On reducing the Erdös-Szekeres problem into a constraint unsatisfiability problem regarding certain multisets

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Abstract

We introduce the theory of *div point set*, which aims to provide a discrete framework for studying the combinatoric nature of any set of points in general position on an Euclidean plane. We then show that the Erdös-Szekeres conjecture can be proved through proving the unsatisfiability of some formulae involving some sets of 5-cardinal multisets over boolean variables under certain constraints.

1 Introduction

More than half a century ago Erdös and Szekeres [1] proved that for all $n \geq 3$, there exists an integer N such that among any N points in general position on an Euclidean plane, there always exists n points forming a convex polygon, and conjectured that the smallest number for N is determined by the function $g(n) = 2^{n-2} + 1$. This was known as the Erdös-Szekeres conjecture (and the problem of determining N was later named the *Happy Ending Problem*, as it led to the marriage of Szekeres and Klein, who first proposed the question). Currently the best known bounds are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1$$

Erdös and Szekeres first derived the lower bound in 1960 [2]. Many improvements for the upper bound have been made throughout the decades. The current upper bound was obtained by Tóth and Valtr [3] in 1998 as an improvement to the previous upper bound by Kleitman and Pachter [4] in the same year.

There are also attempts to verify individual instances of n. In 2002 it was proven that the conjecture holds for $n = 6$ by Szekeres and Peters with the help of an algorithm [5]. Even to this day it remains the best known result. Rather than showing that it holds for some $n \geq 7$ or proving for a smaller upper bound, our aim in this article is to demonstrate that solving some instances of a certain multiset unsatisfiability problem would prove the Erdös and Szekeres conjecture, through the theory of *div point set*.

1.1 preliminary

Throughout the article we would assume Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). The term "class" would be used to denote a collection of sets satisfying some predicate ϕ. A general form of Kuratowski definition would be used to define tuples. 2-tuples would be referred to as ordered pairs. A set of n-tuples would be referred to as relation (and described as binary relation when it is a set of ordered pairs). It would not matter as how natural numbers are defined as long as they satisfy Peano axioms. $\mathbb{N}_{\geq c}$ would be used to refer to the set of natural numbers greater or equal to some $c \in \mathbb{N}$ (e.g. $0 \notin \mathbb{N}_{\geq 1}$). For any natural numbers a, b, $\binom{a}{b}$ denotes the binomial coefficient a *choose* b.
Everything would be formulated under first order logic ($\wedge$,$\vee$,$\neg$,$\Rightarrow$ and $\Leftrightarrow$ would mean *and*,*or*,*not*,*imply* and *iff* respectively). We write $A := B$ if A is defined to be equivalent to B. $\forall x_1 \in A \forall x_2 \in A \forall x_3 \in A ... \forall x_n \in A$ would be shorten as

$$\forall x_1, x_2, x_3 ... x_n \in A$$

and $\exists x_1 \in A \exists x_2 \in A \exists x_3 \in A ... \exists x_n \in A$ as

$$\exists x_1, x_2, x_3 ... x_n \in A$$

For any set V, $|V|$ would be used to denote its cardinality, and $\mathscr{P}(V)$ be used to denote its power set.

$$\mathscr{P}(V) = \{v : v \subseteq V\}$$

We say a set V is totally ordered over certain binary relation $\geq$ iff for all a, b and c in V,

$$\begin{gathered}(a \geq b \wedge b \geq a) \Leftrightarrow (a = b) \\ (a \geq b \wedge c \geq b) \Leftrightarrow (a \geq c) \\ a \geq b \vee b \geq a\end{gathered}$$

The subscript of a set union or set intersection may be omitted to indicate that union or intersection is applied to each element in the set:

$$\begin{aligned}&\text{For any set, } A, \\ &\bigcup A = \bigcup_{a \in A} a = a_1 \cup a_2 \cup .. a_n \\ &\bigcap A = \bigcap_{a \in A} a = a_1 \cap a_2 \cap .. a_n \\ &\text{where } |A| = n \text{ and } a_1, a_2, .. a_n \text{ are all } n \text{ distinct elements of } A\end{aligned}$$

For any k-tuple T, $\pi_i(T)$ would be used to denote the i-th element of T where $i, k \in \mathbb{N}$ and $i \leq k$; $\pi_\cup(T)$ would be used the denote the union of 1st, 2nd ... k-th elements of a k-tuple;

and $\pi_{\cap}(T)$ would be used to denote intersection in such fashion.

$$\text{For any } k\text{-tuple, } T, \quad \begin{aligned} \pi_{\cup}(T) &= \bigcup_{i=1}^{k} \pi_i(T) \\ \pi_{\cap}(T) &= \bigcap_{i=1}^{k} \pi_i(T) \end{aligned}$$

A function is any relation, f, satisfying

$$\begin{aligned}
&\forall x \in X \\
&\qquad \exists r \in f = \pi_1(r) \\
&\forall r \in f \\
&\qquad \pi_1(r) \in X \\
&\qquad \pi_2(r) \in Y \\
&\forall r_1, r_2 \in f \\
&\qquad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2)
\end{aligned}$$

for some none-empty sets X (often referred to as domain) and Y (referred to as co-domain). We often express the relation between f, X, and Y as:

$$f : X \to Y$$

We write $f(x) = y$ iff there exists an ordered pair (x, y) in f. A function f is injective iff

$$\begin{aligned}
&\forall r_1, r_2 \in f \\
&\qquad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2)
\end{aligned}$$

It is subjective iff

$$\begin{aligned}
&\forall y \in Y \\
&\qquad \exists r \in f \quad y = \pi_2(r)
\end{aligned}$$

It is bijective iff it is both injective and surjective. To avoid ambiguity, for any function $f : X \to Y$, $f^{members}$ would be used to denote a new function, from $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ such that

$$f^{members}(x) := \bigcup_{a \in x} \{f(a)\}$$

Here is a generalization of it, $f^{members^n}$, defined recursively:

$$\begin{aligned}
f^{members^n}(x) &:= \bigcup_{a \in x} \{f^{members^{n-1}}(a)\} \text{ where } n \in \mathbb{N}_{\geq 2} \\
f^{members^1}(x) &:= f^{members}(x)
\end{aligned}$$

A multiset is a generalization of set, where the same element can occur more for multiple times. It is defined as an ordered pair (A, m_{m}) where $m_{\text{m}} : A \to \mathbb{N}_{\geq 1}$ is a function used to denote the number of occurrences of some element in the multiset and A is a set of all distinct elements in the multiset. The cardinality of a multiset (A, m_{m}) is defined as the sum of all $m(x)$ for $x \in A$. Multisets are expressed using square brackets, [ ], as compared to sets which use curly brackets, $\{\}$. Here is an example:

$$\begin{aligned}&[f(x) : x \in \mathbb{N}_{\geq 1} : x \leq 3] = [1, 1, 1]\\&\text{where } f(x) = 1\end{aligned}$$

A hypergraph is a generalization of graph, where an edge can contain any number of vertices. It is defined as an ordered pair (V, E) where

$$E \subseteq \mathscr{P}(V) \setminus \varnothing$$

Members of V are referred to as vertices while members of E are referred to as edges or hyperedges. A hypergraph is k-uniformed when

$$\forall e \in E \quad |e| = k$$

where $k \in \mathbb{N}_{\geq 1}$. A full vertex coloring on some graph or hypergraph, (V, E), is defined as a function, $C : V \to cDom$, such that

$$\begin{gathered}|C| = |V|\\\forall c \in C \quad \pi_1(c) \in V \land \pi_2(c) \in cDom\\\forall c_1, c_2 \in C \quad c_1 = c_2 \Leftrightarrow \pi_1(c_1) = \pi_1(c_2)\end{gathered}$$

where $cDom \subset \mathbb{N}$, and it is often referred to as the set of colors. When $|Dom| = 2$, we say the coloring is monochromatic. We would use $FullCol(G, cDom)$ to denote the set of all possible full vertex colorings on a graph G of the set of colors $cDom$. For any graph G of n vertices, and any non-empty $cDom$,

$$|FullCol(G, cDom)| = n^{|cDom|}$$

2 *Div point set* as a representation for any set of points in general position

We start off by introducing an object which we would be referring to as *div point set*.

Definition 1. A *div point set* is any order-pair (P, Θ_P) satisfying

$$|\Theta_P| = \binom{|P|}{2} \wedge P \neq \varnothing \tag{2.1}$$

$$\forall D_n \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ \bigcup \delta_n = P \setminus d_n \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.2}$$

$$\forall D_n, D_m \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ (d_m, \delta_m) \coloneqq D_m \\ d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.3}$$

We would be using $\mathscr{DPS}^*$ to denote the class of all ordered pairs satisfying (2.1), (2.2) and (2.3). Thus $\mathscr{X}$ is a *div point set* iff $\mathscr{X} \in \mathscr{DPS}^*$.

For any n points in general position, where $n \geq 2$, we can always select any 2 arbitrary points and draw a line across them, dividing the rest of the points into 2 disjoint sets. So long as the points are in general position, we can be sure that no 3 points forms a line, and thus each of the remaining $n - 2$ points would always be in one of these sets. Let's refer to these 2 disjoint sets as *divs* produced by a *divider* made up of 2 distinct points, and the points in the *divs* as *TBD points* of the *divider* (short for *to-be-distributed-among-divs points*). The process of selecting 2 distinct points, creating a *divider*, and producing 2 *divs* can be repeated $\binom{|P|}{2}$ times until all sets of 2 points in P are selected.

Any set of points P in general position on an Euclidean plane where $|P| \geq 2$ can be represented by some *div point set* (P, Θ_P). We would refer to each member of $D_n \in \Theta_P$ as a *dividon*, to be interpreted as follows:

$$(d_n, \delta_n) \coloneqq D_n \quad \left| \begin{array}{l} \{a, b\} := d_n \\ a \text{ and } b \text{ represent the 2 points which make up the } \textit{divider} \\ \{div_1, div_2\} := \delta_n \\ div_1 \text{ and } div_2 \text{ represent the 2 } \textit{divs} \text{ produced by the } \textit{divider} \\ \bigcup \delta_n \text{ thus represents the set of } \textit{TBD points} \text{ of the } \textit{divider} \end{array} \right.$$

The sets of points in Figures 1, 2 and 3 can be represented by any div-point set (A, Θ_A)

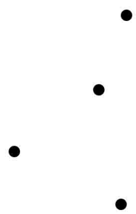

Figure I

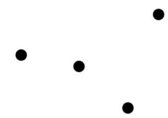

Figure II

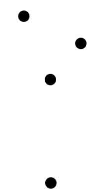

Figure III

as long as $A = \{a, b, c, d\}$ and

$$\begin{aligned}\Theta_A =\{&(\{a, b\}, \{(\{c\}, \{d\}\}),\\ &(\{a, c\}, \{(\{b\}, \{d\}\}),\\ &(\{a, d\}, \{(\{b\}, \{c\}\}),\\ &(\{b, c\}, \{(\{a, d\}, \varnothing\}),\\ &(\{b, d\}, \{(\{a, c\}, \varnothing\}),\\ &(\{c, d\}, \{(\{a, b\}, \varnothing\})\})\end{aligned}$$

To make sense of the *div point set* representation, we label the third point from the bottom in Figure I and the second point from the bottom in Figures II and III as a (note that each of these is the point in the figure that are surrounded by the remaining 3 points). For the rest of the points in each figure we simply label them arbitrarily as b, c, and d.

