On reducing the Erdös-Szekeres problem into a constraint unsatisfiability problem regarding certain multisets

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Abstract

We introduce the theory of *div point set*, which aims to provide a framework to study the combinatoric nature of any set of points in general position on an Euclidean plane. We then show that the Erdös-Szekeres conjecture can be proved through proving the unsatisfiability of some first-order logic formulae concerning some sets of 5-cardinal multisets over boolean variables under certain constraints.

1 Introduction

More than half a century ago Erdös and Szekeres [1] proved that for all $n \geq 3$, there exists an integer N such that among any N points in general position on an Euclidean plane, there always exists n points forming a convex polygon, and conjectured that the smallest number for N is determined by the function $g(n) = 2^{n-2} + 1$. This was known as the Erdös-Szekeres conjecture (and the problem of determining such N was often referred to the *Happy End Problem*, as it led to the marriage of Szekeres and Klein, who first proposed the question). 25 years after the initial paper, Erdös and Szekeres [2] showed that $g(x)$ cannot be less than 2^{n-2}. Currently the best known bounds for $g(x)$ are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1$$

Many improvements for the upper bound have been made throughout the decades. The current upper bound was obtained by Tóth and Valtr [3] in 1998 as an improvement to the previous upper bound by Kleitman and Pachter [4] in the same year.

There are also attempts to verify individual instances of n. In 2002 Szekeres and Peters [5] showed using an exhaustive computer search that the conjecture holds for $n = 6$. Even to this day it remains the best known result. Rather than describing a computer Proof for $n \geq 7$ or improving the upper bound, our aim in this article is to demonstrate that solving some instances of a certain multiset unsatisfiability problem would prove the Erdös and Szekeres conjecture, through the theory of *div point set*.

1.1 preliminary

Throughout the article we would assume Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). The term "class" would be used to denote a collection of sets satisfying some predicate ϕ. A general form of Kuratowski definition would be used to define tuples. 2-tuples would be referred to as ordered pairs. A set of n-tuples would be referred to as relation (and described as binary relation when it is a set of ordered pairs). It would not matter as how natural numbers are defined as long as they satisfy Peano axioms. $\mathbb{N}_{\geq c}$ would be used to refer to the set of natural numbers greater or equal to some $c \in \mathbb{N}$ (e.g. $0 \notin \mathbb{N}_{\geq 1}$). For any 2 natural numbers a, b, $\binom{a}{b}$ denotes the binomial coefficient *a choose b*. Everything would be formulated under first order logic ($\wedge$,$\vee$,$\neg$,$\Rightarrow$ and $\Leftrightarrow$ would mean *and*, *or*, *not*, *imply* and *iff* respectively). We write $A := B$ if A is defined to be equivalent to B. $\forall x_1 \in A \forall x_2 \in A \forall x_3 \in A ... \forall x_n \in A$ would be abbreviated as

$$\forall x_1, x_2, x_3 ... x_n \in A$$

and $\exists x_1 \in A \exists x_2 \in A \exists x_3 \in A ... \exists x_n \in A$ as

$$\exists x_1, x_2, x_3 ... x_n \in A$$

For any set V, $|V|$ would be used to denote its cardinality, and $\mathscr{P}(V)$ be used to denote its power set:

$$\mathscr{P}(V) = \{v : v \subseteq V\}$$

We say a set V is totally ordered over certain binary relation $\geq$ iff for all a, b and c in V,

$$\begin{gathered}(a \geq b \wedge b \geq a) \Leftrightarrow (a = b)\\(a \geq b \wedge c \geq b) \Leftrightarrow (a \geq c)\\(a \geq b) \vee (b \geq a)\end{gathered}$$

The subscript of a set union or set intersection may be omitted to indicate that union or intersection is applied to each element in the set:

$$\begin{aligned}&\text{For any set, } A,\\&\bigcup A = \bigcup_{a \in A} a = a_1 \cup a_2 \cup .. a_n\\&\bigcap A = \bigcap_{a \in A} a = a_1 \cap a_2 \cap .. a_n\\&\text{where } |A| = n \text{ and } a_1, a_2, .. a_n \text{ are all } n \text{ distinct elements in } A\end{aligned}$$

For any k-tuple T, $\pi_i(T)$ would be used to denote the i-th element of T where $i, k \in \mathbb{N}$ and $i \leq k$; $\pi_\cup(T)$ would be used the denote the union of 1st, 2nd ... k-th elements of a k-tuple;

and $\pi_\cap(T)$ would be used to denote intersection in such fashion:

$$\text{For any } k\text{-tuple, } T, \quad \begin{aligned} \pi_\cup(T) &= \bigcup_{i=1}^{k} \pi_i(T) \\ \pi_\cap(T) &= \bigcap_{i=1}^{k} \pi_i(T) \end{aligned}$$

A single-argument function is any binary relation, f, satisfying

$$\begin{aligned} &\forall x \in X \\ &\qquad \exists r \in f = \pi_1(r) \\ &\forall r \in f \\ &\qquad \pi_1(r) \in X \\ &\qquad \pi_2(r) \in Y \\ &\forall r_1, r_2 \in f \\ &\qquad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2) \end{aligned}$$

for some none-empty sets X (often referred to as domain) and Y (referred to as co-domain). We often express the relationship between f, X, and Y as:

$$f : X \longrightarrow Y$$

We write $f(x) = y$ iff there exists an ordered pair (x, y) in f. A function is always assumed to be single-argument, unless otherwise stated. A function f is injective iff

$$\begin{aligned} &\forall r_1, r_2 \in f \\ &\qquad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2) \end{aligned}$$

It is subjective iff

$$\begin{aligned} &\forall y \in Y \\ &\qquad \exists r \in f \quad y = \pi_2(r) \end{aligned}$$

It is bijective iff it is both injective and surjective, in which case $\xrightarrow{1:1}$ would be used to denote such property. To avoid ambiguity, for any function $f : X \longrightarrow Y$, we would use $f^{members}$ to denote a new function, from $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ such that

$$f^{members}(x) := \bigcup_{a \in x} \{f(a)\}$$

Here is a generalization of it, $f^{members^n}$, defined recursively:

$$f^{members^n}(x) := \bigcup_{a \in x} \{f^{members^{n-1}}(a)\} \text{ where } n \in \mathbb{N}_{\geq 2}$$
$$f^{members^1}(x) := f^{members}(x)$$

A multiset is a generalization of set, where the same element can occur multiple times, making a difference. Two multisets are equal iff (1) both multisets contain the same distinct elements and (2) for each distinct element, it occurs the same number of times in both multisets. A multiset is defined as an ordered pair (A, m_{m}) where $m_{\mathrm{m}} : A \longrightarrow \mathbb{N}_{\geq 1}$ is a function that describes the number of occurrences of some element in the multiset, and A is a set of all distinct elements in the multiset. The cardinality of a multiset (A, m_{m}) is defined as the sum of all $m_{\mathrm{m}}(x)$ for $x \in A$. Multisets are expressed using square brackets, $[\,]$, as compared to sets which use curly brackets, $\{\}$. Here is an example:

$$[f(x) : x \in \mathbb{N}_{\geq 1} : x \leq 3] = [1, 1, 1] = (\{1\}, \{(1, 3)\})$$
$$\text{where } f(x) = 1$$

A hypergraph is a generalization of graph, where an edge can contain any number of vertices. It is defined as an ordered pair (V, E) where E is a subset of $\mathcal{P}(V) \setminus \varnothing$. Elements in V are referred to as vertices while elements in E are referred to as edges or hyperedges. A hypergraph is k-uniformed when

$$\forall e \in E \quad |e| = k$$

where $k \in \mathbb{N}_{\geq 1}$. A full vertex coloring on some graph or hypergraph, (V, E), is defined as a function, $C : V \longrightarrow cDom$, such that

$$|C| = |V|$$
$$\forall c \in C \quad \pi_1(c) \in V \wedge \pi_2(c) \in cDom$$
$$\forall c_1, c_2 \in C \quad c_1 = c_2 \Leftrightarrow \pi_1(c_1) = \pi_1(c_2)$$

where $cDom \subset \mathbb{N}$, and it is often referred to as the set of colors. When $|Dom| = 2$, we say the coloring is monochromatic. We would use $FullCol(G, cDom)$ to denote the set of all possible full vertex colorings on a graph G of the set of colors $cDom$. For any graph G of n vertices, and any non-empty $cDom$,

$$|FullCol(G, cDom)| = n^{|cDom|}$$

2 *Div point set* as a representation for any set of points in general position

We start off by introducing an object which we would be referring to as *div point set*.

Definition 1. A *div point set* is any order-pair (P, Θ_P) satisfying

$$|\Theta_P| = \binom{|P|}{2} \land P \neq \varnothing \tag{2.1}$$

$$\forall D_n \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ \bigcup \delta_n = P \setminus d_n \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.2}$$

$$\forall D_n, D_m \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ (d_m, \delta_m) \coloneqq D_m \\ d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.3}$$

We would be using $\mathscr{DPS}^*$ to denote the class of all ordered pairs satisfying (2.1), (2.2) and (2.3). Thus $\mathscr{X}$ is a *div point set* iff $\mathscr{X} \in \mathscr{DPS}^*$.

For any n points in general position, where $n \geq 2$, we can always select any 2 arbitrary points and draw a line across them, dividing the rest of the points into 2 disjoint sets. So long as the points are in general position, we can be sure that no 3 points forms a line, and thus each of the remaining $n - 2$ points would always be in one of these sets. Let's refer to these 2 disjoint sets as *divs* produced by a *divider* made up of 2 distinct points, and the points in the *divs* as *TBD points* of the *divider* (*TBD* is short for *to-be-distributed-among-divs*). The process of selecting 2 distinct points from a set of point P, creating a *divider*, and producing 2 *divs* can be repeated $\binom{|P|}{2}$ times until all sets of 2 points in P are selected.

Any set of points P in general position on an Euclidean plane where $|P| \geq 2$ can be represented by some *div point set* (P, Θ_P). Each member of $D_n \in \Theta_P$ would be referred to as a *dividon*, to be interpreted as follows:

$$(d_n, \delta_n) \coloneqq D_n \quad \left| \begin{array}{l} \{a, b\} \coloneqq d_n \\ \quad a \text{ and } b \text{ represent the 2 points which make up the } \textit{divider} \\ \{div_1, div_2\} \coloneqq \delta_n \\ \quad div_1 \text{ and } div_2 \text{ represent the 2 } \textit{divs} \text{ produced by the } \textit{divider} \\ \quad \bigcup \delta_n \text{ thus represents the set of } \textit{TBD points} \text{ of the } \textit{divider} \end{array} \right. \tag{2.4}$$

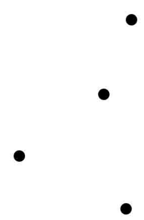

Figure I

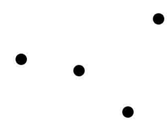

Figure II

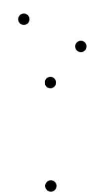

Figure III

The sets of points in *Figures I, II* and *III* can be represented by any *div point set* (A, Θ_A) as long as A is a set of 4 arbitrary elements a, b, c, d and

$$\begin{aligned}\Theta_A = \{ & (\{a,b\}, \{(\{c\},\{d\}\}), \\ & (\{a,c\}, \{(\{b\},\{d\}\}), \\ & (\{a,d\}, \{(\{b\},\{c\}\}), \\ & (\{b,c\}, \{(\{a,d\},\varnothing\}), \\ & (\{b,d\}, \{(\{a,c\},\varnothing\}), \\ & (\{c,d\}, \{(\{a,b\},\varnothing\})\}\end{aligned}$$

To make sense of the *div point set* representation, we label the third point from the bottom in *Figure I* and the second point from the bottom in *Figures II* and *III* as a (note that each of these is the point surrounded by the remaining 3 points in the figure). For the rest of the points in each figure we simply label them arbitrarily as b, c, and d.

Only a handful of *div point sets* can be used to represent points in general position in $\mathbb{E}^2$. For majority of $\mathscr{X} \in \mathscr{DPS}^*$, let $(P, \Theta_P) \coloneqq \mathscr{X}$, there exists no meaningful interpretation for P as some sets of points in $\mathbb{E}^2$ such that each $D \in \Theta_P$ is a *dividon* that describes how *TBD points* are distributed between the 2 *divs* produced by each *divider*. A classical example would be (Q, Θ_Q) where Q is a set of 4 arbitrary elements a, b, c, d and

$$\begin{aligned}\Theta_Q = \{ & (\{a,b\}, \{(\{c,d\},\varnothing\}), \\ & (\{a,c\}, \{(\{b,d\},\varnothing\}), \\ & (\{a,d\}, \{(\{b,c\},\varnothing\}), \\ & (\{b,c\}, \{(\{a,d\},\varnothing\}), \\ & (\{b,d\}, \{(\{a,c\},\varnothing\}), \\ & (\{c,d\}, \{(\{a,b\},\varnothing\})\}\end{aligned}$$

For a *div point set* (P, Θ_P) to have a meaningful interpretation for P as some set of points in $\mathbb{E}^2$, it has to satisfy certain conditions. For any 3 distinct points, x, y, and z in general position in $\mathbb{E}^2$, let $\langle x, y\rangle^z$ denote the *div* containing z produced by the *divider* made up of the point x and y, and $\langle x, y\rangle^{\text{-}z}$ denote the *div* not containing z produced by the *divider*. After some experimentation with points in $\mathbb{E}^2$, we would make the observation that the following formulas always hold true for any distinct points a, b, c, d in $\mathbb{E}^2$. (2.5) is trivially true, while (2.6), (2.7) and (2.8) are demonstrated in *Figures IV*, *V* and *VI* respectively.

