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25.(a) $L=\sum_{i=1}^{m} p_i l_i = \sum_{i=1}^{m} p_i \lceil \log \frac{1}{p_i} \rceil$ since $\log\frac{1}{p_i} \le \lceil \log \frac{1}{p_i} \rceil < \log \frac{1}{p_i} + 1$

we have $H(X) = \sum_{i=1}^{m} p_i \log\frac{1}{p_i} \le \sum_{i=1}^{m} p_i \lceil \log \frac{1}{p_i} \rceil < \sum_{i=1}^{m} p_i \log \frac{1}{p_i} + p_i = \sum_{i=1}^{m} p_i \log\frac{1}{p_i} + \sum_{i=1}^{m} p_i$

Hence $H(X) \le L < H(X)+1$ $\quad = H(X)+1$

since $\log\frac{1}{p_i} \le l_i < \log\frac{1}{p_i}+1$ $\quad 2^{-l_i} \le p_i \le 2^{-l_i+1}$ $\quad \sum_{i=1}^{m} 2^{-l_i} \le \sum_{i=1}^{m} p_i = 1$ by Kraft inequality,

Consider F_i and F_j, where $j>i$, F_j is different from F_j at least from the first i bit. $l_j > l_i$, the code is ① prefix codes,

Hence F_i can't be the prefix of F_j. ②

Hence, by ①② the code constructed by this process is prefix-free.

(b) $\mathcal{X} = \{1, 2, 3, 4\}$

$F_1 = 0$, $F_2 = 0.5$ $F_3 = 0.75$ $F_4 = 0.875$

$=(0)_2$ $\quad =(0.1)_2$ $\quad =(0.11)_2$ $\quad =(0.111)_2$

$l_1 = \log 2 = 1$ $\quad l_2 = \log 4 = 2$ $\quad l_3 = l_4 = \log 8 = 3$

Hence, the code for the probability distribution (0.5, 0.25, 0.125, 0.125) is $C(1) = 0$ $\quad C(2) = 10$ $\quad C(3) = 110$ $\quad C(4) = 111$

verification:

$H(X) = 0.5 + 0.25\times 2 + 0.125 \times 3 \times 2 = \frac{7}{4}$ Here, $H(X) = L$.

$L = E(l(X)) = 0.5 + 0.25 \times 2 + 0.125 \times 3 \times 2 = \frac{7}{4}$

Let $f(D) = 2(D^{-1}+D^{-2}+D^{-3})$ which is monotonic decreasing

2. $\sum_{i=1}^{6} D^{-l_i} = 2(D^{-1}+D^{-2}+D^{-3}) \le 1 \rightarrow D = 3, 4, 5, \ldots$

Hence, a good lower bound on D is $D=3$

explain the title of the problem: since the cardinality of this code alphabet is 3. we guess a Martian has 3 fingers?