Only a handful of *div point sets* can be used to represent points in general position in $\mathbb{E}^2$. For majority of $\mathscr{X} \in \mathscr{DPS}^*$, let $(P, \Theta_P) \coloneqq \mathscr{X}$, there exists no meaningful interpretation for P as some sets of points in $\mathbb{E}^2$ such that Θ_P describe how points are distributed between *divs* produced by different *dividers*. A classical example would be any *div* point set (Q, Θ_Q) where $Q = \{a, b, c, d\}$ and

$$\begin{aligned}\Theta_Q =\{&(\{a, b\}, \{(\{c, d\}, \varnothing\}),\\ &(\{a, c\}, \{(\{b, d\}, \varnothing\}),\\ &(\{a, d\}, \{(\{b, c\}, \varnothing\}),\\ &(\{b, c\}, \{(\{a, d\}, \varnothing\}),\\ &(\{b, d\}, \{(\{a, c\}, \varnothing\}),\\ &(\{c, d\}, \{(\{a, b\}, \varnothing\})\})\end{aligned}$$

For a *div point set* (P, Θ_P) to have a meaningful interpretation for P as some set of points in $\mathbb{E}^2$, it has to be satisfy certain constraints. For any 3 arbitrary points, x, y, and

z in general position in $\mathbb{E}^2$, let's use $\langle x, y\rangle^z$ to denote the *div* containing z produced by the *divider* made up of the point x and y, and $\langle x, y\rangle^{\text{-}z}$ to denote the *div* not containing z produced by the *divider*.

$$\begin{aligned}
&\forall x, y, z \\
&\qquad z \in \langle x, y\rangle^z \\
&\qquad z \notin \langle x, y\rangle^{\text{-}z}
\end{aligned}$$

After some experimentation with points in $\mathbb{E}^2$, we can conclude that the following formulas always hold true for any distinct points a, b, c, d in $\mathbb{E}^2$. (2.4) is trivially true, (2.5) is demonstrated in Figure *IV*, (2.6) is demonstrated in Figure *V* and (2.7) is demonstrated in Figure *VI*.

$$\begin{aligned}
&\forall a, b, c, d \\
&\qquad a \in \langle b, c\rangle^d \Leftrightarrow d \in \langle b, c\rangle^a \\
&\qquad a \in \langle b, c\rangle^{\text{-}d} \Leftrightarrow d \in \langle b, c\rangle^{\text{-}a}
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\forall a, b, c, d \\
&\qquad c \in \langle a, b\rangle^{\text{-}d} \\
&\qquad \Leftrightarrow (a \in \langle b, c\rangle^d \wedge a \in \langle b, d\rangle^c) \\
&\qquad\qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{\text{-}c})
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&\forall a, b, c, d \\
&\qquad c \in \langle a, b\rangle^d \\
&\qquad \Leftrightarrow (a \in \langle b, c\rangle^d \wedge a \in \langle b, d\rangle^{\text{-}c}) \\
&\qquad\qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^c)
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\forall a, b, c, d \\
&\qquad a \in \langle b, c\rangle^{-d} \wedge a \in \langle b, d\rangle^{-c} \Rightarrow a \in \langle c, d\rangle^b
\end{aligned} \tag{2.7}$$

In the context of *div point sets*, (2.4) is always true by (2.2) (recall $\bigcap \delta = \varnothing$), while (2.5), (2.6) and (2.7) can be rewritten as constraints on the *dividons* of a *div point set* as shown in (2.8), (2.9), and (2.10).

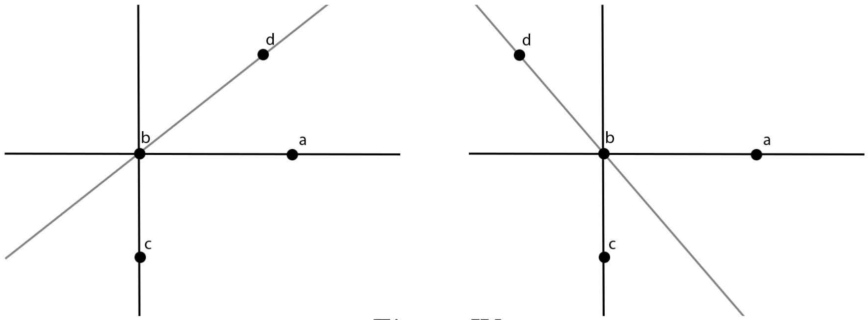

Figure IV

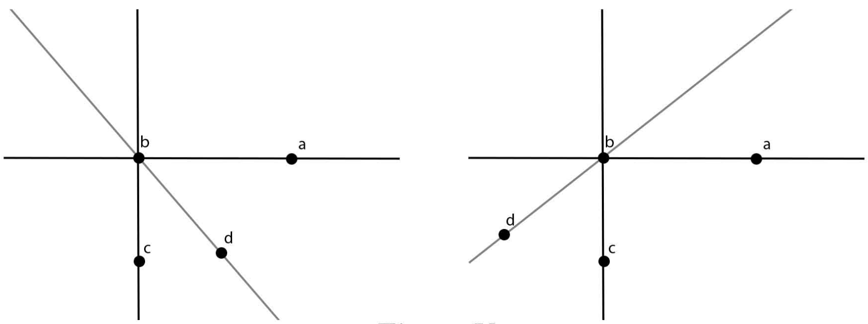

Figure V

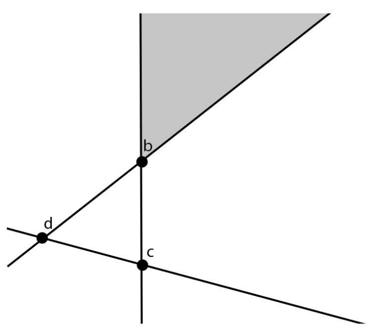

Figure VI

For any *div point set* (P, Θ_P),

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{3} \pi_1(D_n) = R \\
&\qquad\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 1 \\
&\qquad\qquad\qquad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{3} \pi_1(D_n) = R \\
&\qquad\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 0 \\
&\qquad\qquad\qquad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) \neq \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad \bigcap_{n=1}^{2} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{2} \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3) \\
&\qquad\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = 0 \\
&\qquad\qquad\qquad \Rightarrow \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) = 1\ )
\end{aligned}
\tag{2.10}
$$

where ϕ is a function for determining if two arbitrary points are elements of the same *div* in some δ of a *dividon*:

$$\phi(\delta, w) = \begin{cases} 1 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 = div_2 \\ 0 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 \neq div_2 \end{cases} \text{ where } \left| \begin{array}{l} \delta = \{div_1, div_2\} \\ w = \{a, b\} \end{array} \right. \tag{2.11}$$

Axiom. Any set of points in general position in $\mathbb{E}^2$ can be represented by a *div point set*. A *div point sets* (P, Θ_P) has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$ such that $D \in \Theta_P$ each describes the relative positions of the points (in terms of how the *TBD points* of each *divider* is distributed between *divs* it produced) iff it is in $\mathscr{DPS}^+$, the class of *div point sets* of 4 ore more points satisfying (2.8), (2.9), and (2.10).

$$\mathscr{DPS}^+ \subset \mathscr{DPS}^*$$

Definition 2. We say that two *div point sets* (A, Θ_A) and (B, Θ_B) are isomorphic iff there exists a bijective function $f : A \overset{1:1}{\to} B$ such that it preserves the *divions* structure:

$$(A, \Theta_A) \cong (B, \Theta_B) \quad \Leftrightarrow \left| \begin{array}{l} \exists f : A \overset{1:1}{\to} B \\ \qquad \forall D_A \in \Theta_A \\ \qquad\qquad \exists D_B \in \Theta_B \\ \qquad\qquad\qquad (d_a, \delta_a) := D_A \\ \qquad\qquad\qquad (d_b, \delta_b) := D_B \\ \qquad\qquad\qquad f^{members}(d_a) = d_b \Leftrightarrow f^{members^2}(\delta_a) = \delta_b \end{array} \right. \tag{2.12}$$

in which case we would refer to f as the isomorphism between (A, Θ_A) and (B, Θ_B).

Remark. It is trivially true that all div point sets (P, Θ_P) in $\mathscr{DPS}^*$ where $|P| \leq 3$ are isomorphic to any div point sets (Q, Θ_Q) in $\mathscr{DPS}^*$ iff $|Q| = |P|$.

Theorem 1. $\neg\mathscr{X} \cong Conc_4^1 \Leftrightarrow \mathscr{X} \cong Conv_4$ for all $\mathscr{X} \in \mathscr{DPS}_4^+$ where $\mathscr{DPS}_4^+$ denotes the *div point sets* in $\mathscr{DPS}^+$ of 4 points and

$$\begin{aligned} Conc_4^1 =& (Cc_4^1, \Theta_{Cc_4^1}) & Conv_4 =& (Cv_4, \Theta_{Cv_4}) \\ Cc_4^1 =& \{1,2,3,4\} & Cv_4 =& \{1,2,3,4\} \\ \Theta_{Cc_4^1} =& \{(\{1,2\}, \{(\{3\}, \{4\}\}), & \Theta_{Cv_4} =& \{(\{1,2\}, \{(\{3,4\}, \varnothing\}), \\ & (\{1,3\}, \{\{2\}, \{4\}\}), & & (\{1,3\}, \{\{2,4\}, \varnothing\}), \\ & (\{1,4\}, \{\{2,3\}, \varnothing\}), & & (\{1,4\}, \{\{2\}, \{3\}\}), \\ & (\{2,3\}, \{\{1\}, \{4\}\}), & & (\{2,3\}, \{\{1,4\}, \varnothing\}), \\ & (\{2,4\}, \{\{1,3\}, \varnothing\}), & & (\{2,4\}, \{\{1\}, \{3\}\}), \\ & (\{3,4\}, \{\{1,2\}, \varnothing\}) & & (\{3,4\}, \{\{1\}, \{2\}\}) \end{aligned} \tag{2.13}$$

Proof. For all div point sets (P, Θ_P) in $\mathscr{DPS}_4^+$, since $|P| = 4$, we can be certain that

$$\begin{aligned}
&\forall D \in \Theta_P \\
&\qquad \pi_2(D) \in \{type_0, type_1\} \\
&\qquad \{a, b\} = P \setminus \pi_1(D) \\
&\qquad type_0 = \{\{a\}, \{b\}\} \\
&\qquad type_1 = \{\{a, b\}, \varnothing\}
\end{aligned} \tag{2.14}$$

Therefore, every *dividon* D of any $\mathscr{X} \in \mathscr{DPS}_4^+$ satisfies

$$\phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \tag{2.15}$$

where ψ a simpler version of ϕ:

$$\psi(\delta) = \begin{cases} 1 & \text{if } \exists div \in \delta \quad |div| = 2 \\ 0 & \text{if } \forall div \in \delta \quad |div| = 1 \end{cases} \tag{2.16}$$

Let's define $\mathscr{DPS}_4^{\mathbb{N}}$ to be the set of all *div point sets* (P, Θ_P) where $P = \{1, 2, 3, 4\}$. All $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ would have the same *divider*s (Recall the set of *divider*s is just the set of elements in $\mathscr{P}(P)$ whose cardinality is 2. What makes $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ different is the distribution of *TBD points* in each pair of *divs*.) Now let $H = (V, E)$ be a hyper-graph whose vertices are the *dividers* of the div point sets in $\mathscr{DPS}_4^{\mathbb{N}}$

$$V = \{d \in \mathscr{P}(P) : |d| = 2\} \tag{2.17}$$

and we can define a bijective function that transforms the set of *dividons* of a *div point set* in $\mathscr{DPS}_4^{\mathbb{R}}$ into some full vertex monochromatic coloring for H.

