materials as you start to prepare for the test. Take one practice test to make yourself familiar with the exam and to get a baseline score. After you have studied using this book and other study materials, take the second practice test to determine whether you need to shift your focus to other areas you need to strengthen. Note that the free practice tests may include questions that are also published in this book. As your test day approaches, consider taking more official practice tests to help measure your progress and give you a better idea of how you might score on exam day.

2.3 Where Can I Get Additional Practice?

If you would like additional practice, you may want to purchase *GMAT® Official Guide 2019* and/or *GMAT® Official Guide Verbal Review 2019* . You can also find more Quantitative, Verbal, and Integrated Reasoning practice questions, full-length, computer-adaptive practice exams, Analytical Writing Assessment practice prompts, and other helpful study materials at mba.com.

2.4 General Test-Taking Suggestions

Specific test-taking strategies for individual question types are presented later in this book. The following are general suggestions to help you perform your best on the test.

1. **Use your time wisely.**
   Although the GMAT exam stresses accuracy more than speed, it is important to use your time wisely. On average, you will have about 1¾ minutes for each Verbal question, about 2 minutes for each Quantitative question, and about 2½ minutes for each Integrated Reasoning question, some of which have multiple questions. Once you start the test, an onscreen clock will show the time you have left. You can hide this display if you want, but it is a good idea to check the clock periodically to monitor your progress. The clock will automatically alert you when 5 minutes remain for the section you are working on.

2. **Answer practice questions ahead of time.**
   After you become generally familiar with all question types, use the practice questions in this book and online at gmat.wiley.com to prepare for the actual test. It may be useful to time yourself as you answer the practice questions to get an idea of how long you will have for each question when you sit for the actual test, as well as to determine whether you are answering quickly enough to finish the test in the allotted time.

3. **Read all test directions carefully.**
   The directions explain exactly what is required to answer each question type. If you read hastily, you may miss important instructions and impact your ability to answer correctly. To review directions during the test, click on the Help icon. But be aware that the time you spend reviewing directions will count against your time allotment for that section of the test.

4. **Read each question carefully and thoroughly.**
   Before you answer a question, determine exactly what is being asked and then select the best choice. Never skim a question or the possible answers; skimming may cause you to miss important information or nuances.

Myth -vs- FACT

M – It is more important to respond correctly to the test questions than it is to finish the test.

F – There is a significant penalty for not completing the GMAT exam.

Pacing is important. If you are stumped by a question, give it your best guess and move on. If you guess incorrectly, the computer program will likely give you an easier question, which you are likely to answer correctly, and the computer will rapidly return to giving you questions matched to your ability. If you don't finish the test, your score will be reduced. Failing to answer five verbal questions, for example, could reduce your score from the 91st percentile to the 77th percentile.

5. **Do not spend too much time on any one question.**
   If you do not know the correct answer, or if the question is too time consuming, try to eliminate choices you know are wrong, select the best of the remaining answer choices, and move on to the next question.

   Not completing sections and randomly guessing answers to questions at the end of each test section can significantly lower your score. As long as you have worked on each section, you will receive a score even if you do not finish one or more sections in the allotted time. You will not earn points for questions you never get to see.

6. **Confirm your answers ONLY when you are ready to move on.**
   On the Quantitative and Verbal sections, once you have selected your answer to a multiple-choice question, you will be asked to confirm it. Once you confirm your response, you cannot go back and change it. You may not skip questions. In the Integrated Reasoning section, there may be several questions based on information provided in the same question prompt. When there is more than one response on a single screen, you can change your response to any of the questions on the screen before moving on to the next screen. However, you may not navigate back to a previous screen to change any responses.

7. **Plan your essay answer before you begin to write.**
   The best way to approach the Analysis of an Argument section is to read the directions carefully, take a few minutes to think about the question, and plan a response before you begin writing. Take time to organize your ideas and develop them fully but leave time to reread your response and make any revisions that you think would improve it.

Myth -vs- FACT

M – **The first 10 questions are critical and you should invest the most time on those.**

F – **All questions count.**

The computer-adaptive testing algorithm uses each answered question to obtain an *initial* estimate. However, as you continue to answer questions, the algorithm self-corrects by computing an updated estimate on the basis of all the questions you have answered, and then administers items that are closely matched to this new estimate of your ability. Your final score is based on all your responses and considers the difficulty of all the questions you answered. Taking additional time on the first 10 questions will not game the system and can hurt your ability to finish the test.

3.0 Math Review

3.0 Math Review

To answer quantitative reasoning questions on the GMAT exam, you will need to be familiar with basic mathematical concepts and formulas. This chapter contains a list of the basic mathematical concepts, terms, and formulas that may appear or can be useful for answering quantitative reasoning questions on the GMAT exam. This chapter offers only a high-level overview, so if you find unfamiliar terms or concepts, you should consult other resources for a more detailed discussion and explanation.

Keep in mind that this knowledge of basic math, while necessary, is seldom sufficient in answering GMAT questions. Unlike traditional math problems that you may have encountered in school, GMAT quantitative reasoning questions require you to *apply* your knowledge of math. For example, rather than asking you to demonstrate your knowledge of prime factorization by listing the prime factors of a number, a GMAT question may require you to apply your knowledge of prime factorization and properties of exponents to simplify an algebraic expression with a radical.

To prepare for the GMAT Quantitative Reasoning section, we recommend starting with a review of the basic mathematical concepts and formulas to ensure that you have the foundational knowledge necessary for solving the questions, before moving on to practicing the application of this knowledge on real GMAT questions from past exams.

Section 3.1, "Arithmetic," includes the following topics:

1. Properties of Integers
2. Fractions
3. Decimals
4. Real Numbers
5. Ratio and Proportion
6. Percents
7. Powers and Roots of Numbers
8. Descriptive Statistics
9. Sets
10. Counting Methods
11. Discrete Probability

Section 3.2, "Algebra," does not extend beyond what is usually covered in a first-year high school algebra course. The topics included are as follows:

1. Simplifying Algebraic Expressions
2. Equations
3. Solving Linear Equations with One Unknown
4. Solving Two Linear Equations with Two Unknowns
5. Solving Equations by Factoring
6. Solving Quadratic Equations
7. Exponents
8. Inequalities
9. Absolute Value
10. Functions

Section 3.3, "Geometry," is limited primarily to measurement and intuitive geometry or spatial visualization. Extensive knowledge of theorems and the ability to construct proofs, skills that are usually developed in a formal geometry course, are not tested. The topics included in this section are the following:

1. Lines
2. Intersecting Lines and Angles
3. Perpendicular Lines
4. Parallel Lines
5. Polygons (Convex)
6. Triangles
7. Quadrilaterals
8. Circles
9. Rectangular Solids and Cylinders
10. Coordinate Geometry

Section 3.4, "Word Problems," presents examples of and solutions to the following types of word problems:

1. Rate Problems
2. Work Problems
3. Mixture Problems
4. Interest Problems
5. Discount
6. Profit
7. Sets
8. Geometry Problems
9. Measurement Problems
10. Data Interpretation

3.1 Arithmetic

1. Properties of Integers

An *integer* is any number in the set $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$. If x and y are integers and $x \neq 0$, then x is a *divisor* (*factor*) of y provided that $y = xn$ for some integer n. In this case, y is also said to be *divisible* by x or to be a *multiple* of x. For example, 7 is a divisor or factor of 28 since $28 = (7)(4)$, but 8 is not a divisor of 28 since there is no integer n such that $28 = 8n$.

If x and y are positive integers, there exist unique integers q and r, called the *quotient* and *remainder*, respectively, such that $y = xq + r$ and $0 \leq r < x$. For example, when 28 is divided by 8, the quotient is 3 and the remainder is 4 since $28 = (8)(3) + 4$. Note that y is divisible by x if and only if the remainder r is 0; for example, 32 has a remainder of 0 when divided by 8 because 32 is divisible by 8. Also, note that when a smaller integer is divided by a larger integer, the quotient is 0 and the remainder is the smaller integer. For example, 5 divided by 7 has the quotient 0 and the remainder 5 since $5 = (7)(0) + 5$.

Any integer that is divisible by 2 is an *even integer*; the set of even integers is $\{\ldots -4, -2, 0, 2, 4, 6, 8, \ldots\}$. Integers that are not divisible by 2 are *odd integers*; $\{\ldots -3, -1, 1, 3, 5, \ldots\}$ is the set of odd integers.

If at least one factor of a product of integers is even, then the product is even; otherwise the product is odd. If two integers are both even or both odd, then their sum and their difference are even. Otherwise, their sum and their difference are odd.

A *prime* number is a positive integer that has exactly two different positive divisors, 1 and itself. For example, 2, 3, 5, 7, 11, and 13 are prime numbers, but 15 is not, since 15 has four different positive divisors, 1, 3, 5, and 15. The number 1 is not a prime number since it has only one positive divisor. Every integer greater than 1 either is prime or can be uniquely expressed as a product of prime factors. For example, $14 = (2)(7)$, $81 = (3)(3)(3)(3)$, and $484 = (2)(2)(11)(11)$.

