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MULTIPLE OBJECTIVE OPTIMIZATION

Multiobjective portfolio optimization: bridging mathematical theory with asset management practice

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Abstract We attempt to establish an integrated portfolio optimization business framework, in order to bridge the underlying gap between the complex mathematical theory of multiobjective mathematical programming and asset management practice. Our aim is to assist practitioners and portfolio managers in formulating successful investment strategies, by providing them with an effective decision support tool. In particular, we propose a multiobjective portfolio model, able to support the simultaneous optimization of multiple investment objectives. We also manage to integrate a set of sophisticated real-world non-convex investment policy limitations, such as the cardinality constraints, the buy-in thresholds, the transaction costs, along with particular normative rules. The underlying investment management rationale of the proposed managerial protocol is displayed through an illustrative business flowchart, while we also provide an analytical step-by-step portfolio management business routine. The validity of the model is verified through an extended empirical testing application on the Eurostoxx 50. According to the results, a sufficient number of efficient or Pareto optimal portfolios produced by the model, appear to possess superior out-of-sample returns with respect to the underlying benchmark.

Keywords Portfolio optimization · Multiobjective mathematical programming · Asset management · Eurostoxx 50

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1 Problem setting

The typical single-period portfolio selection problem was initially formulated as a quadratic bi-criteria optimization process, where the expected return is maximized for a desired level of risk (Markowitz 1952). On this basis, risk is quantified as the variance of portfolio returns and the graphical interpretation of the problem's Pareto optimal solutions on the risk-return plane results in the so-called *efficient frontier*. Apparently, the choice of the final portfolio over the efficient frontier is a trade-off procedure between risk and return, heavily dependent on the investor's risk tolerance profile.

The conventional mean-variance approach is restricted by the underlying assumptions that investors prefer more to less, are risk averse and either security returns are multivariate normally distributed or utility functions are quadratic. However investors' perception against risk is not symmetric, since they do feel unsatisfied to have small or negative profits, but they also do feel satisfied to have larger gains. In fact, evidence gained throughout global markets implies that security returns are often not normally nor even symmetrically distributed. Another addressed criticism towards the Markowitz model is the lack of consistency with axiomatic models of references for choice under risk (Bell and Raiffa 1988; Ogryczak 2000). Moreover, the mean-variance model does not provide a solution regarding the estimation of risk and the use of one-single point estimate for the random variables, i.e., return of the assets (Bonami and Lejeune 2009; Cornuejóls and Tütüncü 2007).

The mean-variance formulation is often criticized for the intrinsic difficulty of its implementation. As an example, an institution that follows a number of 250 stocks will need 31,125 correlation coefficients for implementing the underlying framework. But it seems quite unlikely that analysts will be able to directly estimate such huge and complex correlation structures, within the feasible organizational limits of a traditional asset management house. Additionally, the computational difficulty associated with solving a large-scale quadratic programming problem, with very dense covariance matrix on a real-time basis is high. However, Mitra et al. (2007) and Lejeune and Filomena (2014) provide treatments for the exact solution of convex quadratic problems (variants of the Markowitz optimization model) of very large size (up to 2500 assets) in very short amount of time.

The aforementioned computational issue was successfully confronted by some very effective techniques developed by Markowitz and Perold (1981), Perold (1984) and DeMiguel et al. (2007, 2009). To avoid manipulations with very dense covariance matrices, these approaches focused on methods for diagonalizing the covariance matrix structure, but with a loss of information. Konno and Suzuki (1992) proposed yet another diagonalization strategy without any loss of information. While this method, when applicable, will produce a true non-dominated frontier, the difficulty is that most large covariance matrices are not invertible (Steuer et al. 2011).

Dealing with the same problem but from another perspective, many authors have tried to heal the relevant difficulties by using various approximation models. For example, the use of index models enables one to reduce the amount of computation by introducing the notion of particular factors that influence stock prices (Sharpe 1963). However, these efforts are systematically devaluated because of the popularity of equilibrium models, such as CAPM and APT, which are computationally less demanding. However, the idea of Markowitz is still present, since CAPM is based upon this model. Moreover, equilibrium models impose several unrealistic assumptions to derive a simple, but usually unstable, relation between the rate of return of individual assets and the market portfolio.

An additional factor that significantly perplexes the portfolio optimization process is the need of integrating a set of sophisticated real-world investment constraints in the models

(Branke et al. 2009). Such constraints may be: (a) the maximum number of securities to be included in the portfolio, (b) certain buy-in thresholds, which demand at least some minimum amount of the stock to be bought, if this stock is included in the portfolio, (c) the transaction costs volume, and (d) particular normative rules, such as the 5-10-40 constraint, which defines upper limits in the weight of each individual security or sum of weights of the same issuer's securities in the portfolio.

The above features can be properly formulated by introducing integer variables (Canakgoz and Beasley 2009). However, solving portfolio optimization problems containing integer variables has been considered as a heavy task. The main reason is that the standard model uses variance (or standard deviation) as the measure of risk. On this basis, many researchers propose the linearization of the portfolio selection problem. Following Sharpe (1971), Konno and Yamazaki (1991), Ogryczak (2000) and Mansini et al. (2003) represent milestone attempts towards this direction, i.e. exploiting absolute deviation as an alternative measure of risk and formulating the portfolio selection problem as a linear one. The stated practical portfolio optimization problems can be solved by the conventional integer linear programming software within a practical amount of time.

Nevertheless, the whole framework is becoming even more complicated, when multiple investment objectives are to be simultaneously optimized (Lee and Chesser 1980; Bana Costa and Soares 2004; Steuer et al. 2007; Xidonas et al. 2011). For a very representative sample of the recent developments in the field of multiobjective portfolio management, see also Köksalan et al. (2014), Bilbao-Terol et al. (2015), Masri (2015), Messaoudi et al. (2015), Qi et al. (2015), Zhao et al. (2015) and Walter et al. (2016).

Putting all things together, in the above case, the portfolio selection process reflects to a mixed-integer multiobjective portfolio optimization problem. The current paper aspires to effectively deal with all the above issues, since few propositions, if any, are concerned with an integrated methodological framework for modeling and analytically solving real-world portfolio optimization problems, in the presence of: (a) multiple investment objectives, (b) cardinality constraints (i.e. binary policy constraints), (c) transaction costs, and (d) compliance norms.

The suggested framework's main objective is to practically assist portfolio managers in formulating successful investment strategies and eventually provide them with an effective investment decision support tool. In particular, we propose an integrated multiobjective portfolio optimization managerial framework, while we provide a step-by-step business routine. The value of the suggested framework is enhanced by the use of two certain concepts in the field of multiobjective portfolio optimization, i.e. the *security impact plane* and the *barycentric portfolio*. The first represents a measure of each security's impact in the efficient surface of Pareto optimal portfolios. The second serves as the vehicle for implementing a balanced strategy of iterative portfolio tuning. Thus, the underlying value of the proposed framework is strongly connected with the managerial effectiveness and standardization of the investment analytics procedure.

In order to demonstrate the practical usability of the proposed *multiobjective managerial protocol*, we provide a detailed flowchart that summarizes the framework's business flow. Also, we highlight the key steps of the proposed protocol as a simple pattern that can be used by practitioners and market specialists. Hence, the proposed approach effectively deals with real-world portfolio analytics issues, such as: (a) multiple investment criteria, (b) the straightforward and hands-on exploitation of analysts' experience, and (c) the introduction of a novel business flow for portfolio engineering.