$$Col(\Omega_P) = \{(\pi_1(D), \psi(\pi_2(D))) : D \in \Omega_P\} \tag{2.18}$$

Col would return a different coloring depending on how different each $\psi(\delta)$ is for the same *dividons* in two different *div point sets*. There are $2^{\binom{4}{2}} = 64$ distinct full vertex monochromatic coloring on H in total, which is also the number of distinct div point sets $\mathscr{DPS}_4^{\mathbb{N}}$ contains.

$$|FullCol(H, \{0, 1\})| = |\mathscr{DPS}_4^{\mathbb{N}}|$$

Now let's define any set of three *divider*s containing 1 element in common to be an edge of H:

$$E = \{e \in \mathscr{P}(V) : |e| = 3 \wedge |\bigcap e| = 1\} \tag{2.19}$$

H is a 3-uniform hypergraph with 4 hyperedges. For $X \in \mathscr{DPS}_4^{\mathbb{N}}$ to satisfy (2.8) and (2.9) is equivalent to having $Col(X)$ of H satisfy the following:

I. If a vertex, V, is colored 0, the other 2 vertices belonging to the same edge as V must have the same coloring.

II. If a vertex, V, is colored 1, the other 2 vertices belonging to the same edge as V must have different colorings.

This is due to the fact, for *div point sets* of 4 points, (2.8) and (2.9) can be rewritten as having some set of *dividers* to satisfy some formulae, namely the following:

$$\begin{aligned} &\forall e \in E \\ &\quad \forall \delta_1, \delta_2, \delta_3 \in e \\ &\quad\quad \delta_1 \neq \delta_2 \neq \delta_3 \\ &\quad\quad \Leftrightarrow (\psi(\delta_1) = 1 \Leftrightarrow \psi(\delta_2) = \psi(\delta_3)) \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\forall e \in E \\ &\quad \forall \delta_1, \delta_2, \delta_3 \in e \\ &\quad\quad \delta_1 \neq \delta_2 \neq \delta_3 \\ &\quad\quad \Leftrightarrow (\psi(\delta_1) = 0 \Leftrightarrow \psi(\delta_2) \neq \psi(\delta_3)) \end{aligned} \tag{2.21}$$

as a result of

$$\begin{aligned} &\forall p_1, p_2, p_3, p_4 \in P \\ &\quad R := \bigcup_{n=1}^{4} \{p_n\} \\ &\quad\quad \forall D \in \Theta_P \\ &\quad\quad\quad \phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D), P \setminus \pi_1(D)) \\ &\quad\quad\quad \phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \end{aligned} \tag{2.22}$$

for any div point set (P, Θ_P) where $|P| = 4$ (recall (2.14)), and by (2.3), there exists a bijective function between set of *dividon* and the set of *dividers*: selecting some *dividons* $D_1, D_2, D_3 \in \Theta_P$ where

$$|\bigcap_{n=1}^{3} \pi_1(D_n)| = 1 \wedge |\bigcup_{n=1}^{3} \pi_1(D_n)| = 4$$

can be done by selecting some *dividers* $d_1, d_2, d_3 \in V$ where

$$|\bigcap_{n=1}^{3} d_n| = 1 \wedge d_1 \neq d_2 \neq d_3$$

and thus a *div point set* of 4 points, $\mathcal{X}$, satisfies (2.8) and (2.9) iff $Col(\mathcal{X})$ satisfies I and II.

By I and II, 3 vertices belonging to the same edge can only be colored $[0,0,0]$ or $[0,1,1]$.

Suppose we give some vertices belonging to the same edge the coloring of $[0,0,0]$, by I this would indicate that the rest of the vertices need to have the same colors (recall (2.19): each vertex belongs to 2 different edges). We can either end up with H having all vertices colored 0 (let's call it scenario 1), or 3 vertices colored 0 and 3 vertices colored 1 (let's call it scenario 2).

Suppose we give some vertices belonging to the same edge the coloring of $[0,1,1]$, by I this would indicate that the remaining 2 vertices of another edge, which the vertex colored 0 belongs to, need to have the same colors. If we give them the coloring of $[0,0]$, we would have an edge with vertices colored $[0,0,0]$, and end up in scenario 2 again. If we give them the coloring of $[1,1]$, we would end up with 1 vertex colored 0 and 4 vertices colored 1, in which case the last uncolored vertex would need to be colored 0, since it belongs to 2 edges both with 2 vertices colored 1. Let's name this scenario 3, where 2 vertices are colored 0 and 4 vertices are colored 1.

Scenario 2 is a coloring isomorphic to $Col(Conc_4^1($ while scenario 3 is a coloring isomorphic to $Col(Conv_4)$. $Conc_4^1$ and $Conv_4$ both satisfy (2.10). Scenario 1 is equivalent to $Col((P,\Theta_\varnothing))$ where

$$\begin{aligned}\Theta_\varnothing =\{&(\{1,2\},\{(\{3,4\},\varnothing\}),\\ &(\{1,3\},\{(\{2,4\},\varnothing\}),\\ &(\{1,4\},\{(\{2,3\},\varnothing\}),\\ &(\{2,3\},\{(\{1,4\},\varnothing\}),\\ &(\{2,4\},\{(\{1,3\},\varnothing\}),\\ &(\{3,4\},\{(\{1,2\},\varnothing\})\})\end{aligned}$$

which it does not satisfy (2.10). Since any div point set of 4 points is isomorphic to some $\mathcal{X} \in \mathcal{DPS}_4^{\mathbb{N}}$, and only $Conc_4^1$ and $Conv_4$ satisfy (2.8), (2.9), and (2.10), we conclude that

$$\forall X \in \mathcal{DPS}_4^+ \quad \exists a \in \{Conc_4^1, Conv_4\} \quad X \cong a$$

■

A pictorial description of the coloring is shown in Figure VII (for illustrative purpose, each edge is colored differently).

Remark. In Euclidean geometry, Theorem 1 can be interpreted as stating: for any set of 4 distinct points in general positions, it is either the case that it forms a structure where 1 point is inside a triangle formed by connecting the rest of 3 points, or the case that a convex polygon can be created by connecting the 4 points in a certain manner, which can be verified rather easily by a human child with a pen, a piece of paper and a love for geometry.

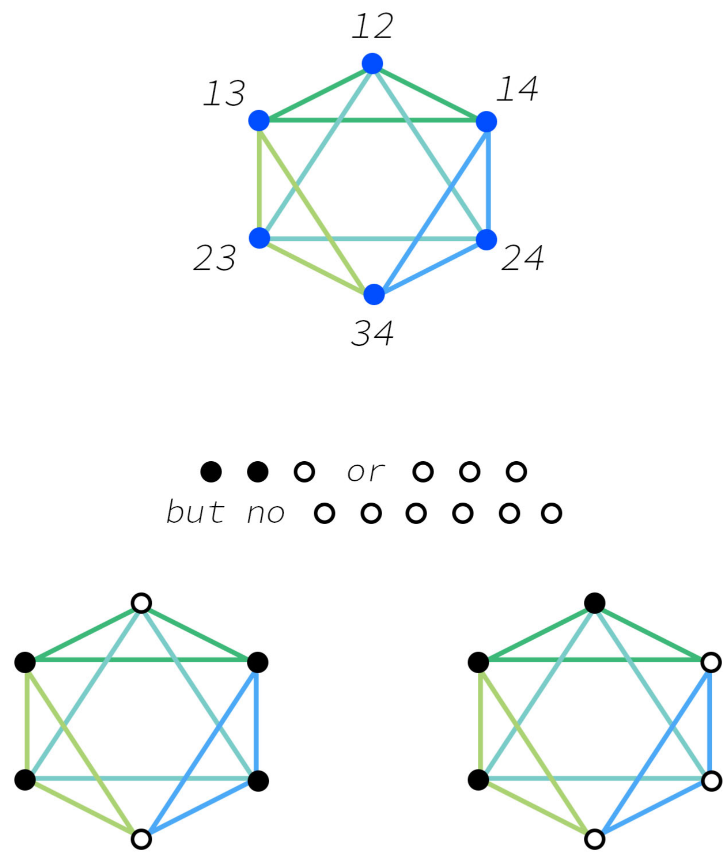

Figure VII

2.1 *unit div point set* and *sub div point set*

For *div point sets* of 5 or more points, the function ψ defined in (2.16) would not be really useful since there would be 3 or more *TBD points* in each *dividon*. That means we cannot apply to same technique above to derive *div point sets* of 5 or more points satisfying (2.8), (2.9) and (2.10). With that in mind, we introduce *units div point set*, a generalization of *div point set* that uses the notion of *unit dividon*.

Definition 3. A *unit div point set* is any order-pair (P, Ω_P) satisfying (2.23), (2.24) and (2.25).

$$|\Omega_P| = \binom{|P|}{2}\binom{|P-2|}{2} \wedge P \neq \varnothing \tag{2.23}$$

$$\forall D_n \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ |\bigcup \delta_n| = 2 \\ \bigcup \delta_n \in \mathscr{P}(P \setminus d_n) \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.24}$$

$$\forall D_n, D_m \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ (d_m, \delta_m) := D_m \\ d_n \cup \bigcup \delta_n = d_m \cup \bigcup \delta_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.25}$$

We would be using $\mathscr{UDPS}^*$ to denote the class of all *unit div point set*.

Remark. Similar to how *div point sets* of 4 points always satisfy (2.14), a *unit div point set* always satisfies (2.26).

$$\begin{array}{l} \forall \mathfrak{X} \in \mathscr{UDPS}^* \\ \quad (P, \Omega_P) := \mathfrak{X} \\ \quad \forall D \in \Omega_P \\ \quad\quad \pi_2(D) \in \{type_0, type_1\} \\ \quad\quad \{a, b\} \subseteq P \setminus \pi_1(D) \\ \quad\quad\quad type_0 = \{\{a\}, \{b\}\} \\ \quad\quad\quad type_1 = \{\{a, b\}, \varnothing\} \end{array} \tag{2.26}$$

For any *unit div point set*, (P, Ω_P), by (2.26), we can use ψ defined in (2.16) to map every $\pi_2(D) \in \Omega_P$ to some $k \in \{0, 1\}$.

Remark. Comparing the definition above with the definition of *div point set* we would immediately notice that

$$\{\mathscr{X}_{udps} \in \mathscr{UDPS}^* \ : |\pi_1(X)| = 4\} = \{\mathscr{X}_{dps} \in \mathscr{DPS}^* \ : |\pi_1(X)| = 4\}$$

due to the fact that

$$\binom{|4|}{2}\binom{|4-2|}{2} = \binom{|4|}{2}$$

$$\forall X \in \mathscr{UDPS}^* \quad \left| \begin{array}{l} (P, \Omega_P) \coloneqq X \\ |P| = 4 \\ \forall D_n \in \Omega_P \\ \qquad \delta_n = P \setminus d_n \\ \forall D_n, D_m \in \Omega_P \\ \qquad d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right.$$

As we can see, the difference between a *div point set* and a *unit div point set* lies in that the former relies on a single *dividon* to describe the distribution of $|P| - 2$ *TBD points* between the 2 *divs* produced by a *divider*, while the later relies on $\binom{|P-2|}{2}$ unit divions for that, as each *unit dividon* only describe the distribution of 2 *TBD points*. For every $\mathscr{X}_{dps} \in \mathscr{DPS}^*$ there exists a unique $\mathscr{X}_{udps} \in \mathscr{UDPS}^*$ which $\mathscr{X}_{dps}$ can be transformed into. To transform a *div point set* into a *unit div point set*, we simply break down each *dividon* into $\binom{|P-2|}{2}$ *unit dividons*, which can be achieved by the function $\mathscr{bd}$ as defined below:

$$\begin{aligned} \mathscr{bd}(D, P) &= \{(\pi_1(D), bd(x, \pi_2(D)) : x \in \mathscr{P}(P \setminus \pi_1(D)) : |x| = 2\} \\ bd(x, divs) &= \begin{cases} \{x, \varnothing\}, & \text{if } x \subseteq divs \\ \{\{a\}, \{b\}\}, & \text{if } a \in div_1 \land b \in div_2 \land \{x_1, x_2\} = divs \\ & \text{where } x = \{a, b\} \end{cases} \end{aligned} \tag{2.27}$$

for some $D \in \Theta_P$ and P of a *div point set* (P, Θ_P).

