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2.2 Let $Y = g(X)$, suppose $X \sim p(x)$, then

$$p(y) = \sum_{x: y=g(x)} p(x), \quad p(y)\log p(y) = \sum_{x: y=g(x)} p(x) \log p(y) \geq \sum_{x: y=g(x)} p(x) \log p(x)$$

$$H(X) = \sum_{x\in\mathcal{X}} p(x) \log \frac{1}{p(x)}$$ ~~since $\sum_{x\in\mathcal{X}}$~~

$$H(Y) = \sum_{y\in\mathcal{Y}} p(y) \log \frac{1}{p(y)} = -\sum_{y\in\mathcal{Y}} \sum_{x: y=g(x)} p(x) \log p(y) \leq -\sum_{y\in\mathcal{Y}} \sum_{x: y=g(x)} p(x) \log p(x)$$

$$= -\sum_{x\in\mathcal{X}} p(x) \log p(x) = H(X)$$

Hence, if $Y = g(X)$, the general inequality relationship of $H(X)$ and $H(Y)$ is that $H(X) \geq H(Y)$ with equality iff g is one-to-one

(a) $Y = 2^x$, $H(X) = H(Y)$ since $g = 2^x$ is an one-to-one function

(b) $Y = \cos X$, $H(X) \geq H(Y)$ since $g = \cos x$ is not necessarily one-to-one with equality iff $\cos x$ is in the ~~Monotonous~~ Monotonous range of X.

2.3 since $p_1 + p_2 + \cdots p_n = 1$ $H(p_1, \cdots p_n) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$

when $p_i = 1$ or 0 $H(\vec{p}) = 0$

Hence $H(p_1, \cdots p_n)_{min} = 0$, when $\vec{p} = (1, 0, \cdots 0), (0, 1, \cdots 0) \ldots$

$(0, 0, \cdots 1)$

2.5 proof. ~~$H(Y|X) = H(X,Y) - H(X) = 0$~~

~~i.e. $H(X,Y) = H(X)$~~

~~$\sum_x \sum_y p(x,y) \log \frac{1}{p(x,y)} = \sum_x p(x) \log \frac{1}{p(x)}$~~

Assume that there exists a x_0 s.t $p(x_0, y_1) > 0$, $p(x_0, y_2) > 0$ for $p(x_0) > 0$, then $p(x_0) \geq p(x_0, y_1) + p(x_0, y_2) > 0$ then $p(y_1|x_0) = \frac{p(x_0, y_1)}{p(x_0)} \neq 0$ and $p(y_1|x_0) \neq 1$. $p(y_2|x_0)$ is the same.

~~$H(Y|X) = \sum_{i=1}^{n} p(x_i) H(Y|X = x_i) = \sum_x \sum_y p(x,y) \log \frac{1}{p(x,y)}$~~

~~$\geq p(x_0, y_1) \log \frac{1}{p(x_0, y_1)} + p(x_0, y_2) \log \frac{1}{p(x_0, y_2)}$~~

$H(Y|X) = \sum_x \sum_y p(x,y) \log \frac{1}{p(y|x)}$ ~~$= \sum_x \sum_y p(x,y) \log \frac{p(x)}{p(x,y)}$~~

~~$= \sum_x \sum_y p(x,y) \log p(x) + \sum_x \sum_y p(x,y) \log \frac{1}{p(x,y)}$~~

$= \sum_x \sum_y p(x) p(y|x) \log \frac{1}{p(y|x)} = \sum_x p(x) \sum_y p(y|x) \log \frac{1}{p(y|x)}$

$\geq p(x_0) p(y_1|x_0) \log \frac{1}{p(y_1|x_0)} + p(x_0) p(y_2|x_0) \log \frac{1}{p(y_2|x_0)} > 0 \neq 0$

contradiction!

Hence, if $H(Y|X) = 0$, ~~th~~ then Y is a function of X.

2.6 (a) since if $X \to Y \to Z$, $I(X;Y) \geq I(X;Y|Z)$

when $I(X;Z)=0$ or X and Z are independent, $I(X;Y)=I(X;Y|Z)$

give an example:

Let $X=\begin{cases}0 & p=\frac{1}{2}\\ 1 & p=\frac{1}{2}\end{cases}$ and $Y=X$, $Z=Y$,

we have $I(X;Y)=H(X)-\underbrace{H(X|Y)}_{\to 0}=H(X)=\frac{1}{2}\log 2+\frac{1}{2}\log 2=1$

$I(X;Y|Z)=\underbrace{H(X|Z)}_{\to 0}-\underbrace{H(X|Y,Z)}_{\to 0}=0$

Hence $I(X;Y|Z)<I(X;Y)$

(b) Let $X=\begin{cases}0 & p=\frac{1}{2}\\ 1 & p=\frac{1}{2}\end{cases}$ and $Y=\begin{cases}0, & q=\frac{1}{2}\\ 1, & q=\frac{1}{2}\end{cases}$, $Z=X+Y$, and X and Y are independent

we have $I(X;Y)=0$

$I(X;Y|Z)=H(X|Z)-H(X|Y,Z)=H(X|Z)=P(Z=1)H(X|Z=1)$

$=\frac{1}{2}\times(\frac{1}{2}\times\log 2+\frac{1}{2}\times\log 2)=\frac{1}{2}$

Hence $I(X;Y|Z)>I(X;Y)$