The paper proceeds as follows: In Sect. 2 we present the proposed managerial protocol and in Sect. 3 we provide the methodological framework for dealing with the type of multiobjective portfolio selection problems, we previously described. In Sect. 4 we meticulously analyse the whole model building procedure. Finally, the empirical results and concluding remarks are given in Sects. 5 and 6.

2 The proposed multiobjective managerial protocol

The proposed multiobjective managerial protocol is a complete methodological routine for dealing with multiobjective portfolio optimization problems. Our intention is to provide a hands-on approach that may be easily applied in practice by portfolio managers, market specialists and practitioners. The scope of the proposed approach is to carry out the decision making process of the portfolio selection problem, using certain mathematical programming tools.

In general, the portfolio management process is an integrated set of steps undertaken in a consistent manner to create and maintain an appropriate portfolio (combination of assets) to meet clients' stated goals (Maginn et al. 2007). The three fundamental elements in managing any business process are: planning, execution and feedback. The same steps form the basis for the portfolio management process. In the planning step, investment objectives and policies are formulated, capital market expectations are formed and strategic asset allocations are established. In the execution step, the manager constructs the portfolio and integrates investment strategies with capital market expectations to select the specific assets for the portfolio. Finally, in the feedback step, the manager monitors and evaluates the portfolio compared with the plan. Any changes suggested by the feedback must be examined carefully to ensure that they represent long-run considerations. Our multiobjective managerial protocol is an integrated process, which includes the following phases: (a) planning and investor policy setting, (b) active portfolio management, through executing a multiobjective portfolio rebalancing process, and (c) results inspection and feedback wrap-up.

As already stressed, the emphasis in this article is laid on the multiobjective portfolio optimization context, with respect to the various investor policy constraints. Our intention is to successfully embody a multiobjective optimization framework into the decision making process of portfolio managers. The major characteristics and business flows of the conventional investment management organization are shown in Fig. 1.

More specifically, projections concerning the economy and the financial markets are made by economists, technicians, fundamentalists or other market experts, within or outside the organization. The projected economic environment is communicated by means of briefings and written reports, usually in a rather implicit and qualitative manner, to the organization's security analysts. Actually, analysts' predictions may be focused on various investment objectives, such as maximization of expected returns, minimization of risk, a certain level of dividend yield etc.

The securities' multi-aspect profile and various written reports constitute the information formally transmitted to an investment committee, which typically includes the senior management of the organization. In addition, analysts occasionally brief the investment committee on their feelings about various securities. The investment committee's primary formal output is often an *approved list* or *authorized list*, which consists of the securities deemed worthy of accumulation in a given portfolio. Apart from the practitioners' investment sentiment, securities may also be selected on the basis of mathematical portfolio optimization, with respect to the various multiple investment objectives. The rules of the organization typically specify that any security in the list may be bought, whereas those not on the list should be either held or sold, depending the circumstances. The presence or absence of a security on the approved

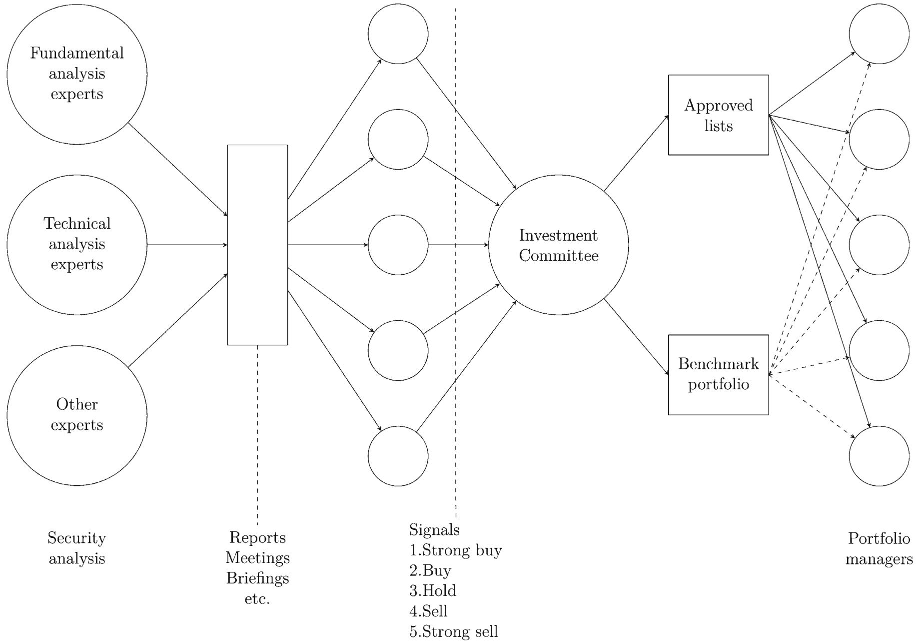


Fig. 1 The classical investment management business framework

list constitutes the major information transmitted explicitly from the investment committee to a portfolio manager. In the proposed managerial protocol, we evaluate the dynamics of securities, using the *security impact plane*.

In Fig. 2, we present the business flowchart of the proposed *multiobjective managerial protocol*. It may provide portfolio managers with a straightforward, tractable and transparent scheme for its incorporation into the portfolio management process. Moreover, we precisely develop the complete grid of details regarding the modelling routine of the empirical testing procedure.

A step-by-step description of how the proposed methodology can be applied to the problem of multiobjective optimization process is provided below:

Step 1 The portfolio manager defines a portfolio management policy, by specifying the parameters of the underlying investment strategy constraints.

Step 2 The corresponding group of analysts collects securities' data for every desired objective function, on the basis of the relevant time frequency (e.g. weekly returns, quarterly dividends etc.).

Step 3 The payoff matrix is calculated, the AUGEMCON routine run and the output of the multiobjective optimization is obtained.

Step 4 The security impact plane is extracted.

Step 5 Portfolios are evaluated with out-of-sample data.

Step 6 The same procedure for consecutive window time frames is repeated.

After running the whole routine, managers may perform a visual evaluation of the securities relying on a wide series of graphs for each iteration. Thus, practitioners are equipped with graphing tools for direct monitoring of the securities behavior over time. Furthermore, in terms of returns evaluation, the out-of-sample performance of the multiobjective rebalancing

