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5. (a) Based on Definition 2,

$$H'(\mathcal{X}) = \lim_{n\to\infty} H(X_n | X_{n-1}, X_{n-2} \cdots X_1) \overset{\text{Markov chain}}{=\!=\!=} \lim_{n\to\infty} H(X_n | X_{n-1}) \overset{\text{As the assumption that Markov chain is time invariant}}{=\!=} \lim_{n\to\infty} H(X_2|X_1)$$

the two-state Markov chain with transition matrix:

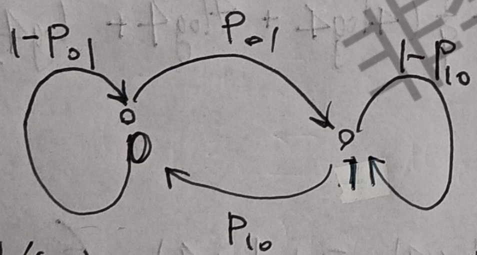
$$= \underset{\text{since independent with 'n'}}{H(X_2|X_1)} = \frac{P_{10}}{P_{01}+P_{10}} H(P_{01}) + \frac{P_{01}}{P_{01}+P_{10}} H(P_{10}) = \underline{\frac{P_{10}H(P_{01}) + P_{01}H(P_{10})}{P_{01}+P_{10}}}$$

(b) $$H'(\mathcal{X}) = \frac{P_{10}\left(P_{01}\log\frac{1}{P_{01}} + (1-P_{01})\log\frac{1}{(1-P_{01})}\right) + P_{01}\left(P_{10}\log\frac{1}{P_{10}} + (1-P_{10})\log\frac{1}{(1-P_{10})}\right)}{P_{01}+P_{10}}$$

where $P_{10} \in [0,1]$, $P_{01} \in [0,1]$. Use differential-evolutionary algorithm (DE) to find the maximun value of $H'(\mathcal{X})$, we have $P_{01} = P_{10} = 0.5$ to

$$P = \begin{bmatrix} 1-P_{01} & P_{01} \\ P_{10} & 1-P_{10} \end{bmatrix}$$

achieve the maximum of $H'(\mathcal{X}) = 1$.



(c) $$H'(\mathcal{X}) = \frac{1}{1+p} H(p) + \frac{p}{1+p} \underset{0}{\underline{H(1)}} = \frac{1}{1+p} H(p) = \frac{1}{1+p}\left(p\log\frac{1}{p} + (1-p)\frac{1}{\log(1-p)}\right)$$

(d) to maxmize $\frac{1}{1+p} H(p) = \frac{-p}{1+p}\log p + \frac{p-1}{1+p}\log(1-p)$

when $p = 0.382$, the maximum value of $H'(\mathcal{X}) = 0.694$

(e) $N(t) = N(t-1) + N(t-2)$ where $N(1) = 2$ $N(2) = 3$ ~~N(3)~~

$$N(t) = \left(\frac{3\sqrt{5}+5}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^t + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^t$$

$$H_0 = \lim_{t\to\infty} \frac{1}{t}\log N(t) = \lim_{t\to\infty}\frac{1}{t}\log \frac{(3\sqrt5+5)(1+\sqrt5)^t + (5-3\sqrt5)(1-\sqrt5)^t}{10\cdot 2^t}$$

$$= \lim_{t\to\infty} \frac{1}{t} \log\left[\left(\frac{3\sqrt5+5}{10}\right)\left(\frac{1+\sqrt5}{2}\right)^t + \left(\frac{5-3\sqrt5}{10}\right)\left(\frac{1-\sqrt5}{2}\right)^t\right] \quad \text{since } \left|\frac{1-\sqrt5}{2}\right|<1$$

$$= \lim_{t\to\infty} \log\left[\left(\frac{3\sqrt5+5}{10}\right)^{\frac{1}{t}}\frac{1+\sqrt5}{2}\right] = \log\frac{1+\sqrt5}{2} = 0.694 \qquad \left|\frac{1+\sqrt5}{2}\right|>1$$

