

INFORMATION THEORY & CODING

Channel Code - 2

Dr. Rui Wang

Department of Electrical and Electronic Engineering
Southern Univ. of Science and Technology (SUSTech)

Email: wang.r@sustech.edu.cn

November 21, 2023



- **Channel capacity.** The logarithm of the number of distinguishable inputs is given by

$$C = \max_{p(x)} I(X; Y).$$

- **Examples**

- Binary symmetric channel: $C = 1 - H(p)$
- Binary erasure channel: $C = 1 - \alpha$
- Symmetric channel: $C = \log |\mathcal{Y}| - H$ (row of trans. matrix)

Definition

An (M, n) code for the channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ consists of :

1. An **index set** $\{1, 2, \dots, M\}$ representing messages.
2. An **encoding function** $X^n : \{1, 2, \dots, M\} \rightarrow \mathcal{X}^n$, yielding codewords $x^n(1), x^n(2), \dots, x^n(M)$. The set of codewords is called **codebook**.
3. A **decoding function** $g : \mathcal{Y}^n \rightarrow \{1, 2, \dots, M\}$.

The **rate** R of an (M, n) code is

$$R = \frac{\log M}{n} \text{ bit per transmission}$$

On the other hand, we usually write

$$M = \lceil 2^{nR} \rceil$$

Definition

An (M, n) code for the channel $(\mathcal{X}, p(y|x), \mathcal{Y})$ consists of :

1. An **index set** $\{1, 2, \dots, M\}$ representing messages.
2. An **encoding function** $X^n : \{1, 2, \dots, M\} \rightarrow \mathcal{X}^n$, yielding codewords $x^n(1), x^n(2), \dots, x^n(M)$. The set of codewords is called **codebook**.
3. A **decoding function** $g : \mathcal{Y}^n \rightarrow \{1, 2, \dots, M\}$.

The **rate** R of an (M, n) code is

$$R = \frac{\log M}{n} \text{ bit per transmission}$$

On the other hand, we usually write

$$M = \lceil 2^{nR} \rceil$$

- Conditional probability of error:

$$\lambda_i = \Pr[g(Y_n) \neq i | X^n = x^n(i)] = \sum_{y^n} p(y^n | x^n(i)) I(g(y^n) \neq i)$$

- Maximal probability of error: $\lambda^{(n)} = \max_{i \in \{1, 2, \dots, M\}} \lambda_i$
- Decoding error probability: $\Pr[W \neq g(Y^n)] = \sum_i \lambda_i \Pr[W = i]$
- Arithmetic average probability of error:

$$P_e^{(n)} = \frac{1}{M} \sum_{i=1}^M \lambda_i, \quad P_e^{(n)} \leq \lambda^{(n)}$$

If W is uniformly distributed:

$$P_e^{(n)} = \Pr[W \neq g(Y^n)] \quad \text{Decoding error probability}$$

Achievable Rate

- A rate R is **achievable**,

if there exists a sequence of codes with rate R and codeword length n , denoted as $(\lceil 2^{nR} \rceil, n)$, such that the maximal probability of error $\lambda^{(n)} \rightarrow 0$ as $n \rightarrow \infty$.

Recall that

The **rate** R of an (M, n) code is

$$R = \frac{\log M}{n} \text{ bit per transmission.}$$

- Joint typicality. Given two i.i.d. random variable sequences X^n and Y^n , the set of jointly typical sequences is

$$A_{\epsilon}^{(n)} = \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \right. \\ \left| -\frac{1}{n} \log p(x^n) - H(X) \right| < \epsilon \\ \left| -\frac{1}{n} \log p(y^n) - H(Y) \right| < \epsilon \\ \left. \left| -\frac{1}{n} \log p(x^n, y^n) - H(X, Y) \right| < \epsilon \right\}$$

where $p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)$.

- **Joint AEP** Let (X^n, Y^n) be the sequences of length n drawn i.i.d. according to $p(x^n, y^n) = \prod_{i=1}^n p(x_i, y_i)$, then:
 1. $\Pr \left[(X^n, Y^n) \in A_\epsilon^{(n)} \right] \rightarrow 1$ as $n \rightarrow \infty$.
 2. $|A_\epsilon^{(n)}| \leq 2^{n(H(X,Y)+\epsilon)}$.
 3. If $(\tilde{X}^n, \tilde{Y}^n) \sim p(x^n)p(y^n)$, then

$$\Pr \left[(\tilde{X}^n, \tilde{Y}^n) \in A_\epsilon^{(n)} \right] \leq 2^{-n(I(X;Y)-3\epsilon)}.$$

Please refer to p196 for the proof (proof of Theorem 7.6.1)

Channel Coding Theorem

Theorem (Channel coding theorem)

For a discrete memoryless channel, *all rates below capacity C are achievable*. Specifically, for every rate $R < C$, there exists a sequence of $(2^{nR}, n)$ codes with maximum probability of error $\lambda^{(n)} \rightarrow 0$.