The numbers $-2, -1, 0, 1, 2, 3, 4, 5$ are *consecutive integers*. Consecutive integers can be represented by $n, n + 1, n + 2, n + 3, \ldots$, where n is an integer. The numbers 0, 2, 4, 6, 8 are *consecutive even integers*, and 1, 3, 5, 7, 9 are *consecutive odd integers*. Consecutive even integers can be represented by $2n, 2n + 2, 2n + 4, \ldots$, and consecutive odd integers can be represented by $2n + 1, 2n + 3, 2n + 5, \ldots$, where n is an integer.

Properties of the integer 1. If n is any number, then $1 \cdot n = n$, and for any number $n \neq 0$, $n \cdot \frac{1}{n} = 1$.

The number 1 can be expressed in many ways; for example, $\frac{n}{n} = 1$ for any number $n \neq 0$.

Multiplying or dividing an expression by 1, in any form, does not change the value of that expression.

Properties of the integer 0. The integer 0 is neither positive nor negative. If n is any number, then $n + 0 = n$ and $n \cdot 0 = 0$. Division by 0 is not defined.

2. Fractions

In a fraction $\frac{n}{d}$, n is the *numerator* and d is the *denominator*. The denominator of a fraction can never be 0, because division by 0 is not defined.

Two fractions are said to be *equivalent* if they represent the same number. For example, $\frac{8}{36}$ and $\frac{14}{63}$ are equivalent since they both represent the number $\frac{2}{9}$. In each case, the fraction is reduced to lowest terms by dividing both numerator and denominator by their *greatest common divisor* (gcd). The gcd of 8 and 36 is 4 and the gcd of 14 and 63 is 7.

Addition and subtraction of fractions.

Two fractions with the same denominator can be added or subtracted by performing the required operation with the numerators, leaving the denominators the same. For example, $\frac{3}{5}+\frac{4}{5}=\frac{3+4}{5}=\frac{7}{5}$ and $\frac{5}{7}-\frac{2}{7}=\frac{5-2}{7}=\frac{3}{7}$. If two fractions do not have the same denominator, express them as equivalent fractions with the same denominator. For example, to add $\frac{3}{5}$ and $\frac{4}{7}$, multiply the numerator and denominator of the first fraction by 7 and the numerator and denominator of the second fraction by 5, obtaining $\frac{21}{35}$ and $\frac{20}{35}$, respectively; $\frac{21}{35}+\frac{20}{35}=\frac{41}{35}$.

For the new denominator, choosing the *least common multiple* (lcm) of the denominators usually lessens the work. For $\frac{2}{3}+\frac{1}{6}$, the lcm of 3 and 6 is 6 (not $3 \times 6 = 18$), so $\frac{2}{3}+\frac{1}{6}=\frac{2}{3}\times\frac{2}{2}+\frac{1}{6}=\frac{4}{6}+\frac{1}{6}=\frac{5}{6}$.

Multiplication and division of fractions.

To multiply two fractions, simply multiply the two numerators and multiply the two denominators.

For example, $\frac{2}{3}\times\frac{4}{7}=\frac{2\times 4}{3\times 7}=\frac{8}{21}$.

To divide by a fraction, invert the divisor (that is, find its *reciprocal*) and multiply. For example, $\frac{2}{3}\div\frac{4}{7}=\frac{2}{3}\times\frac{7}{4}=\frac{14}{12}=\frac{7}{6}$.

In the problem above, the reciprocal of $\frac{4}{7}$ is $\frac{7}{4}$. In general, the reciprocal of a fraction $\frac{n}{d}$ is $\frac{d}{n}$, where n and d are not zero.

Mixed numbers.

A number that consists of a whole number and a fraction, for example, $7\frac{2}{3}$, is a mixed number: $7\frac{2}{3}$ means $7+\frac{2}{3}$.

To change a mixed number into a fraction, multiply the whole number by the denominator of the fraction and add this number to the numerator of the fraction; then put the result over the denominator of the fraction. For example, $7\frac{2}{3}=\frac{(3\times 7)+2}{3}=\frac{23}{3}$.

3. Decimals

In the decimal system, the position of the period or *decimal point* determines the place value of the digits. For example, the digits in the number 7,654.321 have the following place values:

Thousands		Hundreds	Tens	Ones or units		Tenths	Hundredths	Thousandths
7	,	6	5	4	.	3	2	1

Some examples of decimals follow.

$$0.321 = \frac{3}{10} + \frac{2}{100} + \frac{1}{1,000} = \frac{321}{1,000}$$

$$0.0321 = \frac{0}{10} + \frac{3}{100} + \frac{2}{1,000} + \frac{1}{10,000} = \frac{321}{10,000}$$

$$1.56 = 1 + \frac{5}{10} + \frac{6}{100} = \frac{156}{100}$$

Sometimes decimals are expressed as the product of a number with only one digit to the left of the decimal point and a power of 10. This is called *scientific notation.* For example, 231 can be written as 2.31×10^2 and 0.0231 can be written as 2.31×10^{-2}. When a number is expressed in scientific notation, the exponent of the 10 indicates the number of places that the decimal point is to be moved in the number that is to be multiplied by a power of 10 in order to obtain the product. The decimal point is moved to the right if the exponent is positive and to the left if the exponent is negative. For example, 2.013×10^4 is equal to 20,130 and 1.91×10^{-4} is equal to 0.000191.

Addition and subtraction of decimals.

To add or subtract two decimals, the decimal points of both numbers should be lined up. If one of the numbers has fewer digits to the right of the decimal point than the other, zeros may be inserted to the right of the last digit. For example, to add 17.6512 and 653.27, set up the numbers in a column and add:

$$\begin{array}{r} 17.6512 \\ +\,653.2700 \\ \hline 670.9212 \end{array}$$

Likewise for 653.27 minus 17.6512:

$$\begin{array}{r} 653.2700 \\ -17.6512 \\ \hline 635.6188 \end{array}$$

Multiplication of decimals.

To multiply decimals, multiply the numbers as if they were whole numbers and then insert the decimal point in the product so that the number of digits to the right of the decimal point is equal to the sum of the numbers of digits to the right of the decimal points in the numbers being multiplied. For example:

$$\begin{array}{rl}
2.09 & \text{(2 digits to the right)} \\
\underline{\times 1.3} & \text{(1 digit to the right)} \\
627 & \\
\underline{2090} & \\
2.717 & (2+1=3 \text{ digits to the right})
\end{array}$$

Division of decimals.

To divide a number (the dividend) by a decimal (the divisor), move the decimal point of the divisor to the right until the divisor is a whole number. Then move the decimal point of the dividend the same number of places to the right, and divide as you would by a whole number. The decimal point in the quotient will be directly above the decimal point in the new dividend. For example, to divide 698.12 by 12.4:

$$12.4\overline{)698.12}$$

will be replaced by:

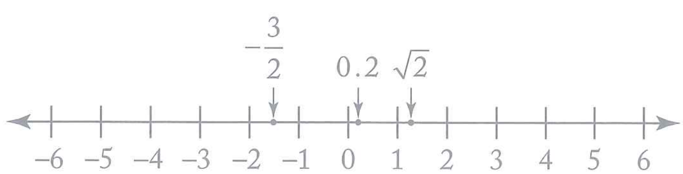
$$124\overline{)6981.2}$$

and the division would proceed as follows:

$$\begin{array}{r}
56.3 \\
124\overline{)6981.2} \\
\underline{620} \\
781 \\
\underline{744} \\
372 \\
\underline{372} \\
0
\end{array}$$

4. Real Numbers

All *real* numbers correspond to points on the number line and all points on the number line correspond to real numbers. All real numbers except zero are either positive or negative.



On a number line, numbers corresponding to points to the left of zero are negative and numbers corresponding to points to the right of zero are positive. For any two numbers on the number line, the number to the left is less than the number to the right; for example, $-4 < -3 < -\frac{3}{2} < -1$, and $1 < \sqrt{2} < 2$.

To say that the number n is between 1 and 4 on the number line means that $n > 1$ and $n < 4$, that is, $1 < n < 4$. If n is "between 1 and 4, inclusive," then $1 \le n \le 4$.

The distance between a number and zero on the number line is called the *absolute value* of the number. Thus 3 and -3 have the same absolute value, 3, since they are both three units from zero. The absolute value of 3 is denoted $|3|$. Examples of absolute values of numbers are

$$|-5| = |5| = 5, \left|-\frac{7}{2}\right| = \frac{7}{2}, \text{ and } |0| = 0.$$

Note that the absolute value of any nonzero number is positive.

Here are some properties of real numbers that are used frequently. If x, y, and z are real numbers, then

(1) $x + y = y + x$ and $xy = yx$.
For example, $8 + 3 = 3 + 8 = 11$, and $(17)(5) = (5)(17) = 85$.

(2) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$.
For example, $(7 + 5) + 2 = 7 + (5 + 2) = 7 + (7) = 14$, and $\left(5\sqrt{3}\right)\left(\sqrt{3}\right) = (5)\left(\sqrt{3}\sqrt{3}\right) = (5)(3) = 15$.

(3) $xy + xz = x(y + z)$.
For example, $718(36) + 718(64) = 718(36 + 64) = 718(100) = 71{,}800$.

(4) If x and y are both positive, then $x + y$ and xy are positive.

(5) If x and y are both negative, then $x + y$ is negative and xy is positive.

(6) If x is positive and y is negative, then xy is negative.

(7) If $xy = 0$, then $x = 0$ or $y = 0$. For example, $3y = 0$ implies $y = 0$.