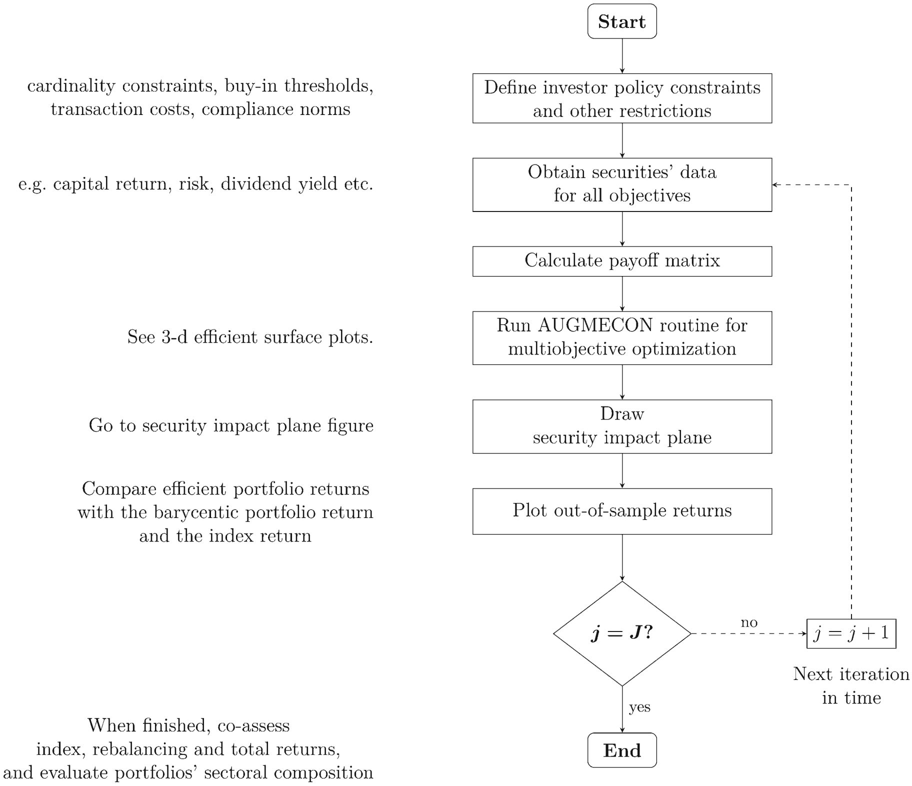


Fig. 2 The proposed multiobjective managerial flowchart

strategy can be inspected and appreciated in relation to the buy-and-hold strategy for the corresponding time period.

3 Underlying modeling framework

The underlying modelling framework is presented in Xidonas et al. (2011) and Xidonas and Mavrotas (2014), thus in this section we will only sketch some fundamental details of it. The proposed methodological approach is based on the theoretical framework of multiobjective mathematical programming (MMP). This section highlights the interconnection between this approach and the major issue of introducing multiple investment objectives in the real-world portfolio management practice.

The general MMP model is defined as follows:

$$\min \left[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_p(\mathbf{x}) \,|\mathbf{x} \in \mathrm{S} \right] \tag{1}$$

where $\mathbf{x}$ is the vector of decision variables, $f_i(\mathbf{x})$, $i = 1, 2, \ldots, p$ are the objective functions and S is the feasible region. As already noted, the concept of optimality in MMP is replaced with that of efficiency or Pareto optimality.

According to Hwang and Masud (1979), the methods for solving MMP problems can be classified into three categories, according to the phase in which the decision maker (DM), i.e.

the investor in the case of portfolio selection, is involved in the decision making process, by expressing his preferences: The a-priori methods, the *interactive* methods and the *generation* or a-posteriori methods. In a-priori methods, the DM expresses his preferences before the solution process, by setting, either goals or weights for the objective functions. The criticism about the a-priori methods is that it is very difficult for the DM to beforehand define and accurately quantify his preferences. In the *interactive* methods, phases of dialogue with the DM are interchanged with phases of calculation and the process usually converges after several iterations to the most preferred solution. The drawback is that the DM never sees the whole picture, i.e. the whole set of Pareto optimal solutions (portfolios) or even an approximation of it. Hence, the most preferred solution is the most preferred in relation to what he has seen and compared so far. Finally, in a-posteriori or *generation* methods the Pareto optimal solutions of the problem (all of them or a sufficient representation) are generated and then the DM is involved, in order to select among them, the most preferred one.

Generation methods are the less popular due to the underlying computational burden and the lack of widely available software. However, they have some significant advantages. The solution process within a generation method is divided in two phases: The first phase concerns the generation of the efficient solutions, while the second phase concerns the involvement of the DM when the aggregate information is on the table. Consequently, the DM is involved only in the second phase, having at hand all the possible alternatives, i.e. the whole Pareto optimal set of portfolios. Hence, generation methods are very favorable in cases where the DM is hardly available and the interaction with him difficult, an extremely crucial issue within the typical process of investment management. Besides, the fact that none of the potential solutions has been left uncovered reinforces the DM's confidence on the final decision.

The most widely used generation methods are the *weighting method* and the ε-constraint method. In the weighting method, a weighted sum of the objective functions is optimized. By varying the weights of the objective functions, different efficient solutions are obtained. In the ε-constraint method, one of the objective functions is optimized using the other objective functions as constraints, incorporating them in the constraint part of the model as shown below (Chankong and Haimes 1983). Without loss of generality, the first objective function is optimized and the rest $p-1$ objective functions are treated as constraints:

$$\max \{ f_1(\mathbf{x}) \mid f_1(\mathbf{x}) \geq e_i, \quad i = 2, 3, \ldots, p \wedge \mathbf{x} \in \mathrm{S} \} \tag{2}$$

By parametrical variation in the right-hand side of the constrained objective functions, the efficient solutions of the problem are obtained.

The ε-constraint method has several advantages over the weighting method, especially in cases where some of the decision variables are integer (Steuer 1989). Several versions of the ε-constraint method have appeared in the literature trying to improve its performance or adapt it to a specific type of problems (Hamacher et al. 2007). However, despite its advantages over the weighting method, the ε-constraint method has three points that need special attention in its implementation: (a) the calculation of the range of the objective functions over the efficient set, (b) the guarantee of efficiency of the obtained solution, and (c) the increased solution time for problems with several (more than two) objective functions.

Mavrotas (2009) effectively addresses the above issues are by introducing a novel version of the ε-constraint method, the so called augmented ε-constraint method (AUGMECON). The main innovations introduced within the AUGMECON method are: (a) the use of *lexicographic optimization*, (b) the guarantee for producing only Pareto optimal solutions (thus avoiding weakly Pareto optimal solutions), (c) the algorithmic acceleration of the computational process, using the *early exit from the nested loops*, and (d) the implementation of the algorithm.

The above technical novelties are briefly described below:

Construction of the payoff matrix
In order to properly apply the ε-constraint method, we must have the range of every objective function, at least for the $p-1$ objective functions that will be used as constraints. The calculation of the range of the objective functions over the efficient set is not a trivial task. While the best value is easily attainable as the optimal of the individual optimization, the worst value over the efficient set (nadir value) is not. The most common approach is to calculate these ranges from the payoff matrix (the table with the results from the individual optimization of the p objective functions). On this basis, the nadir value is usually approximated as the minimum of the corresponding column. However, even in this case, someone must be sure that the obtained solutions from the individual optimization of the objective functions are indeed efficient solutions. In presence of alternative optima, there is no guarantee that a solution obtained by commercial software is an efficient solution. In order to overcome this ambiguity we propose the use of lexicographic optimization for every objective function in order to construct the payoff matrix, exclusively with efficient solutions. It is well known that lexicographic optimization provides only efficient solutions (Miettinen 1998). A simple remedy in order to bypass the difficulty of estimating the nadir values of the objective functions is to define reservation values for the objective functions. A reservation value acts like a lower (or upper for minimization objective functions) bound. Values worse than the reservation value are outside the feasible region and are not allowed.

