

# On reducing the Erdős-Szekeres problem into a constraint unsatisfiability problem regarding certain multisets

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## Abstract

We introduce the theory of *div point set*, which aims to provide a framework to study the combinatoric nature of any set of points in general position on an Euclidean plane. We then show that the Erdős-Szekeres conjecture can be proven through proving the unsatisfiability of some first-order logic formulae concerning some sets of 5-cardinal multisets over boolean variables under certain constraints.

## 1 Introduction

More than half a century ago Erdős and Szekeres [1] proved that for all  $n \geq 3$ , there exists an integer  $N$  such that among any  $N$  points in general position on an Euclidean plane, there always exists  $n$  points forming a convex polygon, and conjectured that the smallest number for  $N$  is determined by the function  $g(n) = 2^{n-2} + 1$ . This was known as the Erdős-Szekeres conjecture (and the problem of determining such  $N$  was often referred to the *Happy End Problem*, as it led to the marriage of Szekeres and Klein, who first proposed the question). 25 years after the initial paper, Erdős and Szekeres [2] showed that  $g(x)$  cannot be less than  $2^{n-2}$ . Currently the best known bounds for  $g(x)$  are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1$$

Many improvements for the upper bound have been made throughout the decades. The current upper bound was obtained by Tóth and Valtr [3] in 1998 as an improvement to the previous upper bound by Kleitman and Pachter [4] in the same year.

There are also attempts to verify individual instances of  $n$ . In 2002 Szekeres and Peters [5] showed using an exhaustive computer search that the conjecture holds for  $n = 6$ . Even to this day it remains the best known result. Rather than describing a computer Proof for  $n \geq 7$  or improving the upper bound, our aim in this article is to demonstrate that solving some instances of a certain multiset unsatisfiability problem would prove the Erdős and Szekeres conjecture, through the theory of *div point set*.

## 1.1 preliminary

Throughout the article we would assume Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). The term "class" would be used to denote a collection of sets satisfying some predicate  $\phi$ . A general form of Kuratowski definition would be used to define tuples. 2-tuples would be referred to as ordered pairs. A set of n-tuples would be referred to as relation (and described as binary relation when it is a set of ordered pairs). It would not matter as how natural numbers are defined as long as they satisfy Peano axioms.  $\mathbb{N}_{\geq c}$  would be used to refer to the set of natural numbers greater or equal to some  $c \in \mathbb{N}$  (e.g.  $0 \notin \mathbb{N}_{\geq 1}$ ). For any 2 natural numbers  $a, b$ ,  $\binom{a}{b}$  denotes the binomial coefficient  $a$  choose  $b$ . Everything would be formulated under FOL, or first order logic, ( $\wedge, \vee, \neg, \Rightarrow$  and  $\Leftrightarrow$  would mean *and*, *or*, *not*, *imply* and *iff* respectively). We write  $A := B$  if  $A$  is defined to be equivalent to  $B$ .  $\forall x_1 \in A \forall x_2 \in A \forall x_3 \in A \dots \forall x_n \in A$  would be abbreviated as

$$\forall x_1, x_2, x_3 \dots x_n \in A$$

and  $\exists x_1 \in A \exists x_2 \in A \exists x_3 \in A \dots \exists x_n \in A$  as

$$\exists x_1, x_2, x_3 \dots x_n \in A$$

For any set  $V$ ,  $|V|$  would be used to denote its cardinality, and  $\mathcal{P}(V)$  be used to denote its power set:

$$\mathcal{P}(V) = \{v : v \subseteq V\}$$

The subscript of a set union or set intersection may be omitted to indicate that union or intersection is applied to each element in the set:

For any set,  $A$ ,

$$\bigcup_{a \in A} A = \bigcup_{a \in A} a = a_1 \cup a_2 \cup \dots a_n$$

$$\bigcap_{a \in A} A = \bigcap_{a \in A} a = a_1 \cap a_2 \cap \dots a_n$$

where  $|A| = n$  and  $a_1, a_2, \dots a_n$  are all  $n$  distinct elements in  $A$

For any  $k$ -tuple  $T$ ,  $\pi_i(T)$  would be used to denote the  $i$ -th element of  $T$  where  $i, k \in \mathbb{N}$  and  $i \leq k$ ;  $\pi_{\cup}(T)$  would be used to denote the union of 1st, 2nd ...  $k$ -th elements of a  $k$ -tuple; and  $\pi_{\cap}(T)$  would be used to denote intersection in such fashion:

$$\begin{aligned} \pi_{\cup}(T) &= \bigcup_{i=1}^k \pi_i(T) \\ \text{For any } k\text{-tuple, } T, \\ \pi_{\cap}(T) &= \bigcap_{i=1}^k \pi_i(T) \end{aligned}$$

A single-argument function is any binary relation,  $f$ , satisfying

$$\begin{aligned}
&\forall x \in X \\
&\quad \exists r \in f = \pi_1(r) \\
&\forall r \in f \\
&\quad \pi_1(r) \in X \\
&\quad \pi_2(r) \in Y \\
&\forall r_1, r_2 \in f \\
&\quad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2)
\end{aligned}$$

for some none-empty sets  $X$  (often referred to as domain) and  $Y$  (referred to as co-domain). We often express the relationship between  $f$ ,  $X$ , and  $Y$  as:

$$f : X \longrightarrow Y$$

We write  $f(x) = y$  iff there exists an ordered pair  $(x, y)$  in  $f$ . A function is always assumed to be single-argument, unless otherwise stated. A function  $f$  is injective iff

$$\begin{aligned}
&\forall r_1, r_2 \in f \\
&\quad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2)
\end{aligned}$$

It is surjective iff

$$\begin{aligned}
&\forall y \in Y \\
&\quad \exists r \in f \quad y = \pi_2(r)
\end{aligned}$$

It is bijective iff it is both injective and surjective, in which case  $\xrightarrow{1:1}$  would be used to denote such property. To avoid ambiguity, for any function  $f : X \longrightarrow Y$ , we would use  $f^{members}$  to denote a new function, from  $\mathcal{P}(X)$  to  $\mathcal{P}(Y)$  such that

$$f^{members}(x) := \bigcup_{a \in x} \{f(a)\}$$

Here is a generalization of it,  $f^{members^n}$ , defined recursively:

$$\begin{aligned}
f^{members^n}(x) &:= \bigcup_{a \in x} \{f^{members^{n-1}}(a)\} \text{ where } n \in \mathbb{N}_{\geq 2} \\
f^{members^1}(x) &:= f^{members}(x)
\end{aligned}$$

A multiset is a generalization of set, where the same element can occur multiple times, making a difference. Two multisets are equal iff (1) both multisets contain the same distinct elements and (2) for each distinct element, it occurs the same number of times in

both multisets. A multiset is defined as an ordered pair  $(A, m_m)$  where  $m_m : A \rightarrow \mathbb{N}_{\geq 1}$  is a function that describes the number of occurrences of some element in the multiset, and  $A$  is a set of all distinct elements in the multiset. The cardinality of a multiset  $(A, m_m)$  is defined as the sum of all  $m_m(x)$  for  $x \in A$ . Multisets are expressed using square brackets,  $[]$ , as compared to sets which use curly brackets,  $\{\}$ . Here is an example:

$$[f(x) : x \in \mathbb{N}_{\geq 1} : x \leq 3] = [1, 1, 1] = (\{1\}, \{(1, 3)\})$$

where  $f(x) = 1$

A hypergraph is a generalization of graph, where an edge can contain any number of vertices. It is defined as an ordered pair  $(V, E)$  where  $E$  is a subset of  $\mathcal{P}(V) \setminus \emptyset$ . Elements in  $V$  are referred to as vertices while elements in  $E$  are referred to as edges or hyperedges. A hypergraph is  $k$ -uniformed when

$$\forall e \in E \quad |e| = k$$

where  $k \in \mathbb{N}_{\geq 1}$ . A full vertex coloring on some graph or hypergraph,  $(V, E)$ , is defined as a function,  $C : V \rightarrow cDom$ , such that

$$\begin{aligned} |C| &= |V| \\ \forall c \in C \quad \pi_1(c) \in V \wedge \pi_2(c) \in cDom \\ \forall c_1, c_2 \in C \quad c_1 = c_2 &\Leftrightarrow \pi_1(c_1) = \pi_1(c_2) \end{aligned}$$

where  $cDom \subset \mathbb{N}$ , and it is often referred to as the set of colors. When  $|Dom| = 2$ , we say the coloring is monochromatic. We would use  $FullCol(G, cDom)$  to denote the set of all possible full vertex colorings on a graph  $G$  of the set of colors  $cDom$ . For any graph  $G$  of  $n$  vertices, and any non-empty  $cDom$ ,

$$|FullCol(G, cDom)| = n^{|cDom|}$$

SAT, or the Boolean Satisfiability Problem, is the problem of determining if there exists some value-assignment for the variables in a propositional logic formula such that it yields *True* (i.e. it is satisfiable).

A propositional logic formula is referred to as a tautology when there exists no value-assignment for the variables in it such that it yields *False* (e.g.  $a \vee \neg a$ ).

A propositional logic formula is in DNF, or Disjunctive Normal Form, when it is a disjunction of conjunctions. A disjunction is a propositional logic formula,  $F$ , that can be expressed as  $\bigwedge_{F_s \in S} F_s$  where  $S$  is a set of formulae, while a conjunction is a propositional logic formula,  $F$ , that can be expressed as  $\bigvee_{F_s \in S} F_s$  where  $S$  is a set of formulae.

## 2 *Div point set* as a representation for any set of points in general position

We start off by introducing an object which we would be referring to as *div point set*.

**Definition 1.** A *div point set* is any order-pair  $(P, \Theta_P)$  satisfying

$$|\Theta_P| = \binom{|P|}{2} \wedge P \neq \emptyset \quad (2.1)$$

$$\forall D_n \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ |d_n| = 2 \\ d_n \in \mathcal{P}(P) \\ |\delta_n| = 2 \\ \bigcup \delta_n = P \setminus d_n \\ \bigcap \delta_n = \emptyset \end{array} \right. \quad (2.2)$$

$$\forall D_n, D_m \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ (d_m, \delta_m) := D_m \\ d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \quad (2.3)$$

We would be using  $\mathcal{DPS}^*$  to denote the class of all ordered pairs satisfying (2.1), (2.2) and (2.3). Thus  $\mathcal{X}$  is a *div point set* iff  $\mathcal{X} \in \mathcal{DPS}^*$ .

For any  $n$  points in general position, where  $n \geq 2$ , we can always select any 2 arbitrary points and draw a line across them, dividing the rest of the points into 2 disjoint sets. So long as the points are in general position, we can be sure that no 3 points forms a line, and thus each of the remaining  $n - 2$  points would always be in one of these sets. Let's refer to these 2 disjoint sets as *divs* produced by a *divider* made up of 2 distinct points, and the points in the *divs* as *TBD points* of the *divider* (*TBD* is short for *to-be-distributed-among-divs*). The process of selecting 2 distinct points from a set of point  $P$ , creating a *divider*, and producing 2 *divs* can be repeated  $\binom{|P|}{2}$  times until all sets of 2 points in  $P$  are selected.

Any set of points  $P$  in general position on an Euclidean plane where  $|P| \geq 2$  can be represented by some *div point set*  $(P, \Theta_P)$ . Each member of  $D_n \in \Theta_P$  would be referred to as a *dividon*, to be interpreted as follows:

$$(d_n, \delta_n) := D_n \quad \left| \begin{array}{l} \{a, b\} := d_n \\ a \text{ and } b \text{ represent the 2 points which make up the divider} \\ \{div_1, div_2\} := \delta_n \\ div_1 \text{ and } div_2 \text{ represent the 2 divs produced by the divider} \\ \bigcup \delta_n \text{ thus represents the set of TBD points of the divider} \end{array} \right. \quad (2.4)$$

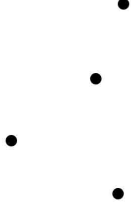


Figure I



Figure II



Figure III

The sets of points in *Figures I, II and III* can be represented by any *div point set*  $(A, \Theta_A)$  as long as  $A$  is a set of 4 arbitrary elements  $a, b, c, d$  and

$$\begin{aligned} \Theta_A = & \{(\{a, b\}, \{(\{c\}, \{d\})\}), \\ & (\{a, c\}, \{(\{b\}, \{d\})\}), \\ & (\{a, d\}, \{(\{b\}, \{c\})\}), \\ & (\{b, c\}, \{(\{a, d\}, \emptyset)\}), \\ & (\{b, d\}, \{(\{a, c\}, \emptyset)\}), \\ & (\{c, d\}, \{(\{a, b\}, \emptyset)\})\} \end{aligned}$$

To make sense of the *div point set* representation, we label the third point from the bottom in *Figure I* and the second point from the bottom in *Figures II and III* as  $a$  (note that each of these is the point surrounded by the remaining 3 points in the figure). For the rest of the points in each figure we simply label them arbitrarily as  $b, c$ , and  $d$ .

