On reducing the Erdös-Szekeres problem into a constraint unsatisfiability problem regarding certain multisets

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Abstract

We introduce the theory of *div point set*, which aims to provide a framework to study the combinatoric nature of any set of points in general position on an Euclidean plane. We then show that the Erdös-Szekeres conjecture can be proven through proving the unsatisfiability of some first-order logic formulae concerning some sets of multisets of uniform cardinality over boolean variables under certain constraints.

1 Introduction

More than half a century ago Erdös and Szekeres [1] proved that for all $n \geq 3$, there exists an integer N such that among any N points in general position on an Euclidean plane, there always exists a set of n points forming a convex polygon, and conjectured that the smallest number for N is determined by the function g where $g(n) = 2^{n-2} + 1$. This is now known as the Erdös-Szekeres conjecture (and the problem of determining such N is often referred to as the *Happy Ending Problem* since it led to the marriage of Szekeres and Klein, who first proposed the question). 25 years after the initial paper, Erdös and Szekeres [2] showed that $g(n)$ cannot be less than 2^{n-2}. Currently the best known bounds for $g(n)$ are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1$$

Many improvements for the upper bound have been made throughout the decades. The current upper bound was obtained by Tóth and Valtr [3] in 1998 as an improvement to the previous upper bound by Kleitman and Pachter [4] in the same year.

There are also attempts to verify individual instances of n. In 2002 Szekeres and Peters [5] showed that the conjecture holds for $n = 6$ with the help of an algorithm that performs an exhaustive search. Even to this day it remains the largest n for which we know for certainty that the conjecture holds.

Our aim in this article is to demonstrate that solving the instances of a certain multiset unsatisfiability problem would prove the Erdös and Szekeres conjecture, through the theory of *div point set*.

1.1 preliminary

Throughout the article we would assume Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). The term "class" would be used to denote a collection of sets satisfying some predicate ϕ. A general form of Kuratowski definition would be used to define tuples. 2-tuples would be referred to as ordered pairs. A set of n-tuples would be referred to as relation. A relation is called binary relation when it is a set of ordered pairs. It would not matter as how natural numbers are defined as long as they satisfy Peano axioms. $\mathbb{N}_{\geq c}$ would be used to refer to the set of natural numbers greater or equal to some $c \in \mathbb{N}$ (i.e. $0 \notin \mathbb{N}_{\geq 1}$). For any 2 natural numbers a, b, $\binom{a}{b}$ denotes the binomial coefficient *a choose b*. Everything would be formulated under first order logic (or FOL). $\wedge$,$\vee$,$\neg$,$\Rightarrow$ and $\Leftrightarrow$ would be used to mean *and*, *or*, *not*, *imply* and *iff* respectively. We write $A := B$ if A is defined to be equivalent to B. $\forall x_1 \in A \, \forall x_2 \in A \, \forall x_3 \in A \, ... \forall x_n \in A$ would be abbreviated to

$$\forall x_1, x_2, x_3 ... x_n \in A$$

and $\exists x_1 \in A \, \exists x_2 \in A \, \exists x_3 \in A \, ... \exists x_n \in A$ to

$$\exists x_1, x_2, x_3 ... x_n \in A$$

For any set V, $|V|$ would be used to denote its cardinality. When a set V has a cardinality of k, for some $k \in \mathbb{N}$, we may describe it as a *k-cardinality set.* $\mathscr{P}(V)$ would be used to denote its power set, notionally,

$$\mathscr{P}(V) = \{v : v \subseteq V\}$$

The subscript of a set union or intersection may be omitted to indicate that the union or intersection is applied to each element in the set. That is to say,

$$\begin{aligned}
&\text{For any set, } A, \\
&\bigcup A = \bigcup_{a \in A} a = \bigcup_{k=1}^{n} a_k = a_1 \cup a_2 \cup .. a_n \\
&\bigcap A = \bigcap_{a \in A} a = \bigcap_{k=1}^{n} a_k = a_1 \cap a_2 \cap .. a_n \\
&\text{where } |A| = n \text{ and } a_x = a_y \Leftrightarrow x = y
\end{aligned}$$

This use of notation applies to $\bigvee$ and $\bigwedge$ as well:

$$\text{For any set of formulae, } A,$$
$$\bigvee A = \bigvee_{a \in A} a = \bigvee_{k=1}^{n} a_k = a_1 \vee a_2 \vee ..a_n$$
$$\bigwedge A = \bigwedge_{a \in A} a = \bigwedge_{k=1}^{n} a_k = a_1 \wedge a_2 \wedge ..a_n$$
$$\text{where } |A| = n \text{ and } a_x = a_y \Leftrightarrow x = y$$

For any k-tuple T, $\pi_i(T)$ would be used to denote the i-th element of T where $i \leq k$; $\pi_\cup(T)$ would be used the denote the union of 1st, 2nd ... k-th elements of a k-tuple; and $\pi_\cap(T)$ would be used to denote the intersection in such fashion. That is to say,

$$\text{For any } k\text{-tuple, } T, \quad \begin{aligned} \pi_\cup(T) &= \bigcup_{i=1}^{k} \pi_i(T) \\ \pi_\cap(T) &= \bigcap_{i=1}^{k} \pi_i(T) \end{aligned}$$

To avoid ambiguity, for any function $f : X \longrightarrow Y$, we would use $f^{members}$ to denote a new function, from $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ such that

$$\forall x \in \mathscr{P}(X)$$
$$f^{members}(x) := \bigcup_{a \in x} \{f(a)\}$$

Here is a generalization of it, $f^{members^n}$, defined recursively:

$$f^{members^1}(x) := f^{members}(x)$$
$$f^{members^n}(x) := \bigcup_{a \in x} \{f^{members^{n-1}}(a)\} \text{ where } n \in \mathbb{N}_{\geq 2}$$

A multiset is a generalization of set, where the same element can occur multiple times, which is to say, two multisets are equal iff both multisets contain the same distinct elements and every distinct element occurs the same number of times in both multisets. A multiset is defined as an ordered pair (A, m_{m}) where $m_{\mathrm{m}} : A \longrightarrow \mathbb{N}_{\geq 1}$ is a function that describes the number of occurrences of each element in the multiset, and A is the set of all distinct elements in the multiset. The cardinality of a multiset (A, m_{m}) is defined as the sum of all $m_{\mathrm{m}}(x)$ for $x \in A$. Multisets are expressed using square brackets, $[\ ]$, as compared to sets which use curly brackets, $\{\}$. One example: let f be a function that always outputs 1

$$[f(x) : x \in \mathbb{N}_{\geq 1} : x \leq 3] = [1, 1, 1] = (\{1\}, \{(1, 3)\})$$
as compared to
$$\{f(x) : x \in \mathbb{N}_{\geq 1} : x \leq 3\} = \{1\}$$

A hypergraph is a generalization of graph, where an edge can contain any number of vertices. It is defined as an ordered pair (V, E) where E is a subset of $\mathscr{P}(V) \setminus \varnothing$. Elements in V are referred to as vertices while elements in E are referred to as edges or hyperedges. A hypergraph is k-uniformed when all of its hyperedges have the same cardinality. A graph in the conventional sense can thus be defined as a 2-uniformed hypergraph.

A full vertex coloring on some hypergraph, (V, E), is defined as a surjective function, $C : V \longrightarrow cDom$, where $cDom$ is a non-empty subset of $\mathbb{N}$, and is often referred to as the set of colors. When $|Dom| = 2$, we say the coloring is monochromatic. We would use $FullCol(G, cDom)$ to denote the set of all possible full vertex colorings on a graph G of the set of colors $cDom$. That is to say, for any graph G of n vertices and any $cDom$,

$$|FullCol(G, cDom)| = n^{|cDom|}$$

SAT, or the Boolean Satisfiability Problem, is the problem of determining if there exists some value-assignment for the variables in a propositional logic formula such that it yields $True$ (i.e. it is satisfiable).

A formula is referred to as a tautology when there exists no value-assignment for the variables in it such that it yields $False$ (e.g. $a \vee \neg a$).

A formula is in Disjunctive Normal Form (or DNF), when it is a disjunction of conjunctions. A disjunction is a formula, F, that can be expressed as $\bigwedge S$ where S is a set of formulae, while a conjunction is a formula, F, that can be expressed as $\bigvee S$ where S is a set of formulae.

2 *Div point set* as a representation for any set of points in general position

We start off by introducing an object which we would be referring to as *div point set*.

Definition 1. A *div point set* is any order-pair (P, Θ_P) satisfying

$$|\Theta_P| = \binom{|P|}{2} \wedge P \neq \varnothing \tag{2.1}$$

$$\forall D_n \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ |d_n| = 2 \\ d_n \in \mathcal{P}(P) \\ |\delta_n| = 2 \\ \bigcup \delta_n = P \setminus d_n \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.2}$$

$$\forall D_n, D_m \in \Theta_P \quad D_n = D_m \Leftrightarrow \pi_1(D_n) = \pi_1(D_m) \tag{2.3}$$

We would be using $\mathcal{DPS}^*$ to denote the class of all ordered pairs satisfying (2.1), (2.2) and (2.3). Thus $\mathfrak{X}$ is a *div point set* iff $\mathfrak{X} \in \mathcal{DPS}^*$.

For any n points in general position, where $n \geq 2$, we can always select any 2 arbitrary points and draw a line across them, dividing the rest of the points into 2 disjoint sets. So long as the points are in general position, we can be sure that no 3 points forms a line, and thus each of the remaining $n-2$ points would always be in one of these sets. Let's refer to these 2 disjoint sets as *divs* produced by a *divider* made up of 2 distinct points, and the points in the *divs* as *TBD points* of the *divider* (*TBD* is short for *to-be-distributed-among-divs*). The process of selecting 2 distinct points from a set of point P, creating a *divider*, and producing 2 *divs* can be repeated $\binom{|P|}{2}$ times until all sets of 2 points in P are selected.

Any set of points P in general position on an Euclidean plane where $|P| \geq 2$ can be represented by some *div point set* (P, Θ_P). Each $D_n \in \Theta_P$ would be referred to as a *dividon*, to be interpreted as follows:

$$(d_n, \delta_n) \coloneqq D_n \quad \left| \begin{array}{l} d_n = \{a, b\} \\ \quad a \text{ and } b \text{ represent the 2 points which make up the } \textit{divider} \\ \delta_n = \{div_1, div_2\} \\ \quad div_1 \text{ and } div_2 \text{ represent the 2 } \textit{divs} \text{ produced by the } \textit{divider} \\ \quad \bigcup \delta_n \text{ thus represents the set of } \textit{TBD points} \text{ of the } \textit{divider} \end{array} \right. \tag{2.4}$$

The sets of points in *Figures I, II* and *III* can be represented by any *div point set*

(A, Θ_A) as long as A is a set of 4 arbitrary elements a, b, c, d and

$$\begin{aligned}\Theta_A = \{&(\{a,b\}, \{(\{c\}, \{d\}\}),\\ &(\{a,c\}, \{(\{b\}, \{d\}\}),\\ &(\{a,d\}, \{(\{b\}, \{c\}\}),\\ &(\{b,c\}, \{(\{a,d\}, \varnothing\}),\\ &(\{b,d\}, \{(\{a,c\}, \varnothing\}),\\ &(\{c,d\}, \{(\{a,b\}, \varnothing\})\})\end{aligned}$$

To make sense of the *div point set* representation, we label the third point from the bottom in *Figure I* and the second point from the bottom in *Figures II* and *III* as a (note that each of these points is surrounded by the remaining 3 points in the figure). For the rest of the points in each figure we label them arbitrarily as b, c, and d. Notice how in all *Figures* for the *dividers* made up of a and any one of the 3 points, we have 2 *divs* of 1 cardinality, and how for the rest of the *dividers*, we have one *div* with the remaining points in it, and another *div* as an empty set - precisely that of what *div point set* (A, Θ_A) describes.

Only a handful of *div point sets* can be used to represent points in general position in $\mathbb{E}^2$. For majority of $\mathscr{X} \in \mathscr{DPS}^*$, there exists no meaningful interpretation for $\pi_1(\mathscr{X})$ as some sets of points in $\mathbb{E}^2$ such that each $D \in \pi_2(\mathscr{X})$ is a *dividon* that describes how *TBD points* are distributed between the 2 *divs* produced by each *divider*. A classical example would be (Q, Θ_Q) where Q is any set of 4 arbitrary elements a, b, c, d and

$$\begin{aligned}\Theta_Q = \{&(\{a,b\}, \{(\{c,d\}, \varnothing\}),\\ &(\{a,c\}, \{(\{b,d\}, \varnothing\}),\\ &(\{a,d\}, \{(\{b,c\}, \varnothing\}),\\ &(\{b,c\}, \{(\{a,d\}, \varnothing\}),\\ &(\{b,d\}, \{(\{a,c\}, \varnothing\}),\\ &(\{c,d\}, \{(\{a,b\}, \varnothing\})\})\end{aligned}$$

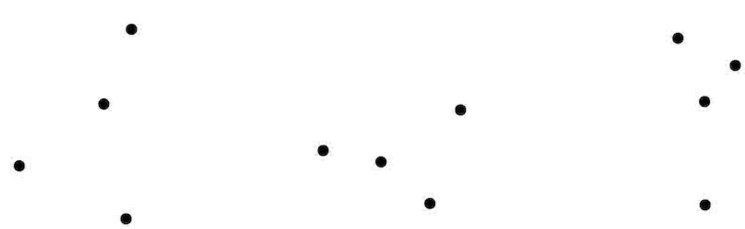

Figure I *Figure II* *Figure III*

For a *div point set* (P, Θ_P) to have a meaningful interpretation for P as some set of points in $\mathbb{E}^2$, it has to satisfy certain conditions. For any 3 distinct points, x, y, and z in general position in $\mathbb{E}^2$, let $\langle x, y\rangle^z$ denote the *div* containing z produced by the *divider* made up of the point x and y, and $\langle x, y\rangle^{\text{-}z}$ denote the *div* not containing z produced by the *divider*. After some experimentation with points in $\mathbb{E}^2$, one would make the observation that the following formulas always hold true for any distinct points a, b, c, d in $\mathbb{E}^2$. (2.5) is trivially true. (2.6), (2.7) and (2.8) are demonstrated in *Figures IV*, *V* and *VI* respectively.

$$a \neq b \neq c \neq d \quad \Rightarrow \quad \left| \begin{array}{l} a \in \langle b, c\rangle^{\text{-}d} \Leftrightarrow d \in \langle b, c\rangle^{\text{-}a} \\ a \in \langle b, c\rangle^{d} \Leftrightarrow d \in \langle b, c\rangle^{a} \end{array} \right. \tag{2.5}$$

$$a \neq b \neq c \neq d \quad \Rightarrow \quad \left| \begin{array}{l} c \in \langle a, b\rangle^{\text{-}d} \\ \quad \Leftrightarrow ((a \in \langle b, c\rangle^{d} \wedge a \in \langle b, d\rangle^{c}) \\ \qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{\text{-}c})) \end{array} \right. \tag{2.6}$$

$$a \neq b \neq c \neq d \quad \Rightarrow \quad \left| \begin{array}{l} c \in \langle a, b\rangle^{d} \\ \quad \Leftrightarrow ((a \in \langle b, c\rangle^{d} \wedge a \in \langle b, d\rangle^{\text{-}c}) \\ \qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{c})) \end{array} \right. \tag{2.7}$$

$$a \neq b \neq c \neq d \quad \Rightarrow \quad \left| \; a \in \langle b, c\rangle^{-d} \wedge a \in \langle b, d\rangle^{-c} \Rightarrow a \in \langle c, d\rangle^{b} \right. \tag{2.8}$$