Avoidance of weakly Pareto optimal solutions
The second point of attention is that the optimal solution of problem (2) is guaranteed to be an efficient solution only if all $p-1$ objective function constraints are binding. If some of the $p-1$ constraints that correspond to the objective functions are non-binding, then problem (2) may have alternative optima that may produce better values for the $p-1$ objective functions (as the relevant constraints are not binding). In this case, the optimal solution of (2) may not be a non-dominated solution of the multiobjective problem. In order to overcome this defect, we propose the transformation of the objective function constraints to equalities by explicitly incorporating the appropriate slack (for minimization objectives) or surplus (for maximization objectives) variables. At the same time, these slack or surplus variables are used as a second term (with lower priority) in the objective function, forcing the program to produce only efficient solutions.
The new problem becomes:

$$\max\left\{ f_1(\mathbf{x})+\delta\times\sum_{i=2}^{p} s_i \mid f_i(\mathbf{x})-s_i=e_i,\quad i=2,3,\ldots,p \wedge \mathbf{x}\in \mathrm{S},\ s_i\in R^+\right\} \tag{3}$$

where δ is a small number, usually between 10^{-3} and 10^{-6}. The above formulation (3) of the ε-constraint method produces only efficient solutions and avoids the generation of weakly efficient solutions. In order to avoid any scaling problems it is also recommended to replace the s_i in the second term of the objective function by s_i/rg_i, where rg_i is the range of the i-th objective function (as calculated from the payoff matrix). Thus, the objective function of the ε-constraint method becomes:

$$\max\left\{ f_1(\mathbf{x})+\delta\times\sum_{i=2}^{p}\frac{s_i}{rg_i}\right\} \tag{4}$$

The proposed version of the ε-constraint method that corresponds to model (3) with the objective function (4) reflects to the so-called augmented ε-constraint method or AUGMECON method.

Early exit from the loops

It must be noted that a similar, to above mentioned, attempt which uses slack and surplus variables in order to transform the ε-constraints into equality constraints is proposed by Ehrgott and Ryan (2002) in their *elastic constraint method*. The main innovation in the elastic constraint method is the way it handles infeasibilities in the ε-constraint problem (surplus variables that are incorporated with penalty coefficients in the objective function). In AUGMECON the case of infeasibilities is treated differently, using the early exit from the nested loops. This is an innovative addition to the algorithm that is activated whenever problem (3) becomes infeasible for some combination of e_i. The early exit from the loops works as follows: The bounding strategy for each one of the objective function starts from the more relaxed formulations (lower bound for a maximization objective function or upper bound for a minimization) and move to strictest (individual optima). In this way, when we arrive at an infeasible solution there is no need to perform the remaining runs of the loop (as the problem will become even stricter and thus remains infeasible) and we force an exit from the loop. In problems with many objective functions (as the portfolio selection problem may have) the early exit from the loops significantly improves the solution time.

Implementation

AUGMECON's implementation is also characterized by a couple of attractive attributes. From the payoff matrix we obtain the range of each one of the $p-1$ objective functions that are going to be used as constraints. Then we divide the range of the i-th objective function to q_i equal intervals using $q_i - 1$ intermediate equidistant *grid points*. Thus we have in total $q_i + 1$ grid points that are used to vary parametrically the right-hand side of the i-th objective function. The total number of runs becomes $(q_2+1)\times(q_3+1)\times\cdots\times\left(q_p+1\right)$. A desirable characteristic of the AUGMECON method (and generally the ε-constraint philosophy) is that we have total control of the density of the efficient set representation, by properly assigning the values to the q_i. The higher the number of grid points the denser is the representation of the efficient set but with the cost of higher computation times. A trade-off between the density of the efficient set and the computation time is always advisable.

Multiobjective optimization results to the acquisition of the efficient surface of Pareto optimal portfolios, from which the final selection has to be made. The effective representation of this information, in such problem types, i.e. selecting portfolios under multiple investment objectives, is a matter of crucial importance. An efficient way to delineate the solutions space is by means of a visualization tool. For this reason we have introduced the concept of the *security impact plane*.

The *security impact plane* is a two-dimensional chart in which the impact of each security in the efficient surface of Pareto optimal portfolios is expressed. More specifically, the average weight of each security in the Pareto optimal portfolios appears on the vertical axis, while the frequency of its participation in them appears on the horizontal axis. It is also stressed that only securities that are present in at least one of the Pareto optimal portfolios are depicted in the *security impact plane*, while the scale of the vertical axis depends on the lower and upper values of the obtained weights.

A *security impact plane* can be theoretically divided in four quadrants. The lower-left quadrant contains securities that are present in few Pareto optimal portfolios and have relatively low average weights. The upper-left quadrant contains securities that are found in few Pareto optimal portfolios but have relatively high average weights. The lower-right quadrant contains securities that are found in the majority of Pareto optimal portfolios but have relatively low average weights. Finally, the upper-right quadrant contains securities that are found in the majority of Pareto optimal portfolios and have relatively high average weights.

On this basis, the plane's four quadrants reflect to following areas: (a) the *light and unlikely* area (lower-left quadrant), (b) the *heavy and unlikely* area (upper-left quadrant), (c) the *light and likely* area (lower-right quadrant), and (d) the *heavy and likely* area (upper-right quadrant). Apparently, the upper-right quadrant is the most attractive and securities that are located in it are the most dominant ones regarding to their impact in the efficient surface of Pareto optimal portfolios. It would be a congenial idea to consider these securities the *approved list* in the investment's committee output, the content of which may be of particular value for the whole portfolio analytics procedure.

4 Modeling process

In this section we precisely develop the whole grid of details regarding the modeling process we introduce for dealing with the multiobjective portfolio optimization problem. For clarity, decision variables are denoted by lowercase letters, while parameters are expressed by capitals.

Assume an existing portfolio P^o at time t_o, where the weight of the i-th security in it is denoted as W_i^o. Then, for the maximization of the portfolio's capital return at time t:

$$\max Z_1 = \sum_{i=1}^{N} \left(\bar{R}_i \times w_i - B_i \times w_i^+ - S_i \times w_i^-\right) \tag{5}$$

where $\bar{R}_i$ and w_i are the average capital return and weight of the i-th security at time t, w_i^+ and w_i^- are the weights of the i-th security that are being bought and sold correspondingly at time t (with regard to the existing weight w_i^o), B_i and S_i are the buying and selling commissions and N is the number of securities. The model's various weight quantities are controlled through the following constraint equation:

$$w_i = W_i^o + w_i^+ - w_i^- \tag{6}$$

For the minimization of portfolio's mean-absolute deviation (MAD), as measure of the non-systematic risk exposure:

$$\min Z_2 = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{i=1}^{N} w_i \times \left(R_{it} - \bar{R}_i\right) \right| \tag{7}$$

where T is the number of historical periods and R_{it} is the return of the i-th security during the t-th historical period. The linearity of the model is persevered through the Konno and Yamazaki (1991) transformation. On this basis, T additional positive continuous variables (y_t) are used for the representation of each period's absolute deviation from the mean, resulting in $2 \times T$ constraints:

$$\sum_{i=1}^{N} w_i \times \left(R_{it} - \bar{R}_i\right) + y_t \geq 0, \quad t = 1, 2, \ldots, T \tag{8}$$

$$\sum_{i=1}^{N} w_i \times \left(R_{it} - \bar{R}_i\right) - y_t \leq 0, \quad t = 1, 2, \ldots, T \tag{9}$$

Then, the objective function is transformed to:

$$\min Z_2 = \frac{1}{T} \sum_{t=1}^{T} y_t \tag{10}$$

For the maximization of portfolio's dividend yield:

$$\max Z_3 = \sum_{i=1}^{N} D_i \times w_i \tag{11}$$

where D_i is the dividend yield of the i-th security.

Moving to the model's set of constraints, the corresponding expressions for capital completeness and no short sales allowance are firstly introduced:

$$\sum_{i=1}^{N} w_i = 1 \quad \text{and} \quad w_i \geq 0 \tag{12}$$

The model is also equipped with binary variables, in order to control the existence ($b_i = 1$) or non-existence ($b_i = 0$) of the i-th security in the portfolio. The use of binary variables produces the following cardinality constraint equation, allowing for the direct determination of the number of securities in the portfolio:

$$S_L \leq \sum_{i=1}^{n} b_i \leq S_U \tag{13}$$

where S_L and S_U are the minimum and maximum number of securities allowed to participate in the portfolio.