Only a handful of *div point sets* can be used to represent points in general position in  $\mathbb{E}^2$ . For majority of  $\mathcal{X} \in \mathcal{DPS}^*$ , let  $(P, \Theta_P) := \mathcal{X}$ , there exists no meaningful interpretation for  $P$  as some sets of points in  $\mathbb{E}^2$  such that each  $D \in \Theta_P$  is a *dividon* that describes how *TBD points* are distributed between the 2 *divs* produced by each *divider*. A classical example would be  $(Q, \Theta_Q)$  where  $Q$  is a set of 4 arbitrary elements  $a, b, c, d$  and

$$\begin{aligned} \Theta_Q = & \{(\{a, b\}, \{(\{c, d\}, \emptyset)\}), \\ & (\{a, c\}, \{(\{b, d\}, \emptyset)\}), \\ & (\{a, d\}, \{(\{b, c\}, \emptyset)\}), \\ & (\{b, c\}, \{(\{a, d\}, \emptyset)\}), \\ & (\{b, d\}, \{(\{a, c\}, \emptyset)\}), \\ & (\{c, d\}, \{(\{a, b\}, \emptyset)\})\} \end{aligned}$$

For a *div point set*  $(P, \Theta_P)$  to have a meaningful interpretation for  $P$  as some set of points in  $\mathbb{E}^2$ , it has to satisfy certain conditions. For any 3 distinct points,  $x$ ,  $y$ , and  $z$  in general position in  $\mathbb{E}^2$ , let  $\langle x, y \rangle^z$  denote the *div* containing  $z$  produced by the *divider* made up of the point  $x$  and  $y$ , and  $\langle x, y \rangle^{-z}$  denote the *div* not containing  $z$  produced by the *divider*. After some experimentation with points in  $\mathbb{E}^2$ , we would make the observation that the following formulas always hold true for any distinct points  $a, b, c, d$  in  $\mathbb{E}^2$ . (2.5) is trivially true, while (2.6), (2.7) and (2.8) are demonstrated in *Figures IV, V and VI* respectively.

$$\begin{aligned} \forall a, b, c, d \\ a \in \langle b, c \rangle^d \Leftrightarrow d \in \langle b, c \rangle^a \\ a \in \langle b, c \rangle^{-d} \Leftrightarrow d \in \langle b, c \rangle^{-a} \end{aligned} \quad (2.5)$$

$$\begin{aligned} \forall a, b, c, d \\ c \in \langle a, b \rangle^{-d} \\ \Leftrightarrow ((a \in \langle b, c \rangle^d \wedge a \in \langle b, d \rangle^c) \\ \vee (a \in \langle b, c \rangle^{-d} \wedge a \in \langle b, d \rangle^{-c})) \end{aligned} \quad (2.6)$$

$$\begin{aligned} \forall a, b, c, d \\ c \in \langle a, b \rangle^d \\ \Leftrightarrow ((a \in \langle b, c \rangle^d \wedge a \in \langle b, d \rangle^{-c}) \\ \vee (a \in \langle b, c \rangle^{-d} \wedge a \in \langle b, d \rangle^c)) \end{aligned} \quad (2.7)$$

$$\begin{aligned} \forall a, b, c, d \\ a \in \langle b, c \rangle^{-d} \wedge a \in \langle b, d \rangle^{-c} \Rightarrow a \in \langle c, d \rangle^b \end{aligned} \quad (2.8)$$

In the context of *div point sets*, (2.5) is always true by (2.2) (recall  $\bigcap \delta = \emptyset$ ), while (2.6), (2.7) and (2.8) can be rewritten as constraints on the *dividons* of a *div point set* as shown in (2.10), (2.11), and (2.12), using a function,  $\phi$ , for determining if two arbitrary points belong to the same *div* in some  $\delta$  of a *dividon*:

$$\phi(\delta, w) = \begin{cases} 1 & \text{if } (a \in \text{div}_1 \wedge b \in \text{div}_2) \Leftrightarrow \text{div}_1 = \text{div}_2 \\ 0 & \text{if } (a \in \text{div}_1 \wedge b \in \text{div}_2) \Leftrightarrow \text{div}_1 \neq \text{div}_2 \end{cases} \quad \text{where} \quad \left| \begin{array}{l} \delta = \{\text{div}_1, \text{div}_2\} \\ w = \{a, b\} \end{array} \right. \quad (2.9)$$

For any *div point set*  $(P, \Theta_P)$ ,

$$\forall p_1, p_2, p_3, p_4 \in P$$

$$R := \bigcup_{n=1}^4 \{p_n\}$$

$$|R| = 4$$

$$\Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P$$

$$(2.10)$$

$$\bigcup_{n=1}^3 \pi_1(D_n) = R \wedge \bigcap_{n=1}^3 \pi_1(D_n) = \{p_4\}$$

$$\Rightarrow (\phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 1$$

$$\Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) )$$

$$\forall p_1, p_2, p_3, p_4 \in P$$

$$R := \bigcup_{n=1}^4 \{p_n\}$$

$$|R| = 4$$

$$\Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P$$

$$(2.11)$$

$$\bigcup_{n=1}^3 \pi_1(D_n) = R \wedge \bigcap_{n=1}^3 \pi_1(D_n) = \{p_4\}$$

$$\Rightarrow (\phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 0$$

$$\Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) \neq \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) )$$

$$\forall p_1, p_2, p_3, p_4 \in P$$

$$R := \bigcup_{n=1}^4 \{p_n\}$$

$$|R| = 4$$

$$\Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P$$

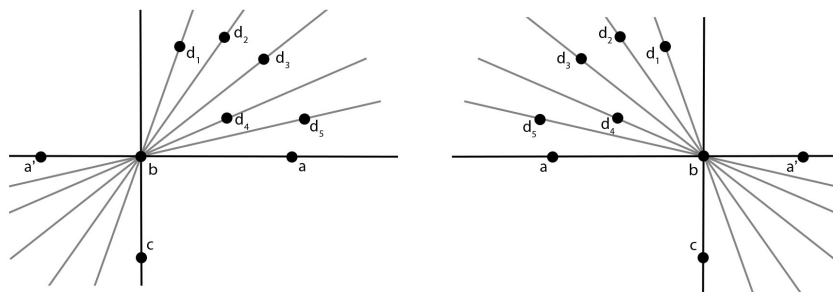
$$(2.12)$$

$$\bigcap_{n=1}^2 \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^2 \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3)$$

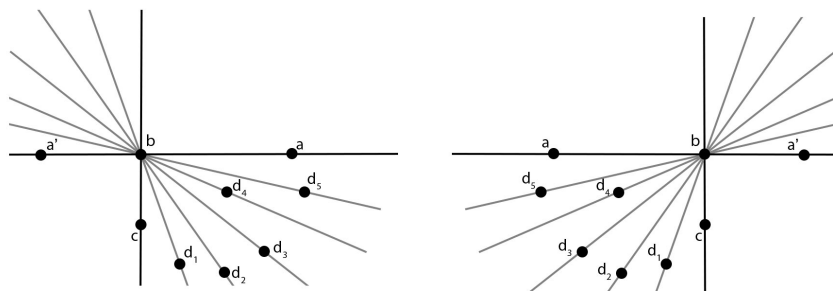
$$\Rightarrow (\phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = 0$$

$$\Rightarrow \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) = 1 )$$

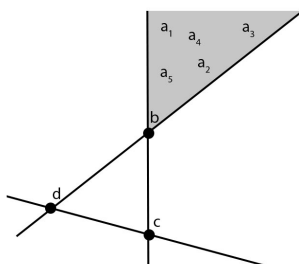




*Figure IV*



*Figure V*



*Figure VI*

**Axiom 1.** A *div point sets*  $(P, \Theta_P)$  has an interpretation for  $P$  as some set of points in  $\mathbb{E}^2$  such that  $D \in \Theta_P$  each describes the relative positions of the points (in terms of how the *TBD points* of each *divider* is distributed between 2 *divs* it produced) iff it is in  $\mathcal{DPS}^+$ , the class of *div point sets* satisfying (2.10), (2.11), and (2.12).

**Remark.** For *div point sets* of 3 or less points, it is vacuously true that they satisfy (2.10), (2.11), and (2.12) and thus they are by default in the class  $\mathcal{DPS}^+$ . This is consistent with Euclidean geometry: any set of 3 points in general position can be represented by any *div point set* of 3 points, and the same goes to any set of 2 points, and any set of 1 point.

**Definition 2.** We say that two *div point sets*  $(A, \Theta_A)$  and  $(B, \Theta_B)$  are isomorphic iff there exists a bijective function  $f : A \xrightarrow{1:1} B$  which preseves the structure of the *divisions*. Notationally,

$$\begin{aligned}
 (A, \Theta_A) \cong (B, \Theta_B) &\Leftrightarrow \exists f : A \xrightarrow{1:1} B \\
 &\quad \forall D_A \in \Theta_A \\
 &\quad \quad \exists D_B \in \Theta_B \\
 &\quad \quad (d_a, \delta_a) := D_A \\
 &\quad \quad (d_b, \delta_b) := D_B \\
 &\quad \quad f^{members}(d_a) = d_b \Leftrightarrow f^{members^2}(\delta_a) = \delta_b
 \end{aligned} \tag{2.13}$$

in which case  $f$  would be referred to as the isomorphism between the two sets.

**Remark.** It is trivially true that all *div point sets*  $(P, \Theta_P)$  in  $\mathcal{DPS}^*$  where  $|P| \leq 3$  are isomorphic to any *div point sets*  $(Q, \Theta_Q)$  in  $\mathcal{DPS}^*$  where  $|Q| = |P|$ .

**Theorem 1.**  $\neg(\mathcal{X} \cong Conc_4^1) \Leftrightarrow (\mathcal{X} \cong Conv_4)$  for all  $\mathcal{X} \in \mathcal{DPS}_4^+$  where  $\mathcal{DPS}_4^+$  denotes the *div point sets* of 4 points in  $\mathcal{DPS}^+$  and

$$\begin{aligned}
 Conc_4^1 &= (Cc_4^1, \Theta_{Cc_4^1}) & Conv_4 &= (Cv_4, \Theta_{Cv_4}) \\
 Cc_4^1 &= \{1, 2, 3, 4\} & Cv_4 &= \{1, 2, 3, 4\} \\
 \Theta_{Cc_4^1} &= \{(\{1, 2\}, \{\{3\}, \{4\}\}), & \Theta_{Cv_4} &= \{(\{1, 2\}, \{\{3, 4\}, \emptyset\}), \\
 &\quad (\{1, 3\}, \{\{2\}, \{4\}\}), & &\quad (\{1, 3\}, \{\{2\}, \{4\}\}), \\
 &\quad (\{1, 4\}, \{\{2\}, \{3\}\}), & &\quad (\{1, 4\}, \{\{2, 3\}, \emptyset\}), \\
 &\quad (\{2, 3\}, \{\{1, 4\}, \emptyset\}), & &\quad (\{2, 3\}, \{\{1, 4\}, \emptyset\}), \\
 &\quad (\{2, 4\}, \{\{1, 3\}, \emptyset\}), & &\quad (\{2, 4\}, \{\{1\}, \{3\}\}), \\
 &\quad (\{3, 4\}, \{\{1, 2\}, \emptyset\})\} & &\quad (\{3, 4\}, \{\{1, 2\}, \emptyset\})\}
 \end{aligned} \tag{2.14}$$

*Proof for Theorem 1.*

**Summary.** In Part 1 of the proof we would define a function  $\psi$  that returns 0 or 1 based on the *divs* of a *dividon* of some *div point set* in  $\mathcal{DPS}_4^+$ . In Part 2 we would define  $\mathcal{DPS}_4^{\mathbb{N}}$  and a function  $Col$  that uses  $\psi$ , and show that for every  $\mathcal{X} \in \mathcal{DPS}_4^{\mathbb{N}}$ , there exists a unique full vertex monochromatic coloring  $Col(\pi_2(\mathcal{X}))$  on some hypergraph  $H$ , where the vertices of  $H$  are the dividers of the *div point sets* in  $\mathcal{DPS}_4^{\mathbb{N}}$ . In Part 3 we would define the edges of  $H$  in such a manner that the coloring  $Col(\pi_2(\mathcal{X}))$  on  $H$  satisfies some conditions iff  $\mathcal{X}$  satisfies (2.10) and (2.11). In Part 4 we demonstrate that for the coloring to satisfy the conditions, there exists only 3 *Scenarios*, and colorings in *Scenario* 2 and 3 are isomorphic to  $Col(\pi_2(Conc_4^1))$  and  $Col(\pi_2(Conv_4))$ , and  $Conc_4^1$  and  $Conv_4$  satisfy (2.12), but not the other *div point set* the coloring in *Scenario* 1 is based on, and thus proving Theorem 1.