(8) $|x + y| \le |x| + |y|$. For example, if $x = 10$ and $y = 2$, then $|x + y| = |12| = 12 = |x| + |y|$; and if $x = 10$ and $y = -2$, then $|x + y| = |8| = 8 < 12 = |x| + |y|$.

5. Ratio and Proportion

The *ratio* of the number a to the number b $(b \ne 0)$ is $\frac{a}{b}$.

A ratio may be expressed or represented in several ways. For example, the ratio of 2 to 3 can be written as 2 to 3, 2:3, or $\frac{2}{3}$. The order of the terms of a ratio is important. For example, the ratio of the number of months with exactly 30 days to the number with exactly 31 days is $\frac{4}{7}$, not $\frac{7}{4}$.

A *proportion* is a statement that two ratios are equal; for example, $\frac{2}{3} = \frac{8}{12}$ is a proportion. One way to solve a proportion involving an unknown is to cross multiply, obtaining a new equality. For example, to solve for n in the proportion $\frac{2}{3} = \frac{n}{12}$, cross multiply, obtaining $24 = 3n$; then divide both sides by 3, to get $n = 8$.

6. Percents

Percent means *per hundred* or *number out of 100*. A percent can be represented as a fraction with a denominator of 100, or as a decimal. For example:

$$37\% = \frac{37}{100} = 0.37.$$

To find a certain percent of a number, multiply the number by the percent expressed as a decimal or fraction. For example:

$$20\% \text{ of } 90 = 0.2 \times 90 = 18$$

or

$$20\% \text{ of } 90 = \frac{20}{100} \times 90 = \frac{1}{5} \times 90 = 18.$$

Percents greater than 100%.

Percents greater than 100% are represented by numbers greater than 1. For example:

$$300\% = \frac{300}{100} = 3$$

$$250\% \text{ of } 80 = 2.5 \times 80 = 200.$$

Percents less than 1%.

The percent 0.5% means $\frac{1}{2}$ of 1 percent. For example, 0.5% of 12 is equal to $0.005 \times 12 = 0.06$.

Percent change.

Often a problem will ask for the percent increase or decrease from one quantity to another quantity. For example, "If the price of an item increases from $24 to $30, what is the percent increase in price?" To find the percent increase, first find the amount of the increase; then divide this increase by the original amount, and express this quotient as a percent. In the example above, the percent increase would be found in the following way: the amount of the increase is $(30 - 24) = 6$. Therefore, the percent increase is $\frac{6}{24} = 0.25 = 25\%$.

Likewise, to find the percent decrease (for example, the price of an item is reduced from $30 to $24), first find the amount of the decrease; then divide this decrease by the original amount, and express this quotient as a percent. In the example above, the amount of decrease is $(30 - 24) = 6$.

Therefore, the percent decrease is $\frac{6}{30} = 0.20 = 20\%$.

Note that the percent increase from 24 to 30 is not the same as the percent decrease from 30 to 24.

In the following example, the increase is greater than 100 percent: If the cost of a certain house in 1983 was 300 percent of its cost in 1970, by what percent did the cost increase?

If n is the cost in 1970, then the percent increase is equal to $\frac{3n-n}{n}=\frac{2n}{n}=2$, or 200%.

7. Powers and Roots of Numbers

When a number k is to be used n times as a factor in a product, it can be expressed as k^n, which means the nth power of k. For example, $2^2=2\times2=4$ and $2^3=2\times2\times2=8$ are powers of 2.

Squaring a number that is greater than 1, or raising it to a higher power, results in a larger number; squaring a number between 0 and 1 results in a smaller number. For example:

$$3^2=9 \qquad (9>3)$$
$$\left(\frac{1}{3}\right)^2=\frac{1}{9} \qquad \left(\frac{1}{9}<\frac{1}{3}\right)$$
$$(0.1)^2=0.01 \qquad (0.01<0.1)$$

A *square root* of a number n is a number that, when squared, is equal to n. The square root of a negative number is not a real number. Every positive number n has two square roots, one positive and the other negative, but $\sqrt{n}$ denotes the positive number whose square is n. For example, $\sqrt{9}$ denotes 3. The two square roots of 9 are $\sqrt{9}=3$ and $-\sqrt{9}=-3$.

Every real number r has exactly one real *cube root,* which is the number s such that $s^3=r$. The real cube root of r is denoted by $\sqrt[3]{r}$. Since $2^3=8$, $\sqrt[3]{8}=2$. Similarly, $\sqrt[3]{-8}=-2$, because $(-2)^3=-8$.

8. Descriptive Statistics

A list of numbers, or numerical data, can be described by various statistical measures. One of the most common of these measures is the *average,* or *(arithmetic) mean,* which locates a type of "center" for the data. The average of n numbers is defined as the sum of the n numbers divided by n. For example, the average of 6, 4, 7, 10, and 4 is $\frac{6+4+7+10+4}{5}=\frac{31}{5}=6.2$.

The *median* is another type of center for a list of numbers. To calculate the median of n numbers, first order the numbers from least to greatest; if n is odd, the median is defined as the middle number, whereas if n is even, the median is defined as the average of the two middle numbers. In the example above, the numbers, in order, are 4, 4, 6, 7, 10, and the median is 6, the middle number.

For the numbers 4, 6, 6, 8, 9, 12, the median is $\frac{6+8}{2}=7$. Note that the mean of these numbers is 7.5.

The median of a set of data can be less than, equal to, or greater than the mean. Note that for a large set of data (for example, the salaries of 800 company employees), it is often true that about half of the data is less than the median and about half of the data is greater than the median; but this is not always the case, as the following data show.

3, 5, 7, 7, 7, 7, 7, 7, 8, 9, 9, 9, 9, 10, 10

Here the median is 7, but only $\frac{2}{15}$ of the data is less than the median.

The *mode* of a list of numbers is the number that occurs most frequently in the list. For example, the mode of 1, 3, 6, 4, 3, 5 is 3. A list of numbers may have more than one mode. For example, the list 1, 2, 3, 3, 3, 5, 7, 10, 10, 10, 20 has two modes, 3 and 10.

The degree to which numerical data are spread out or dispersed can be measured in many ways. The simplest measure of dispersion is the *range,* which is defined as the greatest value in the numerical data minus the least value. For example, the range of 11, 10, 5, 13, 21 is 21 − 5 = 16. Note how the range depends on only two values in the data.

One of the most common measures of dispersion is the *standard deviation.* Generally speaking, the more the data are spread away from the mean, the greater the standard deviation. The standard deviation of n numbers can be calculated as follows: (1) find the arithmetic mean, (2) find the differences between the mean and each of the n numbers, (3) square each of the differences, (4) find the average of the squared differences, and (5) take the nonnegative square root of this average. Shown below is this calculation for the data 0, 7, 8, 10, 10, which have arithmetic mean 7.

x	$x - 7$	$(x - 7)^2$
0	−7	49
7	0	0
8	1	1
10	3	9
10	3	9
	Total	68

Standard deviation $\sqrt{\frac{68}{5}} \approx 3.7$

Notice that the standard deviation depends on every data value, although it depends most on values that are farthest from the mean. This is why a distribution with data grouped closely around the mean will have a smaller standard deviation than will data spread far from the mean. To illustrate this, compare the data 6, 6, 6.5, 7.5, 9, which also have mean 7. Note that the numbers in the second set of data seem to be grouped more closely around the mean of 7 than the numbers in the first set. This is reflected in the standard deviation, which is less for the second set (approximately 1.1) than for the first set (approximately 3.7).

There are many ways to display numerical data that show how the data are distributed. One simple way is with a *frequency distribution,* which is useful for data that have values occurring with varying frequencies. For example, the 20 numbers

−4	0	0	−3	−2	−1	−1	0	−1	−4
−1	−5	0	−2	0	−5	−2	0	0	−1

are displayed on the next page in a frequency distribution by listing each different value x and the frequency f with which x occurs.

Data Value x	Frequency f
−5	2
−4	2
−3	1
−2	3
−1	5
0	7
Total	20

From the frequency distribution, one can readily compute descriptive statistics:

Mean: $= \dfrac{(-5)(2)+(-4)(2)+(-3)(1)+(-2)(3)+(-1)(5)+(0)(7)}{20} = -1.6$

Median: −1 (the average of the 10th and 11th numbers)

Mode: 0 (the number that occurs most frequently)

Range: $0 - (-5) = 5$

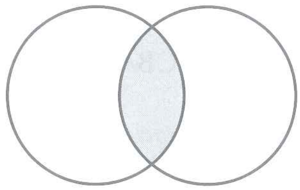
Standard deviation: $\sqrt{\dfrac{(-5+1.6)^2(2)+(-4+1.6)^2(2)+\ldots+(0+1.6)^2(7)}{20}} \approx 1.7$

9. Sets

In mathematics a *set* is a collection of numbers or other objects. The objects are called the *elements* of the set. If S is a set having a finite number of elements, then the number of elements is denoted by $|S|$. Such a set is often defined by listing its elements; for example, $S = \{-5, 0, 1\}$ is a set with $|S| = 3$. The order in which the elements are listed in a set does not matter; thus $\{-5, 0, 1\} = \{0, 1, -5\}$. If all the elements of a set S are also elements of a set T, then S is a *subset* of T; for example, $S = \{-5, 0, 1\}$ is a subset of $T = \{-5, 0, 1, 4, 10\}$.