Two more constraint equations are introduced for the exact calibration of the i-th security's lower and upper weight in the portfolio:

$$w_i - W_L \times b_i \geq 0, \quad i = 1, 2, \ldots, N \tag{14}$$

$$w_i - W_U \times b_i \leq 0, \quad i = 1, 2, \ldots, N \tag{15}$$

where W_L and W_U are the minimum and maximum allowable security weights in the portfolio.

Moreover, for the proper tuning of the portfolio's beta coefficient (market risk), we exploit the inequality:

$$M_L \leq \sum_{i=1}^{N} M_i \times w_i \leq M_U \tag{16}$$

where M_i is the beta coefficient of the i-th security and M_L, M_U are the lower and upper allowable portfolio beta bounds correspondingly.

The proposed modeling approach also incorporates the popular 5-10-40 constraint, a rule that is based on the established legislative framework for mutual funds. According to this specific rule, securities of the same issuer are allowed to weight a maximum of 5 % of the mutual fund's net asset value. However, in the special case that the total weight of all the issuer's securities with amounts between 5 and 10 % is less than 40 %, they are allowed to amount a maximum of 10 %.

More specifically, the 5-10-40 rule is modeled as follows:

$$\sum_{i \in P_j} w_i \leq 0.05 \times \left(1 + o_j\right), \quad j = 1, 2, \ldots, S \tag{17}$$

$$\sum_{i \in P_j} w_i - 0.05 \leq a_j \leq 0.05 \times o_j, \quad j = 1, 2, \ldots, S \tag{18}$$

$$0.05 \times \sum_{j=1}^{S} o_j + \sum_{j=1}^{S} a_j \leq 0.4 \tag{19}$$

where P_j is a portfolio that contains the securities of the j-th issuer, o_j are binary variables which indicate if the j-th issuer's weight is less than 5 % $\left(o_j = 0\right)$ or more than 5 % $\left(o_j = 1\right)$, a_j are the auxiliary continuous variables associated with the j-th issuer (receiving zero value if the issuers' weight is less than or equal to 5 % or the value of the additional weight over 5 % if the issuers' weight is greater than 5 %) and S is the number of issuers. It must be stressed that if there is only one security per issuer, then $S = N$.

The out-of-sample evaluation phase is strongly connected with the exploitation of a novel concept that we have also introduced in the field of multiobjective portfolio optimization. It is the *barycentric portfolio* term, a notion which implies and allows for adopting a rebalancing strategy of equal-weighted investment objectives. More precisely, the *barycentric portfolio* is defined as the solution possessing the minimum distance D_k from the barycenter of a hypothetical equilateral triangle that represents the efficient surface of Pareto optimal portfolios. The edges of this triangle represent the optimum of each one of the three objective functions (see Fig. 3).

The *barycentric portfolio* is formulated by means of a set of normalized scores, i.e. values that are connected with the obtained Pareto optimal solutions. The normalized scores of each one of the k Pareto optimal solutions are calculated through the following expressions:

$$S_{Z_{1k}} = \frac{Z_{1k} - Z_{1\,\min}}{Z_{1\,\max} - Z_{1\,\min}}, \tag{20}$$

$$S_{Z_{2k}} = \frac{Z_{2\,\max} - Z_{2k}}{Z_{2\,\max} - Z_{2\,\min}} \quad \text{and} \tag{21}$$

$$S_{Z_{3k}} = \frac{Z_{3k} - Z_{3\,\min}}{Z_{3\,\max} - Z_{3\,\min}} \tag{22}$$

Then, the Euclidean distance D_k of each one of the Pareto optimal solutions from the barycenter is given by the equation:

$$D_k = \sqrt{\left(S_{Z_{1k}} - 0.5\right)^2 + \left(S_{Z_{2k}} - 0.5\right)^2 + \left(S_{Z_{3k}} - 0.5\right)^2} \tag{23}$$

The *barycentric portfolio* is defined as the Pareto optimal solution with the minimum distance from the barycenter:

$$P_{Barycentric} = \min\{D_1, D_2, \ldots, D_k, \ldots\} \tag{24}$$

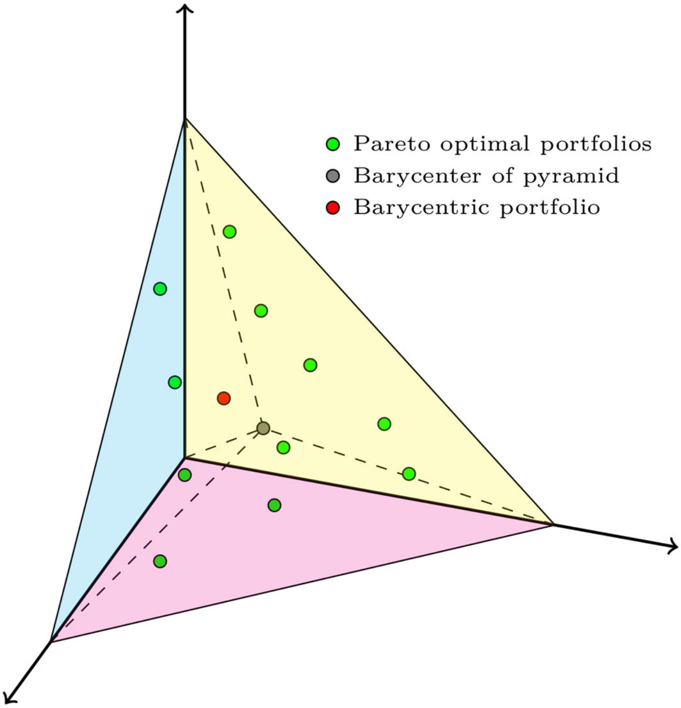


Fig. 3 The barycentric portfolio

Adopting the *barycentric portfolio strategy*, we simulate the characteristics of an investment policy profile containing balanced preferences regarding the performance of the selected portfolio in the underlying objectives. In the same manner, other behaviors can also be simulated and result to the corresponding most preferred portfolio among the efficient surface.

5 Empirical testing

The proposed methodology has been applied to one of the most popular European stock market indices, the Eurostoxx 50 (www.stoxx.com). Eurostoxx 50 is the leading blue-chip index for the Eurozone area, providing a high capitalization representation of supersector leaders in the Eurozone and covering 50 stocks from 12 countries. The empirical testing process has two basic characteristics: (a) a rolling historical optimization horizon of 2-years constant length, starting from 01-01-08, and (b) an out-of-sample evaluation period of one year, with 3 months rebalancing step and more specifically, three rebalancing actions across 2010. Consequently, the validation phase covers in total a 3-years period.

At this point, it has to be stressed that, instead of feeding the proposed model with historical time-series data for calculating the required expected returns and dividend yields, two options can also be considered alternatively: (a) direct exploitation of analysts' estimates, or (b) use of the output of index, multi-index and equilibrium models. This characteristic provides the suggested process with strong flexibility value, while allows for tightly bridging the distance between the well-established models of classical theory of finance and modern global optimization techniques.