Conversely, any sequence of $(2^{nR}, n)$ codes with $\lambda^{(n)} \rightarrow 0$ must have $R \leq C$.

Achievability: when $R < C$, there exists zero-error code.

Converse: zero-error codes must have $R \leq C$.

Random Codebook

- Generate a $(2^{nR}, n)$ code at random according to $p(x)$, where $p(x)$ is the **capacity achieving distribution**. The 2^{nR} are the rows of a matrix:

$$\mathcal{C} = \begin{bmatrix} x_1(1) & x_2(1) & \dots & x_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & \dots & x_n(2^{nR}) \end{bmatrix}.$$

Each entry is generated **i.i.d.** according to $p(x)$.

- Encoding**: map the message $w = \{1, 2, 3, \dots, 2^{nR}\}$ to codeword $[x_1(w), x_2(w), \dots, x_n(w)]$, i.e.

$$\mathcal{C} \rightarrow [x_1(w), x_2(w), \dots, x_n(w)] = x_{\mathcal{C}}^n(w), w = 1, 2, \dots, 2^{nR}$$

- We shall prove the average detection error probability (over all codebooks) tends to zero as n increase, which implies that there must exists one good codebook whose detection error probability tends to zero



Jointly Typical Decoding

- **Decoding**: finds the only \hat{w} such that $(x_{\mathcal{C}}^n(\hat{w}), Y_{\mathcal{C}}^n)$ is jointly typical.
- **Decoding error**: Suppose message 1 is sent to via codeword $x_{\mathcal{C}}^n(1)$ and $Y_{\mathcal{C}}^n$ is the received signal, the possible decoding error events include:
 - $(x_{\mathcal{C}}^n(1), Y_{\mathcal{C}}^n)$ is not joint typical.
 - $(x_{\mathcal{C}}^n(i), Y_{\mathcal{C}}^n)$ is joint typical ($i = 2, 3, \dots, 2^{nR}$).
- **Idea of proof**: According to **joint AEP**, since $x_{\mathcal{C}}^n(1)$ and $Y_{\mathcal{C}}^n$ are generated according to joint distribution $p(x^n, y^n)$, the chance of the first event is small. Moreover, since $Y_{\mathcal{C}}^n$ is generated independently of $x_{\mathcal{C}}^n(i)$, the total chance of the second event is also small.

Proof for achievability

- A message W is chosen according to a uniform distribution

$$\Pr[W = w] = 2^{-nR},$$

for $w = 1, 2, \dots, 2^{nR}$. The w -th codeword $x_{\mathcal{C}}^n(w)$, corresponding to the w -th row of \mathcal{C} , is sent over the channel.

- The receiver receives a sequence $Y_{\mathcal{C}}^n$ according to the distribution according to the distribution

$$\Pr\left(y_{\mathcal{C}}^n | x_{\mathcal{C}}^n(w)\right) = \prod_{i=1}^n \Pr\left(y_{i,\mathcal{C}} | x_{i,\mathcal{C}}(w)\right),$$

and guesses which message was sent using **jointly typical decoding**.

Proof for achievability

- Let $\varepsilon = \{\hat{W}(Y^n) \neq W\}$ denote the error event, $\lambda_w(\mathcal{C})$ be the error probability of the w -th codeword of code \mathcal{C} . The **average probability of error**, over all codewords and all codebooks, is:

$$\begin{aligned}\Pr(\varepsilon) &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) P_e^{(n)}(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \lambda_w(\mathcal{C}) \\ &= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_1(\mathcal{C}),\end{aligned}$$

where $\sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_1(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_w(\mathcal{C}), \forall w \neq 1$.