Definition 4. The function $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ transforms a *div point set* of into a *unit div point set*.

$$\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}) = (P, \Omega_P)$$

$$\text{where} \quad \left| \begin{array}{l} (P, \Theta_P) \coloneqq \mathscr{X} \\ \forall D \in \Theta_P \\ \qquad \exists \Omega_{sub} \subset \Omega_P \\ \qquad \qquad \Omega_{sub} = \mathscr{bd}(D, P) \\ |\Omega_P| = \binom{|P|}{2}\binom{|P-2|}{2} \end{array} \right. \tag{2.28}$$

$\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ can be implemented in Haskell as follow:

```haskell
import Control.Monad
import Data.List ((\\))
powerList = filterM (const [True, False])

f:: ([Int],[([Int],[[Int]])]) -> ([Int],[([Int],[[Int]])])
f (points,dividons) = (points,unit_dividons)
    where
        unit_dividons = foldl (++) [] $ map get_unit_dividons dividons
        get_unit_dividons (d,(delta1:_)) = [(d,(\(a:b:_)->
            if a `in_same_div_as_b` b
                then [[a,b],[]]
                else [[a],[b]])
            x ) |
            x <- powerList (points \\ d), length x == 2,
            let (in_same_div_as_b) a b = (a `elem` delta1) == (b `elem` delta1)]
```

Remark. If we use $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ on *div point sets* of 4 points we would immediately realize that $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ returns the same ordered pair, since for *div point sets* of 4 points, $\Omega_{sub} \subset \Omega_P$ in (2.28) would contain only one element and the element is some $D_\Theta \in \Theta_P$. For 5 or more points Ω_{sub} would contain 3 or more elements, thus

$$
\begin{aligned}
&\forall \mathscr{X} \in \mathscr{DPS}^* \\
&\quad (P, \Theta_P) \coloneqq \mathscr{X} \\
&\quad \mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}((P, \Theta_P)) = (P, \Theta_P) \Leftrightarrow |P| = 4
\end{aligned}
$$

Remark. A *div point set* with 3 or less points on the other hand would result in $(P, \varnothing)$ since $\binom{n-2}{2} = 0$ for $n < 4$ and that is not going to be useful. So it is more sensible to define $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ over *div point sets* of 4 or more points.

$$\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps} : \mathscr{DPS}^*_{\geq 4} \rightarrow \mathscr{UDPS}^*$$

Lemma 1. $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps} : \mathscr{DPS}^*_{\geq 4} \rightarrow \mathscr{UDPS}^*$ *is injective but not surjective. If the codomain is defined to be* $\mathscr{UDPS}^\Theta$*, the set of unit div point sets satisfying* (2.29)*,* $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}$ *is then bijective.*

$$
\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\quad (D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3) \\
&\quad \Rightarrow (\psi(\pi_2(D_1)) = 1 \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))) \\
&\qquad \wedge (\psi(\pi_2(D_1)) = 0 \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3)))
\end{aligned}
\tag{2.29}
$$

where ψ is defined in (2.16).

Proof. It is injective because V differs depending on $D \in \Theta_P$ as a result of $db(a,b)$ being injective. It is surjective over the co-domain $\mathscr{UDPS}^*$, but bijective over the co-domain $\mathscr{UDPS}^{\Theta}$, as a consequence of

I. $|\delta_n| = 2$ in (2.2): *Unit div point sets* with *unit dividons* such as

$$\{(a,b),(\{c\},\{d\})\},\{(a,b),(\{c\},\{e\})\},\{(a,b),(\{e\},\{d\})\}$$

can only be transformed from a *div point set* where $|\delta_n| = 3$ for some dividion, in this case:

$$\{(a,b),(\{c\},\{d\},\{e\})\}$$

Thus we have

$$\begin{aligned} &\forall D_1, D_2, D_3 \in \Omega_P \\ &\qquad \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3 \\ &\qquad \Leftrightarrow \neg((\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = \psi(\pi_2(D_3)) = 0) \end{aligned} \tag{2.30}$$

II. Associativity: if a and b are in the same *div*, and b and c are in the same *div*, a and c must be in the same *div*. So unit *div* points set with *unit dividons* such as

$$\{(a,b),(\{c,d\},\varnothing)\},\{(a,b),(\{c,e\},\varnothing)\},\{(a,b),(\{e\},\{d\})\}$$

can not be transformed from any *div point set*. Thus we have

$$\begin{aligned} &\forall D_1, D_2, D_3 \in \Omega_P \\ &\qquad \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3 \\ &\qquad \Leftrightarrow \neg(\psi(\pi_2(D_1)) = \psi(\pi_2(D_2) = 1 \wedge \psi(\pi_2(D_3)) = 0) \end{aligned} \tag{2.31}$$

Combining (2.31) and (2.30) gives (2.29). ■

Lemma 2. A *unit div point sets* (P, Ω_P) has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$ such that $D \in \Omega_P$ each describes the relative positions of the points (in terms of how 2 *TBD points* of each *divider* is distributed between *divs* it produced) iff it is in $\mathscr{UDPS}^+$, the class of *unit div point sets* of 4 ore more points satisfying (2.29), (2.32), (2.33), and (2.34).

$$\mathscr{UDPS}^+ \subset \mathscr{UDPS}^{\Theta} \subset \mathscr{UDPS}^*$$

For any *unit div point set* (P, Ω_P),

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Omega_P \\
&\qquad\qquad (\ \pi_\cup(D_1) = \pi_\cup(D_2) = \pi_\cup(D_3) = R \\
&\qquad\qquad \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = 1 \\
&\qquad\qquad\qquad \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))\ )
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad (\ \pi_\cup(D_1) = \pi_\cup(D_2) = \pi_\cup(D_3) = R \\
&\qquad\qquad \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = 0 \\
&\qquad\qquad\qquad \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3))\ )
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad (\ \pi_\cup(D_1) = \pi_\cup(D_2) = \pi_\cup(D_3) = R \\
&\qquad\qquad \wedge \bigcap_{n=1}^{2} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{2} \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3)\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 0 \\
&\qquad\qquad\qquad \Rightarrow \psi(\pi_2(D_3)) = 1\ )
\end{aligned}
\tag{2.34}
$$

Proof. For any $\mathscr{X}_{udps}$, where $\mathscr{X}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X}_{dps})$ for some $\mathscr{X}_{dps} \in \mathscr{DPS}^*$, by Lemma 1, $\mathscr{X}_{udps}$ always satisfies (2.29). For any $\mathscr{Y}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{Y}_{dps})$ for some $\mathscr{Y}_{dps} \in \mathscr{DPS}^+$, $\mathscr{Y}_{udps}$ always satisfies (2.32), (2.33), and (2.34), since they are simply a different way of writing (2.8), (2.9), and (2.10) for *unit div point sets*. This can be demonstrated in a similar way as (2.22): for any *unit divdion* D_{udps} of some *unit div point set*, $\mathscr{A}_{udps}$, and its corresponding *divdion* D_{dps} of the *div point set* $\mathscr{A}_{dps}$ where $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{A}_{dps}) = \mathscr{A}_{udps}$ - corresponding in the sense that $D_{udps} \in \mathscr{bd}(D_{dps}, P)$ where $P = \pi_2(mathscrA_{dps})$ - we would have:

$$
\begin{aligned}
&\pi_1(D_\Omega) \cup \bigcup \pi_2(D_\Omega) = \{p_1, p_2, p_3, p_4\} \\
&R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad \phi(\pi_2(D_\Theta), R \setminus \pi_1(D_\Theta)) = \phi(\pi_2(D_\Omega), \bigcup \pi_2(D_\Omega)) \\
&\quad \phi(\pi_2(D_\Omega), \bigcup \pi_2(D_\Omega)) = \psi(\pi_2(D))
\end{aligned}
\tag{2.35}
$$

Therefore $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps} : \mathscr{DPS}^+ \rightarrow \mathscr{UDPS}^+$ is bijective, and we conclude that any *unit div point set* (P, Θ_P) in $\mathscr{UDPS}^+$ has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$, similar to how any *div point set* (P, Θ_P) in $\mathscr{DPS}^+$ has an interpretation for P by the Axiom. ■

Lemma 3. A *unit div point set* is in $\mathscr{UDPS}^+$ iff it is isomorphic to some *unit div point set* (P, Θ_P) in $\mathscr{DPS}^{\mathbb{N}}$ where $Col_{udps}(\Theta_P)$, a full vertex monochromatic coloring on H_{udps}, satisfies (2.39) and (2.40), where $\mathscr{DPS}^{\mathbb{N}}$ is the class of all *unit div point sets* (P, Θ_P) satisfying

$$P = \{x \in \mathbb{N}_{\geq 1} : x \leq n\}$$

for some $n \in \mathbb{N}_{\geq 3}$, and Col_{udps} is a function similar to Col from (2.18)

$$Col_{udps}(\Theta_P) = \{((\pi_1(D), \bigcup \pi_2(D)), \psi(\pi_2(D)) : D \in \Theta_P\} \tag{2.36}$$

and H_{udps} is a 3-and-6-uniform hyper graph with 2 sets of hyperedges, E_1 and E_2 defined as a 3-tuple: $H_{udps} = (V_{udps}, E_1, E_2)$ constructed based on P:

$$
\begin{aligned}
V_{udps} &= \{\bigcup V_{of}(x_{dv}) : x_{dv} \in \mathscr{P}(P) : |x_{dv}| = 2\} \\
E_1 &= \{e \in \mathscr{P}(V) : |e| = 6 \wedge \forall v_1, v_2 \in e\ \pi_\cup(v_1) = \pi_\cup(v_2)\} \\
E_2 &= \{e \in \mathscr{P}(V) : |e| = 3 \wedge \forall v_1, v_2 \in e\ \pi_1(v_1) = \pi_2(v_2) \wedge |\bigcup_{v \in e} \pi_2(v)| = 3\}
\end{aligned}
\tag{2.37}
$$

with $V_{of}(x)$ being a function that returns a set of vertices whose first element is x:

$$V_{of}(x) = \{(x, x_{dp}) : x_{dp} \in \mathscr{P}(P \setminus x) : |x_{dp}| = 2\} \tag{2.38}$$

and, finally, we have

$$\begin{aligned}
&\forall e \in E_1 \\
&\quad \exists v_1, v_2 \in e \quad \left| \begin{array}{l} \pi_1(v_1) \cap \pi_1(v_2) = \pi_2(v_1) \cap \pi_2(v_2) = \varnothing \\ C(v_1) = C(v_2) = 0 \\ C^{members}(e \setminus \{v_1, v_2\}) = \{1\} \end{array} \right. \\
&\quad \Leftrightarrow \neg\exists v_1, v_2, v3 \in e \quad \left| \begin{array}{l} |\pi_1(v_1) \cap \pi_1(v_2) \cap \pi_1(v_3)| = 1 \\ C(v_1) = C(v_2) = C(v_3) = 0 \\ C^{members}(e \setminus \{v_1, v_2, v_3\}) = \{1\} \end{array} \right.
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
&\forall e \in E_2 \\
&\quad \forall v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} v_1 \neq v_2 \neq v_3 \\ \Rightarrow (C(v_1) = 1 \Leftrightarrow C(v_2) = C(v_3)) \\ \qquad \wedge (C(v_1) = 0 \Leftrightarrow C(v_2) \neq C(v_3)) \end{array} \right.
\end{aligned} \tag{2.40}$$

Remark. You may have already noticed, the construction of $H_{\mathscr{udps}}$ depends on the points of a *unit div point set* (i.e. P of some (P, Θ_P)), as different from the coloring $Col_{\mathscr{udps}}$, which depends on the *dividons* (i.e. Θ_P of the (P, Θ_P)). This is similar to how H and Col are defined back in our proof for Theorem 1.

