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REGULAR RESEARCH PAPER

Transfer weight functions for injecting problem information in the multi-objective CMA-ES

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Abstract The covariance matrix adaptation evolution strategy (CMA-ES) is one of the state-of-the-art evolutionary algorithms for optimization problems with continuous representation. It has been extensively applied to single-objective optimization problems, and different variants of CMA-ES have also been proposed for multi-objective optimization problems (MOPs). When applied to MOPs, the traditional steps of CMA-ES have to be modified to accommodate for multiple objectives. This fact is particularly evident when the number of objectives is higher than 3 and, with a high probability, all the solutions produced become non-dominated. An open question is to what extent information about the objective values of the non-dominated solutions can be injected in the CMA-ES model for a more effective search. In this paper, we investigate this general question using several metrics that describe the quality of the solutions already evaluated, different transfer weight functions, and a set of difficult benchmark instances including many-objective problems. We introduce a number of new strategies that modify how the probabilistic model is learned in CMA-ES. By conducting an exhaustive empirical analysis on two difficult benchmarks of many-objective functions we show that the proposed strategies to infuse information about the quality indicators into the learned models can achieve consistent improvements in the quality of the Pareto fronts obtained and enhance the convergence rate of the algorithm. Moreover, we conducted a comparison with a state-of-the-art algorithm from the literature, and achieved competitive results in problems with irregular Pareto fronts.

Keywords Many-objective · Covariance matrix adaptation · Optimization · Probabilistic modeling · Estimation of distribution algorithm

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1 Introduction

The application of model-based evolutionary algorithms to multi-objective optimization problems is one of the active current research lines in this area [14,31]. Models can be used in MOPs in several ways: to encode the regularities of the Pareto set approximations [31], to learn and represent dependencies between variables [3], or between variables and objectives [14], etc. The rationale behind the use of models is that, by being able to capture and exploit the characteristics of the represented domain (i.e. either the shape of the Pareto front, the interactions between variables and objectives or any other characteristic) they can lead to a more informed and therefore efficient search of the space of solutions. There are additional benefits in addressing MOPs with model-based

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MOEAs, such as the ability to extract information about the problem domain [22].

Probabilistic models are among the most used methods for modeling in multi-objective evolutionary algorithms (MOEAs). In this paper we focus on an evolutionary algorithm that uses Gaussian modeling of the best solutions visited by the algorithm. The covariance matrix adaptation evolution strategy (CMA-ES) [10] combines characteristics of both evolution strategies [2] and estimation of distribution algorithms (EDAs) [17,21]. It is considered a state-of-the-art optimizer for single-objective continuous functions.

One of the distinguishing features of CMA-ES is the use of a covariance matrix to estimate the probability of the moves that lead to more promising solutions. This matrix is updated throughout the evolution in a process that injects information from the best solutions found so far into the matrix. CMA-ES has several invariance properties i.e., it has uniform performance on a class of functions, thus allowing the generalization of empirical results. The following invariances are found in CMA-ES: invariance against order preserving transformations of objective function values; invariance against angle preserving transformations of the search space (rotation, reflection and translation); scale invariance; invariance against invertible linear transformation of the search space.

Different variants of CMA-ES have been proposed for MOPs [12,13,26,29]. Among other components, a multi objective CMA-ES (MO-CMA-ES) usually differs from others in the methods employed to rank the solutions that are used to update the covariance matrix. Once the solutions have been ranked, the linear or logarithmic function of the ranks is used as weights for updating the matrix.

In this paper we propose the idea of using transfer weight functions (TWFs) as flexible components to manage the incorporation of information into a Pareto-based MO-CMA-ES. A TWF is composed by a ranking method and a weighting function, hence new TWFs can be easily incorporated into the framework. As examples we created 10 TWFs and discuss the rationale behind their choice, their strengths and weaknesses. Moreover, an empirical study was conducted to assess the impact of each one in the performance of MO-CMA-ES. Finally, another experimental study was performed in order to validate our framework, where we compared it using the best performing TWFs to the state-of-the-art NSGA-III algorithm.

Our work is inspired on previous results [24,25] in EDAs that show how the direct infusion of fitness information into the probabilistic models can lead to improved results on these algorithms. Some of the TWFs proposed incorporate information about the fitness of the functions, this purposely violates some of the attractive properties of CMA-ES, in particular its invariance under order preserving transformations of the objective function values. We hypothesize however, that TWFs can be an effective way to cope with many-objective problems (MaOPs). In this particular context, the invariance properties may be more difficult to fulfill and, in any case, less important than the needs for search efficiency.

To summarize the main contributions of this paper, we: (1) Investigate the behavior of an informed MO-CMA-ES, whose main characteristic is the use of information about the quality of the solutions to update the covariance matrix. (2) Introduce TWFs as an effective and simple way to incorporate information from the solutions into the algorithm. Different TWFs are explained and compared in terms of their effect. (3) Conduct extensive experiments using difficult benchmarks that include MOPs of up to 20 objectives and present empirical evidence of the gains in performance that can be achieved by the introduced strategies. (4) Empirically compare the best performing TWFs to a state-of-the-art optimizer to validate our framework.

The remainder of this paper is organized as follows: the next section presents some basic notation, necessary concepts and the CMA-ES algorithm. Section 3 reviews related work on MO-CMA-ES, and its connection with the proposal introduced in this paper. Section 4 describes in detail the characteristics of our approach to define transfer weight functions. The experimental framework and the numerical results of our experiments are presented in Sect. 5. We conclude the paper in Sect. 6, where avenues for future research are also discussed.

2 Preliminary concepts

In this section, we introduce the notation and present some concepts required for the presentation of the algorithm.

2.1 Many-objective optimization

Multi-objective optimization problems (MOPs) require the simultaneous optimization (maximization or minimization) of two or more objective functions. These objectives are usually in conflict, so these problems don't have only one optimal solution (as in single objective optimization problems), but a set of them. This set of solutions is usually found using the Pareto optimality theory.

A general MOP without constraints can be defined as optimizing $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, where $\mathbf{x} \in \Omega$ is an n-dimensional decision variable vector $\mathbf{x} = (x_1, \ldots, x_n)$ from a universe Ω, and m is the number of objective functions.

An objective vector $\mathbf{u} = \mathbf{f}(\mathbf{x})$ dominates a vector $\mathbf{v} = \mathbf{f}(\mathbf{y})$, denoted by $\mathbf{u} \preceq \mathbf{v}$ (in case of minimization) if $\mathbf{u}$ is partially less than $\mathbf{v}$ i.e., $\forall i \in \{1, \ldots, m\}, \mathbf{u_i} \leq \mathbf{v_i} \wedge \exists j \in \{1, \ldots, m\} : \mathbf{u_j} < \mathbf{v_j}$.

A vector $\mathbf{u}$ is non-dominated if there is no $\mathbf{v}$ that dominates $\mathbf{u}$. Given that $\mathbf{u} = \mathbf{f}(\mathbf{x})$, if $\mathbf{u}$ is non-dominated, then $\mathbf{x}$ is Pareto optimal. The set of Pareto optimal solutions is called the Pareto optimal set, and the image of these solutions in the objective space is called the Pareto front [6].

Many-objective optimization problems (MaOPs) are a type of MOPs that present more than three objective functions to be optimized simultaneously. Several studies have indicated that Pareto based algorithms scale poorly in MaOPs [7]. The main reason for this is the number of non-dominated solutions which increases greatly with the number of objectives, consequently, the search ability is deteriorated because it is not possible to impose preferences for selection purposes.

2.2 CMA-ES

The covariance matrix adaptation evolution strategy (CMA-ES), firstly introduced in [10] is a state-of-the-art optimizer for single-objective continuous functions.

CMA-ES works by sampling solutions from a multivariate normal distribution based on a mean vector $\mathbf{m}$, an $n \times n$ covariance matrix $\mathbf{C}$, and a step size σ. The search progresses by iteratively adapting these parameters in order to obtain better solutions. To accomplish this, the generated solutions are ranked based on their quality, originally determined by their fitness. Then, these ranked solutions are used in weighted equations to update the parameters of the distribution for the next generation.

The CMA-ES starts with the mean being initialized in the average of the normalized search space $[0, 1]$ as $\mathbf{m} = (0.5, \ldots, 0.5)$, the covariance matrix $\mathbf{C}$ initialized as the identity matrix, the evolution paths $\mathbf{p}_\sigma = \mathbf{p_c} = (0, \ldots, 0)$, and the step size $\sigma = 0.3$.

In an iteration g of CMA-ES, the following operations are performed:

1. Sample a set of $\lambda \geq 1$ solutions from the multivariate normal distribution following Eq. (1):

$$\mathbf{x_i}^{(g+1)} \sim N(\mathbf{m}^{(g)}, (\sigma^{(g)})^2\mathbf{C}^{(g)}), \quad i = 1, \ldots, \lambda \tag{1}$$

2. Evaluate the solutions and order them according to their quality such that $\mathbf{x}_{i:\lambda}$ represents the i-th best solution.
3. Calculate the new mean vector ($\mathbf{m}$) through a weighted recombination of the μ best solutions as in Eq. (2):

$$\mathbf{m}^{(g+1)} = \sum_{i=1}^{\mu} w_i \mathbf{x}_{i:\lambda}^{(g+1)}, \quad \sum_{i=1}^{\mu} w_i = 1, \ w_i > 0 \tag{2}$$

4. Update the evolution path $\mathbf{p}_\sigma$ through Eq. (3):

$$\mathbf{p}_\sigma = (1-c_\sigma)\mathbf{p}_\sigma + \sqrt{c_\sigma(2-c_\sigma)\mu_{eff}}\,\mathbf{C}^{(g)^{-\frac{1}{2}}} \frac{\mathbf{m}^{(g+1)} - \mathbf{m}^{(g)}}{\sigma^{(g)}} \tag{3}$$

where the variance effective selection mass is $\mu_{eff} = \frac{1}{\sum_{i=1}^{\mu} w_i^2}$, $\frac{1}{c_\sigma}$ is the backward time horizon of the evolution path $\mathbf{p}_\sigma$, set as $c_\sigma = \frac{\mu_{eff}+2}{n+\mu_{eff}+5}$.

5. Based on the previously calculated evolution path, update the step size σ as shown in Eq. (4):

$$\sigma = \sigma \exp\left(\frac{c_\sigma}{d_\sigma}\left(\frac{||\mathbf{p}_\sigma||}{E||N(0, I)||} - 1\right)\right) \tag{4}$$

where the damping parameter is set as $d_\sigma = 1 + 2\max(0, \sqrt{\frac{\mu_{eff}-1}{n+1}} - 1)$, and the expectation of the norm of an $N(0, I)$ random vector is set as $E||N(0, I)|| \approx \sqrt{n}(1 - \frac{1}{4n} + \frac{1}{21n^2})$.

6. Update the evolution path $\mathbf{p}_c$ through Eq. (5):

$$\mathbf{p_c} = (1 - c_c)\mathbf{p_c} + h_\sigma\sqrt{c_c(2 - c_c)\mu_{eff}}\,\frac{\mathbf{m}^{(g+1)} - \mathbf{m}^{(g)}}{\sigma^{(g)}} \tag{5}$$

where $\frac{1}{c_c}$ is the backward time horizon of the path $\mathbf{p}_c$ set as $c_c = \frac{4+\mu_{eff}/n}{n+4+2\mu_{eff}/n}$. The Heaviside function can stall the update of $\mathbf{p}_c$ if $||\mathbf{p}_\sigma||$ is large, preventing a too fast increase of axis of $\mathbf{C}$ in a linear surrounding (when the step size is far too small), it is defined as $h_\sigma = 1$ if $\frac{||\mathbf{p}_\sigma||}{\sqrt{1-(1-c_\sigma)^{2(g+1)}}} < (1.5 + \frac{1}{n-0.5}E||N(0, I)||)$, or 0 otherwise.

7. Finally, update the covariance matrix based on the previously calculated parameters through Eq. (6):

$$\begin{aligned}\mathbf{C}^{(g+1)} &= (1 - c_{cov})\mathbf{C}^{(g)} + \frac{c_{cov}}{\mu_{cov}}\left(\mathbf{p_c}\mathbf{p_c}^T + \delta(h_\sigma)\mathbf{C}^{(g)}\right)\\ &\quad + c_{cov}\left(1 - \frac{1}{\mu_{cov}}\right)\sum_{i=1}^{\mu} w_i\, OP\left(\frac{\mathbf{x}_{i:\lambda}^{(g+1)} - \mathbf{m}^{(g)}}{\sigma^{(g)}}\right)\end{aligned} \tag{6}$$

where the relative weighting $c_{cov} = \frac{1}{\mu_{cov}}\frac{2}{(n+\sqrt{2})^2} + (1 - \frac{1}{\mu_{cov}})\min(1, \frac{2\mu_{eff}-1}{(n+2)^2+\mu_{eff}})$, $\delta(h_\sigma^{(g+1)}) = (1 - h_\sigma^{(g+1)})c_c(2 - c_c) \leq 1$ is of minor relevance and can be set to 0. The outer product of a vector with itself is denoted as OP: $\mathbb{R}^n \rightarrow \mathbb{R}^{n\times n}, \mathbf{x} \mapsto \mathbf{x}\mathbf{x}^T$, which results in a matrix of rank one with eigenvector $\mathbf{x}$ and eigenvalue $||\mathbf{x}||^2$.

In this work, we start using the single-objective version outlined above and implement a multi-objective version. To accomplish this, we change steps 2 and 3 in the above schema, more specifically, the ranking and weighting of the solutions. In the next sections we show that the choice of these components can affect the performance of the algorithm.

3 Related work

In this section we analyze work related to the use of methods based on CMA-ES for MOPs. First, we review in detail some multi-objective variants of CMA-ES proposed so far. Then, we emphasize the characteristics of these approaches that make them different from our proposal, in particular the way they use information about the search for the selection process and for building the model. Finally, we also cover in the section some related work on EDAs that explicitly use selection probabilities when constructing the probabilistic models in EDAs.

Algorithm 1: MO-CMA-ES

$g = 0$, initialize $a_k^{(g)}$ for $k = 1, ..., \lambda$
repeat
  for *k=1,...,λ* **do**
    $a'^{(g+1)}_k \leftarrow a_k^{(g)}$
    $x'^{(g+1)}_k \sim N(x_k^{(g)}, {\sigma_k^{(g)}}^2 C_k^{(g)})$
  endfor
  $Q^{(g)} = \{a'^{(g+1)}_k, a_k^{(g)} | 1 \leq k \leq \lambda\}$
  for *k=1,...,λ* **do**
    updateStepSize($a_k^{(g)}, \lambda^{(g+1)}_{succ,Q^{(g)},k}$)
    updateStepSize($a'^{(g+1)}_k, \lambda^{(g+1)}_{succ,Q^{(g)},k}$)
    updateCovariance($a'^{(g+1)}_k, \frac{x'^{(g+1)}_k - x_k^{(g)}}{\sigma_k^{(g)}}$)
  endfor
  for *i=1,...,λ* **do**
    $a_i^{(g+1)} \leftarrow Q^{(g)}_{\prec:i}$
  endfor
  $g \leftarrow g + 1$
until stopping criterion is met;

The first multi-objective version of CMA-ES was proposed in [12] by Igel et. al. and is called MO-CMA-ES algorithm. This algorithm is based on the NSGA-II [8] but each individual of its population is a full parameterized $(1 + \lambda)$-CMA-ES. In this elitist $(1 + \lambda)$-CMA-ES the parent population consists of a single individual generating λ offspring and the best individual out of the parents and offspring becomes the parent of the next generation [12]. Since they considered $\lambda = 1$, in practice a $(1 + 1)$-CMA-ES was used. MO-CMA-ES is presented in Algorithm 1 and begins by randomly generating a set of λ individuals $a_k^{(g)}$ to be the first parents, then for a predefined number of iterations, each parent samples λ individuals from its model. Parents and offspring form a set Q. The step sizes σ of all solutions in Q are updated depending on whether the new solutions were successful.[1] The covariance matrix of the offspring is updated taking into account the mutation that has led to its genotype. The best individuals in Q are sorted to form the next parent generation by using the same success rule, where $Q_{\prec:i}$ means the i-th best offspring Q.

An efficient covariance matrix update based on Cholesky factorization is proposed by Igel et al. in [13]. The authors claim that this new update rule presents a slower (in terms of generations) covariance adaptation, however its computational complexity is smaller ($\mathcal{O}(n^3)$ to $\mathcal{O}(n^2)$) where n is the number of decision variables. Two steady state variants of MO-CMA-ES are proposed using the new matrix update. Both are based on the selection criterion from MO-CMA-ES, that uses domination rank of the solutions as the first measure and contributing hypervolume to break ties.

A scalarized version of the steady-state MO-CMA-ES is proposed by Voß et al. in [26]. In this version, a $(1+1)$-CMA-ES is executed for each subproblem individually during a predefined number of generations. Each solution is evaluated according to the quality regarding its scalarized weight (as in MOEA/D [30].

An additional update of the covariance matrix for each individual is proposed in [27] by Voß et al. This additional update adds a recombination of the neighboring solutions to speed up the parameter adaptation of CMA-ES. This recombination update uses information from the successful neighbors of an individual and weights them according to a distance measure that takes into consideration the shape of the search distribution and the step size of the individual being updated.

A new success rule for MO-CMA-ES is proposed by Voß et al. in [28]. Instead of considering successful an offspring individual that is better than its parent (considering dominance ranks and hypervolume), a mutation is considered successful if it is selected for the next parent population.

In order to allow the MO-CMA-ES to be used in many-objective problems, Bringmann et al. proposed in [4] two approaches to calculate the approximate contributing hypervolume. The two approaches are the *Probably Approximately Correct Approximation* and *Fast Approximate Computation*. These approximate approaches are incorporated in [28] and compared to the exact calculation on the DTLZ [9] problems up to seven objectives, and on two real-world problems.

Rostami and Shenfield proposed in [20] the CMA-PAES algorithm, which is a multi-objective CMA that uses the archiving strategy of the Pareto Archived Evolution Strategy [15] to store the best non-dominated solutions found so far during the search.

Recently Zapotecas-Martínez et al. proposed in [29] a new algorithm called MOEA/D-CMA that combines MOEA/D with CMA-ES. This algorithm maintains a CMA-ES instance for each different subproblem. In one iteration and for each subproblem, the algorithm samples a set of solutions that can be used to update the best solutions of its neighbors as

[1] A newly generated solution is considered successful if the offspring is better than the parent according to the dominance ranks or a secondary measure, in the original paper they used both crowding distance and contributing hypervolume.

in MOEA/D.[2] Then, the CMA-ES parameters of each subproblem are updated with the best solutions generated by its own distribution and with solutions injected from its neighbors.

In a brief review of the works presented, most of them are based on Pareto optimality, except for [26,29], which use scalarization. Moreover, except for these two and [20] (based on PAES), all algorithms are inspired by the NSGA-II framework, and hence use a selection scheme based on dominance ranking and a secondary measure (crowding distance or contributing hypervolume) to break ties. Except for [20], all the variants presented here kept a full parametrized CMA-ES for each individual of the population (or sub-problem in [29]). Since in most works there is no communication among the solutions and only one solution is used to update the covariance matrix, rank is not needed. The approaches that use information from more than one solution, and thus rank them, are presented in [28], where for each solution a rank and set of weights is calculated based on its covariance matrix. In [20] the solutions are ranked based on the information given by the archiver (weighting is not specified). Finally in [29] the solutions are ranked based on the scalarizing weight vectors.