Proof for achievability

- Let $Y_{\mathcal{C}}^n$ be the received signal for $x_{\mathcal{C}}^n(1)$

$$e_i(\mathcal{C}) = \{(x_{\mathcal{C}}^n(i), Y_{\mathcal{C}}^n) \in A_{\epsilon}^{(n)}\}, i \in \{1, 2, \dots, 2^{nR}\},$$

and $e_i^c(\mathcal{C}) = \neg e_i(\mathcal{C})$. Thus,

$$\begin{aligned}\Pr[\varepsilon] &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \lambda_1(\mathcal{C}) = \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr \left[e_1^c(\mathcal{C}) \cup \left(\bigcup_{i=2}^{2^{nR}} e_i(\mathcal{C}) \right) \middle| W = 1 \right] \\ &\leq \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1^c(\mathcal{C}) | W = 1] + \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{i=2}^{2^{nR}} \Pr[e_i(\mathcal{C}) | W = 1] \\ &= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1^c(\mathcal{C}) | W = 1] + \sum_{i=2}^{2^{nR}} \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_i(\mathcal{C}) | W = 1]\end{aligned}$$

Proof for achievability

$$\begin{aligned}& \sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1^c(\mathcal{C})|W = 1] \\&= \sum_{\mathcal{C}} \left(\prod_{i=1}^{2^{nR}} \Pr(x_{\mathcal{C}}^n(i)) \right) \Pr[e_1^c(\mathcal{C})|W = 1] \\&= \sum_{x_1^n} \sum_{\mathcal{C}: x_{\mathcal{C}}^n(1)=x_1^n} \prod_{i=1}^{2^{nR}} \Pr(x_{\mathcal{C}}^n(i)) \Pr(x_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1) \\&= \sum_{x_1^n} \Pr(x_1^n) \Pr(x_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1) \\&\quad \times \sum_{\mathcal{C}: x_{\mathcal{C}}^n(1)=x_1^n} \prod_{i=2}^{2^{nR}} \Pr(x_{\mathcal{C}}^n(i)) \\&= \sum_{x_1^n} \Pr(x_1^n) \Pr(x_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1) \\&= \Pr(X_1^n \text{ and } Y^n \text{ are not joint typical}|W = 1) = \Pr(E_1^c|W = 1)\end{aligned}$$



Proof for achievability

- Similarly,

$$\begin{aligned}\sum_{\mathcal{C}} \Pr(\mathcal{C}) \Pr[e_1(\mathcal{C})|W=1] &= \Pr(X_i^n \text{ and } Y^n \text{ are joint typical} | W=1) \\ &= \Pr(E_i | W=1)\end{aligned}$$

- As a result,

$$\Pr[\varepsilon] \leq \Pr[E_1^c | W=1] + \sum_{i=2}^{2^{nR}} \Pr[E_i | W=1]$$

Proof for achievability

- By the joint AEP, $\Pr[E_1^c | W = 1] \leq \epsilon$ for n sufficiently large. By the code generation process, $X^n(1)$ and $X^n(i)$ are independent for $i \neq 1$, so are Y^n and $X^n(i)$. Hence the probability that $X^n(i)$ and Y^n are jointly typical is $\leq 2^{-n(I(X;Y)-3\epsilon)}$ by the joint AEP.

$$\begin{aligned}\Pr[\varepsilon] &\leq \epsilon + \sum_{i=2}^{2^{nR}} 2^{-n(I(X;Y)-3\epsilon)} \\ &= \epsilon + (2^{nR} - 1)2^{-n(I(X;Y)-3\epsilon)} \\ &\leq \epsilon + 2^{3n\epsilon} 2^{-n(I(X;Y)-R)} \\ &\leq 2\epsilon \quad \text{for } R \leq I(X;Y) - 4\epsilon \text{ and sufficiently large } n\end{aligned}$$

Hence, if $R < I(X;Y)$, we can choose ϵ and n so that the average probability of error, over codebooks and codewords, is less than 2ϵ .

- Since $p(x)$ is the capacity achieving distribution, $R < I(X;Y)$ becomes $R < C$.

Proof for achievability

- **Get rid of the average over codebooks.** Since the average probability of error is $\leq 2\epsilon$, there exists **at least one** codebook \mathcal{C}^* with a small average probability of error ($\Pr(\varepsilon|\mathcal{C}^*) \leq 2\epsilon$). Since we have chosen \hat{W} according to a uniform distribution, we have

$$\Pr(\varepsilon|\mathcal{C}^*) = \frac{1}{2^{nR}} \sum_{i=1}^{2^{nR}} \lambda_i(\mathcal{C}^*).$$

- **Throw away the worst half of the codewords in the best codebook \mathcal{C}^* .** We have $\Pr(\varepsilon|\mathcal{C}^*) \leq \frac{1}{2^{nR}} \sum \lambda_i(\mathcal{C}^*) \leq 2\epsilon$. This implies that **at least half** the indices i and their associated codewords $X^n(I)$ must have conditional probability of error $\lambda_i \leq 4\epsilon$. If we reindex the codewords, we have 2^{nR-1} codewords. The rate now is $R' = R - \frac{1}{n}$ with maximal probability of error $\lambda^{(n)} \leq 4\epsilon$.

