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3. $X \longrightarrow \overset{Z}{\oplus} \longrightarrow Y$ $\quad C = \max\limits_{p(x)} I(X;Y) = \max\limits_{p(x)} [H(Y) - H(Y|X)]$

Case I: if $a=0$, then

X: $0 \xrightarrow{1} 0$, $1 \xrightarrow{1} 1$: Y

$C = \max\limits_{p(x)} H(Y) = 1$ bit/transmission

when $p(x) = (\frac{1}{2}, \frac{1}{2})$

Case II: if $a=1$, then

X: $0 \xrightarrow{\frac{1}{2}} 0$, $0 \xrightarrow{\frac{1}{2}} 1$, $1 \xrightarrow{\frac{1}{2}} 1$, $1 \xrightarrow{\frac{1}{2}} 2$: Y

$C = \max\limits_{p(x)} H(Y) - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

Assume $P(X=0) = p$, $P(X=1) = 1-p$.

we have $P(Y=0) = \frac{1}{2}p$, $P(Y=1) = \frac{1}{2}$, $P(Y=2) = \frac{1}{2}(1-p)$

$H(Y) = \frac{1}{2}p \log \frac{2}{p} + \frac{1}{2}\log 2 + \frac{1}{2}(1-p)\log \frac{2}{1-p}$

$\leq \frac{1}{4}\log 4 + \frac{1}{2}\log 2 + \frac{1}{4}\log 4 = \frac{3}{2}$, when $p(x) = (\frac{1}{2}, \frac{1}{2})$

Case III: if $a=-1$, then

X: $0 \xrightarrow{\frac{1}{2}} -1$, $0 \xrightarrow{\frac{1}{2}} 0$, $1 \xrightarrow{\frac{1}{2}} 0$, $1 \xrightarrow{\frac{1}{2}} 1$: Y

the same as case II, $C = \frac{1}{2}$

Case IV: if $a \neq 0, \pm 1$, then

X: $0 \xrightarrow{\frac{1}{2}} 0$, $0 \xrightarrow{\frac{1}{2}} a$, $1 \xrightarrow{\frac{1}{2}} 1$, $1 \xrightarrow{\frac{1}{2}} 1+a$: Y

$C = \max\limits_{p(x)} H(Y) - 1$

$= \max\limits_{p(x)} [H(X) - H(X|Y)]$

$= \max\limits_{p(x)} H(X) = 1$

when $p(x) = (\frac{1}{2}, \frac{1}{2})$

Hence, when $a = \pm 1$, $C = \frac{1}{2}$ bit/transmission

otherwise $C = 1$. bit/transmission

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5. (a) $C = \max_{p(x)} I(X;Y) = \max_{p(x)} \left[H(Y) - H(Y|X)\right] = \max_{p(x)} H(Y) - \log_2 3$

$= \log_2 11 - \log_2 3 = \log_2 \frac{11}{3}$ bits/transmission

(b) $C = \log_2 \frac{11}{3}$ when $p^*(x) = \left(\frac{1}{11}, \frac{1}{11}, \cdots \frac{1}{11}\right)$ which is a uniform distribution.

6. $C = \max_{p(x_1,x_2)} I(X_1,X_2;Y_1,Y_2) = \max_{p(x_1,x_2)} \left[H(Y_1,Y_2) - H(Y_1,Y_2|X_1,X_2)\right]$

$C_1 = \max_{p(x_1)} I(X_1;Y_1)$ $\quad C_2 = \max_{p(x_2)} I(X_2;Y_2)$

since $p(x_1,x_2,y_1,y_2) = p(x_1,x_2)\,p(y_1,y_2|x_1,x_2)$ since we form a new channel.

$= p(x_1,x_2)\,p(y_1|x_1)\,p(y_2|x_2)$

we can rewrite the above formula: $\cancel{p(y_1,y_2|x_1,x_2)}$ $p(x_1,x_2,y_1,y_2)$

$= \cancel{p(y_2|x_1,x_2)\,p(y_1|y_2,x_1,x_2)}$

$= \cancel{p(y_1|x_1,x_2)\,p(y_2|y_1,x_1,x_2)}$

$= \cancel{p(y_1|x_1)\,p(y_2|x_2)}$

$= p(x_1)\,p(x_2|x_1)\,p(y_1|x_1)\,p(y_2|x_2)$

$= p(x_1)\,p(y_1|x_1)\,p(x_2|x_1)\,p(y_2|x_2)$

$= p(y_1|x_1)\,p(x_1)\,p(x_2|x_1)\,p(y_2|x_2)$

$=$ ~~so, we have the Markov chain:~~

$p(y_1|x_1)p(x_1) = p(x_1,y_1) = p(x_1|y_1)\,p(y_1)$

$= p(y_1)\,p(x_1|y_1)\,p(x_2|x_1)\,p(y_2|x_2)$

$\cancel{X_2 \to X_1}$ $\quad \cancel{X_1 \to X_2 \to Y_2}$

Hence $p(y_1,x_1,x_2,y_2) = p(y_1)\,p(x_1|y_1)\,p(x_2|x_1)\,p(y_2|x_2)$

$= p(x_1,x_2,y_1,y_2)$ we have the Markov chain $Y_1 \to X_1 \to X_2 \to Y_2$

$C = \max_{p(x_1,x_2)} I(X_1,X_2;Y_1,Y_2) = \max_{p(x_1,x_2)} \left[H(Y_1,Y_2) - H(Y_1,Y_2|X_1,X_2)\right]$

since $Y_1 \to X_1 \to X_2 \to Y_2$, Y_1 and Y_2 are conditional independent for given X_1, X_2

we have $H(Y_1,Y_2|X_1,X_2) = H(Y_1|X_1,X_2) + H(Y_2|X_1,X_2)$

also, we have $Y_2 \to X_2 \to X_1 \to Y_1$

$= H(Y_1|X_1) + H(Y_2|X_2)$

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$$H(Y_1, Y_2) - H(Y_1, Y_2 \mid X_1, X_2) = H(Y_1, Y_2) - H(Y_1 \mid X_1) - H(Y_2 \mid X_2)$$

$$\leq H(Y_1) + H(Y_2) - H(Y_1 \mid X_1) - H(Y_2 \mid X_2)$$

$$= [H(Y_1) - H(Y_1 \mid X_1)] + [H(Y_2) - H(Y_2 \mid X_2)]$$

$$= I(X_1 ; Y_1) + I(X_2 ; Y_2)$$

Hence, we have $I(X_1, X_2 ; Y_1, Y_2) \leq I(X_1 ; Y_1) + I(X_2 ; Y_2)$

$$C = \max_{p(x_1, x_2)} I(X_1, X_2 ; Y_1, Y_2) \leq \max_{p(x_1, x_2)} [I(X_1 ; Y_1) + I(X_2 ; Y_2)]$$

$$= \max_{p(x_1, x_2)} I(X_1 ; Y_1) + \max_{p(x_1, x_2)} I(X_2 ; Y_2)$$

here we assume that $p(x_1, x_2) = p(x_1)\, p(x_2)$

$$= \max_{p(x_1)} I(X_1 ; Y_1) + \max_{p(x_2)} I(X_2 ; Y_2) = C_1 + C_2$$

Hence the capacity of the Channel $(\mathcal{X}_1 \times \mathcal{X}_2,\ p(y_1 \mid x_1) \times p(y_2 \mid x_2),\ \mathcal{Y}_1 \times \mathcal{Y}_2)$ is $C = C_1 + C_2$, when $p(x_1)$ maximizes $I(X_1 ; Y_1)$ and $p(x_2)$ maximizes $I(X_2 ; Y_2)$

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