**Part 1.** For any *div point set*  $(P, \Theta_P)$  in  $\mathcal{DPS}_4^+$ , since  $|P| = 4$ , we can be certain that

$$\begin{aligned} \forall D \in \Theta_P \\ \pi_2(D) &\in \{type_0, type_1\} \\ \{a, b\} &= P \setminus \pi_1(D) \\ type_0 &= \{\{a\}, \{b\}\} \\ type_1 &= \{\{a, b\}, \emptyset\} \end{aligned} \tag{2.15}$$

Recall that in (2.9), we define a function  $\phi$  that takes in some  $\pi_2(D)$  and a set of 2 *TBD points*, and returns 1 if the *TBD points* belong to the same *div* in  $\pi_2(D)$ , or 0 if they belong to different *divs* in  $\pi_2(D)$ . For  $\mathcal{X} \in \mathcal{DPS}_4^+$ , we can define a new function  $\psi$ , a simpler version of  $\phi$  that does basically the same thing by exploiting (2.15), namely the fact every  $\pi_2(D)$  is either  $type_0$  or  $type_1$  (since that there are only 2 *TBD points* for each divider):

$$\psi(\delta) = \begin{cases} 1 & \text{if } \exists div \in \delta \quad |div| = 2 \\ 0 & \text{if } \forall div \in \delta \quad |div| = 1 \end{cases} \tag{2.16}$$

For every *dividon*  $D$  of any  $\mathcal{X} \in \mathcal{DPS}_4^+$ , we have

$$\phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \tag{2.17}$$

**Part 2.** Let's define  $\mathcal{DPS}_4^{\mathbb{N}}$  to be the set of all *div point sets*  $(P, \Theta_P)$  for which  $P = \{1, 2, 3, 4\}$ . All  $\mathcal{X} \in \mathcal{DPS}_4^{\mathbb{N}}$  would have the same *dividers* (Recall the set of *dividers* is just the set of elements in  $\mathcal{P}(P)$  whose cardinality is 2.) Now let  $H = (V, E)$  be a hypergraph whose vertices are the *dividers* of *div point sets* in  $\mathcal{DPS}_4^{\mathbb{N}}$ . Using  $\psi$ , we can define a bijective function,  $Col$ , that transforms the set of *dividons* of a *div point set* in  $\mathcal{DPS}_4^{\mathbb{R}}$  into some full vertex monochromatic coloring for  $H$ .

$$\begin{aligned} Col : \{\pi_2(\mathcal{X}) : \mathcal{X} \in \mathcal{DPS}_4^{\mathbb{N}}\} &\longrightarrow FullCol(H, \{0, 1\}) \\ Col(\Omega_P) &= \{(\pi_1(D), \psi(\pi_2(D))) : D \in \Omega_P\} \end{aligned} \tag{2.18}$$

It is bijective because

$$\begin{aligned} \forall \mathfrak{X}_1, \mathfrak{X}_2 \in \mathcal{DPS}^{\mathbb{R}} \\ Col(\pi_2(\mathfrak{X}_1)) = Col(\pi_2(\mathfrak{X}_2)) \Leftrightarrow \mathfrak{X}_1 = \mathfrak{X}_2 \end{aligned} \quad (2.19)$$

due to the fact that  $\psi$  is bijective for every dividon of any *div point set* of 4 points.

**Part 3.** Now let's define any set of three *dividers* containing 1 element in common to be an edge of  $H$  (recall that the vertices are the *dividers*), notationally,

$$E = \{e \in \mathcal{P}(V) : |e| = 3 \wedge |\bigcap e| = 1\} \quad (2.20)$$

$H$  is a 3-uniform hypergraph with 4 hyperedges. For  $\mathfrak{X} \in \mathcal{DPS}_4^{\mathbb{N}}$  to satisfy (2.10) and (2.11) is equivalent to having  $Col(\pi_2(\mathfrak{X})) \in FullCol(H, \{0, 1\})$  to satisfy the following:

- I. If a vertex,  $V$ , is colored 0, the other 2 vertices belonging to the same edge as  $V$  must have the same coloring.
- II. If a vertex,  $V$ , is colored 1, the other 2 vertices belonging to the same edge as  $V$  must have different colorings.

This is due to the fact, for any  $\mathfrak{X} \in \mathcal{DPS}_4^{\mathbb{N}}$ , (2.10) and (2.11) can be rewritten as having the colors on the vertices of each edge to satisfy some formulae, namely the following:

$$\begin{aligned} \forall e \in E \\ \forall d_1, d_2, d_3 \in e \\ d_1 \neq d_2 \neq d_3 \\ \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_1) = 1 \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_2) = Col(\pi_2(\mathfrak{X}))(d_3) \end{aligned} \quad (2.21)$$

$$\begin{aligned} \forall e \in E \\ \forall d_1, d_2, d_3 \in e \\ d_1 \neq d_2 \neq d_3 \\ \Leftrightarrow (Col(\pi_2(\mathfrak{X}))(d_1) = 0 \Leftrightarrow Col(\pi_2(\mathfrak{X}))(d_2) \neq Col(\pi_2(\mathfrak{X}))(d_3)) \end{aligned} \quad (2.22)$$

(recall that  $Col$  is the function to transform some  $\pi_2(\mathfrak{X})$  into a coloring, while  $Col(\pi_2(\mathfrak{X}))$  is the actual coloring, which is defined as a function in *Preliminary*) This is a result of

$$\begin{aligned} \forall p_1, p_2, p_3, p_4 \in P \\ R := \bigcup_{n=1}^4 \{p_n\} \\ \forall D \in \Theta_P \\ \phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \end{aligned} \quad (2.23)$$

for any div point set  $(P, \Theta_P)$  for which  $|P| = 4$  (recall (2.15)), and any *dividons*  $D_1, D_2$  and  $D_3$  where

$$|\bigcap_{n=1}^3 \pi_1(D_n)| = 1 \wedge |\bigcup_{n=1}^3 \pi_1(D_n)| = 4 \quad (2.24)$$

would always have the *dividers*  $d_1, d_2$  and  $d_3$  respectively, where

$$|\bigcap_{n=1}^3 d_n| = 1 \wedge d_1 \neq d_2 \neq d_3 \quad (2.25)$$

which are precisely what makes up an edge in  $E$  (recall (2.20)). Therefore a *div point set* of 4 points,  $\mathfrak{X}$ , satisfies (2.10) and (2.11) iff  $Col(\pi_2(\mathfrak{X}))$  satisfies *I* and *II*.

**Part 4.** To satisfy I and II, 3 vertices belonging to the same edge must either be colored  $[0, 0, 0]$  or  $[0, 1, 1]$ .

Suppose we start off by giving some vertices belonging to the same edge the coloring of  $[0, 0, 0]$ , by I this would indicate that the rest of the vertices need to have the same colors (recall that each vertex belongs to 2 different edges). We can either end up with H having all vertices colored 0 (let's call it *Scenario 1*), or 3 vertices colored 0 and 3 vertices colored 1 (let's call it *Scenario 2*).

Now suppose we start off by giving some vertices belonging to the same edge the coloring of  $[0, 1, 1]$ , by I this would indicate that the remaining 2 vertices of another edge, which the vertex colored 0 belongs to, must have the same colors. If we give them the coloring of  $[0, 0]$ , we would have an edge with vertices colored  $[0, 0, 0]$ , and the last uncolored vertex must then be colored 1, so we end up in *Scenario 2* again. If we give them the coloring of  $[1, 1]$ , we would end up with 1 vertex colored 0 and 4 vertices colored 1, in which case the last uncolored vertex would need to be colored 0, since it belongs to 2 edges both with 2 vertices colored 1. Let's name this *Scenario 3*, where 2 vertices are colored 0 and 4 vertices are colored 1.

A pictorial description of the colorings is shown in Figure VII.

Scenario 1 describes a coloring isomorphic to  $Col(\pi_2(\mathfrak{X}_\emptyset))$  where  $\mathfrak{X}_\emptyset \in \mathcal{DPS}_4^{\mathbb{N}}$  and

$$\begin{aligned} \pi_2(\mathfrak{X}_\emptyset) = & \{(\{1, 2\}, \{(\{3, 4\}, \emptyset)\}), \\ & (\{1, 3\}, \{(\{2, 4\}, \emptyset)\}), \\ & (\{1, 4\}, \{(\{2, 3\}, \emptyset)\}), \\ & (\{2, 3\}, \{(\{1, 4\}, \emptyset)\}), \\ & (\{2, 4\}, \{(\{1, 3\}, \emptyset)\}), \\ & (\{3, 4\}, \{(\{1, 2\}, \emptyset)\}) \} \end{aligned}$$

while Scenario 2 describes a coloring isomorphic to  $Col(\pi_2(Conc_4^1))$  and scenario 3 describes a coloring isomorphic to  $Col(\pi_2(Conv_4))$ .  $Conc_4^1$  and  $Conv_4$  both satisfy (2.12), and  $\mathfrak{X}_\emptyset$

does not. Since any div point set of 4 points is isomorphic to some  $\mathfrak{X} \in \mathcal{DPS}_4^N$ , and only  $\text{Conc}_4^1$  and  $\text{Conv}_4$  satisfy all (2.10), (2.11), and (2.12), we conclude that

$$\forall X \in \mathcal{DPS}_4^+ \quad \exists a \in \{\text{Conc}_4^1, \text{Conv}_4\} \quad X \cong a$$

□

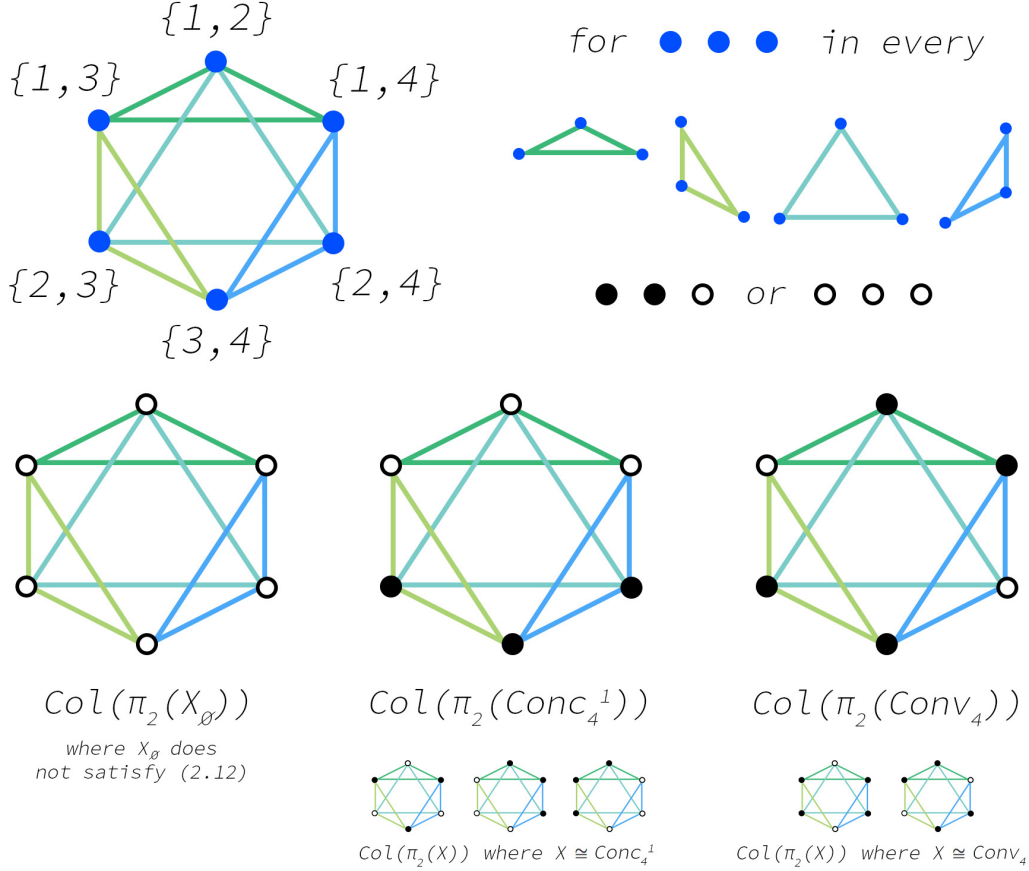


Figure VII

**Remark.** In Euclidean geometry, Theorem 1 can be interpreted as stating that for any set of 4 distinct points in general positions, it is either the case that it forms a structure where 1 point is inside a triangle formed by connecting the rest of 3 points, or the case that a convex polygon can be created by connecting the 4 points in a certain manner, which can be verified rather easily by a human child with a pen, a piece of paper and a love for Euclidean geometry.

## 2.1 *unit div point set and sub div point set*

For *div point sets* of 5 or more points, the function  $\psi$  defined in (2.16) would not be really useful since there would be 3 or more *TBD points* in each *dividon*. That means we cannot apply to same technique above to derive *div point sets* of 5 or more points satisfying (2.10), (2.11) and (2.12). With that in mind, we introduce the object *unit div point set* which makes use of *unit dividons*.

**Definition 3.** A *unit div point set* is any order-pair  $(P, \Omega_P)$  satisfying (2.26), (2.27) and (2.28).

$$|\Omega_P| = \binom{|P|}{2} \binom{|P|-2}{2} \wedge P \neq \emptyset \quad (2.26)$$

$$\forall D_n \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ |d_n| = 2 \\ d_n \in \mathcal{P}(P) \\ |\delta_n| = 2 \\ |\bigcup \delta_n| = 2 \\ \bigcup \delta_n \in \mathcal{P}(P \setminus d_n) \\ \bigcap \delta_n = \emptyset \end{array} \right. \quad (2.27)$$

$$\forall D_n, D_m \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ (d_m, \delta_m) := D_m \\ d_n \cup \bigcup \delta_n = d_m \cup \bigcup \delta_m \Leftrightarrow D_n = D_m \end{array} \right. \quad (2.28)$$

We would be using  $\mathcal{UDPS}^*$  to denote the class of all *unit div point set*.

**Remark.** Similar to how *div point sets* of 4 points always satisfy (2.15), a *unit div point set* always satisfies (2.29).

$$\begin{aligned} \forall \mathfrak{X} \in \mathcal{UDPS}^* \\ (P, \Omega_P) &:= \mathfrak{X} \\ \forall D \in \Omega_P \\ \pi_2(D) &\in \{type_0, type_1\} \\ \{a, b\} &\subseteq P \setminus \pi_1(D) \\ type_0 &= \{\{a\}, \{b\}\} \\ type_1 &= \{\{a, b\}, \emptyset\} \end{aligned} \quad (2.29)$$

For any *unit div point set*,  $(P, \Omega_P)$ , we can use  $\psi$  (defined in (2.16)) to map every  $\pi_2(D) \in \Omega_P$  to some  $k \in \{0, 1\}$ .