The first two objective functions, Z_1 and Z_2, i.e. capital return and MAD are fed with weekly data, while the third objective function, Z_3, i.e. dividend yield is fed with quarterly data. The portfolio's beta coefficient was constrained between 0.7 and 1.3, while the buying and selling commissions correspondingly were set to 0.1 and 0.15 % correspondingly. Regarding the cardinality constraints and due to the 5-10-40 norm, which forces the minimum number of securities that participate in the portfolio to 16, only an upper bound was determined to 20. Moreover, the minimum buy-in threshold was set to 3 %. It is stressed that the above model finally consists of 304 continuous variables, 100 binary variables and 514 constraints.

The computational process of the empirical testing phase is now described. More precisely, we exploit the model presented in Sect. 3, using a 2-years historical optimization horizon, from 01-01-08 to 31-12-09, in order to obtain the Pareto optimal portfolios. At the end of the first quarter of 2010, we use the securities' actual returns from 01-01-10 to 31-03-10, for calculating the actual returns of the obtained Pareto optimal portfolios, including the return of the barycentric portfolio. We also calculate the Eurostoxx 50 return for the same quarter. The results are shown in Fig. 4a, where the green line represents the benchmark return and the yellow column the barycentric portfolio return. The obtained barycentric portfolio is used as initial portfolio for the second run. The previous procedure is repeated for the rest of the 2010 quarters, each time using a rolling historical optimization horizon of 2-years constant length. The remaining of the out-of-sample results are presented in Fig. 4b–d, while Fig. 4e refers to the case of a hypothetical buy-and-hold strategy, where the initial barycentric portfolio is being held for the whole year.

Eventually, three rebalancing decisions have been made, at the end of each quarter during 2010. With regard to the above analysis, the corresponding efficient surfaces are also shown in Fig. 5a–d, via 3D graphs. The barycentric portfolio is appropriately indicated. An overall summary of the out-of-sample empirical testing results is provided in Table 1 and Fig. 6.

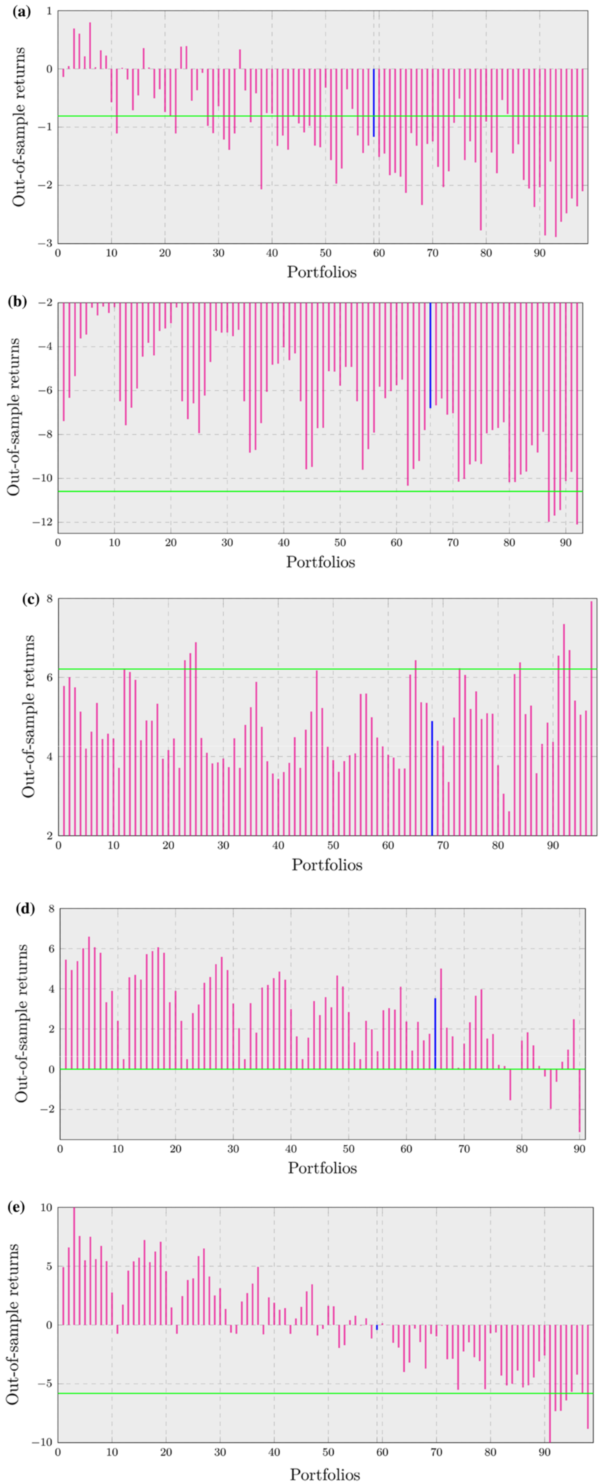
The major finding to be stressed is that in all four runs, a sufficient number of Pareto optimal portfolios that outperform the index do exist. On average, the 59.25 % of the generated Pareto optimal portfolios from both the four runs do better than the underlying benchmark. It is also observed that whenever the benchmark return deteriorates from quarter to quarter, the vast majority of the generated Pareto optimal outperforms the index (93 % in Q2 and 91 % in Q4). The above fact may offer an indication of the additional resistance that multiobjective portfolio optimization provides to market downward trends.

According to Table 1, the annual return for the benchmark is −5.81 %, while the annual return for the barycentric portfolio is 0.03 %. It is also noted that without rebalancing, i.e. buy on 01-01-10 and hold until 31-12-10, the obtained barycentric portfolio attains an annual return of −0.41 %. This is still better than the benchmark, but worse than the application of the rebalancing strategy. Moreover, the successfulness of rebalancing is also verified by the fact that the issue of transaction costs did not affect at all the achieved total return, comparing to that of the buy-and-hold strategy.

As analyzed in Sect. 2, the impact of each security in the efficient surface, is expressed by means of two measures: (a) the average weight of each security in the Pareto optimal portfolios, and (b) the frequency of its participation in them. These measures are visualized through the *security impact plane*. These charts have been created for each one of the four runs (Fig. 7a–d).

It is observed that 33 different securities (out of 50) appear in the Pareto optimal portfolios for both the four runs. As an example, regarding the allocation of the first run: (a) 16 securities appear in the *light and unlikely* area, (b) 0 securities appear in the *heavy and unlikely* area, (c) 8 securities appear in the *light and likely* area, and (d) 9 securities appear in the *heavy and*