In the context of *div point sets*, (2.5) is always true by (2.2) (recall $\bigcap \delta = \varnothing$), while (2.6), (2.7) and (2.8) can be rewritten as constraints on the *dividons* of a *div point set* (P, Θ_P) as shown in (2.10), (2.11), and (2.12), using a function, ϕ, for determining if two distinct *TBD* points, a and b, belong to the same *div* in some δ of a *dividon*:

$$\phi(\delta, w) = \begin{cases} 1 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 = div_2 \\ 0 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 \neq div_2 \end{cases} \text{where} \left| \begin{array}{l} \delta = \{div_1, div_2\} \\ w = \{a, b\} \end{array} \right. \tag{2.9}$$

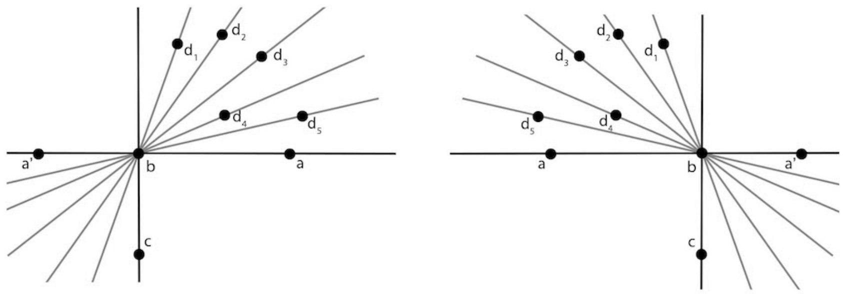

Figure IV

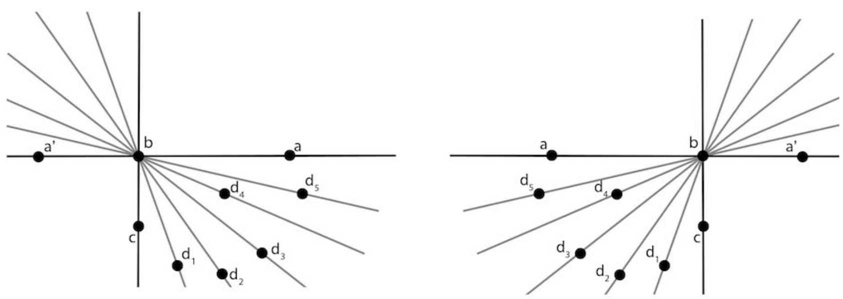

Figure V

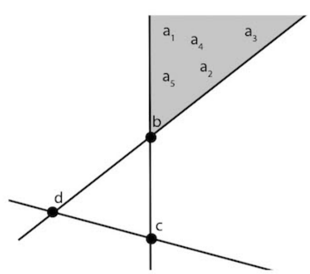

Figure VI

$$
\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge |\bigcap_{n=1}^{3} \pi_1(D_n)| = 1 \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 0 \\
&\qquad\qquad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge |\bigcap_{n=1}^{3} \pi_1(D_n)| = 1 \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 1 \\
&\qquad\qquad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) \neq \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) \subset R \wedge |\bigcup_{n=1}^{3} \pi_1(D_n)| = 3 \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = 0 \\
&\qquad\qquad \Rightarrow \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) = 1\ )
\end{aligned}
\tag{2.12}
$$

Remark. In (2.10) and (2.11), it is not necessary to write down $D_1 \neq D_2 \neq D_3$ explicitly as a part of the conjunction in the antecedent since $\bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge |\bigcap_{n=1}^{3} \pi_1(D_n)| = 1$ ensures that D_1, D_2, and D_3 are distinct.

Axiom 1. A *div point set* (P, Θ_P) has an interpretation for P as some set of points in $\mathbb{E}^2$ such that $D \in \Theta_P$ each describes the relative positions of the points (in terms of how the *TBD points* of each *divider* is distributed between 2 *divs* it produced) iff it is in $\mathscr{DPS}^+$, the class of *div point sets* satisfying (2.10), (2.11), and (2.12).

Remark. For *div point sets* of 3 or less points, it is vacuously true that they satisfy (2.10), (2.11), and (2.12) and thus they are by default in the class $\mathscr{DPS}^+$. This is consistent with Euclidean geometry: any set of 3 points in general position can be represented by any *div point set* of 3 points, and the same goes to any set of 2 or 1 point.

Definition 2. We say that two *div point sets* (A, Θ_A) and (B, Θ_B) are isomorphic iff there exists a bijective function $f : A \xrightarrow{1:1} B$ which preseves the structure of the *divions.*

Notationally,

$$\begin{aligned}(A, \Theta_A) \cong (B, \Theta_B) \Leftrightarrow \exists f : A \xrightarrow{1:1} B \\ \forall D_A \in \Theta_A \\ \exists D_B \in \Theta_B \\ (d_a, \delta_a) := D_A \\ (d_b, \delta_b) := D_B \\ f^{members}(d_a) = d_b \Leftrightarrow f^{members^2}(\delta_a) = \delta_b\end{aligned} \tag{2.13}$$

in which case f would be referred to as the isomorphism between the two *div point sets*.

Remark. It is trivially true that all div point sets (P, Θ_P) in $\mathscr{DPS}^*$ where $|P| \leq 3$ are isomorphic to any div point sets (Q, Θ_Q) in $\mathscr{DPS}^*$ where $|Q| = |P|$.

Theorem 1. $\neg(\mathfrak{X} \cong Conc_4^1) \Leftrightarrow (\mathfrak{X} \cong Conv_4)$ for all $\mathfrak{X} \in \mathscr{DPS}_4^+$ where $\mathscr{DPS}_4^+$ denotes the *div point sets* of 4 points in $\mathscr{DPS}^+$ and

$$\begin{aligned} Conc_4^1 &= (Cc_4^1, \Theta_{Cc_4^1}) & Conv_4 &= (Cv_4, \Theta_{Cv_4}) \\ Cc_4^1 &= \{1,2,3,4\} & Cv_4 &= \{1,2,3,4\} \\ \Theta_{Cc_4^1} &= \{(\{1,2\},\{\{3\},\{4\}\}), & \Theta_{Cv_4} &= \{(\{1,2\},\{\{3,4\},\varnothing\}), \\ &\quad (\{1,3\},\{\{2\},\{4\}\}), && \quad (\{1,3\},\{\{2\},\{4\}\}), \\ &\quad (\{1,4\},\{\{2\},\{3\}\}), && \quad (\{1,4\},\{\{2,3\},\varnothing\}), \\ &\quad (\{2,3\},\{\{1,4\},\varnothing\}), && \quad (\{2,3\},\{\{1,4\},\varnothing\}), \\ &\quad (\{2,4\},\{\{1,3\},\varnothing\}), && \quad (\{2,4\},\{\{1\},\{3\}), \\ &\quad (\{3,4\},\{\{1,2\},\varnothing\})\} && \quad (\{3,4\},\{\{1,2\},\varnothing\})\} \end{aligned} \tag{2.14}$$

Proof for Theorem 1.

Summary. In Part 1 of the proof we would define a function ψ that returns 0 or 1 based on the *divs* of a *dividon* of some *div point set* in $\mathscr{DPS}_4^+$. In Part 2 we would define the class $\mathscr{DPS}_4^{\mathbb{N}}$, the vertices of a hypergraph H, and a function Col that uses ψ, and show that for every $\mathfrak{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, there exists a unique full vertex monochromatic coloring $Col(\pi_2(\mathfrak{X}))$ on H. In Part 3 we would define the edges of H in such a manner that the coloring $Col(\pi_2(\mathfrak{X}))$ on H satisfies some conditions iff $\mathfrak{X}$ satisfies (2.10) and (2.11). In Part 4 we demonstrate that for the coloring to satisfy the conditions, there exists only 3 $\mathscr{Scenarios}$. The colorings described in $\mathscr{Scenarios}$ II and III are isomorphic to $Col(\pi_2(Conc_4^1))$ and $Col(\pi_2(Conv_4))$ respectively, and $Conc_4^1$ and $Conv_4$ both satisfy (2.12), but the *div point set* the coloring in $\mathscr{Scenario}$ *I* is based on does not satisfy (2.12). Therefore we conclude that *div point sets* of 4 points satisfying (2.10), (2.11) and (2.12) are either isomorphic to $Conc_4^1$ or $Conv_4$, thus proving *Theorem 1*.

Part 1. For any *div point set* (P, Θ_P) in $\mathscr{DPS}_4^+$, since $|P| = 4$, we can be certain that each *divider* has $4 - 2 = 2$ *TBD* points, and so

$$
\begin{aligned}
&\forall D \in \Theta_P \\
&\qquad \text{Let } a \text{ and } b \text{ be the TBD points i.e. } \{a, b\} = P \setminus \pi_1(D) \\
&\qquad type_0 \coloneqq \{\{a\}, \{b\}\} \\
&\qquad type_1 \coloneqq \{\{a, b\}, \varnothing\} \\
&\qquad \pi_2(D) \in \{type_0, type_1\}
\end{aligned} \tag{2.15}
$$

Recall that in (2.9) we have defined a function ϕ that takes in some $\pi_2(D)$ and a set of 2 *TBD points*, and returns 1 if the *TBD points* belong to the same *div* in $\pi_2(D)$, or 0 if they belong to different *divs* in $\pi_2(D)$. For $\mathscr{X} \in \mathscr{DPS}_4^+$, we can define a new function ψ, a simpler version of ϕ that does basically the same thing by exploiting (2.15), namely the fact every $\pi_2(D)$ is either $type_0$ or $type_1$:

$$
\psi(\delta) = \begin{cases} 1 & \text{if } \exists div \in \delta \quad |div| = 2 \\ 0 & \text{if } \forall div \in \delta \quad |div| = 1 \end{cases} \tag{2.16}
$$

For every *dividon* D of any $\mathscr{X} \in \mathscr{DPS}_4^+$, we have

$$
\phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \tag{2.17}
$$

Part 2. Let's define $\mathscr{DPS}_4^{\mathbb{N}}$ to be the class of all *div point sets* (P, Θ_P) for which $P = \{1, 2, 3, 4\}$. All $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ would have the same *dividers* (recall the set of *dividers* is just the set of elements in $\mathscr{P}(P)$ whose cardinality is 2). Now let $H = (V, E)$ be a hyper-graph whose vertices are the *dividers* of *div point sets* in $\mathscr{DPS}_4^{\mathbb{N}}$. Using ψ, we can define a bijective function, Col, that transforms the set of *dividons* of any $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ into some full vertex monochromatic coloring for H.

$$
\begin{aligned}
Col : \{\pi_2(\mathscr{X}) : \mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}\} &\xrightarrow{1:1} FullCol(H, \{0, 1\}) \\
Col(\Omega_P) &= \{(\pi_1(D), \psi(\pi_2(D))) : D \in \Omega_P\}
\end{aligned} \tag{2.18}
$$

It is bijective since

$$
\begin{aligned}
&\forall \mathscr{X}_1, \mathscr{X}_2 \in \mathscr{DPS}^{\mathbb{N}} \\
&\qquad Col(\pi_2(\mathscr{X}_1)) = Col(\pi_2(\mathscr{X}_2)) \Leftrightarrow \mathscr{X}_1 = \mathscr{X}_2
\end{aligned} \tag{2.19}
$$

due to the fact that for any 2 *dividons*, D_1 and D_2, made up of the same dividers (i.e. $\pi_1(D_1) = \pi_1(D_2)$), of 2 *div point set* respectively, $\psi(\pi_2(D_1)) = \psi(\pi_2(D_2))$ iff $D_1 = D_2$.

Part 3. Let's define any set of 3 *dividers* containing 1 element in common to be an edge of H, notationally (recall that the vertices are the *dividers*),

$$E = \{e \in \mathscr{P}(V) : |e| = 3 \wedge |\bigcap e| = 1\} \tag{2.20}$$

H is a 3-uniform hypergraph with 4 hyperedges. For some $\mathfrak{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ to satisfy (2.10) and (2.11) is equivalent to having $Col(\pi_2(\mathfrak{X})) \in FullCol(H, \{0, 1\})$ to satisfy the I and II.

I. For any vertex v colored 0, the other 2 vertices belonging to the same edge as v must be colored the same.

II. For any vertex v is colored 1, the other 2 vertices belonging to the same edge as v must be colored differently.

This is in virtue of fact that for any $\mathfrak{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, (2.10) and (2.11) can be rewritten as having the coloring $Col(\pi_2(\mathfrak{X}))$ to satisfy some formulae, namely (2.21) and (2.22).

$$\begin{aligned}
&\forall e \in E \\
&\quad \forall d_1, d_2, d_3 \in e \\
&\quad\quad d_1 \neq d_2 \neq d_3 \\
&\quad\quad \Rightarrow (Col(\pi_2(\mathfrak{X}))(d_1) = 0 \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_2) = Col(\pi_2(\mathfrak{X}))(d_3))
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
&\forall e \in E \\
&\quad \forall d_1, d_2, d_3 \in e \\
&\quad\quad d_1 \neq d_2 \neq d_3 \\
&\quad\quad \Rightarrow (Col(\pi_2(\mathfrak{X}))(d_1) = 1 \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_2) \neq Col(\pi_2(\mathfrak{X}))(d_3))
\end{aligned} \tag{2.22}$$

(Just a remainder: Col is the function that transforms some $\pi_2(\mathfrak{X})$ into a coloring while $Col(\pi_2(\mathfrak{X}))$ is the actual coloring, which is also a function as defined in *Preliminary*.) This is a result of the fact that

$$\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D \in \Theta_P \\
&\quad\quad \phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D))
\end{aligned} \tag{2.23}$$

for any *div point set* (P, Θ_P) for which $|P| = 4$ (recall (2.15)), combined with the fact that any *dividons* D_1, D_2 and D_3 where

$$|\bigcap_{n=1}^{3} \pi_1(D_n)| = 1 \wedge |\bigcup_{n=1}^{3} \pi_1(D_n)| = 4 \tag{2.24}$$

would respectively have three *dividers* d_1, d_2 and d_3 where

$$|\bigcap_{n=1}^{3} d_n| = 1 \wedge d_1 \neq d_2 \neq d_3 \tag{2.25}$$

which are precisely what makes up an edge of H (recall (2.20)). Therefore some $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ satisfies (2.10) and (2.11) iff $Col(\pi_2(\mathscr{X}))$ satisfies *I* and *II*.

Part 4. To satisfy I and II, 3 vertices belonging to the same edge must either be colored $[0, 0, 0]$ or $[0, 1, 1]$.

Suppose we start off by giving three arbitrary vertices belonging to the same edge the coloring of $[0, 0, 0]$, to satisfy I the rest of the vertices have to be colored the same (recall that each vertex belongs to 2 different edges). We either end up with H having all vertices colored 0 (let's call it $\mathscr{S}cenario\ I$), or 3 vertices colored 0 and 3 vertices colored 1 (let's call it $\mathscr{S}cenario\ II$).

Now suppose we start off by giving three arbitrary vertices belonging to the same edge the coloring of $[0, 1, 1]$, to satisfy I the remaining 2 vertices of another edge, which the vertex colored 0 belongs to, have to be colored the same. If we colored them both 0, the last uncolored vertex must then be colored 1, since it belongs to 2 edges where the other 2 vertices are colored differently. We would end up in $\mathscr{S}cenario\ II$ again. On the other hand, if we colored them both 1, the last uncolored vertex must then be colored 0, since it belongs to 2 edges where the other 2 vertices are colored the same. Let's call this $\mathscr{S}cenario\ III$, where 2 vertices are colored 0 and 4 vertices are colored 1.

A pictorial description of the colorings is shown in Figure VII.