H_0 is an upper bound on the ~~entroy~~ entropy rate of the Markov chain

because $N(t)=|\mathcal{X}^t|$, $H(X_1,X_2\cdots X_t)\leq \log|\mathcal{X}^t| = \log N(t)$

hecence $H'(\mathcal{X}) = \lim_{t\to\infty} \frac{H(X_1,X_2,\cdots X_t)}{t} \leq \lim_{t\to\infty} \frac{\log N(t)}{t} = \log\frac{1+\sqrt5}{2} = 0.694$

10. (a) $H(X_0,X_1,X_2\cdots X_n) = H(X_n|X_0,X_1,X_2\cdots X_{n-1}) + H(X_0,X_1,X_2\cdots X_{n-1})$

$= H(X_n|X_{n-1},X_{n-2}) + H(X_{n-1}|X_{n-2},X_{n-3}) + \cdots + H(X_2|X_1,X_0) + H(X_1|X_0) + H(X_0)$

$= (n-1)H(X_2|X_1,X_0)+1 = (n-1)\left[0.1\log\frac{1}{0.1} + 0.9\log\frac{1}{0.9}\right]+1 = 0.469(n-1)+1$

(b) $H(\mathcal{X}) = \lim_{n\to\infty} \frac{H(X_0,X_1,X_2\cdots X_n)}{n+1} = \lim_{n\to\infty} \frac{0.469(n-1)+1}{n+1} = 0.469$

(c) expected number of steps $=\lim_{n\to\infty}\sum_{i=1}^{n} i(0.9)^{i-1}(0.1)$ (*)

$0.9(*) = \frac{1}{9}\sum 0.9^2 + 2\times0.9^3+\cdots$

$(*) = \frac{1}{9}\sum 0.9+2\times0.9^2+3\times0.9^3+\cdots$

$(*) = \frac{(*)-0.9(*)}{0.1} = \frac{\frac{1}{9}\sum 0.9+0.9^2+0.9^3}{0.1} = \frac{10}{9}\frac{0.9}{1-0.9} = 10$

11.

Let $X_0 X_1 \cdots X_n$ be the state sequence of the random walk of a king, then

$$H(X_0 X_1, \cdots X_n) = H(X_n | X_0, X_1, \cdots X_{n-1}) + H(X_0, X_1 \cdots X_{n-1})$$

$$= H(X_n | X_{n-1}) + \cdots H(X_1 | X_0) + H(X_0)$$

Assume $X_0 \sim$ stationary distribution, then $H(X_0) = H\left(\frac{3}{40}, \frac{5}{40}, \frac{3}{40}, \frac{5}{40}, \frac{8}{40}, \frac{5}{40}, \frac{3}{40}, \frac{5}{40}, \frac{3}{40}\right)$

$$H(X_0 X_1 X_2 \cdots X_n) = n H(X_1 | X_0) + H(X_0).$$

$$H(X_1 | X_0) = \sum_{i=1}^{9} H(X_1 | X_0 = i) P(X_0 = i) = \frac{3\times 4}{40}\log 3 + \frac{5\times 4}{40}\log 5 + \frac{8}{40}\log 8 = 2.236$$

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{n(H_1(X_0) + H(X_0)}{n+1} = 2.236.$$

rooks: $H(X_0) = H\left(\frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}, \frac{4}{36}\right)$ $H(X_1|X_0) = \frac{1}{9}\times 9\log 4 = 2$

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{n(H_1|X_0) + H(X_0)}{n+1} = 2$$

bishops: ① type 1 $H(X_0) = H\left(\frac{2}{12}, 0, \frac{2}{12}, 0, \frac{4}{12}, 0, \frac{2}{12}, 0, \frac{2}{12}\right)$

$H(X_1|X_0) = \frac{1}{6}\times 4\log 2 + \frac{1}{3}\log 4 = \frac{4}{3} = 1.333$ $H(\mathcal{X}) = 1.333$

② type2: $H(X_0) = H\left(0, \frac{2}{8}, 0, \frac{2}{8}, 0, \frac{2}{8}, 0, \frac{2}{8}, 0\right)$

$H(X_1|X_0) = \frac{1}{4}\times 4\log 2 = 1$ $H(\mathcal{X}) = 1$

queens: $H(X_0) = \left(\frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{8}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}, \frac{6}{56}\right)$

$H(X_1|X_0) = \frac{3}{28} 8\log 6 + \frac{1}{7}\log 8 = 2.644$ $H(\mathcal{X}) = 2.644$