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3.4 3.8 3.13

3.4 (a) Yes. By AEP, $X^n \in A^n$ tends to 1 as $n \to \infty$.

(b) Yes. Since $Pr(X^n \in B^n) \to 1$ by Law of Big Numbers.

$\exists \varepsilon > 0, N_1 > 0, N_2 > 0$ s.t $Pr(X^n \in A^n) > 1-\varepsilon$, $Pr(X^n \in B^n) > 1-\varepsilon$
for $n > N_1$ for $n > N_2$

Hence, $Pr(X^n \in A^n \cap B^n)$

$= Pr(X^n \in A^n) + Pr(X^n \in B^n) - \underbrace{Pr(X^n \in A^n \cup B^n)}_{\leq 1}$

$> 1-\varepsilon + 1-\varepsilon - 1 = 1-2\varepsilon = 1-\varepsilon'$

Let $\varepsilon' = 2\varepsilon > 0$ so for $\varepsilon' > 0$ $\exists$ $n > \max(N_1, N_2)$ s.t

$Pr(X^n \in A^n \cap B^n) > 1-\varepsilon'$, $Pr(X^n \in A^n \cap B^n) \to 1$ in probability.

(c) proof: $1 = \sum_{x^n \in \mathcal{X}^n} p(x^n) \geq \sum_{x^n \in A^n \cap B^n} p(x^n)$

Since ~~$|A^n| \leq 2^{n(H+\varepsilon)}$~~ $X^n \in A^n$, we have $2^{-n(H+\varepsilon)} \leq P(X^n) \leq 2^{-n(H-\varepsilon)}$

So $\sum_{x^n \in A^n \cap B^n} P(X^n) \geq \sum_{X^n \in A^n \cap B^n} 2^{-n(H+\varepsilon)} \geq 2^{-n(H+\varepsilon)} |A^n \cap B^n|$

Hence $|A^n \cap B^n| \leq 2^{n(H+\varepsilon)}$ for all n

(d) Based on (b) since when n is large sufficiently $Pr(X^n \in A^n \cap B^n) > 1-\varepsilon'$, we have $1-\varepsilon' < Pr(X^n \in A^n \cap B^n)$

since $X^n \in A^n$, $P(X^n) \leq 2^{-n(H-\varepsilon)}$ $\leq \sum_{X^n \in A^n \cap B^n} P(X^n)$

Hence, $1-\varepsilon' \leq \sum_{X^n \in A^n \cap B^n} P(X^n) \leq |A^n \cap B^n| \, 2^{-n(H-\varepsilon)}$ $\quad |A^n \cap B^n| \geq (1-\varepsilon') 2^{n(H-\varepsilon)}$

Let $\varepsilon' = \frac{1}{2}$, we have $\geq (\frac{1}{2}) 2^{n(H-\varepsilon)}$

3.8 $\lim_{n\to\infty}(X_1 X_2 \cdots X_n)^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = EX = \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{4}\times 3$

$= \frac{7}{4}$

method 1:

$\lim_{n\to\infty}(X_1 X_2 \cdots X_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1^{\frac{n}{2}} \cdot 2^{\frac{n}{4}} 3^{\frac{n}{4}}\right)^{\frac{1}{n}} = 1^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 3^{\frac{1}{4}} = 6^{\frac{1}{4}}$

when n is large sufficiently almost all events are almost equally.

method 2:

$\lim_{n\to\infty}(X_1 X_2 \cdots X_n)^{\frac{1}{n}} = \lim_{n\to\infty} 2^{\log (X_1 X_2 \cdots X_n)^{\frac{1}{n}}} = \lim_{n\to\infty} 2^{\frac{1}{n}\sum_{i=1}^{n}\log X_i}$

$= 2^{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log X_i} = 2^{E\log X_i} = 2^{\frac{1}{2}\log 1 + \frac{1}{4}\log 2 + \frac{1}{4}\log 3}$

$= 2^{\frac{1}{4}\log 6} = 6^{\frac{1}{4}}$

3.13 (a) $H(X) = 0.6\log\frac{5}{3} + 0.4\log\frac{5}{2} = 0.6\log 5 - 0.6\log 3 + 0.4\log 5 - 0.4\log 2$

$= \log 5 - 0.6\log 3 - 0.4 \approx 0.970951$

(b) 对 $X^n \in A_\varepsilon^{(n)}$, then $H(X) - \varepsilon \leq -\frac{1}{n}\log p(X^n) \leq H(X) + \varepsilon$

Let $H(X) = 0.970951$, $\varepsilon = 0.1$, $n = 25$, we have $0.870951 \leq -\frac{1}{n}\log p(X^n) \leq 1.070951$

check the table, ~~when~~ $11 \leq k \leq 19$ (k is the number of $X_i = 1$)

the sequence that fall in the typical set $A_\varepsilon^{(n)}$

$Pr(A_\varepsilon^{(n)}) = 0.970638 - 0.034391 = 0.936247$, there are totally $\sum_{k=11}^{19}\binom{n}{k} = 26366510$ elements in the typical set

(c)

(c) $\frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k}} = p^k (1-p)^{n-k} = 0.6^k 0.4^{25-k} = 0.4^{25} \cdot \left(\frac{3}{2}\right)^k$

since $\frac{3}{2} > 1$ $\quad 0.4^{25} \left(\frac{3}{2}\right)^k$ is increasing with k.

hence, we check the table and find that $\sum_{k=13}^{25} \binom{n}{k} P(X^n) = 0.846232$

$\sum_{k=12}^{25} \binom{n}{k} P(X^n) = 0.922199$ so $\frac{0.90 - 0.846232}{0.075967} \times 5200300 + \sum_{k=13}^{25} \binom{n}{k}$

$= 20457900$ sequences in the smallest set that has probability 0.9

(d) Based on (b) and (c)

$\frac{0.9 - 0.846232}{0.075967} \times 520030 + \sum_{k=13}^{19} \binom{n}{k} = 20389484$ sequences in the intersection

of this intersection

$P = 0.9 - 0.846232 + \sum_{k=13}^{19} \binom{n}{k} P(X^n) = 0.053768 + 0.816870 = 0.870638$