$\mathscr{S}cenario$ I describes a coloring isomorphic to $Col(\pi_2(\mathscr{X}_\varnothing))$ where $\mathscr{X}_\varnothing \in \mathscr{DPS}_4^{\mathbb{N}}$ and

$$\begin{aligned}\pi_2(\mathscr{X}_\varnothing) = \{&(\{1,2\}, \{(\{3,4\}, \varnothing\}),\\ &(\{1,3\}, \{(\{2,4\}, \varnothing\}),\\ &(\{1,4\}, \{(\{2,3\}, \varnothing\}),\\ &(\{2,3\}, \{(\{1,4\}, \varnothing\}),\\ &(\{2,4\}, \{(\{1,3\}, \varnothing\}),\\ &(\{3,4\}, \{(\{1,2\}, \varnothing\})\}\end{aligned}$$

while $\mathscr{S}cenario$ II describes a coloring isomorphic to $Col(\pi_2(Conc_4^1))$ and $\mathscr{S}cenario$ III descrbies a coloring isomorphic to $Col(\pi_2(Conv_4))$. $Conc_4^1$ and $Conv_4$ both satisfy (2.12), and $\mathscr{X}_\varnothing$ does not. Since any *div point set* of 4 points is isomorphic to some $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, and in $\mathscr{DPS}_4^{\mathbb{N}}$ only $Conc_4^1$ and $Conv_4$ satisfy all (2.10), (2.11), and (2.12), we conclude that

$$\forall X \in \mathscr{DPS}_4^+ \quad \exists a \in \{Conc_4^1, Conv_4\} \quad X \cong a \tag{2.26}$$

□

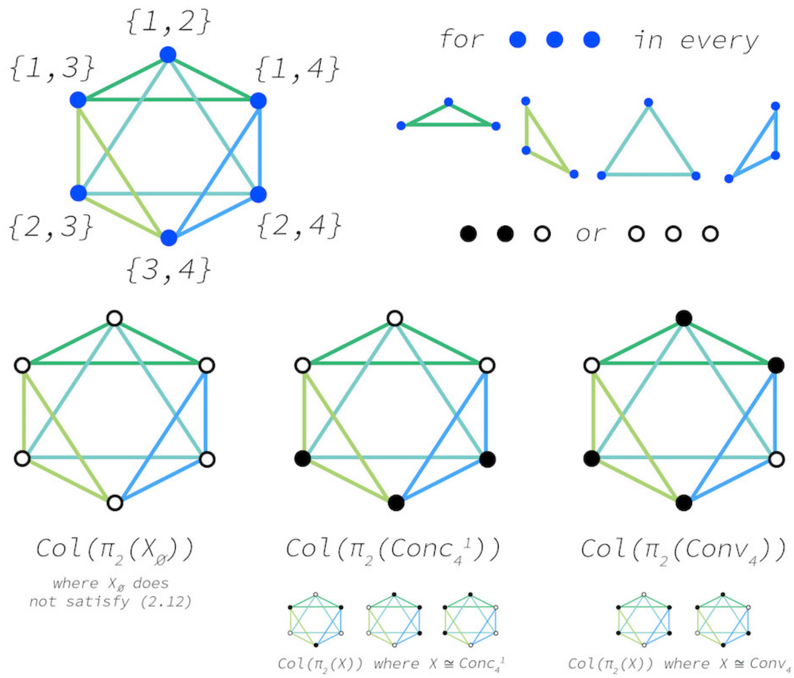

Figure VII

Remark. In Euclidean geometry, *Theorem 1* can be interpreted as stating that for any set of 4 distinct points in general positions, it is either the case that it forms a structure where 1 point is inside a triangle formed by connecting the rest of 3 points, or the case that a convex polygon can be created by connecting the 4 points in a certain manner, which can be verified rather easily by a human child with a pen, a piece of paper and a love for Euclidean geometry.

2.1 *unit div point set* and *sub div point set*

For *div point sets* of 5 or more points, the function ψ defined in (2.16) would not be really useful since there would be 3 or more *TBD points* in each *dividon*. That means we cannot apply to same technique above to derive *div point sets* of 5 or more points satisfying (2.10), (2.11) and (2.12). With that in mind, we introduce the object *unit div point set* which makes use of *unit dividons*.

Definition 3. A *unit div point set* is any order-pair (P, Ω_P) satisfying (2.27), (2.28) and (2.29).

$$|\Omega_P| = \binom{|P|}{2}\binom{|P|-2}{2} \wedge P \neq \varnothing \tag{2.27}$$

$$\forall D_n \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ |\bigcup \delta_n| = 2 \\ \bigcup \delta_n \in \mathscr{P}(P \setminus d_n) \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.28}$$

$$\forall D_n, D_m \in \Omega_P \quad \left| \; \pi_1(D_n) \cup \bigcup \pi_2(D_n) = \pi_1(D_m) \cup \bigcup \pi_2(D_m) \Leftrightarrow D_n = D_m \right. \tag{2.29}$$

We would be using $\mathscr{UDPS}^*$ to denote the class of all *unit div point set.*

Remark. Similar to how *div point sets* of 4 points always satisfy (2.15), a *unit div point set* (P, Ω_P) always satisfies (2.30).

$$\begin{array}{l} \forall D \in \Omega_P \\ \quad \exists a, b \in P \setminus \pi_1(D) \\ \quad\quad type_0 \coloneqq \{\{a\}, \{b\}\} \\ \quad\quad type_1 \coloneqq \{\{a, b\}, \varnothing\} \\ \quad\quad \pi_2(D) \in \{type_0, type_1\} \end{array} \tag{2.30}$$

Remark. One may immediately notice that any *div point sets* of 4 points also satisfy (2.27), (2.28) and (2.29), and any *unit div point set* of 4 points also satisfy (2.1), (2.2) and (2.3), which is to say,

$$\{\mathscr{X}_{udps} \in \mathscr{UDPS}^* : |\pi_1(\mathscr{X}_{udps})| = 4\} = \{\mathscr{X}_{dps} \in \mathscr{DPS}^* : |\pi_1(\mathscr{X}_{dps})| = 4\} \tag{2.31}$$

by virtue of the fact that

$$\binom{|4|}{2}\binom{|4-2|}{2} = \binom{|4|}{2} \tag{2.32}$$

and

$$\begin{array}{l} \forall \mathscr{X} \in \mathscr{UDPS}^* \\ \quad |\pi_1(\mathscr{X})| = 4 \quad \Rightarrow \left| \begin{array}{l} \forall D_n \in \pi_2(\mathscr{X}) \\ \quad \bigcup \delta_n = P \setminus d_n \\ \forall D_n, D_m \in \pi_2(\mathscr{X}) \\ \quad \pi_1(D_n) = \pi_1(D_m) \Leftrightarrow D_n = D_m \end{array} \right. \end{array} \tag{2.33}$$

As we can see, the difference between a *div point set* and a *unit div point set* lies in that the former relies on a single *dividon* to describe the distribution of the $|P|-2$ *TBD points* between 2 *divs* for each *divider*, while the later relies on $\binom{|P|-2}{2}$ *unit dividons* for that (since each *unit dividon* only describes the distribution of 2 *TBD points*). For every $\mathscr{X}_{dps} \in \mathscr{DPS}^*$ there exists a unique $\mathscr{X}_{udps} \in \mathscr{UDPS}^*$ which $\mathscr{X}_{dps}$ can be transformed into, by breaking down each *dividon* into $\binom{|P|-2}{2}$ *unit dividons* containing the same *divider*, achievable using the function $\mathscr{bd}$ defined in (2.34).

$$\begin{aligned} \mathscr{bd}(D,P) &= \{(\pi_1(D), \mathscr{d}_u(x, \pi_2(D)) : x \in \mathscr{P}(P \setminus \pi_1(D)) : |x| = 2\} \\ \mathscr{d}_u(x, divs) &= \begin{cases} \{x, \varnothing\}, & \text{if } \exists div \in divs\ x \subseteq div \\ \{i \in \mathscr{P}(x) : |i| = 1\}, & \text{if } \neg(\exists div \in divs\ x \subseteq div) \wedge x \subseteq \bigcup divs \end{cases} \end{aligned} \tag{2.34}$$

$\mathscr{bd}$ takes in a *dividon* D and a set of points P, and returns a set of *unit dividons*. It uses $\mathscr{d}_u$, which takes in a set of 2 point x and a set of *divs* $divs$, and returns a set of *divs* for a *unit dividon* whose set of *TBD* points is x.

Definition 4. The function $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ transforms a *div point set* into a *unit div point set*.

$$\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}_{dps}) = (\pi_1(\mathscr{X}_{dps}), \bigcup\{\mathscr{bd}(D, \pi_1(\mathscr{X}_{dps})) : D \in \pi_2(\mathscr{X}_{dps})\}) \tag{2.35}$$

$\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ can be implemented in Haskell as follows:

```haskell
import Control.Monad
import Data.List ((\\))
powerList = filterM (const [True, False])

f:: ([Int],[([Int],[[Int]])]) -> ([Int],[([Int],[[Int]])])
f (points,dividons) = (points,unit_dividons)
   where
      unit_dividons = foldl (++) [] $ map get_unit_dividons dividons
      get_unit_dividons (d,(delta1:_)) = [(d,(\(a:b:_)->
         if a `in_same_div_as_b` b
            then [[a,b],[]]
            else [[a],[b]])
         x ) |
         x <- powerList (points \\ d), length x == 2,
         let (in_same_div_as_b) a b = (a `elem` delta1) == (b `elem` delta1)]
```

Remark. It is no surprise that

$$\forall \mathscr{X} \in \mathscr{DPS}^* \qquad \mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}) = \mathscr{X} \Leftrightarrow |\pi_1(\mathscr{X})| = 4 \tag{2.36}$$

For each *dividon* D in any *div point set* of 4 points, $\mathcal{bd}(D, \pi_1(\mathcal{X}_{dps})$ in (2.35) contains only one element and the element is D itself. For *div point sets* of 5 or more points $\mathcal{bd}(D, \pi_1(\mathcal{X}_{dps})$ would contain 3 or more elements. On the other hand,

$$\forall \mathcal{X} \in \mathcal{DPS}^* \qquad \mathcal{F}_{\overrightarrow{u}dps}^{\mathcal{DPS}}(\mathcal{X}) = (\pi_1(\mathcal{X}), \varnothing) \Leftrightarrow |\pi_1(\mathcal{X})| \leq 3 \tag{2.37}$$

since $\binom{n-2}{2} = 0$ for $n \leq 3$ and that is not going to be useful. So it is more sensible to define $\mathcal{F}_{\overrightarrow{u}dps}^{\mathcal{DPS}}$ over *div point sets* of 4 or more points i.e.

$$\mathcal{F}_{\overrightarrow{u}dps}^{\mathcal{DPS}} : \mathcal{DPS}^*_{\geq 4} \longrightarrow \mathcal{UDPS}^* \tag{2.38}$$

Lemma 1. $\mathcal{F}_{\overrightarrow{u}dps}^{\mathcal{DPS}} : \mathcal{DPS}^*_{\geq 4} \longrightarrow \mathcal{UDPS}^*$ is injective but not surjective.

Proof for Lemma 1. It is injective because for every *dividon* D of any *div point set* of 4 or more points, $\mathcal{bd}(D, \pi_1(\mathcal{X}_{dps})$ in (2.35) differs depending on D. It is not surjective onto the co-domain $\mathcal{UDPS}^*$ as a consequence of *I* and *II* below.

I. As a consequence of $|\delta_n| = 2$ in (2.2), there exists $\mathcal{X}_{udps} \in \mathcal{UDPS}^*$, where $\mathcal{F}_{\overrightarrow{u}dps}^{\mathcal{DPS}}(\mathcal{W}') = \mathcal{X}_{udps}$ iff $\mathcal{W}'$ is an ordered pair satisfying all the conditions to be a *div point set* except that it has $|\pi_2(D)| > 2$ for some *dividon* D, and as a result it is not in $\mathcal{DPS}^*$. E.g. *unit div point sets* with *unit dividons* such as

$$\{(a,b),(\{c\},\{d\})\},\{(a,b),(\{c\},\{e\})\},\{(a,b),(\{e\},\{d\})\}$$

can only be transformed from a *div-point-set*-like ordered pair where $|\delta_n| = 3$ for some *dividion*, in this case:

$$\{(a,b),(\{c\},\{d\},\{e\})\}$$

which is to say, for any $\mathcal{X}_{udps'} \in \mathcal{UDPS}^*$, where $\mathcal{X}_{udps'} = \mathcal{F}_{\overrightarrow{u}dps}^{\mathcal{DPS}}(\mathcal{X}_{dps})$ for some $\mathcal{X}_{dps} \in \mathcal{DPS}^*$, $\mathcal{X}_{udps'}$ satisfies (2.39).

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \pi_2(\mathcal{X}_{udps'}) \\
&\qquad D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \bigcup \pi_2(D_n)| = 3 \\
&\qquad \Rightarrow \neg((\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = \psi(\pi_2(D_3)) = 0)
\end{aligned} \tag{2.39}$$

II. As a consequence of $\bigcap \delta_n = \varnothing$ in (2.2), for any distinct *TBD* points c, d, and e, of some *divider* of a *div point set*, if c and d are in the same *div*, and d and e are in the same *div*, it is certainly the case for c and e to be found in the same *div*. So *unit div point set* with *unit dividons* such as

$$\{(a,b),(\{c,d\},\varnothing)\},\{(a,b),(\{c,e\},\varnothing)\},\{(a,b),(\{e\},\{d\})\}$$

can not be transformed from any *div point set*, which is to say, for any $\mathscr{X}_{udps'} \in \mathscr{UDPS}^*$, where $\mathscr{X}_{udps'} = \mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_{dps})$ for some $\mathscr{X}_{dps} \in \mathscr{DPS}^*$, $\mathscr{X}_{udps'}$ satisfies (2.40).

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \pi_2(\mathscr{X}_{udps'}) \\
&\qquad D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \bigcup \pi_2(D_n)| = 3 \\
&\qquad \Rightarrow \neg(\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 1 \wedge \psi(\pi_2(D_3)) = 0)
\end{aligned} \tag{2.40}$$

□

Remark. Combining (2.40) and (2.39) above gives (2.41).

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\qquad (D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \bigcup \pi_2(D_n)| = 3) \\
&\qquad \Rightarrow (\psi(\pi_2(D_1)) = 1 \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))) \\
&\qquad\qquad \wedge (\psi(\pi_2(D_1)) = 0 \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3)))
\end{aligned} \tag{2.41}$$

Let's define $\mathscr{UDPS}^{\Theta}$ to be a subclass of $\mathscr{UDPS}^*$ for which $\mathscr{F}^{\mathscr{DPS}}_{\to udps} : \mathscr{DPS}^*_{\geq 4} \longrightarrow \mathscr{UDPS}^{\Theta}$ is bijective. We can be certain that $\mathscr{UDPS}^{\Theta} \subseteq \mathscr{UDPS}^{\Theta\prime}$, where $\mathscr{UDPS}^{\Theta\prime}$ is the class of *unit div point sets* of 4 or more points satisfying (2.41). It is likely the case that (2.41) is all that a *unit div point set* must satisfy to be in the class $\mathscr{UDPS}^{\Theta}$ (i.e. $\mathscr{UDPS}^{\Theta\prime} = \mathscr{UDPS}^{\Theta}$), but that is not important in the current discussion.