For any two sets A and B, the *union* of A and B is the set of all elements that are in A *or* in B *or* in both. The *intersection* of A and B is the set of all elements that are both in A *and* in B. The union is denoted by $A \cup B$ and the intersection is denoted by $A \cap B$. As an example, if $A = \{3, 4\}$ and $B = \{4, 5, 6\}$, then $A \cup B = \{3, 4, 5, 6\}$ and $A \cap B = \{4\}$. Two sets that have no elements in common are said to be *disjoint* or *mutually exclusive*.

The relationship between sets is often illustrated with a *Venn diagram* in which sets are represented by regions in a plane. For two sets S and T that are not disjoint and neither is a subset of the other, the intersection $S \cap T$ is represented by the shaded region of the diagram below.



This diagram illustrates a fact about any two finite sets S and T: the number of elements in their union equals the sum of their individual numbers of elements minus the number of elements in their intersection (because the latter are counted twice in the sum); more concisely,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

This counting method is called the general addition rule for two sets. As a special case, if S and T are disjoint, then

$$|S \cup T| = |S| + |T|$$

since $|S \cap T| = 0$.

10. Counting Methods

There are some useful methods for counting objects and sets of objects without actually listing the elements to be counted. The following principle of multiplication is fundamental to these methods.

If an object is to be chosen from a set of m objects and a second object is to be chosen from a different set of n objects, then there are mn ways of choosing both objects simultaneously.

As an example, suppose the objects are items on a menu. If a meal consists of one entree and one dessert and there are 5 entrees and 3 desserts on the menu, then there are $5 \times 3 = 15$ different meals that can be ordered from the menu. As another example, each time a coin is flipped, there are two possible outcomes, heads and tails. If an experiment consists of 8 consecutive coin flips, then the experiment has 2^8 possible outcomes, where each of these outcomes is a list of heads and tails in some order.

A symbol that is often used with the multiplication principle is the *factorial.* If n is an integer greater than 1, then n factorial, denoted by the symbol $n!$, is defined as the product of all the integers from 1 to n. Therefore,

$$2! = (1)(2) = 2,$$
$$3! = (1)(2)(3) = 6,$$
$$4! = (1)(2)(3)(4) = 24, \text{ etc.}$$

Also, by definition, $0! = 1! = 1$.

The factorial is useful for counting the number of ways that a set of objects can be ordered. If a set of n objects is to be ordered from 1st to nth, then there are n choices for the 1st object, $n - 1$ choices for the 2nd object, $n - 2$ choices for the 3rd object, and so on, until there is only 1 choice for the nth object.

Thus, by the multiplication principle, the number of ways of ordering the n objects is

$n(n-1)(n-2)\cdots(3)(2)(1)=n!$.

For example, the number of ways of ordering the letters A, B, and C is 3!, or 6:

ABC, ACB, BAC, BCA, CAB, and CBA.

These orderings are called the *permutations* of the letters A, B, and C.

A permutation can be thought of as a selection process in which objects are selected one by one in a certain order. If the order of selection is not relevant and only k objects are to be selected from a larger set of n objects, a different counting method is employed.

Specifically, consider a set of n objects from which a complete selection of k objects is to be made without regard to order, where $0 \le k \le n$. Then the number of possible complete selections of k objects is called the number of *combinations* of n objects taken k at a time and is denoted by $\binom{n}{k}$.

The value of $\binom{n}{k}$ is given by $\binom{n}{k}=\dfrac{n!}{k!(n-k)!}$.

Note that $\binom{n}{k}$ is the number of k-element subsets of a set with n elements. For example, if $S=\{A, B, C, D, E\}$, then the number of 2-element subsets of S, or the number of combinations of 5 letters taken 2 at a time, is $\binom{5}{2}=\dfrac{5!}{2!3!}=\dfrac{120}{(2)(6)}=10$.

The subsets are $\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}$, and $\{D, E\}$. Note that $\binom{5}{2}=10=\binom{5}{3}$ because every 2-element subset chosen from a set of 5 elements corresponds to a unique 3-element subset consisting of the elements *not* chosen.

In general, $\binom{n}{k}=\binom{n}{n-k}$.

11. Discrete Probability

Many of the ideas discussed in the preceding three topics are important to the study of discrete probability. Discrete probability is concerned with *experiments* that have a finite number of *outcomes*. Given such an experiment, an *event* is a particular set of outcomes. For example, rolling a number cube with faces numbered 1 to 6 (similar to a 6-sided die) is an experiment with 6 possible outcomes: 1, 2, 3, 4, 5, or 6. One event in this experiment is that the outcome is 4, denoted $\{4\}$; another event is that the outcome is an odd number: $\{1, 3, 5\}$.

The probability that an event E occurs, denoted by $P(E)$, is a number between 0 and 1, inclusive. If E has no outcomes, then E is *impossible* and $P(E)=0$; if E is the set of all possible outcomes of the experiment, then E is *certain* to occur and $P(E)=1$. Otherwise, E is possible but uncertain, and $0<P(E)<1$. If F is a subset of E, then $P(F)\le P(E)$. In the example above, if the probability of each of the 6 outcomes is the same, then the probability of each outcome is $\frac{1}{6}$, and the outcomes are said to be *equally likely.* For experiments in which all the individual outcomes are equally likely, the probability of an event E is

$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The total number of possible outcomes}}.$$

In the example, the probability that the outcome is an odd number is

$$P(\{1,3,5\}) = \frac{|\{1,3,5\}|}{6} = \frac{3}{6} = \frac{1}{2}.$$

Given an experiment with events E and F, the following events are defined:
"*not* E" is the set of outcomes that are not outcomes in E;
"E *or* F" is the set of outcomes in E or F or both, that is, $E \cup F$;
"E *and* F" is the set of outcomes in both E and F, that is, $E \cap F$.

The probability that E does not occur is $P(\text{not } E) = 1 - P(E)$. The probability that "$E$ or F" occurs is $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$, using the general addition rule at the end of section 4.1.9 ("Sets"). For the number cube, if E is the event that the outcome is an odd number, $\{1, 3, 5\}$, and F is the event that the outcome is a prime number, $\{2, 3, 5\}$, then $P(E \text{ and } F) = P(\{3,5\}) = \frac{2}{6} = \frac{1}{3}$ and so $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$.
Note that the event "E or F" is $E \cup F = \{1, 2, 3, 5\}$, and hence $P(E \text{ or } F) = \frac{|\{1,2,3,5\}|}{6} = \frac{4}{6} = \frac{2}{3}$.

If the event "E and F" is impossible (that is, $E \cap F$ has no outcomes), then E and F are said to be *mutually exclusive* events, and $P(E \text{ and } F) = 0$. Then the general addition rule is reduced to $P(E \text{ or } F) = P(E) + P(F)$.

This is the special addition rule for the probability of two mutually exclusive events.

Two events A and B are said to be *independent* if the occurrence of either event does not alter the probability that the other event occurs. For one roll of the number cube, let $A = \{2, 4, 6\}$ and let $B = \{5, 6\}$. Then the probability that A occurs is $P(A) = \frac{|A|}{6} = \frac{3}{6} = \frac{1}{2}$, while, *presuming B occurs,* the probability that A occurs is

$$\frac{|A \cap B|}{|B|} = \frac{|\{6\}|}{|\{5,6\}|} = \frac{1}{2}.$$

Similarly, the probability that B occurs is $P(B) = \frac{|B|}{6} = \frac{2}{6} = \frac{1}{3}$, while, *presuming A occurs,* the probability that B occurs is

$$\frac{|B \cap A|}{|A|} = \frac{|\{6\}|}{|\{2,4,6\}|} = \frac{1}{3}.$$

Thus, the occurrence of either event does not affect the probability that the other event occurs. Therefore, A and B are independent.

The following multiplication rule holds for any independent events E and F: $P(E \text{ and } F) = P(E)P(F)$.

For the independent events A and B above, $P(A \text{ and } B) = P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \left(\frac{1}{6}\right)$.

Note that the event "A and B" is $A \cap B = \{6\}$, and hence $P(A \text{ and } B) = P(\{6\}) = \frac{1}{6}$. It follows from the general addition rule and the multiplication rule above that if E and F are independent, then

$$P(E \text{ or } F) = P(E) + P(F) - P(E)P(F).$$

For a final example of some of these rules, consider an experiment with events A, B, and C for which $P(A) = 0.23$, $P(B) = 0.40$, and $P(C) = 0.85$. Also, suppose that events A and B are mutually exclusive and events B and C are independent. Then

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) \text{ (since } A \text{ or } B \text{ are mutually exclusive)} \\ &= 0.23 + 0.40 \\ &= 0.63 \\ P(B \text{ or } C) &= P(B) + P(C) - P(B)P(C) \text{ (by independence)} \\ &= 0.40 + 0.85 - (0.40)(0.85) \\ &= 0.91 \end{aligned}$$

Note that $P(A \text{ or } C)$ and $P(A \text{ and } C)$ cannot be determined using the information given. But it can be determined that A and C are *not* mutually exclusive since $P(A) + P(C) = 1.08$, which is greater than 1, and therefore cannot equal $P(A \text{ or } C)$; from this it follows that $P(A \text{ and } C) \geq 0.08$. One can also deduce that $P(A \text{ and } C) \leq P(A) = 0.23$, since $A \cap C$ is a subset of A, and that $P(A \text{ or } C) \geq P(C) = 0.85$ since C is a subset of $A \cup C$. Thus, one can conclude that $0.85 \leq P(A \text{ or } C) \leq 1$ and $0.08 \leq P(A \text{ and } C) \leq 0.23$.