Besides the use of CMA-ES for multi-objective problems, our work relates to the conception of methods that explicitly use selection probabilities at the time of constructing the probabilistic models in EDAs. The idea of these approaches, instead of sampling the individuals according to their selection probabilities and then learning a model from the sampled solutions, is to directly incorporate the selection probabilities in the construction of the model. In the approach introduced in this paper, the weights used to update the covariance matrix of CMA-ES play a similar role to selection probabilities in EDAs.

Methods that explicitly use the fitness information in the construction of the probabilistic models have been proposed for problems with discrete representation [24,25] and continuous representation [25]. They mainly differ in the type of models (e.g. univariate models, Bayesian and Gaussian networks, etc.) in which the selection probabilities are infused. Other related approaches explicitly model fitness information as part of the probabilistic model [14].

4 Learning strategies for MO-CMA-ES

CMA-ES was proposed to learn from good solutions of a given problem and model the path that leads to them, so it can sample new solutions closer to the optimum. It was created with single-objective problems in mind, and in these problems it is trivial to determine good and bad solutions, one just has to look at the fitness values.

Our MO-CMA-ES variant is inspired in the original single objective CMA-ES and in the first MO-CMA-ES proposed in [12] and depicted in Algorithm 1. As in the single objective version, we keep only one probabilistic model for the entire search and update it using the best solutions found so far, ranked and weighted. However, since we are dealing with multi and many-objective problems, there is a clear notion that non-dominated solutions can be considered superior to the others, however it is hard to rank between them in the same way as is done for single-objective problems.

In summary, our framework works as follows: first a set of solutions is generated randomly, evaluated and the non-dominated solutions are stored in the external archive.[3] Next, for a predefined number of iterations, a TWF is used to rank and weight all the non-dominated solutions available (from the population and the archive) that are used to update the probabilistic model. Following, a new set of solutions is sampled from the model, evaluated and the best non-dominated solutions are stored in the archive. After the maximum number of iterations is reached, the content of the repository is returned as result of the algorithm solution of the problem.

There are several ways of ranking and weighting non-dominated solutions in the literature, and all of them present specific strengths and weaknesses. In this work we propose the use of TWFs as a flexible component of the MO-CMA-ES that are composed by a combination of ranking and weighting methods to be used before learning the model. The rest of this section is devoted to presenting and discussing some possibilities of components to create a TWF.

4.1 Which non-dominated solutions should CMA-ES model?

The variant of MO-CMA-ES used in this paper works with two sets of solutions: the population and the repository. The repository, or external archive, maintains the "best" non-dominated solutions found so far during the search. This repository is bounded by a predefined maximum size, and, until it is full, all non-dominated solutions are added. Once it is full, when a new non-dominated solution is found, another criterion is needed to decide if it is kept and if so, which solution will be discarded. In this work the criterion is based on the crowding distance (CD) [8]. When a new solution is inserted, temporarily surpassing the capacity of the reposi-

[2] In MOEA/D and MOEA/D-CMA a subproblem is considered a neighbor of itself, hence when updating the solutions of the neighbors it updates its own solution.

[3] An external archive or repository is used to store a predefined number of non-dominated solutions, when the archive is full and a new non-dominated solution is found, it is temporarily added and the solution that has the smallest crowding distance is removed.

tory in one solution, the CD of all solutions is computed and the one having the smallest value (more crowded region) is removed. This is a way of ensuring some level of diversity in the solutions contained in the repository.

(i) A first option for selecting the solutions to update the CMA-ES model is directly using the solutions in the repository. This approach presents some advantages:

  – There are always non-dominated solutions in the repository;
  – The solutions in the repository are as good and diverse as possible;
  – In the beginning of the search (until the repository is full) all the non-dominated solutions found so far are in the repository.

  However, this approach also has a few drawbacks:

  – The repository can be the same for several iterations (no new solution enters), leading to over-fitting. This situation is more usual when dealing with a small objective number;
  – If there are many non-dominated solutions (usual scenario in many-objective), several of them will not enter the repository, thus they will not be used for learning, wasting (perhaps valuable) information gathered during the search;
  – Some bias can be introduced by the archiver in the search (i.e., preference to solutions in the extremes of the front).

  The other set of solutions kept during the search is the population itself. These are the newest solutions to enter the search process. Among these solutions there are usually dominated and non-dominated solutions (regarding the repository), and at the end of the iteration some of them can be present in the repository as well.

(ii) A second option to select the solutions used to update the CMA-ES model, is using the non-dominated solutions (regarding the repository) from the population. This approach has some advantages:

  – There are always different solutions at each iteration, so there is no risk of over-fitting;
  – All the non-dominated solutions found during the search are used in the learning process once;
  – It is less likely to have bias introduced by the archiver (some is still possible because the domination relation of the solutions is compared to the repository).

  However, this approach has a few drawbacks as well:

  – It is possible to have only dominated solutions in some iterations, especially using few objective functions;
  – If the problem is biased, most of the solutions found will be concentrated in one region so there is no diversity preservation mechanism introduced, as in the repository;
  – The "best" solutions found are used in the learning only once, so there is no elitism.

  In order to use this second option for learning the model, the first drawback must be dealt with, because since we established that the algorithm will only learn from non-dominated solutions and there are none, there is no learning and the algorithm can be stuck for several iterations. Hence, in this work, when there are no non-dominated solutions in the population, the solutions in the repository are used. This strategy also alleviates the third issue, introducing some elitism when the algorithm is not able to generate good solutions.

(iii) A third option for selecting the solutions, is to merge both sets. This way we use all the non-dominated solutions at hand in a given iteration to learn the model. This approach can combine some advantages of both methods as:

  – There are always non-dominated solutions to learn from;
  – There are usually some different solutions at each iteration, so there is a smaller risk of over-fitting;
  – All the non-dominated solutions found during the search are used in the learning process at least once.

  However, this approach also exhibits some of the drawbacks of both approaches:

  – Some bias can be introduced by the archiver in the search. Nevertheless, since we are using more solutions to learn, this problem is alleviated;
  – The diversity preserving mechanism of the archiver will have less effect because more solutions are used for learning. In biased problems it is possible to have more solutions in some regions;
  – If there are few non-dominated solutions per iteration, there is a risk of over-fitting.

  Although an empirical study comparing the effects of using each of these three sources of information for CMA-ES is interesting, this is out of the scope of this paper and can be considered for future works. Here we used the third option because it combines some advantages and alleviates some drawbacks of the other approaches. Moreover, by combining the non-dominated solutions of the two sets we have a higher number of solutions to rank and weight properly, hence the differences between the methods compared become more evident.

4.2 How to set the weights for the modeled solutions?

In the single-objective CMA-ES, all the operations involving the solutions are weighted, so the best solutions contribute more to the update of the model, and this contribution decreases as the ranks of the solutions get worse. This ranking is based solely on the fitness value.

When considering multi-objective optimization, at first all the non-dominated solutions are incomparable and can be considered equally good. So setting equal weights for all solutions is the most trivial option.

Another path to explore is to use a quality indicator in order to rank the solutions, giving more weight to the solutions that present desirable characteristics according to a given metric. Different quality indicators will be presented in the next section.

Once having ranked the solutions, the classic weighting strategies of CMA-ES can be used. Here we study two: log and linear, defined by Eqs. (7) and (8), respectively.

$$w_i = \frac{w_i'}{\sum_{j=1}^{\mu} w_j'}, \quad w_i' = \ln(\mu + 0.5) - \ln i, \quad \text{for } i = 1, \ldots, \mu \tag{7}$$

$$w_i = \frac{w_i'}{\sum_{j=1}^{\mu} w_j'}, \quad w_i' = \mu - i, \quad \text{for } i = 1, \ldots, \mu \tag{8}$$

These two approaches give higher weights to the best ranked solutions and this weight decreases at different rates as the ranks get worse.

Besides these three approaches, we also propose to use another one, where the weights are set based on the quality of a solution measured by an indicator. This approach is defined in Eq. (9).

$$\begin{aligned} & x_{max} = \max(x^{i:\lambda}), x_{min} = \min(x^{i:\lambda}) - \frac{\min(x^{i:\lambda})}{100} \\ & w_i' = \frac{x_i - x_{min}}{x_{max} - x_{min}} \\ & \text{if } w_i' \leq 0 : w_i' = \frac{\min(x^{i:\lambda}) - x_{min}}{x_{max} - x_{min}} \\ & \text{if } w_i' > 1 : w_i' = 1, \quad \text{for } i = 1, \ldots, \mu \\ & w_i = \frac{w_i'}{\sum_{j=1}^{\mu} w_j'} \end{aligned} \tag{9}$$

where $\max(x^{i:\lambda})$ is the maximum indicator value for all solutions, except for ∞, which is used in the crowding distance, and likewise $\min(x^{i:\lambda})$ is the smallest value except for zero. Instead of directly using the smallest value $\min(x^{i:\lambda})$ for x_{min}, we use a slightly smaller value. This is done to set a nonzero weight to a solution when it achieves an indicator value of zero.

In this last strategy the importance given to a solution is proportional to the value given to it by the indicator. Hence a good solution will have a bigger influence on the model than a worse one. By means of this weight-assignment strategy, we expect the indicator chosen to have a higher impact on the search.

4.3 Which quality indicators to use?

In the previous section we discussed different weighting strategies where solutions might influence the search more and others less according to their ranking or measured quality. In multi-objective search there are many metrics for evaluating non-dominated solutions in order to rank them. We now present three popular indicators.

The first one is the crowding distance (CD), initially proposed in [8], which is a popular diversity estimator from the MOEA literature. This metric is used to estimate the density of solutions surrounding a particular point. To calculate the CD, the average distance of the two neighbors of a solution in each objective is used. This value is an estimation of the size of the largest cuboid enclosing the solution **u** without including any other solution.

Algorithm 2: Crowding Distance calculation.

```
N = |PF|
for i=1 to N do
    CD_i = 0
endfor
for j = 1 to m do
    PF = sort(PF, j)
    CD_1 = CD_N = ∞
    for i = 2 to (N − 1) do
        CD_i = CD_i + (u_{j,(i+1)} − u_{j,i−1})
    endfor
endfor
```

The CD for each solution **u** in the set *PF* is calculated through Algorithm 2, where m is the number of objectives, N is the number of solutions in *PF*, and $sort(a, b)$ is used to sort the elements of a in ascending order in relation to its b-th objective value. $u_{j,i}$ represents the j-th objective value of the i-th solution of the set *PF*.

In the first step of the algorithm, N stores the number of solutions contained in *PF*, then the *CD* of each solution is initialized to 0. In the next step, for each objective, the population is sorted in an ascending order in relation to this objective. The first and last solutions (extreme solutions of the axis) have their *CD* values set to ∞. Next, the *CD* of the other solutions is calculated, where at each objective, the CD_i is increased by the difference between the objective values of the neighbors of solution i.

The second indicator used here is another extensively used metric called hypervolume [6]. The hypervolume of a set of solutions measures the size of the portion of the objective space that is dominated by those solutions. Hypervolume captures in one scalar both the closeness of the solutions to the optimal set and their spread across the objective space. It also has nicer mathematical properties than other metrics, being the only type of indicator that is strictly monotonic.[4] However, the hypervolume is very sensitive to the scaling of the objectives and to the presence of extremal points [1,6].

Given an arbitrary decision space Ω, we want to minimize a function $F : \Omega \rightarrow \mathbb{R}^m_{\geq 0}$. A solution $\mathbf{x}$ is an element of the decision space Ω but we will typically identify it with the corresponding $\mathbf{f}(\mathbf{x}) = \mathbf{u}$ in the objective space. A population-based algorithm maintains a set of solutions PS whose image in the objective space is represented by $PF \subseteq \mathbb{R}^m_{\geq 0}$ of size μ . Then the hypervolume is defined as follows [4]

$$HV(PF) := VOL\left(\bigcup_{(u_1,\dots,u_m)\in PF} [r_1, u_1] \times \cdots \times [r_m, u_m]\right) \quad (10)$$

where $VOL(.)$ is the usual Lebesgue measure and the reference point $\mathbf{r}$ is the nadir (anti-optimal or "worst possible") point in space. The greater the hypervolume value of a set is, the better that set is taken to be.

The main disadvantage of the hypervolume indicator is that its calculation is computationally expensive, even the best known algorithms for computing the hypervolume have running times exponential in the number of objectives, which restricts the use of hypervolume-based methods to problems with less than five or six objectives [1].

In order to use the hypervolume indicator to evaluate the quality of a single solution, usually the contributing hypervolume is used, which is defined according to Eq. (11) [5].

$$C_{HV}(\mathbf{u}, PF, \mathbf{r}) = HV(PF, \mathbf{r}) - HV(PF\backslash\{\mathbf{u}\}, \mathbf{r}) \quad (11)$$

where $C_{HV}(\mathbf{u}, PF, \mathbf{r})$ reflects the influence of a single point on the quality of the approximation set.

Due to the high computational cost of calculating the hypervolume for many objectives, we used a third popular indicator called $R2$ [5]. It was originally proposed to assess the relative quality of two approximation sets. It is an indicator that simultaneously evaluates the convergence and diversity of a Pareto front approximation and presents a low computational cost [5].

In its most usual form, the $R2$ indicator uses as utility function a standard weighted Tchebycheff function defined as: $\max_{j\in\{1\dots m\}} w_j \times |z^*_j - u_j|$, based on a reference point $\mathbf{z}^*$ and a set of weight vectors $\mathbf{w} = (w_1, \dots, w_m) \in W$ chosen uniformly distributed over the weighting space. In this form, the $R2$ indicator can be defined as:

$$R2(PF, W, \mathbf{z}^*) = \frac{1}{|W|}\sum_{\mathbf{w}\in W} \min_{\mathbf{u}\in PF}\left\{\max_{j\in\{1,\dots,m\}}\{w_j \times |\mathbf{z}^*{}_j - u_j|\}\right\} \quad (12)$$

A utopian point is usually used as the reference point $\mathbf{z}^*$. A utopian point is a point that is never dominated by any feasible solution in the objective space, for instance (0, 0) for a bi-dimensional objective space where all the objective values are greater than or equal to 0. The set of weight vectors is generated using the approach presented in [7], which places points on a normalized hyper-plane which is equally inclined to all objective axes and has an intercept of one on each axis (a $m-1$ dimensional unit simplex).

A lower $R2$ value indicates that an individual set is closer to the reference point. $R2 = 0$ when an individual is positioned on the reference point.

Similarly to hypervolume, a contributing $R2$ can be computed through Eq. (13) [5].

$$C_{R2}(\mathbf{u}, PF, W, \mathbf{z}^*) = R2(PF, W, \mathbf{z}^*) - R2(PF\backslash\{\mathbf{u}\}, W, \mathbf{z}^*) \quad (13)$$

where $C_{R2}(\mathbf{u}, PF, W, \mathbf{z}^*)$ reflects the influence of a single point on the quality of the approximation set.

4.4 Transfer weight functions

We define a transfer weight function (TWF) as a tuple composed by a ranking strategy and a weight function that determines a mapping from the "characteristics" of the ranked solutions to the weight that is used for learning.

In the previous two subsections we have discussed the two components of TWFs. Now we summarize the alternatives presented in this paper. We consider three ranking strategies:

1. CD: Based on the crowding distance
2. C_{HV}: Based on the contributing hypervolume measure
3. C_{R2}: Based on the contributing $R2$ measure

We have used three different weighting functions:

1. Linear
2. Logarithmic
3. Metric (value defined by the metric used for ranking)

A special case of TWF is when no preference is given to the solutions and all receive equal weights. In this case, the ranking strategy is not needed and the weighting function can

[4] When a Pareto set approximation dominates another, the indicator value of the former will be greater than the latter.

Table 1 Summary of the TWFs investigated

Variant	Metric	Weighting
Equal	n/a	Equal
Linear (CD)	Crowding distance	Linear
Linear (C_{R2})	R2	Linear
Linear (C_{HV})	Hypervolume	Linear
Log (CD)	Crowding distance	Logarithmic
Log (C_{R2})	R2	Logarithmic
Log (C_{HV})	Hypervolume	Logarithmic
Metric (CD)	Crowding distance	Metric
Metric (C_{R2})	R2	Metric
Metric (C_{HV})	Hypervolume	Metric

be simply defined as the constant function. The feasible combination of the ranking strategies and weighting functions produces ten different TWFs that are empirically evaluated in the section of experiments. All the TWFs investigated here are summarized in Table 1.

We emphasize that one of the contributions of the paper is introducing TWF as a means of manipulating the behavior of MO-CMA-ES, particularly in situations where all solutions are non-dominated. However, in addition to the TWFs discussed in this paper, we envision other possible strategies to define the weights in such a way that relevant information about the quality of the solutions is *directly* infused into the model.

5 Experiments

The main goal of our experiments is to investigate how a MO-CMA-ES which uses TWFs to incorporate information about the quality of the solutions in the computation of the model, behaves in comparison to classical models of updating the covariance matrix.

We also investigate a number of related questions: (1) What is the influence of the type of quality indicator used to rank the solutions? (2) What is the influence of the weighting function? (3) How the different variants of TWFs behave when the number of objectives is increased?

5.1 Experimental setup

This section presents the experimental framework used to evaluate and compare the algorithms.

5.1.1 Benchmark problems

In this paper we use the DTLZ [9] and WFG [11] benchmarks. These benchmark problems have been characterized according to different complexity criteria (e.g. bias, deception, multimodality, etc.). In addition, they can scale both in number of objectives and in the number of decision variables. Finally, the true Pareto optimal front for these functions is known.

A summary of the characteristics of the DTLZ and WFG problems used is shown in Table 2.

Table 2 Characteristics of the functions included in the DTLZ and WFG benchmarks [18]

Problem	Characteristics
DTLZ1	Linear, multi-modal
DTLZ2	Concave
DTLZ3	Concave, multi-modal
DTLZ4	Concave, biased
DTLZ5	Concave, degenerate
DTLZ6	Concave, degenerate, multi-modal
DTLZ7	Disconnected, multi-modal
WFG1	Mixed, biased
WFG2	Convex, disconnected, multi-modal, non-separable
WFG3	Linear, degenerate, non-separable
WFG4	Concave, multi-modal
WFG5	Concave, deceptive
WFG6	Concave, non-separable
WFG7	Concave, biased
WFG8	Concave, biased, non-separable
WFG9	Concave, biased, multi-modal, deceptive, non-separable

5.1.2 Performance metrics and statistical tests

To assess the quality of the Pareto fronts generated by each algorithm, we use a modified version of the inverted generational distance (IGD) known as IGD_p [23] and the hypervolume indicator (Eq. (10)) as quality indicators.

IGD is a widely used metric, especially in the many-objective community due to its low computational cost and its ability to measure the convergence and diversity of a solution set at the same time. The modified version used here and proposed in [23] makes the indicator more fair without altering its main properties.

The results measured with the quality indicators in 30 independent runs of the algorithms are submitted to the Kruskal–Wallis [16] statistical test at 5 % significance level. When significant differences were found we conducted a post-hoc analysis using the Nemenyi [19] test to identify particular differences between samples.

The comparison between the different TWFs is presented in tables showing the mean ranks of the indicators as used in the statistical tests. The final ranks, presented in parentheses, are assigned to the algorithms according to their mean ranks.

Table 3 Parameters of the functions and MO-CMA-ES

Number of objectives (m)	3, 5, 8, 10, 15 and 20
Number of decision variables for WFG	$k + l$ where $k = 2 \times (m - 1)$ and $l = 20$
Number of decision variables for DTLZ	$m + k - 1$ where $k = 5$ for DTLZ1, $k = 20$ for DTLZ7 and $k = 10$ otherwise
Initial population	100 solutions
Number of iterations	100
Repository maximum size	100 solutions

In case of a statistical tie (algorithms presenting no statistically significant difference), the final rank of each of the tied algorithms is equal to the average of the ranks that would be assigned to them. The algorithm(s) with the smallest final rankings are highlighted.