However, each vertex of $H_{\mathscr{udps}}$ is an ordered pair, as compared to each vertex of H which is a set with cardinality of 2. Such definition of vertices for $H_{\mathscr{udps}}$ that depends not only on the *divider* of a *unit dividon* but also its *TBD points* is necessary. This is because for any *unit div point set* (P, Ω_P), there exists $\binom{|P|-2}{2}$ distinct *unit dividons* who share a common *divider*, where $\binom{|P|-2}{2} > 1$ when $|P| \geq 5$. We would need not only the *divider* but the *TBD points* to distinguish *unit dividons* from one another.

Proof. [For Lemma 3] Every *unit div point set* is isomorphic to some *unit div point set* in $\mathscr{DPS}^{\mathbb{N}}$. For a *unit div point set* to be in $\mathscr{UDPS}^+$, by Lemma 2 it has to satisfy (2.29), (2.32), (2.33), and (2.34). It is clear that a *unit div point set*, (P, Ω_P), satisfies (2.29) iff $Col(\Omega_P)$ on the $H_{\mathscr{udps}}$ constructed based on P satisfies (2.40): (2.40) is simply a different way of writing (2.29) by first defining D_1, D_2, D_3 as vertices of an edge in E_2 in (2.37). On the other hand (P, Ω_P) would also satisfy (2.32), (2.33), and (2.34) iff $Col(\Omega_P)$ on $H_{\mathscr{udps}}$ constructed based on P satisfies (2.39).

(2.32), (2.33), and (2.34) can be summarized as formulae with universal quantification of 4 points in P, where if these points are distinct, some conditional proposition regarding *unit dividons* in Ω_P must be true. One common property about the conditional proposition in all 3 formulae is that $\pi_{\cup}(D_1) = \pi_{\cup}(D_2) = \pi_{\cup}(D_3) = R$ is always a part of the conjunction that makes up the antecedent. There are a total of 6 *unit dividons* D where $\pi_{\cup}(D) = R$ for any $R \subset P$ where $|R| = 4$, obtainable using $\mathscr{UD}s$, a function which takes in R and returns a set of such *unit dividons*:

$$\begin{aligned} &\mathscr{UD}s(R) = \{ud(x, y) : x, y \in R : x \neq y\} \\ &ud(x, y) = (\{x, y\}, \{R \setminus \{x, y\}\}) \end{aligned} \tag{2.41}$$

Now let's look back at Theorem 1, which states that a unit div point sets of 4 point (recall that div point sets of 4 points are their own unit div point sets) satisfies (2.8), (2.9), and (2.10) iff it is isomorphic to $Conc^1_4$ or $Conv_4$, or more fundamentally, for any $R \subseteq ?P$ where $|R| = 4$, $\mathscr{UD}s(R)$ has to be isomorphic to the set of unit divdions of $Conc^1_4$ or $Conv_4$, which is precisely what is expressed in (2.39). The set of edges E_1 defined in (2.37) for some hypergraph H_{udps} constructed based on a certain unit div point set, $(P, \Omega(P))$, is equivalent to $\{\mathscr{UD}s(R) : R \subseteq P : |R| = 4\}$. ■

Definition 5. We say that *div point set* $\mathcal{X}_1$ is a *sub div point set* of *div point set* $\mathcal{X}_2$ (denoted by $\leq$) iff the set of *unit divdion* of the corresponding *unit div points set* of $\mathcal{X}_1$ is a subset of that of $\mathcal{X}_2$.

$$\begin{aligned} &\forall \mathcal{X}_1, \mathcal{X}_2 \in \mathscr{DPS}^* \\ &\quad (A, \Omega_A) := \vec{\mathscr{F}}^{\mathscr{DPS}}_{udps}(\mathcal{X}_1) \\ &\quad (B, \Omega_B) := \vec{\mathscr{F}}^{\mathscr{DPS}}_{udps}(\mathcal{X}_2) \\ &\quad \mathcal{X}_1 \leq \mathcal{X}_2 \Leftrightarrow \Omega_A \subseteq \Omega_B \end{aligned} \tag{2.42}$$

For clarification, we say that 2 sub *div point sets* of some *div point set*, (S_1, Θ_{S_1}) and (S_2, Θ_{S_2}), are distinct sub *div point sets* if $S_1 \neq S_2$. That is to say, distinctness here is not defined in terms of isomorphism, but equality (i.e. by the axiom of extensionality).

Remark. It is obvious that some *div point set* $\mathscr{A}_{dps}$ is in $\mathscr{DPS}^+$, iff all its *sub div point set* are also in $\mathscr{DPS}^+$. However, we need to keep in mind that not all *unit div point set* (P, P_Ω) is in $\mathscr{UDPS}^+$, even if all subsets of P_Ω of some cardinality are isomorphic to some sets of *unit dividons* of $\vec{\mathscr{F}}^{\mathscr{DPS}}_{udps}(\mathcal{X}_{dps})$ for some $\mathcal{X}_{dps} \in \mathscr{DPS}^+$. It is because such *unit div point set* may not necessarily satisfy (2.29) and be a memeber of $\mathscr{UDPS}^\Theta$.

Definition 6. $\mathcal{S}dps_{of}$ is a function that returns the set of all *sub div point sets* of m points for some div point set where $m \in \mathbb{N}_{\geq 4}$.

$$\mathcal{S}dps_{of}(\mathcal{X}_{dps}, m) = \{\mathcal{S}dps(\mathcal{X}_{dps}, P_s) : P_s \subseteq P : |P_s| = m\} \tag{2.43}$$

where $\mathcal{S}\mathscr{dps}$ is a function that returns the *sub div point set* of a set of points, P_s, of a *div point set* of the set of points, P, where $P_s \subseteq P$:

$$
\begin{aligned}
&\mathcal{S}\mathscr{dps}(\mathcal{X}_{\mathscr{dps}}, P_s) = \underset{\rightarrow \mathscr{dps}}{\mathscr{F}}^{\mathscr{UDPS}}((P_s, \Omega_{P_s}))\\
&\text{where}\\
&\quad (P, \Omega_P) \coloneqq \underset{\rightarrow \mathscr{udps}}{\mathscr{F}}^{\mathscr{DPS}}(\mathcal{X}_{\mathscr{dps}})\\
&\quad \Omega_{P_s} = \{D : D \in \Omega_P : \pi_1(D) \cup \bigcup \pi_2(D) \subseteq P_s\}\\
&\quad \underset{\rightarrow \mathscr{dps}}{\mathscr{F}}^{\mathscr{UDPS}} \text{ is an inverse of } \underset{\rightarrow \mathscr{udps}}{\mathscr{F}}^{\mathscr{DPS}}
\end{aligned}
\tag{2.44}
$$

A *div point set* of n points always has $\binom{n}{m}$ distinct *sub div points sets* of m points, where $m \leq n$ and $m \geq 4$, thus

$$
|\mathcal{S}\mathscr{dps}_{\mathscr{of}}(\mathcal{X}_{\mathscr{dps}}, m)| = \binom{|\pi_1(\mathcal{X}_{\mathscr{dps}})|}{m}
$$

Theorem 2. For all *div point sets* of 5 points in $\mathscr{DPS}^+$, it either has 4, 2 or 0 distinct *sub div points set* of 4 points isomorphic to $Conc_4^1$ (with the remaining *sub div point sets* isomorphic to $Conv_4$).

Proof. Firstly, let's take note no two *sub div point sets* of 4 points of any *div point set* have a *unit dividon* in common, notationally:

$$
\begin{aligned}
&\forall \mathcal{X} \in \mathscr{DPS}^*\\
&\quad \forall \mathscr{A}, \mathscr{B} \in \mathcal{S}\mathscr{dps}_{\mathscr{of}}(\mathcal{X}, 4)\\
&\qquad \pi_2(\mathscr{A}) \cap \pi_2(\mathscr{B}) = \varnothing \Leftrightarrow \pi_1(\mathscr{A}) \neq \pi_1(\mathscr{B})
\end{aligned}
\tag{2.45}
$$

which can be proven by considering the cardinalities of $\pi_2 \underset{\rightarrow \mathscr{udps}}{\mathscr{F}}^{\mathscr{DPS}}(\mathcal{X})$, $\mathcal{S}\mathscr{dps}_{\mathscr{of}}(\mathcal{X}, 4)$ and the number of *unit dividon* $\mathscr{A}$ has for any $\mathcal{X} \in \mathscr{DPS}^*$ and $\mathscr{A} \in \mathscr{DPS}_4^*$ (the set of all div point sets of 4 points):

$$
\begin{aligned}
&\forall \mathcal{X} \in \mathscr{DPS}^*\\
&\quad \pi_2(\underset{\rightarrow \mathscr{udps}}{\mathscr{F}}^{\mathscr{DPS}}(\mathcal{X})) = \binom{|\pi_1(\mathcal{X})|}{2}\binom{|\pi_1(\mathcal{X})| - 2}{2}\\
&\quad |\mathcal{S}\mathscr{dps}_{\mathscr{of}}(\mathcal{X}, 4)| = \binom{|\pi_1(\mathcal{X})|}{4}\\
&\forall \mathscr{A} \in \mathscr{DPS}_4^*\\
&\quad |\pi_2(\mathscr{A})| = 6
\end{aligned}
$$

$\forall a \in \mathbb{N}_{\geq 4}$, the following is always true

$$
\binom{a}{2}\binom{a-2}{2} = 6\binom{a}{4}
$$

we conclude that it is impossible for any two distinct div point sets to share a common *unit dividon* in $\mathcal{Sdps}_{of}(\mathfrak{X}, 4)$ for any div point set $\mathfrak{X}$, or the equation above would not hold for the case when $a = \pi_1(\mathfrak{X})$.

As a consequence of (2.45), for any *unit div point set* of 5 or more points, (P, Θ_P), there always exists $\frac{|\Theta_P|}{6}$ disjoint subsets $\Theta_{of\_4\_points} \subset \Theta_P$ where

$$\begin{aligned} &\forall D_1, D_2 \in \Theta_{of\_4\_points} \\ &\quad \pi_1(D_1) \cup \bigcup \pi_2(D_1) = \pi_1(D_2) \cup \bigcup \pi_2(D_2) \end{aligned} \tag{2.46}$$

and such $\Theta_{of\_4\_points}$ is obtainable using the function $\mathcal{UDs}$ defined in (2.41). Let's refer to these subsets as *four-points unit dividon sets*. By Lemma 3, Theorem 2 can be expressed as follows:

A *unit div point sets* of 5 points, (P, Θ_P), has either 4,2 or 0 distinct *four-points unit dividon sets* $\Theta_{of\_4\_points} \subset \Theta_P$ where all $\Theta_{of\_4\_points}$ is isomorphic to $\pi_2(Conc_4^1)$ (with the remaining *four-points unit dividon sets* isomorphic to $\pi_2(Conc_4^1)$) if $Col_{udps}(\Theta_P)$ satisfies (2.39) and (2.40) for some H_{udps} defined in (2.37) constructed based on P.