**Remark.** One may immediately notice that any *div point sets* of 4 points also satisfy (2.26), (2.27) and (2.28), and any *unit div point set* of 4 points also satisfy (2.1), (2.2) and (2.3), and that is the say

$$\{\mathcal{X}_{udps} \in \mathcal{UDPS}^* : |\pi_1(X)| = 4\} = \{\mathcal{X}_{dps} \in \mathcal{DPS}^* : |\pi_1(X)| = 4\} \quad (2.30)$$

by virtue of the fact that

$$\binom{|4|}{2} \binom{|4-2|}{2} = \binom{|4|}{2} \quad (2.31)$$

and

$$\forall \mathcal{X} \in \mathcal{UDPS}^* \quad (2.32)$$

$$|\pi_1(\mathcal{X})| = 4 \Rightarrow \left\{ \begin{array}{l} \forall D_n \in \pi_2(\mathcal{X}) \\ \delta_n = P \setminus d_n \\ \forall D_n, D_m \in \pi_2(\mathcal{X}) \\ d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \quad (2.33)$$

As we can see, the difference between a *div point set* and a *unit div point set* lies in that the former relies on a single *dividon* to describe the distribution of  $|P| - 2$  *TBD points* between the 2 *divs* produced by a *divider*, while the later relies on  $\binom{|P|-2}{2}$  *unit dividons* for that (since each *unit dividon* only describes the distribution of 2 *TBD points*). For every  $\mathcal{X}_{dps} \in \mathcal{DPS}^*$  there exists a unique  $\mathcal{X}_{udps} \in \mathcal{UDPS}^*$  which  $\mathcal{X}_{dps}$  can be transformed into, by breaking down each *dividon* into  $\binom{|P|-2}{2}$  *unit dividons*, achievable using the function  $\mathfrak{b-d}$  defined below.

$$\begin{aligned} \mathfrak{b-d}(D, P) &= \{(\pi_1(D), d_u(x, \pi_2(D)) : x \in \mathcal{P}(P \setminus \pi_1(D)) : |x| = 2\} \\ d_u(x, divs) &= \begin{cases} \{x, \emptyset\}, & \text{if } x \subseteq divs \\ \{\{a\}, \{b\}\}, & \text{if } a \in div_1 \wedge b \in div_2 \\ \text{where } x = \{a, b\} \text{ and } divs = \{div_1, div_2\} \end{cases} \end{aligned} \quad (2.34)$$

**Definition 4.** The function  $\mathfrak{F}_{udps}^{\mathcal{DPS}}$  transforms a *div point set* into a *unit div point set*.

$$\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathcal{X}_{dps}) = (\pi_1(\mathcal{X}_{dps}), \bigcup \{\mathfrak{b-d}(D, \pi_1(\mathcal{X}_{dps})) : D \in \pi_2(\mathcal{X}_{dps})\}) \quad (2.35)$$

$\mathfrak{F}_{udps}^{\mathcal{DPS}}$  can be implemented in Haskell as follows:

---

```
import Control.Monad
import Data.List ((\\))
powerList = filterM (const [True, False])
```



```

f:: ([Int],[([Int],[[Int]])]) -> ([Int],[([Int],[[Int]])])
f (points,dividons) = (points,unit_dividons)
  where
    unit_dividons = foldl (++) [] $ map get_unit_dividons dividons
    get_unit_dividons (d,(delta1:_)) = [(d,(\(a:b:_)->
      if a 'in_same_div_as_b' b
        then [[a,b],[]]
        else [[a],[b]])
      x ) |
      x <- powerList (points \ d), length x == 2,
      let (in_same_div_as_b) a b = (a 'elem' delta1) == (b 'elem' delta1)]

```

---

**Remark.** If we apply  $\mathcal{F}_{u d p s}^{\mathcal{D P S}}$  on *div point sets* of 4 points we would immediately realize that  $\mathcal{F}_{u d p s}^{\mathcal{D P S}}$  returns the same ordered pair, since for *div point sets* of 4 points,  $\Omega_{sub} \subset \Omega_P$  in (2.35) would contain only one element and the element is some  $D_\Theta \in \Theta_P$ . For *div point sets* of 5 or more points  $\Omega_{sub}$  would contain 3 or more elements, thus

$$\forall \mathcal{X} \in \mathcal{D P S}^* \quad \mathcal{F}_{u d p s}^{\mathcal{D P S}}(\mathcal{X}) = \mathcal{X} \Leftrightarrow |\pi_1(\mathcal{X})| = 4 \quad (2.36)$$

On the other hand, applying  $\mathcal{F}_{u d p s}^{\mathcal{D P S}}$  on *div point sets* with 3 or less points would result in  $(P, \emptyset)$  since  $\binom{n-2}{2} = 0$  for  $n < 4$  and that is not going to be useful. So it is more sensible to define  $\mathcal{F}_{u d p s}^{\mathcal{D P S}}$  over *div point sets* of 4 or more points i.e.

$$\mathcal{F}_{u d p s}^{\mathcal{D P S}} : \mathcal{D P S}_{\geq 4}^* \longrightarrow \mathcal{U D P S}^* \quad (2.37)$$

**Lemma 1.**  $\mathcal{F}_{u d p s}^{\mathcal{D P S}} : \mathcal{D P S}_{\geq 4}^* \longrightarrow \mathcal{U D P S}^*$  is injective but not surjective. If the codomain is defined to be  $\mathcal{U D P S}^\Theta$ , the set of *unit div point sets* of 4 or more points satisfying (2.38),  $\mathcal{F}_{u d p s}^{\mathcal{D P S}}$  is then bijective.

$$\forall D_1, D_2, D_3 \in \Omega_P$$

$$\begin{aligned}
& (D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge \left| \bigcup_{n=1}^3 \pi_2(D_n) \right| = 3) \\
& \Rightarrow (\psi(\pi_2(D_1)) = 1 \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))) \\
& \quad \wedge (\psi(\pi_2(D_1)) = 0 \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3)))
\end{aligned} \quad (2.38)$$

*Proof for Lemma 1.* It is injective because in (2.35)  $\Omega_{sub}$  differs depending on  $D \in \Theta_P$  as a result of  $d_u$  in  $\mathfrak{b}d$  being injective (recall (2.34)). It is not surjective onto the co-domain  $\mathcal{U D P S}^*$ , but surjective onto the co-domain  $\mathcal{U D P S}^\Theta$ , as a consequence of

- I.  $|\delta_n| = 2$  in (2.2) and there exists  $\mathcal{X}_{u dp3} \in \mathcal{UDPS}^*$ , where  $\mathcal{F}_{u dp3}^{\mathcal{DPS}}(\mathcal{W}') = \mathcal{X}_{u dp3}$  iff  $\mathcal{W}'$  is not in  $\mathcal{DPS}^*$ . E.g. *Unit div point sets* with *unit dividons* such as

$$\{(a, b), (\{c\}, \{d\})\}, \{(a, b), (\{c\}, \{e\})\}, \{(a, b), (\{e\}, \{d\})\}$$

can only be transformed from a *div-point-set*-like ordered pair where  $|\delta_n| = 3$  for some *division*, in this case:

$$\{(a, b), (\{c\}, \{d\}, \{e\})\}$$

which is to say, for any  $\mathcal{X}_{u dp3} \in \mathcal{UDPS}^*$ , where  $\mathcal{X}_{u dp3} = \mathcal{F}_{u dp3}^{\mathcal{DPS}}(\mathcal{X}_{dp3})$  for some  $\mathcal{X}_{dp3} \in \mathcal{DPS}^*$ ,  $\mathcal{X}_{u dp3}$  satisfies

$$\forall D_1, D_2, D_3 \in \Omega_P$$

$$\begin{aligned} D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge \left| \bigcup_{n=1}^3 \pi_2(D_n) \right| = 3 \quad (2.39) \\ \Rightarrow \neg(\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = \psi(\pi_2(D_3)) = 0) \end{aligned}$$

- II. Associativity: if  $a$  and  $b$  are in the same *div*, and  $b$  and  $c$  are in the same *div*,  $a$  and  $c$  must be in the same *div*. So unit *div* points set with *unit dividons* such as

$$\{(a, b), (\{c, d\}, \emptyset)\}, \{(a, b), (\{c, e\}, \emptyset)\}, \{(a, b), (\{e\}, \{d\})\}$$

can not be transformed from any *div point set*. Thus for any  $\mathcal{X}_{u dp3} \in \mathcal{UDPS}^*$ , where  $\mathcal{X}_{u dp3} = \mathcal{F}_{u dp3}^{\mathcal{DPS}}(\mathcal{X}_{dp3})$  for some  $\mathcal{X}_{dp3} \in \mathcal{DPS}^*$ ,  $\mathcal{X}_{u dp3}$  satisfies

$$\forall D_1, D_2, D_3 \in \Omega_P$$

$$\begin{aligned} D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge \left| \bigcup_{n=1}^3 \pi_2(D_n) \right| = 3 \quad (2.40) \\ \Rightarrow \neg(\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 1 \wedge \psi(\pi_2(D_3)) = 0) \end{aligned}$$

Combining (2.40) and (2.39) gives (2.38).  $\square$

**Lemma 2.** A *unit div point set*  $(P, \Omega_P)$  has an interpretation for  $P$  as some set of 4 or more points in  $\mathbb{E}^2$  such that  $D \in \Omega_P$  each describes the relative positions of the points (in terms of how 2 *TBD points* of each *divider* is distributed between *divs* it produced) iff it is in  $\mathcal{UDPS}^+$ , the class of *unit div point sets* of 4 or more points satisfying (2.38), (2.42), (2.43), and (2.44), in which  $\xi$  is a function that returns the union of the *divider* and the *TBD points* in a *unit dividon*,  $D$ , notationally,

$$\xi(D) = \pi_1(D) \cup \bigcup \pi_2(D) \quad (2.41)$$

For any *unit div point set*  $(P, \Omega_P)$ ,

$$\forall p_1, p_2, p_3, p_4 \in P$$

$$R := \bigcup_{n=1}^4 \{p_n\}$$

$$|R| = 4$$

$$\Leftrightarrow \forall D_1, D_2, D_3 \in \Omega_P$$

$$(\xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3$$

(2.42)

$$\wedge \bigcap_{n=1}^3 \pi_1(D_n) = \{p_4\} )$$

$$\Rightarrow (\psi(\pi_2(D_1)) = 1$$

$$\Rightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3)) )$$

$$\forall p_1, p_2, p_3, p_4 \in P$$

$$R := \bigcup_{n=1}^4 \{p_n\}$$

$$|R| = 4$$

$$\Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P$$

$$(\xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3$$

(2.43)

$$\wedge \bigcap_{n=1}^3 \pi_1(D_n) = \{p_4\} )$$

$$\Rightarrow (\psi(\pi_2(D_1)) = 0$$

$$\Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3)) )$$

$$\forall p_1, p_2, p_3, p_4 \in P$$

$$R := \bigcup_{n=1}^4 \{p_n\}$$

$$|R| = 4$$

$$\Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P$$

$$(\xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3$$

(2.44)

$$\wedge \bigcap_{n=1}^2 \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^2 \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3) )$$

$$\Rightarrow (\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 0$$

$$\Rightarrow \psi(\pi_2(D_3)) = 1 )$$

*Proof for Lemma 2.* A *div point set*  $\mathcal{X}_{dps}$  satisfying (2.10), (2.11), and (2.12), iff the *unit div point set*  $\mathcal{F}_{udps}^{\mathcal{DP}\mathcal{S}}(\mathcal{X}_{dps})$  satisfies (2.42), (2.43), and (2.44). This can be demonstrated in a similar way as (2.23): for any *unit dividion*  $D_u$  of some *unit div point set*,  $\mathcal{A}_{udps}$ , and its corresponding *divdion*  $D$  of the *div point set*  $\mathcal{A}_{dps}$  where  $\mathcal{F}_{udps}^{\mathcal{DP}\mathcal{S}}(\mathcal{A}_{dps}) = \mathcal{A}_{udps}$  - corresponding in the sense that  $D_u \in \mathcal{D}(D, \pi_2(\mathcal{A}_{dps}))$  and so  $\pi_1(D_u) = \pi_1(D)$  - let  $R := \xi(D_u)$ , we would have

$$\phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D_u), \bigcup \pi_2(D_u)) = \psi(\pi_2(D_u)) \quad (2.45)$$

By restricting the *unit dividions* into  $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$ , we can then replace the predicate  $\bigcap_{n=1}^3 \pi_1(D_n) = \{p_4\}$  with  $D_1 \neq D_2 \neq D_3$ , and every occurrence of  $\phi(\pi_2(D_n), R \setminus \pi_1(D_n))$  with  $\psi(\pi_2(D_n))$  (for  $n \in \{1, 2, 3\}$ ) in (2.10), (2.11), and (2.12). This would gives (2.42), (2.43), and (2.44): they are basically a different way of expressing (2.10), (2.11), and (2.12) in the case of *unit div point sets*. Thus any *unit div point set*  $(P, \Omega_P)$  in  $\mathcal{UDPS}^+$  has an interpretation for  $P$  as some set of 4 or more points in  $\mathbb{E}^2$ , similar to how any *div point set*  $(P, \Theta_P)$  in  $\mathcal{DP}\mathcal{S}^+$  has an interpretation for  $P$  by *Axiom 1*.  $\square$