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad a \in \langle b, c\rangle^d \Leftrightarrow d \in \langle b, c\rangle^a\\ &\qquad a \in \langle b, c\rangle^{\text{-}d} \Leftrightarrow d \in \langle b, c\rangle^{\text{-}a}\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad c \in \langle a, b\rangle^{\text{-}d}\\ &\qquad \Leftrightarrow ((a \in \langle b, c\rangle^d \wedge a \in \langle b, d\rangle^c)\\ &\qquad\qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{\text{-}c}))\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad c \in \langle a, b\rangle^{d}\\ &\qquad \Leftrightarrow ((a \in \langle b, c\rangle^d \wedge a \in \langle b, d\rangle^{\text{-}c})\\ &\qquad\qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{c}))\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad a \in \langle b, c\rangle^{-d} \wedge a \in \langle b, d\rangle^{-c} \Rightarrow a \in \langle c, d\rangle^b\end{aligned} \tag{2.8}$$

In the context of *div point sets*, (2.5) is always true by (2.2) (recall $\bigcap \delta = \varnothing$), while (2.6), (2.7) and (2.8) can be rewritten as constraints on the *dividons* of a *div point set* as shown in (2.10), (2.11), and (2.12), using a function, ϕ, for determining if two arbitrary points belong to the same *div* in some δ of a *dividon*:

$$\phi(\delta, w) = \begin{cases} 1 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 = div_2 \\ 0 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 \neq div_2 \end{cases} \text{where} \left| \begin{array}{l} \delta = \{div_1, div_2\} \\ w = \{a, b\} \end{array} \right. \tag{2.9}$$

For any *div point set* (P, Θ_P),

$$\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\quad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad |R| = 4 \\
&\quad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\} \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 1 \\
&\qquad\quad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\quad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad |R| = 4 \\
&\quad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\} \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 0 \\
&\qquad\quad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) \neq \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\quad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad |R| = 4 \\
&\quad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcap_{n=1}^{2} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{2} \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3) \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = 0 \\
&\qquad\quad \Rightarrow \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) = 1\ )
\end{aligned} \tag{2.12}$$

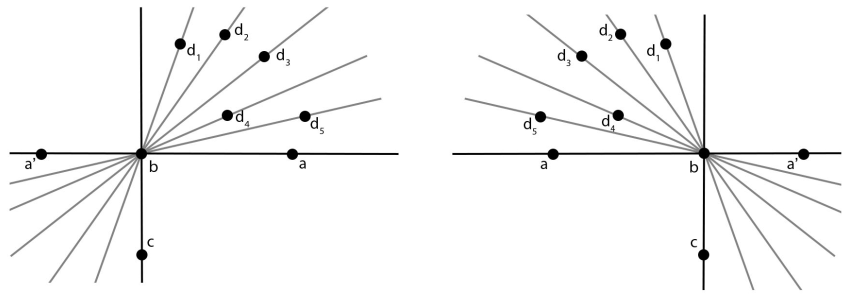

Figure IV

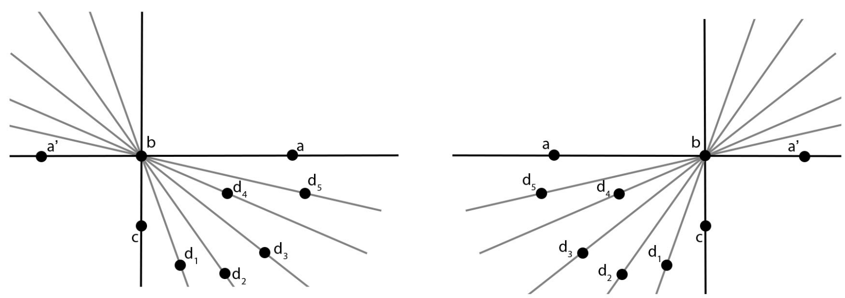

Figure V

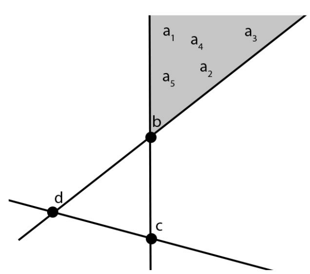

Figure VI

Axiom 1. A *div point sets* (P, Θ_P) has an interpretation for P as some set of points in $\mathbb{E}^2$ such that $D \in \Theta_P$ each describes the relative positions of the points (in terms of how the *TBD points* of each *divider* is distributed between 2 *divs* it produced) iff it is in $\mathscr{DPS}^+$, the class of *div point sets* satisfying (2.10), (2.11), and (2.12).

Remark. For *div point sets* of 3 or less points, it is vacuously true that they satisfy (2.10), (2.11), and (2.12) and thus they are by default in the class $\mathscr{DPS}^+$. This is consistent with Euclidean geometry: any set of 3 points in general position can be represented by any *div point set* of 3 points, and the same goes to any set of 2 points, and any set of 1 point.

Definition 2. We say that two *div point sets* (A, Θ_A) and (B, Θ_B) are isomorphic iff there exists a bijective function $f : A \xrightarrow{1:1} B$ which preseves the structure of the *divions*. Notationally,

$$
\begin{aligned}
(A, \Theta_A) \cong (B, \Theta_B) \Leftrightarrow \exists f : A \xrightarrow{1:1} B & \\
\forall D_A \in \Theta_A & \\
\exists D_B \in \Theta_B & \\
(d_a, \delta_a) := D_A & \\
(d_b, \delta_b) := D_B & \\
f^{members}(d_a) = d_b \Leftrightarrow f^{members^2}(\delta_a) = \delta_b &
\end{aligned} \tag{2.13}
$$

in which case f would be referred to as the isomorphism between the two sets.

Remark. It is trivially true that all div point sets (P, Θ_P) in $\mathscr{DPS}^*$ where $|P| \leq 3$ are isomorphic to any div point sets (Q, Θ_Q) in $\mathscr{DPS}^*$ where $|Q| = |P|$.

Theorem 1. $\neg(\mathfrak{X} \cong Conc_4^1) \Leftrightarrow (\mathfrak{X} \cong Conv_4)$ for all $\mathfrak{X} \in \mathscr{DPS}_4^+$ where $\mathscr{DPS}_4^+$ denotes the *div point sets* of 4 points in $\mathscr{DPS}^+$ and

$$
\begin{aligned}
Conc_4^1 =& (Cc_4^1, \Theta_{Cc_4^1}) & Conv_4 =& (Cv_4, \Theta_{Cv_4}) \\
Cc_4^1 =& \{1,2,3,4\} & Cv_4 =& \{1,2,3,4\} \\
\Theta_{Cc_4^1} =& \{(\{1,2\},\{\{3\},\{4\}\}), & \Theta_{Cv_4} =& \{(\{1,2\},\{\{3,4\},\varnothing\}), \\
& (\{1,3\},\{\{2\},\{4\}\}), & & (\{1,3\},\{\{2\},\{4\}\}), \\
& (\{1,4\},\{\{2\},\{3\}\}), & & (\{1,4\},\{\{2,3\},\varnothing\}), \\
& (\{2,3\},\{\{1,4\},\varnothing\}), & & (\{2,3\},\{\{1,4\},\varnothing\}), \\
& (\{2,4\},\{\{1,3\},\varnothing\}), & & (\{2,4\},\{\{1\},\{3\}), \\
& (\{3,4\},\{\{1,2\},\varnothing\})\} & & (\{3,4\},\{\{1,2\},\varnothing\})\}
\end{aligned} \tag{2.14}
$$

Proof for Theorem 1.

Summary. In Part 1 of the proof we would define a function ψ that returns 0 or 1 based on the *divs* of a *dividon* of some *div point set* in $\mathscr{DPS}_4^+$. In Part 2 we would define $\mathscr{DPS}_4^{\mathbb{N}}$ and a function Col that uses ψ, and show that for every $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, there exists a unique full vertex monochromatic coloring $Col(\pi_2(\mathscr{X}))$ on some hypergraph H, where the vertices of H are the dividers of the *div points sets* in $\mathscr{DPS}_4^{\mathbb{N}}$. In Part 3 we would define the edges of H in such a manner that the coloring $Col(\pi_2(\mathscr{X}))$ on H satisfies some conditions iff $\mathscr{X}$ satisfies (2.10) and (2.11). In Part 4 we demonstrate that for the coloring to satisfy the conditions, there exists only 3 $\mathscr{Scenarios}$, and colorings in $\mathscr{Scenario}$ 2 and 3 are isomorphic to $Col(\pi_2(Conc_4^1))$ and $Col(\pi_2(Conv_4))$, and $Conc_4^1$ and $Conv_4$ satisfy (2.12), but not the other *div point set* the coloring in $\mathscr{Scenario}$ 1 is based on, and thus proving Theorem 1.

Part 1. For any div point set (P, Θ_P) in $\mathscr{DPS}_4^+$, since $|P| = 4$, we can be certain that

$$\begin{aligned}
&\forall D \in \Theta_P \\
&\qquad \pi_2(D) \in \{type_0, type_1\} \\
&\qquad \{a, b\} = P \setminus \pi_1(D) \\
&\qquad type_0 = \{\{a\}, \{b\}\} \\
&\qquad type_1 = \{\{a, b\}, \varnothing\}
\end{aligned} \tag{2.15}$$

Recall that in (2.9), we define a function ϕ that takes in some $\pi_2(D)$ and a set of 2 *TBD points*, and returns 1 if the *TBD points* belong to the same *div* in $\pi_2(D)$, or 0 if they belong to different *divs* in $\pi_2(D)$. For $\mathscr{X} \in \mathscr{DPS}_4^+$, we can define a new function ψ, a simpler version of ϕ that does basically the same thing by exploiting (2.15), namely the fact every $\pi_2(D)$ is either $type_0$ or $type_1$ (since that there are only 2 *TBD points* for each divider):

$$\psi(\delta) = \begin{cases} 1 & \text{if } \exists div \in \delta \quad |div| = 2 \\ 0 & \text{if } \forall div \in \delta \quad |div| = 1 \end{cases} \tag{2.16}$$

For every *dividon* D of any $\mathscr{X} \in \mathscr{DPS}_4^+$, we have

$$\phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \tag{2.17}$$

Part 2. Let's define $\mathscr{DPS}_4^{\mathbb{N}}$ to be the set of all *div point sets* (P, Θ_P) for which $P = \{1, 2, 3, 4\}$. All $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ would have the same *dividers* (Recall the set of *dividers* is just the set of elements in $\mathscr{P}(P)$ whose cardinality is 2.) Now let $H = (V, E)$ be a hypergraph whose vertices are the *dividers* of *div point sets* in $\mathscr{DPS}_4^{\mathbb{N}}$. Using ψ, we can define a bijective function, Col, that transforms the set of *dividons* of a *div point set* in $\mathscr{DPS}_4^{\mathbb{R}}$ into some full vertex monochromatic coloring for H.

$$\begin{gathered}
Col : \{\pi_2(\mathscr{X}) : \mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}\} \longrightarrow FullCol(H, \{0, 1\}) \\
Col(\Omega_P) = \{(\pi_1(D), \psi(\pi_2(D))) : D \in \Omega_P\}
\end{gathered} \tag{2.18}$$

It is bijective because

$$\begin{aligned}&\forall \mathfrak{X}_1, \mathfrak{X}_2 \in \mathscr{DPS}^{\mathbb{R}}\\&\quad Col(\pi_2(\mathfrak{X}_1)) = Col(\pi_2(\mathfrak{X}_2)) \Leftrightarrow \mathfrak{X}_1 = \mathfrak{X}_2\end{aligned} \tag{2.19}$$

due to the fact that ψ is bijective for every dividon of any *div point set* of 4 points.