A summarized table is also presented containing the global results of the algorithms for all the problems of a benchmark family and all numbers of objectives. In the upper part of this table there is an $n_{alg} \times n_{alg}$ matrix showing how many times an algorithm i outperformed another algorithm j with statistically significant difference, where n_{alg} represents the number of CMA-ES variants being compared. In the lower part there are the titles and indexes of the aforementioned algorithms and a sum of how many times an algorithm outperformed any other with statistical difference.

5.1.3 Parameters of the algorithms

The basic parameters used for the MO-CMA-ES are summarized in Table 3.

5.2 Example of weight distribution according to each TWF

We start by showing an example of how the weights can be set for 100 ranked solutions using different TWFs through Fig. 1. These solutions were generated when optimizing the problem DTLZ2 with $m = 3$. In this figure we can see that each TWF distributes the weights differently across the solutions, by doing this, the influence of a solution to the update of the model can be increased or decreased, which impacts the model learned and consequently the entire search.

The strategies log, linear and equal always follow the same distribution since they do not consider the value of the metric to compose the weights. On the other hand C_{R2}, CD and C_{HV} can present different distributions based on the differences of the indicator values obtained by each solution. The larger the differences in weight between the solutions, the higher is the selection pressure towards the best ranked solutions. Equal does not introduce any selection pressure on the algorithm, and the selection pressure towards the best ranked solutions applied by the other weighting strategies can be expressed as: linear $<$ log $<$ metric.

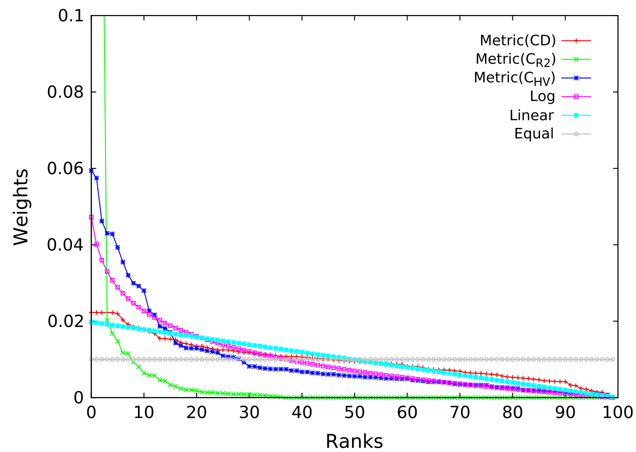

Fig. 1 Example of the computation of different weighting functions for a generation of MO-CMA-ES

5.3 Comparison between the algorithms up to 20 objectives

We now present the results of the experiments conducted using the two families of benchmark problems for 3, 5, 8, 10, 15 and 20 objectives. For these experiments, we used seven TWFs, six of them represent the combinations of the three weighting strategies with the two computationally cheap metrics CD and C_{R2}. The seventh TWF is equal and is added alone because it does not take into consideration the ranking methods. The TWFs using C_{HV} as metric were not compared here due to the high computational cost needed to calculate the exact hypervolume or to approximate it reliably for more than eight objectives.

The results obtained by the algorithms, measured by IGD and Hypervolume, are presented on Tables 4 and 5, respectively, for all the problems and numbers of objectives considered. In these tables, each column represents relative performance obtained by MO-CMA-ES using one of the TWFs and each line indicates a specific number of objectives of a problem. The relative performance of a TWF is presented as the mean rank obtained by the algorithm when considering the indicator values and in parentheses is presented the final ranking attributed according to the mean rankings and statistical differences detected by the Kruskal–Wallis test. The algorithm having the smallest final ranking is highlighted.

Since it can be hard to visualize the results in the general tables and the problem families present different characteristics, summarized results are presented in Tables 6 and 7 for

Table 4 Mean ranks of the IGD obtained by each MO-CMA-ES variant as used in the Kruskal–Wallis test for all problems up to 20 objectives

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Log (CD)	Log (C_{R2})	Metric (CD)	Metric (C_{R2})
DTLZ1	3	122.57 (4.00)	92.00 (4.00)	**81.60 (3.50)**	104.93 (4.00)	101.60 (4.00)	102.50 (4.00)	133.30 (4.50)
	5	111.57 (4.50)	124.73 (4.50)	108.43 (4.00)	111.20 (4.50)	94.90 (4.00)	125.43 (4.50)	**62.23 (2.00)**
	8	123.00 (4.50)	132.03 (4.50)	111.93 (4.50)	123.23 (4.50)	98.93 (4.50)	128.27 (4.50)	**21.10 (1.00)**
	10	127.03 (4.50)	133.50 (4.50)	115.23 (4.50)	116.17 (4.50)	96.50 (4.50)	132.20 (4.50)	**17.87 (1.00)**
	15	109.97 (4.50)	133.47 (4.50)	113.67 (4.50)	116.77 (4.50)	94.83 (4.00)	150.20 (5.00)	**19.60 (1.00)**
	20	86.00 (3.50)	124.90 (4.50)	119.73 (4.50)	133.83 (5.00)	100.83 (4.00)	155.80 (5.50)	**17.40 (1.00)**
DTLZ2	3	**110.83 (3.50)**	**91.37 (3.50)**	**66.47 (3.50)**	**103.73 (3.50)**	**69.57 (3.50)**	**101.03 (3.50)**	195.50 (7.00)
	5	100.43 (4.00)	131.07 (4.50)	96.87 (4.00)	134.50 (5.00)	85.70 (3.50)	122.90 (4.50)	**67.03 (2.50)**
	8	124.53 (5.00)	140.00 (5.00)	105.80 (4.50)	143.47 (5.00)	66.40 (2.00)	135.50 (5.00)	**22.80 (1.50)**
	10	127.73 (5.00)	128.23 (5.00)	105.30 (4.50)	152.90 (5.50)	**59.03 (1.50)**	141.67 (5.00)	**23.63 (1.50)**
	15	122.13 (5.00)	134.60 (5.00)	108.83 (4.50)	130.87 (5.00)	75.20 (2.50)	147.60 (5.00)	**19.27 (1.00)**
	20	128.07 (5.00)	148.43 (5.50)	91.53 (4.00)	131.77 (5.00)	81.30 (2.50)	136.60 (5.00)	**20.80 (1.00)**
DTLZ3	3	**85.87 (4.00)**	**99.87 (4.00)**	**104.57 (4.00)**	**115.63 (4.00)**	**105.63 (4.00)**	**105.53 (4.00)**	**121.40 (4.00)**
	5	82.67 (3.00)	122.57 (4.50)	130.13 (5.00)	110.47 (4.00)	135.80 (5.00)	91.07 (4.00)	**65.80 (2.50)**
	8	124.97 (4.50)	122.93 (4.50)	100.00 (4.50)	119.17 (4.50)	110.70 (4.50)	138.30 (4.50)	**22.43 (1.00)**
	10	124.60 (4.50)	132.80 (4.50)	116.50 (4.50)	115.60 (4.50)	88.17 (4.00)	141.80 (5.00)	**19.03 (1.00)**
	15	116.03 (4.50)	127.43 (4.50)	104.77 (4.50)	130.47 (4.50)	121.10 (4.50)	121.03 (4.50)	**17.67 (1.00)**
	20	104.07 (4.50)	118.53 (4.50)	119.83 (4.50)	121.40 (4.50)	116.07 (4.50)	141.50 (4.50)	**17.10 (1.00)**
DTLZ4	3	**95.22 (4.00)**	**126.40 (4.00)**	**90.23 (4.00)**	**121.13 (4.00)**	**85.13 (4.00)**	**92.02 (4.00)**	**128.37 (4.00)**
	5	103.17 (4.00)	91.27 (3.50)	**72.80 (3.00)**	136.80 (5.00)	**88.30 (3.00)**	102.20 (4.00)	143.97 (5.50)
	8	**72.10 (2.50)**	103.60 (3.50)	**76.60 (2.50)**	129.67 (5.50)	81.53 (3.00)	122.87 (5.00)	152.13 (6.00)
	10	**60.43 (2.00)**	125.03 (5.00)	86.77 (3.00)	133.07 (5.50)	74.83 (2.50)	116.20 (4.50)	142.17 (5.50)
	15	111.80 (4.00)	117.37 (4.00)	**77.60 (3.50)**	124.30 (4.50)	91.67 (4.00)	111.07 (4.00)	104.70 (4.00)
	20	87.93 (4.00)	129.63 (5.00)	**71.47 (2.00)**	132.00 (5.00)	**72.73 (2.00)**	123.77 (5.00)	120.97 (5.00)
DTLZ5	3	**76.73 (3.50)**	**95.33 (3.50)**	**86.20 (3.50)**	**92.30 (3.50)**	**80.10 (3.50)**	**112.33 (3.50)**	195.50 (7.00)
	5	143.03 (5.00)	110.00 (4.00)	102.47 (4.00)	116.47 (4.50)	**70.00 (3.00)**	82.23 (3.50)	114.30 (4.00)
	8	144.73 (5.50)	156.83 (6.00)	107.27 (4.00)	113.17 (4.50)	**56.23 (2.00)**	83.30 (3.00)	76.97 (3.00)
	10	165.03 (6.00)	133.90 (5.50)	124.10 (5.50)	108.83 (4.00)	**58.30 (2.00)**	74.27 (2.50)	74.07 (2.50)
	15	156.63 (6.00)	128.83 (5.00)	124.07 (4.50)	83.30 (3.50)	**66.10 (2.50)**	98.37 (3.50)	81.20 (3.00)
	20	184.00 (7.00)	123.13 (4.50)	122.60 (4.50)	**73.80 (2.50)**	**61.00 (2.50)**	78.60 (3.50)	95.37 (3.50)
DTLZ6	3	130.73 (5.00)	104.77 (4.50)	**55.60 (1.50)**	128.13 (5.00)	**50.57 (1.50)**	105.07 (4.50)	163.63 (6.00)
	5	113.23 (4.00)	135.97 (4.50)	105.57 (4.00)	103.07 (4.00)	102.93 (4.00)	108.57 (4.00)	**69.17 (3.50)**
	8	114.83 (4.50)	147.10 (5.50)	99.23 (4.00)	133.60 (5.00)	80.33 (3.00)	131.43 (5.00)	**31.97 (1.00)**
	10	122.67 (4.50)	133.93 (4.50)	111.37 (4.50)	110.43 (4.50)	105.23 (4.50)	130.73 (4.50)	**24.13 (1.00)**
	15	123.30 (4.50)	136.73 (4.50)	105.07 (4.50)	129.10 (4.50)	98.83 (4.50)	129.73 (4.50)	**15.73 (1.00)**
	20	121.97 (4.50)	127.00 (4.50)	104.50 (4.50)	134.67 (4.50)	104.57 (4.50)	130.23 (4.50)	**15.57 (1.00)**
DTLZ7	3	138.00 (5.50)	87.80 (3.50)	125.30 (5.00)	**63.00 (2.00)**	121.63 (4.50)	76.77 (2.50)	126.00 (5.00)
	5	132.10 (5.00)	74.67 (2.00)	123.70 (5.00)	97.27 (4.50)	123.00 (5.00)	**50.67 (1.50)**	137.10 (5.00)
	8	**76.83 (2.00)**	127.27 (5.50)	**45.50 (2.00)**	155.20 (5.50)	**55.00 (2.00)**	139.97 (5.50)	138.73 (5.50)
	10	95.47 (3.50)	150.93 (5.50)	76.37 (3.00)	117.20 (4.50)	**38.47 (1.50)**	153.20 (6.00)	106.87 (4.00)
	15	191.60 (7.00)	135.27 (5.50)	130.17 (5.50)	**70.03 (2.50)**	**51.03 (2.50)**	**78.57 (2.50)**	**81.83 (2.50)**
	20	193.13 (6.50)	**88.67 (3.00)**	**99.90 (3.00)**	**78.03 (3.00)**	**67.50 (3.00)**	**63.67 (3.00)**	147.60 (6.50)
WFG1	3	**86.17 (3.00)**	**93.53 (3.00)**	**59.20 (3.00)**	**95.17 (3.00)**	**63.83 (3.00)**	186.73 (6.50)	153.87 (6.50)
	5	**67.73 (2.50)**	87.07 (3.00)	76.80 (3.00)	117.43 (4.50)	96.73 (3.50)	152.90 (6.00)	139.83 (5.50)
	8	**60.70 (2.50)**	114.40 (4.50)	87.87 (3.50)	118.90 (4.50)	105.53 (4.00)	148.27 (5.00)	102.83 (4.00)
	10	**78.77 (3.00)**	95.70 (3.50)	101.77 (4.00)	129.33 (5.00)	112.90 (4.00)	144.20 (5.50)	**75.83 (3.00)**

Table 4 continued

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Log (CD)	Log (C_{R2})	Metric (CD)	Metric (C_{R2})
	15	110.07 (4.50)	111.43 (4.50)	106.20 (4.50)	123.90 (4.50)	138.53 (4.50)	101.13 (4.50)	**47.23** (**1.00**)
	20	103.83 (4.00)	100.37 (4.00)	109.90 (4.00)	115.30 (4.50)	120.33 (4.50)	121.77 (4.50)	**67.00** (**2.50**)
WFG2	3	**66.17** (**3.00**)	**70.37** (**3.00**)	**64.73** (**3.00**)	108.90 (3.50)	**97.87** (**3.00**)	148.50 (6.00)	181.97 (6.50)
	5	110.33 (4.00)	134.43 (5.00)	**45.53** (**1.50**)	99.60 (4.00)	81.57 (3.00)	94.37 (4.00)	172.67 (6.50)
	8	128.87 (5.00)	118.20 (5.00)	64.13 (2.00)	106.67 (4.50)	**56.37** (**1.50**)	132.17 (5.00)	132.10 (5.00)
	10	110.93 (4.00)	116.40 (4.00)	**75.23** (**3.50**)	117.60 (4.00)	**74.80** (**3.50**)	145.87 (5.50)	**97.67** (**3.50**)
	15	**67.27** (**3.00**)	102.80 (4.00)	105.83 (4.00)	146.47 (5.50)	96.83 (3.50)	138.00 (5.00)	**81.30** (**3.00**)
	20	94.53 (3.50)	136.27 (5.00)	94.53 (3.50)	145.40 (6.00)	83.43 (3.00)	151.73 (6.00)	**32.60** (**1.00**)
WFG3	3	78.37 (3.00)	105.93 (4.50)	**56.07** (**2.00**)	119.13 (4.50)	**48.53** (**2.00**)	144.60 (5.50)	185.87 (6.50)
	5	156.90 (5.50)	122.60 (5.00)	74.37 (3.00)	111.40 (4.50)	**27.10** (**1.00**)	137.40 (5.00)	108.73 (4.00)
	8	146.20 (5.50)	131.00 (5.00)	104.33 (4.00)	75.43 (3.00)	**68.53** (**2.50**)	93.30 (3.50)	119.70 (4.50)
	10	142.50 (5.50)	107.43 (4.00)	100.50 (4.00)	**93.67** (**3.00**)	**82.77** (**3.00**)	**67.40** (**3.00**)	144.23 (5.50)
	15	122.07 (4.50)	87.73 (3.50)	79.47 (3.50)	104.87 (3.50)	**72.67** (**3.00**)	114.67 (4.00)	157.03 (6.00)
	20	138.27 (4.50)	101.03 (4.00)	94.00 (4.00)	95.93 (4.00)	**70.13** (**2.50**)	117.30 (4.50)	121.83 (4.50)
WFG4	3	**110.90** (**3.50**)	**88.43** (**3.50**)	**77.43** (**3.50**)	**96.63** (**3.50**)	**73.83** (**3.50**)	**98.43** (**3.50**)	192.83 (7.00)
	5	85.37 (3.50)	90.57 (3.50)	**65.00** (**3.00**)	105.50 (3.50)	90.97 (3.50)	119.90 (4.00)	181.20 (7.00)
	8	**58.93** (**2.00**)	**50.80** (**2.00**)	**45.60** (**2.00**)	143.57 (5.50)	144.23 (5.50)	153.90 (5.50)	141.47 (5.50)
	10	**41.13** (**2.00**)	**61.90** (**2.00**)	**66.77** (**2.00**)	149.27 (5.50)	138.13 (5.50)	161.77 (5.50)	119.53 (5.50)
	15	**33.50** (**1.50**)	128.27 (5.00)	122.03 (5.00)	139.00 (5.00)	136.60 (5.00)	131.33 (5.00)	**47.77** (**1.50**)
	20	**39.37** (**1.50**)	101.87 (4.50)	120.60 (5.00)	154.40 (5.50)	141.80 (5.00)	145.23 (5.00)	**35.23** (**1.50**)
WFG5	3	**61.27** (**2.00**)	112.17 (4.50)	**42.90** (**2.00**)	152.20 (5.50)	85.67 (3.00)	116.57 (4.50)	167.73 (6.50)
	5	**71.80** (**2.50**)	96.00 (4.00)	86.67 (3.50)	135.87 (5.00)	98.77 (4.00)	129.57 (4.50)	119.83 (4.50)
	8	**43.77** (**1.00**)	108.57 (4.50)	103.40 (4.50)	130.40 (4.50)	110.70 (4.50)	130.33 (4.50)	111.33 (4.50)
	10	**58.53** (**2.00**)	102.13 (4.00)	110.47 (4.50)	125.17 (4.50)	114.37 (4.50)	146.13 (5.00)	81.70 (3.50)
	15	103.13 (4.50)	132.57 (4.50)	103.67 (4.50)	120.90 (4.50)	109.93 (4.50)	126.90 (4.50)	**41.40** (**1.00**)
	20	128.30 (4.50)	125.53 (4.50)	116.37 (4.50)	128.67 (4.50)	97.33 (4.50)	105.40 (4.50)	**36.90** (**1.00**)
WFG6	3	65.07 (3.00)	94.73 (3.50)	**56.77** (**2.50**)	130.23 (4.50)	91.70 (3.50)	104.90 (4.00)	195.10 (7.00)
	5	**92.37** (**3.50**)	**89.03** (**3.50**)	**61.67** (**3.50**)	**105.20** (**3.50**)	**94.13** (**3.50**)	**107.00** (**3.50**)	189.10 (7.00)
	8	**50.00** (**2.00**)	**58.73** (**2.00**)	**45.00** (**2.00**)	125.87 (5.00)	140.87 (5.50)	131.87 (5.00)	186.17 (6.50)
	10	**48.50** (**2.00**)	**47.60** (**2.00**)	**55.23** (**2.00**)	124.23 (5.00)	149.70 (5.50)	129.03 (5.00)	184.20 (6.50)
	15	**18.93** (**1.00**)	71.10 (3.00)	94.90 (3.50)	111.87 (4.00)	144.67 (5.50)	130.50 (5.00)	166.53 (6.00)
	20	**17.47** (**1.00**)	72.30 (3.00)	101.20 (4.00)	102.87 (4.00)	141.13 (5.00)	130.00 (5.00)	173.53 (6.00)
WFG7	3	**72.37** (**3.00**)	86.03 (3.50)	**61.47** (**3.00**)	129.83 (4.50)	93.43 (3.50)	99.90 (3.50)	195.47 (7.00)
	5	101.87 (3.50)	96.87 (3.50)	**72.20** (**3.00**)	121.47 (4.50)	**66.80** (**3.00**)	94.10 (3.50)	185.20 (7.00)
	8	135.60 (5.50)	**80.90** (**3.00**)	**73.63** (**3.00**)	**81.97** (**3.00**)	**76.13** (**3.00**)	96.07 (3.50)	194.20 (7.00)
	10	128.40 (4.50)	102.60 (3.50)	86.40 (3.50)	**72.20** (**3.00**)	**60.93** (**3.00**)	93.23 (3.50)	194.73 (7.00)
	15	123.03 (5.00)	105.67 (3.50)	**68.67** (**3.00**)	**74.33** (**3.00**)	**70.03** (**3.00**)	101.27 (3.50)	195.50 (7.00)
	20	105.57 (3.50)	**66.37** (**3.00**)	**64.70** (**3.00**)	92.90 (3.50)	131.60 (5.00)	**81.87** (**3.00**)	195.50 (7.00)
WFG8	3	73.23 (3.00)	91.40 (3.50)	**52.53** (**2.50**)	133.67 (5.00)	73.57 (3.00)	118.73 (4.00)	195.37 (7.00)
	5	**47.37** (**1.50**)	98.93 (4.00)	50.03 (2.00)	152.83 (6.50)	96.17 (3.50)	100.87 (4.00)	192.30 (6.50)
	8	**34.10** (**1.00**)	83.17 (3.50)	116.17 (4.50)	104.37 (4.00)	130.33 (5.00)	109.70 (4.00)	160.67 (6.00)
	10	**85.17** (**3.00**)	**73.63** (**3.00**)	**89.33** (**3.00**)	**81.30** (**3.00**)	138.07 (6.00)	100.40 (3.50)	170.60 (6.50)
	15	139.37 (5.50)	105.40 (4.00)	101.30 (3.50)	**58.30** (**2.50**)	89.20 (3.00)	76.70 (3.00)	168.23 (6.50)
	20	125.17 (4.00)	**102.50** (**3.50**)	**82.27** (**3.50**)	**85.07** (**3.50**)	**82.13** (**3.50**)	111.43 (4.00)	149.93 (6.00)
WFG9	3	**102.83** (**4.00**)	**91.37** (**4.00**)	**98.90** (**4.00**)	**97.80** (**4.00**)	**94.60** (**4.00**)	**122.10** (**4.00**)	**130.90** (**4.00**)
	5	**80.10** (**3.00**)	100.77 (4.00)	91.17 (4.00)	110.13 (4.00)	97.90 (4.00)	127.43 (4.50)	131.00 (4.50)
	8	**77.67** (**3.50**)	**86.23** (**3.50**)	**95.67** (**3.50**)	**116.00** (**3.50**)	**87.33** (**3.50**)	**108.37** (**3.50**)	167.23 (7.00)