One may have immediately noticed that satisfying (2.39) is equivalent to having all of its *four-points unit dividon sets* isomorphic to either $\pi_2(Conc_4^1)$ or $\pi_2(Conv_4)$. In (2.13), we can see that $Conc_4^1$ has an odd number of *unit dividons* D where $\psi(D) = 1$, while $Conv_4$ has an even number for such *unit dividons*.

$$\begin{aligned} &\Theta_{Conc} \coloneqq \pi_2(Conc_4^1) \\ &\Theta_{Conv} \coloneqq \pi_2(Conv_4) \\ &\exists A \subset \Theta_{Conc_4^1} \quad \left| \begin{array}{l} |A| = 3 \\ \forall D \in A \\ \quad \psi(D) = 1 \\ \forall D' \in \Theta_{Conc_4^1} \setminus A \\ \quad \psi(D') = 0 \end{array} \right. \\ &\exists A \subset \Theta_{Conv} \quad \left| \begin{array}{l} |A| = 4 \\ \forall D \in A \\ \quad \psi(D) = 1 \\ \forall D' \in \Theta_{Conv} \setminus A \\ \quad \psi(D') = 0 \end{array} \right. \end{aligned} \tag{2.47}$$

By (2.40), *unit div point sets* of 5 points in $\mathscr{DPS}^+$ always have an even number of *unit dividon* D where $\phi(D) = 0$. For that to be true, there must be an even number of *four-points unit dividon sets* that are isomorphic to $\pi_2(Conc_4^1)$, and therefore it is impossible for (P, Θ_P) to have 5,3 or 1 *four-points unit dividon sets* isomorphic to $\pi_2(Conc_4^1)$.

On the other hand, it is possible for a *unit div point sets* of 5 points, (P, Θ_P) to have either 4,2 or 0 distinct *four-points unit dividon sets* $\Theta_{of\_4\_points} \subset \Theta_P$ where $\Theta_{of\_4\_points} \cong \pi_2(Conc_4^1)$, here are the proofs for all three cases:

I. There exists a *unit div point set*, (P, Θ_P) of 5 points with 0 *four-points unit dividon sets* isomorphic to $\pi_2(Conc_4^1)$ and 5 four-points unit divdion sets isomorphic to $\pi_2(Conv_4)$, such that $Col_{\mathfrak{udps}}((P, \Theta_P))$ satisfies (2.39).

   For any unit divdion sets $A \subset \Theta_P$ there exists 2 *unit dividons* $D \in A$ where $\phi(D) = 0$ (while $\phi(D') = 1$ for the rest of the *dividons* D' in A). Let's denote the set of all these *dividons* as D^*, the 5 distinct *four-points unit dividon sets* as A_1, A_2, A_3, A_4, A_5 and each pair of such unit divdions as D_n^1 and D_n^2 for $n \in \mathbb{N}_{\geq 1}$ where $n \leq 5$ and

$$\{D_n^1, D_n^2\} = A_n \cap D^*$$

   To satisfy (2.39), we simply let any 2 *unit dividons* D_n^x, D_m^x where $x \in \{1, 2\}$ and $n \neq m$ to have a common *divider*, while avoiding any 3 distinct *unit dividon* in D^* to have a common *divider*, but the same time ensuring that

$$\begin{aligned} \pi_1(D_n^1) &= \bigcup \pi_2(D_n^2) \\ \pi_1(D_n^2) &= \bigcup \pi_2(D_n^1) \end{aligned} \tag{2.48}$$

   and, no two distinct *four-points unit dividon sets* have *unit dividon* in common (recall (2.45)). That is to say, for some sets of 2 cardinality $A, B, C, D, E, F \subset P$ of the *unit div point sets* (P, Θ_P), we have

$$\begin{aligned} \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) = A \\ \pi_1(D_1^2) = \bigcup \pi_2(D_1^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) = B \\ \pi_1(D_2^2) = \bigcup \pi_2(D_2^1) = \pi_1(D_4^2) = \bigcup \pi_2(D_4^1) = C \\ \pi_1(D_4^1) = \bigcup \pi_2(D_4^2) = \pi_1(D_5^1) = \bigcup \pi_2(D_5^2) = D \\ \pi_1(D_5^2) = \bigcup \pi_2(D_5^1) = \pi_1(D_6^2) = \bigcup \pi_2(D_6^1) = E \\ \pi_1(D_3^1) = \bigcup \pi_2(D_3^2) = \pi_1(D_6^1) = \bigcup \pi_2(D_6^2) = F \end{aligned}$$

   where

$$\begin{gathered} A \neq B \neq C \neq D \neq E \neq F \\ (A \cap B) = (A \cap C) = (B \cap F) = (C \cap D) = (D \cap E) = (E \cap F) = \varnothing \end{gathered}$$

II. There exists a *unit div point set*, (P, Θ_P) of 5 points with 2 *four-points unit dividon sets* isomorphic to $\pi_2(Conc_4^1)$ and 3 four-points unit divdion sets isomorphic to $\pi_2(Conv_4)$, such that $Col_{udps}((P, \Theta_P))$ satisfies (2.39). Let's use the same notations as I: this time we have

$$\forall n \in \{1,2,3\} \qquad \{D_n^1, D_n^2\} = A_n \cap D^*$$
$$\forall n \in \{4,5\} \qquad \{D_n^1, D_n^2, D_n^3\} = A_n \cap D^*$$

The *divider* of *unit dividons* D_n^1, D_n^2, D_n^3 for $n \in \{4,5\}$ has 1 element in common (recall (2.39)), while (2.48) still applies to D_n^1, D_n^2 for $n \in \{1,2,3\}$. To satisfy (2.39), we can let D_4^x to share the same *divider* as D_5^x for $x \in \{1,2\}$, while letting the remaining unit dividions in D_4 and D_5, namely D_4^3 and D_5^3, to share the same *divider* as D_1^1 and D_2^1 respectively, and the remaining unit dividions in D_1 and D_2, namely D_1^2 and D_2^2 to share the same *dividers* as the two *dividons* in D_3 respectively. That is to say, for some distinct points $a, b, c, d, e \in P$ of the *unit div point sets* (P, Θ_P), we have

$$\begin{aligned}
\pi_1(D_4^1) = \pi_2(D_5^1) &= \{a, b\} \\
\pi_1(D_4^2) = \pi_2(D_5^2) &= \{a, c\} \\
\pi_1(D_4^3) = \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) &= \{a, d\} \\
\pi_1(D_5^3) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) &= \{a, e\} \\
\pi_1(D_1^2) = \bigcup \pi_2(D_2^1) = \pi_1(D_3^1) = \bigcup \pi_2(D_3^2) &\subset P \setminus \{a, d\} \\
\pi_1(D_2^2) = \bigcup \pi_2(D_2^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) &\subset P \setminus \{a, e\}
\end{aligned}$$

III. There exists a *unit div point set*, (P, Θ_P) of 5 points with 4 *four-points unit dividon sets* isomorphic to $\pi_2(Conc_4^1)$ and 1 four-points unit divdion set isomorphic to $\pi_2(Conv_4)$, such that $Col_{udps}((P, \Theta_P))$ satisfies (2.39). Let's use the same notations as II, this time we have

$$\forall n \in \{1\} \qquad \{D_n^1, D_n^2\} = A_n \cap D^*$$
$$\forall n \in \{2,3,4,5\} \qquad \{D_n^1, D_n^2, D_n^3\} = A_n \cap D^*$$

To satisfy (2.39), we can let D_4^x to share the same *divider* as D_5^x,and D_2^x to share the same *divider* as D_3^x, for $x \in \{1,2\}$. And then we let the remaining unit dividions in D_4 and D_5, namely D_4^3 and D_5^3, to share the same *divider* as D_3^3 and D_1^1 respectively, while the remaining unit dividions in D_2, namely D_2^3 to share the same *divider* asD_1^2. That is to say, for some distinct points $a, b, c, d, e \in P$ of the *unit div point sets*

(P, Θ_P), we need to have

$$\pi_1(D_4^1) = \pi_2(D_5^1) = \{a, b\}$$
$$\pi_1(D_4^2) = \pi_2(D_5^2) = \{a, c\}$$
$$\pi_1(D_4^3) = \pi_1(D_3^3) = \{a, d\}$$
$$\pi_1(D_3^1) = \pi_1(D_2^1) = \{e, d\}$$
$$\pi_1(D_3^2) = \pi_1(D_2^2) = \{c, d\}$$
$$\pi_1(D_5^3) = \pi_1(D_1^1) \bigcup \pi_2(D_1^2) = \{a, e\}$$
$$\pi_1(D_2^3) = \pi_1(D_1^2) \bigcup \pi_2(D_1^1) = \{b, d\}$$

■

Remark. A stronger version of Theorem 2 would state that for all $\mathfrak{X}_{dps} \in \mathcal{DPS}_5^+$, X is either isomorphic to $Conv_5$, $Conc_5^1$ or $Conc_5^2$, where

$$\begin{aligned}
Conv_5 =& (Cv_5, \Theta_{Cv_5}) \\
Cv_5 =& \{1,2,3,4,5\} \\
\Theta_{Cv_5} =& \{(\{1,2\},\{(\{3,4\},\varnothing\}), \\
& (\{1,3\},\{\{2\},\{4,5\}\}), \\
& (\{1,4\},\{\{2,3\},\{5\}\}), \\
& (\{1,5\},\{\{2,3,4\},\varnothing\}), \\
& (\{2,3\},\{\{1,4,5\},\varnothing\}), \\
& (\{2,4\},\{\{1,5\},\{3\}\}), \\
& (\{2,5\},\{\{1\},\{3,4\}\}), \\
& (\{3,4\},\{\{1,2,5\},\varnothing\}) \\
& (\{3,5\},\{\{1,2\},\{4\}\}) \\
& (\{3,4\},\{\{1,2,5\},\varnothing\})
\end{aligned}
\quad
\begin{aligned}
Conc_5^1 =& (Cc_5^1, \Theta_{Cc_5^1}) \\
Cc_5^1 =& \{1,2,3,4,5\} \\
\Theta_{Cc_5^1} =& \{(\{1,2\},\{(\{3,4,5\},\varnothing\}), \\
& (\{1,3\},\{\{2,5\},\{4\}\}), \\
& (\{1,4\},\{\{2,3,5\},\varnothing\}), \\
& (\{1,5\},\{\{2\},\{3,4\}\}), \\
& (\{2,3\},\{\{1,4,5\},\varnothing\}), \\
& (\{2,4\},\{\{1\},\{3,5\}\}), \\
& (\{2,5\},\{\{1,4\},\{3\}\}), \\
& (\{3,4\},\{\{1,2,5\},\varnothing\}) \\
& (\{3,5\},\{\{1,2\},\{4\}\}) \\
& (\{4,5\},\{\{1,2,5\},\varnothing\})
\end{aligned}
\quad
\begin{aligned}
Conc_5^2 =& (Cc_5^2, \Theta_{Cc_5^2}) \\
Cc_5^2 =& \{1,2,3,4,5\} \\
\Theta_{Cc_5^2} =& \{(\{1,2\},\{(\{3,4,5\},\varnothing\}), \\
& (\{1,3\},\{\{2,4,5\},\varnothing\}), \\
& (\{1,4\},\{\{2\},\{3,5\}\}), \\
& (\{1,5\},\{\{2,4\},\{3\}\}), \\
& (\{2,3\},\{\{1,4,5\},\varnothing\}), \\
& (\{2,4\},\{\{1\},\{3,5\}\}), \\
& (\{2,5\},\{\{1,4\},\{3\}\}), \\
& (\{3,4\},\{\{1\},\{2,5\}\}) \\
& (\{3,5\},\{\{1,4\},\{2\}\}) \\
& (\{4,5\},\{\{1,3\},\{2\}\})
\end{aligned}$$

To prove this version of Theorem 2 we would need to prove that for any $\mathfrak{X}_{dps} \in \mathcal{DPS}_5^+$, $\mathcal{F}_{\to udps}^{\mathcal{DPS}}(\mathfrak{X}_{dps})$ is one of the *unit div point sets* described case I, II, and III, and furthermore, it is of the one described in case I, II, or III iff $\mathfrak{X}_{dps}$ is isomorphic to $Conv_5$, $Conc_5^1$ or $Conc_5^2$ respectively.