Lemma 2. A *unit div point set* (P, Ω_P) has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$ such that $D \in \Omega_P$ each describes the relative positions of the points (in terms of how 2 *TBD points* of each *divider* is distributed between *divs* it produced) iff it is in $\mathscr{UDPS}^+$, a subclass of $\mathscr{UDPS}^{\Theta}$, wherein each *unit div point set* satisfies (2.43), (2.44), and (2.45), in which ξ is a function that returns the union of the *divider* and the *TBD points* in a *unit dividon*, D, notationally,

$$\xi(D) = \pi_1(D) \cup \bigcup \pi_2(D) \tag{2.42}$$

$$\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D_1, D_2, D_3 \in \Omega_P \\
&\quad\quad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\quad\quad \wedge |\bigcap_{n=1}^{3} \pi_1(D_n)| = 1\ ) \\
&\quad\quad \Rightarrow (\ \psi(\pi_2(D_1)) = 0 \\
&\quad\quad\quad \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))\ )
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D_1, D_2, D_3 \in \Omega_P \\
&\quad\quad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\quad\quad \wedge |\bigcap_{n=1}^{3} \pi_1(D_n)| = 1\ ) \\
&\quad\quad \Rightarrow (\ \psi(\pi_2(D_1)) = 1 \\
&\quad\quad\quad \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3))\ )
\end{aligned} \tag{2.44}$$

$$\begin{aligned}
&\forall R \in \{S \in \mathscr{P}(P) : |S| = 4\} \\
&\quad \forall D_1, D_2, D_3 \in \Omega_P \\
&\quad\quad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\quad\quad \wedge |\bigcup_{n=1}^{3} \pi_1(D_n)| = 3\ ) \\
&\quad\quad \Rightarrow (\ \psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 0 \\
&\quad\quad\quad \Rightarrow \psi(\pi_2(D_3)) = 1\ )
\end{aligned} \tag{2.45}$$

Proof for Lemma 2. A *div point set* $\mathscr{X}_{dps}$ satisfying (2.10), (2.11), and (2.12), iff the *unit div point set* $\underset{\rightarrow}{\mathscr{F}}{}^{\mathscr{DPS}}_{udps}(\mathscr{X}_{dps})$ satisfies (2.43), (2.44), and (2.45). This can be demonstrated in a similar way as (2.23): for any *unit divdion* D_u of some *unit div point set*, $\mathscr{A}_{udps}$, and its corresponding *divdion* D of the *div point set* $\mathscr{A}_{dps}$ where $\underset{\rightarrow}{\mathscr{F}}{}^{\mathscr{DPS}}_{udps}(\mathscr{A}_{dps}) = \mathscr{A}_{udps}$ - corresponding in the sense that $D_u \in \mathscr{bd}(D, \pi_2(\mathscr{A}_{dps}))$ and so $\pi_1(D_u) = \pi_1(D)$ - let $R := \xi(D_u)$, we would have

$$\phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D_u), \bigcup \pi_2(D_u)) = \psi(\pi_2(D_u)) \tag{2.46}$$

By restricting D_1, D_2 and D_3 into satisfying $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$, we can replace every occurrence of $\phi(\pi_2(D_n), R \setminus \pi_1(D_n))$ with $\psi(\pi_2(D_n))$ (for $n \in \{1, 2, 3\}$) in (2.10), (2.11), and (2.12), and ensure the satisfiability of $\bigcup_{n=1}^{3} \pi_1(D_n) = R$ (in (2.10) and (2.11)) by further restricting these *unit dividons* to be distinct (i.e $D_1 \neq D_2 \neq D_3$). This would give (2.43), (2.44), and (2.45): they are basically a different way of expressing (2.10), (2.11), and (2.12) in the case of *unit div point sets*. Thus any $\mathcal{X}_{udps}$ in $\mathcal{UDPS}^+$ has an interpretation for $\pi_1(\mathcal{X}_{udps})$ as some set of 4 or more points in $\mathbb{E}^2$ similar to how any $\mathcal{X}_{dps}$ in $\mathcal{DPS}^+$ has an interpretation for $\pi_1(\mathcal{X}_{dps})$ (by *Axiom 1*).

□

Lemma 3. If (P, Ω_P) is in $\mathcal{UDPS}^+$, $Col_{udps}(\Omega_P)$, a full vertex monochromatic coloring on H_{udps}, satisfies (2.50) and (2.51). Here Col_{udps} is a function similar to Col in (2.18),

$$Col_{udps}(\Omega_P) = \{((\pi_1(D), \bigcup \pi_2(D)), \psi(\pi_2(D)) : D \in \Omega_P\} \tag{2.47}$$

and H_{udps} is a 3-and-6-uniform hypergraph with 2 sets of hyperedges, E_1 and E_2, defined as a 3-tuple $H_{udps} = (V_{udps}, E_1, E_2)$, constructed based on P:

$$\begin{aligned} V_{udps} &= \bigcup \{V_{of}(d, P) : d \in \mathcal{P}(P) : |d| = 2\} \\ E_1 &= \{e \in \mathcal{P}(V_{udps}) : |e| = 6 \wedge \forall v_1, v_2 \in e \;\; \pi_\cup(v_1) = \pi_\cup(v_2)\} \\ E_2 &= \{e \in \mathcal{P}(V_{udps}) : |e| = 3 \wedge |\bigcup_{v \in e} \pi_2(v)| = 3 \wedge \forall v_1, v_2 \in e \;\; \pi_1(v_1) = \pi_1(v_2)\} \end{aligned} \tag{2.48}$$

with V_{of} being a function that returns a set of ordered pairs consisting of *divider* and *TBD points* of *unit dividons* of the same *divider*,

$$V_{of}(d, P) = \{(d, P_{TBD}) : P_{TBD} \in \mathcal{P}(P \setminus x) : |P_{TBD}| = 2\} \tag{2.49}$$

and, finally, we have

$$\begin{aligned} &\forall e \in E_1 \\ &\qquad \exists v_1, v_2 \in e \quad \left| \begin{array}{l} \pi_1(v_1) = \pi_2(v_2) \\ \pi_1(v_2) = \pi_2(v_1) \\ C(v_1) = C(v_2) = 0 \\ C^{members}(e \setminus \{v_1, v_2\}) = \{1\} \end{array} \right. \\ &\qquad \Leftrightarrow \neg\exists v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} |\pi_1(v_1) \cap \pi_1(v_2) \cap \pi_1(v_3)| = 1 \\ C(v_1) = C(v_2) = C(v_3) = 0 \\ C^{members}(e \setminus \{v_1, v_2, v_3\}) = \{1\} \end{array} \right. \end{aligned} \tag{2.50}$$

$$\begin{aligned} &\forall e \in E_2 \\ &\qquad \forall v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} v_1 \neq v_2 \neq v_3 \\ \Rightarrow (C(v_1) = 1 \Leftrightarrow C(v_2) = C(v_3)) \\ \qquad \wedge (C(v_1) = 0 \Leftrightarrow C(v_2) \neq C(v_3)) \end{array} \right. \end{aligned} \tag{2.51}$$

wherein $C = Col_{udps}(\Omega_P)$.

Remark. If $\mathscr{UDPS}^{\Theta\prime} = \mathscr{UDPS}^{\Theta}$, a stronger version of *Lemma 3* is then true: (P, Ω_P) is in $\mathscr{UDPS}^{+}$ iff $Col_{udps}(\Omega_P)$ satisfies (2.50) and (2.51).

Remark. One may notice that the construction of H_{udps} depends solely on $\pi_1(\mathscr{X}_{upds})$ (i.e. the points of a *unit div point set*), as different from the coloring, which depends solely on $\pi_2(\mathscr{X}_{upds})$ (i.e. the set of *unit dividons*), similar to how the hypergraph H and the coloring on its vertices are defined back in the proof for *Theorem 1*. However, the vertices of H_{udps} are ordered pairs, structurally different from vertices of H which are sets with cardinality of 2.

Such definition for the vertices of H_{udps} in terms of not only the *divider* of a *unit dividon* but also its *TBD points* is necessary. This is because for any *unit div point set*, (P, Ω_P), there exists $\binom{|P|-2}{2}$ distinct *unit dividons* sharing a common *divider*. In order to distinguish *unit dividons* from one another in a *unit div point set* of 5 or more points, we would need to know both its *divider* and its *TBD points*.

Remark. For any *unit div point set* $\mathscr{X}$ where $|\pi_1(\mathscr{X})| = 4$, the second set of edges, E_2, of H_{upds} constructed based on $\pi_1(\mathscr{X})$ is an empty set, and thus (2.51) is vacuously true for any coloring on such H_{upds}. On the other hand, there would only be 1 edge in E_1 and the coloring $Col_{udps}(\pi_2(\mathscr{X}))$ satisfies (2.50) iff $\mathscr{X}$ is isomorphic to either $Conv^4$ or $Conc_1^4$: in (2.50), the existential predicate before the logical operator $\Leftrightarrow$ is true iff $\mathscr{X}$ is isomorphic to $Conv^4$, while the existential predicate after the unary logical operator $\neg$ after $\Leftrightarrow$ is true iff $\mathscr{X}$ is isomorphic to $Conc_1^4$.

Proof for Lemma 3. By *Lemma 2* it is clear that all *unit div point sets* in $\mathscr{UDPS}^{+}$ satisfy (2.41), (2.43), (2.44), and (2.45). We would now demonstrate that some $\mathscr{X} \in \mathscr{UDPS}^{*}$ to satisfy (2.41), (2.43), (2.44), and (2.45) iff the coloring $Col(\pi_2(\mathscr{X}))$ on the hypergraph H_{udps} constructed based on $\pi_1(\mathscr{X})$ satisfies (2.50) and (2.51).

(2.51) is simply a different way of expressing (2.41). It does so by first defining the order pairs $(d_n, \bigcup \delta_n)$ of some *unit dividons* $D_n = (d_n, \delta_n)$ that satisfy the necessary conditions (namely $(D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \bigcup \pi_2(D_n)| = 3)$) to be vertices of an edge in E_2 (recall (2.48)).

On the other hand, the coloring $Col(\pi_2(\mathscr{X}))$ satisfies (2.50) iff $\mathscr{X}$ satisfies (2.43), (2.44), and (2.45), which are basically conditional constraints on any 3 distinct *unit dividons* D_1, D_2, and D_3 where $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$ and R is some 4-cardinality subset of P. In any *unit div point set* there are a total of $\binom{4}{2} = 6$ such *unit dividons* for every 4 distinct points in P. By *Theorem 1*, a *unit div point set* of 4 points (recall that *div point sets* of 4 points are their own *unit div point sets*) satisfies (2.43), (2.44), and (2.45) iff it is isomorphic to either $Conc_4^1$ or $Conc_4$. More fundamentally, this means that any *unit div point set* (P, Ω_P) satisfies (2.43), (2.44), and (2.45) iff each 6-cardinality subset Ω_{of6} of

Ω_P, containing *unit dividon* D where $\xi(D)$ is equivalent to some $R \subseteq P$ (a set of such Ω_{of6} is construable using a function $All_{\Omega_{of6}}$ defined in (2.62) below), is isomorphic[1] to either $\pi_2(Conc_4^1)$ or $\pi_2(Conv_4)$. Therefore a *unit div point set* (P, Ω_P), satisfies (2.43), (2.44), and (2.45) iff for all 4-cardinality $R \subseteq P$, there exists a subset C' of $Col_{\mathcal{udps}}(\Omega_P)$ where for all $c \in C'$, $\xi(\pi_1(c)) = R$ and C' is isomorphic[2] to either $Col(\pi_2(Conc_4^1))$ or $Col(\pi_2(Conv_4))$. Notationally,

$$\begin{aligned}
&\forall R \in \{P' \in \mathscr{P}(P) : |P'| = 4\} \\
&\quad \exists C' \in \{S \in \mathscr{P}(Col_{\mathcal{udps}}(\Omega_P)) : \forall c \in S \; \xi(\pi_1(c)) = R\} \\
&\qquad C' \cong Col(\pi_2(Conc_4^1)) \Leftrightarrow \neg(C' \cong Col(\pi_2(Conv_4)))
\end{aligned} \tag{2.52}$$

which is what is expressed in (2.50). Note that E_1 of the $H_{\mathcal{udps}}$ can be constructed in a slightly different manner:

$$E_1 = \{\mathcal{UDs}(R) : R \in \mathscr{P}(P) : |R| = 4\} \tag{2.53}$$

where $\mathcal{UDs}$ is a function that returns a set of ordered pairs each consisting of the *divider* and the *TBD* points of every such *unit dividon* for each R:

$$\begin{aligned}
&\mathcal{UDs}(R) = \{\mathcal{ud}(d) : d \in \mathscr{P}(R) : |d| = 2\} \\
&\mathcal{ud}(d) = (d, \; R \setminus d)
\end{aligned} \tag{2.54}$$

□

Note. *isomorphic*[1]: The definition of isomorphism in (2.56) is that of *div point sets*, but the isomorphism we are talking about here is that of sets of *unit dividons*, which can be defined as follows:

$$\begin{aligned}
\Omega_1 \cong^1 \Omega_2 \Leftrightarrow\; & |\Omega_1| = |\Omega_2| \\
& \wedge \exists f_\Omega : \textstyle\bigcup_{D \in \Omega_1} \pi_1(D) \xrightarrow{1:1} \bigcup_{D \in \Omega_2} \pi_1(D) \\
& \quad \forall D_1 \in \Omega_1 \;\; \exists D_2 \in \Omega_2 \\
& \qquad f_\Omega^{members}(\pi_1(D_1)) = \pi_1(D_2) \Leftrightarrow f_\Omega^{members^2}(\pi_2(D_1)) = \pi_2(D_2)
\end{aligned} \tag{2.55}$$

It is necessary to specify Ω_1 and Ω_2 to have the same cardinality, since it is possible for a bijective function f_Ω satisfying the condition to exist in the case when $|\Omega_1| \neq |\Omega_2|$.

isomorphic[2]: The isomorphism we are talking about here is that of colorings, which can be defined as follows:

$$\begin{aligned}
C_1 \cong^2 C_2 \Leftrightarrow\; & \exists f_C : \{\pi_1(c) : c \in C_1\} \xrightarrow{1:1} \{\pi_1(c) : c \in C_2\} \\
& \quad \forall c_1 \in C_1 \\
& \qquad \exists c_2 \in C_2 \\
& \qquad\quad f_C(\pi_1(c_1)) = \pi_1(c_2) \Rightarrow \pi_2(c_1) = \pi_2(c_2)
\end{aligned} \tag{2.56}$$

Definition 5. We say that $\mathscr{X}_1 \in \mathscr{DPS}^*$ is a *sub div point set* of $\mathscr{X}_2 \in \mathscr{DPS}^*$ (denoted by $\leq$) iff the set of *unit divdion* of the corresponding *unit div point set* of $\mathscr{X}_1$ is a subset of that of $\mathscr{X}_2$. Notationally,

$$\begin{aligned}&\forall \mathscr{X}_1, \mathscr{X}_2 \in \mathscr{DPS}^* \\ &\quad \mathscr{X}_1 \leq \mathscr{X}_2 \Leftrightarrow \pi_2(\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_1)) \subseteq \pi_2(\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_2))\end{aligned} \tag{2.57}$$

Definition 6. $\mathscr{S}dps_{of}$ is a function that returns the set of all *sub div point sets* of m points for some *div point set* where $m \in \mathbb{N}_{\geq 4}$.

$$\mathscr{S}dps_{of}(\mathscr{X}_{dps}, m) = \{\mathscr{S}dps(\mathscr{X}_{dps}, P_s) : P_s \in \mathscr{P}(\pi_1(\mathscr{X}_{dps})) : |P_s| = m\} \tag{2.58}$$

where $\mathscr{S}dps$ is a function that returns the *sub div point set* of a set of points, P_s, of a *div point set* of $\mathscr{X}_{dps}$:

$$\begin{aligned}&\mathscr{S}dps(\mathscr{X}_{dps}, P_s) = \mathscr{F}^{\mathscr{UDPS}}_{\to dps}((P_s, \{D : D \in \pi_2(\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_{dps})) : \xi(D) \subseteq P_s\})) \\ &\text{where } \mathscr{F}^{\mathscr{UDPS}}_{\to dps} \text{ is the inverse of } \mathscr{F}^{\mathscr{DPS}}_{\to udps} \text{ and } \xi \text{ is defined in (2.42).}\end{aligned} \tag{2.59}$$

Since a *div point set* of n points always has $\binom{n}{m}$ distinct *sub div point sets* of m points and $\mathscr{F}^{\mathscr{DPS}}_{\to udps}$ is defined over *div point sets* of 4 or more points $\mathscr{S}dps_{of}(\mathscr{X}_{dps}, m)$ has the cardinality of $\binom{|\pi_1(\mathscr{X}_{dps})|}{m}$, for all $\mathscr{X}_{dps}$ in $\mathscr{DPS}^*$ and $m \geq 4$.