Table 4 continued

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Log (CD)	Log (C_{R2})	Metric (CD)	Metric (C_{R2})
	10	**93.63 (3.50)**	**83.57 (3.50)**	**88.63 (3.50)**	**87.87 (3.50)**	**101.77 (3.50)**	**110.83 (3.50)**	172.20 (7.00)
	15	**76.60 (3.50)**	**93.90 (3.50)**	**89.43 (3.50)**	**119.37 (3.50)**	**80.17 (3.50)**	**110.03 (3.50)**	169.00 (7.00)
	20	87.40 (3.50)	110.53 (4.00)	88.60 (3.50)	117.53 (4.50)	**65.63 (2.50)**	115.67 (4.50)	153.13 (5.50)

Final ranks, presented in parentheses assigned according to mean ranks
The bold values refer to the smallest final rankings obtained in each group (problem/objective number), hence the algorithm variants whose values are highlighted are those who obtained the best results

Table 5 Mean ranks of the IGD obtained by each MO-CMA-ES variant as used in the Kruskal–Wallis test for all problems up to 20 objectives

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Log (CD)	Log (C_{R2})	Metric (CD)	Metric (C_{R2})
DTLZ1	3	92.90 (3.50)	**62.77 (3.00)**	**76.73 (3.00)**	97.20 (3.50)	137.37 (5.00)	**85.00 (3.00)**	186.53 (7.00)
	5	85.87 (3.50)	88.00 (3.50)	98.50 (3.50)	**64.00 (3.00)**	123.83 (4.00)	84.27 (3.50)	194.03 (7.00)
	8	**91.07 (3.00)**	120.30 (4.00)	**75.37 (3.00)**	**82.03 (3.00)**	**78.30 (3.00)**	138.97 (6.00)	152.47 (6.00)
	10	**94.50 (4.00)**	**101.27 (4.00)**	**100.55 (4.00)**	**98.92 (4.00)**	**97.90 (4.00)**	**114.62 (4.00)**	**130.75 (4.00)**
	15	**87.30 (4.00)**	**98.08 (4.00)**	**118.00 (4.00)**	**94.23 (4.00)**	**106.77 (4.00)**	**125.85 (4.00)**	**108.27 (4.00)**
	20	**85.43 (3.50)**	97.78 (4.00)	116.13 (4.00)	112.87 (4.00)	119.73 (4.00)	132.27 (5.00)	**74.28 (3.50)**
DTLZ2	3	98.43 (4.00)	85.70 (3.50)	**49.17 (2.00)**	119.10 (4.00)	87.00 (3.50)	103.60 (4.00)	195.50 (7.00)
	5	103.50 (4.00)	127.80 (4.50)	90.13 (4.00)	110.00 (4.00)	**71.63 (3.00)**	111.67 (4.00)	123.77 (4.50)
	8	117.17 (5.00)	151.30 (5.50)	101.20 (3.50)	123.07 (5.00)	**63.23 (2.00)**	126.40 (5.00)	**56.13 (2.00)**
	10	124.80 (5.00)	138.77 (5.00)	104.07 (5.00)	145.80 (5.00)	**38.93 (1.50)**	128.37 (5.00)	**57.77 (1.50)**
	15	132.07 (4.50)	141.58 (5.00)	106.02 (4.50)	105.53 (4.50)	87.53 (4.00)	129.93 (4.50)	**35.83 (1.00)**
	20	141.43 (4.50)	110.17 (4.50)	97.63 (4.00)	104.50 (4.00)	105.47 (4.00)	116.50 (4.50)	**62.80 (2.50)**
DTLZ3	3	**53.77 (3.00)**	**78.17 (3.00)**	**90.23 (3.00)**	99.80 (3.50)	143.53 (6.00)	**94.10 (3.00)**	178.90 (6.50)
	5	**53.20 (2.00)**	106.77 (4.00)	103.47 (4.00)	84.23 (3.00)	138.90 (5.50)	72.33 (3.00)	179.60 (6.50)
	8	**103.57 (4.00)**	**124.73 (4.00)**	**85.30 (4.00)**	**119.77 (4.00)**	**84.20 (4.00)**	**129.13 (4.00)**	**91.80 (4.00)**
	10	101.70 (4.00)	113.75 (4.00)	95.42 (3.50)	125.40 (4.50)	83.40 (3.50)	143.60 (5.50)	**75.23 (3.00)**
	15	101.10 (4.50)	131.13 (4.50)	112.65 (4.50)	118.63 (4.50)	110.17 (4.50)	132.07 (4.50)	**32.75 (1.00)**
	20	104.03 (4.50)	115.77 (4.50)	119.53 (4.50)	121.50 (4.50)	117.37 (4.50)	133.93 (4.50)	**26.37 (1.00)**
DTLZ4	3	**106.12 (4.00)**	**122.60 (4.00)**	**96.60 (4.00)**	**118.27 (4.00)**	**81.10 (4.00)**	**100.88 (4.00)**	**112.93 (4.00)**
	5	97.87 (3.50)	101.20 (3.50)	**69.20 (3.00)**	150.30 (6.50)	**80.27 (3.00)**	98.73 (3.50)	140.93 (5.00)
	8	**73.63 (3.00)**	104.50 (3.50)	**72.83 (3.00)**	128.10 (5.50)	**80.90 (3.00)**	113.30 (3.50)	165.23 (6.50)
	10	**52.87 (2.00)**	119.43 (4.50)	85.17 (3.50)	130.15 (5.00)	78.10 (3.00)	113.70 (4.50)	159.08 (5.50)
	15	103.83 (4.00)	114.05 (4.00)	**68.28 (3.00)**	130.17 (4.50)	92.47 (4.00)	98.25 (4.00)	131.45 (4.50)
	20	81.82 (2.50)	130.65 (5.50)	**63.07 (2.00)**	132.12 (5.50)	**75.33 (2.00)**	121.65 (5.00)	133.87 (5.50)
DTLZ5	3	110.47 (4.00)	89.57 (3.50)	73.90 (3.00)	93.03 (3.50)	**52.33 (2.50)**	123.70 (4.50)	195.50 (7.00)
	5	155.63 (5.50)	123.10 (5.00)	75.73 (2.50)	115.37 (4.50)	**50.73 (2.00)**	77.33 (3.00)	140.60 (5.50)
	8	153.80 (6.00)	160.50 (6.00)	94.80 (3.50)	131.10 (5.00)	**31.70 (1.50)**	107.30 (4.00)	59.30 (2.00)
	10	167.17 (6.00)	153.60 (5.50)	112.03 (4.50)	122.80 (5.00)	**33.60 (1.50)**	107.00 (4.00)	**42.30 (1.50)**
	15	161.33 (6.00)	140.50 (5.00)	109.50 (4.50)	112.27 (4.50)	**52.50 (1.50)**	134.73 (5.00)	**27.67 (1.50)**
	20	182.23 (6.50)	147.20 (5.00)	108.25 (4.50)	107.43 (4.50)	**52.27 (1.50)**	119.02 (4.50)	**22.10 (1.50)**
DTLZ6	3	125.60 (4.50)	97.87 (3.50)	**53.00 (2.00)**	126.43 (4.50)	**56.77 (2.00)**	103.10 (4.50)	175.73 (7.00)
	5	111.80 (4.00)	143.40 (5.00)	100.40 (4.00)	101.90 (4.00)	**89.23 (3.50)**	113.63 (4.00)	**78.13 (3.50)**
	8	105.37 (4.50)	142.90 (5.50)	93.90 (3.50)	134.10 (5.00)	70.10 (2.50)	135.20 (5.00)	**56.93 (2.00)**
	10	122.07 (4.50)	129.33 (4.50)	106.80 (4.50)	114.90 (4.50)	103.17 (4.50)	128.63 (4.50)	**33.60 (1.00)**
	15	124.23 (4.50)	127.20 (4.50)	109.20 (4.50)	123.00 (4.50)	108.20 (4.50)	131.13 (4.50)	**15.53 (1.00)**
	20	125.07 (4.50)	118.50 (4.50)	108.50 (4.50)	134.73 (4.50)	101.87 (4.50)	134.33 (4.50)	**15.50 (1.00)**

Table 5 continued

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Log (CD)	Log (C_{R2})	Metric (CD)	Metric (C_{R2})
DTLZ7	3	105.37 (4.00)	83.30 (3.50)	**68.40** (**3.00**)	112.63 (4.00)	104.70 (4.00)	116.13 (4.50)	147.97 (5.00)
	5	**67.23** (**2.50**)	147.93 (6.00)	**47.13** (**2.50**)	180.60 (6.00)	**49.90** (**2.50**)	152.60 (6.00)	**93.10** (**2.50**)
	8	132.90 (5.00)	131.77 (5.00)	113.53 (5.00)	142.33 (5.00)	64.00 (2.00)	136.33 (5.00)	**17.63** (**1.00**)
	10	121.00 (4.50)	139.10 (5.00)	115.10 (4.50)	136.57 (5.00)	76.83 (3.00)	133.30 (5.00)	**16.60** (**1.00**)
	15	147.95 (5.50)	129.15 (5.00)	94.85 (4.00)	126.68 (4.50)	81.03 (3.00)	140.00 (5.00)	**18.83** (**1.00**)
	20	106.92 (4.50)	133.05 (5.00)	119.98 (4.50)	135.72 (5.00)	74.28 (3.00)	141.67 (5.00)	**26.88** (**1.00**)
WFG1	3	**84.70** (**3.00**)	**89.07** (**3.00**)	**55.73** (**3.00**)	**97.67** (**3.00**)	**63.57** (**3.00**)	187.13 (6.50)	160.63 (6.50)
	5	**46.60** (**2.50**)	**81.03** (**2.50**)	**57.60** (**2.50**)	134.97 (6.00)	**80.30** (**2.50**)	181.17 (6.00)	156.83 (6.00)
	8	**29.80** (**1.50**)	130.73 (5.00)	62.70 (2.00)	140.57 (5.50)	118.50 (5.00)	164.17 (5.50)	92.03 (3.50)
	10	**32.37** (**1.50**)	120.82 (5.00)	93.13 (3.50)	153.38 (5.50)	112.75 (4.50)	164.60 (6.00)	61.45 (2.00)
	15	88.20 (3.50)	123.98 (4.50)	108.25 (4.50)	146.97 (5.00)	109.57 (4.50)	142.38 (5.00)	**19.15** (**1.00**)
	20	98.67 (4.50)	121.47 (4.50)	114.98 (4.50)	144.77 (4.50)	98.52 (4.50)	141.42 (4.50)	**18.68** (**1.00**)
WFG2	3	**52.40** (**2.00**)	72.83 (3.00)	**51.27** (**2.00**)	116.27 (4.50)	103.97 (4.00)	154.23 (6.00)	187.53 (6.50)
	5	78.10 (2.50)	108.50 (4.50)	**38.00** (**2.00**)	127.57 (5.00)	**60.50** (**2.00**)	146.80 (5.50)	179.03 (6.50)
	8	74.67 (2.50)	113.73 (4.00)	**53.97** (**2.00**)	131.20 (5.50)	69.90 (2.50)	164.07 (6.00)	130.97 (5.50)
	10	101.47 (4.00)	127.40 (4.50)	**47.70** (**2.00**)	147.60 (5.50)	55.37 (2.50)	176.50 (6.50)	82.47 (3.00)
	15	79.10 (2.50)	131.77 (5.50)	86.80 (3.50)	159.80 (6.00)	69.97 (2.50)	171.30 (6.00)	**39.77** (**2.00**)
	20	97.83 (3.00)	145.47 (6.00)	83.77 (3.00)	156.30 (6.00)	69.73 (3.00)	164.30 (6.00)	**21.10** (**1.00**)
WFG3	3	78.43 (3.00)	91.50 (3.00)	**52.03** (**2.50**)	122.23 (4.50)	**54.33** (**2.50**)	145.73 (5.50)	194.23 (7.00)
	5	122.53 (4.50)	97.27 (4.00)	**48.00** (**1.50**)	113.73 (4.50)	**25.73** (**1.50**)	151.03 (5.50)	180.20 (6.50)
	8	107.70 (4.50)	113.00 (4.50)	**54.30** (**1.50**)	139.47 (5.00)	**34.10** (**1.50**)	163.17 (6.00)	126.77 (5.00)
	10	125.77 (5.00)	126.37 (5.00)	59.17 (2.00)	143.23 (5.50)	**38.43** (**1.50**)	154.43 (5.50)	91.10 (3.50)
	15	115.97 (4.00)	117.70 (4.00)	94.20 (3.50)	142.63 (5.50)	75.53 (3.50)	167.67 (6.50)	**24.80** (**1.00**)
	20	115.18 (4.50)	126.27 (4.50)	100.20 (4.00)	139.15 (5.00)	81.70 (3.50)	159.33 (5.50)	**16.67** (**1.00**)
WFG4	3	**111.47** (**3.50**)	**89.37** (**3.50**)	**73.03** (**3.50**)	**102.30** (**3.50**)	**69.57** (**3.50**)	**98.20** (**3.50**)	194.57 (7.00)
	5	100.40 (3.50)	97.57 (3.50)	**57.43** (**2.50**)	110.27 (4.00)	81.80 (3.50)	108.93 (4.00)	182.10 (7.00)
	8	64.83 (2.50)	64.87 (2.50)	**48.70** (**2.00**)	142.53 (6.00)	96.07 (3.00)	176.13 (6.00)	145.37 (6.00)
	10	**39.43** (**2.00**)	69.30 (2.50)	**59.63** (**2.00**)	159.57 (5.50)	115.35 (4.50)	178.00 (6.50)	117.22 (5.00)
	15	**37.50** (**2.00**)	106.77 (3.50)	78.80 (3.00)	167.40 (6.50)	119.13 (4.00)	166.13 (6.50)	62.77 (2.50)
	20	**49.93** (**2.00**)	110.30 (3.50)	96.73 (3.50)	163.40 (6.50)	89.87 (3.00)	159.63 (6.50)	68.63 (3.00)
WFG5	3	**62.27** (**2.00**)	114.43 (4.50)	**41.80** (**2.00**)	152.20 (5.50)	77.67 (3.00)	115.10 (4.50)	175.03 (6.50)
	5	81.00 (3.00)	107.40 (4.50)	110.00 (4.50)	142.67 (5.00)	105.13 (4.50)	134.37 (5.00)	**57.93** (**1.50**)
	8	92.73 (4.00)	143.87 (5.00)	111.53 (4.50)	125.30 (4.50)	105.80 (4.50)	127.30 (4.50)	**31.97** (**1.00**)
	10	101.30 (4.50)	106.37 (4.50)	108.87 (4.50)	127.20 (4.50)	125.53 (4.50)	141.43 (4.50)	**27.80** (**1.00**)
	15	116.87 (4.50)	118.13 (4.50)	118.00 (4.50)	119.60 (4.50)	111.53 (4.50)	138.87 (4.50)	**15.50** (**1.00**)
	20	135.03 (4.50)	109.75 (4.50)	118.02 (4.50)	130.03 (4.50)	106.72 (4.50)	123.45 (4.50)	**15.50** (**1.00**)
WFG6	3	60.03 (3.00)	94.93 (3.50)	**55.23** (**2.50**)	134.87 (4.50)	94.37 (3.50)	103.63 (4.00)	195.43 (7.00)
	5	104.00 (3.50)	96.90 (3.50)	**58.10** (**3.00**)	101.47 (3.50)	**67.93** (**3.00**)	115.20 (4.50)	194.90 (7.00)
	8	**51.93** (**2.00**)	80.63 (2.50)	**37.20** (**2.00**)	146.80 (6.00)	98.97 (3.50)	150.90 (6.00)	172.07 (6.00)
	10	**66.68** (**2.50**)	**83.70** (**2.50**)	**45.60** (**2.50**)	161.80 (6.00)	**76.92** (**2.50**)	163.37 (6.00)	140.43 (6.00)
	15	134.57 (5.00)	113.57 (5.00)	**54.60** (**1.50**)	143.50 (5.00)	**47.13** (**1.50**)	139.10 (5.00)	106.03 (5.00)
	20	127.67 (5.00)	114.17 (5.00)	**60.57** (**1.50**)	120.10 (5.00)	**44.57** (**1.50**)	137.37 (5.00)	134.07 (5.00)
WFG7	3	61.20 (3.00)	84.10 (3.00)	**56.67** (**2.50**)	135.83 (5.00)	98.03 (3.50)	107.17 (4.00)	195.50 (7.00)
	5	102.77 (4.00)	100.43 (4.00)	**51.67** (**2.00**)	123.03 (4.50)	69.13 (3.00)	96.67 (3.50)	194.80 (7.00)
	8	65.33 (2.50)	82.33 (2.50)	**55.03** (**2.00**)	133.60 (5.50)	132.13 (5.50)	105.33 (4.00)	164.73 (6.00)
	10	**54.13** (**1.50**)	100.60 (4.00)	79.23 (3.50)	151.47 (5.50)	118.63 (4.50)	123.17 (4.50)	111.27 (4.50)
	15	126.83 (5.00)	145.57 (5.50)	92.43 (3.00)	143.33 (5.50)	56.87 (2.00)	147.00 (5.50)	**26.47** (**1.50**)
	20	137.42 (5.50)	160.35 (5.50)	86.27 (2.50)	139.13 (5.50)	46.23 (2.00)	143.00 (5.50)	**26.10** (**1.50**)