**Lemma 3.** A *unit div point set* is in  $\mathcal{UDPS}^+$  iff it is isomorphic to some *unit div point set*  $(P, \Omega_P)$  in  $\mathcal{UDPS}^{\mathbb{N}}$  where  $Col_{udps}(\Omega_P)$ , a full vertex monochromatic coloring on  $H_{udps}$ , satisfies (2.49) and (2.50), while  $\mathcal{UDPS}^{\mathbb{N}}$  is the class of all *unit div point sets*  $(P, \Theta_P)$  where  $P \subset \mathbb{N}$  and  $|P| \geq 4$ , and  $Col_{udps}$  is a function similar to  $Col$  in (2.18),

$$Col_{udps}(\Omega_P) = \{((\pi_1(D), \bigcup \pi_2(D)), \psi(\pi_2(D))) : D \in \Omega_P\} \quad (2.46)$$

and  $H_{udps}$  is a 3-and-6-uniform hypergraph with 2 sets of hyperedges,  $E_1$  and  $E_2$ , defined as a 3-tuple  $H_{udps} = (V_{udps}, E_1, E_2)$ , constructed based on  $P$ :

$$\begin{aligned} V_{udps} &= \bigcup \{V_{of}(d) : d \in \mathcal{P}(P) : |d| = 2\} \\ E_1 &= \{e \in \mathcal{P}(V) : |e| = 6 \wedge \forall v_1, v_2 \in e \pi_1(v_1) = \pi_1(v_2)\} \\ E_2 &= \{e \in \mathcal{P}(V) : |e| = 3 \wedge \forall v_1, v_2 \in e \pi_1(v_1) = \pi_2(v_2) \wedge |\bigcup_{v \in e} \pi_2(v)| = 3\} \end{aligned} \quad (2.47)$$

with  $V_{of}(d)$  being a function that returns a set of ordered pair consists of *divider* and *TBD points* of *unit dividions* of the same *divider*,

$$V_{of}(d) = \{(d, P_{TBD}) : P_{TBD} \in \mathcal{P}(P \setminus x) : |P_{TBD}| = 2\} \quad (2.48)$$

and, finally, here are the conditions that the coloring needs to satisfy:

$$\begin{aligned}
& \forall e \in E_1 \\
& \quad \exists v_1, v_2 \in e \quad \left| \begin{array}{l} \pi_1(v_1) = \pi_2(v_2) \\ \pi_1(v_2) = \pi_2(v_1) \\ C(v_1) = C(v_2) = 0 \\ C^{members}(e \setminus \{v_1, v_2\}) = \{1\} \end{array} \right. \quad (2.49) \\
& \Leftrightarrow \neg \exists v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} |\pi_1(v_1) \cap \pi_1(v_2) \cap \pi_1(v_3)| = 1 \\ C(v_1) = C(v_2) = C(v_3) = 0 \\ C^{members}(e \setminus \{v_1, v_2, v_3\}) = \{1\} \end{array} \right. \\
& \forall e \in E_2 \\
& \quad \forall v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} v_1 \neq v_2 \neq v_3 \\ \Rightarrow (C(v_1) = 1 \Leftrightarrow C(v_2) = C(v_3)) \\ \quad \wedge (C(v_1) = 0 \Leftrightarrow C(v_2) \neq C(v_3)) \end{array} \right. \quad (2.50)
\end{aligned}$$

wherein  $C$  is the coloring based on the set of *unit dividons* of some  $\mathfrak{X}_{upds} \in \mathcal{UDPS}^{\mathbb{N}}$ , i.e.  $C = Col_{upds}(\pi_2(\mathfrak{X}_{upds}))$ .

**Remark.** One may notice that the construction of  $H_{upds}$  depends solely on  $\pi_1(\mathfrak{X}_{upds})$  (i.e. the points of a *unit div point set*), as different from the coloring, which depends solely on  $\pi_2(\mathfrak{X}_{upds})$  (i.e. the set of *unit dividons*). This is similar to how the 3-uniform hypergraph  $H$  and the coloring on its vertices are defined back in the Proof for Theorem 1.

However, each vertex of  $H_{upds}$  is an ordered pair, structurally different from each vertex of  $H$  which is a set with cardinality of 2. Such definition for the vertices of  $H_{upds}$  in terms of not only the *divider* of a *unit dividon* but also its *TBD points* is necessary. This is because for any *unit div point set*,  $(P, \Omega_P)$ , there exists  $\binom{|P|-2}{2}$  distinct *unit dividons* sharing a common *divider*, where  $\binom{|P|-2}{2} > 1$  when  $|P| \geq 5$ . In order to distinguish *unit dividons* from one another in a *unit div point set* of 5 or more points, we would need to know both its *divider* and its *TBD points*.

**Remark.** For any *unit div point set*,  $(P, \Omega_P)$  where  $|P| = 4$ ,  $E_2$  of  $H_{upds}$  constructed based on  $P$  is an empty set, and thus (2.50) is trivially true for any coloring on such  $H_{upds}$ . On the other hand, there would only be 1 edge in  $E_1$  and the coloring  $C_{upds}(\Omega_P)$  satisfies (2.49) iff  $(P, \Omega_P)$  is isomorphic to  $\mathfrak{X} \in \{Conv^4, Conc_1^4\}$ : in (2.49), the first-order predicate before the logical connective  $\Leftrightarrow$  is true iff  $\mathfrak{X}$  is isomorphic to  $Conv^4$ , while the first-order predicate after  $\Leftrightarrow$  is true iff  $\mathfrak{X}$  is isomorphic to  $Conc_1^4$ .

*Proof for Lemma 3.* Every *unit div point set* of 4 or more points is isomorphic to some *unit div point set* in  $\mathcal{UDPS}^N$ . For a *unit div point set* to be in  $\mathcal{UDPS}^+$ , it has to satisfy (2.38), (2.42), (2.43), and (2.44). It is clear that a *unit div point set*,  $(P, \Omega_P)$ , satisfies (2.38) iff  $Col(\Omega_P)$  on the  $H_{u d p s}$  constructed based on  $P$  satisfies (2.50): (2.50) is simply a different way of writing (2.38) by first defining the order pairs  $(d_n, \bigcup \delta_n)$  of some *unit dividons*  $D_n = (d_n, \delta_n)$  that satisfy the necessary conditions (namely  $(D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^3 \pi_2(D_n)| = 3)$ ) to be vertices of an edge in  $E_2$  (recall (2.47)). On the other hand  $(P, \Omega_P)$  would satisfy (2.42), (2.43), and (2.44) iff  $Col(\Omega_P)$  on  $H_{u d p s}$  constructed based on  $P$  satisfies (2.49).

(2.42), (2.43), and (2.44) can be summarized as formulae with universal quantification of 4 points in  $P$ , where if these points are distinct, some conditional proposition regarding certain distinct *unit dividons* in  $\Omega_P$  must be true. The common characteristic of the conditional proposition in all 3 formulae is that  $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$  is a part of the conjunction that makes up the antecedent. For any unit div point sets of 4 or more points, there are a total of 6 *unit dividons*  $D$  where  $\xi(D) = R$  for any  $R \subseteq P$  where  $|R| = 4$ , obtainable using  $\mathcal{UDs}$ , a function which takes in a set of 4 points,  $R$ , and returns a set of such ordered pair:

$$\begin{aligned} \mathcal{UDs}(R) &= \{u d(d) : d \in \mathcal{P}(R) : |d| = 2\} \\ u d(d) &= (d, R \setminus d) \end{aligned} \quad (2.51)$$

One may notice that  $\mathcal{UDs}(R)$  always has a cardinality of  $\binom{4}{2} = 6$  and that  $E_1$  of the  $H_{u d p s}$  constructed based on some  $P$  can be expressed in terms of  $\mathcal{UDs}$ .

$$E_1 = \{\mathcal{UDs}(R) : R \in \mathcal{P}(P) : |R| = 4\} \quad (2.52)$$

By *Theorem 1*, a *unit div point set* of 4 points (recall that *div point sets* of 4 points are their own *unit div point sets*) satisfies (2.42), (2.43), and (2.44) iff it is isomorphic to either  $Conc_4^1$  or  $Conc_4$ . More fundamentally, this means that any *unit div point set*,  $(P, \Omega_P)$ , satisfies (2.42), (2.43), and (2.44) iff each set of 6 *unit dividons*,  $\Omega' \subseteq \Omega_P$ , where for all  $D \in \Omega'$ ,  $\xi(D)$  is equivalent to a subset of 4 cardinality of  $P$ , is isomorphic<sup>1</sup> to either  $\pi_2(Conc_4^1)$  or  $\pi_2(Conv_4)$ . The set of all such  $\Omega'$  for any  $\mathcal{X} \in \mathcal{UDPS}^*$ , can be expressed as a function  $All_{\Omega'}$  where:

$$\begin{aligned} All_{\Omega'}(\mathcal{X}) &= \{\Omega'_{basedOn}(R, \pi_2(\mathcal{X})) : R \in \mathcal{P}(\pi_1(\mathcal{X})) : |R| = 4\} \\ \Omega'_{basedOn}(R, \Omega_P) &= \{D : D \in \Omega_P : \xi(D) = R\} \end{aligned} \quad (2.53)$$

One may then realize that the following equation holds true for  $E_1$  of any  $H_{u d p s}$  constructed based on  $\pi_1(\mathcal{X})$ :

$$E_1 = \{(\pi_1(D), \bigcup \pi_2(D)) : D \in \Omega'\} : \Omega' \in All_{\Omega'}(\mathcal{X}) \quad (2.54)$$

Therefore any *unit div point set* of  $n$  points in  $\mathcal{UDPS}^{\mathbb{N}}$ ,  $(P, \Omega_P)$ , satisfies (2.42), (2.43), and (2.44) iff for all 4-cardinal  $R \subseteq P$ , a subset  $C'$  of  $Col_{udps}(\Omega_P)$ , the monochromatic vertex coloring on  $H_{uPd}$  constructed based on  $P$ , where for all  $c \in C'$ ,  $\xi(\pi_1(c)) = R$ , is isomorphic<sup>2</sup> to either the coloring  $Col(\pi_2(Conc_4^1))$  or  $Col(\pi_2(Conv_4))$ . Notationally,

$$\begin{aligned} \forall R \in \{P' \in \mathcal{P}(P) : |P'| = 4\} \\ C' &:= C'_{of}(R, Col_{udps}(\Omega_P)) \\ C' &\cong Col(\pi_2(Conc_4^1)) \Leftrightarrow \neg(C' \cong Col(\pi_2(Conv_4))) \\ \text{where } C'_{of}(R, C) &= \{c \in C : \xi(\pi_1(c)) = R\} \end{aligned} \tag{2.55}$$

which is what is expressed in (2.49).  $\square$

**Note.** *isomorphic*<sup>1</sup>: The definition of isomorphism in (2.57) is that of *div point sets*, but the isomorphism we are talking about here is that of sets of *unit dividons*, which can be defined as follows:

$$\begin{aligned} \Omega_1 \cong^1 \Omega_2 &\Leftrightarrow |\Omega_1| = |\Omega_2| \\ \wedge \exists f_{\Omega} : \bigcup_{D \in \Omega_1} \pi_1(D) &\xrightarrow{1:1} \bigcup_{D \in \Omega_2} \pi_1(D) \\ \forall D_1 \in \Omega_1 & \\ \exists D_2 \in \Omega_2 & \\ (d_1, \delta_1) &:= D_1 \\ (d_2, \delta_2) &:= D_2 \\ f^{members}(d_1) = d_2 &\Leftrightarrow f^{members^2}(\delta_1) = \delta_2 \end{aligned} \tag{2.56}$$

It is necessary to specify  $\Omega_1$  and  $\Omega_2$  to have the same cardinality, since it is possible for  $f_{\Omega}$ , a bijective function satisfying the condition, to exist in the case when  $|\Omega_1| \neq |\Omega_2|$ .

*isomorphic*<sup>2</sup>: The isomorphism we are talking about here is that of colors, which can be defined as follows:

$$\begin{aligned} C_1 \cong^2 C_2 &\Leftrightarrow \exists f_C : \{\pi_1(c) : c \in C_1\} \xrightarrow{1:1} \{\pi_1(c) : c \in C_2\} \\ \forall c_1 \in C_1 & \\ \exists c_2 \in C_2 & \\ (v_1, color_1) &:= c_1 \\ (v_2, color_2) &:= c_2 \\ f(v_1) = v_2 &\Rightarrow color_1 = color_2 \end{aligned} \tag{2.57}$$

**Definition 5.** We say that  $\mathcal{X}_1 \in \mathcal{UDPS}^*$  is a *sub div point set* of  $\mathcal{X}_2 \in \mathcal{UDPS}^*$  (denoted by  $\leq$ ) iff the set of *unit dividon* of the corresponding *unit div point set* of  $\mathcal{X}_1$  is a subset of

that of  $\mathfrak{X}_2$ . Notationally,

$$\begin{aligned}
\forall \mathfrak{X}_1, \mathfrak{X}_2 \in \mathcal{DPS}^* \\
(A, \Omega_A) &:= \mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathfrak{X}_1) \\
(B, \Omega_B) &:= \mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathfrak{X}_2) \\
\mathfrak{X}_1 \leq \mathfrak{X}_2 &\Leftrightarrow \Omega_A \subseteq \Omega_B
\end{aligned} \tag{2.58}$$

For clarification, 2 *sub div point sets* of some *div point set*,  $(S_1, \Omega_{S_1})$  and  $(S_2, \Omega_{S_2})$ , are distinct *sub div point sets* if  $S_1 \neq S_2$ , which is to say, distinctness here is not defined in terms of isomorphism, but equality (i.e. by the axiom of extensionality in ZFC).