3.2 Algebra

Algebra is based on the operations of arithmetic and on the concept of an *unknown quantity*, or *variable*. Letters such as x or n are used to represent unknown quantities. For example, suppose Pam has 5 more pencils than Fred. If F represents the number of pencils that Fred has, then the number of pencils that Pam has is $F + 5$. As another example, if Jim's present salary S is increased by 7%, then his new salary is $1.07S$. A combination of letters and arithmetic operations, such as $F + 5, \frac{3x^2}{2x-5}$, and $19x^2 - 6x + 3$, is called an *algebraic expression*.

The expression $19x^2 - 6x + 3$ consists of the *terms* $19x^2, -6x$, and 3, where 19 is the *coefficient* of x^2, -6 is the coefficient of x^1, and 3 is a *constant term* (or coefficient of $x^0 = 1$). Such an expression is called a *second degree* (or *quadratic*) *polynomial in* x since the highest power of x is 2. The expression $F + 5$ is a *first degree* (or *linear*) *polynomial in* F since the highest power of F is 1. The expression $\frac{3x^2}{2x-5}$ is not a polynomial because it is not a sum of terms that are each powers of x multiplied by coefficients.

1. Simplifying Algebraic Expressions

Often when working with algebraic expressions, it is necessary to simplify them by factoring or combining *like* terms. For example, the expression $6x + 5x$ is equivalent to $(6 + 5)x$, or $11x$. In the expression $9x - 3y$, 3 is a factor common to both terms: $9x - 3y = 3(3x - y)$. In the expression $5x^2 + 6y$, there are no like terms and no common factors.

If there are common factors in the numerator and denominator of an expression, they can be divided out, provided that they are not equal to zero.

For example, if $x \neq 3$, then $\frac{x-3}{x-3}$ is equal to 1; therefore,

$$\begin{aligned}\frac{3xy-9y}{x-3} &= \frac{3y(x-3)}{x-3}\\ &= (3y)(1)\\ &= 3y\end{aligned}$$

To multiply two algebraic expressions, each term of one expression is multiplied by each term of the other expression. For example:

$$\begin{aligned}(3x-4)(9y+x) &= 3x(9y+x)-4(9y+x)\\ &= (3x)(9y)+(3x)(x)+(-4)(9y)+(-4)(x)\\ &= 27xy+3x^2-36y-4x\end{aligned}$$

An algebraic expression can be evaluated by substituting values of the unknowns in the expression. For example, if $x = 3$ and $y = -2$, then $3xy - x^2 + y$ can be evaluated as

$$3(3)(-2)-(3)^2+(-2) = -18-9-2 = -29$$

2. Equations

A major focus of algebra is to solve equations involving algebraic expressions. Some examples of such equations are

$$5x-2 = 9-x \quad \text{(a linear equation with one unknown)}$$

$$3x+1 = y-2 \quad \text{(a linear equation with two unknowns)}$$

$$5x^2+3x-2 = 7x \quad \text{(a quadratic equation with one unknown)}$$

$$\frac{x(x-3)(x^2+5)}{x-4} = 0 \quad \text{(an equation that is factored on one side with 0 on the other)}$$

The *solutions* of an equation with one or more unknowns are those values that make the equation true, or "satisfy the equation," when they are substituted for the unknowns of the equation. An equation may have no solution or one or more solutions. If two or more equations are to be solved together, the solutions must satisfy all the equations simultaneously.

Two equations having the same solution(s) are *equivalent equations.* For example, the equations

$$2+x=3$$
$$4+2x=6$$

each have the unique solution $x=1$. Note that the second equation is the first equation multiplied by 2. Similarly, the equations

$$3x-y=6$$
$$6x-2y=12$$

have the same solutions, although in this case each equation has infinitely many solutions. If any value is assigned to x, then $3x-6$ is a corresponding value for y that will satisfy both equations; for example, $x=2$ and $y=0$ is a solution to both equations, as is $x=5$ and $y=9$.

3. Solving Linear Equations with One Unknown

To solve a linear equation with one unknown (that is, to find the value of the unknown that satisfies the equation), the unknown should be isolated on one side of the equation. This can be done by performing the same mathematical operations on both sides of the equation. Remember that if the same number is added to or subtracted from both sides of the equation, this does not change the equality; likewise, multiplying or dividing both sides by the same nonzero number does not change the equality. For example, to solve the equation $\frac{5x-6}{3}=4$ for x, the variable x can be isolated using the following steps:

$$5x-6=12 \quad \text{(multiplying by 3)}$$
$$5x=18 \quad \text{(adding 6)}$$
$$x=\frac{18}{5} \quad \text{(dividing by 5)}$$

The solution, $\frac{18}{5}$, can be checked by substituting it for x in the original equation to determine whether it satisfies that equation:

$$\frac{5\left(\frac{18}{5}\right)-6}{3}=\frac{18-6}{3}=\frac{12}{3}=4$$

Therefore, $x=\frac{18}{5}$ is the solution.

4. Solving Two Linear Equations with Two Unknowns

For two linear equations with two unknowns, if the equations are equivalent, then there are infinitely many solutions to the equations, as illustrated at the end of section 4.2.2 ("Equations"). If the equations are not equivalent, then they have either one unique solution or no solution. The latter case is illustrated by the two equations:

$$3x+4y=17$$
$$6x+8y=35$$

Note that $3x + 4y = 17$ implies $6x + 8y = 34$, which contradicts the second equation. Thus, no values of x and y can simultaneously satisfy both equations.

There are several methods of solving two linear equations with two unknowns. With any method, if a contradiction is reached, then the equations have no solution; if a trivial equation such as $0 = 0$ is reached, then the equations are equivalent and have infinitely many solutions. Otherwise, a unique solution can be found.

One way to solve for the two unknowns is to express one of the unknowns in terms of the other using one of the equations, and then substitute the expression into the remaining equation to obtain an equation with one unknown. This equation can be solved and the value of the unknown substituted into either of the original equations to find the value of the other unknown. For example, the following two equations can be solved for x and y.

$$\begin{aligned} &(1)\quad 3x+2y=11 \\ &(2)\quad x-y=2 \end{aligned}$$

In equation (2), $x = 2 + y$. Substitute $2 + y$ in equation (1) for x:

$$\begin{aligned} 3(2+y)+2y&=11 \\ 6+3y+2y&=11 \\ 6+5y&=11 \\ 5y&=5 \\ y&=1 \end{aligned}$$

If $y = 1$, then $x - 1 = 2$ and $x = 2 + 1 = 3$.

There is another way to solve for x and y by eliminating one of the unknowns. This can be done by making the coefficients of one of the unknowns the same (disregarding the sign) in both equations and either adding the equations or subtracting one equation from the other. For example, to solve the equations

$$\begin{aligned} &(1)\quad 6x+5y=29 \\ &(2)\quad 4x-3y=-6 \end{aligned}$$

by this method, multiply equation (1) by 3 and equation (2) by 5 to get

$$\begin{aligned} 18x+15y&=87 \\ 20x-15y&=-30 \end{aligned}$$

Adding the two equations eliminates y, yielding $38x = 57$, or $x = \frac{3}{2}$. Finally, substituting $\frac{3}{2}$ for x in one of the equations gives $y = 4$. These answers can be checked by substituting both values into both of the original equations.

5. Solving Equations by Factoring

Some equations can be solved by factoring. To do this, first add or subtract expressions to bring all the expressions to one side of the equation, with 0 on the other side. Then try to factor the nonzero side into a product of expressions. If this is possible, then using property (7) in section 4.1.4 ("Real Numbers") each of the factors can be set equal to 0, yielding several simpler equations that possibly can be solved. The solutions of the simpler equations will be solutions of the factored equation. As an example, consider the equation $x^3 - 2x^2 + x = -5(x - 1)^2$:

$$
\begin{aligned}
x^3 - 2x^2 + x + 5(x-1)^2 &= 0 \\
x(x^2 - 2x + 1) + 5(x-1)^2 &= 0 \\
x(x-1)^2 + 5(x-1)^2 &= 0 \\
(x+5)(x-1)^2 &= 0 \\
x + 5 = 0 \text{ or } (x-1)^2 &= 0 \\
x = -5 \text{ or } x &= 1.
\end{aligned}
$$

For another example, consider $\dfrac{x(x-3)(x^2+5)}{x-4} = 0$. A fraction equals 0 if and only if its numerator equals 0. Thus, $x(x - 3)(x^2 + 5) = 0$:

$$
\begin{aligned}
&x = 0 \text{ or } x - 3 = 0 \text{ or } x^2 + 5 = 0 \\
&x = 0 \text{ or } x = 3 \text{ or } x^2 + 5 = 0.
\end{aligned}
$$

But $x^2 + 5 = 0$ has no real solution because $x^2 + 5 > 0$ for every real number. Thus, the solutions are 0 and 3.

The solutions of an equation are also called the *roots* of the equation. These roots can be checked by substituting them into the original equation to determine whether they satisfy the equation.