Table 5 continued

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Log (CD)	Log (C_{R2})	Metric (CD)	Metric (C_{R2})
WFG8	3	**70.90 (2.50)**	92.03 (3.50)	**50.07 (2.50)**	137.30 (5.00)	75.40 (3.00)	117.30 (4.50)	195.50 (7.00)
	5	81.73 (3.00)	112.67 (4.50)	**48.83 (2.00)**	153.23 (6.00)	**44.30 (2.00)**	104.73 (4.00)	193.00 (6.50)
	8	**34.80 (1.00)**	113.60 (4.00)	101.03 (4.00)	165.23 (6.50)	84.70 (3.50)	145.50 (5.50)	93.63 (3.50)
	10	69.10 (2.50)	144.03 (5.50)	71.00 (2.50)	158.37 (6.00)	98.80 (3.50)	161.53 (6.00)	**35.67 (2.00)**
	15	115.13 (5.00)	139.83 (5.00)	110.57 (5.00)	146.27 (5.00)	**57.37 (1.50)**	151.30 (5.00)	**18.03 (1.50)**
	20	121.02 (5.00)	143.78 (5.50)	81.20 (3.00)	154.87 (5.50)	56.37 (2.00)	164.00 (5.50)	**17.27 (1.50)**
WFG9	3	**89.20 (3.50)**	97.20 (4.00)	94.17 (4.00)	104.93 (4.00)	93.73 (4.00)	123.33 (4.00)	135.93 (4.50)
	5	**65.30 (2.50)**	117.10 (4.50)	99.80 (4.00)	131.37 (4.50)	86.80 (3.50)	136.20 (5.00)	101.93 (4.00)
	8	104.00 (4.00)	139.30 (5.50)	89.37 (3.00)	166.23 (6.00)	71.77 (2.50)	139.70 (5.50)	**28.13 (1.50)**
	10	88.50 (3.50)	133.90 (5.00)	108.57 (4.50)	154.47 (5.50)	82.40 (3.00)	146.10 (5.50)	**24.57 (1.00)**
	15	81.87 (3.00)	133.63 (5.00)	108.40 (4.00)	147.63 (5.50)	90.03 (3.50)	158.83 (6.00)	**18.10 (1.00)**
	20	101.90 (4.50)	134.50 (4.50)	97.13 (3.50)	146.20 (5.50)	92.17 (3.50)	145.13 (5.50)	**21.47 (1.00)**

Final ranks, presented in parentheses assigned according to mean ranks

The bold values refer to the smallest final rankings obtained in each group (problem/objective number), hence the algorithm variants whose values are highlighted are those who obtained the best results

Table 6 Summarized table of the IGD results obtained by each MO-CMA-ES variant for the DTLZ problem up to 20 objectives

Alg.	1	2	3	4	5	6	7
1	n/a	5	4	6	13	9	22
2	3	n/a	7	2	15	3	24
3	1	1	n/a	3	6	4	19
4	4	0	8	n/a	16	1	18
5	1	1	0	1	n/a	1	15
6	5	0	5	0	12	n/a	19
7	5	7	10	4	10	6	n/a

Alg. #	Outperformed the others	Title
1	19	Equal
2	14	Linear (CD)
3	34	Linear (C_{R2})
4	16	Log (CD)
5	72	Log (C_{R2})
6	24	Metric (CD)
7	117	Metric (C_{R2})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

Table 7 Summarized table of the Hypervolume results obtained by each MO-CMA-ES variant for the DTLZ problem up to 20 objectives

Alg.	1	2	3	4	5	6	7
1	n/a	0	8	2	11	4	18
2	4	n/a	6	0	17	2	20
3	1	0	n/a	0	6	0	14
4	5	1	7	n/a	17	1	19
5	2	2	2	2	n/a	3	10
6	4	0	8	1	17	n/a	21
7	11	9	15	8	12	9	n/a

Alg. #	Outperformed the others	Title
1	27	Equal
2	12	Linear (CD)
3	46	Linear (C_{R2})
4	13	Log (CD)
5	80	Log (C_{R2})
6	19	Metric (CD)
7	102	Metric (C_{R2})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

the DTLZ family of problems and in Tables 8 and 9 for the WFG family.

When considering the DTLZ problems, the first feature that can be appreciated in the data presented is that, in general, other TWFs produce better results than equal. This behavior can be easily seen in the summarized Tables 6 and 7, where all the TWFs based on the C_{R2} indicator outperformed other TWFs more times than equal. In a more detailed analysis based on Tables 4 and 5, we can see that equal was among the best algorithms only in 18 out of 84 cases, from which 13 cases were for eight or fewer objectives, where the proportion of non-dominated solutions from the population is small.

Considering the ranking strategies, in all cases the TWFs based on the C_{R2} indicator outperformed other algorithms more times than those based on the CD indicator. In a more detailed view we can see that for more than five objectives,

Table 8 Summarized table of the IGD results obtained by each MO-CMA-ES variant for the WFG problem up to 20 objectives

Alg.	1	2	3	4	5	6	7
1	n/a	1	5	6	10	3	5
2	9	n/a	6	2	6	0	6
3	7	0	n/a	0	1	0	6
4	22	6	14	n/a	10	1	9
5	13	9	7	1	n/a	1	7
6	23	10	19	1	10	n/a	11
7	33	31	33	28	29	23	n/a

Alg. #	Outperformed the others	Title
1	107	Equal
2	57	Linear (CD)
3	84	Linear (C_{R2})
4	38	Log (CD)
5	66	Log (C_{R2})
6	28	Metric (CD)
7	44	Metric (C_{R2})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

Table 9 Summarized table of the Hypervolume results obtained by each MO-CMA-ES variant for the WFG problem up to 20 objectives

Alg.	1	2	3	4	5	6	7
1	n/a	0	8	0	9	0	17
2	13	n/a	18	0	16	0	21
3	3	0	n/a	0	1	0	19
4	29	11	38	n/a	33	1	27
5	9	1	6	0	n/a	0	16
6	29	15	39	1	36	n/a	28
7	25	20	24	12	23	11	n/a

Alg. #	Outperformed the others	Title
1	108	Equal
2	47	Linear (CD)
3	133	Linear (C_{R2})
4	13	Log (CD)
5	118	Log (C_{R2})
6	12	Metric (CD)
7	128	Metric (C_{R2})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

in the few times a TWF based on CD is among the best, it is statistically tied with another one based on C_{R2}, which indicates that CD does not scale well with the number of objectives.

Comparing the weighting functions, we can identify a pattern where the number of times they outperform others (using the same ranking strategy) can be expressed as linear $<$ log $<$ metric. Again in more detail, this trend becomes more evident as the number of objectives increase.

In summary, among the TWFs compared, the most indicated for many-objective instances of the DTLZ problems is the metric(C_{R2}).

Next we present our considerations with respect to the results obtained for the WFG problems. From the summarized IGD table, we can appreciate that the equal TWF outperformed the other algorithms more times, followed by linear(C_{R2}) and log(C_{R2}). Considering the HV, this trend is reversed, and all the TWFs using C_{R2} as ranking method outperformed the other algorithms more times than equal. Another trend identified is that the weighting strategies that introduce less selection pressure in the search present good results when considering a low number of objectives, but when this number increases their performance deteriorate. This can be seen in both Tables 4 and 5 where equal and linear appear among the best algorithms more times for three and five objectives, while metric(C_{R2}) is usually among the best for fifteen and twenty objectives.

5.4 Comparison between the algorithms up to 8 objectives

For eight or fewer objectives, it is possible to calculate exactly C_{HV} as a metric for ranking the algorithms in a reasonable time, hence, we conducted experiments to compare the behavior of this rank strategy associated with the three weighting strategies that consider the ranking (linear, log and metric) to the previous results for 3, 5 and 8 objectives. We hypothesize that by using a powerful indicator such as C_{HV}, the results of the weighting functions that introduce more selection pressure in the model will improve, following the trend seen in the last section, where C_{R2} in general performed better than CD.

Tables 10 and 11 presents the extended comparison results measured by the IGD and HV indicators respectively for all problems. The summarized results for the DTLZ family are presented in Tables 12 and 13 for the IGD and HV. For the WFG family the summarized results are presented in Tables 14 and 15

First, we introduce the results obtained for the DTLZ problems, where we can observe in the tables that equal never outperformed other TWFs more times than any strategy using C_{HV}. Moreover, equal is seldom among the best algorithms, just a total of 8 times considering both indicators, from which on two cases all the algorithms tied, so it can not be considered "better" than any other.

Comparing the algorithms in terms of ranking strategy, the number of times the TWFs outperformed others can be

Table 10 Mean ranks of the IGD obtained by each MO-CMA-ES variant as used in the Kruskal–Wallis test for all problems up to 20 objectives

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Linear (C_{HV})	Log (CD)	Log (C_{R2})	Log (C_{HV})	Metric (CD)	Metric (C_{R2})	Metric (C_{HV})
DTLZ1	3	171.77 (5.50)	129.47 (5.50)	**114.60 (5.00)**	145.60 (5.50)	146.30 (5.50)	143.07 (5.50)	141.17 (5.50)	144.50 (5.50)	187.27 (6.00)	181.27 (5.50)
	5	161.77 (5.50)	180.20 (6.50)	157.60 (5.50)	148.77 (5.50)	161.90 (5.50)	138.83 (5.50)	188.17 (6.50)	181.20 (6.50)	**94.03 (4.00)**	**92.53 (4.00)**
	8	196.93 (7.00)	207.10 (7.00)	182.80 (6.50)	138.87 (6.50)	196.50 (7.00)	164.23 (6.50)	125.57 (4.50)	202.20 (7.00)	**47.93 (1.50)**	**42.87 (1.50)**
DTLZ2	3	176.93 (6.00)	147.50 (5.00)	110.83 (5.00)	132.83 (5.00)	165.63 (5.00)	115.83 (5.00)	**102.37 (4.50)**	163.13 (5.00)	285.50 (10.00)	**104.43 (4.50)**
	5	189.87 (7.00)	220.93 (7.00)	186.30 (7.00)	**76.73 (2.00)**	224.17 (7.00)	173.37 (7.00)	**43.00 (2.00)**	212.57 (7.00)	155.03 (7.00)	**23.03 (2.00)**
	8	210.30 (7.50)	227.20 (8.00)	188.17 (7.00)	133.73 (4.50)	231.00 (8.00)	143.23 (5.00)	**50.20 (2.00)**	221.77 (8.00)	81.37 (3.00)	**18.03 (2.00)**
DTLZ3	3	**115.00 (5.00)**	134.63 (5.50)	140.57 (5.50)	166.53 (5.50)	156.03 (5.50)	142.20 (5.50)	148.97 (5.50)	141.90 (5.50)	165.30 (5.50)	193.87 (6.00)
	5	124.53 (4.50)	188.73 (6.50)	197.70 (7.00)	143.00 (5.50)	173.40 (6.50)	203.40 (7.00)	142.40 (5.50)	135.90 (5.50)	**96.13 (3.50)**	**99.80 (3.50)**
	8	205.57 (7.00)	203.03 (7.00)	172.67 (7.00)	141.20 (6.00)	196.27 (7.00)	187.77 (7.00)	83.30 (2.50)	219.90 (7.50)	**53.07 (2.00)**	**42.23 (2.00)**
DTLZ4	3	**134.43 (5.50)**	**179.68 (5.50)**	**126.30 (5.50)**	**155.97 (5.50)**	**170.28 (5.50)**	**120.40 (5.50)**	**144.30 (5.50)**	**129.83 (5.50)**	**184.18 (5.50)**	**159.62 (5.50)**
	5	145.20 (5.50)	129.60 (5.00)	**98.67 (4.00)**	145.63 (5.50)	195.83 (6.50)	122.97 (4.50)	145.67 (5.50)	145.30 (5.50)	202.80 (7.00)	173.33 (6.00)
	8	**107.23 (3.50)**	152.60 (5.50)	113.93 (4.50)	108.30 (4.00)	188.67 (7.00)	120.70 (5.00)	132.33 (5.00)	181.20 (6.50)	220.93 (8.00)	179.10 (6.00)
DTLZ5	3	**85.67 (3.50)**	109.40 (4.00)	97.50 (4.00)	161.17 (5.00)	103.50 (4.00)	90.67 (4.00)	185.40 (7.50)	131.77 (4.50)	285.50 (9.50)	254.43 (9.00)
	5	227.70 (8.00)	187.33 (6.50)	178.87 (6.50)	142.43 (5.50)	193.13 (6.50)	137.40 (5.50)	73.17 (2.50)	150.87 (6.00)	191.73 (6.50)	**22.37 (1.50)**
	8	233.17 (8.50)	246.23 (8.50)	193.87 (7.00)	108.97 (4.00)	198.67 (7.00)	135.27 (5.50)	45.00 (2.00)	159.90 (5.50)	155.20 (5.50)	**28.73 (1.50)**
DTLZ6	3	210.83 (8.00)	178.47 (7.50)	**106.73 (3.00)**	114.70 (4.00)	207.67 (8.00)	**98.37 (3.00)**	**88.80 (3.00)**	180.20 (7.50)	248.93 (8.00)	**70.30 (3.00)**
	5	200.67 (7.00)	225.57 (7.50)	193.37 (7.00)	97.87 (2.50)	189.67 (7.00)	190.30 (7.00)	**36.57 (2.00)**	196.33 (7.00)	145.03 (6.00)	**29.63 (2.00)**
	8	188.17 (7.00)	228.50 (7.50)	170.17 (7.00)	185.83 (7.00)	211.90 (7.00)	144.37 (6.00)	**72.73 (2.00)**	208.63 (7.00)	74.13 (2.50)	**20.57 (2.00)**
DTLZ7	3	183.10 (7.00)	109.67 (4.50)	161.87 (6.00)	172.73 (6.50)	**75.50 (2.00)**	159.27 (6.00)	173.13 (6.50)	95.23 (3.00)	166.13 (6.50)	208.37 (7.00)
	5	143.97 (5.00)	75.40 (3.50)	131.00 (4.50)	206.13 (8.00)	97.93 (4.00)	128.63 (4.50)	255.80 (9.00)	**50.67 (2.00)**	155.80 (5.50)	259.67 (9.00)
	8	98.80 (4.00)	151.77 (6.00)	**64.53 (2.50)**	**71.03 (2.50)**	181.27 (6.50)	**73.37 (2.50)**	270.80 (9.50)	165.10 (6.00)	163.07 (6.00)	265.27 (9.50)
WFG1	3	**129.47 (4.00)**	**141.20 (4.00)**	**91.33 (4.00)**	**93.67 (4.00)**	**143.07 (4.00)**	**97.83 (4.00)**	**88.33 (4.00)**	273.90 (9.00)	226.23 (9.00)	219.97 (9.00)
	5	131.07 (4.50)	154.93 (6.00)	137.70 (4.50)	106.60 (4.00)	195.03 (7.00)	167.27 (6.50)	**77.37 (3.00)**	237.90 (8.50)	217.43 (8.00)	**79.70 (3.00)**
	8	**101.73 (4.00)**	177.37 (6.00)	139.63 (5.00)	121.97 (5.00)	183.87 (6.50)	165.63 (5.50)	109.67 (4.50)	222.90 (8.00)	160.00 (5.50)	122.23 (5.00)

Table 10 continued

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Linear (C_{HV})	Log (CD)	Log (C_{R2})	Log (C_{HV})	Metric (CD)	Metric (C_{R2})	Metric (C_{HV})
WFG2	3	91.43 (3.50)	96.93 (3.50)	90.03 (3.50)	**72.57** (**3.00**)	146.73 (5.00)	132.70 (4.50)	174.93 (7.00)	198.70 (7.50)	244.53 (8.50)	256.43 (9.00)
	5	148.43 (5.50)	179.53 (6.00)	**63.37** (**2.50**)	69.67 (3.00)	134.57 (5.00)	111.63 (4.50)	175.20 (6.00)	127.97 (4.50)	235.67 (8.50)	258.97 (9.50)
	8	198.17 (7.50)	185.60 (7.50)	110.60 (3.00)	**69.73** (**2.50**)	170.30 (6.50)	102.10 (3.00)	**73.17** (**2.50**)	201.80 (7.50)	200.30 (7.50)	193.23 (7.50)
WFG3	3	141.33 (5.00)	173.90 (6.50)	112.43 (4.00)	72.13 (3.00)	189.53 (7.00)	99.93 (3.50)	**54.43** (**2.50**)	220.27 (8.00)	270.60 (9.50)	170.43 (6.00)
	5	244.60 (8.00)	207.07 (7.50)	151.30 (5.50)	63.60 (2.50)	194.63 (7.00)	82.80 (3.00)	**23.30** (**2.00**)	223.10 (8.00)	190.17 (7.00)	124.43 (4.50)
	8	218.67 (7.50)	201.70 (7.00)	170.10 (6.50)	66.37 (2.50)	136.13 (5.00)	130.43 (4.50)	**17.67** (**1.50**)	158.27 (6.50)	188.87 (6.50)	216.80 (7.50)
WFG4	3	176.83 (6.00)	147.90 (5.50)	132.13 (5.00)	86.10 (3.00)	157.57 (6.00)	126.27 (5.00)	**66.77** (**2.50**)	162.87 (6.00)	281.33 (10.00)	167.23 (6.00)
	5	127.73 (4.50)	134.27 (4.50)	**101.17** (**4.00**)	**95.37** (**4.00**)	153.93 (4.50)	135.07 (4.50)	**98.03** (**4.00**)	173.77 (6.50)	254.87 (9.50)	230.80 (9.00)
	8	**73.23** (**2.50**)	**64.67** (**2.50**)	**58.63** (**2.50**)	**81.87** (**2.50**)	198.00 (7.50)	197.27 (7.50)	188.53 (7.50)	212.13 (7.50)	190.17 (7.50)	240.50 (7.50)
WFG5	3	110.27 (4.00)	169.23 (6.50)	86.33 (3.00)	**50.17** (**2.50**)	216.47 (8.00)	136.60 (4.50)	71.90 (3.00)	174.70 (6.50)	234.13 (8.00)	255.20 (9.00)
	5	131.10 (5.00)	161.67 (6.00)	152.63 (6.00)	**38.90** (**1.50**)	213.43 (7.00)	168.57 (6.50)	92.87 (3.50)	204.00 (7.00)	189.73 (6.50)	152.10 (6.00)
	8	76.30 (2.00)	166.20 (6.50)	159.60 (6.50)	**70.93** (**1.50**)	196.00 (6.50)	169.57 (6.50)	144.03 (6.00)	196.00 (6.50)	168.93 (6.50)	157.43 (6.50)
WFG6	3	113.90 (4.00)	150.80 (5.50)	101.70 (4.00)	**59.33** (**2.50**)	195.47 (7.00)	145.57 (5.50)	**57.50** (**2.50**)	163.33 (5.50)	283.07 (9.50)	234.33 (9.00)
	5	151.87 (5.50)	146.97 (5.50)	116.10 (4.50)	**52.07** (**2.00**)	165.23 (5.50)	151.10 (5.50)	**47.47** (**2.00**)	167.30 (6.00)	269.87 (9.50)	237.03 (9.00)
	8	**59.53** (**2.50**)	**69.97** (**2.50**)	**53.73** (**2.50**)	89.37 (3.00)	160.20 (6.00)	184.60 (6.50)	198.57 (7.50)	168.50 (6.50)	256.70 (9.00)	263.83 (9.00)
WFG7	3	130.40 (5.50)	143.57 (5.50)	115.83 (4.50)	**32.27** (**1.50**)	193.10 (6.50)	152.10 (5.50)	46.17 (2.00)	160.20 (5.50)	282.83 (9.50)	248.53 (9.00)
	5	165.50 (6.00)	161.93 (6.00)	131.33 (5.50)	**50.47** (**1.50**)	190.67 (6.00)	124.33 (5.50)	65.03 (2.50)	157.53 (6.00)	269.07 (10.00)	189.13 (6.00)
	8	161.47 (5.50)	98.13 (4.00)	**89.83** (**3.50**)	**85.67** (**3.50**)	99.97 (4.00)	92.20 (4.00)	229.67 (8.50)	116.20 (4.00)	256.97 (9.00)	274.90 (9.00)
WFG8	3	130.97 (5.50)	152.90 (5.50)	104.87 (3.50)	**43.03** (**2.00**)	201.10 (6.50)	130.60 (5.50)	**45.37** (**2.00**)	184.27 (6.50)	284.17 (9.50)	227.73 (8.50)
	5	95.03 (4.00)	160.17 (6.00)	97.97 (4.00)	**42.67** (**2.50**)	225.07 (7.50)	155.87 (6.00)	**66.83** (**2.50**)	162.03 (6.00)	277.67 (9.00)	221.70 (7.50)
	8	66.63 (2.50)	136.20 (5.00)	178.77 (7.00)	**50.23** (**2.00**)	160.37 (6.00)	194.97 (7.00)	79.50 (2.50)	169.77 (7.00)	233.57 (8.00)	235.00 (8.00)
WFG9	3	163.93 (5.50)	147.07 (5.50)	156.07 (5.50)	**107.03** (**4.50**)	152.17 (5.50)	147.00 (5.50)	118.67 (5.00)	186.90 (6.00)	198.37 (6.50)	127.80 (5.50)
	5	152.10 (6.50)	178.07 (7.00)	166.63 (7.00)	**76.27** (**2.00**)	188.07 (7.00)	172.23 (7.00)	86.37 (2.50)	209.70 (7.00)	209.97 (7.00)	**65.60** (**2.00**)
	8	132.27 (5.00)	141.77 (5.00)	155.63 (5.00)	**88.70** (**4.00**)	182.07 (6.50)	144.80 (5.00)	**94.07** (**4.00**)	172.13 (6.00)	248.27 (9.50)	145.30 (5.00)