**Definition 6.**  $\mathcal{SDps}_{of}$  is a function that returns the set of all *sub div point sets* of  $m$  points for some div point set where  $m \in \mathbb{N}_{\geq 4}$ .

$$\mathcal{SDps}_{of}(\mathfrak{X}_{dps}, m) = \{\mathcal{SDps}(\mathfrak{X}_{dps}, P_s) : P_s \in \pi_1(\mathfrak{X}_{dps})(P) : |P_s| = m\} \tag{2.59}$$

where  $\mathcal{SDps}$  is a function that returns the *sub div point set* of a set of points,  $P_s$ , of a *div point set* of  $\mathfrak{X}_{dps}$ :

$$\begin{aligned}
\mathcal{SDps}(\mathfrak{X}_{dps}, P_s) &= \mathfrak{F}_{dps}^{\mathcal{UDPS}}((P_s, \{D : D \in \pi_2(\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathfrak{X}_{dps})) : \xi(D) \subseteq P_s\})) \\
\text{where } \mathfrak{F}_{dps}^{\mathcal{UDPS}} &\text{ is the inverse of } \mathfrak{F}_{udps}^{\mathcal{DPS}}
\end{aligned} \tag{2.60}$$

Since a *div point set* of  $n$  points always has  $\binom{n}{m}$  distinct *sub div point sets* of  $m$  points, where  $m \leq n$  and  $m \geq 4$ ,  $\mathcal{SDps}_{of}(\mathfrak{X}_{dps}, m)$  has the cardinality of  $\binom{|\pi_1(\mathfrak{X}_{dps})|}{m}$ .

**Lemma 4.** For any *div point set*,  $\mathfrak{X}$ , and any natural number  $m$  greater or equal to 4, let  $\mathcal{A}$  and  $\mathcal{B}$  to be any 2 distinct *sub div point sets* of  $m$  points of  $\mathfrak{X}$ , and  $k$  be the number of points  $\mathcal{A}$  and  $\mathcal{B}$  have in common,  $\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathcal{A})$  and  $\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathcal{B})$  always have  $6\binom{k}{4}$  *unit dividons* in common. Notationally,

$$\begin{aligned}
\forall \mathfrak{X} \in \mathcal{DPS}^* \\
\forall m \in \mathbb{N}_{\geq 1} \\
\forall \mathcal{A}, \mathcal{B} \in \mathcal{SDps}_{of}(\mathfrak{X}, m) \\
|\pi_2(\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathcal{A})) \cap \pi_2(\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathcal{B}))| &= 6 \binom{|\pi_1(\mathcal{A}) \cap \pi_1(\mathcal{B})|}{4}
\end{aligned} \tag{2.61}$$

*Proof for Lemma 4.* For any  $m > |\pi_1(\mathfrak{X})|$ , the proposition on all elements  $\mathcal{A}, \mathcal{B} \in \mathcal{SDps}_{of}(\mathfrak{X}, m)$  is vacuously true. For  $m = |\pi_1(\mathfrak{X})|$ , it is obvious that the proposition is true: since every *dividon* of a *div point set* can be broken down into  $\binom{|P|-2}{2}$  *unit dividon*, any unit div point set of  $n$  points would have  $\binom{n-2}{2} \binom{n}{2} = 6\binom{n}{4}$  *unit dividons* in total. For  $m < 4$ , the proposition is trivially true because  $\binom{m}{4} = 0$  and *unit div point sets* of 3 or less points have no



*unit dividons* (recall (2.26)). For any  $m < |\pi_1(\mathcal{X})|$  but greater than 3, the proposition can be proven true by first observing that  $\mathcal{UDS}(R) \cap \mathcal{UDS}(R') = \emptyset \Leftrightarrow R \neq R'$  (recall (2.51)) for any sets  $R$  and  $R'$  with cardinality of 4, which indicates no 2 distinct *unit div point set* of 4 points have a *unit dividon* in common. Notationally,

$$\begin{aligned} \forall \mathcal{A}, \mathcal{B} \in \{\mathcal{X} : \mathcal{X} \in \mathcal{UDPS}^* : |\pi_1(\mathcal{X})| = 4\} \\ \pi_2(\mathcal{A}) \cap \pi_2(\mathcal{B}) = \emptyset \Leftrightarrow \pi_1(\mathcal{A}) \neq \pi_1(\mathcal{B}) \end{aligned} \quad (2.62)$$

However, for any 2 *unit div point sets* of 5 or more points,  $\mathcal{A}_{udps}$  and  $\mathcal{B}_{udps}$ , if they have 4 points in common, let the set of such 4 points be  $R$  (i.e.  $R \in \mathcal{P}(\pi_1(\mathcal{A}_{udps}) \cap \pi_2(\mathcal{A}_{udps})) \wedge |R| = 4$ ) each  $D'$  in  $\mathcal{UDS}(R)$  would be equivalent to  $\xi(D_a)$  and  $\xi(D_b)$  where  $D_a$  and  $D_b$  are respectively *unit dividon* of  $\mathcal{A}_{udps}$  and  $\mathcal{B}_{udps}$  having the same *divider* and *TBD points*. In the case when  $\mathcal{A}_{udps} = \mathcal{F}_{udps}^{\mathcal{UDPS}}(\mathcal{A}_{dps})$  and  $\mathcal{B}_{udps} = \mathcal{F}_{udps}^{\mathcal{UDPS}}(\mathcal{B}_{dps})$  for some  $\mathcal{A}_{dps}$  and  $\mathcal{B}_{dps}$  that are both *sub div point sets* of a certain  $\mathcal{X} \in \mathcal{UDPS}^*$ ,  $D_a = D_b$  for every such respective *unit dividons* of  $\mathcal{A}_{udps}$  and  $\mathcal{B}_{udps}$ . This implies that for every such distinct  $R$ ,  $\mathcal{A}_{udps}$  and  $\mathcal{B}_{udps}$  have 6 *unit dividons* in common. Let  $k$  be the number of points  $\mathcal{A}_{udps}$  and  $\mathcal{B}_{udps}$  have in common, the number of such distinct  $R$  is precisely  $k$  chooses 4 i.e.  $\binom{\pi_1(\mathcal{A}_{udps}) \cap \pi_1(\mathcal{B}_{udps})}{4}$ .  $\square$

**Theorem 2.** Let  $\mathcal{UDPS}_5^+$  denote the class of all *div point sets* of 5 points in  $\mathcal{UDPS}^+$ . All  $\mathcal{X} \in \mathcal{UDPS}_5^+$  either have 4, 2 or 0 distinct *sub div point set* of 4 points isomorphic to  $\text{Conc}_4^1$  (with the remaining *sub div point sets* of 4 points isomorphic to  $\text{Conv}_4$ ).

*Proof for Theorem 2.*

**Summary.** In *Part 1* we prove that there exists no  $\mathcal{X} \in \mathcal{UDPS}_5^+$  where  $\mathcal{SDPS}_{of}(\mathcal{X}, 4)$  has precisely 1, 3 or 5 elements isomorphic to  $\text{Conc}_4^1$ . In *Part 2* we prove that there exists  $\mathcal{X} \in \mathcal{UDPS}_5^+$  where  $\mathcal{SDPS}_{of}(\mathcal{X}, 4)$  has precisely 0, 2 or 4 elements isomorphic to  $\text{Conc}_4^1$ .

**Part 1.** A *div point set*  $\mathcal{X}_{dps}$  is in  $\mathcal{UDPS}^+$  iff  $\mathcal{F}_{udps}^{\mathcal{UDPS}}(\mathcal{X}_{dps})$  is in  $\mathcal{UDPS}^+$ . Any *unit div point set* of 5 points in  $\mathcal{UDPS}^+$  always has an even number of *unit dividons*  $D$  where  $\psi(\pi_2(D)) = 0$ , since it is isomorphic to some  $(P, \Omega_P)$  in  $\mathcal{UDPS}^{\mathbb{N}}$  where the coloring  $\text{Col}_{udps}(\Omega_P)$  satisfies (2.50). For  $\text{Col}_{udps}(\Omega_P)$  to satisfy (2.50), every  $e$  in  $E_2$  must have its vertices colored  $[1, 0, 0]$  or  $[1, 1, 1]$ . Since in any *unit div point set* of 5 points, there exists only  $\binom{5-2}{2} = 3$  distinct *unit dividon* with the same *divider*, edges in  $E_2$  are disjoint (recall (2.47)), and therefore any coloring satisfying (2.50) would have an even number of vertices colored 0.  $\text{Conc}_4^1$  has an odd number of *unit dividons*  $D$  where  $\psi(\pi_2(D)) = 0$ , while  $\text{Conv}_4$  has an even number for such *unit dividons*. Therefore there exists no *unit div point sets* of 5 points,  $\mathcal{X}_{udps}$ , in  $\mathcal{UDPS}^+$  such that  $\text{All}_{\Omega'}(\mathcal{X}_{udps})$  (defined (2.53)) contains an odd number of elements isomorphic<sup>1</sup> to  $\pi_2(\text{Conc}_4^1)$ . We thereby conclude that there exists no  $\mathcal{X} \in \mathcal{UDPS}_5^+$  where  $\mathcal{SDPS}_{of}(\mathcal{X}, 4)$  has precisely 1, 3 or 5 elements isomorphic to  $\text{Conc}_4^1$ .

**Part 2.** There exists *unit div point sets* of 5 points in  $\mathcal{UDPS}^+$  with precisely 4, 2, or 0 *sub div point sets* of 4 points isomorphic to  $\text{Conc}_4^1$ . We would prove it by demonstrating

that it is possible to construct *unit div point sets* of 5 points,  $\mathfrak{X}_{uds}$ , isomorphic to some  $(P, \Omega_P)$  in  $\mathcal{UDPS}^{\mathbb{N}}$  where the coloring  $Col_{uds}(\Omega_P)$  satisfies (2.49) and (2.50) and there are precisely 4, 2, or 0 distinct  $\Omega' \in All_{\Omega'}(\mathfrak{X}_{uds})$  isomorphic<sup>1</sup> to  $\pi_2(Conc_4^1)$ , (with the remaining  $\Omega'$  isomorphic to  $Conv_4$ ).

- I. To construct such *unit div point sets*  $\mathfrak{X}_{uds}$  where  $All_{\Omega'}(\mathfrak{X}_{uds})$  contains 0 elements isomorphic to  $Conc_4^1$  and 5 elements isomorphic to  $\pi_2(Conv_4)$ , we would need to make sure there are only 2 *unit dividons*  $D \in \Omega'$  where  $\phi(D) = 0$  for all  $\Omega'$  in  $All_{\Omega'}(\mathfrak{X}_{uds})$ . Let's denote the set of all such *unit dividons* as  $D^*$ , the 5 elements in  $All_{\Omega'}(\mathfrak{X}_{uds})$  as  $\Omega'_1, \Omega'_2, \Omega'_3, \Omega'_4, \Omega'_5$  and each 2 such unit dividons in  $\Omega'_n$  as  $D_n^1$  and  $D_n^2$  for  $n \in \{1, 2, 3, 4, 5\}$ , i.e.  $\{D_n^1, D_n^2\} = \Omega'_n \cap D^*$ . For the coloring to satisfy (2.49) and (2.50), we simply let any 2 *unit dividons*  $D_n^x, D_m^x$  where  $x \in \{1, 2\}$  and  $n \neq m$  to have a common *divider*, while avoiding to have 3 distinct *unit dividon* in  $D^*$  to a common *divider*, and the same time ensuring that

$$\begin{aligned}\pi_1(D_n^1) &= \bigcup \pi_2(D_n^2) \\ \pi_1(D_n^2) &= \bigcup \pi_2(D_n^1)\end{aligned}\tag{2.63}$$

(recall (2.49)). That is to say, for some subsets of 2 cardinality,  $A, B, C, D, E, F$  of  $\pi_1(\mathfrak{X}_{uds})$ , we have

$$\begin{aligned}\pi_1(D_1^1) &= \bigcup \pi_2(D_1^2) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) = A \\ \pi_1(D_1^2) &= \bigcup \pi_2(D_1^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) = B \\ \pi_1(D_2^2) &= \bigcup \pi_2(D_2^1) = \pi_1(D_4^2) = \bigcup \pi_2(D_4^1) = C \\ \pi_1(D_4^1) &= \bigcup \pi_2(D_4^2) = \pi_1(D_5^1) = \bigcup \pi_2(D_5^2) = D \\ \pi_1(D_5^2) &= \bigcup \pi_2(D_5^1) = \pi_1(D_6^2) = \bigcup \pi_2(D_6^1) = E \\ \pi_1(D_3^1) &= \bigcup \pi_2(D_3^2) = \pi_1(D_6^1) = \bigcup \pi_2(D_6^2) = F\end{aligned}\tag{2.64}$$

where

$$\begin{aligned}A &\neq B \neq C \neq D \neq E \neq F \\ (A \cap B) &= (A \cap C) = (B \cap F) = (C \cap D) = (D \cap E) = (E \cap F) = \emptyset\end{aligned}$$

Suppose  $\mathfrak{X}$  is a *div point set* where  $\mathfrak{F}_{uds}^{\mathcal{UDPS}}(\mathfrak{X})$  is isomorphic to the *unit div point set* described above:  $\mathfrak{X}$  would be isomorphic to  $Conv_5$  defined in (2.70) below.