6. Solving Quadratic Equations

The standard form for a *quadratic equation* is

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$; for example:

$$
\begin{aligned}
x^2 + 6x + 5 &= 0 \\
3x^2 - 2x &= 0, \text{ and} \\
x^2 + 4 &= 0
\end{aligned}
$$

Some quadratic equations can easily be solved by factoring. For example:

$$
\begin{aligned}
(1) \qquad x^2 + 6x + 5 &= 0 \\
(x+5)(x+1) &= 0 \\
x + 5 = 0 \text{ or } x + 1 &= 0 \\
x = -5 \text{ or } x &= -1
\end{aligned}
$$

$$
\begin{aligned}
(2) \qquad 3x^2 - 3 &= 8x \\
3x^2 - 8x - 3 &= 0 \\
(3x+1)(x-3) &= 0 \\
3x+1 = 0 \text{ or } x - 3 &= 0 \\
x = -\frac{1}{3} \text{ or } x &= 3
\end{aligned}
$$

A quadratic equation has at most two real roots and may have just one or even no real root. For example, the equation $x^2 - 6x + 9 = 0$ can be expressed as $(x - 3)^2 = 0$, or $(x - 3)(x - 3) = 0$; thus the only root is 3. The equation $x^2 + 4 = 0$ has no real root; since the square of any real number is greater than or equal to zero, $x^2 + 4$ must be greater than zero.

An expression of the form $a^2 - b^2$ can be factored as $(a - b)(a + b)$.

For example, the quadratic equation $9x^2 - 25 = 0$ can be solved as follows.

$$
\begin{aligned}
(3x-5)(3x+5) &= 0 \\
3x - 5 = 0 \text{ or } 3x + 5 &= 0 \\
x = \frac{5}{3} \text{ or } x &= -\frac{5}{3}
\end{aligned}
$$

If a quadratic expression is not easily factored, then its roots can always be found using the *quadratic formula:* If $ax^2 + bx + c = 0$ $(a \neq 0)$, then the roots are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

These are two distinct real numbers unless $b^2 - 4ac \leq 0$. If $b^2 - 4ac = 0$, then these two expressions for x are equal to $-\frac{b}{2a}$, and the equation has only one root. If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number and the equation has no real roots.

7. Exponents

A positive integer exponent of a number or a variable indicates a product, and the positive integer is the number of times that the number or variable is a factor in the product. For example, x^5 means $(x)(x)(x)(x)(x)$; that is, x is a factor in the product 5 times.

Some rules about exponents follow.

Let x and y be any positive numbers, and let r and s be any positive integers.

(1) $(x^r)(x^s) = x^{(r+s)}$; for example, $(2^2)(2^3) = 2^{(2+3)} = 2^5 = 32$.

(2) $\frac{x^r}{x^s} = x^{(r-s)}$; for example, $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$.

(3) $(x^r)(y^r) = (xy)^r$; for example, $(3^3)(4^3) = 12^3 = 1{,}728$.

(4) $\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}$; for example, $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$.

(5) $(x^r)^s = x^{rs} = (x^s)^r$; for example, $(x^3)^4 = x^{12} = (x^4)^3$.

(6) $x^{-r} = \frac{1}{x^r}$; for example, $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

(7) $x^0 = 1$; for example, $6^0 = 1$.

(8) $x^{\frac{r}{s}} = \left(x^{\frac{1}{s}}\right)^r = \left(x^r\right)^{\frac{1}{s}} = \sqrt[s]{x^r}$; for example, $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(8^2\right)^{\frac{1}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ and $9^{\frac{1}{2}} = \sqrt{9} = 3$.

It can be shown that rules 1 – 6 also apply when r and s are not integers and are not positive, that is, when r and s are any real numbers.

8. Inequalities

An *inequality* is a statement that uses one of the following symbols:

$\neq$ not equal to

$>$ greater than

$\geq$ greater than or equal to

$<$ less than

$\leq$ less than or equal to

Some examples of inequalities are $5x - 3 < 9$, $6x \geq y$, and $\frac{1}{2} < \frac{3}{4}$. Solving a linear inequality with one unknown is similar to solving an equation; the unknown is isolated on one side of the inequality. As in solving an equation, the same number can be added to or subtracted from both sides of the inequality, or both sides of an inequality can be multiplied or divided by a positive number without changing the truth of the inequality. However, multiplying or dividing an inequality by a negative number reverses the order of the inequality. For example, $6 > 2$, but $(-1)(6) < (-1)(2)$.

To solve the inequality $3x - 2 > 5$ for x, isolate x by using the following steps:

$$3x - 2 > 5$$
$$3x > 7 \quad \text{(adding 2 to both sides)}$$
$$x > \frac{7}{3} \quad \text{(dividing both sides by 3)}$$

To solve the inequality $\frac{5x-1}{-2} < 3$ for x, isolate x by using the following steps:

$$\frac{5x-1}{-2} < 3$$
$$5x - 1 > -6 \quad \text{(multiplying both sides by } -2\text{)}$$
$$5x > -5 \quad \text{(adding 1 to both sides)}$$
$$x > -1 \quad \text{(dividing both sides by 5)}$$

9. Absolute Value

The absolute value of x, denoted $|x|$, is defined to be x if $x \geq 0$ and $-x$ if $x < 0$. Note that $\sqrt{x^2}$ denotes the nonnegative square root of x^2, and so $\sqrt{x^2} = |x|$.

10. Functions

An algebraic expression in one variable can be used to define a *function* of that variable. A function is denoted by a letter such as f or g along with the variable in the expression. For example, the expression $x^3 - 5x^2 + 2$ defines a function f that can be denoted by

$$f(x) = x^3 - 5x^2 + 2.$$

The expression $\frac{2z+7}{\sqrt{z+1}}$ defines a function g that can be denoted by

$$g(z) = \frac{2z+7}{\sqrt{z+1}}.$$

The symbols "$f(x)$" or "$g(z)$" do not represent products; each is merely the symbol for an expression, and is read "f of x" or "g of z."

Function notation provides a short way of writing the result of substituting a value for a variable. If $x = 1$ is substituted in the first expression, the result can be written $f(1) = -2$, and $f(1)$ is called the "value of f at $x = 1$." Similarly, if $z = 0$ is substituted in the second expression, then the value of g at $z = 0$ is $g(0) = 7$.

Once a function $f(x)$ is defined, it is useful to think of the variable x as an input and $f(x)$ as the corresponding output. In any function there can be no more than one output for any given input. However, more than one input can give the same output; for example, if $h(x) = |x + 3|$, then $h(-4) = 1 = h(-2)$.

The set of all allowable inputs for a function is called the *domain* of the function. For f and g defined above, the domain of f is the set of all real numbers and the domain of g is the set of all numbers greater than -1. The domain of any function can be arbitrarily specified, as in the function defined by "$h(x) = 9x - 5$ for $0 \leq x \leq 10$." Without such a restriction, the domain is assumed to be all values of x that result in a real number when substituted into the function.

The domain of a function can consist of only the positive integers and possibly 0. For example, $a(n) = n^2 + \frac{n}{5}$ for $n = 0, 1, 2, 3, \ldots\ldots$

Such a function is called a *sequence* and $a(n)$ is denoted by a_n. The value of the sequence a_n at $n = 3$ is $a_3 = 3^2 + \frac{3}{5} = 9.60$. As another example, consider the sequence defined by $b_n = (-1)^n (n!)$ for $n = 1, 2, 3, \ldots$ A sequence like this is often indicated by listing its values in the order $b_1, b_2, b_3, \ldots, b_n, \ldots$ as follows:

$-1, 2, -6, \ldots, (-1)^n(n!), \ldots,$ and $(-1)^n(n!)$ is called the nth term of the sequence.

3.3 Geometry

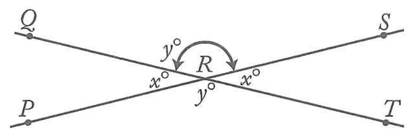
1. Lines

In geometry, the word “line” refers to a straight line that extends without end in both directions.

The line above can be referred to as line PQ or line ℓ. The part of the line from P to Q is called a *line segment.* P and Q are the *endpoints* of the segment. The notation $\overline{PQ}$ is used to denote line segment PQ and PQ is used to denote the length of the segment.

2. Intersecting Lines and Angles

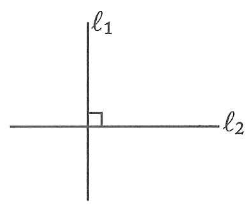
If two lines intersect, the opposite angles are called *vertical angles* and have the same measure. In the figure



$\angle PRQ$ and $\angle SRT$ are vertical angles and $\angle QRS$ and $\angle PRT$ are vertical angles. Also, $x + y = 180°$ since PRS is a straight line.

3. Perpendicular Lines

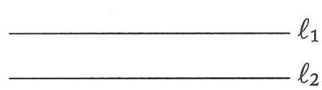
An angle that has a measure of 90° is a *right angle.* If two lines intersect at right angles, the lines are *perpendicular.* For example:



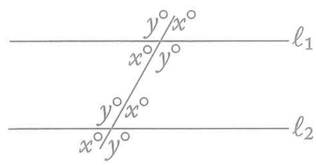
ℓ_1 and ℓ_2 above are perpendicular, denoted by $\ell_1 \perp \ell_2$. A right angle symbol in an angle of intersection indicates that the lines are perpendicular.