Final ranks, presented in parentheses assigned according to mean ranks

The bold values refer to the smallest final rankings obtained in each group (problem/objective number), hence the algorithm variants whose values are highlighted are those who obtained the best results

Table 11 Mean ranks of the hypervolume obtained by each MO-CMA-ES variant as used in the Kruskal–Wallis test for all problems up to 20 objectives

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Linear (C_{HV})	Log (CD)	Log (C_{R2})	Log (C_{HV})	Metric (CD)	Metric (C_{R2})	Metric (C_{HV})
DTLZ1	3	110.87 (4.00)	**73.77** (**3.50**)	**92.37** (**3.50**)	138.13 (4.50)	116.67 (4.00)	171.40 (5.50)	190.03 (7.50)	101.03 (4.00)	262.80 (9.50)	247.93 (9.00)
	5	106.37 (4.50)	113.10 (4.50)	125.37 (4.50)	123.00 (4.50)	**80.07** (**3.50**)	162.17 (5.00)	168.00 (5.00)	105.83 (4.50)	274.17 (9.50)	246.93 (9.50)
	8	155.93 (6.00)	188.60 (6.50)	133.67 (4.50)	**81.57** (**3.00**)	140.67 (4.50)	136.87 (4.50)	**71.93** (**3.00**)	215.27 (8.00)	227.97 (9.00)	152.53 (6.00)
DTLZ2	3	176.57 (6.50)	154.73 (6.00)	108.60 (4.00)	93.83 (3.50)	199.33 (7.00)	157.80 (6.00)	**67.43** (**2.50**)	182.27 (7.00)	285.50 (10.00)	**78.93** (**2.50**)
	5	192.37 (7.00)	217.37 (7.00)	178.47 (7.00)	**72.77** (**2.00**)	199.00 (7.00)	158.00 (7.00)	**40.17** (**2.00**)	200.53 (7.00)	213.57 (7.00)	**32.77** (**2.00**)
	8	205.37 (7.00)	240.90 (8.00)	187.37 (7.00)	98.37 (3.50)	210.37 (7.50)	143.60 (6.00)	41.87 (2.00)	213.83 (7.50)	138.30 (5.00)	**25.03** (**1.50**)
DTLZ3	3	**62.10** (**3.00**)	92.40 (3.50)	107.10 (3.50)	136.83 (4.50)	119.57 (4.00)	181.17 (7.00)	208.50 (8.50)	111.53 (4.00)	239.50 (8.50)	246.30 (8.50)
	5	**75.00** (**3.50**)	149.37 (5.00)	145.10 (4.50)	99.40 (4.00)	118.67 (4.00)	190.93 (7.50)	106.80 (4.00)	103.17 (4.00)	248.33 (9.00)	268.23 (9.50)
	8	175.87 (6.50)	200.47 (6.50)	152.47 (6.50)	**75.77** (**1.50**)	195.43 (6.50)	149.83 (6.50)	**30.43** (**1.50**)	207.17 (6.50)	161.37 (6.50)	156.20 (6.50)
DTLZ4	3	**141.90** (**5.50**)	**167.78** (**5.50**)	**127.80** (**5.50**)	**166.03** (**5.50**)	**164.45** (**5.50**)	**108.23** (**5.50**)	**162.73** (**5.50**)	**135.37** (**5.50**)	**156.82** (**5.50**)	**173.88** (**5.50**)
	5	130.93 (5.00)	136.60 (5.00)	**91.30** (**4.00**)	151.83 (5.50)	208.53 (8.00)	**110.83** (**4.00**)	155.87 (5.50)	133.50 (5.00)	196.83 (6.50)	188.77 (6.50)
	8	**108.20** (**4.00**)	152.13 (5.00)	**106.90** (**4.00**)	**106.50** (**4.00**)	186.57 (7.00)	118.87 (5.00)	131.57 (5.00)	166.60 (5.00)	239.60 (9.00)	188.07 (7.00)
DTLZ5	3	160.53 (5.00)	130.00 (4.50)	106.93 (4.00)	91.13 (4.00)	133.97 (4.50)	**74.03** (**3.50**)	107.57 (4.50)	177.90 (6.50)	285.50 (9.50)	237.43 (9.00)
	5	245.37 (8.50)	208.93 (7.50)	156.73 (6.00)	82.63 (2.50)	198.80 (7.50)	124.10 (4.00)	**35.90** (**2.00**)	157.93 (6.50)	228.73 (8.00)	65.87 (2.50)
	8	243.80 (8.00)	250.50 (8.00)	184.30 (7.50)	86.33 (3.00)	221.10 (8.00)	108.70 (3.50)	**17.33** (**2.00**)	194.70 (7.50)	139.77 (5.00)	58.47 (2.50)
DTLZ6	3	203.90 (8.00)	166.93 (5.50)	99.80 (3.50)	111.67 (4.00)	206.17 (8.00)	104.90 (3.50)	104.33 (3.50)	175.87 (7.00)	262.03 (9.00)	**69.40** (**3.00**)
	5	201.37 (7.00)	232.97 (7.50)	189.43 (7.00)	**85.90** (**2.00**)	190.93 (7.00)	177.37 (7.00)	**33.17** (**2.00**)	203.17 (7.00)	161.20 (6.50)	**29.50** (**2.00**)
	8	185.13 (7.00)	229.40 (8.00)	170.70 (6.50)	145.20 (5.00)	219.00 (8.00)	141.10 (5.00)	65.10 (2.00)	219.77 (8.00)	110.93 (4.00)	**18.67** (**1.50**)
DTLZ7	3	163.53 (5.50)	135.43 (5.00)	117.43 (5.00)	**95.90** (**3.50**)	168.37 (6.50)	159.30 (5.50)	**96.60** (**3.50**)	171.53 (6.50)	209.53 (7.50)	187.37 (6.50)
	5	123.13 (4.00)	232.07 (8.50)	92.17 (3.50)	**52.53** (**3.00**)	270.40 (9.00)	94.80 (3.50)	**73.60** (**3.00**)	241.47 (9.00)	167.37 (6.50)	157.47 (5.00)
	8	222.07 (8.00)	221.07 (8.00)	202.37 (7.50)	111.03 (4.00)	231.87 (8.00)	146.47 (4.50)	**29.87** (**2.00**)	225.73 (8.00)	77.80 (3.00)	**36.73** (**2.00**)
WFG1	3	**126.33** (**4.00**)	**133.47** (**4.00**)	**86.37** (**4.00**)	**98.83** (**4.00**)	**145.33** (**4.00**)	**96.50** (**4.00**)	**82.23** (**4.00**)	275.40 (9.00)	235.10 (9.00)	225.43 (9.00)
	5	101.70 (4.00)	147.47 (5.00)	114.50 (4.00)	**62.53** (**2.50**)	219.67 (9.00)	146.50 (5.00)	**62.93** (**2.50**)	270.03 (9.00)	243.57 (9.00)	136.10 (5.00)
	8	**52.27** (**3.00**)	211.97 (8.00)	111.23 (3.50)	95.63 (3.50)	223.43 (8.50)	192.33 (8.00)	101.57 (3.50)	249.73 (8.50)	152.10 (5.00)	114.73 (3.50)

Table 11 continued

Prob.	Obj.	Equal	Linear (CD)	Linear (C_{R2})	Linear (C_{HV})	Log (CD)	Log (C_{R2})	Log (C_{HV})	Metric (CD)	Metric (C_{R2})	Metric (C_{HV})
WFG2	3	75.27 (3.00)	102.03 (4.00)	74.70 (3.00)	**61.53 (2.50)**	157.13 (6.00)	142.17 (5.00)	168.90 (6.00)	205.17 (7.50)	253.23 (9.00)	264.87 (9.00)
	5	117.83 (4.00)	154.03 (5.50)	**60.37 (2.50)**	**55.80 (2.50)**	176.93 (6.00)	92.50 (3.50)	136.40 (5.50)	200.43 (7.50)	248.60 (9.00)	262.10 (9.00)
	8	125.20 (4.50)	178.40 (6.50)	92.70 (3.00)	**51.03 (2.50)**	201.23 (8.00)	115.40 (4.00)	114.80 (4.00)	244.20 (8.00)	196.93 (8.00)	185.10 (6.50)
WFG3	3	147.27 (6.00)	159.93 (6.00)	115.07 (4.00)	61.87 (2.50)	192.47 (7.00)	116.07 (4.00)	**28.27 (1.50)**	216.53 (7.50)	275.70 (9.50)	191.83 (7.00)
	5	199.37 (7.50)	169.27 (6.50)	112.00 (3.50)	50.67 (2.50)	189.43 (7.00)	82.30 (2.50)	**16.80 (2.00)**	233.17 (7.50)	266.97 (9.00)	185.03 (7.00)
	8	196.90 (7.50)	202.63 (7.50)	139.80 (5.00)	46.67 (2.50)	229.23 (8.00)	114.67 (3.50)	**16.07 (1.50)**	253.13 (8.00)	214.10 (8.00)	91.80 (3.50)
WFG4	3	185.23 (6.00)	158.93 (6.00)	134.07 (5.50)	79.77 (3.00)	172.60 (6.00)	128.87 (5.50)	**47.13 (1.50)**	169.50 (6.00)	284.23 (10.00)	144.67 (5.50)
	5	160.73 (6.00)	157.30 (6.00)	101.00 (4.00)	**71.83 (2.50)**	173.77 (6.50)	134.33 (5.00)	**68.23 (2.50)**	170.37 (6.00)	268.03 (9.50)	199.40 (7.00)
	8	116.80 (4.50)	116.13 (4.50)	93.13 (3.50)	**39.30 (2.00)**	221.00 (8.00)	161.50 (6.00)	81.23 (3.00)	263.67 (9.00)	227.60 (8.00)	184.63 (6.50)
WFG5	3	113.67 (4.00)	171.63 (6.50)	86.00 (3.00)	**45.03 (2.50)**	217.20 (8.00)	128.93 (4.50)	69.77 (3.00)	173.93 (6.50)	241.40 (8.00)	257.43 (9.00)
	5	153.10 (5.50)	184.60 (7.00)	188.83 (7.50)	**37.90 (2.00)**	226.27 (8.00)	184.67 (7.00)	92.13 (3.00)	216.07 (7.50)	117.77 (4.50)	103.67 (3.00)
	8	150.77 (6.00)	219.67 (7.50)	178.40 (6.50)	105.97 (3.50)	197.27 (7.50)	169.47 (6.00)	114.87 (4.00)	198.87 (7.50)	**56.60 (2.50)**	113.13 (4.00)
WFG6	3	114.67 (4.50)	156.77 (5.50)	108.03 (4.00)	46.03 (2.50)	203.97 (7.00)	153.77 (5.50)	**42.27 (2.00)**	165.53 (6.00)	284.37 (9.50)	229.60 (8.50)
	5	169.43 (6.00)	161.90 (6.00)	117.90 (5.50)	**46.50 (1.50)**	167.07 (6.00)	128.53 (5.50)	**32.43 (1.50)**	181.73 (6.00)	282.03 (9.50)	217.47 (7.50)
	8	112.20 (4.50)	150.70 (5.00)	93.77 (3.00)	**29.80 (2.00)**	231.53 (8.00)	175.20 (6.50)	52.77 (2.50)	235.97 (8.00)	257.30 (9.00)	165.77 (6.50)
WFG7	3	**117.83 (5.50)**	**141.83 (5.50)**	**110.53 (5.50)**	**26.60 (5.50)**	**198.43 (5.50)**	**157.10 (5.50)**	**49.93 (5.50)**	**166.77 (5.50)**	**284.67 (5.50)**	**251.30 (5.50)**
	5	**174.27 (5.50)**	**171.70 (5.50)**	**115.10 (5.50)**	**29.70 (5.50)**	**196.33 (5.50)**	**136.80 (5.50)**	**36.17 (5.50)**	**167.73 (5.50)**	**283.60 (5.50)**	**193.60 (5.50)**
	8	136.30 (5.00)	155.30 (6.00)	121.90 (5.00)	**36.17 (1.50)**	212.50 (7.50)	211.30 (7.50)	**35.17 (1.50)**	180.97 (6.50)	250.27 (8.50)	165.13 (6.00)
WFG8	3	130.90 (5.00)	153.23 (5.50)	105.50 (3.50)	**35.13 (2.00)**	203.63 (7.00)	134.67 (5.50)	**42.20 (2.00)**	181.93 (6.50)	284.23 (9.50)	233.57 (8.50)
	5	**151.47 (5.50)**	**189.27 (5.50)**	**113.33 (5.50)**	**40.40 (5.50)**	**237.80 (5.50)**	**109.03 (5.50)**	**21.03 (5.50)**	**180.43 (5.50)**	**283.03 (5.50)**	**179.20 (5.50)**
	8	117.53 (4.50)	201.60 (7.50)	189.97 (7.50)	**39.43 (2.00)**	254.57 (8.50)	173.83 (6.50)	**21.57 (2.00)**	234.07 (7.50)	183.10 (6.50)	89.33 (2.50)
WFG9	3	151.83 (5.50)	156.93 (5.50)	152.03 (5.50)	**93.67 (4.50)**	161.83 (5.50)	148.00 (5.50)	**117.90 (4.50)**	189.43 (6.50)	204.37 (7.00)	129.00 (5.00)
	5	144.37 (6.00)	203.33 (7.00)	184.13 (7.00)	**65.57 (2.00)**	218.20 (7.50)	169.40 (7.00)	**59.27 (2.00)**	224.50 (7.50)	181.80 (7.00)	**54.43 (2.00)**
	8	192.40 (7.50)	228.97 (7.50)	177.80 (7.00)	**85.80 (2.50)**	256.03 (8.50)	159.20 (6.50)	**34.30 (2.50)**	228.83 (7.50)	101.63 (3.00)	**40.03 (2.50)**

Final ranks, presented in parentheses assigned according to mean ranks

The bold values refer to the smallest final rankings obtained in each group (problem/objective number), hence the algorithm variants whose values are highlighted are those who obtained the best results

Table 12 Summarized table of the IGD results obtained by each MO-CMA-ES variant for the DTLZ problem up to 8 objectives

Alg.	1	2	3	4	5	6	7	8	9	10
1	n/a	1	1	6	1	3	10	4	5	10
2	0	n/a	2	5	0	5	9	1	8	11
3	1	0	n/a	3	1	0	7	1	5	9
4	1	1	1	n/a	2	1	2	2	3	6
5	2	0	4	7	n/a	4	9	0	5	10
6	1	0	0	2	1	n/a	6	1	3	9
7	3	3	3	1	4	3	n/a	3	4	2
8	1	0	2	6	0	3	9	n/a	5	10
9	3	4	7	6	3	6	8	4	n/a	6
10	5	4	4	2	4	3	0	4	2	n/a

Alg. #	Outperformed the others	Title
1	17	Equal
2	13	Linear (CD)
3	24	Linear (C_{R2})
4	38	Linear (C_{HV})
5	16	Log (CD)
6	28	Log (C_{R2})
7	60	Log (C_{HV})
8	20	Metric (CD)
9	40	Metric (C_{R2})
10	73	Metric (C_{HV})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

Table 13 Summarized table of the Hypervolume results obtained by each MO-CMA-ES variant for the DTLZ problem up to 8 objectives

Alg.	1	2	3	4	5	6	7	8	9	10
1	n/a	0	2	10	0	5	11	1	3	9
2	2	n/a	1	10	0	6	11	0	5	10
3	0	0	n/a	7	0	1	8	0	1	7
4	1	0	0	n/a	0	0	2	0	0	3
5	3	1	5	13	n/a	7	12	1	5	10
6	2	2	2	4	2	n/a	10	1	0	6
7	2	2	2	1	3	0	n/a	2	0	0
8	1	0	5	13	1	7	13	n/a	4	10
9	8	9	13	15	7	10	16	8	n/a	9
10	6	5	7	10	5	5	7	4	0	n/a

Alg. #	Outperformed the others	Title
1	25	Equal
2	19	Linear (CD)
3	37	Linear (C_{R2})
4	83	Linear (C_{HV})
5	18	Log (CD)
6	41	Log (C_{R2})
7	90	Log (C_{HV})
8	17	Metric (CD)
9	18	Metric (C_{R2})
10	64	Metric (C_{HV})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

expressed as CD $< C_{R2} < C_{HV}$ in both tables, which indicates that the use of a more powerful metric had a good impact on the results. This trend becomes more evident as the number of objectives increases.

When considering the weighting functions, the results obtained using the IGD indicator present clear patterns, where in general it is best to use the metric approach, especially in the many-objective scenario. If we consider the results measured by the HV indicator, the patterns regarding the weighting function are less clear, possibly due to the higher influence of the C_{HV} as ranking method, where in many cases different weighting functions using this indicator are statistically tied.

Next, we present the results obtained for the WFG problems, where both summarized Tables 14 and 15 show a better performance for the TWFs linear(C_{HV}) and log(C_{HV}) over the others for these problems. The same trend can be seen in the general tables, where for all the numbers of objectives, these two TWFs are highlighted most of the times. These results indicate that using a powerful metric to rank the solutions can, in fact, improve the performance of MO-CMA-ES.

Regarding the weighting strategies, the number of times any of them outperformed the other TWFs can be expressed as linear > log > metric considering both indicators.

