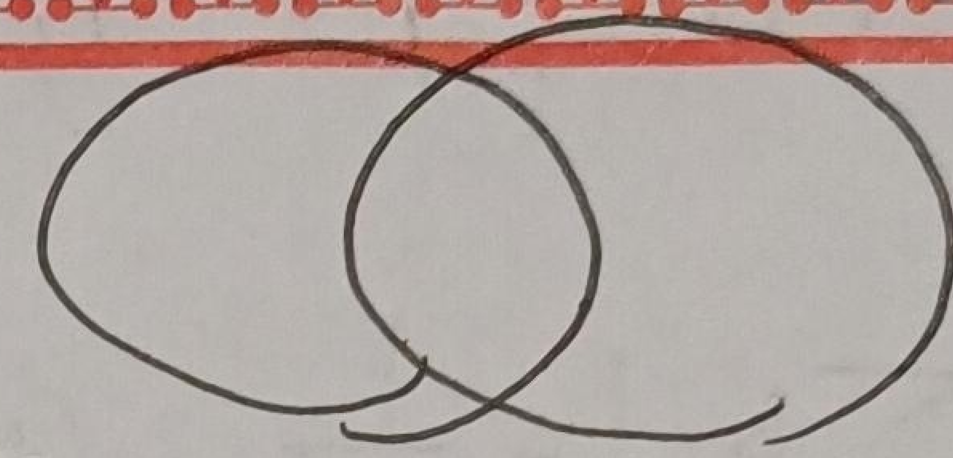
Homework 2 郑祖彬 12112328



12. 28. 42.

12. (a) $H(X) = H(\frac{2}{3}, \frac{1}{3}) = \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 = \log 3 - \frac{2}{3}\log 2 = \log 3 - \frac{2}{3}$

$H(Y) = H(\frac{1}{3}, \frac{2}{3}) = \log 3 - \frac{2}{3}\log 2 = \log 3 - \frac{2}{3}$

(b) $H(X|Y) = \frac{1}{3}H(X|Y=0) + \frac{2}{3}H(X|Y=1) = \frac{1}{3}H(1,0) + \frac{2}{3}H(\frac{1}{2},\frac{1}{2})$

$= 0 + \frac{2}{3}(\frac{1}{2}\log 2 + \frac{1}{2}\log 2) = \frac{2}{3}$

$H(Y|X) = \frac{2}{3}H(Y|X=0) + \frac{1}{3}H(Y|X=1) = \frac{2}{3}H(\frac{1}{2},\frac{1}{2}) + \frac{1}{3}H(0,1)$