Part 3. Now let's define any set of three *dividers* containing 1 element in common to be an edge of H (recall that the vertices are the *dividers*), notationally,

$$E = \{e \in \mathscr{P}(V) : |e| = 3 \wedge |\bigcap e| = 1\} \tag{2.20}$$

H is a 3-uniform hypergraph with 4 hyperedges. For $\mathfrak{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ to satisfy (2.10) and (2.11) is equivalent to having $Col(\pi_2(\mathfrak{X})) \in FullCol(H, \{0,1\})$ to satisfy the following:

I. If a vertex, V, is colored 0, the other 2 vertices belonging to the same edge as V must have the same coloring.

II. If a vertex, V, is colored 1, the other 2 vertices belonging to the same edge as V must have different colorings.

This is due to the fact, for any $\mathfrak{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, (2.10) and (2.11) can be rewritten as having the colors on the vertices of each edge to satisfy some formulae, namely the following:

$$\begin{aligned}&\forall e \in E\\&\quad \forall d_1, d_2, d_3 \in e\\&\quad\quad d_1 \neq d_2 \neq d_3\\&\quad\quad \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_1) = 1 \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_2) = Col(\pi_2(\mathfrak{X}))(d_3))\end{aligned} \tag{2.21}$$

$$\begin{aligned}&\forall e \in E\\&\quad \forall d_1, d_2, d_3 \in e\\&\quad\quad d_1 \neq d_2 \neq d_3\\&\quad\quad \Leftrightarrow (Col(\pi_2(\mathfrak{X}))(d_1) = 0 \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_2) \neq Col(\pi_2(\mathfrak{X}))(d_3))\end{aligned} \tag{2.22}$$

(recall that Col is the function to transform some $\pi_2(\mathfrak{X})$ into a coloring, while $Col(\pi_2(\mathfrak{X}))$ is the actual coloring, which is defined as a function in *Preliminary*) This is a result of

$$\begin{aligned}&\forall p_1, p_2, p_3, p_4 \in P\\&\quad R := \bigcup_{n=1}^{4} \{p_n\}\\&\quad \forall D \in \Theta_P\\&\quad\quad \phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D))\end{aligned} \tag{2.23}$$

for any div point set (P, Θ_P) for which $|P| = 4$ (recall (2.15)), and any *dividons* D_1, D_2 and D_3 where

$$|\bigcap_{n=1}^{3} \pi_1(D_n)| = 1 \wedge |\bigcup_{n=1}^{3} \pi_1(D_n)| = 4 \tag{2.24}$$

would always have the *dividers* d_1, d_2 and d_3 respectively, where

$$|\bigcap_{n=1}^{3} d_n| = 1 \wedge d_1 \neq d_2 \neq d_3 \tag{2.25}$$

which are precisely what makes up an edge in E (recall (2.20)). Therefore a *div point set* of 4 points, $\mathcal{X}$, satisfies (2.10) and (2.11) iff $Col(\pi_2(\mathcal{X}))$ satisfies *I* and *II*.

Part 4. To satisfy I and II, 3 vertices belonging to the same edge must either be colored $[0, 0, 0]$ or $[0, 1, 1]$.

Suppose we start off by giving some vertices belonging to the same edge the coloring of $[0, 0, 0]$, by I this would indicate that the rest of the vertices need to have the same colors (recall that each vertex belongs to 2 different edges). We can either end up with H having all vertices colored 0 (let's call it $\mathscr{Scenario}$ 1), or 3 vertices colored 0 and 3 vertices colored 1 (let's call it $\mathscr{Scenario}$ 2).

Now suppose we start off by giving some vertices belonging to the same edge the coloring of $[0, 1, 1]$, by I this would indicate that the remaining 2 vertices of another edge, which the vertex colored 0 belongs to, must have the same colors. If we give them the coloring of $[0, 0]$, we would have an edge with vertices colored $[0, 0, 0]$, and the last uncolored vertex must then be colored 1, so we end up in $\mathscr{Scenario}$ 2 again. If we give them the coloring of $[1, 1]$, we would end up with 1 vertex colored 0 and 4 vertices colored 1, in which case the last uncolored vertex would need to be colored 0, since it belongs to 2 edges both with 2 vertices colored 1. Let's name this $\mathscr{Scenario}$ 3, where 2 vertices are colored 0 and 4 vertices are colored 1.

A pictorial description of the colorings is shown in Figure VII.

Scenario 1 describes a coloring isomorphic to $Col(\pi_2(\mathcal{X}_\varnothing))$ where $\mathcal{X}_\varnothing \in \mathscr{DPS}_4^{\mathbb{N}}$ and

$$\begin{aligned}\pi_2(\mathcal{X}_\varnothing) = \{&(\{1, 2\}, \{(\{3, 4\}, \varnothing\}),\\ &(\{1, 3\}, \{(\{2, 4\}, \varnothing\}),\\ &(\{1, 4\}, \{(\{2, 3\}, \varnothing\}),\\ &(\{2, 3\}, \{(\{1, 4\}, \varnothing\}),\\ &(\{2, 4\}, \{(\{1, 3\}, \varnothing\}),\\ &(\{3, 4\}, \{(\{1, 2\}, \varnothing\})\}\end{aligned}$$

while Scenario 2 describes a coloring isomorphic to $Col(\pi_2(Conc_4^1))$ and scenario 3 descrbies a coloring isomorphic to $Col(\pi_2(Conv_4))$. $Conc_4^1$ and $Conv_4$ both satisfy (2.12), and $\mathcal{X}_\varnothing$

does not. Since any div point set of 4 points is isomorphic to some $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, and only $Conc_4^1$ and $Conv_4$ satisfy all (2.10), (2.11), and (2.12), we conclude that

$$\forall X \in \mathscr{DPS}_4^+ \quad \exists a \in \{Conc_4^1, Conv_4\} \quad X \cong a$$

□

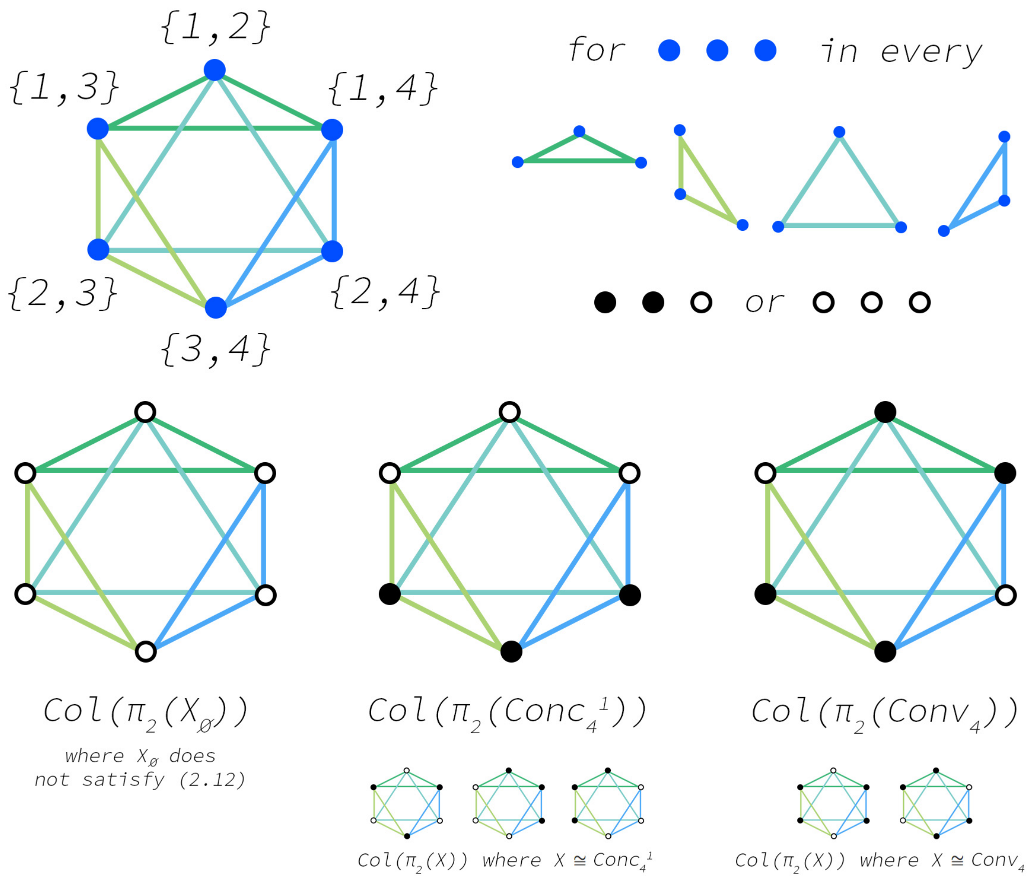

Figure VII

Remark. In Euclidean geometry, Theorem 1 can be interpreted as stating the follows: For any set of 4 distinct points in general positions, it is either the case that it forms a structure where 1 point is inside a triangle formed by connecting the rest of 3 points, or the case that a convex polygon can be created by connecting the 4 points in a certain manner, which can be quite rather easily by a human child with a pen, a piece of paper and a love for Euclidean geometry.

2.1 *unit div point set* and *sub div point set*

For *div point sets* of 5 or more points, the function ψ defined in (2.16) would not be really useful since there would be 3 or more *TBD points* in each *dividon*. That means we cannot apply to same technique above to derive *div point sets* of 5 or more points satisfying (2.10), (2.11) and (2.12). With that in mind, we introduce the object *unit div point set* which makes use of *unit dividons*.

Definition 3. A *unit div point set* is any order-pair (P, Ω_P) satisfying (2.26), (2.27) and (2.28).

$$|\Omega_P| = \binom{|P|}{2}\binom{|P|-2}{2} \land P \neq \varnothing \tag{2.26}$$

$$\forall D_n \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ |\bigcup \delta_n| = 2 \\ \bigcup \delta_n \in \mathscr{P}(P \setminus d_n) \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.27}$$

$$\forall D_n, D_m \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ (d_m, \delta_m) := D_m \\ d_n \cup \bigcup \delta_n = d_m \cup \bigcup \delta_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.28}$$

We would be using $\mathscr{UDPS}^*$ to denote the class of all *unit div point set*.

Remark. Similar to how *div point sets* of 4 points always satisfy (2.15), a *unit div point set* always satisfies (2.29).

$$\begin{array}{l} \forall \mathfrak{X} \in \mathscr{UDPS}^* \\ \quad (P, \Omega_P) := \mathfrak{X} \\ \quad \forall D \in \Omega_P \\ \quad\quad \pi_2(D) \in \{type_0, type_1\} \\ \quad\quad \{a, b\} \subseteq P \setminus \pi_1(D) \\ \quad\quad\quad type_0 = \{\{a\}, \{b\}\} \\ \quad\quad\quad type_1 = \{\{a, b\}, \varnothing\} \end{array} \tag{2.29}$$

For any *unit div point set*, (P, Ω_P), we can use ψ (defined in (2.16)) to map every $\pi_2(D) \in \Omega_P$ to some $k \in \{0, 1\}$.

Remark. One may immediately notice that any *div point sets* of 4 points also satisfy (2.26), (2.27) and (2.28), and any *unit div point set* of 4 points also satisfy (2.1), (2.2) and (2.3), and that is the say

$$\{\mathcal{X}_{udps} \in \mathcal{UDPS}^* : |\pi_1(X)| = 4\} = \{\mathcal{X}_{dps} \in \mathcal{DPS}^* : |\pi_1(X)| = 4\} \tag{2.30}$$

by virtue of the fact that

$$\binom{|4|}{2}\binom{|4-2|}{2} = \binom{|4|}{2} \tag{2.31}$$

and

$$\forall \mathcal{X} \in \mathcal{UDPS}^* \quad \left| \begin{array}{l} (P, \Omega_P) \coloneqq \mathcal{X} \\ |P| = 4 \\ \forall D_n \in \Omega_P \\ \qquad \delta_n = P \setminus d_n \\ \forall D_n, D_m \in \Omega_P \\ \qquad d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.32}$$

As we can see, the difference between a *div point set* and a *unit div point set* lies in that the former relies on a single *dividon* to describe the distribution of $|P| - 2$ *TBD points* between the 2 *divs* produced by a *divider*, while the later relies on $\binom{|P|-2}{2}$ *unit dividons* for that (since each *unit dividon* only describes the distribution of 2 *TBD points*). For every $\mathcal{X}_{dps} \in \mathcal{DPS}^*$ there exists a unique $\mathcal{X}_{udps} \in \mathcal{UDPS}^*$ which $\mathcal{X}_{dps}$ can be transformed into, by breaking down each *dividon* into $\binom{|P|-2}{2}$ *unit dividons*, achievable using the function $\mathcal{bd}$ defined below.

$$\begin{aligned} \mathcal{bd}(D, P) &= \{(\pi_1(D), \mathcal{d}_u(x, \pi_2(D))) : x \in \mathcal{P}(P \setminus \pi_1(D)) : |x| = 2\} \\ \mathcal{d}_u(x, divs) &= \begin{cases} \{x, \varnothing\}, & \text{if } x \subseteq divs \\ \{\{a\}, \{b\}\}, & \text{if } a \in div_1 \wedge b \in div_2 \\ & \text{where } x = \{a, b\} \text{ and } divs = \{div_1, div_2\} \end{cases} \end{aligned} \tag{2.33}$$