5.5 Comparison with the state-of-the-art

To validate our approaches, in this section we compare the results obtained by MO-CMA-ES using the best performing TWFs from previous sections with the state-of-the-art NSGA-III [7] algorithm. We selected weighting strategies linear and metric, since each of them was the best in one family of problems, and the ranking strategies C_{R2} and C_{HV}, that were the best for up to 20 and up to 8 objectives respectively.

As in the previous sections, we present the results in two sets. The first is composed of experiments up to 20 objectives without the TWFs ranked by C_{HV} whose results evaluated by IGD and HV are presented in Tables 16A and B respectively. The second set of results considering all TWFs presented and including experiments up to 8 objectives is shown in Tables 17A and B for the IGD and HV indicators.

Table 14 Summarized table of the IGD results obtained by each MO-CMA-ES variant for the WFG problem up to 8 objectives

Alg.	1	2	3	4	5	6	7	8	9	10
1	n/a	0	4	12	1	3	9	0	0	2
2	2	n/a	3	16	0	4	14	0	0	3
3	2	0	n/a	8	0	0	4	0	0	1
4	0	0	0	n/a	0	0	0	0	0	0
5	9	2	9	19	n/a	3	18	0	0	2
6	4	2	2	13	0	n/a	10	0	0	2
7	3	4	5	6	1	1	n/a	1	0	0
8	10	5	12	25	1	4	20	n/a	0	4
9	20	16	19	26	14	18	20	12	n/a	6
10	15	13	14	21	9	13	17	6	0	n/a

Alg. #	Outperformed the others	Title
1	65	Equal
2	42	Linear (CD)
3	68	Linear (C_{R2})
4	146	Linear (C_{HV})
5	26	Log (CD)
6	46	Log (C_{R2})
7	112	Log (C_{HV})
8	19	Metric (CD)
9	0	Metric (C_{R2})
10	20	Metric (C_{HV})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

Table 15 Summarized table of the Hypervolume results obtained by each MO-CMA-ES variant for the WFG problem up to 8 objectives

Alg.	1	2	3	4	5	6	7	8	9	10
1	n/a	0	1	15	0	2	12	0	2	3
2	2	n/a	4	21	0	2	18	0	2	7
3	1	0	n/a	8	0	0	10	0	3	4
4	0	0	0	n/a	0	0	0	0	0	0
5	13	3	16	22	n/a	9	20	0	5	8
6	2	0	3	14	0	n/a	15	0	1	5
7	1	0	2	2	0	0	n/a	0	0	0
8	10	5	13	24	1	8	22	n/a	4	9
9	16	14	17	21	10	15	20	5	n/a	10
10	6	6	13	15	3	9	16	1	0	n/a

Alg. #	Outperformed the others	Title
1	51	Equal
2	28	Linear (CD)
3	69	Linear (C_{R2})
4	142	Linear (C_{HV})
5	14	Log (CD)
6	45	Log (C_{R2})
7	133	Log (C_{HV})
8	6	Metric (CD)
9	17	Metric (C_{R2})
10	46	Metric (C_{HV})

The upper part shows the number of times the algorithm outperformed the others. The lower part shows the sum of the times an algorithm outperformed all the others

In an overall comparison, NSGA-III outperforms MO-CMA-ES in most of the problems investigated, however in a closer look some interesting patterns of behavior can be observed.

As seen in the previous sections, the TWFs that rank the solutions using C_{HV} present more competitive results to NSGA-III than those that rank the solutions using C_{R2}.

When observing the behavior of the algorithms per problem, a pattern emerges, where MO-CMA-ES is able to outperform NSGA-III more frequently in problems DTLZ5, DTLZ6, DTLZ7, WFG1, WFG2 and WFG3. These six problems share the characteristic of presenting Pareto front shapes different from the rest of the problems, since DTLZ5, DTLZ6 and WFG3 are degenerate, DTLZ7 and WFG2 are disconnected and WFG1 is mixed (concave, convex). This type of problem can be challenging to algorithms based on reference points or weight vectors like NSGA-III. Unlike NSGA-III, MO-CMA-ES is only based on Pareto dominance, so it can concentrate the efforts in a small area for degenerate or disconnected problems or spread the solutions in rugged objective spaces of mixed problems without making any assumptions based on predefined reference points.

Table 16 Mean ranks of the IGD and hypervolume obtained by each MO-CMA-ES variant and NSGA-III as used in the Kruskal–Wallis test for the problems up to 20 objectives

Prob.	Obj.	Linear (C_{R2})	Metric (C_{R2})	NSGA-III
(A) IGD ranks				
DTLZ1	3	53.67 (2.50)	67.33 (2.50)	**15.50 (1.00)**
	5	67.50 (2.50)	53.50 (2.50)	**15.50 (1.00)**
	8	74.47 (3.00)	46.53 (2.00)	**15.50 (1.00)**
	10	75.20 (3.00)	45.80 (2.00)	**15.50 (1.00)**
	15	75.20 (3.00)	45.80 (2.00)	**15.50 (1.00)**
	20	75.20 (3.00)	45.80 (2.00)	**15.50 (1.00)**
DTLZ2	3	**17.73 (1.00)**	75.50 (3.00)	43.27 (2.00)
	5	59.87 (2.50)	61.13 (2.50)	**15.50 (1.00)**
	8	61.80 (3.00)	**34.57 (1.50)**	**40.13 (1.50)**
	10	68.27 (3.00)	42.07 (2.00)	**26.17 (1.00)**
	15	72.77 (3.00)	**35.93 (1.50)**	**27.80 (1.50)**
	20	71.33 (3.00)	48.27 (2.00)	**16.90 (1.00)**
DTLZ3	3	58.00 (2.50)	63.00 (2.50)	**15.50 (1.00)**
	5	69.70 (3.00)	51.30 (2.00)	**15.50 (1.00)**
	8	72.93 (3.00)	48.07 (2.00)	**15.50 (1.00)**
	10	74.93 (3.00)	46.07 (2.00)	**15.50 (1.00)**

Table 16 continued

Prob.	Obj.	Linear (C_{R2})	Metric (C_{R2})	NSGA-III
	15	74.80 (3.00)	46.17 (2.00)	**15.53** (**1.00**)
	20	75.17 (3.00)	45.83 (2.00)	**15.50** (**1.00**)
DTLZ4	3	**43.22** (**2.00**)	**53.58** (**2.00**)	**39.70** (**2.00**)
	5	48.33 (2.00)	72.33 (3.00)	**15.83** (**1.00**)
	8	54.20 (2.50)	65.63 (2.50)	**16.67** (**1.00**)
	10	58.50 (2.50)	62.50 (2.50)	**15.50** (**1.00**)
	15	60.67 (2.50)	58.77 (2.50)	**17.07** (**1.00**)
	20	67.63 (2.50)	53.37 (2.50)	**15.50** (**1.00**)
DTLZ5	3	**15.50** (**1.00**)	69.63 (3.00)	51.37 (2.00)
	5	**33.20** (**1.50**)	**40.07** (**1.50**)	63.23 (3.00)
	8	**37.00** (**1.50**)	**28.43** (**1.50**)	71.07 (3.00)
	10	39.40 (2.00)	**23.13** (**1.00**)	73.97 (3.00)
	15	**36.33** (**1.50**)	**24.70** (**1.50**)	75.47 (3.00)
	20	**35.37** (**1.50**)	**25.63** (**1.50**)	75.50 (3.00)
DTLZ6	3	**17.80** (**1.00**)	64.57 (2.50)	54.13 (2.50)
	5	64.53 (3.00)	45.10 (2.00)	**26.87** (**1.00**)
	8	62.53 (2.50)	**24.77** (**1.00**)	49.20 (2.50)
	10	63.67 (2.50)	**18.00** (**1.00**)	54.83 (2.50)
	15	57.87 (2.50)	**15.60** (**1.00**)	63.03 (2.50)
	20	58.27 (2.50)	**15.57** (**1.00**)	62.67 (2.50)
DTLZ7	3	59.47 (2.50)	60.73 (2.50)	**16.30** (**1.00**)
	5	47.97 (2.50)	56.93 (2.50)	**31.60** (**1.00**)
	8	**21.83** (**1.00**)	39.20 (2.00)	75.47 (3.00)
	10	**29.13** (**1.50**)	**31.97** (**1.50**)	75.40 (3.00)
	15	38.60 (2.00)	**22.40** (**1.00**)	75.50 (3.00)
	20	**22.53** (**1.00**)	38.47 (2.00)	75.50 (3.00)
WFG1	3	**17.80** (**1.00**)	43.20 (2.00)	75.50 (3.00)
	5	**21.03** (**1.00**)	39.97 (2.00)	75.50 (3.00)
	8	**25.67** (**1.50**)	**35.33** (**1.50**)	75.50 (3.00)
	10	**31.70** (**1.50**)	**29.30** (**1.50**)	75.50 (3.00)
	15	39.23 (2.00)	**21.77** (**1.00**)	75.50 (3.00)
	20	39.33 (2.00)	**21.67** (**1.00**)	75.50 (3.00)
WFG2	3	**23.70** (**1.00**)	72.30 (3.00)	40.50 (2.00)
	5	**33.40** (**1.00**)	50.60 (2.50)	52.50 (2.50)
	8	**21.90** (**1.00**)	51.00 (2.50)	63.60 (2.50)
	10	**35.67** (**1.50**)	46.67 (2.00)	54.17 (2.50)
	15	62.53 (3.00)	**37.47** (**1.50**)	**36.50** (**1.50**)
	20	73.67 (3.00)	**27.23** (**1.50**)	**35.60** (**1.50**)
WFG3	3	**24.23** (**1.50**)	74.53 (3.00)	**37.73** (**1.50**)
	5	48.67 (2.50)	58.07 (2.50)	**29.77** (**1.00**)
	8	55.97 (2.50)	52.33 (2.50)	**28.20** (**1.00**)
	10	**41.07** (**2.00**)	**47.40** (**2.00**)	**48.03** (**2.00**)
	15	64.90 (3.00)	47.50 (2.00)	**24.10** (**1.00**)
	20	68.87 (3.00)	**35.13** (**1.50**)	**32.50** (**1.50**)
WFG4	3	45.03 (2.00)	75.50 (3.00)	**15.97** (**1.00**)
	5	44.13 (2.00)	**26.63** (**1.00**)	65.73 (3.00)
	8	52.10 (2.50)	**23.80** (**1.00**)	60.60 (2.50)
	10	**40.67** (**2.00**)	**46.27** (**2.00**)	**49.57** (**2.00**)

Table 16 continued

Prob.	Obj.	Linear (C_{R2})	Metric (C_{R2})	NSGA-III
	15	51.03 (2.00)	67.37 (3.00)	**18.10** (**1.00**)
	20	59.13 (2.50)	61.70 (2.50)	**15.67** (**1.00**)
WFG5	3	45.50 (2.00)	75.50 (3.00)	**15.50** (**1.00**)
	5	73.57 (3.00)	47.43 (2.00)	**15.50** (**1.00**)
	8	72.27 (3.00)	**34.77** (**1.50**)	**29.47** (**1.50**)
	10	71.10 (3.00)	44.60 (2.00)	**20.80** (**1.00**)
	15	75.50 (3.00)	41.33 (2.00)	**19.67** (**1.00**)
	20	75.50 (3.00)	44.80 (2.00)	**16.20** (**1.00**)
WFG6	3	45.37 (2.00)	75.50 (3.00)	**15.63** (**1.00**)
	5	45.53 (2.00)	75.47 (3.00)	**15.50** (**1.00**)
	8	**17.40** (**1.00**)	45.17 (2.00)	73.93 (3.00)
	10	**17.90** (**1.00**)	50.73 (2.00)	67.87 (3.00)
	15	**26.23** (**1.50**)	74.77 (3.00)	**35.50** (**1.50**)
	20	45.50 (2.00)	75.50 (3.00)	**15.50** (**1.00**)
WFG7	3	45.13 (2.00)	75.50 (3.00)	**15.87** (**1.00**)
	5	61.47 (2.50)	59.53 (2.50)	**15.50** (**1.00**)
	8	48.27 (2.00)	**34.20** (**1.50**)	54.03 (2.50)
	10	67.13 (3.00)	44.20 (2.00)	**25.17** (**1.00**)
	15	63.70 (2.50)	54.97 (2.50)	**17.83** (**1.00**)
	20	71.10 (3.00)	49.90 (2.00)	**15.50** (**1.00**)
WFG8	3	45.50 (2.00)	75.50 (3.00)	**15.50** (**1.00**)
	5	56.57 (2.50)	64.43 (2.50)	**15.50** (**1.00**)
	8	60.67 (2.50)	**29.97** (**1.00**)	45.87 (2.50)
	10	60.20 (2.50)	44.60 (2.00)	**31.70** (**1.50**)
	15	54.07 (2.50)	64.93 (2.50)	**17.50** (**1.00**)
	20	56.43 (2.50)	64.57 (2.50)	**15.50** (**1.00**)
WFG9	3	46.07 (2.00)	72.30 (3.00)	**18.13** (**1.00**)
	5	60.00 (2.50)	61.00 (2.50)	**15.50** (**1.00**)
	8	58.90 (2.50)	**32.60** (**1.50**)	45.00 (2.00)
	10	70.00 (3.00)	46.70 (2.00)	**19.80** (**1.00**)
	15	61.60 (2.50)	55.40 (2.50)	**19.50** (**1.00**)
	20	57.27 (2.50)	63.73 (2.50)	**15.50** (**1.00**)
(B) Hypervolume ranks				
DTLZ1	3	47.03 (2.00)	73.97 (3.00)	**15.50** (**1.00**)
	5	45.57 (2.00)	75.43 (3.00)	**15.50** (**1.00**)
	8	52.67 (2.50)	68.00 (2.50)	**15.83** (**1.00**)
	10	60.53 (2.50)	60.47 (2.50)	**15.50** (**1.00**)
	15	66.83 (2.50)	54.17 (2.50)	**15.50** (**1.00**)
	20	72.13 (3.00)	48.87 (2.00)	**15.50** (**1.00**)
DTLZ2	3	45.50 (2.00)	75.50 (3.00)	**15.50** (**1.00**)
	5	56.10 (2.50)	64.90 (2.50)	**15.50** (**1.00**)
	8	68.87 (3.00)	52.13 (2.00)	**15.50** (**1.00**)
	10	68.67 (3.00)	52.27 (2.00)	**15.57** (**1.00**)
	15	72.23 (3.00)	48.63 (2.00)	**15.63** (**1.00**)
	20	65.77 (2.50)	55.23 (2.50)	**15.50** (**1.00**)
DTLZ3	3	47.70 (2.00)	73.30 (3.00)	**15.50** (**1.00**)
	5	48.43 (2.00)	72.57 (3.00)	**15.50** (**1.00**)
	8	59.83 (2.50)	61.17 (2.50)	**15.50** (**1.00**)

Table 16 continued

Prob.	Obj.	Linear (C_{R2})	Metric (C_{R2})	NSGA-III
	10	64.77 (2.50)	56.23 (2.50)	**15.50 (1.00)**
	15	72.43 (3.00)	48.57 (2.00)	**15.50 (1.00)**
	20	73.63 (3.00)	47.37 (2.00)	**15.50 (1.00)**
DTLZ4	3	52.85 (2.50)	58.38 (2.50)	**25.27 (1.00)**
	5	50.73 (2.00)	70.27 (3.00)	**15.50 (1.00)**
	8	**36.20 (1.50)**	71.57 (3.00)	**28.73 (1.50)**
	10	49.47 (2.00)	70.33 (3.00)	**16.70 (1.00)**
	15	**35.50 (1.50)**	57.63 (2.50)	43.37 (2.00)
	20	**40.43 (1.50)**	65.87 (3.00)	**30.20 (1.50)**
DTLZ5	3	**15.50 (1.00)**	73.43 (3.00)	47.57 (2.00)
	5	48.97 (2.00)	70.13 (3.00)	**17.40 (1.00)**
	8	64.90 (3.00)	**41.53 (1.50)**	**30.07 (1.50)**
	10	69.90 (3.00)	**34.70 (1.50)**	**31.90 (1.50)**
	15	55.90 (2.50)	**17.00 (1.00)**	63.60 (2.50)
	20	52.10 (2.00)	**16.17 (1.00)**	68.23 (3.00)
DTLZ6	3	**23.07 (1.00)**	71.87 (3.00)	41.57 (2.00)
	5	64.43 (2.50)	56.57 (2.50)	**15.50 (1.00)**
	8	68.00 (3.00)	45.83 (2.00)	**22.67 (1.00)**
	10	73.27 (3.00)	**39.50 (1.50)**	**23.73 (1.50)**
	15	75.33 (3.00)	**19.30 (1.00)**	41.87 (2.00)
	20	75.50 (3.00)	**26.47 (1.50)**	**34.53 (1.50)**
DTLZ7	3	51.13 (2.00)	69.87 (3.00)	**15.50 (1.00)**
	5	41.90 (2.00)	70.23 (3.00)	**24.37 (1.00)**
	8	75.50 (3.00)	39.83 (2.00)	**21.17 (1.00)**
	10	75.47 (3.00)	41.03 (2.00)	**20.00 (1.00)**
	15	74.63 (3.00)	46.30 (2.00)	**15.57 (1.00)**
	20	74.18 (3.00)	46.82 (2.00)	**15.50 (1.00)**
WFG1	3	**16.33 (1.00)**	44.67 (2.00)	75.50 (3.00)
	5	**16.63 (1.00)**	44.37 (2.00)	75.50 (3.00)
	8	**26.57 (1.50)**	**34.43 (1.50)**	75.50 (3.00)
	10	**36.95 (1.50)**	**28.98 (1.50)**	70.57 (3.00)
	15	44.50 (2.00)	**16.50 (1.00)**	75.50 (3.00)
	20	50.17 (2.00)	**19.80 (1.00)**	66.53 (3.00)
WFG2	3	**31.50 (1.50)**	74.10 (3.00)	**30.90 (1.50)**
	5	45.53 (2.00)	74.30 (3.00)	**16.67 (1.00)**
	8	**38.10 (1.50)**	63.87 (3.00)	**34.53 (1.50)**
	10	50.03 (2.00)	66.70 (3.00)	**19.77 (1.00)**
	15	66.93 (2.50)	52.43 (2.50)	**17.13 (1.00)**
	20	73.17 (3.00)	45.03 (2.00)	**18.30 (1.00)**
WFG3	3	**32.23 (1.50)**	75.50 (3.00)	**28.77 (1.50)**
	5	43.83 (2.00)	75.50 (3.00)	**17.17 (1.00)**
	8	50.90 (2.50)	63.67 (2.50)	**21.93 (1.00)**
	10	45.50 (2.00)	59.60 (2.50)	**31.40 (1.50)**
	15	71.77 (3.00)	**36.43 (1.50)**	**28.30 (1.50)**
	20	72.40 (3.00)	**26.37 (1.50)**	**37.73 (1.50)**
WFG4	3	45.60 (2.00)	75.40 (3.00)	**15.50 (1.00)**
	5	45.93 (2.00)	75.07 (3.00)	**15.50 (1.00)**
	8	46.27 (2.00)	74.73 (3.00)	**15.50 (1.00)**