Proof for the converse

- The index W is uniformly distributed on the set $\mathcal{W} = \{1, 2, \dots, 2^{nR}\}$, and the sequence Y^n is related to W . From Y^n , we estimate the index W as $\hat{W} = g(Y^n)$. Thus, $W \rightarrow X^n(W) \rightarrow Y^n \rightarrow \hat{W}$ forms a Markov chain.

Data processing inequality: $I(W; \hat{W}) \leq I(X^n(W); Y^n)$

Lemma (Fano's inequality)

For a discrete memoryless channel with a codebook \mathcal{C} and the input message W uniformly distributed over 2^{nR} , we have

$$H(W|\hat{W}) \leq 1 + P_e^{(n)} nR.$$

Proof for the converse

- The index W is uniformly distributed on the set $\mathcal{W} = \{1, 2, \dots, 2^{nR}\}$, and the sequence Y^n is related to W . From Y^n , we estimate the index W as $\hat{W} = g(Y^n)$. Thus, $W \rightarrow X^n(W) \rightarrow Y^n \rightarrow \hat{W}$ forms a Markov chain.

Data processing inequality: $I(W; \hat{W}) \leq I(X^n(W); Y^n)$

Lemma (Fano's inequality)

For a discrete memoryless channel with a codebook \mathcal{C} and the input message W uniformly distributed over 2^{nR} , we have

$$H(W|\hat{W}) \leq 1 + P_e^{(n)} nR.$$

Lemma

Let Y^n be the result of passing X^n through a discrete memoryless channel of capacity C . Then

$$I(X^n; Y^n) \leq nC, \quad \text{for all } p(x^n).$$

Proof.

$$\begin{aligned} I(X^n; Y^n) &= H(Y^n) - H(Y^n|X^n) = H(Y^n) - \sum_{i=1}^n H(Y_i|Y_1, \dots, Y_{i-1}, X^n) \\ &= H(Y^n) - \sum_{i=1}^n H(Y_i|X_i) \quad \text{memoryless} \\ &\leq \sum_{i=1}^n H(Y_i) - \sum_{i=1}^n H(Y_i|X_i) \quad \text{independence bound} \\ &= \sum_{i=1}^n I(X_i|Y_i) \leq nC \end{aligned}$$

Lemma

Let Y^n be the result of passing X^n through a discrete memoryless channel of capacity C . Then

$$I(X^n; Y^n) \leq nC, \quad \text{for all } p(x^n).$$

Proof.

$$\begin{aligned} I(X^n; Y^n) &= H(Y^n) - H(Y^n|X^n) = H(Y^n) - \sum_{i=1}^n H(Y_i|Y_1, \dots, Y_{i-1}, X^n) \\ &= H(Y^n) - \sum_{i=1}^n H(Y_i|X_i) \quad \text{memoryless} \\ &\leq \sum_{i=1}^n H(Y_i) - \sum_{i=1}^n H(Y_i|X_i) \quad \text{independence bound} \\ &= \sum_{i=1}^n I(X_i|Y_i) \leq nC \end{aligned}$$

Proof for the converse

Proof.

Converse to channel coding theorem: Since W has a uniform distribution, we have

$$\begin{aligned} nR &= H(W) = H(W|\hat{W}) + I(W; \hat{W}) \\ &\leq 1 + P_e^{(n)} nR + I(W; \hat{W}) \quad \text{Fano's inequality} \\ &\leq 1 + P_e^{(n)} nR + I(X^n; Y^n) \quad \text{data-processing inequality} \\ &\leq 1 + P_e^{(n)} nR + nC \quad \text{Lemma 7.9.2} \end{aligned}$$

We obtain $R \leq \frac{1}{n(1+P_e^{(n)})} + \frac{C}{1+P_e^{(n)}} \rightarrow \frac{1}{n} + C$.

Letting $n \rightarrow \infty$, we have $R \leq C$.



Reading & Homework

- **Reading:** Chapter 7: 7.6-7.10
- **Homework:** Problems 7.15, 7.31.