Definition 4. The function $\mathcal{F}^{\mathcal{DPS}}_{\rightarrow udps}$ transforms a *div point set* into a *unit div point set*.

$$\mathcal{F}^{\mathcal{DPS}}_{\rightarrow udps}(\mathcal{X}_{dps}) = (\pi_1(\mathcal{X}_{dps}), \bigcup \{\mathcal{bd}(D, \pi_1(\mathcal{X}_{dps})) : D \in \pi_2(\mathcal{X}_{dps})\} \tag{2.34}$$

$\mathcal{F}^{\mathcal{DPS}}_{\rightarrow udps}$ can be implemented in Haskell as follows:

```haskell
import Control.Monad
import Data.List ((\\))
```

```
powerList = filterM (const [True, False])

f:: ([Int],[([Int],[[Int]])]) -> ([Int],[([Int],[[Int]])])
f (points,dividons) = (points,unit_dividons)
  where
    unit_dividons = foldl (++) [] $ map get_unit_dividons dividons
    get_unit_dividons (d,(delta1:_)) = [(d,(\(a:b:_)->
        if a `in_same_div_as_b` b
          then [[a,b],[]]
          else [[a],[b]])
        x ) |
        x <- powerList (points \\ d), length x == 2,
        let (in_same_div_as_b) a b = (a `elem` delta1) == (b `elem` delta1)]
```

Remark. If we apply $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ on *div point sets* of 4 points we would immediately realize that $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ returns the same ordered pair, since for *div point sets* of 4 points, $\Omega_{sub} \subset \Omega_P$ in (2.34) would contain only one element and the element is some $D_\Theta \in \Theta_P$. For *div point sets* of 5 or more points Ω_{sub} would contain 3 or more elements, thus

$$\forall \mathscr{X} \in \mathscr{DPS}^* \qquad \mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}(\mathscr{X}) = \mathscr{X} \Leftrightarrow |\pi_1(\mathscr{X})| = 4 \tag{2.35}$$

On the other hand, applying $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ on *div point sets* with 3 or less points would result in $(P, \varnothing)$ since $\binom{n-2}{2} = 0$ for $n < 4$ and that is not going to be useful. So it is more sensible to define $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ over *div point sets* of 4 or more points.

$$\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}} : \mathscr{DPS}^*_{\geq 4} \longrightarrow \mathscr{UDPS}^* \tag{2.36}$$

Lemma 1. $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}} : \mathscr{DPS}^*_{\geq 4} \longrightarrow \mathscr{UDPS}^*$ is injective but not surjective. If the codomain is defined to be $\mathscr{UDPS}^\Theta$, the set of *unit div point sets* of 4 or more points satisfying (2.37), $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ is then bijective.

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\qquad (D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3) \\
&\qquad \Rightarrow (\psi(\pi_2(D_1)) = 1 \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))) \\
&\qquad\quad \wedge (\psi(\pi_2(D_1)) = 0 \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3)))
\end{aligned} \tag{2.37}$$

Proof for Lemma 1. It is injective because in (2.34) Ω_{sub} differs depending on $D \in \Theta_P$ as a result of d_u in $d\mathscr{b}$ being injective. It is not surjective onto the co-domain $\mathscr{UDPS}^*$, but surjective onto the co-domain $\mathscr{UDPS}^\Theta$, as a consequence of

I. $|\delta_n| = 2$ in (2.2): *Unit div point sets* with *unit dividons* such as

$$\{(a,b),(\{c\},\{d\})\},\{(a,b),(\{c\},\{e\})\},\{(a,b),(\{e\},\{d\})\}$$

can only be transformed from a *div point set* where $|\delta_n| = 3$ for some *dividion*, in this case:

$$\{(a,b),(\{c\},\{d\},\{e\})\}$$

Thus we have

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\quad D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3 \\
&\quad \Leftrightarrow \neg((\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = \psi(\pi_2(D_3)) = 0)
\end{aligned} \tag{2.38}$$

II. Associativity: if a and b are in the same *div*, and b and c are in the same *div*, a and c must be in the same *div*. So unit *div* points set with *unit dividons* such as

$$\{(a,b),(\{c,d\},\varnothing)\},\{(a,b),(\{c,e\},\varnothing)\},\{(a,b),(\{e\},\{d\})\}$$

can not be transformed from any *div point set*. Thus we have

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\quad D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3 \\
&\quad \Leftrightarrow \neg(\psi(\pi_2(D_1)) = \psi(\pi_2(D_2) = 1 \wedge \psi(\pi_2(D_3)) = 0)
\end{aligned} \tag{2.39}$$

Combining (2.39) and (2.38) gives (2.37). □

Lemma 2. A *unit div point set* (P, Ω_P) has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$ such that $D \in \Omega_P$ each describes the relative positions of the points (in terms of how 2 *TBD points* of each *divider* is distributed between *divs* it produced) iff it is in $\mathscr{UDPS}^+$, the class of *unit div point sets* of 4 ore more points satisfying (2.37), (2.41), (2.42), and (2.43), in which ξ is a function that returns the union of the *divider* and the *TBD points* in a *unit dividon*, D, notationally,

$$\xi(D) = \pi_1(D) \cup \bigcup \pi_2(D) \tag{2.40}$$

For any *unit div point set* (P, Ω_P),

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Omega_P \\
&\qquad\qquad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad\qquad \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = 1 \\
&\qquad\qquad\qquad \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))\ )
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad\qquad \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = 0 \\
&\qquad\qquad\qquad \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3))\ )
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad\qquad \wedge \bigcap_{n=1}^{2} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{2} \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3)\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 0 \\
&\qquad\qquad\qquad \Rightarrow \psi(\pi_2(D_3)) = 1\ )
\end{aligned}
\tag{2.43}
$$

Proof for Lemma 2. A *div point set* $\mathscr{X}_{dps}$ satisfying (2.10), (2.11), and (2.12), iff the *unit div point set* $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_{dps})$ satisfies (2.41), (2.42), and (2.43). This can be demonstrated in a similar way as (2.23): for any *unit divdion* D_u of some *unit div point set*, $\mathscr{A}_{udps}$, and its corresponding *divdion* D of the *div point set* $\mathscr{A}_{dps}$ where $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{A}_{dps}) = \mathscr{A}_{udps}$ - corresponding in the sense that $D_u \in \mathscr{b}d(D, \pi_2(\mathscr{A}_{dps}))$ and so $\pi_1(D_u) = \pi_1(D)$ - let $R := \xi(D_u)$, we would have

$$\phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D_u), \bigcup \pi_2(D_u)) = \psi(\pi_2(D_u)) \tag{2.44}$$

By restricting the *unit dividons* into $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$, we can then replace the predicate $\bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}$ with $D_1 \neq D_2 \neq D_3$, and every occurrence of $\phi(\pi_2(D_n), R \setminus \pi_1(D_n))$ with $\psi(\pi_2(D_n))$ (for $n \in \{1, 2, 3\}$) in (2.10), (2.11), and (2.12). This would gives (2.41), (2.42), and (2.43): they are basically a different way of expressing (2.10), (2.11), and (2.12) in the case of *unit div point sets.* Thus any *unit div point set* (P, Ω_P) in $\mathscr{UDPS}^+$ has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$, similar to how any *div point set* (P, Θ_P) in $\mathscr{DPS}^+$ has an interpretation for P by *Axiom 1.* □

Lemma 3. A *unit div point set* is in $\mathscr{UDPS}^+$ iff it is isomorphic to some *unit div point set* (P, Ω_P) in $\mathscr{UDPS}^{\mathbb{N}}$ where $Col_{udps}(\Omega_P)$, a full vertex monochromatic coloring on H_{udps}, satisfies (2.48) and (2.49), while $\mathscr{UDPS}^{\mathbb{N}}$ is the class of all *unit div point sets* (P, Θ_P) where $P \subset \mathbb{N}$ and $|P| \geq 4$, and Col_{udps} is a function similar to Col in (2.18),

$$Col_{udps}(\Omega_P) = \{((\pi_1(D), \bigcup \pi_2(D)), \psi(\pi_2(D)) : D \in \Omega_P\} \tag{2.45}$$

and H_{udps} is a 3-and-6-uniform hypergraph with 2 sets of hyperedges, E_1 and E_2, defined as a 3-tuple $H_{udps} = (V_{udps}, E_1, E_2)$, constructed based on P:

$$\begin{aligned} V_{udps} &= \bigcup \{V_{of}(d) : d \in \mathscr{P}(P) : |d| = 2\} \\ E_1 &= \{e \in \mathscr{P}(V) : |e| = 6 \wedge \forall v_1, v_2 \in e\ \pi_\cup(v_1) = \pi_\cup(v_2)\} \\ E_2 &= \{e \in \mathscr{P}(V) : |e| = 3 \wedge \forall v_1, v_2 \in e\ \pi_1(v_1) = \pi_2(v_2) \wedge |\bigcup_{v \in e} \pi_2(v)| = 3\} \end{aligned} \tag{2.46}$$

with $V_{of}(d)$ being a function that returns a set of ordered pair consists of *divider* and *TBD points* of *unit dividons* of the same *divider*,

$$V_{of}(d) = \{(d, P_{TBD}) : P_{TBD} \in \mathscr{P}(P \setminus x) : |P_{TBD}| = 2\} \tag{2.47}$$

and, finally, here are the conditions that the coloring needs to satisfy:

$$
\begin{aligned}
&\forall e \in E_1 \\
&\qquad \exists v_1, v_2 \in e \quad \left|\; \begin{array}{l} \pi_1(v_1) = \pi_2(v_2) \\ \pi_1(v_2) = \pi_2(v_1) \\ C(v_1) = C(v_2) = 0 \\ C^{members}(e \setminus \{v_1, v_2\}) = \{1\} \end{array} \right. \\
&\qquad \Leftrightarrow \neg\exists v_1, v_2, v_3 \in e \quad \left|\; \begin{array}{l} |\pi_1(v_1) \cap \pi_1(v_2) \cap \pi_1(v_3)| = 1 \\ C(v_1) = C(v_2) = C(v_3) = 0 \\ C^{members}(e \setminus \{v_1, v_2, v_3\}) = \{1\} \end{array} \right.
\end{aligned} \tag{2.48}
$$

$$
\begin{aligned}
&\forall e \in E_2 \\
&\qquad \forall v_1, v_2, v_3 \in e \quad \left|\; \begin{array}{l} v_1 \neq v_2 \neq v_3 \\ \Rightarrow (C(v_1) = 1 \Leftrightarrow C(v_2) = C(v_3)) \\ \qquad \wedge (C(v_1) = 0 \Leftrightarrow C(v_2) \neq C(v_3)) \end{array} \right.
\end{aligned} \tag{2.49}
$$

wherein C is the coloring based on the set of *unit divdons* of some $\mathscr{X}_{updś} \in \mathscr{UDPS}^{\mathbb{N}}$, i.e. $C = Col_{udpś}(\pi_2(\mathscr{X}_{updś}))$.

Remark. One may notice that the construction of $H_{udpś}$ depends solely on $\pi_1(\mathscr{X}_{updś})$ (i.e. the points of a *unit div point set*), as different from the coloring, which depends solely on $\pi_2(\mathscr{X}_{updś})$ (i.e. the set of *unit dividons*). This is similar to how the 3-uniform hypergraph H and the coloring on its vertices are defined back in the Proof for Theorem 1.

However, each vertex of $H_{udpś}$ is an ordered pair, structurally different from each vertex of H which is a set with cardinality of 2. Such definition for the vertices of $H_{udpś}$ in terms of not only the *divider* of a *unit dividon* but also its *TBD points* is necessary. This is because for any *unit div point set*, (P, Ω_P), there exists $\binom{|P|-2}{2}$ distinct *unit dividons* sharing a common *divider*, where $\binom{|P|-2}{2} > 1$ when $|P| \geq 5$. In order to distinguish *unit dividons* from one another in a *unit div point set* of 5 or more points, we would need to know both its *divider* and its *TBD points*.

Remark. For any *unit div point set*, (P, Ω_P) where $|P| = 4$, E_2 of $H_{updś}$ constructed based on P is an empty set, and thus (2.49) is trivially true for any coloring on such $H_{updś}$. On the other hand, there would only be 1 edge in E_1 and the coloring $C_{udpś}(\Omega_P)$ satisfies (2.48) iff (P, Ω_P) is isomorphic to $\mathscr{X} \in \{Conv^4, Conc_1^4\}$: in (2.48), the first-order predicate before the logical connective $\Leftrightarrow$ is true iff $\mathscr{X}$ is isomorphic to $Conv^4$, while the first-order predicate after $\Leftrightarrow$ is true iff $\mathscr{X}$ is isomorphic to $Conc_1^4$.