- II. To construct such *unit div point sets*  $\mathfrak{X}_{uds}$  where  $All_{\Omega'}(\mathfrak{X}_{uds})$  contains 2 elements isomorphic to  $Conc_4^1$  and 3 elements isomorphic to  $\pi_2(Conv_4)$  - let's use the same

notations in I - we would need to make sure that

$$\begin{aligned} \forall n \in \{1, 2, 3\} \quad & \{D_n^1, D_n^2\} = \Omega'_n \cap D^* \\ \forall n \in \{4, 5\} \quad & \{D_n^1, D_n^2, D_n^3\} = \Omega'_n \cap D^* \end{aligned} \quad (2.65)$$

The *divider* of *unit dividons*  $D_n^1, D_n^2, D_n^3$  for  $n \in \{4, 5\}$  need to have 1 element in common:

$$|\pi_1(D_n^1) \cap \pi_1(D_n^2) \cap \pi_1(D_n^3)| = 1 \quad (2.66)$$

(recall (2.49)), while (2.63) continues to apply to  $D_n^1, D_n^2$  for  $n \in \{1, 2, 3\}$ . For the coloring to satisfy (2.49) and (2.50), we can have  $D_4^x$  to share the same *divider* as  $D_5^x$  for all  $x \in \{1, 2\}$ , while letting the remaining *unit dividons* in  $D_4$  and  $D_5$ , namely  $D_4^3$  and  $D_5^3$ , to respectively share the same *divider* as  $D_1^1$  and  $D_2^1$ , and the remaining *unit dividons* in  $D_1$  and  $D_2$ , namely  $D_1^2$  and  $D_2^2$ , to respectively share the same *dividers* as the two *dividons* in  $D_3$ . That is to say, for the distinct points  $a, b, c, d, e \in \pi_1(X_{udps})$ , we have

$$\begin{aligned} \pi_1(D_4^1) &= \pi_1(D_5^1) = \{a, b\} \\ \pi_1(D_4^2) &= \pi_1(D_5^2) = \{a, c\} \\ \pi_1(D_4^3) &= \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) = \{a, d\} \\ \pi_1(D_5^3) &= \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) = \{a, e\} \\ \pi_1(D_1^2) &= \bigcup \pi_2(D_1^1) = \pi_1(D_3^1) = \bigcup \pi_2(D_3^2) \subset P \setminus \{a, d\} \\ \pi_1(D_2^2) &= \bigcup \pi_2(D_2^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) \subset P \setminus \{a, e\} \end{aligned} \quad (2.67)$$

Suppose  $\mathfrak{X}$  is a *div point set* where  $\mathfrak{X}_{udps}^{\mathfrak{DPS}}(\mathfrak{X})$  is isomorphic to the *unit div point set* described above:  $\mathfrak{X}$  would be isomorphic to  $Conc_5^1$  defined in (2.70) below.

- III. To construct such *unit div point sets*  $\mathfrak{X}_{udps}$  where  $All_{\Omega'}(\mathfrak{X}_{udps})$  contains 4 elements isomorphic to  $Conc_4^1$  and 1 element isomorphic to  $\pi_2(Conv_4)$  - let's use the same notations in II - this time we would need to make sure that

$$\begin{aligned} \forall n \in \{1\} \quad & \{D_n^1, D_n^2\} = \Omega'_n \cap D^* \\ \forall n \in \{2, 3, 4, 5\} \quad & \{D_n^1, D_n^2, D_n^3\} = \Omega'_n \cap D^* \end{aligned} \quad (2.68)$$

where (2.66) applies to  $D_n^1, D_n^2$  and  $D_n^3$  for  $n \in \{2, 3, 4, 5\}$  and (2.63) continues to apply to  $D_n^1, D_n^2$  for  $n \in \{1\}$ . For the coloring to satisfy (2.49) and (2.50), we can let  $D_4^x$  to share the same *divider* as  $D_5^x$ , and  $D_2^x$  to share the same *divider* as  $D_3^x$ , for  $x \in \{1, 2\}$ . And then we let the remaining *unit dividons* in  $D_4$  and  $D_5$ , namely  $D_4^3$  and  $D_5^3$ , to respectively share the same *divider* as  $D_3^3$  and  $D_1^1$ , while the remaining *unit dividons* in  $D_2$ , namely  $D_2^2$  to share the same *divider* as  $D_1^2$ . That is to say, for

the distinct points  $a, b, c, d, e \in \pi_1(X_{udps})$ , we have

$$\begin{aligned}
\pi_1(D_4^1) &= \pi_2(D_5^1) = \{a, b\} \\
\pi_1(D_4^2) &= \pi_2(D_5^2) = \{a, c\} \\
\pi_1(D_4^3) &= \pi_1(D_3^3) = \{a, d\} \\
\pi_1(D_3^1) &= \pi_1(D_2^1) = \{e, d\} \\
\pi_1(D_3^2) &= \pi_1(D_2^2) = \{c, d\} \\
\pi_1(D_5^3) &= \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) = \{a, e\} \\
\pi_1(D_2^3) &= \pi_1(D_1^2) = \bigcup \pi_2(D_1^1) = \{b, d\}
\end{aligned} \tag{2.69}$$

Suppose  $\mathfrak{X}$  is a *div point set* where  $\mathfrak{F}_{udps}^{\mathcal{DPS}}(\mathfrak{X})$  is isomorphic to the *unit div point set* described above:  $\mathfrak{X}$  would be isomorphic to  $\text{Conc}_5^2$  defined in (2.70) below.

□

**Remark.** A stronger version of *Theorem 2* would state that for all  $\mathfrak{X}_{dps} \in \mathcal{DPS}_5^+$ ,  $\mathfrak{X}_{dps}$  is either isomorphic to  $\text{Conv}_5$ ,  $\text{Conc}_5^1$  or  $\text{Conc}_5^2$ , where

$$\begin{array}{lll}
\text{Conv}_5 = (Cv_5, \Theta_{Cv_5}) & \text{Conc}_5^1 = (Cc_5^1, \Theta_{Cc_5^1}) & \text{Conc}_5^2 = (Cc_5^2, \Theta_{Cc_5^2}) \\
Cv_5 = \{1, 2, 3, 4, 5\} & Cc_5^1 = \{1, 2, 3, 4, 5\} & Cc_5^2 = \{1, 2, 3, 4, 5\} \\
\Theta_{Cv_5} = \{(\{1, 2\}, \{\{3, 4\}, \emptyset\}), & \Theta_{Cc_5^1} = \{(\{1, 2\}, \{\{3, 4, 5\}, \emptyset\}), & \Theta_{Cc_5^2} = \{(\{1, 2\}, \{\{3, 4, 5\}, \emptyset\}), \\
(\{1, 3\}, \{\{2\}, \{4, 5\}\}), & (\{1, 3\}, \{\{2, 5\}, \{4\}\}), & (\{1, 3\}, \{\{2, 4, 5\}, \emptyset\}), \\
(\{1, 4\}, \{\{2, 3\}, \{5\}\}), & (\{1, 4\}, \{\{2, 3, 5\}, \emptyset\}), & (\{1, 4\}, \{\{2\}, \{3, 5\}\}), \\
(\{1, 5\}, \{\{2, 3, 4\}, \emptyset\}), & (\{1, 5\}, \{\{2\}, \{3, 4\}\}), & (\{1, 5\}, \{\{2, 4\}, \{3\}\}), \\
(\{2, 3\}, \{\{1, 4, 5\}, \emptyset\}), & (\{2, 3\}, \{\{1, 4, 5\}, \emptyset\}), & (\{2, 3\}, \{\{1, 4, 5\}, \emptyset\}), \\
(\{2, 4\}, \{\{1, 5\}, \{3\}\}), & (\{2, 4\}, \{\{1\}, \{3, 5\}\}), & (\{2, 4\}, \{\{1\}, \{3, 5\}\}), \\
(\{2, 5\}, \{\{1\}, \{3, 4\}\}), & (\{2, 5\}, \{\{1, 4\}, \{3\}\}), & (\{2, 5\}, \{\{1, 4\}, \{3\}\}), \\
(\{3, 4\}, \{\{1, 2, 5\}, \emptyset\}) & (\{3, 4\}, \{\{1, 2, 5\}, \emptyset\}) & (\{3, 4\}, \{\{1\}, \{2, 5\}\}) \\
(\{3, 5\}, \{\{1, 2\}, \{4\}\}) & (\{3, 5\}, \{\{1, 2\}, \{4\}\}) & (\{3, 5\}, \{\{1, 4\}, \{2\}\}) \\
(\{3, 4\}, \{\{1, 2, 5\}, \emptyset\}) & (\{4, 5\}, \{\{1, 2, 5\}, \emptyset\}) & (\{4, 5\}, \{\{1, 3\}, \{2\}\})
\end{array} \tag{2.70}$$

To prove this version of *Theorem 2* we would need to prove that there exists no *div point sets* in  $\mathcal{DPS}_5^+$  not isomorphic to  $\text{Conv}_5$ ,  $\text{Conc}_5^1$  or  $\text{Conc}_5^2$ .

**Remark.** Let  $\text{All}_{of\Omega'}(\mathfrak{X}, n)$  be a generalization of  $\text{All}_{\Omega'}(\mathfrak{X})$  such that

$$\text{All}_{of\Omega'}(\mathfrak{X}, n) = \{\Omega'_{basedOn}(R, \pi_2(\mathfrak{X})) : R \in \mathcal{P}(\pi_1(\mathfrak{X})) : |R| = n\} \tag{2.71}$$

Using *Theorem 2*, we can prove that the proposition below about *unit div point set* is false:

A *unit div point set* of 5 or more points,  $\mathcal{X}_{u\mathcal{DPS}}$ , is in  $\mathcal{UDPS}^+$  iff all elements in  $All_{of\Omega'}(\mathcal{X}_{u\mathcal{DPS}}, n)$  are isomorphic to the set of *unit dividons* of some *unit div point set* in  $\mathcal{UDPS}^+$ , for any  $n \in \mathbb{N}_{\geq 4}$  less than  $|\pi_1(\mathcal{X}_{u\mathcal{DPS}})|$ .

However, the following weaker version of it still holds true.

If  $\mathcal{X}_{u\mathcal{DPS}}$  is in  $\mathcal{UDPS}^+$ , all members of  $All_{of\Omega'}(\mathcal{X}_{u\mathcal{DPS}}, n)$  are also in  $\mathcal{UDPS}^+$  for any  $n \in \mathbb{N}_{\geq 4}$  less than  $|\pi_1(\mathcal{X}_{u\mathcal{DPS}})|$ .

This is equivalent as stating that for any *unit div point set*,  $\mathcal{X}_3$ , whose *unit dividons* is a subset of that of some  $\mathcal{X} \in \mathcal{UDPS}^+$ , there certainly exists an interpretation for  $\pi_1(\mathcal{X}_3)$  as some set of points in  $\mathbb{E}^2$ , which is just a subset of the interpretation for  $\pi_1(\mathcal{X})$  as some set of points in  $\mathbb{E}^2$ , since  $\pi_1(\mathcal{X}_3) \subseteq \pi_1(\mathcal{X})$ .

There is undoubtedly some similarity between the false proposition above, and the following proposition which is too false:

A *div point set* of 4 or more points,  $\mathcal{X}_{\mathcal{DPS}}$ , is in  $\mathcal{DPS}^+$  iff all elements in  $\mathcal{SDPS}_{of}(\mathcal{X}_{\mathcal{DPS}}, n)$  are also in  $\mathcal{DPS}^+$ , for any  $n \in \mathbb{N}_{\geq 3}$  less than  $|\pi_1(\mathcal{X}_{\mathcal{DPS}})|$ . If this is true, it would imply that  $\mathcal{DPS}^* = \mathcal{DPS}^+$ , which is obviously false.

However, if we closely examine this proposition, we would realize that it would be true if not for the case when  $n = 3$ : since it is vacuously true that any *div point sets* of 3 points satisfy (2.10), (2.11), and (2.12), we cannot conclude that a certain *div point set* satisfies (2.10), (2.11), and (2.12), even if all its *sub div point sets* of 3 points satisfy them. Now recall *Lemma 3* where  $E_2$  of the hypergraph based on  $P$  is an empty set in the case when  $|P| = 4$  and as a result, it is vacuously true that such  $E_2$  always satisfies (2.50). This is why the proposition regarding *unit div point sets* above is false: we cannot conclude that a certain *unit div point set* is isomorphic to some *unit div point set*,  $(P, \Omega_P)$ , in  $\mathcal{UDPS}^{\mathbb{N}}$  where  $Col_{u\mathcal{DPS}}(\Omega_P)$  satisfies (2.49) and (2.50), even if all elements in  $All_{of\Omega'}(\mathcal{X}_{u\mathcal{DPS}}, 4)$  are isomorphic to some *unit div point set*,  $(P, \Omega_P)$ , in  $\mathcal{UDPS}^{\mathbb{N}}$  where  $Col_{u\mathcal{DPS}}(\Omega_P)$  satisfies them.