Lemma 4. For any *div point set* $\mathscr{X}$ and any natural number m greater or equal to 4, let $\mathscr{A}$ and $\mathscr{B}$ to be any 2 distinct *sub div point sets* of m points of $\mathscr{X}$, and k be the number of points $\mathscr{A}$ and $\mathscr{B}$ have in common, $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{A})$ and $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{B})$ always have $6\binom{k}{4}$ *unit dividons* in common. Notationally,

$$\begin{aligned}&\forall \mathscr{X} \in \mathscr{DPS}^* \\ &\quad \forall m \in \mathbb{N}_{\geq 4} \\ &\quad\quad \forall \mathscr{A}, \mathscr{B} \in \mathscr{S}dps_{of}(\mathscr{X}, m) \\ &\quad\quad\quad |\pi_2(\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{A})) \cap \pi_2(\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{B}))| = 6\binom{|\pi_1(\mathscr{A}) \cap \pi_1(\mathscr{B})|}{4} \Leftrightarrow \mathscr{B} \neq \mathscr{A}\end{aligned} \tag{2.60}$$

Proof for Lemma 4. For any $m \geq |\pi_1(\mathscr{X})|$, the proposition on 2 distinct elements in $\mathscr{S}dps_{of}(\mathscr{X}, m)$ is vacuously true. For any $m < 4$, the proposition is trivially true because $\binom{m}{4} = 0$ and *unit div point sets* of 3 or less points have 0 *unit dividon* (recall (2.27)). For any m less than $|\pi_1(\mathscr{X})|$ but greater than or equal to 4, the proposition can be proven by first observing that $\mathscr{UD}s(R) \cap \mathscr{UD}s(R') = \varnothing \Leftrightarrow R \neq R'$ (recall (2.54)) for any sets R and R' with cardinality of 4, indicating that no 2 distinct *unit div point set* of 4 points have a *unit dividon* in common. Notationally,

$$\begin{aligned}&\forall \mathscr{A}, \mathscr{B} \in \{\mathscr{X} : \mathscr{X} \in \mathscr{UDPS}^* : |\pi_1(\mathscr{X})| = 4\} \\ &\quad \pi_2(\mathscr{A}) \cap \pi_2(\mathscr{B}) = \varnothing \Leftrightarrow \pi_1(\mathscr{A}) \neq \pi_1(\mathscr{B})\end{aligned} \tag{2.61}$$

For any 2 *unit div point sets* of 5 or more points, $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$, if they have 4 points in common, let the set of such 4 points be R i.e. $R = \pi_1(\mathscr{A}_{udps}) \cap \pi_1(\mathscr{B}_{udps})$, each D' in $\mathscr{UDs}(R)$ would be equivalent to $\xi(D_a)$ and $\xi(D_b)$ where D_a is some *unit dividon* in $\pi_2(\mathscr{A}_{udps})$ and D_b is some *unit dividon* in $\pi_2(\mathscr{B}_{udps})$. In the case when $\mathscr{A}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{A}_{dps})$ and $\mathscr{B}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{B}_{dps})$ for some $\mathscr{A}_{dps}$ and $\mathscr{B}_{dps}$ that are both *sub div point sets* of a certain $\mathscr{X} \in \mathscr{DPS}^*$, $D_a = D_b$ for every such respective *unit dividons* of $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$. This implies that for every set of 4 points $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$ have in common, they have 6 *unit dividons* in common. Let k be the number of points $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$ have in common, the number of such distinct set of 4 points is precisely k *chooses* 4 i.e. $\binom{|\pi_1(\mathscr{A}_{udps})\cap\pi_1(\mathscr{B}_{udps})|}{4}$. By (2.61), we can conclude that the number of *unit dividons* any 2 distinct *sub div point sets* $\mathscr{A}$ and $\mathscr{B}$ of m points of some *div point set* have in common is $6\binom{|\pi_1(\mathscr{A})\cap\pi_1(\mathscr{B})|}{4}$.

□

Theorem 2. Let $\mathscr{DPS}^+_5$ denote the class of all *div point sets* of 5 points in $\mathscr{DPS}^+$, all $\mathscr{X} \in \mathscr{DPS}^+_5$ either have 4, 2 or 0 distinct *sub div point set* of 4 points isomorphic to $Conc^1_4$ (with the remaining *sub div point sets* of 4 points isomorphic to $Conv_4$).

Proof for Theorem 2.

Summary. In *Part 1* we show that there exists no $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{Sdps}_{of}(\mathscr{X}, 4)$ has precisely 1, 3 or 5 elements isomorphic to $Conc^1_4$. In *Part 2* we show that there exists $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{Sdps}_{of}(\mathscr{X}, 4)$ has precisely 0, 2 or 4 elements isomorphic to $Conc^1_4$.

Part 1. By *Lemma 2* it is clear that a *div point set* $\mathscr{X}_{dps}$ is in $\mathscr{DPS}^+$ iff $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}_{dps})$ is in $\mathscr{UDPS}^+$, which, by *Lemma 3*, implies that $Col_{udps}(\pi_2(\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}_{dps}))$ satisfies (2.51). For any full vertex monochromatic coloring C on some hypergraph H_{udps} to satisfy (2.51), every e in E_2 of H_{udps} must has its vertices colored $[1, 0, 0]$ or $[1, 1, 1]$. Since in any *unit div point set* of 5 points, there exists exactly $\binom{5-2}{2} = 3$ distinct *unit dividon* with the same *divider*, edges in E_2 are disjoint (recall (2.48)), and therefore any coloring satisfying (2.51) would have an even number of vertices colored 0. $Conc^1_4$ has an odd number of *unit dividons* D where $\psi(\pi_2(D)) = 0$, while $Conv_4$ has an even number for such *unit dividons*. By *Lemma 4* we can see that no 2 distinct *sub div point set* of 4 points of any *div point set* shares a *unit dividon* in common. Therefore, for any *div point set* of 5 points $\mathscr{X}_{dps}$, if $Col_{udps}(\pi_2(\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}_{dps}))$ satisfies (2.51), $\mathscr{X}_{dps}$ would not have an odd number of *sub div point sets* of 4 points isomorphic to $Conc^1_4$. We thereby conclude that there exists no $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{Sdps}_{of}(\mathscr{X}, 4)$ has precisely 1, 3 or 5 elements isomorphic to $Conc^1_4$.

Part 2. There exists *div point sets* of 5 points in $\mathscr{DPS}^+$ with precisely 4, 2, or 0 *sub div point sets* of 4 points isomorphic to $Conc^1_4$. We would prove it by demonstrating that it is possible to construct *unit div point sets* of 5 points $\mathscr{X}_{upds}$, where $Col_{udps}(\pi_2(\mathscr{X}_{upds}))$ satisfies (2.50) and (2.51) and there are precisely 4, 2, or 0 distinct $\Omega_{of6} \in All_{\Omega_{of6}}(\mathscr{X}_{udps})$ isomorphic[1] to $\pi_2(Conc^1_4)$, (with the remaining Ω_{of6} isomorphic to $Conv_4$), and that such

$\mathcal{X}_{udps}$ is in $\mathcal{UDPS}^{\Theta}$. Here $All_{\Omega_{of6}}$ is a function that returns a set of 6-cardinality sets of *unit dividons* where for any 2 *unit dividons* D_1 and D_2 in such set, $\xi(D_1) = \xi(D_2)$.

$$\begin{aligned} &All_{\Omega_{of6}}(\mathcal{X}_{udps}) = \{\Omega_{of_{based\_on}}(R, \pi_2(\mathcal{X}_{udps})) : R \in \mathcal{P}(\pi_1(\mathcal{X}_{udps})) : |R| = 4\} \\ &\Omega_{of_{based\_on}}(R, \Omega) = \{D \in \Omega : \xi(D) = R\} \end{aligned} \tag{2.62}$$

I. To construct such *unit div point sets* $\mathcal{X}_{udps}$ where no Ω_{of6} in $All_{\Omega_{of6}}(\mathcal{X}_{udps})$ is isomorphic[1] to $\pi_2(Conc_4^1)$, we would need to make sure there are only 2 *unit dividons* $D \in \Omega_{of6}$ where $\phi(D) = 0$ for every Ω_{of6} in $All_{\Omega_{of6}}(\mathcal{X}_{udps})$, notationally

$$\begin{aligned} &\forall \Omega_{of6} \in All_{\Omega_{of6}}(\mathcal{X}_{udps}) \\ &\quad |\{D \in \Omega_{of6} : \phi(D) = 0\}| = 2 \end{aligned} \tag{2.63}$$

Since $All_{\Omega_{of6}}(\mathcal{X}_{udps})$ is a countable set, (2.63) can be expressed as

$$|\{\Omega_{of6} \in All_{\Omega_{of6}}(\mathcal{X}_{udps}) : |\{D \in \Omega_{of6} : \phi(D) = 0\}| = 2\}| = |All_{\Omega_{of6}}(\mathcal{X}_{udps})| \tag{2.64}$$

Let's denote the set of all such *unit dividons* as D^*, the 5 elements in $All_{\Omega_{of6}}(\mathcal{X}_{udps})$ as Ω_{of6_1}, Ω_{of6_2}, Ω_{of6_3}, Ω_{of6_4}, Ω_{of6_5} and each 2 such unit divdions in Ω_{of6_n} as D_n^1 and D_n^2 for $n \in \{1,2,3,4,5\}$, i.e. $\{D_n^1, D_n^2\} = \Omega_{of6_n} \cap D^*$. For the coloring to satisfy (2.50) and (2.51), we need to ensure that

$$\begin{aligned} \pi_1(D_n^1) &= \bigcup \pi_2(D_n^2) \\ \pi_1(D_n^2) &= \bigcup \pi_2(D_n^1) \end{aligned} \tag{2.65}$$

holds for all $n \in \{1,2,3,4,5\}$, while letting every *unit dividons* D_n^x to have a common *divider* as some D_m^x where $n \neq m$ for all $x \in \{1,2\}$, and the same time avoiding the scenario where 3 distinct *unit dividon* in D^* having a *divider* in common. That is to say, for some subsets of 2 cardinality, A, B, C, D, E, F of $\pi_1(X_{udps})$, we have

$$\begin{aligned} \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) = A \\ \pi_1(D_1^2) = \bigcup \pi_2(D_1^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) = B \\ \pi_1(D_2^2) = \bigcup \pi_2(D_2^1) = \pi_1(D_4^2) = \bigcup \pi_2(D_4^1) = C \\ \pi_1(D_4^1) = \bigcup \pi_2(D_4^2) = \pi_1(D_5^1) = \bigcup \pi_2(D_5^2) = D \\ \pi_1(D_5^2) = \bigcup \pi_2(D_5^1) = \pi_1(D_6^2) = \bigcup \pi_2(D_6^1) = E \\ \pi_1(D_3^1) = \bigcup \pi_2(D_3^2) = \pi_1(D_6^1) = \bigcup \pi_2(D_6^2) = F \end{aligned} \tag{2.66}$$

where

$$\begin{gathered} A \neq B \neq C \neq D \neq E \neq F \\ (A \cap B) = (A \cap C) = (B \cap F) = (C \cap D) = (D \cap E) = (E \cap F) = \varnothing \end{gathered}$$

$\mathscr{X}_{updp}$ described above is in $\mathscr{UDPS}^{\Theta}$ because there exists $\mathscr{X} \in \mathscr{DPS}^{+}$ where $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}) = \mathscr{X}_{updp}$. Such $\mathscr{X}$ would be isomorphic to $Conv_5$ defined in (2.72).

II. To construct such *unit div point sets* $\mathscr{X}_{udps}$ where precisely 2 Ω_{of6} in $All_{\Omega_{of6}}(\mathscr{X}_{udps})$ are isomorphic[1] to $\pi_2(Conc^1_4)$, we would need to make sure that there are precisely 3 *unit dividons* $D \in \Omega_{of6}$ where $\phi(D) = 1$ for 2 Ω_{of6} in $All_{\Omega_{of6}}(\mathscr{X}_{udps})$, and only 2 *unit dividons* $D \in \Omega_{of6}$ where $\phi(D) = 0$ for 3 Ω_{of6} in $All_{\Omega_{of6}}(\mathscr{X}_{udps})$, notationally

$$\begin{aligned} |\{\Omega_{of6} \in All_{\Omega_{of6}}(\mathscr{X}_{udps}) : |\{D \in \Omega_{of6} : \phi(D) = 0\}| = 3\}| = 2 \\ |\{\Omega_{of6} \in All_{\Omega_{of6}}(\mathscr{X}_{udps}) : |\{D \in \Omega_{of6} : \phi(D) = 0\}| = 2\}| = 3 \end{aligned} \tag{2.67}$$

Using the same notation above, this time we would have $\{D^1_n, D^2_n\} = \Omega_{of6_n} \cap D^*$ for $n \in \{1, 2, 3\}$ and $\{D^1_n, D^2_n, D^3_n\} = \Omega_{of6_n} \cap D^*$ for $n \in \{4, 5\}$. For the coloring to satisfy (2.50) and (2.51), we need to ensure that (2.65) holds for $n \in \{1, 2, 3\}$ and

$$|\pi_1(D^1_n) \cap \pi_1(D^2_n) \cap \pi_1(D^3_n)| = 1 \tag{2.68}$$

holds for $n \in \{4, 5\}$, while letting every *unit dividons* D^x_n to have a common *divider* as some D^x_m where $n \neq m$ for all $x \in \{1, 2\}$, and the same time avoiding the scenario where 3 distinct *unit dividon* in D^* having a *divider* in common. One way to go about satisfying these conditions is to let D^x_4 to have a common *divider* as D^x_5 for all $x \in \{1, 2\}$, while letting the remaining *unit dividons* in D_4 and D_5, namely D^3_4 and D^3_5, to have a common *divider* as D^1_1 and D^1_2 respectively, and the remaining *unit dividons* in D_1 and D_2, namely D^2_1 and D^2_2, to have a common *dividers* as the two *dividons* in D_3 respectively. That is to say, for some subsets of 2 cardinality, A, B, C, D, E, F of $\pi_1(X_{udps})$, we have

$$\begin{gathered} \pi_1(D^1_4) = \pi_1(D^1_5) = A \\ \pi_1(D^2_4) = \pi_1(D^2_5) = B \\ \pi_1(D^3_4) = \pi_1(D^1_1) = \bigcup \pi_2(D^2_1) = C \\ \pi_1(D^3_5) = \pi_1(D^1_2) = \bigcup \pi_2(D^2_2) = D \\ \pi_1(D^2_1) = \bigcup \pi_2(D^1_1) = \pi_1(D^1_3) = \bigcup \pi_2(D^2_3) = E \\ \pi_1(D^2_2) = \bigcup \pi_2(D^1_2) = \pi_1(D^2_3) = \bigcup \pi_2(D^1_3) = F \end{gathered} \tag{2.69}$$

where

$$\begin{gathered} A \neq B \neq C \neq D \neq E \neq F \\ |A \cap B \cap C| = 1 \\ |A \cap B \cap D| = 1 \\ (C \cap E) = (D \cap F) = (E \cap F) = \varnothing \end{gathered}$$

$\mathscr{X}_{updps}$ described above is in $\mathscr{UDPS}^{\Theta}$ because there exists $\mathscr{X} \in \mathscr{DPS}^{+}$ where $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}) = \mathscr{X}_{updps}$. Such $\mathscr{X}$ would be isomorphic to $Conc^1_5$ defined in (2.72).