Proof for Lemma 3. Every *unit div point set* of 4 or more points is isomorphic to some *unit div point set* in $\mathscr{UDPS}^{\mathbb{N}}$. For a *unit div point set* to be in $\mathscr{UDPS}^{+}$, it has to satisfy (2.37), (2.41), (2.42), and (2.43). It is clear that a *unit div point set*, (P, Ω_P), satisfies (2.37) iff $Col(\Omega_P)$ on the H_{udps} constructed based on P satisfies (2.49): (2.49) is simply a different way of writing (2.37) by first defining the order pairs $(d_n, \bigcup \delta_n)$ of some *unit dividons* $D_n = (d_n, \delta_n)$ that satisfy the necessary conditions (namely $(D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3)$) to be vertices of an edge in E_2 (recall (2.46)). On the other hand (P, Ω_P) would satisfy (2.41), (2.42), and (2.43) iff $Col(\Omega_P)$ on H_{udps} constructed based on P satisfies (2.48).

(2.41), (2.42), and (2.43) can be summarized as formulae with universal quantification of 4 points in P, where if these points are distinct, some conditional proposition regarding certain distinct *unit dividons* in Ω_P must be true. The common characteristic of the conditional proposition in all 3 formulae is that $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$ is a part of the conjunction that makes up the antecedent. For any unit div point sets of 4 or more points, there are a total of 6 *unit dividons* D where $\xi(D) = R$ for any $R \subseteq P$ where $|R| = 4$, obtainable using $\mathscr{UDs}$, a function which takes in a set of 4 points, R, and returns a set of such ordered pair:

$$\begin{aligned} \mathscr{UDs}(R) &= \{\mathscr{ud}(d) : d \in \mathscr{P}(R) : |d| = 2\} \\ \mathscr{ud}(d) &= (d,\ R \setminus d) \end{aligned} \tag{2.50}$$

One may notice that $\mathscr{UDs}(R)$ always has a cardinality of $\binom{4}{2} = 6$ and that E_1 of the H_{udps} constructed based on some P can be expressed in terms of $\mathscr{UDs}$.

$$E_1 = \{\mathscr{UDs}(R) : R \in \mathscr{P}(P) : |R| = 4\} \tag{2.51}$$

By *Theorem 1*, a *unit div point set* of 4 points (recall that *div point sets* of 4 points are their own *unit div point sets*) satisfies (2.41), (2.42), and (2.43) iff it is isomorphic to either $Conc_4^1$ or $Conc_4$. More fundamentally, this means that any *unit div point set*, (P, Ω_P), satisfies (2.41), (2.42), and (2.43) iff each set of 6 *unit dividons*, $\Omega' \subseteq \Omega_P$, where for all $D \in \Omega'$, $\xi(D)$ is equivalent to a subset of 4 cardinality of P, is isomorphic[1] to either $\pi_2(Conc_4^1)$ or $\pi_2(Conv_4)$. The set of all such Ω' for any $\mathscr{X} \in \mathscr{UDPS}^{*}$, can be expressed as a function $All_{\Omega'}$ where:

$$\begin{aligned} All_{\Omega'}(\mathscr{X}) &= \{\Omega'_{basedOn}(R, \pi_2(\mathscr{X})) : R \in \mathscr{P}(\pi_1(\mathscr{X})) : |R| = 4\} \\ \Omega'_{basedOn}(R, \Omega_P) &= \{D : D \in \Omega_P : \xi(D) = R\} \end{aligned} \tag{2.52}$$

One may then realize that the following equation holds true for E_1 of any H_{upds} constructed based on $\pi_1(\mathscr{X})$:

$$E_1 = \{\{(\pi_1(D), \bigcup \pi_2(D)) : D \in \Omega'\} : \Omega' \in All_{\Omega'}(\mathscr{X})\} \tag{2.53}$$

Therefore any *unit div point set* of n points in $\mathscr{UDPS}^{\mathbb{N}}$, (P, Ω_P), satisfies (2.41), (2.42), and (2.43) iff for all 4-cardinal $R \subseteq P$, a subset C' of $Col_{\mathscr{udps}}(\Omega_P)$, the monochromatic vertex coloring on $H_{\mathscr{upds}}$ constructed based on P, where for all $c \in C'$, $\xi(\pi_1(c)) = R$, is isomorphic[2] to either the coloring $Col(\pi_2(Conc_4^1))$ or $Col(\pi_2(Conv_4))$. Notationally,

$$
\begin{aligned}
&\forall R \in \{P' \in \mathscr{P}(P) : |P'| = 4\} \\
&\qquad C' := C'_{of}(R, Col_{\mathscr{udps}}(\Omega_P)) \\
&\qquad C' \cong Col(\pi_2(Conc_4^1)) \Leftrightarrow \neg(C' \cong Col(\pi_2(Conv_4))) \\
&\text{where } C'_{of}(R, C) = \{c \in C : \xi(\pi_1(c)) = R\}
\end{aligned} \tag{2.54}
$$

which is what is expressed in (2.48). □

Note. *isomorphic*[1]: The definition of isomorphism in (2.56) is that of *div point sets*, but the isomorphism we are talking about here is that of sets of *unit dividons*, which can be defined as follows:

$$
\begin{aligned}
\Omega_1 \cong^1 \Omega_2 \Leftrightarrow\ & |\Omega_1| = |\Omega_2| \\
& \wedge \exists f_\Omega : \textstyle\bigcup_{D \in \Omega_1} \pi_1(D) \xrightarrow{1:1} \bigcup_{D \in \Omega_2} \pi_1(D) \\
& \quad \forall D_1 \in \Omega_1 \\
& \quad\quad \exists D_2 \in \Omega_2 \\
& \quad\quad\quad (d_1, \delta_1) := D_1 \\
& \quad\quad\quad (d_2, \delta_2) := D_2 \\
& \quad\quad\quad f^{members}(d_1) = d_2 \Leftrightarrow f^{members^2}(\delta_1) = \delta_2
\end{aligned} \tag{2.55}
$$

It is necessary to specify Ω_1 and Ω_2 to have the same cardinality, since it is possible for f_Ω, a bijective function satisfying the condition, to exist in the case when $|\Omega_1| \neq |\Omega_2|$.

isomorphic[2]: The isomorphism we are talking about here is that of colors, which can be defined as follows:

$$
\begin{aligned}
C_1 \cong^2 C_2 \Leftrightarrow\ & \exists f_C : \{\pi_1(c) : c \in C_1\} \xrightarrow{1:1} \{\pi_1(c) : c \in C_2\} \\
& \quad \forall c_1 \in C_1 \\
& \quad\quad \exists c_2 \in C_2 \\
& \quad\quad\quad (v_1, color_1) := C_1 \\
& \quad\quad\quad (v_2, color_2) := C_2 \\
& \quad\quad\quad f(v_1) = v_2 \Rightarrow color_1 = color_2
\end{aligned} \tag{2.56}
$$

Definition 5. We say that $\mathscr{X}_1 \in \mathscr{DPS}^*$ is a *sub div point set* of $\mathscr{X}_2 \in \mathscr{DPS}^*$ (denoted by $\leq$) iff the set of *unit divdion* of the corresponding *unit div point set* of $\mathscr{X}_1$ is a subset of that of $\mathscr{X}_2$. Notationally,

$$\begin{aligned}
&\forall \mathscr{X}_1, \mathscr{X}_2 \in \mathscr{DPS}^* \\
&\quad (A, \Omega_A) := \mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{X}_1) \\
&\quad (B, \Omega_B) := \mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{X}_2) \\
&\quad \mathscr{X}_1 \leq \mathscr{X}_2 \Leftrightarrow \Omega_A \subseteq \Omega_B
\end{aligned} \tag{2.57}$$

For clarification, 2 *sub div point sets* of some *div point set*, (S_1, Ω_{S_1}) and (S_2, Ω_{S_2}), are distinct *sub div point sets* if $S_1 \neq S_2$, which is to say, distinctness here is not defined in terms of isomorphism, but equality (i.e. by the axiom of extensionality in ZFC).

Definition 6. $\mathscr{S}dps_{of}$ is a function that returns the set of all *sub div point sets* of m points for some div point set where $m \in \mathbb{N}_{\geq 4}$.

$$\mathscr{S}dps_{of}(\mathscr{X}_{dps}, m) = \{\mathscr{S}dps(\mathscr{X}_{dps}, P_s) : P_s \in \pi_1(\mathscr{X}_{dps})(P) : |P_s| = m\} \tag{2.58}$$

where $\mathscr{S}dps$ is a function that returns the *sub div point set* of a set of points, P_s, of a *div point set* of $\mathscr{X}_{dps}$:

$$\begin{aligned}
&\mathscr{S}dps(\mathscr{X}_{dps}, P_s) = \mathscr{F}^{\mathscr{UDPS}}_{\vec{d}ps}((P_s, \{D : D \in \pi_2(\mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{X}_{dps})) : \xi(D) \subseteq P_s\})) \\
&\text{where } \mathscr{F}^{\mathscr{UDPS}}_{\vec{d}ps} \text{ is the inverse of } \mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}
\end{aligned} \tag{2.59}$$

Since a *div point set* of n points always has $\binom{n}{m}$ distinct *sub div points sets* of m points, where $m \leq n$ and $m \geq 4$, $\mathscr{S}dps_{of}(\mathscr{X}_{dps}, m)$ has the cardinality of $\binom{|\pi_1(\mathscr{X}_{dps})|}{m}$.

Lemma 4. For any *div point set*, $\mathscr{X}$, and any natural number m greater or equal to 4, let $\mathscr{A}$ and $\mathscr{B}$ to be any 2 distinct *sub div point sets* of m points of $\mathscr{X}$, and k be the number of points $\mathscr{A}$ and $\mathscr{B}$ have in common, $\mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{A})$ and $\mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{B})$ always have $6\binom{k}{4}$ *unit dividons* in common. Notationally,

$$\begin{aligned}
&\forall \mathscr{X} \in \mathscr{DPS}^* \\
&\quad \forall m \in \mathbb{N}_{\geq 1} \\
&\quad\quad \forall \mathscr{A}, \mathscr{B} \in \mathscr{S}dps_{of}(\mathscr{X}, m) \\
&\quad\quad\quad |\pi_2(\mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{A})) \cap \pi_2(\mathscr{F}^{\mathscr{DPS}}_{\vec{u}dps}(\mathscr{B}))| = 6\binom{|\pi_1(\mathscr{A}) \cap \pi_1(\mathscr{B})|}{4}
\end{aligned} \tag{2.60}$$

Proof for Lemma 4. For any $m > |\pi_1(\mathscr{X})|$, the proposition on elements in $\mathscr{S}dps_{of}(\mathscr{X}, m)$ is vacuously true. For $m = |\pi_1(\mathscr{X})|$, it is obvious that the proposition is true: since every *dividon* of a *div point set* can be broken down into $\binom{|P|-2}{2}$ *unit dividon*, any unit div point set of n points would have $\binom{n-2}{2}\binom{n}{2} = 6\binom{n}{4}$ *unit dividons* in total. For $m < 4$, the proposition is trivially true because $\binom{m}{4} = 0$ and *unit div point sets* of 3 or less points have

0 *unit dividons* (recall (2.26)). For any $m < |\pi_1(\mathscr{X})|$ but greater than 3, the proposition can be proven by first observing that $\mathscr{UD}\mathscr{s}(R) \cap \mathscr{UD}\mathscr{s}(R') = \varnothing \Leftrightarrow R \neq R'$ (recall (2.50)) for any sets R and R' with cardinality of 4, which indicates no 2 distinct *unit div point set* of 4 points have a *unit dividon* in common. Notationally,

$$\begin{aligned} &\forall \mathscr{A}, \mathscr{B} \in \{\mathscr{X} : \mathscr{X} \in \mathscr{UDPS}^* : |\pi_1(\mathscr{X})| = 4\} \\ &\pi_2(\mathscr{A}) \cap \pi_2(\mathscr{B}) = \varnothing \Leftrightarrow \pi_1(\mathscr{A}) \neq \pi_1(\mathscr{B}) \end{aligned} \tag{2.61}$$

However, for any 2 *unit div point sets* of 5 or more points, $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$, if they have 4 points in common, let the set of such 4 points be R, each D' in $\mathscr{UD}\mathscr{s}(R)$ would be equivalent to $\xi(D_a)$ and $\xi(D_b)$ where D_a and D_b are *unit dividon* of $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$ respectively, having the same *divider* and *TBD points*. In the case when $\mathscr{A}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{A}_{dps})$ and $\mathscr{B}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{B}_{dps})$ for some $\mathscr{A}_{dps}$ and $\mathscr{B}_{dps}$ that are both *sub div point sets* of a certain $\mathscr{X} \in \mathscr{DPS}^*$, $D_a = D_b$ for every such respective *unit dividons* of $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$. This implies that for every such distinct R, $\mathscr{A}_{udps}$ and $\mathscr{B}_{dps}$ have 6 *unit dividons* in common. Let k be the number of points $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$ have in common, the number of such distinct R is precisely k *chooses* 4 i.e. $\binom{\pi_1(\mathscr{A}_{udps}) \cap \pi_1(\mathscr{B}_{udps})}{4}$. $\square$

Theorem 2. Let $\mathscr{DPS}^+_5$ denotes the class of all *div point sets* of 5 points in $\mathscr{DPS}^+$. All $\mathscr{X} \in \mathscr{DPS}^+_5$ either have 4, 2 or 0 distinct *sub div points set* of 4 points isomorphic to $Conc^1_4$ (with the remaining *sub div point sets* of 4 points isomorphic to $Conv_4$).