Table 16 continued

Prob.	Obj.	Linear (C_{R2})	Metric (C_{R2})	NSGA-III
	10	49.97 (2.00)	71.03 (3.00)	**15.50 (1.00)**
	15	59.50 (2.50)	61.50 (2.50)	**15.50 (1.00)**
	20	66.68 (2.50)	54.32 (2.50)	**15.50 (1.00)**
WFG5	3	45.50 (2.00)	75.50 (3.00)	**15.50 (1.00)**
	5	69.07 (3.00)	51.93 (2.00)	**15.50 (1.00)**
	8	72.93 (3.00)	48.07 (2.00)	**15.50 (1.00)**
	10	73.40 (3.00)	47.60 (2.00)	**15.50 (1.00)**
	15	75.50 (3.00)	45.50 (2.00)	**15.50 (1.00)**
	20	75.50 (3.00)	45.50 (2.00)	**15.50 (1.00)**
WFG6	3	45.50 (2.00)	75.50 (3.00)	**15.50 (1.00)**
	5	45.50 (2.00)	75.50 (3.00)	**15.50 (1.00)**
	8	46.77 (2.00)	73.40 (3.00)	**16.33 (1.00)**
	10	47.57 (2.00)	73.43 (3.00)	**15.50 (1.00)**
	15	50.97 (2.00)	70.03 (3.00)	**15.50 (1.00)**
	20	54.67 (2.50)	66.33 (2.50)	**15.50 (1.00)**
WFG7	3	45.47 (2.00)	75.50 (3.00)	**15.53 (1.00)**
	5	45.50 (2.00)	75.50 (3.00)	**15.50 (1.00)**
	8	48.33 (2.00)	72.67 (3.00)	**15.50 (1.00)**
	10	55.03 (2.50)	65.97 (2.50)	**15.50 (1.00)**
	15	73.63 (3.00)	47.37 (2.00)	**15.50 (1.00)**
	20	74.07 (3.00)	46.93 (2.00)	**15.50 (1.00)**
WFG8	3	45.50 (2.00)	75.50 (3.00)	**15.50 (1.00)**
	5	45.50 (2.00)	75.50 (3.00)	**15.50 (1.00)**
	8	65.00 (2.50)	56.00 (2.50)	**15.50 (1.00)**
	10	67.97 (2.50)	53.03 (2.50)	**15.50 (1.00)**
	15	75.00 (3.00)	46.00 (2.00)	**15.50 (1.00)**
	20	75.03 (3.00)	45.97 (2.00)	**15.50 (1.00)**
WFG9	3	54.33 (2.50)	66.67 (2.50)	**15.50 (1.00)**
	5	60.30 (2.50)	60.70 (2.50)	**15.50 (1.00)**
	8	72.80 (3.00)	48.20 (2.00)	**15.50 (1.00)**
	10	73.87 (3.00)	47.13 (2.00)	**15.50 (1.00)**
	15	74.97 (3.00)	46.03 (2.00)	**15.50 (1.00)**
	20	74.20 (3.00)	46.80 (2.00)	**15.50 (1.00)**

Final ranks, presented in parentheses assigned according to mean ranks
The bold values refer to the smallest final rankings obtained in each group (problem/objective number), hence the algorithm variants whose values are highlighted are those who obtained the best results

Table 17 Mean ranks of the IGD and Hypervolume obtained by each MO-CMA-ES variant and NSGA-III as used in the Kruskal–Wallis test for the problems up to 8 objectives

Prob.	Obj.	Linear (C_{HV})	Metric (C_{HV})	NSGA-III
(A) IGD ranks				
DTLZ1	3	57.97 (2.50)	62.93 (2.50)	**15.60 (1.00)**
	5	65.63 (2.50)	55.37 (2.50)	**15.50 (1.00)**
	8	71.33 (3.00)	49.67 (2.00)	**15.50 (1.00)**
DTLZ2	3	**31.37 (1.50)**	**32.77 (1.50)**	72.37 (3.00)
	8	63.37 (2.50)	**15.70 (1.00)**	57.43 (2.50)

Table 17 continued

Prob.	Obj.	Linear (C_{HV})	Metric (C_{HV})	NSGA-III
	5	75.17 (3.00)	**27.93** (**1.50**)	**33.40** (**1.50**)
DTLZ3	3	57.80 (2.50)	63.20 (2.50)	**15.50** (**1.00**)
	5	65.30 (2.50)	55.70 (2.50)	**15.50** (**1.00**)
	8	72.33 (3.00)	48.67 (2.00)	**15.50** (**1.00**)
DTLZ4	3	**44.78** (**2.00**)	**43.88** (**2.00**)	**47.83** (**2.00**)
	5	59.10 (2.50)	58.43 (2.50)	**18.97** (**1.00**)
	8	54.13 (2.50)	60.07 (2.50)	**22.30** (**1.00**)
DTLZ5	3	**15.53** (**1.00**)	45.47 (2.00)	75.50 (3.00)
	5	47.50 (2.00)	**16.53** (**1.00**)	72.47 (3.00)
	8	44.67 (2.00)	**16.50** (**1.00**)	75.33 (3.00)
DTLZ6	3	**37.20** (**1.50**)	**25.77** (**1.50**)	73.53 (3.00)
	5	61.10 (2.50)	**16.80** (**1.00**)	58.60 (2.50)
	8	71.47 (3.00)	**15.50** (**1.00**)	49.53 (2.00)
DTLZ7	3	55.20 (2.50)	65.43 (2.50)	**15.87** (**1.00**)
	5	44.50 (2.00)	71.73 (3.00)	**20.27** (**1.00**)
	8	**18.13** (**1.00**)	45.90 (2.00)	72.47 (3.00)
WFG1	3	**17.27** (**1.00**)	43.73 (2.00)	75.50 (3.00)
	5	**25.87** (**1.50**)	**35.13** (**1.50**)	75.50 (3.00)
	8	**29.83** (**1.50**)	**31.17** (**1.50**)	75.50 (3.00)
WFG2	3	**19.73** (**1.00**)	74.17 (3.00)	42.60 (2.00)
	5	**31.30** (**1.00**)	47.67 (2.50)	57.53 (2.50)
	8	**22.20** (**1.00**)	45.50 (2.00)	68.80 (3.00)
WFG3	3	**27.13** (**1.00**)	54.50 (2.50)	54.87 (2.50)
	5	**40.93** (**1.50**)	**28.13** (**1.50**)	67.43 (3.00)
	8	51.13 (2.50)	**26.40** (**1.00**)	58.97 (2.50)
WFG4	3	51.97 (2.50)	67.57 (2.50)	**16.97** (**1.00**)
	5	42.00 (2.00)	**25.93** (**1.00**)	68.57 (3.00)
	8	49.43 (2.50)	**24.00** (**1.00**)	63.07 (2.50)
WFG5	3	41.67 (2.00)	75.27 (3.00)	**19.57** (**1.00**)
	5	54.20 (2.50)	66.80 (2.50)	**15.50** (**1.00**)
	8	66.90 (3.00)	45.87 (2.00)	**23.73** (**1.00**)
WFG6	3	43.70 (2.00)	75.13 (3.00)	**17.67** (**1.00**)
	5	57.97 (2.50)	63.03 (2.50)	**15.50** (**1.00**)
	8	**24.17** (**1.50**)	**36.87** (**1.50**)	75.47 (3.00)
WFG7	3	**32.53** (**1.50**)	75.50 (3.00)	**28.47** (**1.50**)
	5	68.80 (3.00)	48.83 (2.00)	**18.87** (**1.00**)
	8	49.10 (2.50)	**22.70** (**1.00**)	64.70 (2.50)
WFG8	3	45.67 (2.00)	75.33 (3.00)	**15.50** (**1.00**)
	5	61.30 (2.50)	59.70 (2.50)	**15.50** (**1.00**)
	8	42.97 (2.00)	**36.17** (**1.50**)	57.37 (2.50)
WFG9	3	**36.67** (**1.50**)	64.40 (3.00)	**35.43** (**1.50**)
	5	63.23 (2.50)	53.57 (2.50)	**19.70** (**1.00**)
	8	55.93 (2.50)	**17.87** (**1.00**)	62.70 (2.50)
(B) Hypervolume ranks				
DTLZ1	3	47.43 (2.00)	73.57 (3.00)	**15.50** (**1.00**)
	5	47.60 (2.00)	73.40 (3.00)	**15.50** (**1.00**)
	8	55.87 (2.50)	65.07 (2.50)	**15.57** (**1.00**)

Table 17 continued

Prob.	Obj.	Linear (C_{HV})	Metric (C_{HV})	NSGA-III
DTLZ2	3	62.97 (2.50)	58.03 (2.50)	**15.50** (**1.00**)
	5	72.60 (3.00)	48.40 (2.00)	**15.50** (**1.00**)
	8	74.93 (3.00)	43.10 (2.00)	**18.47** (**1.00**)
DTLZ3	3	48.70 (2.00)	72.30 (3.00)	**15.50** (**1.00**)
	5	45.90 (2.00)	75.10 (3.00)	**15.50** (**1.00**)
	8	53.27 (2.50)	67.73 (2.50)	**15.50** (**1.00**)
DTLZ4	3	54.78 (2.50)	56.98 (2.50)	**24.73** (**1.00**)
	5	56.33 (2.50)	64.67 (2.50)	**15.50** (**1.00**)
	8	**42.63** (**1.50**)	62.17 (3.00)	**31.70** (**1.50**)
DTLZ5	3	**24.13** (**1.50**)	**36.87** (**1.50**)	75.50 (3.00)
	5	58.57 (2.50)	56.30 (2.50)	**21.63** (**1.00**)
	8	47.53 (2.00)	**35.93** (**1.50**)	53.03 (2.50)
DTLZ6	3	44.73 (2.00)	**26.43** (**1.00**)	65.33 (3.00)
	5	75.27 (3.00)	**28.17** (**1.50**)	**33.07** (**1.50**)
	8	74.77 (3.00)	**19.60** (**1.00**)	42.13 (2.00)
DTLZ7	3	54.03 (2.50)	66.67 (2.50)	**15.80** (**1.00**)
	5	**30.97** (**1.50**)	73.03 (3.00)	**32.50** (**1.50**)
	8	73.23 (3.00)	**16.50** (**1.00**)	46.77 (2.00)
WFG1	3	**17.43** (**1.00**)	43.57 (2.00)	75.50 (3.00)
	5	**21.63** (**1.00**)	39.37 (2.00)	75.50 (3.00)
	8	**28.73** (**1.50**)	**32.27** (**1.50**)	75.50 (3.00)
WFG2	3	**31.47** (**1.50**)	75.20 (3.00)	**29.83** (**1.50**)
	5	44.73 (2.00)	75.27 (3.00)	**16.50** (**1.00**)
	8	**31.90** (**1.50**)	72.50 (3.00)	**32.10** (**1.50**)
WFG3	3	**29.93** (**1.00**)	57.77 (2.50)	48.80 (2.50)
	5	**30.07** (**1.50**)	74.63 (3.00)	**31.80** (**1.50**)
	8	**36.27** (**1.50**)	**39.20** (**1.50**)	61.03 (3.00)
WFG4	3	53.13 (2.50)	67.83 (2.50)	**15.53** (**1.00**)
	5	49.70 (2.00)	71.30 (3.00)	**15.50** (**1.00**)
	8	45.73 (2.00)	75.03 (3.00)	**15.73** (**1.00**)
WFG5	3	45.70 (2.00)	75.30 (3.00)	**15.50** (**1.00**)
	5	50.00 (2.00)	71.00 (3.00)	**15.50** (**1.00**)
	8	61.50 (2.50)	59.50 (2.50)	**15.50** (**1.00**)
WFG6	3	45.57 (2.00)	75.43 (3.00)	**15.50** (**1.00**)
	5	47.17 (2.00)	73.83 (3.00)	**15.50** (**1.00**)
	8	45.70 (2.00)	75.30 (3.00)	**15.50** (**1.00**)
WFG7	3	44.20 (2.00)	75.50 (3.00)	**16.80** (**1.00**)
	5	45.87 (2.00)	75.13 (3.00)	**15.50** (**1.00**)
	8	48.57 (2.00)	72.43 (3.00)	**15.50** (**1.00**)
WFG8	3	45.63 (2.00)	75.37 (3.00)	**15.50** (**1.00**)
	5	45.50 (2.00)	75.50 (3.00)	**15.50** (**1.00**)
	8	45.50 (2.00)	75.50 (3.00)	**15.50** (**1.00**)
WFG9	3	58.23 (2.50)	62.77 (2.50)	**15.50** (**1.00**)
	5	66.40 (2.50)	54.60 (2.50)	**15.50** (**1.00**)
	8	71.33 (3.00)	49.67 (2.00)	**15.50** (**1.00**)

Final ranks, presented in parentheses assigned according to mean ranks
The bold values refer to the smallest final rankings obtained in each group (problem/objective number), hence the algorithm variants whose values are highlighted are those who obtained the best results

5.6 Discussion

In conclusion, we can split the discussion of the results achieved by each TWF in three groups, based on their characteristics: the equal, which is unique since it does not use a ranking strategy; the functions using the three distinct metrics, and the functions using the three remaining weighting schemes.

Starting by the equal, if we consider the summarized Tables 6, 7, 8 and 9 equal only outperformed the other TWFs more times in Table 8, where C_{HV} was not used as ranking method. More specifically through Table 4, we can see that most of the times equal appeared among the best algorithms was up to eight objectives (17 times). When we include the TWFs ranked by the C_{HV} metric, this number drops to 4 times, from which, despite not having statistical differences from the best, it has the absolute smallest rank in only one case (WFG1 $m = 8$). This analysis indicates that using an equal weight distribution can be advantageous only when we do not have reliable metrics to rank the solutions.

Among the functions using the three different ranking indicators, the results in all tested cases show that C_{HV} performs the best, since within all blocks of weighting strategies C_{HV} statistically outperforms the other algorithms more times. The relevance of these results is twofold: first it confirms that if we use a powerful quality indicator to rank the solutions, the model learns these preferences introduced by the indicator and generates solutions following this criterion; secondly, it was expected (although not guaranteed) that if the model learns through the preferences of hypervolume, it would generate good solutions according to this metric. However we compared the final approximations using the IGD indicator and C_{HV} still obtained the best results, which means that the model is really generating better solutions, instead of simply exploiting information about the metric to sample solutions which are biased towards the preferred regions of the hypervolume.

However, using the exact hypervolume calculation for problems with more than eight objectives is computationally very expensive. Approximate versions do exist, but generating enough samples in order to achieve reliable results in each iteration is computationally expensive as well, this is why we used C_{HV} only up to eight objectives and used the approximate hypervolume only to evaluate the final front generated by each variant of the algorithm. When considering the other metrics used (C_{R2} and CD), we can observe that in most of the cases it is preferable to use C_{R2} over CD. This result can be explained by the fact that R2 is a metric that measures both convergence and diversity, while CD is only an estimator of diversity. This difference can be seen in most of the tables, where CD is among the best algorithms more times for three objectives (where the pressure towards the Pareto front is stronger) than for other numbers of objectives.

A problem often seen in C_{R2} is that several solutions present a value of zero. This happens when more than one solution is close to a single weight vector, which makes the closer solution contribute to the general R2, but the farthest has a contribution of zero. A possible way to ease this problem would be to have more weight vectors, but this can change the relation convergence versus diversity of the algorithm. This would happen because if we have more weight vectors, more solutions in a same region will contribute to the R2 and consequently to the model update. This would generate even more solutions in that region, but a detailed analysis on the effect of the density of weight vectors for the C_{R2} indicator is material for future works.

The third characteristic to be observed on the TWFs is the strategy used to assign weights to the solutions ranked (equal does not use the ranks), where we have the three variants: linear, log and metric. Among these three variants, we can state that linear introduces less selection pressure towards the best ranked solutions, since the differences in weights are smaller. The same way, log is an intermediate, and in general, metric applies more selection pressure since the differences among the first and last solutions usually are larger (see Fig. 1). This trend is reflected in the results obtained, where for the DTLZ problems the number of times each of these schemes outperforms the others (including themselves) can be expressed as linear $<$ log $<$ metric in all the summarized tables except in hypervolume when considering up to 8 objectives. An inverse trend can be observed when considering the WFG problems, where these numbers can be expressed as linear $>$ log $>$ metric in all the summarized tables except in hypervolume when considering up to 20 objectives. This inverse impact observed about the selection pressure applied by both functions is an important fact, however so far we have not been able to identify a particular reason for this behavior. Additional research will be conducted to investigate this difference and gain further insight into the effects of using TWFs.

In a comparison of the MO-CMA-ES using the best TWFs to the state-of-the-art NSGA-III algorithm, in general NSGA-III presented better results. However, MO-CMA-ES was particularly competitive in the discontinuous, degenerate and mixed instances from both families of benchmarking problems. This behavior suggests that by not relying on reference points to achieve diversity, fewer assumptions are made about the shape and continuity of the Pareto front, hence the search is more flexible to adapt to unusual Pareto shapes.

6 Conclusion

The injection of information from good solutions is a fundamental step for the CMA-ES algorithm, since this information will be used to create and evolve a model of the problem being explored. In Pareto-based algorithms, several methods can be used to determine how "good" a non-dominated solution is and, thus, can be used to rank the solutions of a Pareto set. However, each method can value different characteristics of the solutions, leading to different models being learned. Moreover, the pressure introduced by a ranking method on the model can be adjusted by appropriately selecting a weighting function over the ranked solutions.

As far as we know, this is the first work to present the idea of using transfer weight functions (TWFs) as flexible components to manage the incorporation of information into a Pareto-based MO-CMA-ES, moreover, we compared the performance of different TWFs. The TWF components used here included three different methods to rank the non-dominated solutions, each one based on a well-known quality indicator. Furthermore, four different ways of distributing weights to the solutions were used, one of the options was giving them equal weights, so there is no influence from the ranking method, the other three alternatives give higher weights to better ranked solutions. Each combination of weighting strategy and ranking metric composed as TWF, except the equal weight distribution which was treated separately, hence, we end up with 10 different TWFs.

In order to assess the influence of the proposed TWFs on the MO-CMA-ES algorithm, we conducted an experimental study comparing the TWFs on two sets of popular benchmark problems on a total of 16 functions. Additionally, we used different numbers of objectives, ranging from 3 to 20, to observe the scalability of these TWFs in the many-objective scenario. The results obtained were evaluated using two different quality indicators and the values measured by them were submitted to a statistical test to look for statistically significant differences.

Our results indicate that, in general, the equal TWF does not produce the best results, therefore, it is useful to apply a ranking method and unequal weights for the solutions according to its quality. When comparing the remaining strategies to weight the ranked solutions, we saw that a good choice can be problem-dependent, since opposed trends were found when analyzing both families of problems: for WFG, in most cases, it is best to use the linear approach, while for DTLZ the metric approach performed better.

When considering the three different ranking indicators, we clearly obtained the best results by using C_{HV}, however, due to the high computational cost to calculate it exactly or approximate it reliably when using many-objectives, we limited its use up to eight objectives. In the scenarios where the C_{HV} was absent, in most cases C_{R2} outperformed CD, this difference was smaller when we considered three objectives, but as the number of objectives increased and, consequently, the influence of the Pareto dominance weakened, the differences in favor of the C_{R2} indicator increased, since it takes into consideration both the convergence and the diversity, while CD is only a diversity estimator.

In a comparison with the state-of-the-art algorithm NSGA-III, the results obtained by MO-CMA-ES where particularly competitive on discontinuous, degenerate and mixed problem instances from both the families of benchmark problems investigated. This result indicates that MO-CMA-ES is more flexible to deal with these irregularities than NSGA-III since it does not use reference points and thus does not make any assumption about the shape of the Pareto fronts.

Future works include a study on the effects of using different sets of solutions in the model building of CMA-ES, a detailed study on the number and density of weight vectors in the C_{R2} metric and its impact on MO-CMA-ES, other studies may be concentrated on the use of alternative many-objective quality indicators to rank the solutions.

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