Fig. 4 Out-of-sample empirical testing



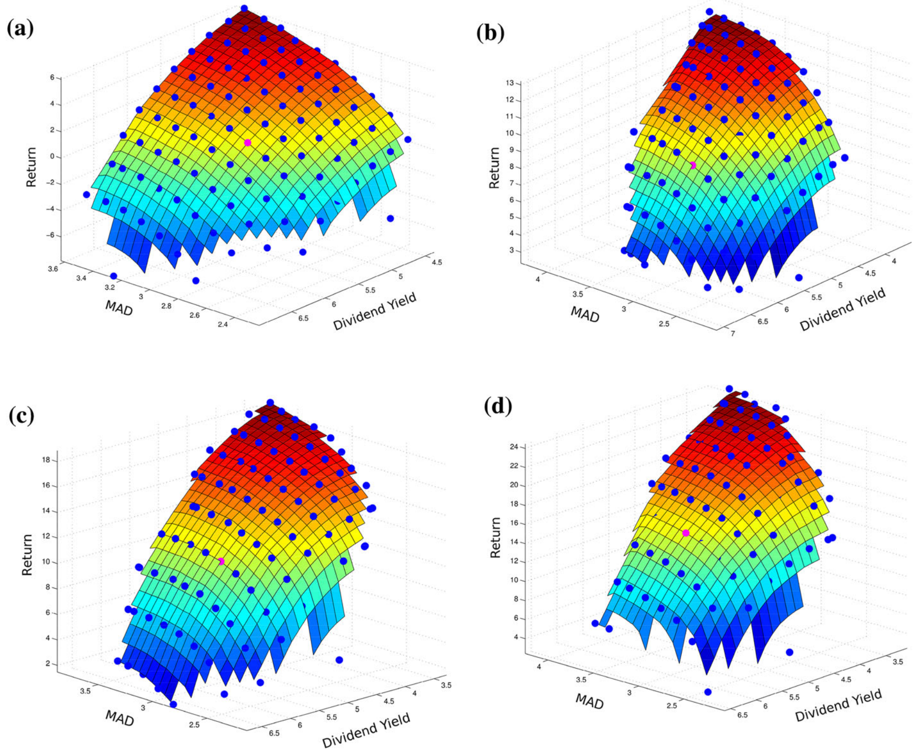


Fig. 5 The obtained efficient surfaces

Table 1 Results

Historical optimization horizon	Out-of-sample period	Benchmark Eurostoxx 50 returns (%)	Rebalancing Barycentric portfolio returns (%)	Buy and hold Barycentric portfolio returns (%)	% of Pareto optimal portfolios that beat the benchmark (%)
Jan 08–Dec 09	Jan 10–Mar 10	−0.81	−1.17		38
Apr 08–Mar 10	Apr 10–Jun 10	−10.59	−6.80		93
Jul 08–Jun 10	Jul 10–Sep 10	6.21	4.89		15
Oct 08–Sep 10	Oct 10–Dec 10	0.01	3.53		91
Total return		−5.81	0.03	−0.41	92

likely area. In general, an elaborate examination of all *security impact planes* may enrich the analysis by concluding with: (a) a possible subset of dominant securities, i.e. those appear in the *heavy and likely* area (upper-right quadrant), (b) a possible subset of supportive securities, i.e. those appear in the *light and likely* area (lower-right quadrant), and (c) a possible subset of securities that never participate in any of the Pareto optimal portfolios (17 securities). Finally, Table 2a, b offer details regarding the composition and beta coefficients of the barycentric portfolios for the four runs, while Fig. 8 provides a view of the underlying sectoral synthesis.

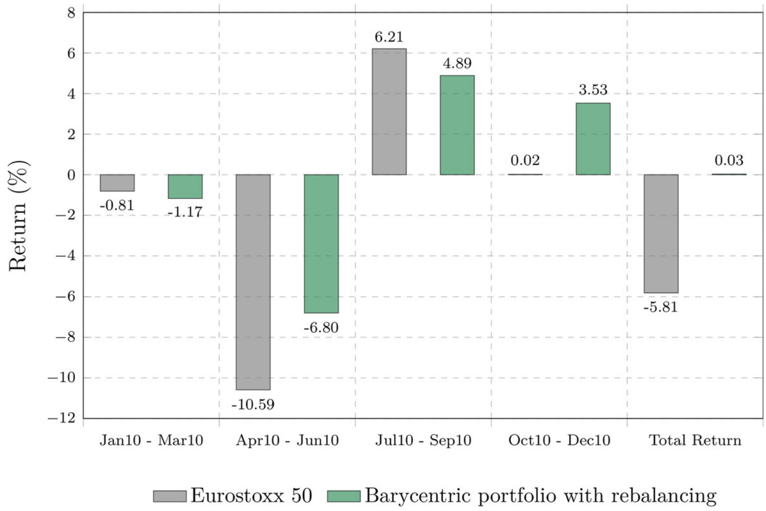


Fig. 6 Overall empirical testing results

According to this information, all four portfolios consist of 17 securities (an upper bound was set to 20), while its beta coefficients vary within a rather small range, i.e. from 0.852 to 0.934 (the lower and upper bounds were set to 0.7 and 1.3). The relatively low portfolios betas imply conservative investment approaches and are in direct accordance with Euro area's crisis in progress during 2010, despite 3Q's impermanent recovery. Moreover, the presence of the *communications* and *non-cyclical consumer* sectors, i.e. sectors of high demand, are the most dominant for the four runs, having weights of 25 , 20 , 28.8 , 25 and 20 , 19.6 , 20 , 18 % correspondingly.

At the same time, the *financial* sector appears with remarkably decreasing weights in both four portfolios, i.e. 29.5 , 19.6 , 13.7 and 9.3 %. This finding clearly explains the barycentric portfolio's excess returns in 2Q and 4Q. Moreover, the diminishing exposure in financial components constitutes an additional proof of the severity of problems that still haunt the financial sector in Europe during the last four years and also indicates the proposed model's sensitivity in capturing certain market situations.

6 Conclusions

We have focused on the theoretical establishment and managerial positioning of a practical and unified multiobjective portfolio optimization business framework. The underlying gap is a critical one, since there is a clear need for practitioners to be supported in handling the connected complexity of Pareto optimality, on an applied specific basis.

On this basis, our aim in this article is to expand the limited mean-variance framework, within which the portfolio selection process is typically confronted. Thus, we attempt to provide portfolio managers with an effective decision support tool for assessing multiple investment objectives. More precisely, focusing on the multiobjective portfolio management process, we propose a straightforward *multiobjective managerial protocol*.

The core of the proposed approach consists of an integrated mixed-integer mathematical programming model, able to cope with multiple investment objectives and real-world