2.2 *convexity*

The notion that some set of points in $\mathbb{E}^2$ consists of n points that can be connected together in a certain manner and form a convex n-gon can be expressed through *convexity*.

Definition 7. A *div point set* (P, Θ_P) has a *convexity* of n if there exists (Q, Θ_Q) such that $(Q, \Theta_Q) \leq (P, \Theta_P)$ and (Q, Θ_Q) is isomorphic to $Conv_n$, defined as follow

$$
\begin{aligned}
&Conv_n \in \mathscr{DPS}^* \\
&Conv_n = (P, \Theta_P) \\
&\text{where} \quad \left| \begin{aligned}
&P = \{x \in \mathbb{N}_{\geq 1} : x \leq n\} \\
&\forall D \in \Theta_P \\
&\quad (divider, divs) := D \\
&\quad \exists a, b \in divider \\
&\quad\quad a \neq b \\
&\quad\quad \exists div_1 \in divs \\
&\quad\quad\quad x \in div_1 \Leftrightarrow x > a \wedge x < b
\end{aligned} \right.
\end{aligned} \tag{2.49}
$$

for $n \geq 3$. Here is an implementation of it as a function in Haskell:

```haskell
import Data.List

combine :: Int -> [a] -> [[a]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                        , ys <- combine (n-1) xs']

convex:: Int -> ([Int],[([Int],[[Int]])])
convex n = (points, dividons)
  where
    points = [1..n]
    dividers = combine 2 points
    dividons = [(divider,[div1,div2])
      | divider@(a:b:_) <- dividers,
      let divs = points \\ divider,
      let div1 = [ x | x <- divs, x > a, x < b ],
      let div2 = divs \\ div1 ]
```

Lemma 4. All sub *div point sets* of n points of $Conv_{n+1}$ are isomorphic to $Conv_n$ for all $n \geq 3$.

Proof. We would arrive at $Conv_n$ by removing all *dividons* in $Conv_{n+1}$ with divider made up of $(n + 1)$ and removing $(n + 1)$ from the *TBD points* in the remaining *dividons*. We would arrive at a *div point set* isomorphic to $Conv_n$ if we replace P in (2.49) with a set P' where $|P| = |P'|$ and P' is also totally ordered under a certain operation, and if necessary, replace $>$ and $<$ in $x \in div_1 \Leftrightarrow x > a \wedge x < b$ with the corresponding strictly comparison relations for elements in P'.

Since for all sub div point sets (Q, Θ_Q) of n points of $Conv_{n+1}$, Q is a subset of P and is totally ordered, these subsets are all isomorphic to $Conv_n$. ■

Remark. In Euclidean geometry, Lemma 4 can be viewed as stating that, for $n \geq 3$, removing any one point from a set of $n+1$ points that are the vertices of a convex polygon would result in n points that too form a convex polygon, which is trivially true.

Now that we have defined *convexity*, we can conclude from Theorem 2 that all *div point sets* of 5 points in $\mathscr{DPS}^+$ has a *convexity* of 4, since it always has a *sub div point set* isomorphic to $Conv_4$. In Euclidean geometry that can be interpreted as stating that any set of 5 points in general position always contains a subset of 4 points that form a convex polygon, which was proven by Klein [1] in just five sentences.

3 *a reduction to a multiset unsatisfiability problem*

In the theory of *div point set*, Erdös-Szekeres conjecture can be expressed as follows:

$$\exists A \in \mathscr{DPS}^+ \quad |\pi_1(A)| = n^2 \ \wedge \exists A_s \leq A \quad A_s \ncong Conv_n \tag{3.1}$$

$$\forall A \in \mathscr{DPS}^+ \quad |\pi_1(A)| > n^2 \Leftrightarrow \exists A_s \leq A \quad A_s \cong Conv_n \tag{3.2}$$

for all $n > 3$.

Definition 8. Let's define $UNSAT_{multiset}$ to be the problem of determining if there does not exist any value-assignment for some set of variables V, such that each mutliset in M satisfies certain constraints, over some domain D (the set of values for which a variable can be assigned to).

The particular instances of $UNSAT_{multiset}$ problem we are interested in are of a set of variables V, where $|V| = \binom{2^{n-2}+1}{4}$ for some $n \in \mathbb{N}_{\geq 5}$ over the domain $\{0, 1\}$, distributed in multisets in set M where $M = A \cup B$ such that

$$|A| = \binom{2^{n-2}+1}{5} \wedge |B| = \binom{2^{n-2}+1}{n} \tag{3.3}$$

$$\forall a \in A \qquad a \in \{[1,1,1,1,0], [1,1,0,0,0], [0,0,0,0,0]\} \tag{3.4}$$

$$\forall b \in B \qquad |b| = \binom{n}{4} \wedge b \neq \underbrace{[0,0,0,...,0,0]}_{\binom{n}{4}\ 0's} \tag{3.5}$$

and for the distribution of variables in multisets we have

I. No multiset in A or B contains 2 same variable

II. For any $v \in V$ there are precisely 5 multisets in A and precisely $\frac{\binom{2^{n-2}+1}{n}\binom{n}{4}}{\binom{2^{n-2}+1}{4}}$ multisets in B that contain it.

III. Any 5 multiset in A and $\frac{\binom{2^{n-2}+1}{n}\binom{n}{4}}{\binom{2^{n-2}+1}{4}}$ multisets in B have at most 1 variable in common

We would be referring to such case of $UNSAT_{multiset}$ as $UNSAT^{+}_{multiset}$.

Here is the simplest instance of $UNSAT^{+}_{multiset}$ (when $n = 5$): we have the variables as v_n where $1 \leq k \leq u$ and $u = \binom{2^{5-2}+1}{4} = 126$, $A = B = M$, which is the set below:

$$
\begin{gathered}
\{[v_1, v_2, v_7, v_{22}, v_{57}], [v_1, v_3, v_8, v_{23}, v_{58}], [v_1, v_4, v_9, v_{24}, v_{59}], [v_1, v_5, v_{10}, v_{25}, v_{60}], [v_1, v_6, v_{11}, v_{26}, v_{61}], [v_2, v_3, v_{12}, v_{27}, v_{62}], \\
[v_2, v_4, v_{13}, v_{28}, v_{63}], [v_2, v_5, v_{14}, v_{29}, v_{64}], [v_2, v_6, v_{15}, v_{30}, v_{65}], [v_3, v_4, v_{16}, v_{31}, v_{66}], [v_3, v_5, v_{17}, v_{32}, v_{67}], [v_3, v_6, v_{18}, v_{33}, v_{68}], \\
[v_4, v_5, v_{19}, v_{34}, v_{69}], [v_4, v_6, v_{20}, v_{35}, v_{70}], [v_5, v_6, v_{21}, v_{36}, v_{71}], [v_7, v_8, v_{12}, v_{37}, v_{72}], [v_7, v_9, v_{13}, v_{38}, v_{73}], [v_7, v_{10}, v_{14}, v_{39}, v_{74}], \\
[v_7, v_{11}, v_{15}, v_{40}, v_{75}], [v_8, v_9, v_{16}, v_{41}, v_{76}], [v_8, v_{10}, v_{17}, v_{42}, v_{77}], [v_8, v_{11}, v_{18}, v_{43}, v_{78}], [v_9, v_{10}, v_{19}, v_{44}, v_{79}], [v_9, v_{11}, v_{20}, v_{45}, v_{80}], \\
[v_{10}, v_{11}, v_{21}, v_{46}, v_{81}], [v_{12}, v_{13}, v_{16}, v_{47}, v_{82}], [v_{12}, v_{14}, v_{17}, v_{48}, v_{83}], [v_{12}, v_{15}, v_{18}, v_{49}, v_{84}], [v_{13}, v_{14}, v_{19}, v_{50}, v_{85}], [v_{13}, v_{15}, v_{20}, v_{51}, v_{86}], \\
[v_{14}, v_{15}, v_{21}, v_{52}, v_{87}], [v_{16}, v_{17}, v_{19}, v_{53}, v_{88}], [v_{16}, v_{18}, v_{20}, v_{54}, v_{89}], [v_{17}, v_{18}, v_{21}, v_{55}, v_{90}], [v_{19}, v_{20}, v_{21}, v_{56}, v_{91}], [v_{22}, v_{23}, v_{27}, v_{37}, v_{92}], \\
[v_{22}, v_{24}, v_{28}, v_{38}, v_{93}], [v_{22}, v_{25}, v_{29}, v_{39}, v_{94}], [v_{22}, v_{26}, v_{30}, v_{40}, v_{95}], [v_{23}, v_{24}, v_{31}, v_{41}, v_{96}], [v_{23}, v_{25}, v_{32}, v_{42}, v_{97}], [v_{23}, v_{26}, v_{33}, v_{43}, v_{98}], \\
[v_{24}, v_{25}, v_{34}, v_{44}, v_{99}], [v_{24}, v_{26}, v_{35}, v_{45}, v_{100}], [v_{25}, v_{26}, v_{36}, v_{46}, v_{101}], [v_{27}, v_{28}, v_{31}, v_{47}, v_{102}], [v_{27}, v_{29}, v_{32}, v_{48}, v_{103}], [v_{27}, v_{30}, v_{33}, v_{49}, v_{104}], \\
[v_{28}, v_{29}, v_{34}, v_{50}, v_{105}], [v_{28}, v_{30}, v_{35}, v_{51}, v_{106}], [v_{29}, v_{30}, v_{36}, v_{52}, v_{107}], [v_{31}, v_{32}, v_{34}, v_{53}, v_{108}], [v_{31}, v_{33}, v_{35}, v_{54}, v_{109}], [v_{32}, v_{33}, v_{36}, v_{55}, v_{110}], \\
[v_{34}, v_{35}, v_{36}, v_{56}, v_{111}], [v_{37}, v_{38}, v_{41}, v_{47}, v_{112}], [v_{37}, v_{39}, v_{42}, v_{48}, v_{113}], [v_{37}, v_{40}, v_{43}, v_{49}, v_{114}], [v_{38}, v_{39}, v_{44}, v_{50}, v_{115}], [v_{38}, v_{40}, v_{45}, v_{51}, v_{116}], \\
[v_{39}, v_{40}, v_{46}, v_{52}, v_{117}], [v_{41}, v_{42}, v_{44}, v_{53}, v_{118}], [v_{41}, v_{43}, v_{45}, v_{54}, v_{119}], [v_{42}, v_{43}, v_{46}, v_{55}, v_{120}], [v_{44}, v_{45}, v_{46}, v_{56}, v_{121}], [v_{47}, v_{48}, v_{50}, v_{53}, v_{122}], \\
[v_{47}, v_{49}, v_{51}, v_{54}, v_{123}], [v_{48}, v_{49}, v_{52}, v_{55}, v_{124}], [v_{50}, v_{51}, v_{52}, v_{56}, v_{125}], [v_{53}, v_{54}, v_{55}, v_{56}, v_{126}], [v_{57}, v_{58}, v_{62}, v_{72}, v_{92}], [v_{57}, v_{59}, v_{63}, v_{73}, v_{93}], \\
[v_{57}, v_{60}, v_{64}, v_{74}, v_{94}], [v_{57}, v_{61}, v_{65}, v_{75}, v_{95}], [v_{58}, v_{59}, v_{66}, v_{76}, v_{96}], [v_{58}, v_{60}, v_{67}, v_{77}, v_{97}], [v_{58}, v_{61}, v_{68}, v_{78}, v_{98}], [v_{59}, v_{60}, v_{69}, v_{79}, v_{99}], \\
[v_{59}, v_{61}, v_{70}, v_{80}, v_{100}], [v_{60}, v_{61}, v_{71}, v_{81}, v_{101}], [v_{62}, v_{63}, v_{66}, v_{82}, v_{102}], [v_{62}, v_{64}, v_{67}, v_{83}, v_{103}], [v_{62}, v_{65}, v_{68}, v_{84}, v_{104}], [v_{63}, v_{64}, v_{69}, v_{85}, v_{105}], \\
[v_{63}, v_{65}, v_{70}, v_{86}, v_{106}], [v_{64}, v_{65}, v_{71}, v_{87}, v_{107}], [v_{66}, v_{67}, v_{69}, v_{88}, v_{108}], [v_{66}, v_{68}, v_{70}, v_{89}, v_{109}], [v_{67}, v_{68}, v_{71}, v_{90}, v_{110}], [v_{69}, v_{70}, v_{71}, v_{91}, v_{111}], \\
[v_{72}, v_{73}, v_{76}, v_{82}, v_{112}], [v_{72}, v_{74}, v_{77}, v_{83}, v_{113}], [v_{72}, v_{75}, v_{78}, v_{84}, v_{114}], [v_{73}, v_{74}, v_{79}, v_{85}, v_{115}], [v_{73}, v_{75}, v_{80}, v_{86}, v_{116}], [v_{74}, v_{75}, v_{81}, v_{87}, v_{117}], \\
[v_{76}, v_{77}, v_{79}, v_{88}, v_{118}], [v_{76}, v_{78}, v_{80}, v_{89}, v_{119}], [v_{77}, v_{78}, v_{81}, v_{90}, v_{120}], [v_{79}, v_{80}, v_{81}, v_{91}, v_{121}], [v_{82}, v_{83}, v_{85}, v_{88}, v_{122}], [v_{82}, v_{84}, v_{86}, v_{89}, v_{123}], \\
[v_{83}, v_{84}, v_{87}, v_{90}, v_{124}], [v_{85}, v_{86}, v_{87}, v_{91}, v_{125}], [v_{88}, v_{89}, v_{90}, v_{91}, v_{126}], [v_{92}, v_{93}, v_{96}, v_{102}, v_{112}], [v_{92}, v_{94}, v_{97}, v_{103}, v_{113}], [v_{92}, v_{95}, v_{98}, v_{104}, v_{114}], \\
[v_{93}, v_{94}, v_{99}, v_{105}, v_{115}], [v_{93}, v_{95}, v_{100}, v_{106}, v_{116}], [v_{94}, v_{95}, v_{101}, v_{107}, v_{117}], [v_{96}, v_{97}, v_{99}, v_{108}, v_{118}], [v_{96}, v_{98}, v_{100}, v_{109}, v_{119}], \\
[v_{97}, v_{98}, v_{101}, v_{110}, v_{120}], [v_{99}, v_{100}, v_{101}, v_{111}, v_{121}], [v_{102}, v_{103}, v_{105}, v_{108}, v_{122}], [v_{102}, v_{104}, v_{106}, v_{109}, v_{123}], [v_{103}, v_{104}, v_{107}, v_{110}, v_{124}], \\
[v_{105}, v_{106}, v_{107}, v_{111}, v_{125}], [v_{108}, v_{109}, v_{110}, v_{111}, v_{126}], [v_{112}, v_{113}, v_{115}, v_{118}, v_{122}], [v_{112}, v_{114}, v_{116}, v_{119}, v_{123}], [v_{113}, v_{114}, v_{117}, v_{120}, v_{124}], \\
[v_{115}, v_{116}, v_{117}, v_{121}, v_{125}], [v_{118}, v_{119}, v_{120}, v_{121}, v_{126}], [v_{122}, v_{123}, v_{124}, v_{125}, v_{126}]\}
\end{gathered}
\tag{3.6}
$$