Proof for Theorem 2.

Summary. In *Part 1* we prove that there exists no $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ has precisely 1, 3 or 5 elements isomorphic to $Conc^1_4$. In *Part 2* we prove that there exists $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ has precisely 0, 2 or 4 elements isomorphic to $Conc^1_4$.

Part 1. A *div point set* $\mathscr{X}_{dps}$ is in $\mathscr{DPS}^+$ iff $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_{dps})$ is in $\mathscr{UDPS}^+$. Any *unit div point set* of 5 points in $\mathscr{UDPS}^+$ always has an even number of *unit dividons* D where $\psi(\pi_2(D)) = 0$, since it is isomorphic to some (P, Ω_P) in $\mathscr{UDPS}^{\mathbb{N}}$ where the coloring $Col_{udps}(\Omega_P)$ satisfies (2.49). For $Col_{udps}(\Omega_P)$ to satisfy (2.49), every e in E_2 must has its vertices colored $[1, 0, 0]$ or $[1, 1, 1]$. Since in any *unit div point set* of 5 points, there exists only $\binom{5-2}{2} = 3$ distinct *unit dividon* with the same *divider*, edges in E_2 are disjoint (recall (2.46)), and therefore any coloring satisfying (2.49) would have an even number of vertices colored 0. $Conc^1_4$ has an odd number of *unit dividons* D where $\psi(\pi_2(D)) = 0$, while $Conv_4$ has an even number for such *unit dividons*. Therefore there exists no *unit div point sets* of 5 points, $\mathscr{X}_{udps}$, in $\mathscr{UDPS}^+$ such that $All_{\Omega'}(\mathscr{X}_{udps})$ (defined (2.52)) contains an odd number of elements isomorphic[1] to $\pi_2(Conc^1_4)$. We thereby conclude that there exists no $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ has precisely 1, 3 or 5 elements isomorphic to $Conc^1_4$.

Part 2. There exists *unit div point sets* of 5 points in $\mathscr{UDPS}^+$ with precisely 4, 2, or 0 *sub div point sets* of 4 points isomorphic to $Conc^1_4$, since it is possible to construct *unit div point sets* of 5 points, $\mathscr{X}_{upds}$, isomorphic to some (P, Ω_P) in $\mathscr{UDPS}^{\mathbb{N}}$ where the coloring $Col_{udps}(\Omega_P)$ satisfies (2.48) and (2.49) and there are precisely 4, 2, or 0 distinct $\Omega' \in All_{\Omega'}(\mathscr{X}_{udps})$ isomorphic[1] to $\pi_2(Conc_4^1)$, (with the remaining Ω' isomorphic to $Conv_4$).

I. To construct such *unit div point sets* $\mathscr{X}_{udps}$ where $All_{\Omega'}(\mathscr{X}_{udps})$ contains 0 elements isomorphic to $Conc_4^1$ and 5 elements isomorphic to $\pi_2(Conv_4)$, we would need to make sure there are only 2 *unit dividons* $D \in \Omega'$ where $\phi(D) = 0$ for all Ω' in $All_{\Omega'}(\mathscr{X}_{udps})$. Let's denote the set of all such *unit dividons* as D^*, the 5 elements in $All_{\Omega'}(\mathscr{X}_{udps})$ as Ω_1', Ω_2', Ω_3', Ω_4', Ω_5' and each 2 such unit divdions in Ω_n' as D_n^1 and D_n^2 for $n \in \{1,2,3,4,5\}$, i.e. $\{D_n^1, D_n^2\} = \Omega_n' \cap D^*$. For the coloring to satisfy (2.48) and (2.49), we simply let any 2 *unit dividons* D_n^x, D_m^x where $x \in \{1,2\}$ and $n \neq m$ to have a common *divider*, while avoiding to have 3 distinct *unit dividon* in D^* to a common *divider* , and the same time ensuring that

$$\begin{aligned} \pi_1(D_n^1) &= \bigcup \pi_2(D_n^2) \\ \pi_1(D_n^2) &= \bigcup \pi_2(D_n^1) \end{aligned} \tag{2.62}$$

(recall (2.48)). That is to say, for some subsets of 2 cardinality, A, B, C, D, E, F of $\pi_1(X_{udps})$, we have

$$\begin{aligned} \pi_1(D_1^1) &= \bigcup \pi_2(D_1^2) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) = A \\ \pi_1(D_1^2) &= \bigcup \pi_2(D_1^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) = B \\ \pi_1(D_2^2) &= \bigcup \pi_2(D_2^1) = \pi_1(D_4^2) = \bigcup \pi_2(D_4^1) = C \\ \pi_1(D_4^1) &= \bigcup \pi_2(D_4^2) = \pi_1(D_5^1) = \bigcup \pi_2(D_5^2) = D \\ \pi_1(D_5^2) &= \bigcup \pi_2(D_5^1) = \pi_1(D_6^2) = \bigcup \pi_2(D_6^1) = E \\ \pi_1(D_3^1) &= \bigcup \pi_2(D_3^2) = \pi_1(D_6^1) = \bigcup \pi_2(D_6^2) = F \end{aligned} \tag{2.63}$$

where

$$A \neq B \neq C \neq D \neq E \neq F$$
$$(A \cap B) = (A \cap C) = (B \cap F) = (C \cap D) = (D \cap E) = (E \cap F) = \varnothing$$

II. To construct such *unit div point sets* $\mathscr{X}_{udps}$ where $All_{\Omega'}(\mathscr{X}_{udps})$ contains 2 elements isomorphic to $Conc_4^1$ and 3 elements isomorphic to $\pi_2(Conv_4)$ - let's use the same notations in I - we would need to make sure that

$$\begin{aligned} \forall n \in \{1,2,3\} \qquad & \{D_n^1, D_n^2\} = \Omega_n' \cap D^* \\ \forall n \in \{4,5\} \qquad & \{D_n^1, D_n^2, D_n^3\} = \Omega_n' \cap D^* \end{aligned} \tag{2.64}$$

The *divider* of *unit dividons* D_n^1, D_n^2, D_n^3 for $n \in \{4,5\}$ need to have 1 element in common:

$$|\pi_1(D_n^1) \cap \pi_1(D_n^1) \cap \pi_1(D_n^1)| = 1 \tag{2.65}$$

(recall (2.48)), while (2.62) continues to apply to D_n^1, D_n^2 for $n \in \{1,2,3\}$. For the coloring to satisfy (2.48) and (2.49), we can have D_4^x to share the same *divider* as D_5^x for all $x \in \{1,2\}$, while letting the remaining *unit dividons* in D_4 and D_5, namely D_4^3 and D_5^3, to respectively share the same *divider* as D_1^1 and D_2^1, and the remaining *unit dividons* in D_1 and D_2, namely D_1^2 and D_2^2, to respectively share the same *dividers* as the two *dividons* in D_3. That is to say, for the distinct points $a, b, c, d, e \in \pi_1(X_{udps})$, we have

$$\begin{aligned}
\pi_1(D_4^1) = \pi_1(D_5^1) &= \{a, b\} \\
\pi_1(D_4^2) = \pi_1(D_5^2) &= \{a, c\} \\
\pi_1(D_4^3) = \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) &= \{a, d\} \\
\pi_1(D_5^3) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) &= \{a, e\} \\
\pi_1(D_1^2) = \bigcup \pi_2(D_1^1) = \pi_1(D_3^1) = \bigcup \pi_2(D_3^2) &\subset P \setminus \{a, d\} \\
\pi_1(D_2^2) = \bigcup \pi_2(D_2^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) &\subset P \setminus \{a, e\}
\end{aligned} \tag{2.66}$$

III. To construct such *unit div point sets* $\mathfrak{X}_{udps}$ where $All_{\Omega'}(\mathfrak{X}_{udps})$ contains 4 elements isomorphic to $Conc_4^1$ and 1 element isomorphic to $\pi_2(Conv_4)$ - let's use the same notations in II - this time we would need to make sure that

$$\begin{aligned}
\forall n \in \{1\} \qquad & \{D_n^1, D_n^2\} = \Omega'_n \cap D^* \\
\forall n \in \{2,3,4,5\} \qquad & \{D_n^1, D_n^2, D_n^3\} = \Omega'_n \cap D^*
\end{aligned} \tag{2.67}$$

where (2.65) applies to D_n^1, D_n^2 and D_n^3 for $n \in \{2,3,4,5\}$ and (2.62) continues to apply to D_n^1, D_n^2 for $n \in \{1\}$. For the coloring to satisfy (2.48) and (2.49), we can let D_4^x to share the same *divider* as D_5^x,and D_2^x to share the same *divider* as D_3^x, for $x \in \{1,2\}$. And then we let the remaining *unit dividons* in D_4 and D_5, namely D_4^3 and D_5^3, to respectively share the same *divider* as D_3^3 and D_1^1, while the remaining *unit dividons* in D_2, namely D_2^3 to share the same *divider* as D_1^2. That is to say, for the distinct points $a, b, c, d, e \in \pi_1(X_{udps})$, we have

$$\begin{aligned}
\pi_1(D_4^1) &= \pi_2(D_5^1) = \{a, b\} \\
\pi_1(D_4^2) &= \pi_2(D_5^2) = \{a, c\} \\
\pi_1(D_4^3) &= \pi_1(D_3^3) = \{a, d\} \\
\pi_1(D_3^1) &= \pi_1(D_2^1) = \{e, d\} \\
\pi_1(D_3^2) &= \pi_1(D_2^2) = \{c, d\} \\
\pi_1(D_5^3) &= \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) = \{a, e\} \\
\pi_1(D_2^3) &= \pi_1(D_1^2) = \bigcup \pi_2(D_1^1) = \{b, d\}
\end{aligned} \tag{2.68}$$

□

Remark. A stronger version of *Theorem 2* would state that for all $\mathcal{X}_{dps} \in \mathcal{DPS}_5^+$, $\mathcal{X}_{dps}$ is either isomorphic to $Conv_5$, $Conc_5^1$ or $Conc_5^2$, where

$$\begin{aligned}
Conv_5 =& (Cv_5, \Theta_{Cv_5}) \\
Cv_5 =& \{1,2,3,4,5\} \\
\Theta_{Cv_5} =& \{(\{1,2\}, \{(\{3,4\}, \varnothing\}), \\
& (\{1,3\}, \{\{2\}, \{4,5\}\}), \\
& (\{1,4\}, \{\{2,3\}, \{5\}\}), \\
& (\{1,5\}, \{\{2,3,4\}, \varnothing\}), \\
& (\{2,3\}, \{\{1,4,5\}, \varnothing\}), \\
& (\{2,4\}, \{\{1,5\}, \{3\}\}), \\
& (\{2,5\}, \{\{1\}, \{3,4\}\}), \\
& (\{3,4\}, \{\{1,2,5\}, \varnothing\}) \\
& (\{3,5\}, \{\{1,2\}, \{4\}\}) \\
& (\{3,4\}, \{\{1,2,5\}, \varnothing\})
\end{aligned}$$

$$\begin{aligned}
Conc_5^1 =& (Cc_5^1, \Theta_{Cc_5^1}) \\
Cc_5^1 =& \{1,2,3,4,5\} \\
\Theta_{Cc_5^1} =& \{(\{1,2\}, \{(\{3,4,5\}, \varnothing\}), \\
& (\{1,3\}, \{\{2,5\}, \{4\}\}), \\
& (\{1,4\}, \{\{2,3,5\}, \varnothing\}), \\
& (\{1,5\}, \{\{2\}, \{3,4\}\}), \\
& (\{2,3\}, \{\{1,4,5\}, \varnothing\}), \\
& (\{2,4\}, \{\{1\}, \{3,5\}\}), \\
& (\{2,5\}, \{\{1,4\}, \{3\}\}), \\
& (\{3,4\}, \{\{1,2,5\}, \varnothing\}) \\
& (\{3,5\}, \{\{1,2\}, \{4\}\}) \\
& (\{4,5\}, \{\{1,2,5\}, \varnothing\})
\end{aligned}$$

$$\begin{aligned}
Conc_5^2 =& (Cc_5^2, \Theta_{Cc_5^2}) \\
Cc_5^2 =& \{1,2,3,4,5\} \\
\Theta_{Cc_5^2} =& \{(\{1,2\}, \{(\{3,4,5\}, \varnothing\}), \\
& (\{1,3\}, \{\{2,4,5\}, \varnothing\}), \\
& (\{1,4\}, \{\{2\}, \{3,5\}\}), \\
& (\{1,5\}, \{\{2,4\}, \{3\}\}), \\
& (\{2,3\}, \{\{1,4,5\}, \varnothing\}), \\
& (\{2,4\}, \{\{1\}, \{3,5\}\}), \\
& (\{2,5\}, \{\{1,4\}, \{3\}\}), \\
& (\{3,4\}, \{\{1\}, \{2,5\}\}) \\
& (\{3,5\}, \{\{1,4\}, \{2\}\}) \\
& (\{4,5\}, \{\{1,3\}, \{2\}\})
\end{aligned}$$

To prove this version of *Theorem 2* we would need to prove that there exists no *div point sets* in $\mathcal{DPS}_5^+$ not isomorphic to $Conv_5$, $Conc_5^1$ or $Conc_5^2$.