III. To construct such *unit div point sets* $\mathscr{X}_{udps}$ where precisely 4 Ω_{of6} in $All_{\Omega_{of6}}(\mathscr{X}_{udps})$ are isomorphic[1] to $\pi_2(Conc^1_4)$, we would need to make sure that there are precisely 3 *unit dividons* $D \in \Omega_{of6}$ where $\phi(D) = 1$ for 4 Ω_{of6} in $All_{\Omega_{of6}}(\mathscr{X}_{udps})$, and only 2 *unit dividons* $D \in \Omega_{of6}$ where $\phi(D) = 0$ for 1 Ω_{of6} in $All_{\Omega_{of6}}(\mathscr{X}_{udps})$, notationally

$$\begin{aligned} &|\{\Omega_{of6} \in All_{\Omega_{of6}}(\mathscr{X}_{udps}) : |\{D \in \Omega_{of6} : \phi(D) = 0\}| = 3\}| = 4 \\ &|\{\Omega_{of6} \in All_{\Omega_{of6}}(\mathscr{X}_{udps}) : |\{D \in \Omega_{of6} : \phi(D) = 0\}| = 2\}| = 1 \end{aligned} \tag{2.70}$$

Using the same notation above, this time we would have $\{D^1_n, D^2_n\} = \Omega_{of6_n} \cap D^*$ for $n \in \{1\}$ and $\{D^1_n, D^2_n, D^3_n\} = \Omega_{of6_n} \cap D^*$ for $n \in \{2, 3, 4, 5\}$. For the coloring to satisfy (2.50) and (2.51), we need to ensure that (2.65) holds for $n \in \{1\}$ and (2.68) holds for $n \in \{2, 3, 4, 5\}$, while letting every *unit dividons* D^x_n to have a common *divider* as some D^x_m where $n \neq m$ for all $x \in \{1, 2\}$, and the same time avoiding the scenario where 3 distinct *unit dividon* in D^* having a *divider* in common. One way to go about satisfying these conditions is to let D^x_4 and D^x_2 to have a common *divider* as D^x_5 and D^x_3 respecitvely, for $x \in \{1, 2\}$, while letting the remaining *unit dividons* in D_2, D_4 and D_5, namely D^3_2, D^3_4 and D^3_5, to have a common *divider* as D^2_1, D^3_3 and D^1_1 respectively. That is to say, for some subsets of 2 cardinality, A, B, C, D, E, F of $\pi_1(X_{udps})$, we have

$$\begin{aligned} \pi_1(D^1_4) = \pi_1(D^1_5) &= A \\ \pi_1(D^2_4) = \pi_1(D^2_5) &= B \\ \pi_1(D^3_4) = \pi_1(D^3_3) &= C \\ \pi_1(D^1_3) = \pi_1(D^1_2) &= D \\ \pi_1(D^2_3) = \pi_1(D^2_2) &= E \\ \pi_1(D^3_5) = \pi_1(D^1_1) = \bigcup \pi_2(D^2_1) &= F \\ \pi_1(D^3_2) = \pi_1(D^2_1) = \bigcup \pi_2(D^1_1) &= G \end{aligned} \tag{2.71}$$

where

$$\begin{gathered} A \neq B \neq C \neq D \neq E \neq F \neq G \\ |A \cap B \cap C| = 1 \\ |A \cap B \cap F| = 1 \\ |C \cap D \cap E| = 1 \\ |D \cap E \cap G| = 1 \\ F \cap G = \varnothing \end{gathered}$$

$\mathscr{X}_{upds}$ described above is in $\mathscr{UDPS}^{\Theta}$ because there exists $\mathscr{X} \in \mathscr{DPS}^{+}$ where $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}) = \mathscr{X}_{upds}$. Such $\mathscr{X}$ would be isomorphic to $Conc^2_5$ defined in (2.72).

$\square$

Remark. A stronger version of *Theorem 2* would state that for all $\mathscr{X}_{dps} \in \mathscr{DPS}^{+}_{5}$, $\mathscr{X}_{dps}$ is either isomorphic to $Conv_5$, $Conc^1_5$ or $Conc^2_5$, where

$$
\begin{aligned}
Conv_5 =&(Cv_5, \Theta_{Cv_5}) & Conc^1_5 =&(Cc^1_5, \Theta_{Cc^1_5}) & Conc^2_5 =&(Cc^2_5, \Theta_{Cc^2_5})\\
Cv_5 =&\{1,2,3,4,5\} & Cc^1_5 =&\{1,2,3,4,5\} & Cc^2_5 =&\{1,2,3,4,5\}\\
\Theta_{Cv_5} =&\{(\{1,2\},\{(\{3,4\},\varnothing\}), & \Theta_{Cc^1_5} =&\{(\{1,2\},\{(\{3,4,5\},\varnothing\}), & \Theta_{Cc^2_5} =&\{(\{1,2\},\{(\{3,4,5\},\varnothing\}),\\
&(\{1,3\},\{\{2\},\{4,5\}\}), && (\{1,3\},\{\{2,5\},\{4\}\}), && (\{1,3\},\{\{2,4,5\},\varnothing\}),\\
&(\{1,4\},\{\{2,3\},\{5\}\}), && (\{1,4\},\{\{2,3,5\},\varnothing\}), && (\{1,4\},\{\{2\},\{3,5\}\}),\\
&(\{1,5\},\{\{2,3,4\},\varnothing\}), && (\{1,5\},\{\{2\},\{3,4\}\}), && (\{1,5\},\{\{2,4\},\{3\}\}),\\
&(\{2,3\},\{\{1,4,5\},\varnothing\}), && (\{2,3\},\{\{1,4,5\},\varnothing\}), && (\{2,3\},\{\{1,4,5\},\varnothing\}),\\
&(\{2,4\},\{\{1,5\},\{3\}\}), && (\{2,4\},\{\{1\},\{3,5\}\}), && (\{2,4\},\{\{1\},\{3,5\}\}),\\
&(\{2,5\},\{\{1\},\{3,4\}\}), && (\{2,5\},\{\{1,4\},\{3\}\}), && (\{2,5\},\{\{1,4\},\{3\}\}),\\
&(\{3,4\},\{\{1,2,5\},\varnothing\}) && (\{3,4\},\{\{1,2,5\},\varnothing\}) && (\{3,4\},\{\{1\},\{2,5\}\})\\
&(\{3,5\},\{\{1,2\},\{4\}\}) && (\{3,5\},\{\{1,2\},\{4\}\}) && (\{3,5\},\{\{1,4\},\{2\}\})\\
&(\{3,4\},\{\{1,2,5\},\varnothing\}) && (\{4,5\},\{\{1,2,5\},\varnothing\}) && (\{4,5\},\{\{1,3\},\{2\}\})
\end{aligned}
\tag{2.72}
$$

To prove this version of *Theorem 2* we would need to prove that there exists no *div point sets* in $\mathscr{DPS}^{+}_{5}$ not isomorphic to $Conv_5$, $Conc^1_5$ or $Conc^2_5$.

Remark. Let $All_{of\Omega}$ be a generalization of $All_{of\Omega_6}$ such that

$$All_{of\Omega}(\mathscr{X}, n) = \{\Omega_{of_{based\_on}}(R, \pi_2(\mathscr{X})) : R \in \mathscr{P}(\pi_1(\mathscr{X})) : |R| = n\} \tag{2.73}$$

by *Theorem 2*, it is clear that following proposition is false:

> A *unit div point set* of 5 or more points $\mathscr{X}_{udps}$ is in $\mathscr{UDPS}^{+}$ iff all elements in $All_{of\Omega_{of6}}(\mathscr{X}_{udps}, n)$ are isomorphic to the set of *unit dividons* of some *unit div point set* in $\mathscr{UDPS}^{+}$, for any $n \in \mathbb{N}_{\geq 4}$ less than $|\pi_1(\mathscr{X}_{udps})|$.

However, this weaker version of it still holds true:

> If $\mathscr{X}_{udps}$ is in $\mathscr{UDPS}^{+}$, all members of $All_{of\Omega_{of6}}(\mathscr{X}_{udps}, n)$ are also in $\mathscr{UDPS}^{+}$ for any $n \in \mathbb{N}_{\geq 4}$ less than $|\pi_1(\mathscr{X}_{udps})|$.

There is undoubtedly some similarity between the false proposition above, and the following proposition which is too false:

> A *div point set* of 4 or more points, $\mathscr{X}_{dps}$, is in $\mathscr{DPS}^{+}$ iff all elements in $\mathscr{S}dps_{of}(\mathscr{X}_{dps}, n)$ are also in $\mathscr{DPS}^{+}$, for any $n \in \mathbb{N}_{\geq 3}$ less than $|\pi_1(X_{dps})|$.

Since it is vacuously true that any *div point sets* of 3 points satisfy (2.10), (2.11), and (2.12), we cannot conclude that a certain *div point set* satisfies (2.10), (2.11), and (2.12) just because all its *sub div point sets* of 3 points satisfy them. Now recall *Lemma 3* where E_2 of the hypergraph based on P is an empty set in the case when $|P| = 4$ and as a result, it is vacuously true that such E_2 always satisfies (2.51), and so we cannot conclude that a certain *unit div point set* $\mathcal{X}_{udps}$ where $Col_{udps}(\pi_2(\mathcal{X}_{udps}))$ satisfies (2.51), just because all elements in $All_{of\Omega_{of6}}(\mathcal{X}_{udps}, 4)$ are isomorphic to some *unit div point set*, $\mathcal{A}_{udps}$, in where $Col_{udps}(\pi_2(\mathcal{X}_{udps}))$ satisfies it.

It can be proven that in the case when $n \in \mathbb{N}_{\geq 5}$, assuming the stronger version of *Lemma 3*, the proposition regarding *unit div point sets* above is then true, similar to how the proposition regarding *div point sets* is true in the case when $n \in \mathbb{N}_{\geq 4}$.

2.2 *convexity*

The notion that there exists n points forming a convex polygon among some set of points in $\mathbb{E}^2$ can be expressed through *convexity* in the context of *div point sets*.

Definition 7. A *div point set* (P, Θ_P) has a *convexity* of n iff there exists (Q, Θ_Q) such that $(Q, \Theta_Q) \leq (P, \Theta_P)$ and (Q, Θ_Q) is isomorphic to $Conv_n$, defined as follow

$$
\begin{aligned}
&Conv_n = (P, \{(d, \delta_{conv}(d, P)) : d \in \mathcal{P}(P) : |d| = 2\}) \\
&\text{where } \begin{cases}
P \coloneqq \{x \in \mathbb{N}_{\geq 1} : x \leq n\} \\
\delta_{conv}(d, P) = \{\{p : p \in P : inside(p, d)\}, \{p : p \in P : outside(p, d)\}\} \\
inside(p, d) = (p > min(d) \wedge p < max(d)) \\
outside(p, d) = (p < min(d) \vee p > max(d)) \\
min(d) \text{ returns the smallest number in } d \\
max(d) \text{ returns the biggest number in } d.
\end{cases}
\end{aligned}
\tag{2.74}
$$

where n is in $\mathbb{N}_{\geq 3}$. Here is an implementation of $Conv_n$ as a function in Haskell:

```
import Data.List

combine :: Int -> [a] -> [[a]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                      , ys <- combine (n-1) xs']

convex:: Int -> ([Int],[([Int],[[Int]])])
convex n = (points, dividons)
   where
      points = [1..n]
      dividers = combine 2 points
```

```
dividons = [(divider,[div1,div2])
    | divider@(a:b:_) <- dividers,
    let divs = points \\ divider,
    let div1 = [ x | x <- divs, x > a, x < b ],
    let div2 = divs \\ div1 ]
```

Axiom 2. For any $\mathscr{X}$ in $\mathscr{DPS}^+$, $\mathscr{X}$ has an interpretation for $\pi_1(\mathscr{X})$ as some set of points in E^2 among which there exists n points forming a convex polygon, iff $\mathscr{X}$ has a convexity of n. More precisely, there exists an interpretation for $P' \subseteq \pi_1(\mathscr{X})$ as some set of n points in E^2 forming a convex polygon iff $\mathscr{Sdps}(\mathscr{X}, P')$ is isomorphic to $Conv_n$, for all $n \geq 3$.

Remark. One may notice that for $n \geq 4$, all *sub div point sets* of $n-1$ points of $Conv_n$ are isomorphic to $Conv_{n-1}$, and as a consequence, a *div point set* with a *convexity* of k would also have a convexity of m, for all k, m in $\mathbb{N}_{\geq 3}$ where $m < k$. By *Axiom 2*, that is equivalent to the following proposition: for any $n \geq 4$, after removing any one point from a set of n points that are the vertices of a convex polygon on an Euclidean plane, the remaining points too forms a convex polygon, and as a consequence, any set of points in general position containing k points forming a convex polygon would also contain m points forming a convex polygon, for all k, m in $\mathbb{N}_{\geq 3}$ where $m < k$.

Remark. We can conclude from *Theorem 2* that a *div point set* of 5 or more points always have a convexity of 4. By *Axiom 2*, this means that we can always find 4 points forming a convex polygon in any set of 5 or more points in general position on an Euclidean plane.

3 A reduction to a *multiset unsatisfiability problem*

The Erdös-Szekeres conjecture can be expressed as a conjunction of (3.1) and (3.2) in the theory of *div point set*.

$$\begin{aligned}&\forall n \in \mathbb{N}_{\geq 3}\\ &\quad \exists \mathscr{A} \in \mathscr{DPS}^+ \quad |\pi_1(A)| = 2^{n-2} \ \wedge \exists \mathscr{A}_s \leq \mathscr{A} \quad \mathscr{A}_s \ncong Conv_n\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\forall n \in \mathbb{N}_{\geq 3}\\ &\quad \forall \mathscr{A} \in \mathscr{DPS}^+ \quad |\pi_1(A)| > 2^{n-2} \Leftrightarrow \exists \mathscr{A}_s \leq \mathscr{A} \quad \mathscr{A}_s \cong Conv_n\end{aligned} \tag{3.2}$$

Since the lower bound has been proven to be $2^{n-2} + 1$, all is left is to prove (3.2) and the conjecture would be proven.