4. Parallel Lines

If two lines that are in the same plane do not intersect, the two lines are *parallel.* In the figure



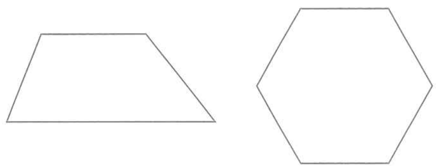
lines ℓ_1 and ℓ_2 are parallel, denoted by $\ell_1 \parallel \ell_2$. If two parallel lines are intersected by a third line, as shown below, then the angle measures are related as indicated, where $x + y = 180°$.



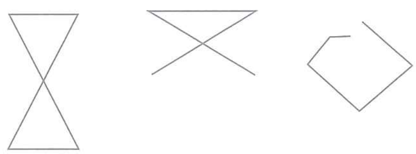
5. Polygons (Convex)

A *polygon* is a closed plane figure formed by three or more line segments, called the *sides* of the polygon. Each side intersects exactly two other sides at their endpoints. The points of intersection of the sides are *vertices*. The term "polygon" will be used to mean a convex polygon, that is, a polygon in which each interior angle has a measure of less than 180°.

The following figures are polygons:

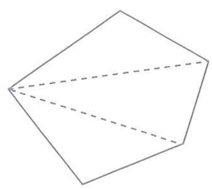


The following figures are not polygons:



A polygon with three sides is a *triangle*; with four sides, a *quadrilateral*; with five sides, a *pentagon*; and with six sides, a *hexagon*.

The sum of the interior angle measures of a triangle is 180°. In general, the sum of the interior angle measures of a polygon with n sides is equal to $(n - 2)180°$. For example, this sum for a pentagon is $(5 - 2)180° = (3)180° = 540°$.



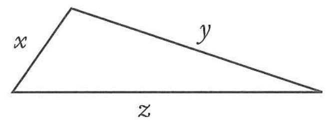
Note that a pentagon can be partitioned into three triangles and therefore the sum of the angle measures can be found by adding the sum of the angle measures of three triangles.

The *perimeter* of a polygon is the sum of the lengths of its sides.

The commonly used phrase "area of a triangle" (or any other plane figure) is used to mean the area of the region enclosed by that figure.

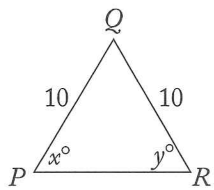
6. Triangles

There are several special types of triangles with important properties. But one property that all triangles share is that the sum of the lengths of any two of the sides is greater than the length of the third side, as illustrated below.

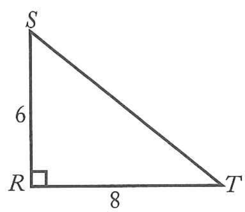


$x + y > z$, $x + z > y$, and $y + z > x$

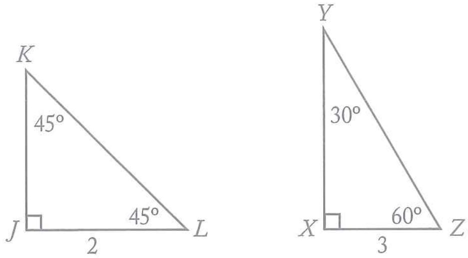
An *equilateral* triangle has all sides of equal length. All angles of an equilateral triangle have equal measure. An *isosceles* triangle has at least two sides of the same length. If two sides of a triangle have the same length, then the two angles opposite those sides have the same measure. Conversely, if two angles of a triangle have the same measure, then the sides opposite those angles have the same length. In isosceles triangle PQR below, $x = y$ since $PQ = QR$.



A triangle that has a right angle is a *right* triangle. In a right triangle, the side opposite the right angle is the *hypotenuse,* and the other two sides are the *legs.* An important theorem concerning right triangles is the *Pythagorean theorem,* which states: In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.



In the figure above, ΔRST is a right triangle, so $(RS)^2 + (RT)^2 = (ST)^2$. Here, $RS = 6$ and $RT = 8$, so $ST = 10$, since $6^2 + 8^2 = 36 + 64 = 100 = (ST)^2$ and $ST = \sqrt{100}$. Any triangle in which the lengths of the sides are in the ratio 3:4:5 is a right triangle. In general, if a, b, and c are the lengths of the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

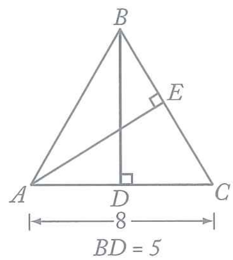


In 45°– 45°– 90° triangles, the lengths of the sides are in the ratio $1{:}1{:}\sqrt{2}$. For example, in ΔJKL, if $JL = 2$, then $JK = 2$ and $KL = 2\sqrt{2}$. In 30°– 60°– 90° triangles, the lengths of the sides are in the ratio $1{:}\sqrt{3}{:}2$. For example, in ΔXYZ, if $XZ = 3$, then $XY = 3\sqrt{3}$ and $YZ = 6$.

The *altitude* of a triangle is the segment drawn from a vertex perpendicular to the side opposite that vertex. Relative to that vertex and altitude, the opposite side is called the *base*.

The area of a triangle is equal to:

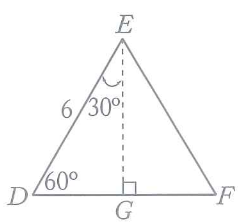
$$\frac{(\text{the length of the altitude}) \times (\text{the length of the base})}{2}$$



In ΔABC, $\overline{BD}$ is the altitude to base $\overline{AC}$ and $\overline{AE}$ is the altitude to base $\overline{BC}$. The area of ΔABC is equal to

$$\frac{BD \times AC}{2} = \frac{5 \times 8}{2} = 20.$$

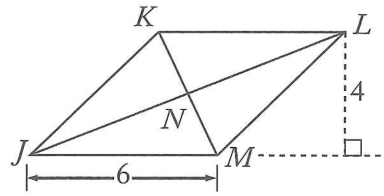
The area is also equal to $\frac{AE \times BC}{2}$. If ΔABC above is isosceles and $AB = BC$, then altitude $\overline{BD}$ bisects the base; that is, $AD = DC = 4$. Similarly, any altitude of an equilateral triangle bisects the side to which it is drawn.



In equilateral triangle DEF, if $DE = 6$, then $DG = 3$ and $EG = 3\sqrt{3}$. The area of ΔDEF is equal to $\frac{3\sqrt{3} \times 6}{2} = 9\sqrt{3}$.

7. Quadrilaterals

A polygon with four sides is a *quadrilateral.* A quadrilateral in which both pairs of opposite sides are parallel is a *parallelogram.* The opposite sides of a parallelogram also have equal length.



In parallelogram $JKLM$, $\overline{JK} \parallel \overline{LM}$ and $JK = LM$; $\overline{KL} \parallel \overline{JM}$ and $KL = JM$.

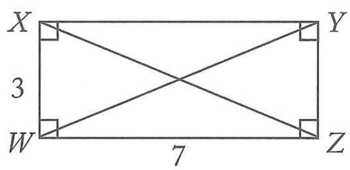
The diagonals of a parallelogram bisect each other (that is, $KN = NM$ and $JN = NL$).

The area of a parallelogram is equal to

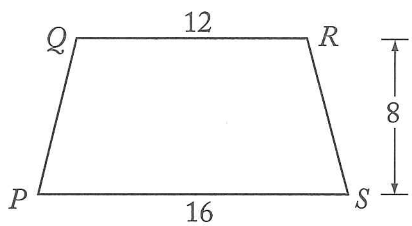
(the length of the altitude) × (the length of the base).

The area of $JKLM$ is equal to $4 \times 6 = 24$.

A parallelogram with right angles is a *rectangle,* and a rectangle with all sides of equal length is a *square.*



The perimeter of $WXYZ = 2(3) + 2(7) = 20$ and the area of $WXYZ$ is equal to $3 \times 7 = 21$. The diagonals of a rectangle are equal; therefore $WY = XZ = \sqrt{9+49} = \sqrt{58}$.



A quadrilateral with two sides that are parallel, as shown above, is a *trapezoid.* The area of trapezoid $PQRS$ may be calculated as follows:

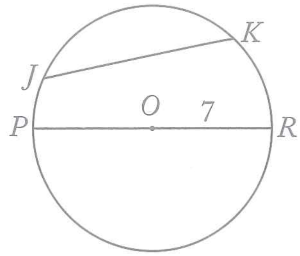
$$\frac{1}{2}(\text{the sum of the lengths of the bases})(\text{the height}) = \frac{1}{2}(QR + PS)(8) = \frac{1}{2}(28 \times 8) = 112.$$

8. Circles

A *circle* is a set of points in a plane that are all located the same distance from a fixed point (the *center* of the circle).

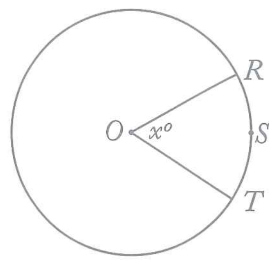
A *chord* of a circle is a line segment that has its endpoints on the circle. A chord that passes through the center of the circle is a *diameter* of the circle. A *radius* of a circle is a segment from the center of the circle to a point on the circle. The words "diameter" and "radius" are also used to refer to the lengths of these segments.