Remark. Let $All_{of\Omega'}(\mathcal{X}, n)$ be a generalization of $All_{\Omega'}(\mathcal{X})$ such that

$$All_{of\Omega'}(\mathcal{X}, n) = \{\Omega'_{basedOn}(R, \pi_2(\mathcal{X})) : R \in \mathcal{P}(\pi_1(\mathcal{X})) : |R| = n\} \tag{2.69}$$

The proposition that a *unit div point set* of 5 or more points, $\mathcal{X}_{udps}$, is in $\mathcal{UDPS}^+$ iff all elements in $All_{of\Omega'}(\mathcal{X}_{udps}, n)$ are isomorphic to the set of *unit dividons* of some *unit div point set* in $\mathcal{UDPS}^+$, for any $n \in \mathbb{N}_{\geq 4}$ less than $|\pi_1(\mathcal{X}_{udps})|$, can be proven to be false using *Theorem 2*. However, a weaker version of it still holds true: if $\mathcal{X}_{udps}$ is in $\mathcal{UDPS}^+$, all members of $All_{of\Omega'}(\mathscr{X}_{udps}, n)$ are also in $\mathscr{UDPS}^+$ for any $n \in \mathbb{N}_{\geq 4}$ less than $|\pi_1(\mathscr{X}_{udps})|$. This is equivalent as stating: for any *unit div point set*, $\mathscr{X}_1$, whose *unit dividons* is a subset of that of some $\mathscr{X}_2 \in \mathscr{UDPS}^+$, there certainly exists an interpretation for $\pi_1(\mathscr{X}_1)$ as some set of points in $\mathbb{E}^2$, which is just a subset of the interpretation for $\pi_1(\mathscr{X}_2)$ as some set of points in $\mathbb{E}^2$, since $\pi_1(\mathscr{X}_1) \subseteq \pi_1(\mathscr{X}_2)$.

There is undoubtedly some similarity between the false proposition above, and the following proposition which is too false: a *div point set* of 4 or more points, $\mathscr{X}_{dps}$, is in $\mathscr{DPS}^+$ iff all elements in $\mathscr{S}dps_{of}(\mathscr{X}_{dps}, n)$ are also in $\mathscr{DPS}^+$, for any $n \in \mathbb{N}_{\geq 3}$ less than $|\pi_1(X_{dps})|$. If this is true, it would imply that $\mathscr{DPS}^* = \mathscr{DPS}^+$, which is obviously false. However, if we closely examine this proposition, we would realize that it would be true if not for the case when $n = 3$: it is vacuously true that any *div point sets* of 3 points satisfy (2.10), (2.11), and (2.12), thus we can't conclude that a certain *div point set* satisfies (2.10), (2.11), and (2.12), even if all its *sub div point sets* of 3 points satisfy them. Now recall Lemma 3 where E_2 of the hypergraph based on P is an empty set in the case when $|P| = 4$ and as a result, it is vacuously true that such E_2 always satisfies (2.49). This is why the proposition regarding *unit div point sets* above is false: we cannot conclude that a certain *unit div point set* is isomorphic to some *unit div point set*, (P, Ω_P), in $\mathscr{UDPS}^{\mathbb{N}}$ where $Col_{udps}(\Omega_P)$ satisfies (2.48) and (2.49), even if all elements in $All_{of\Omega'}(\mathscr{X}_{udps}, 4)$ are isomorphic to some *unit div point set*, (P, Ω_P), in $\mathscr{UDPS}^{\mathbb{N}}$ where $Col_{udps}(\Omega_P)$ satisfies them.

It can be proven that in the case when $n \in \mathbb{N}_{\geq 5}$, the proposition regarding *unit div point sets* above is true.

2.2 *convexity*

The notion that there exists n points forming a convex polygon among some set of points in $\mathbb{E}^2$ can be expressed through *convexity* in the context of *div point sets*.

Definition 7. A *div point set* (P, Θ_P) has a *convexity* of n if there exists (Q, Θ_Q) such that $(Q, \Theta_Q) \leq (P, \Theta_P)$ and (Q, Θ_Q) is isomorphic to $Conv_n$, defined as follow

$$Conv_n = (P, \{(d, \delta_{conv}(d, P)) : d \in \mathscr{P}(P) : |d| = 2\})$$
$$\text{where} \begin{cases} P \coloneqq \{x \in \mathbb{N}_{\geq 1} : x \leq n\} \\ \delta_{conv}(d, P) = \{\{p : p \in P : inside(p, d)\}, \{p : p \in P : outside(p, d)\}\} \\ inside(p, d) = (p > min(d) \wedge p < max(d)) \\ outside(p, d) = (p < min(d) \vee p > max(d)) \\ min(d) \text{ returns the smallest number in } d \\ max(d) \text{ returns the biggest number in } d. \end{cases} \tag{2.70}$$

where n is a natural number ≥ 3. Here is an implementation of it as a function in Haskell:

```haskell
import Data.List

combine :: Int -> [a] -> [[a]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                        , ys <- combine (n-1) xs']

convex:: Int -> ([Int],[([Int],[[Int]])])
convex n = (points, dividons)
   where
      points = [1..n]
      dividers = combine 2 points
      dividons = [(divider,[div1,div2])
         | divider@(a:b:_) <- dividers,
         let divs = points \\ divider,
         let div1 = [ x | x <- divs, x > a, x < b ],
         let div2 = divs \\ div1 ]
```

Axiom 2. For any $\mathscr{X}$ in $\mathscr{DPS}^+$, there exists n points forming a convex polygon among some set of points in E^2 for which $\pi_1(\mathscr{X})$ can be interpreted as, iff $\mathscr{X}$ has a convexity of n. More precisely, there exists an interpretation for $P' \subseteq \pi_1(\mathscr{X})$ as some set of n points forming a convex polygon iff $Sdps(\mathscr{X}, P')$ is isomorphic to $Conv_n$, for all $n \geq 3$.

Remark. One may notice that for any $n \geq 4$, all *sub div point sets* of $n-1$ points of $Conv_n$ are isomorphic to $Conv_{n-1}$. By *Axiom 2*, that is equivalent to the following proposition: for any $n \geq 4$, after removing any one point from a set of n points that are the vertices of a convex polygon on an Euclidean plane, the remaining points too forms a convex polygon, which is trivially true.

Remark. By *Axiom 2*, we can conclude from *Theorem 2* that for any 5 points in general position on an Euclidean plane, there always exists 4 points forming a convex polygon.

3 *A reduction to a multiset unsatisfiability problem*

The Erdös-Szekeres conjecture can be expressed as a conjunction of (3.1) and (3.2) in the theory of *div point set*.

$$\exists \mathscr{A} \in \mathscr{DPS}^+ \quad |\pi_1(A)| = 2^{n-2} \wedge \exists \mathscr{A}_s \leq \mathscr{A} \quad \mathscr{A}_s \ncong Conv_n \tag{3.1}$$

$$\forall \mathscr{A} \in \mathscr{DPS}^+ \quad |\pi_1(A)| > 2^{n-2} \Leftrightarrow \exists \mathscr{A}_s \leq \mathscr{A} \quad \mathscr{A}_s \cong Conv_n \tag{3.2}$$

for all $n \geq 3$. Since the lower bound has already been proven to be $2^{n-2}+1$, all we are left is to prove (3.2).

Let's define a function *assign* where

$$Assign(\mathscr{X}) = \begin{cases} 1 & \text{if} \quad \mathscr{X} \cong Conc_4^1 \\ 0 & \text{if} \quad \mathscr{X} \cong Conv_4 \end{cases} \tag{3.3}$$

(3.2) can be rewritten as

$$\begin{aligned} &\forall \mathscr{A} \in \mathscr{DPS}^+ \\ &\quad |\pi_1(A)| > 2^{n-2} \\ &\quad \Leftrightarrow \exists \mathscr{A}_s \in \mathcal{S}dps_{of}(\mathscr{A}, n) \\ &\qquad \forall \mathscr{A}_{ss} \in \mathcal{S}dps_{of}(\mathscr{A}_s, 4) \quad Assign(\mathscr{A}_{ss}) = 0 \end{aligned} \tag{3.4}$$

By *Theorem 2*, for any $\mathscr{X}_5 \in \mathscr{DPS}_5^+$,

$$[Assign(\mathscr{X}) : \mathscr{X} \in \mathcal{S}dps_{of}(\mathscr{X}_5, 4)] \in \{[1,1,1,1,0], [1,1,0,0,0], [0,0,0,0,0]\} \tag{3.5}$$

which is to say, for any *div point set* of 5 or more points in $\mathscr{DPS}^+$, all $\mathscr{X}_5$ in $\mathcal{S}dps_{of}(\mathscr{X}, 5)$ satisfies (3.5). We now present a multiset unsatisfiability problem, for which, if solved, would prove that there exists no *div point set* of $2^{n-2} + 1$ points, $\mathscr{X}$, where

$$\begin{aligned} &\forall \mathscr{X}_5 \in \mathcal{S}dps_{of}(\mathscr{X}, 5) \\ &\qquad \mathscr{X}_5 \text{ satisfies } (3.5) \wedge \mathscr{X}_5 \text{ does not satisfies } (3.4) \end{aligned}$$

and thus proving (3.2).

Let's define $UNSAT_{multiset}$ to be the problem of determining if there exists no value-assignments for all variables in V, distributed in a certain manner among the multisets in M, that satisfy certain constraints, over some domain D, the set of values for which a variable can be assigned to. The particular instances of $UNSAT_{multiset}$ we are interested in are of a set of variables V over the domain $\{0, 1\}$, where each variable represents whether a particular element in $\mathcal{S}dps_{of}(\mathscr{X}, 4)$ is isomorphic to $Conc_4^1$ or $Conv_4$, for some div point set $\mathscr{X}$ of $2^{n-2} + 1$ points where $n \in \mathbb{N}_{\geq 5}$, ans so $|V| = \mathcal{S}dps_{of}(\mathscr{X}, 4) = \binom{2^{n-2}+1}{4}$. In these instances of $UNSAT_{multiset}$, $M = A \cup B$, where A is a set of 5-cardinal multisets, and B is a set of n-cardinal multisets. The variables shall be distributed in A the same way as how elements in $\mathcal{S}dps_{of}(\mathscr{X}, 4)$ are distributed in $\mathcal{S}dps_{of}(\mathscr{X}, 5)$, while the variables shall be distributed in B the same way as how elements in $\mathcal{S}dps_{of}(\mathscr{X}, 4)$ are distributed in $\mathcal{S}dps_{of}(\mathscr{X}, n)$. $A = B$ in the case when $n = 5$. Here are the constraints the value assignments for the variables must satisfy

$$\begin{aligned} &\forall a \in A \qquad a \in \{[1,1,1,1,0], [1,1,0,0,0], [0,0,0,0,0]\} \\ &\forall b \in B \qquad b \neq \underbrace{[0,0,0,...,0,0]}_{\binom{n}{4}\ 0's} \end{aligned} \tag{3.6}$$

We would be referring to such instances of $UNSAT_{multiset}$ as $UNSAT^{+}_{multiset}$.