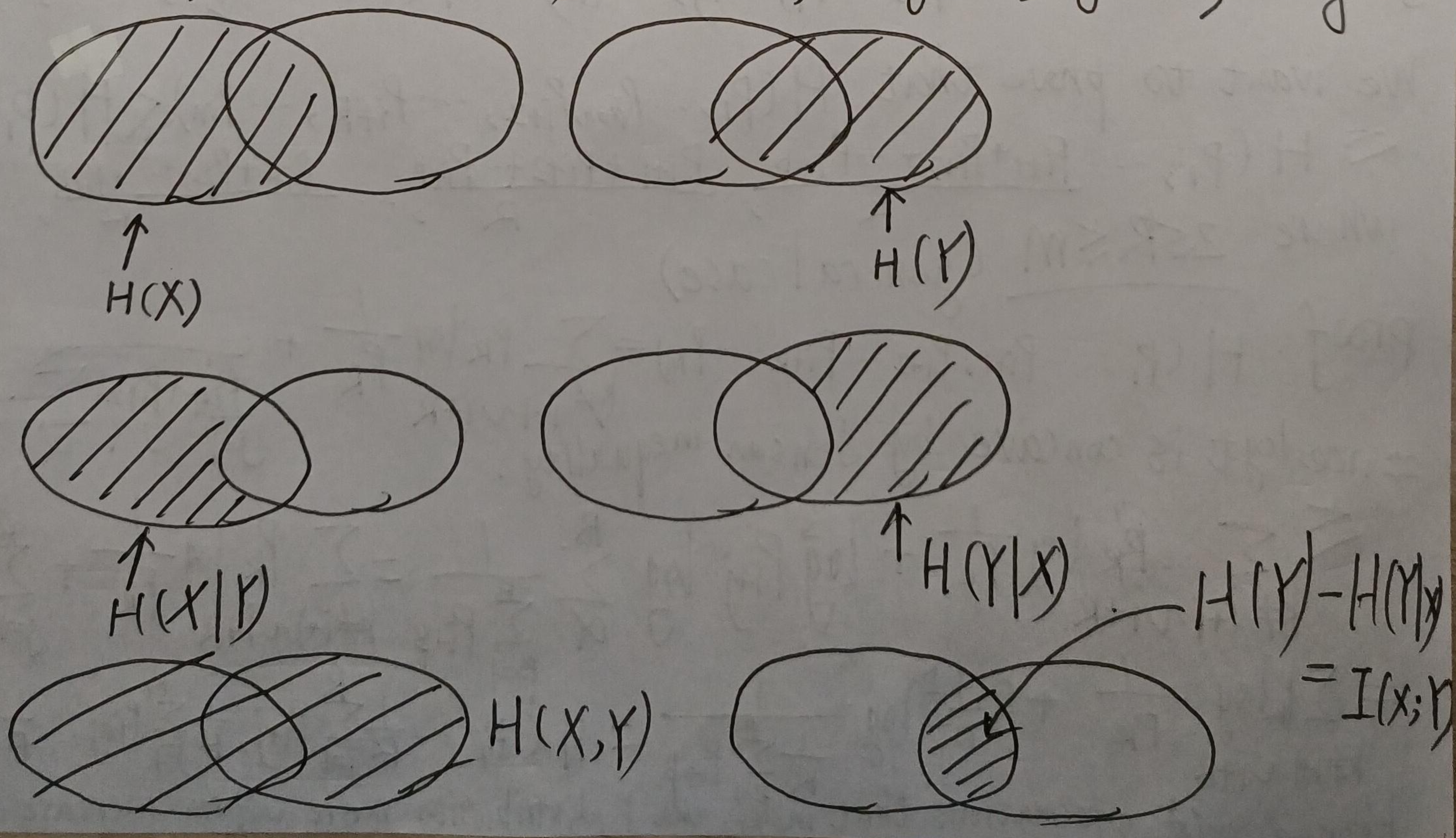
$= \frac{2}{3}$

(c) $H(X,Y) = H(X) + H(Y|X) = \log 3 - \frac{2}{3}\log 2 + \frac{2}{3} = \log 3$

(d) $H(Y) - H(Y|X) = \log 3 - \frac{2}{3}\log 2 - \frac{2}{3} = \log 3 - \frac{4}{3}$

(e) $I(X;Y) = H(X) + H(Y) - H(X,Y) = \log 3 - \frac{2}{3}\log 2 - \frac{2}{3} = \log 3 - \frac{4}{3}$

(f)



28. $H(p_1,\dots p_i,\dots p_j,\dots p_m)=\sum_{k=1}^{m}p_k\log\frac{1}{p_k}=\sum_{\substack{k=\\1\sim i-1\\i+1\sim j-1\\j+1\sim m}}p_k\log\frac{1}{p_k}+p_i\log\frac{1}{p_i}+p_j\log\frac{1}{p_j}$

proof.

$=\sum_{\substack{k=\\1\sim i-1\\i+1\sim j-1\\j+1\sim m}}p_k\log\frac{1}{p_k}+(p_i+p_j)\left(\frac{p_i}{p_i+p_j}\log\frac{1}{p_i}+\frac{p_j}{p_i+p_j}\log\frac{1}{p_j}\right)$ since $\log t$ is concave, by Jensen inequality

$\leq\sum_{\substack{k=\\1\sim i-1\\i+1\sim j-1\\j+1\sim m}}p_k\log\frac{1}{p_k}+(p_i+p_j)\log\frac{2}{p_i+p_j}=\sum_{\substack{k=\\1\sim i-1\\i+1\sim j-1\\j+1\sim m}}p_k\log\frac{1}{p_k}+2\left(\frac{p_i+p_j}{2}\right)\log\frac{1}{\frac{p_i+p_j}{2}}$

$=\sum_{\substack{k=\\1\sim i-1\\i+1\sim j-1\\j+1\sim m}}p_k\log\frac{1}{p_k}+\frac{p_i+p_j}{2}\log\frac{1}{\frac{p_i+p_j}{2}}+\frac{p_i+p_j}{2}\log\frac{1}{\frac{p_i+p_j}{2}}=H\left(p_1,\dots\frac{p_i+p_j}{2},\dots\frac{p_i+p_j}{2}\dots p_m\right)$

Hence $H(p_1,\dots p_i,\dots p_j,\dots p_m)\leq H\left(p_1,\dots\frac{p_i+p_j}{2},\dots\frac{p_i+p_j}{2},\dots p_m\right)$

We want to prove that $H(p_1,\dots p_{i+1},p_{i+2},\dots p_{i+R},\dots p_m)$
$\leq H\left(p_1,\dots\frac{p_{i+1}+p_{i+2}+\dots p_{i+R}}{R},\frac{p_{i+1}+p_{i+2}+\dots p_{i+R}}{R},\dots\frac{p_{i+1}+p_{i+2}+\dots+p_{i+R}}{R},\dots p_m\right)$,
where $\underline{2\leq R\leq m}$ (general case)

proof. $H(p_1,\dots p_{i+1},p_{i+2}\dots p_{i+R}\dots p_m)=\sum_{k\neq i+1\sim i+R}p_k\log\frac{1}{p_k}+\sum_{j=1}^{R}p_{i+j}\left(\sum_{\alpha}^{R}\frac{p_{i+\alpha}}{\sum_{\beta=1}^{R}p_{i+\beta}}\log\frac{1}{p_{i+\alpha}}\right)$

since $\log t$ is concave by Jensen inequality:

$\leq\sum_{k\neq i+1\sim i+R}p_k\log\frac{1}{p_k}+\sum_{j=1}^{R}p_{i+j}\log\sum_{\alpha}^{R}\frac{1}{\sum_{\beta=1}^{R}p_{i+\beta}}=\sum_{k\neq i+1\sim i+R}p_k\log\frac{1}{p_k}+\sum_{j=1}^{R}p_{i+j}\log\frac{R}{\sum_{\beta=1}^{R}p_{i+\beta}}$

$=\sum_{k\neq i+1\sim i+R}p_k\log\frac{1}{p_k}+R\frac{1}{R}\sum_{j=1}^{R}p_{i+j}\frac{1}{\frac{1}{R}\sum_{\beta=1}^{R}p_{i+\beta}}=H\left(p_1,\dots\frac{1}{R}\sum_{j=1}^{R}p_{i+j},\frac{1}{R}\sum_{j=1}^{R}p_{i+j}\dots\frac{1}{R}\sum_{j=1}^{R}p_{i+j},p_m\right)$

Hence, transformations that make the p distribution more uniform increase H.

42. (a) $H(5X) = H(X)$ since $X \to 5X$ is a one-to-one mapping

(b) $I(g(X);Y) \le I(X;Y)$

(c) $H(X_0 | X_{-1}) \ge H(X_0 | X_{-1}, X_1)$

(d) $$\frac{H(X,Y)}{H(X)+H(Y)} \le 1.$$