The *circumference* of a circle is the distance around the circle. If r is the radius of the circle, then the circumference is equal to $2\pi r$, where π is approximately $\frac{22}{7}$ or 3.14. The *area* of a circle of radius r is equal to πr^2.



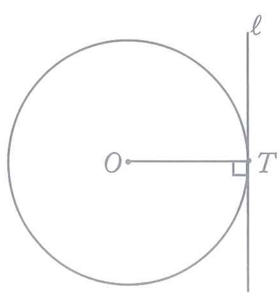
In the circle above, O is the center of the circle and $\overline{JK}$ and $\overline{PR}$ are chords. $\overline{PR}$ is a diameter and $\overline{OR}$ is a radius. If $OR = 7$, then the circumference of the circle is $2\pi(7) = 14\pi$ and the area of the circle is $\pi(7)^2 = 49\pi$.

The number of degrees of arc in a circle (or the number of degrees in a complete revolution) is 360.



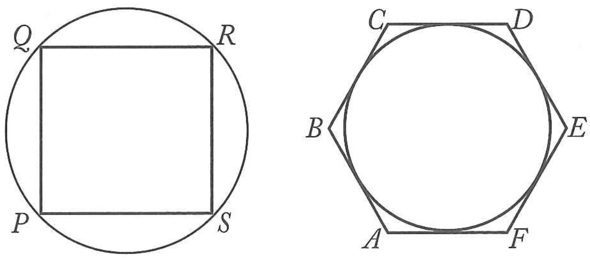
In the circle with center O above, the length of arc RST is $\frac{x}{360}$ of the circumference of the circle; for example, if $x = 60$, then arc RST has length $\frac{1}{6}$ of the circumference of the circle.

A line that has exactly one point in common with a circle is said to be *tangent* to the circle, and that common point is called the *point of tangency*. A radius or diameter with an endpoint at the point of tangency is perpendicular to the tangent line, and, conversely, a line that is perpendicular to a radius or diameter at one of its endpoints is tangent to the circle at that endpoint.



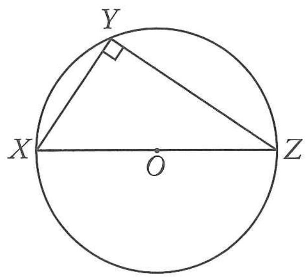
The line ℓ above is tangent to the circle and radius $\overline{OT}$ is perpendicular to ℓ.

If each vertex of a polygon lies on a circle, then the polygon is *inscribed* in the circle and the circle is *circumscribed* about the polygon. If each side of a polygon is tangent to a circle, then the polygon is *circumscribed* about the circle and the circle is *inscribed* in the polygon.



In the figure above, quadrilateral *PQRS* is inscribed in a circle and hexagon *ABCDEF* is circumscribed about a circle.

If a triangle is inscribed in a circle so that one of its sides is a diameter of the circle, then the triangle is a right triangle.



In the circle above, $\overline{XZ}$ is a diameter and the measure of $\angle XYZ$ is 90°.

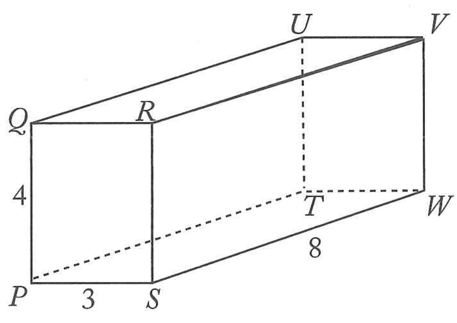
9. Rectangular Solids and Cylinders

A *rectangular solid* is a three-dimensional figure formed by 6 rectangular surfaces, as shown below. Each rectangular surface is a *face*. Each solid or dotted line segment is an *edge,* and each point at which the edges meet is a *vertex.* A rectangular solid has 6 faces, 12 edges, and 8 vertices. Opposite faces are parallel rectangles that have the same dimensions. A rectangular solid in which all edges are of equal length is a *cube.*

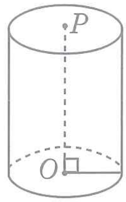
The *surface area* of a rectangular solid is equal to the sum of the areas of all the faces. The *volume* is equal to

$$(\text{length}) \times (\text{width}) \times (\text{height});$$

$$\text{in other words, } (\text{area of base}) \times (\text{height}).$$



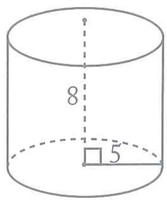
In the rectangular solid above, the dimensions are 3, 4, and 8. The surface area is equal to $2(3 \times 4) + 2(3 \times 8) + 2(4 \times 8) = 136$. The volume is equal to $3 \times 4 \times 8 = 96$.



The figure above is a right circular *cylinder.* The two bases are circles of the same size with centers O and P, respectively, and altitude (height) $\overline{OP}$ is perpendicular to the bases. The surface area of a right circular cylinder with a base of radius r and height h is equal to $2(\pi r^2) + 2\pi rh$ (the sum of the areas of the two bases plus the area of the curved surface).

The volume of a cylinder is equal to $\pi r^2 h$, that is,

$$(\text{area of base}) \times (\text{height}).$$



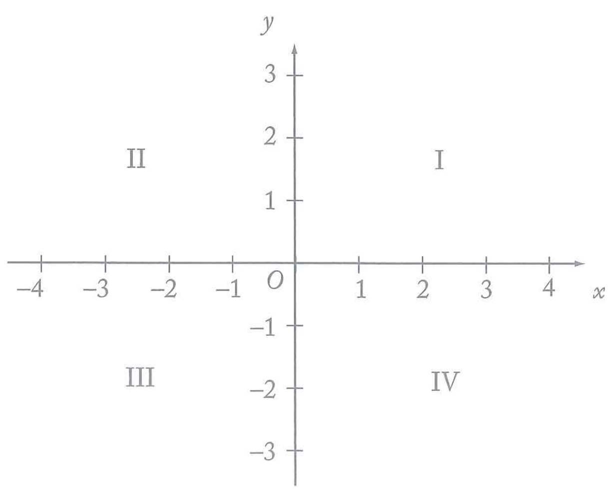
In the cylinder above, the surface area is equal to

$$2(25\pi) + 2\pi(5)(8) = 130\pi,$$

and the volume is equal to

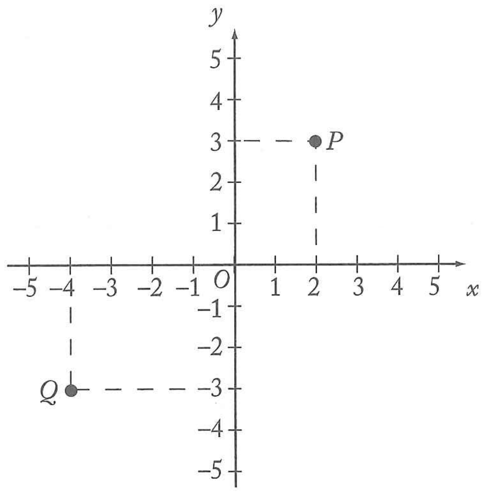
$$25\pi(8) = 200\pi.$$

10. Coordinate Geometry



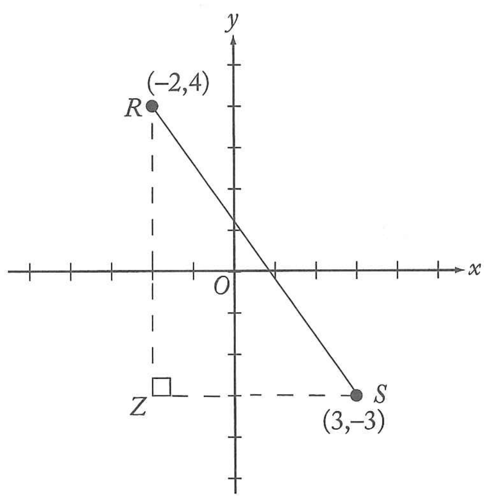
The figure above shows the (rectangular) *coordinate plane.* The horizontal line is called the *x-axis* and the perpendicular vertical line is called the *y-axis.* The point at which these two axes intersect, designated O, is called the *origin.* The axes divide the plane into four quadrants, I, II, III, and IV, as shown.

Each point in the plane has an *x-coordinate* and a *y-coordinate.* A point is identified by an ordered pair (x,y) of numbers in which the x-coordinate is the first number and the y-coordinate is the second number.



In the graph above, the (x,y) coordinates of point P are (2,3) since P is 2 units to the right of the y-axis (that is, $x = 2$) and 3 units above the x-axis (that is, $y = 3$). Similarly, the (x,y) coordinates of point Q are (−4,−3). The origin O has coordinates (0,0).

One way to find the distance between two points in the coordinate plane is to use the Pythagorean theorem.



To find the distance between points R and S using the Pythagorean theorem, draw the triangle as shown. Note that Z has (x,y) coordinates (−2,−3), $RZ = 7$, and $ZS = 5$. Therefore, the distance between R and S is equal to

$$\sqrt{7^2+5^2}=\sqrt{74}.$$

For a line in the coordinate plane, the coordinates of each point on the line satisfy a linear equation of the form $y = mx + b$ (or the form $x = a$ if the line is vertical). For example, each point on the line on the next page satisfies the equation $y=-\frac{1}{2}x+1$. One can verify this for the points (−2,2), (2,0), and (0,1) by substituting the respective coordinates for x and y in the equation.