We can generate the set A and B for any $n \in \mathbb{N}_{\geq 5}$ in Haskell as follow (note: $A \cap B = \varnothing$ for all $n > 5$):

```haskell
import Data.List
import Data.Maybe
type Multiset = [Integer]

merge (a:x) (b:y) = (a,b) : merge x y
merge [] _ = []

choose :: Integer -> Integer -> Integer
n `choose` k
   | k < 0    = 0
   | k > n    = 0
```

```
  | otherwise = factorial n `div` (factorial k * factorial (n-k))

factorial :: Integer -> Integer
factorial n = foldl (*) 1 [1..n]

combine :: Integer -> [Integer] -> [[Integer]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                      , ys <- combine (n-1) xs']

number_of_points = (\n->(2^(n-2)+1))

n_setOf_m_Multisets:: Integer -> Integer -> [Multiset]
n_setOf_m_Multisets m n = [ map fromJust $ map ((flip lookup) encoding)
    (combine 4 m_points) | m_points <- combine n [1..m] ]
    where
        encoding = merge (combine 4 [1..m]) [1..(m `choose` 4)]

setA :: Integer -> [Multiset]
setA n = n_setOf_m_Multisets (number_of_points n) 5

setB :: Integer -> [Multiset]
setB n = [ x | x <- n_setOf_m_Multisets (number_of_points n) n, 2 `elem` x ]
```

Theorem 3. For $k \geq 5$, proving the unsatisfiability of an instance of $UNSAT^{+}_{multiset}$ of $n = k$ would prove that for the case when $x = k$ the upper bound of $g(x)$ is $2^{n-2} + 1$ in the Erdös-Szekeres problem, and thus proving the unsatisfiability of all instances of $UNSAT^{+}_{multiset}$ would prove the Erdös-Szekeres conjecture.

Proof. Let's define a function $Assign : Sub(X, 4) \to \{0, 1\}$ where

$$Assign(V) = \begin{cases} 1 & \text{if} \quad V \cong Conc^1_4 \\ 0 & \text{if} \quad V \cong Conv_4 \end{cases} \tag{3.7}$$

For any $X \in \mathscr{DPS}^+$, by Theorem 1, we can see that $Assign(V)$ is defined on all $V \in Sub(X, 4)$. By Theorem 2, X is in $\mathscr{DPS}^*$ would imply that all $T \in Sub(X, 5)$ has either 4, 2, or 0 distinct $V_T \in Sub(T, 4)$ isomorphic to $Conc^1_4$, which is to say

$$[Assign(V) : V_T \in Sub(T, 4) : V_T] \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0], [0, 0, 0, 0, 0]\}$$

Suppose (3.2) is false, we obtain

$$\forall A \in \mathscr{DPS}^+ \quad |\pi_1(A)| > n^2 \Leftrightarrow \nexists A_s \leq A \quad A_s \cong Conv_n$$

which is to say that, for the case when n is some constant $v \in mathbbN_{\geq 3}$, there exists some *div point set* of v^2+1 points, X, in $\mathscr{DPS}^+$ such that there exists some $V \in Sub(X, n)$,

$$[Assign(V) : V \in Sub(V, 4)] \neq \underbrace{[0, 0, 0, ..., 0, 0]}_{\binom{|V|}{4}\ 0's} \tag{3.8}$$

Note in Theorem 3, each variable $v \in V$ is simply some $Assign(V)$ for $V \in Sub(X, 4)$ and the sets A and B are

$$\{[Assign(V) : V_T \in Sub(T, 4)] : V \in Sub(X, 5)\}$$

and

$$\{[Assign(V) : V_T \in Sub(T, 4)] : V \in Sub(X, n)\}$$

respectively, where X is a *div point set* of c points in $\mathscr{DPS}^*$ and $c = 2^{n-2} + 1$.

Thus proving the unsatisfiability of an instance of $UNSAT^+_{multiset}$ for some n would show that there exists no boolean assignment for all $V \in Sub(X, 4)$ to satisfy the constraints, for all $X \in \mathscr{DPS}^*$. This disproves (3.8) and therefore proving (3.2), which is equivalent to stating that the upper bound of $g(n)$ is $2^{n-2} + 1$ in the Erdös-Szekeres problem for some n.

■

Remark. You may have already noticed, $UNSAT^+_{multiset}$ can be reduced into the Boolean Satisfiability Problem (SAT) by first converting each multiset in A into the formula:

$$\bigvee_{v_x \in V} (\neg v_x \wedge \bigwedge_{v_z in V \setminus \{v_x\}} v_z) \vee \bigvee_{V_3 \subset V : |V_3| = 3} (\bigwedge_{v_x \in V3} \neg v_x \wedge \bigwedge_{v_z \in V \setminus V_3} v_z) \vee (\bigwedge_{v_x \in V} v_x)$$

where V is a set of meta-variables in each A, and converting each mutliset in B into the formula

$$\bigvee_{u_x \in U} u_x$$

where U is a set of meta-variables in each B, then joining all the formulae from multisets in both A and B conjunctively.

One may immediately realize that the conjunction of $\bigvee_{u_x \in U} u_x$ and $\bigwedge_{v_x \in V} v_x$ gives a tautology in the case when $V = U$, and thus for the instance of $UNSAT^+_{multiset}$ when $n = 5$ the propositional formula of the SAT instance would be in a simpler form. This can also be observed in the constraints on this instance of $UNSAT^+_{multiset}$ where in order to satisfy (3.5), we have

$$\forall a \in A \qquad a \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0]\}$$

We thereby conclude that a practical approach to proving the Erdös-Szekeres problem is by first proving the unsatisfiability for the instance of $UNSAT^{+}_{multiset}$ when $n = 5$ - apparently accomplishable with a modern SAT solver in a high performance computer system- and then proving that the unsatisfiability of an instance of $UNSAT^{+}_{multiset}$ for all $n \in \mathbb{N}_{\geq 5}$ implies the unsatisfiability of that for $n + 1$.

Remark. One thing we may want to take note, however, is that the Erdös-Szekeres conjecture would not be disproven even if one of the instance of $UNSAT^{+}_{multiest}$ turns out to be satisfiable, since merely satisfying the constraints implies that there exists a *div point set* of $2^{n-2} + 1$ points for some $n \in \mathbb{Z}_{\geq 5}$ where none of its sub *div point sets* of n points is isomorphic to $Conv_n$ and each of its sub *div point sets* of 5 points has 4, 2 or 0 distinct *sub div points set* of 4 points isomorphic to $Conc^1_4$, which does not make it a member of $\mathscr{DPS}^*$ unless it too satisfies the stronger version of Theorem 2 (i.e. it is possible that one of its *sub div point set* of 5 points is not in $\mathscr{DPS}^*$). Furthermore, even if we can show that a set that (3.2) is false, we would still need to somehow demonstrate that there exists no other rules besides (2.8), (2.9), and (2.10) that a *div* point, (P, Θ_P), has to satisfy in order to have an interpretation for P as some set of points in $\mathbb{E}^2$ such that $D \in \Theta_P$ each describes some relative positions of the points (i.e. Axiom 1's consistency with Euclidean geometry).

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