Fig. 7 The security impact planes

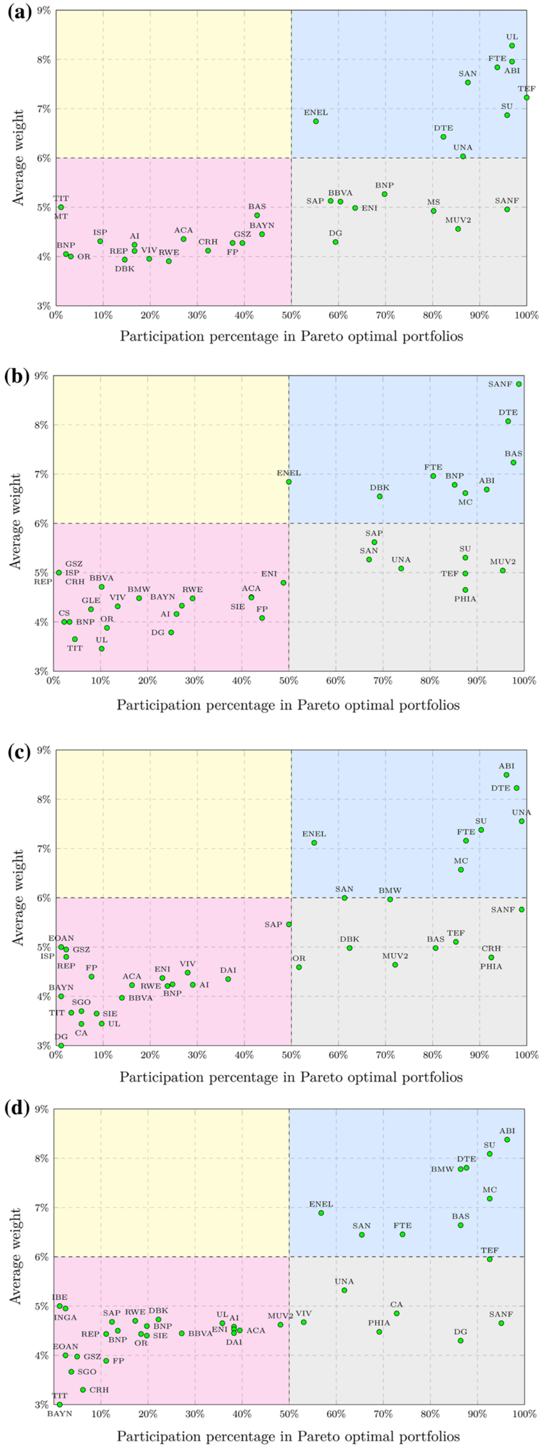


Table 2 Portfolios' composition

No.	Symbol	Security	Industry	Weight (%)
(*a*)				
Jan 10–Mar 10, Portfolio 59 ($\beta = 0.872$)				
1	TEF	Telefonica	Communications	10.0
2	SANB	Banco Santander	Financial	5.0
3	ENI	ENI	Energy	5.0
4	UNA	Unilever	Consumer, Non-cyclical	5.0
5	SANS	Sanofi-Aventis	Consumer, Non-cyclical	5.0
6	ABI	Anheuser-Busch	Consumer, Non-cyclical	10.0
7	BNP	BNP Paribas	Financial	5.0
8	BAS	BASF	Basic Materials	5.0
9	SAP	SAP	Technology	3.0
10	DTE	Deutsche Telekom	Communications	5.0
11	ENEL	Enel	Utilities	4.6
12	FTE	France Telecom	Communications	10.0
13	BBVA	Banco Bilbao Vizcaya Argentari	Financial	5.0
14	SU	Schneider Electric	Industrial	5.0
15	DG	Vinci	Industrial	3.0
16	MUV2	Muenchener Rueckversicherungs	Financial	4.5
17	UL	Unibail-Rodamco	Financial	10.0
Apr 10–Jun 10, Portfolio 66 ($\beta = 0.934$)				
1	TEF	Telefonica	Communications	5.0
2	SANB	Banco Santander	Financial	5.0
3	ENI	ENI	Energy	5.0
4	UNA	Unilever	Consumer, Non-cyclical	4.6
5	SANS	Sanofi-Aventis	Consumer, Non-cyclical	10.0
6	ABI	Anheuser-Busch	Consumer, Non-cyclical	5.0
7	BNP	BNP Paribas	Financial	5.0
8	BAS	BASF	Basic Materials	10.0
9	MC	LVMH Moet Hennessy Louis Vuitton	Diversified	4.8
10	DTE	Deutsche Telekom	Communications	10.0
11	ENEL	Enel	Utilities	5.0
12	FTE	France Telecom	Communications	5.0
13	DBK	Deutsche Bank	Financial	4.6
14	SU	Schneider Electric	Industrial	10.0
15	RWE	RWE	Utilities	3.0
16	PHIA	Koninklijke Philips Electronic	Industrial	3.0
17	MUV2	Muenchener Rueckversicherungs	Financial	5.0
(*b*)				
Jul 10–Sep 10, Portfolio 68 ($\beta = 0.852$)				
1	TEF	Telefonica	Communications	5.0
2	SANB	Banco Santander	Financial	5.0
3	UNA	Unilever	Consumer, Non-cyclical	5.0

Table 2 continued

No	Symbol	Security	Industry	Weight (%)
4	SANS	Sanofi-Aventis	Consumer, Non-cyclical	5.0
5	ABI	Anheuser-Busch	Consumer, Non-cyclical	10.0
6	BAS	BASF	Basic Materials	5.0
7	MC	LVMH Moet Hennessy Louis Vuitton	Diversified	4.5
8	DTE	Deutsche Telekom	Communications	10.0
9	ENEL	Enel	Utilities	5.0
10	FTE	France Telecom	Communications	10.0
11	DBK	Deutsche Bank	Financial	3.7
12	BMW	Bayerische Motoren Werke	Consumer, Cyclical	3.0
13	SU	Schneider Electric	Industrial	10.0
14	VIV	Vivendi	Communications	3.8
15	PHIA	Koninklijke Philips Electronic	Industrial	5.0
16	MUV2	Muenchener Rueckversicherungs	Financial	5.0
17	CRH	CRH	Industrial	5.0
Oct 10–Dec 10, Portfolio 65 ($\beta = 0.905$)				
1	TEF	Telefonica	Communications	5.0
2	SANB	Banco Santander	Financial	5.0
3	ENI	ENI	Energy	5.0
4	UNA	Unilever	Consumer, Non-cyclical	3.0
5	SANS	Sanofi-Aventis	Consumer, Non-cyclical	5.0
6	ABI	Anheuser-Busch	Consumer, Non-cyclical	10.0
7	BAS	BASF	Basic Materials	10.0
8	MC	LVMH Moet Hennessy Louis Vuitton	Diversified	5.0
9	DTE	Deutsche Telekom	Communications	10.0
10	ENEL	Enel	Utilities	5.0
11	FTE	France Telecom	Communications	5.0
12	BMW	Bayerische Motoren Werke	Consumer, Cyclical	5.0
13	SU	Schneider Electric	Industrial	10.0
14	ACA	Credit Agricole	Financial	4.3
15	VIV	Vivendi	Communications	5.0
16	DG	Vinci	Industrial	3.2
17	PHIA	Koninklijke Philips Electronic	Industrial	4.6

non-convex policy limitations, such as: (a) cardinality constraints, (b) buy-in thresholds, (c) transaction costs, and (d) compliance norms. It was then solved by use of the AUGMECON method, in which a number of significant computational innovations are embedded. The contribution of the proposed approach is also founded on the introduction of two novel concepts in the field of multiobjective portfolio optimization, i.e. the *security impact plane* and the *barycentric portfolio*.

The validity of the model has been verified through an illustrative application on the Eurostoxx 50. Taking into account the highly technical complexity of the implemented investment policy statement, the results reported are characterized as interesting enough, since

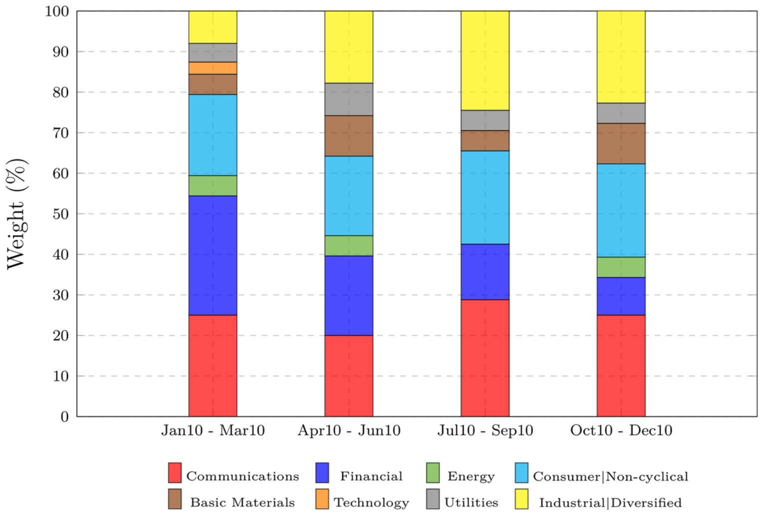


Fig. 8 The sectoral synthesis of the barycentric portfolios

a sufficient number of efficient or Pareto optimal portfolios that produced by the model, outperform the benchmark. Further research has to be focused on the issue of even more strengthening the observed positive consistency of empirical testing results.

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