3.1 a combinatorial characteristics of *sub div point sets*

As we examine *div point sets* of v points for $v > 5$, we would notice this pretty interesting fact about *sub div point sets*: for any natural number $a \geq 1$, let $\mathscr{SDPSS}$ be the set of all *sub div point set* of $v-a$ points of any *div point sets* of v points, for any $\mathscr{X}_{\mathscr{SDPS}}$ in $\mathscr{SDPSS}$, we can always select $v-a$ distinct sets of $a+1$ cardinality in the powerset of $\mathscr{SDPSS}$ such that each of these subsets of $\mathscr{SDPSS}$ contains $\mathscr{X}_{\mathscr{SDPS}}$ and other *div point sets* all of which have $\binom{v-a-1}{t}$ common *sub div point sets* of t points, for all $t \in \mathbb{N}_{\geq 1}$. What is cool about this is that it can be generalized from $a+1$ to $a+b$ for any $b \in \mathbb{N}_{\geq 1}$ as long as $a+b$ is smaller than v (and in which case the *div point sets* would have $\binom{v-a-b}{t}$ *sub div point sets* of t points in common). Notationally,

$$
\begin{aligned}
&\forall \mathscr{X} \in \mathscr{DPS}^{+} \\
&\quad v := |\pi_1(\mathscr{X})| \\
&\quad \forall a \in \mathbb{N}_{\geq 1} \\
&\quad\quad \mathscr{SDPSS} := \mathscr{S}dps_{of}(\mathscr{X}, v-a) \\
&\quad\quad \forall \mathscr{X}_{\mathscr{SDPS}} \in \mathscr{SDPSS} \\
&\quad\quad\quad \forall b \in \{x : x \in \mathbb{N}_{\geq 1} : x < v-a\} \\
&\quad\quad\quad\quad \exists \mathscr{S} \in \mathscr{P}_n(\mathscr{P}_n(\mathscr{SDPSS}, a+b), v-a) \\
&\quad\quad\quad\quad\quad \forall s \in \mathscr{S} \\
&\quad\quad\quad\quad\quad\quad \mathscr{X}_{\mathscr{SDPS}} \in s \\
&\quad\quad\quad\quad\quad\quad \forall t \in \mathbb{N}_{\geq 1} \\
&\quad\quad\quad\quad\quad\quad\quad |\bigcap_{\ell \in s} \mathscr{S}dps_{of}(\ell, t)| = \binom{v-a-b}{t}
\end{aligned}
\tag{3.3}
$$

where

$$
\mathscr{P}_n(S, n) = \{x : x \in \mathscr{P}(S) : |x| = n\} \tag{3.4}
$$

To understand why such combinatorial characteristics exists, consider this: any 2 *sub div point sets*, $\mathscr{S}_1$ and $\mathscr{S}_2$ of a certain *div point set* is distinct iff they are of distinct points i.e. $\mathscr{S}_1 \neq \mathscr{S}_2 \Leftrightarrow \pi_1(\mathscr{S}_1) \neq \pi_1(\mathscr{S}_2)$, and thus (3.3) is equivalent as stating that for any set $\mathcal{n}$ with the same cardinality as $\mathbb{N}$,

$$
\begin{aligned}
&\forall X \in \mathscr{P}(\mathcal{N}) \\
&\quad v := |X| \\
&\quad \forall a \in \mathbb{N}_{\geq 1} \\
&\quad\quad X_{subset-set} := \mathscr{P}_n(X, v-a) \\
&\quad\quad \forall X_{subset} \in X_{subset-set} \\
&\quad\quad\quad \forall b \in \{x : x \in \mathbb{N}_{\geq 1} : x < v-a\} \\
&\quad\quad\quad\quad \exists S \in \mathscr{P}_n(\mathscr{P}_n(X_{subset-set}, a+b), v-a) \\
&\quad\quad\quad\quad\quad \forall s \in S \\
&\quad\quad\quad\quad\quad\quad X_{subset} \in s \\
&\quad\quad\quad\quad\quad\quad \forall n \in \mathbb{N}_{\geq 1} \\
&\quad\quad\quad\quad\quad\quad\quad |\bigcap_{l \in s} \mathscr{P}_n(l, n)| = \binom{v-a-b}{n}
\end{aligned}
\tag{3.5}
$$

For the purpose of illustration, suppose we have some *div point set* of 9 pointst $\mathcal{X}_9$, let $isom$ be a bijective function from $\mathcal{S}dps_{of}(\mathcal{X}_9, 4)$ to a set of natural numbers, N, where $N = \{n : n \in \mathbb{N}_{\geq 1} : n \leq |\mathcal{S}dps_{of}(\mathcal{X}_9, 4)|\}$, the set

$$\{\{iso(\mathcal{X}_4) : \mathcal{X}_4 \in \mathcal{X}_5\} : \mathcal{X}_5 \in \mathcal{S}dps_{of}(\mathcal{X}_9, 5)\} \tag{3.6}$$

shows how *sub div point sets* of 4 points of $\mathcal{X}_9$ (each represented by a distinct natural number) would be disturbed among *sub div point sets* of 5 points of $\mathcal{X}_9$ and is isomorphic[3] to:

$$
\begin{aligned}
&\{\{1,2,7,22,57\},\{1,3,8,23,58\},\{1,4,9,24,59\},\{1,5,10,25,60\},\{1,6,11,26,61\},\{2,3,12,27,62\},\\
&\{2,4,13,28,63\},\{2,5,14,29,64\},\{2,6,15,30,65\},\{3,4,16,31,66\},\{3,5,17,32,67\},\{3,6,18,33,68\},\\
&\{4,5,19,34,69\},\{4,6,20,35,70\},\{5,6,21,36,71\},\{7,8,12,37,72\},\{7,9,13,38,73\},\{7,10,14,39,74\},\\
&\{7,11,15,40,75\},\{8,9,16,41,76\},\{8,10,17,42,77\},\{8,11,18,43,78\},\{9,10,19,44,79\},\{9,11,20,45,80\},\\
&\{10,11,21,46,81\},\{12,13,16,47,82\},\{12,14,17,48,83\},\{12,15,18,49,84\},\{13,14,19,50,85\},\{13,15,20,51,86\},\\
&\{14,15,21,52,87\},\{16,17,19,53,88\},\{16,18,20,54,89\},\{17,18,21,55,90\},\{19,20,21,56,91\},\{22,23,27,37,92\},\\
&\{22,24,28,38,93\},\{22,25,29,39,94\},\{22,26,30,40,95\},\{23,24,31,41,96\},\{23,25,32,42,97\},\{23,26,33,43,98\},\\
&\{24,25,34,44,99\},\{24,26,35,45,100\},\{25,26,36,46,101\},\{27,28,31,47,102\},\{27,29,32,48,103\},\{27,30,33,49,104\},\\
&\{28,29,34,50,105\},\{28,30,35,51,106\},\{29,30,36,52,107\},\{31,32,34,53,108\},\{31,33,35,54,109\},\{32,33,36,55,110\},\\
&\{34,35,36,56,111\},\{37,38,41,47,112\},\{37,39,42,48,113\},\{37,40,43,49,114\},\{38,39,44,50,115\},\{38,40,45,51,116\},\\
&\{39,40,46,52,117\},\{41,42,44,53,118\},\{41,43,45,54,119\},\{42,43,46,55,120\},\{44,45,46,56,121\},\{47,48,50,53,122\},\\
&\{47,49,51,54,123\},\{48,49,52,55,124\},\{50,51,52,56,125\},\{53,54,55,56,126\},\{57,58,62,72,92\},\{57,59,63,73,93\},\\
&\{57,60,64,74,94\},\{57,61,65,75,95\},\{58,59,66,76,96\},\{58,60,67,77,97\},\{58,61,68,78,98\},\{59,60,69,79,99\},\\
&\{59,61,70,80,100\},\{60,61,71,81,101\},\{62,63,66,82,102\},\{62,64,67,83,103\},\{62,65,68,84,104\},\{63,64,69,85,105\},\\
&\{63,65,70,86,106\},\{64,65,71,87,107\},\{66,67,69,88,108\},\{66,68,70,89,109\},\{67,68,71,90,110\},\{69,70,71,91,111\},\\
&\{72,73,76,82,112\},\{72,74,77,83,113\},\{72,75,78,84,114\},\{73,74,79,85,115\},\{73,75,80,86,116\},\{74,75,81,87,117\},\\
&\{76,77,79,88,118\},\{76,78,80,89,119\},\{77,78,81,90,120\},\{79,80,81,91,121\},\{82,83,85,88,122\},\{82,84,86,89,123\},\\
&\{83,84,87,90,124\},\{85,86,87,91,125\},\{88,89,90,91,126\},\{92,93,96,102,112\},\{92,94,97,103,113\},\{92,95,98,104,114\},\\
&\{93,94,99,105,115\},\{93,95,100,106,116\},\{94,95,101,107,117\},\{96,97,99,108,118\},\{96,98,100,109,119\},\\
&\{97,98,101,110,120\},\{99,100,101,111,121\},\{102,103,105,108,122\},\{102,104,106,109,123\},\{103,104,107,110,124\},\\
&\{105,106,107,111,125\},\{108,109,110,111,126\},\{112,113,115,118,122\},\{112,114,116,119,123\},\{113,114,117,120,124\},\\
&\{115,116,117,121,125\},\{118,119,120,121,126\},\{122,123,124,125,126\}\}
\end{aligned}
\tag{3.7}
$$

Notice how for every $\mathscr{X}_5 \in \mathscr{Sdps}_{of}(\mathscr{X}_9, 5)$, there exists a set in $\mathscr{P}_n(\mathscr{P}_n(\mathscr{Sdps}_{of}(\mathscr{X}_9, 5), 5), 5)$ where its elements are distinct subsets of $\mathscr{Sdps}_{of}(\mathscr{X}_9, 5)$, each containing $\mathscr{X}_5$ and other *div point sets* all of which have 5 common *sub div point sets* of 4 points.

We believe that (3.2) is simply an elegant result of having a structure, whose sub-structures possess the combinatorial characteristics described above, that satisfies a certain constraint, which, in this case, is that described in (3.8) below.

Note. *isomorphic*[3]: by A *is isomorphic*[3] *to* B we mean that there exists a function f from N to N such that $f^{members^2}(A) = B$.

3.2 the problem $UNSAT^{\mathscr{DPS}^+}_{multiset}$

Definition 8. $UNSAT_{multiset}$ is the decision problem of determining if there exists no value-assignment for all variables in V, distributed in a certain manner among the multisets in M, such that it satisfies the FOL formulae in C, where the value-assignment is defined to be a function, Z: for all v in V, $Z(v) = x$ for some $x \in D$. Here D, the set of values a variable can be assigned to, is often referred to as the domain. An instance of $UNSAT_{multiset}$ can thus be represented as a 4-tuple (V, D, M, C).

We shall now present the problem $UNSAT^{\mathscr{DPS}^+}_{multiset}$, a special case of $UNSAT_{multiset}$, of which, if an instance is solved (*solved* in the sense that it is demonstrated that the formulae F are unsatisfiable (e.g. by a Turing machine)), it would prove that, for a particular $n \in \mathbb{N}_{\geq 5}$ (depending on which instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ is solved), there exists no *div point set* of $2^{n-2}+1$ points $\mathscr{X}$ which satisfies

$$\begin{aligned}&\forall \mathscr{X}_5 \in \mathscr{Sdps}_{of}(\mathscr{X}, 5)\\ &\qquad [Assign(\mathscr{X}_4) : \mathscr{X}_4 \in \mathscr{Sdps}_{of}(\mathscr{X}_5, 4)] \in \{[1,1,1,1,0], [1,1,0,0,0], [0,0,0,0,0]\}\end{aligned} \tag{3.8}$$

but does not satisfy

$$\begin{aligned}&\exists \mathscr{A}_s \in \mathscr{Sdps}_{of}(\mathscr{X}, n)\\ &\qquad \forall \mathscr{A}_{ss} \in \mathscr{Sdps}_{of}(\mathscr{X}, 4) \quad Assign(\mathscr{A}_{ss}) = 0\end{aligned} \tag{3.9}$$

where

$$Assign(\mathscr{X}) = \begin{cases} 1 & \text{if} \quad \mathscr{X} \cong Conc^1_4 \\ 0 & \text{if} \quad \mathscr{X} \cong Conv_4 \end{cases} \tag{3.10}$$

consequently proving that

$$\forall \mathscr{A} \in \mathscr{DPS}^+ \quad |\pi_1(A)| > 2^{n-2} \Leftrightarrow \exists \mathscr{A}_s \leq \mathscr{A} \quad \mathscr{A}_s \cong Conv_n \tag{3.11}$$

since if a *div point set* of 5 or more points, $\mathscr{X}$, is in $\mathscr{DPS}^+$, by *Theorem 2*, $\mathscr{X}$ satisfy (3.8), and if there exists no *div point set* of $2^{n-2}+1$ points that satisfies (3.8) but not (3.9), it would indicate that every *div point set* of $2^{n-2}+1$ or more points in $\mathscr{DPS}^+$ has a convexity of n (note that (3.2) can be rewritten as follows

$$
\begin{aligned}
&\forall n \in \mathbb{N}_{\geq 3} \\
&\quad \forall \mathscr{A} \in \mathscr{DPS}^+ \\
&\quad\quad |\pi_1(A)| > 2^{n-2} \\
&\quad\quad \Leftrightarrow \exists \mathscr{A}_s \in \mathscr{S}dps_{of}(\mathscr{A}, n) \\
&\quad\quad\quad \forall \mathscr{A}_{ss} \in \mathscr{S}dps_{of}(\mathscr{A}_s, 4) \quad Assign(\mathscr{A}_{ss}) = 0
\end{aligned}
\tag{3.12}
$$

since for all $k \in \mathbb{N}_{\geq 3}$ and $n \geq k$, any *sub div point set* of k points of any $Conv_n$ is isomorphic to some $Conv_k$).

Definition 9. $UNSAT_{multiset}^{\mathscr{DPS}^+}$ is a special case, or subproblem, of $UNSAT_{multiset}$ (*subproblem* in the sense that all instances of $UNSAT_{multiset}^{\mathscr{DPS}^+}$ are instances of $UNSAT_{multiset}$). An instance of $UNSAT_{multiset}$, (V, D, M, C), is an instance of $UNSAT_{multiset}^{\mathscr{DPS}^+}$ iff for some $n \geq 5$,

$$
\begin{gathered}
|V| = \binom{2^{n-2}+1}{4} \\
D = \{0, 1\} \\
M = A \cup B \\
C \text{ consists of formulae 3.14 and 3.15.}
\end{gathered}
\tag{3.13}
$$

and A is a set of 5-cardinality multisets while B is a set of n-cardinality multisets, and the variables in V are distributed in $m \in A$ the same way as how elements in $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ are distributed in $\mathscr{X}_{\mathscr{SPDPS}} \in \mathscr{S}dps_{of}(\mathscr{X}, 5)$, while the variables are distributed in $m \in B$ the same way as how elements in $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ are distributed in $\mathscr{X}_{\mathscr{SPDPS}} \in \mathscr{S}dps_{of}(\mathscr{X}, n)$, where $\mathscr{X}$ is any *div point set* of n points.

$$
\forall a \in A \qquad a \in \{[1,1,1,1,0], [1,1,0,0,0], [0,0,0,0,0]\} \tag{3.14}
$$

$$
\forall b \in B \qquad b \neq \underbrace{[0,0,0,...,0,0]}_{\binom{n}{4}\ 0's} \tag{3.15}
$$

The distribution of variables in A and B can be implement in Haskell as follows:

```
import Data.List
import Data.Maybe
type Multiset = [Integer]

merge (a:x) (b:y) = (a,b) : merge x y
merge [] _ = []

choose :: Integer -> Integer -> Integer
```

```
n `choose` k
    | k < 0     = 0
    | k > n     = 0
    | otherwise = factorial n `div` (factorial k * factorial (n-k))

factorial :: Integer -> Integer
factorial n = foldl (*) 1 [1..n]

combine :: Integer -> [Integer] -> [[Integer]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                         , ys <- combine (n-1) xs']

number_of_points = (\n->(2^(n-2)+1))

n_setOf_m_Multisets:: Integer -> Integer -> [Multiset]
n_setOf_m_Multisets m n = [ map fromJust $ map ((flip lookup) encoding)
    (combine 4 m_points) | m_points <- combine n [1..m] ]
    where
        encoding = merge (combine 4 [1..m]) [1..(m `choose` 4)]

setA :: Integer -> [Multiset]
setA n = n_setOf_m_Multisets (number_of_points n) 5

setB :: Integer -> [Multiset]
setB n = [ x | x <- n_setOf_m_Multisets (number_of_points n) n, 2 `elem` x ]
```

Remark. A different implementation may result in a different M for the same n. Nonetheless, the different M obtained from a different implementation would be isomorphic[3] to the M obtained from this implementation, in which case we would consider that distribution to be the same. Thus as far as unsatisfiability is concerned, for every $n \in \mathbb{N}_{\geq 5}$, there exists exactly one instance of $UNSAT_{multiset}^{\mathscr{DPS}^+}$.