Suppose $\mathscr{Y}$ is a *div point set* in $\mathscr{DPS}^{+}$, all elements in $\mathscr{Sdps}_{of}(\mathscr{X}, 5)$ would have precisely either 0, 2, or 4 *sub div point set* of 4 points isomorphic to $Conc^{1}_{4}$, with the remaining *sub div point sets* isomorphic is $Conv_4$ (recall *Theorem 2*). If the Erdös-Szekeres conjecture is true and $\pi_1(\mathscr{Y}) = 2^{n-2} + 1$ for some $n \in \mathbb{N}_{\geq 5}$, there would be some *sub div point set* of n points isomorphic to $Conv_n$ i.e. such *sub div point set* with have all its *sub div point sets* of 4 points isomorphic to $Conv_4$. So if we show that

The distribution of variables in A and B can be implement in Haskell as follows:

```haskell
import Data.List
import Data.Maybe
type Multiset = [Integer]

merge (a:x) (b:y) = (a,b) : merge x y
merge [] _ = []

choose :: Integer -> Integer -> Integer
n `choose` k
    | k < 0     = 0
    | k > n     = 0
    | otherwise = factorial n `div` (factorial k * factorial (n-k))

factorial :: Integer -> Integer
factorial n = foldl (*) 1 [1..n]

combine :: Integer -> [Integer] -> [[Integer]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                         , ys <- combine (n-1) xs']

number_of_points = (\n->(2^(n-2)+1))

n_setOf_m_Multisets:: Integer -> Integer -> [Multiset]
n_setOf_m_Multisets m n = [ map fromJust $ map ((flip lookup) encoding)
    (combine 4 m_points) | m_points <- combine n [1..m] ]
    where
        encoding = merge (combine 4 [1..m]) [1..(m `choose` 4)]

setA :: Integer -> [Multiset]
setA n = n_setOf_m_Multisets (number_of_points n) 5

setB :: Integer -> [Multiset]
setB n = [ x | x <- n_setOf_m_Multisets (number_of_points n) n, 2 `elem` x ]
```

Here is the simplest instance of $UNSAT^{+}_{multiset}$ (when $n = 5$): $A = B$, and so we have

$|M| = |A| = |B| = \binom{2^{5-2}+1}{5} = 126$ multisets, and, interestingly, $|V| = \binom{2^{5-2}+1}{4} = 126$ variables (with each denoted by v_n below), distribute in the multisets in M as follows:

$$
\begin{aligned}
&\{[v_1,v_2,v_7,v_{22},v_{57}],[v_1,v_3,v_8,v_{23},v_{58}],[v_1,v_4,v_9,v_{24},v_{59}],[v_1,v_5,v_{10},v_{25},v_{60}],[v_1,v_6,v_{11},v_{26},v_{61}],[v_2,v_3,v_{12},v_{27},v_{62}],\\
&[v_2,v_4,v_{13},v_{28},v_{63}],[v_2,v_5,v_{14},v_{29},v_{64}],[v_2,v_6,v_{15},v_{30},v_{65}],[v_3,v_4,v_{16},v_{31},v_{66}],[v_3,v_5,v_{17},v_{32},v_{67}],[v_3,v_6,v_{18},v_{33},v_{68}],\\
&[v_4,v_5,v_{19},v_{34},v_{69}],[v_4,v_6,v_{20},v_{35},v_{70}],[v_5,v_6,v_{21},v_{36},v_{71}],[v_7,v_8,v_{12},v_{37},v_{72}],[v_7,v_9,v_{13},v_{38},v_{73}],[v_7,v_{10},v_{14},v_{39},v_{74}],\\
&[v_7,v_{11},v_{15},v_{40},v_{75}],[v_8,v_9,v_{16},v_{41},v_{76}],[v_8,v_{10},v_{17},v_{42},v_{77}],[v_8,v_{11},v_{18},v_{43},v_{78}],[v_9,v_{10},v_{19},v_{44},v_{79}],[v_9,v_{11},v_{20},v_{45},v_{80}],\\
&[v_{10},v_{11},v_{21},v_{46},v_{81}],[v_{12},v_{13},v_{16},v_{47},v_{82}],[v_{12},v_{14},v_{17},v_{48},v_{83}],[v_{12},v_{15},v_{18},v_{49},v_{84}],[v_{13},v_{14},v_{19},v_{50},v_{85}],[v_{13},v_{15},v_{20},v_{51},v_{86}],\\
&[v_{14},v_{15},v_{21},v_{52},v_{87}],[v_{16},v_{17},v_{19},v_{53},v_{88}],[v_{16},v_{18},v_{20},v_{54},v_{89}],[v_{17},v_{18},v_{21},v_{55},v_{90}],[v_{19},v_{20},v_{21},v_{56},v_{91}],[v_{22},v_{23},v_{27},v_{37},v_{92}],\\
&[v_{22},v_{24},v_{28},v_{38},v_{93}],[v_{22},v_{25},v_{29},v_{39},v_{94}],[v_{22},v_{26},v_{30},v_{40},v_{95}],[v_{23},v_{24},v_{31},v_{41},v_{96}],[v_{23},v_{25},v_{32},v_{42},v_{97}],[v_{23},v_{26},v_{33},v_{43},v_{98}],\\
&[v_{24},v_{25},v_{34},v_{44},v_{99}],[v_{24},v_{26},v_{35},v_{45},v_{100}],[v_{25},v_{26},v_{36},v_{46},v_{101}],[v_{27},v_{28},v_{31},v_{47},v_{102}],[v_{27},v_{29},v_{32},v_{48},v_{103}],[v_{27},v_{30},v_{33},v_{49},v_{104}],\\
&[v_{28},v_{29},v_{34},v_{50},v_{105}],[v_{28},v_{30},v_{35},v_{51},v_{106}],[v_{29},v_{30},v_{36},v_{52},v_{107}],[v_{31},v_{32},v_{34},v_{53},v_{108}],[v_{31},v_{33},v_{35},v_{54},v_{109}],[v_{32},v_{33},v_{36},v_{55},v_{110}],\\
&[v_{34},v_{35},v_{36},v_{56},v_{111}],[v_{37},v_{38},v_{41},v_{47},v_{112}],[v_{37},v_{39},v_{42},v_{48},v_{113}],[v_{37},v_{40},v_{43},v_{49},v_{114}],[v_{38},v_{39},v_{44},v_{50},v_{115}],[v_{38},v_{40},v_{45},v_{51},v_{116}],\\
&[v_{39},v_{40},v_{46},v_{52},v_{117}],[v_{41},v_{42},v_{44},v_{53},v_{118}],[v_{41},v_{43},v_{45},v_{54},v_{119}],[v_{42},v_{43},v_{46},v_{55},v_{120}],[v_{44},v_{45},v_{46},v_{56},v_{121}],[v_{47},v_{48},v_{50},v_{53},v_{122}],\\
&[v_{47},v_{49},v_{51},v_{54},v_{123}],[v_{48},v_{49},v_{52},v_{55},v_{124}],[v_{50},v_{51},v_{52},v_{56},v_{125}],[v_{53},v_{54},v_{55},v_{56},v_{126}],[v_{57},v_{58},v_{62},v_{72},v_{92}],[v_{57},v_{59},v_{63},v_{73},v_{93}],\\
&[v_{57},v_{60},v_{64},v_{74},v_{94}],[v_{57},v_{61},v_{65},v_{75},v_{95}],[v_{58},v_{59},v_{66},v_{76},v_{96}],[v_{58},v_{60},v_{67},v_{77},v_{97}],[v_{58},v_{61},v_{68},v_{78},v_{98}],[v_{59},v_{60},v_{69},v_{79},v_{99}],\\
&[v_{59},v_{61},v_{70},v_{80},v_{100}],[v_{60},v_{61},v_{71},v_{81},v_{101}],[v_{62},v_{63},v_{66},v_{82},v_{102}],[v_{62},v_{64},v_{67},v_{83},v_{103}],[v_{62},v_{65},v_{68},v_{84},v_{104}],[v_{63},v_{64},v_{69},v_{85},v_{105}],\\
&[v_{63},v_{65},v_{70},v_{86},v_{106}],[v_{64},v_{65},v_{71},v_{87},v_{107}],[v_{66},v_{67},v_{69},v_{88},v_{108}],[v_{66},v_{68},v_{70},v_{89},v_{109}],[v_{67},v_{68},v_{71},v_{90},v_{110}],[v_{69},v_{70},v_{71},v_{91},v_{111}],\\
&[v_{72},v_{73},v_{76},v_{82},v_{112}],[v_{72},v_{74},v_{77},v_{83},v_{113}],[v_{72},v_{75},v_{78},v_{84},v_{114}],[v_{73},v_{74},v_{79},v_{85},v_{115}],[v_{73},v_{75},v_{80},v_{86},v_{116}],[v_{74},v_{75},v_{81},v_{87},v_{117}],\\
&[v_{76},v_{77},v_{79},v_{88},v_{118}],[v_{76},v_{78},v_{80},v_{89},v_{119}],[v_{77},v_{78},v_{81},v_{90},v_{120}],[v_{79},v_{80},v_{81},v_{91},v_{121}],[v_{82},v_{83},v_{85},v_{88},v_{122}],[v_{82},v_{84},v_{86},v_{89},v_{123}],\\
&[v_{83},v_{84},v_{87},v_{90},v_{124}],[v_{85},v_{86},v_{87},v_{91},v_{125}],[v_{88},v_{89},v_{90},v_{91},v_{126}],[v_{92},v_{93},v_{96},v_{102},v_{112}],[v_{92},v_{94},v_{97},v_{103},v_{113}],[v_{92},v_{95},v_{98},v_{104},v_{114}],\\
&[v_{93},v_{94},v_{99},v_{105},v_{115}],[v_{93},v_{95},v_{100},v_{106},v_{116}],[v_{94},v_{95},v_{101},v_{107},v_{117}],[v_{96},v_{97},v_{99},v_{108},v_{118}],[v_{96},v_{98},v_{100},v_{109},v_{119}],\\
&[v_{97},v_{98},v_{101},v_{110},v_{120}],[v_{99},v_{100},v_{101},v_{111},v_{121}],[v_{102},v_{103},v_{105},v_{108},v_{122}],[v_{102},v_{104},v_{106},v_{109},v_{123}],[v_{103},v_{104},v_{107},v_{110},v_{124}],\\
&[v_{105},v_{106},v_{107},v_{111},v_{125}],[v_{108},v_{109},v_{110},v_{111},v_{126}],[v_{112},v_{113},v_{115},v_{118},v_{122}],[v_{112},v_{114},v_{116},v_{119},v_{123}],[v_{113},v_{114},v_{117},v_{120},v_{124}],\\
&[v_{115},v_{116},v_{117},v_{121},v_{125}],[v_{118},v_{119},v_{120},v_{121},v_{126}],[v_{122},v_{123},v_{124},v_{125},v_{126}]\}
\end{aligned}
$$

Remark. One may have noticed, $UNSAT^+_{multiset}$ can be reduced into the Boolean Unsatisfiability Problem, the complement of SAT, by first converting each multiset in A into the formula:

$$\bigvee_{v_0 \in V} (\neg v_0 \wedge \bigwedge_{v_1 \in V \setminus \{v_0\}} v_1) \vee \bigvee_{V_{|3|} \in V^*_{|3|}} (\bigwedge_{v_0 \in V_3} \neg v_0 \wedge \bigwedge_{v_1 \in V \setminus V_{|3|}} v_1) \vee (\bigwedge_{v_0 \in V} \neg v_0) \tag{3.7}$$

where $V^*_{|3|} = \{V_{|3|} \in \mathbb{P}(V) : |V_{|3|}| = 3\}$ and V is the set of meta-variables in each the multiset, and converting each multiset in B into the formula

$$\bigvee_{u \in U} u \tag{3.8}$$

where U is the set of meta-variables in each the multiset, then joining all the formulae from multisets in both A and B conjunctively.

One may realize that the conjunction of $\bigvee_{u_x \in U} u_x$ and $\bigwedge_{v_x \in V} v_x$ gives a tautology in the case when $V = U$, and thus for the instance of $UNSAT^+_{multiset}$ when $n = 5$, we would have a simpler propositional formula for the $UNSAT$ instance. This can also be observed in (3.6) where in order to satisfy $\forall b \in B \quad b \neq [0, 0, 0, 0, 0]$, we would have

$$\forall a \in A \qquad a \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0]\}$$

We thereby conclude that a plausible approach to proving the Erdös-Szekeres conjecture is by first solving for the instance of $UNSAT^+_{multiset}$ when $n = 5$ - apparently accomplishable with a modern SAT solver running on a high performance computer - and then proving that the unsatisfiability of an instance of $UNSAT^{+}_{multiset}$ where $n = k$ can be derived from the unsatisfiability of the instance of $UNSAT^{+}_{multiset}$ where $n = k - 1$ for all $k \in \mathbb{N}_{\geq 6}$.

Remark. One thing we may want to take note is that the Erdös-Szekeres conjecture would not be disproven even if one instance of $UNSAT^{+}_{multiest}$ turns out to be satisfiable. This is because satisfying the constraints only implies that there exists a *div point set* of $2^{n-2} + 1$ points for some $n \in \mathbb{N}_{\geq 5}$ where

I. none of its *sub div point sets* of n points is isomorphic to $Conv_n$

II. each of its *sub div point sets* of 5 points has 4, 2 or 0 distinct *sub div points set* of 4 points isomorphic to $Conc^1_4$

This does not mean we can be certain that it is an element in $\mathscr{DPS}^{+}$ unless it too satisfies the stronger version of *Theorem 2* (i.e. it is possible that one of its *sub div point set* of 5 points is not in $\mathscr{DPS}^{*}$, despite having 4, 2 or 0 distinct *sub div points set* of 4 points isomorphic to $Conc^1_4$).
Furthermore, even if we can show that (3.2) is false, we would still need to somehow demonstrate that there exists no other rules besides (2.10), (2.11), and (2.12) that a *div* point, (P, Θ_P), has to satisfy in order to have an interpretation for P (i.e. *Axiom 1*'s consistency with Euclidean geometry).

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