Remark. Each variable in V represents $Assign(\mathscr{X}_4)$ for a particular element $\mathscr{X}_4 \in \mathscr{S}dps_{of}(\mathscr{A}, 4)$ where $\mathscr{A}$ is a *div point set* of $2^{n-2}+1$ points for some $n \in \mathbb{N}_{\geq 5}$. If there exists no value-assignment Z satisfying formulae in C, we can be certain that there exists no *div point set* of $2^{n-2}+1$ points $\mathscr{X}$ satisfying (3.8) but not (3.9) as mentioned above and subsequently proving the n-instance of conjecture.

Remark. Here is the simplest instance of $UNSAT_{multiset}^{\mathscr{DPS}^+}$ (when $n = 5$): since $A = B$, we have $|M| = |A| = |B| = \binom{2^{5-2}+1}{5} = 126$ multisets, and $|V| = \binom{2^{5-2}+1}{4} = 126$ variables as well (with each denoted by v_n below), distributed among the multisets in M as follows:

$\{[v_1, v_2, v_7, v_{22}, v_{57}], [v_1, v_3, v_8, v_{23}, v_{58}], [v_1, v_4, v_9, v_{24}, v_{59}], [v_1, v_5, v_{10}, v_{25}, v_{60}], [v_1, v_6, v_{11}, v_{26}, v_{61}], [v_2, v_3, v_{12}, v_{27}, v_{62}],$
$[v_2, v_4, v_{13}, v_{28}, v_{63}], [v_2, v_5, v_{14}, v_{29}, v_{64}], [v_2, v_6, v_{15}, v_{30}, v_{65}], [v_3, v_4, v_{16}, v_{31}, v_{66}], [v_3, v_5, v_{17}, v_{32}, v_{67}], [v_3, v_6, v_{18}, v_{33}, v_{68}],$
$[v_4, v_5, v_{19}, v_{34}, v_{69}], [v_4, v_6, v_{20}, v_{35}, v_{70}], [v_5, v_6, v_{21}, v_{36}, v_{71}], [v_7, v_8, v_{12}, v_{37}, v_{72}], [v_7, v_9, v_{13}, v_{38}, v_{73}], [v_7, v_{10}, v_{14}, v_{39}, v_{74}],$
$[v_7, v_{11}, v_{15}, v_{40}, v_{75}], [v_8, v_9, v_{16}, v_{41}, v_{76}], [v_8, v_{10}, v_{17}, v_{42}, v_{77}], [v_8, v_{11}, v_{18}, v_{43}, v_{78}], [v_9, v_{10}, v_{19}, v_{44}, v_{79}], [v_9, v_{11}, v_{20}, v_{45}, v_{80}],$
$[v_{10}, v_{11}, v_{21}, v_{46}, v_{81}], [v_{12}, v_{13}, v_{16}, v_{47}, v_{82}], [v_{12}, v_{14}, v_{17}, v_{48}, v_{83}], [v_{12}, v_{15}, v_{18}, v_{49}, v_{84}], [v_{13}, v_{14}, v_{19}, v_{50}, v_{85}], [v_{13}, v_{15}, v_{20}, v_{51}, v_{86}],$
$[v_{14}, v_{15}, v_{21}, v_{52}, v_{87}], [v_{16}, v_{17}, v_{19}, v_{53}, v_{88}], [v_{16}, v_{18}, v_{20}, v_{54}, v_{89}], [v_{17}, v_{18}, v_{21}, v_{55}, v_{90}], [v_{19}, v_{20}, v_{21}, v_{56}, v_{91}], [v_{22}, v_{23}, v_{27}, v_{37}, v_{92}],$
$[v_{22}, v_{24}, v_{28}, v_{38}, v_{93}], [v_{22}, v_{25}, v_{29}, v_{39}, v_{94}], [v_{22}, v_{26}, v_{30}, v_{40}, v_{95}], [v_{23}, v_{24}, v_{31}, v_{41}, v_{96}], [v_{23}, v_{25}, v_{32}, v_{42}, v_{97}], [v_{23}, v_{26}, v_{33}, v_{43}, v_{98}],$
$[v_{24}, v_{25}, v_{34}, v_{44}, v_{99}], [v_{24}, v_{26}, v_{35}, v_{45}, v_{100}], [v_{25}, v_{26}, v_{36}, v_{46}, v_{101}], [v_{27}, v_{28}, v_{31}, v_{47}, v_{102}], [v_{27}, v_{29}, v_{32}, v_{48}, v_{103}], [v_{27}, v_{30}, v_{33}, v_{49}, v_{104}],$
$[v_{28}, v_{29}, v_{34}, v_{50}, v_{105}], [v_{28}, v_{30}, v_{35}, v_{51}, v_{106}], [v_{29}, v_{30}, v_{36}, v_{52}, v_{107}], [v_{31}, v_{32}, v_{34}, v_{53}, v_{108}], [v_{31}, v_{33}, v_{35}, v_{54}, v_{109}], [v_{32}, v_{33}, v_{36}, v_{55}, v_{110}],$
$[v_{34}, v_{35}, v_{36}, v_{56}, v_{111}], [v_{37}, v_{38}, v_{41}, v_{47}, v_{112}], [v_{37}, v_{39}, v_{42}, v_{48}, v_{113}], [v_{37}, v_{40}, v_{43}, v_{49}, v_{114}], [v_{38}, v_{39}, v_{44}, v_{50}, v_{115}], [v_{38}, v_{40}, v_{45}, v_{51}, v_{116}],$
$[v_{39}, v_{40}, v_{46}, v_{52}, v_{117}], [v_{41}, v_{42}, v_{44}, v_{53}, v_{118}], [v_{41}, v_{43}, v_{45}, v_{54}, v_{119}], [v_{42}, v_{43}, v_{46}, v_{55}, v_{120}], [v_{44}, v_{45}, v_{46}, v_{56}, v_{121}], [v_{47}, v_{48}, v_{50}, v_{53}, v_{122}],$
$[v_{47}, v_{49}, v_{51}, v_{54}, v_{123}], [v_{48}, v_{49}, v_{52}, v_{55}, v_{124}], [v_{50}, v_{51}, v_{52}, v_{56}, v_{125}], [v_{53}, v_{54}, v_{55}, v_{56}, v_{126}], [v_{57}, v_{58}, v_{62}, v_{72}, v_{92}], [v_{57}, v_{59}, v_{63}, v_{73}, v_{93}],$
$[v_{57}, v_{60}, v_{64}, v_{74}, v_{94}], [v_{57}, v_{61}, v_{65}, v_{75}, v_{95}], [v_{58}, v_{59}, v_{66}, v_{76}, v_{96}], [v_{58}, v_{60}, v_{67}, v_{77}, v_{97}], [v_{58}, v_{61}, v_{68}, v_{78}, v_{98}], [v_{59}, v_{60}, v_{69}, v_{79}, v_{99}],$
$[v_{59}, v_{61}, v_{70}, v_{80}, v_{100}], [v_{60}, v_{61}, v_{71}, v_{81}, v_{101}], [v_{62}, v_{63}, v_{66}, v_{82}, v_{102}], [v_{62}, v_{64}, v_{67}, v_{83}, v_{103}], [v_{62}, v_{65}, v_{68}, v_{84}, v_{104}], [v_{63}, v_{64}, v_{69}, v_{85}, v_{105}],$
$[v_{63}, v_{65}, v_{70}, v_{86}, v_{106}], [v_{64}, v_{65}, v_{71}, v_{87}, v_{107}], [v_{66}, v_{67}, v_{69}, v_{88}, v_{108}], [v_{66}, v_{68}, v_{70}, v_{89}, v_{109}], [v_{67}, v_{68}, v_{71}, v_{90}, v_{110}], [v_{69}, v_{70}, v_{71}, v_{91}, v_{111}],$
$[v_{72}, v_{73}, v_{76}, v_{82}, v_{112}], [v_{72}, v_{74}, v_{77}, v_{83}, v_{113}], [v_{72}, v_{75}, v_{78}, v_{84}, v_{114}], [v_{73}, v_{74}, v_{79}, v_{85}, v_{115}], [v_{73}, v_{75}, v_{80}, v_{86}, v_{116}], [v_{74}, v_{75}, v_{81}, v_{87}, v_{117}],$
$[v_{76}, v_{77}, v_{79}, v_{88}, v_{118}], [v_{76}, v_{78}, v_{80}, v_{89}, v_{119}], [v_{77}, v_{78}, v_{81}, v_{90}, v_{120}], [v_{79}, v_{80}, v_{81}, v_{91}, v_{121}], [v_{82}, v_{83}, v_{85}, v_{88}, v_{122}], [v_{82}, v_{84}, v_{86}, v_{89}, v_{123}],$
$[v_{83}, v_{84}, v_{87}, v_{90}, v_{124}], [v_{85}, v_{86}, v_{87}, v_{91}, v_{125}], [v_{88}, v_{89}, v_{90}, v_{91}, v_{126}], [v_{92}, v_{93}, v_{96}, v_{102}, v_{112}], [v_{92}, v_{94}, v_{97}, v_{103}, v_{113}], [v_{92}, v_{95}, v_{98}, v_{104}, v_{114}],$
$[v_{93}, v_{94}, v_{99}, v_{105}, v_{115}], [v_{93}, v_{95}, v_{100}, v_{106}, v_{116}], [v_{94}, v_{95}, v_{101}, v_{107}, v_{117}], [v_{96}, v_{97}, v_{99}, v_{108}, v_{118}], [v_{96}, v_{98}, v_{100}, v_{109}, v_{119}],$
$[v_{97}, v_{98}, v_{101}, v_{110}, v_{120}], [v_{99}, v_{100}, v_{101}, v_{111}, v_{121}], [v_{102}, v_{103}, v_{105}, v_{108}, v_{122}], [v_{102}, v_{104}, v_{106}, v_{109}, v_{123}], [v_{103}, v_{104}, v_{107}, v_{110}, v_{124}],$
$[v_{105}, v_{106}, v_{107}, v_{111}, v_{125}], [v_{108}, v_{109}, v_{110}, v_{111}, v_{126}], [v_{112}, v_{113}, v_{115}, v_{118}, v_{122}], [v_{112}, v_{114}, v_{116}, v_{119}, v_{123}], [v_{113}, v_{114}, v_{117}, v_{120}, v_{124}],$
$[v_{115}, v_{116}, v_{117}, v_{121}, v_{125}], [v_{118}, v_{119}, v_{120}, v_{121}, v_{126}], [v_{122}, v_{123}, v_{124}, v_{125}, v_{126}]\}$

It is no surprise that the distribution of variables in $m \in M$ above is exactly that of *sub div point sets* of 4 points in $\mathscr{X}_5 \in \mathscr{S}dps_{of}(\mathscr{X}_9, 5)$ where $\mathscr{X}_9$ is any *div point set* of 9 points as shown in (3.7).

Remark. $UNSAT^{\mathscr{DPS}^+}_{multiset}$ can be reduced into the boolean unsatisfiability problem, the complement of SAT, rather straightforwardly by first converting each multiset in A into the DNF formula below:

$$\bigvee_{v_0 \in \mathscr{A}} (\neg v_0 \wedge \bigwedge_{v_1 \in \mathscr{A} \setminus \{v_0\}} v_1) \vee \bigvee_{\mathscr{A}_{|3|} \in \mathscr{A}^*_{|3|}} (\bigwedge_{v_0 \in \mathscr{A}_3} \neg v_0 \wedge \bigwedge_{v_1 \in V \setminus \mathscr{A}_{|3|}} v_1) \vee (\bigwedge_{v_0 \in \mathscr{A}} \neg v_0) \tag{3.16}$$

where $\mathscr{A}^*_{|3|} = \{\mathscr{A}_{|3|} \in \mathbb{P}(\mathscr{A}) : |\mathscr{A}_{|3|}| = 3\}$ and $\mathscr{A}$ denotes the set of variables in each multiset, and each multiset in B into the DNF formula below:

$$\bigvee_{v \in \mathscr{B}} v \tag{3.17}$$

where $\mathscr{B}$ denotes the set of variables in each multiset, then joining all these DNF formulae conjunctively. One may realize that, in the case when $\mathscr{B} = \mathscr{A}$,the conjunction of $\bigvee_{u \in \mathscr{B}} u$ and $\bigwedge_{v_0 \in \mathscr{A}} \neg v$ gives a tautology, and thus for the instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ where $n = 5$, we would have a simpler propositional formula. The same observation can be made in the FOL formulae of such $UNSAT^{\mathscr{DPS}^+}_{multiset}$ instance where in order to satisfy (3.15), we would need to restrict (3.14) to $\forall a \in A \;\; a \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0]\}$.

We thereby conclude that a plausible approach to proving the upper-bound of the Erdös-Szekeres conjecture through $UNSAT^{\mathscr{DPS}^+}_{multiset}$ is by induction i.e. we first solve for the instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ where $n = 5$ - apparently accomplishable with a modern SAT solver running on a high performance computer - and then we prove the inductive hypothesis that $\forall m \in \mathbb{N}_{\geq 5}\ \mathcal{UNSAT}(m) \Rightarrow \mathcal{UNSAT}(m+1)$ where $\mathcal{UNSAT}(k)$ denotes the unsatisfiability of the instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ in which $n = k$.

Remark. The Erdös-Szekeres conjecture would not be disproven even if a certain instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ turns out to yield $False$ (i.e. it is satisfiable), since satisfying the constraints only implies that there exists a *div point set* of $2^{n-2}+1$ points for a particular $n \in \mathbb{N}_{\geq 5}$ where

I. none of its *sub div point sets* of n points is isomorphic to $Conv_n$

II. each of its *sub div point sets* of 5 points has 4, 2 or 0 distinct *sub div point sets* of 4 points isomorphic to $Conc^1_4$

from which we cannot conclude that such *div point set* is in $\mathscr{DPS}^+$, unless it too satisfies the stronger version of *Theorem 2* i.e. unless proven so, we should not rule out the possibility for some of its *sub div point sets* of 5 points to not be in $\mathscr{DPS}^+$ despite themselves having 4, 2 or 0 distinct *sub div point sets* of 4 points isomorphic to $Conc^1_4$ (with the remaining isomorphic to $Conv_4$).

To disprove the Erdös-Szekeres conjecture, not only do we need to show that (3.2) is false, we need to demonstrate there exists no other constraints besides (2.10), (2.11), and (2.12) $\mathscr{X} \in \mathscr{DPS}^*$ has to satisfy such that there exists an interpretation for $\pi_1(\mathscr{X})$ as some set of points in $\mathbb{E}^2$ i.e. *Axiom 1*'s consistency with Euclidean geometry.

References

[1] Erdös, P. and Szekeres, G. *A Combinatorial Problem in Geometry*, Compositio Math. 2, 463-470, 1935.

[2] Erdös P. and Szekeres, G. *On some extremum problems in elementary geometry*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 3-4 (1961), 53-62. Reprinted in: *Paul Erdös: The Art of Counting. Selected Writings (J. Spencer, ed.)*, 680-689, MIT Press, Cambridge, MA, 1973.

[3] Töh, G. and Valtr, P. *Note on the Erdös-Szekeres Theorem*, Discr. Comput. Geom. 19, 457-459, 1998.

[4] Kleitman, D. and Pachter, L. *Finding Convex Sets among Points in the Plane*, Discr. Comput. Geom. 19, 405-410, 1998.

[5] Szekeres, G. and Peters, L. *Computer Solution to the 17-Point Erdös-Szekeres Problem.* ANZIAM J. 48, 151-164, 2006.