It can be proven that in the case when  $n \in \mathbb{N}_{\geq 5}$ , the proposition regarding *unit div point sets* above is true.

## 2.2 convexity

The notion that there exists  $n$  points forming a convex polygon among some set of points in  $\mathbb{E}^2$  can be expressed through *convexity* in the context of *div point sets*.

**Definition 7.** A *div point set*  $(P, \Theta_P)$  has a *convexity* of  $n$  iff there exists  $(Q, \Theta_Q)$  such

that  $(Q, \Theta_Q) \leq (P, \Theta_P)$  and  $(Q, \Theta_Q)$  is isomorphic to  $Conv_n$ , defined as follow

$$Conv_n = (P, \{(d, \delta_{conv}(d, P)) : d \in \mathcal{P}(P) : |d| = 2\})$$

$$\text{where } \begin{cases} P := \{x \in \mathbb{N}_{\geq 1} : x \leq n\} \\ \delta_{conv}(d, P) = \{\{p : p \in P : \text{inside}(p, d)\}, \{p : p \in P : \text{outside}(p, d)\}\} \\ \text{inside}(p, d) = (p > \min(d) \wedge p < \max(d)) \\ \text{outside}(p, d) = (p < \min(d) \vee p > \max(d)) \\ \min(d) \text{ returns the smallest number in } d \\ \max(d) \text{ returns the biggest number in } d. \end{cases} \quad (2.72)$$

where  $n$  is a natural number  $\geq 3$ . Here is an implementation of it as a function in Haskell:

---

```
import Data.List

combine :: Int -> [a] -> [[a]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                    , ys <- combine (n-1) xs' ]

convex :: Int -> ([Int], [[Int], [[Int]]])
convex n = (points, dividons)
  where
    points = [1..n]
    dividers = combine 2 points
    dividons = [(divider, [div1, div2])
                 | divider@(a:b:_) <- dividers,
                   let divs = points \\ divider,
                   let div1 = [ x | x <- divs, x > a, x < b ],
                   let div2 = divs \\ div1 ]
```

---

**Axiom 2.** For any  $\mathcal{X}$  in  $\mathcal{DPS}^+$ ,  $\mathcal{X}$  has an interpretation for  $\pi_1(\mathcal{X})$  as some set of points in  $E^2$  among which there exists  $n$  points forming a convex polygon, iff  $\mathcal{X}$  has a convexity of  $n$ . More precisely, there exists an interpretation for  $P' \subseteq \pi_1(\mathcal{X})$  as some set of  $n$  points in  $E^2$  forming a convex polygon iff  $Sdps(\mathcal{X}, P')$  is isomorphic to  $Conv_n$ , for all  $n \geq 3$ .

**Remark.** One may notice that for any  $n \geq 4$ , all *sub div point sets* of  $n-1$  points of  $Conv_n$  are isomorphic to  $Conv_{n-1}$ . By *Axiom 2*, that is equivalent to the following proposition: for any  $n \geq 4$ , after removing any one point from a set of  $n$  points that are the vertices of a convex polygon on an Euclidean plane, the remaining points too forms a convex polygon, which is trivially true.

**Remark.** By *Axiom 2*, we can conclude from *Theorem 2* that for any 5 points in general position on an Euclidean plane, there always exists 4 points forming a convex polygon.

### 3 A reduction to a multiset unsatisfiability problem

The Erdős-Szekeres conjecture can be expressed as a conjunction of (3.1) and (3.2) in the theory of *div point set*.

$$\begin{aligned} \forall n \in \mathbb{N}_{\geq 3} \\ \exists \mathcal{A} \in \mathcal{DP}\mathcal{S}^+ \quad |\pi_1(\mathcal{A})| = 2^{n-2} \wedge \exists \mathcal{A}_\delta \leq \mathcal{A} \quad \mathcal{A}_\delta \not\cong \text{Conv}_n \end{aligned} \quad (3.1)$$

$$\begin{aligned} \forall n \in \mathbb{N}_{\geq 3} \\ \forall \mathcal{A} \in \mathcal{DP}\mathcal{S}^+ \quad |\pi_1(\mathcal{A})| > 2^{n-2} \Leftrightarrow \exists \mathcal{A}_\delta \leq \mathcal{A} \quad \mathcal{A}_\delta \cong \text{Conv}_n \end{aligned} \quad (3.2)$$

Since the lower bound has been proven to be  $2^{n-2} + 1$ , all is left is to prove (3.2) and the conjecture would be proven.

#### 3.1 a combinatorial characteristics of *sub div point sets*

As we examine *div point sets* of  $v$  points for  $v > 5$ , we would notice this rather interesting fact about *sub div point sets*: for any natural number  $a > 1$ , let  $\mathcal{SDP}\mathcal{SS}$  be the set of all *sub div point set* of  $v - a$  points of any *div point sets* of  $v$  points, for all  $\mathcal{X}_{\mathcal{SDP}\mathcal{S}}$  in  $\mathcal{SDP}\mathcal{SS}$ , we can always select  $v - a$  distinct sets of  $a + 1$  *div point sets* in  $\mathcal{SDP}\mathcal{SS}$  such that for each of these sets, it contains  $\mathcal{X}_{\mathcal{SDP}\mathcal{S}}$  and all *div points sets* in it have  $\binom{v-a-1}{t}$  *sub div point sets* of  $t$  points in common, for all  $t \in \mathbb{N}_{\geq 1}$ . What is cool about this is that it doesn't just apply to  $a + 1$  but  $a + b$  for any  $b \in \mathbb{N}_{\geq 1}$  as long as  $a + b$  is smaller than  $v$  (and in which case we would have  $\binom{v-a-b}{t}$  *sub div point sets* of  $t$  points in common).

Notationally,

$$\begin{aligned} \forall \mathcal{X} \in \mathcal{DP}\mathcal{S}^+ \\ v := |\pi_1(\mathcal{X})| \\ \forall a \in \mathbb{N}_{\geq 1} \\ \mathcal{SDP}\mathcal{SS} := \mathcal{SDP}\mathcal{S}_{\text{of}}(\mathcal{X}, v - a) \\ \forall \mathcal{X}_{\mathcal{SDP}\mathcal{S}} \in \mathcal{SDP}\mathcal{SS} \\ \forall b \in \{x : x \in \mathbb{N}_{\geq 1} : x < v - a\} \\ \exists \mathcal{S} \in \mathcal{P}_n(\mathcal{P}_n(\mathcal{SDP}\mathcal{SS}, a + b), v - a) \\ \forall \mathfrak{s} \in \mathcal{S} \\ \mathcal{X}_{\mathcal{SDP}\mathcal{S}} \in \mathfrak{s} \\ \forall t \in \mathbb{N}_{\geq 1} \\ \left| \bigcap_{\ell \in \mathfrak{s}} \mathcal{SDP}\mathcal{S}_{\text{of}}(\ell, t) \right| = \binom{v - a - b}{t} \end{aligned} \quad (3.3)$$

where

$$\mathcal{P}_n(S, n) = \{x : x \in \mathcal{P}(S) : |x| = n\} \quad (3.4)$$

To understand why such combinatorial characteristics exists, consider this: any 2 *sub div point sets*,  $\mathcal{S}_1$  and  $\mathcal{S}_2$  of a certain div point set is distinct as long as they are of distinct points i.e.  $\mathcal{S}_1 \neq \mathcal{S}_2 \Leftrightarrow \pi_1(\mathcal{S}_1) \neq \pi_1(\mathcal{S}_2)$  and are the same as long as they are of the same points i.e.  $\mathcal{S}_1 = \mathcal{S}_2 \Leftrightarrow \pi_1(\mathcal{S}_1) = \pi_1(\mathcal{S}_2)$ , thus (3.3) is equivalent as stating: for any set  $\mathcal{N}$  with the same cardinality as  $\mathbb{N}$ ,

$$\begin{aligned}
& \forall X \in \mathcal{P}(\mathcal{N}) \\
& v := |X| \\
& \forall a \in \mathbb{N}_{\geq 1} \\
& X_{\text{subset-set}} := \mathcal{P}_n(X, v - a) \\
& \forall X_{\text{subset}} \in X_{\text{subset-set}} \\
& \forall b \in \{x : x \in \mathbb{N}_{\geq 1} : x < v - a\} \\
& \exists S \in \mathcal{P}_n(\mathcal{P}_n(X_{\text{subset-set}}, a + b), v - a) \\
& \forall s \in S \\
& X_{\text{subset}} \in s \\
& \forall n \in \mathbb{N}_{\geq 1} \\
& \left| \bigcap_{l \in s} \mathcal{P}_n(l, n) \right| = \binom{v - a - b}{n}
\end{aligned} \tag{3.5}$$

For the purpose of illustration, suppose we have some div point set  $\mathcal{X}_9$  of 9 points, let *isom* be an bijective function from  $\mathcal{S}dp\mathcal{S}_{of}(\mathcal{X}_9, 4)$  to a set of natural numbers,  $\mathcal{N}$ , where  $\mathcal{N} = \{n : n \in \mathbb{N}_{\geq 1} : n \leq |\mathcal{S}dp\mathcal{S}_{of}(\mathcal{X}_9, 4)|\}$ , the set  $\{\{iso(\mathcal{X}_4) : \mathcal{X}_4 \in \mathcal{X}_5\} : \mathcal{X}_5 \in \mathcal{S}dp\mathcal{S}_{of}(\mathcal{X}_9, 5)\}$  would be isomorphic to:

$$\begin{aligned}
& \{\{1, 2, 7, 22, 57\}, \{1, 3, 8, 23, 58\}, \{1, 4, 9, 24, 59\}, \{1, 5, 10, 25, 60\}, \{1, 6, 11, 26, 61\}, \{2, 3, 12, 27, 62\}, \\
& \{2, 4, 13, 28, 63\}, \{2, 5, 14, 29, 64\}, \{2, 6, 15, 30, 65\}, \{3, 4, 16, 31, 66\}, \{3, 5, 17, 32, 67\}, \{3, 6, 18, 33, 68\}, \\
& \{4, 5, 19, 34, 69\}, \{4, 6, 20, 35, 70\}, \{5, 6, 21, 36, 71\}, \{7, 8, 12, 37, 72\}, \{7, 9, 13, 38, 73\}, \{7, 10, 14, 39, 74\}, \\
& \{7, 11, 15, 40, 75\}, \{8, 9, 16, 41, 76\}, \{8, 10, 17, 42, 77\}, \{8, 11, 18, 43, 78\}, \{9, 10, 19, 44, 79\}, \{9, 11, 20, 45, 80\}, \\
& \{10, 11, 21, 46, 81\}, \{12, 13, 16, 47, 82\}, \{12, 14, 17, 48, 83\}, \{12, 15, 18, 49, 84\}, \{13, 14, 19, 50, 85\}, \{13, 15, 20, 51, 86\}, \\
& \{14, 15, 21, 52, 87\}, \{16, 17, 19, 53, 88\}, \{16, 18, 20, 54, 89\}, \{17, 18, 21, 55, 90\}, \{19, 20, 21, 56, 91\}, \{22, 23, 27, 37, 92\}, \\
& \{22, 24, 28, 38, 93\}, \{22, 25, 29, 39, 94\}, \{22, 26, 30, 40, 95\}, \{23, 24, 31, 41, 96\}, \{23, 25, 32, 42, 97\}, \{23, 26, 33, 43, 98\}, \\
& \{24, 25, 34, 44, 99\}, \{24, 26, 35, 45, 100\}, \{25, 26, 36, 46, 101\}, \{27, 28, 31, 47, 102\}, \{27, 29, 32, 48, 103\}, \{27, 30, 33, 49, 104\}, \\
& \{28, 29, 34, 50, 105\}, \{28, 30, 35, 51, 106\}, \{29, 30, 36, 52, 107\}, \{31, 32, 34, 53, 108\}, \{31, 33, 35, 54, 109\}, \{32, 33, 36, 55, 110\}, \\
& \{34, 35, 36, 56, 111\}, \{37, 38, 41, 47, 112\}, \{37, 39, 42, 48, 113\}, \{37, 40, 43, 49, 114\}, \{38, 39, 44, 50, 115\}, \{38, 40, 45, 51, 116\}, \\
& \{39, 40, 46, 52, 117\}, \{41, 42, 44, 53, 118\}, \{41, 43, 45, 54, 119\}, \{42, 43, 46, 55, 120\}, \{44, 45, 46, 56, 121\}, \{47, 48, 50, 53, 122\}, \\
& \{47, 49, 51, 54, 123\}, \{48, 49, 52, 55, 124\}, \{50, 51, 52, 56, 125\}, \{53, 54, 55, 56, 126\}, \{57, 58, 62, 72, 92\}, \{57, 59, 63, 73, 93\}, \\
& \{57, 60, 64, 74, 94\}, \{57, 61, 65, 75, 95\}, \{58, 59, 66, 76, 96\}, \{58, 60, 67, 77, 97\}, \{58, 61, 68, 78, 98\}, \{59, 60, 69, 79, 99\}, \\
& \{59, 61, 70, 80, 100\}, \{60, 61, 71, 81, 101\}, \{62, 63, 66, 82, 102\}, \{62, 64, 67, 83, 103\}, \{62, 65, 68, 84, 104\}, \{63, 64, 69, 85, 105\}, \\
& \{63, 65, 70, 86, 106\}, \{64, 65, 71, 87, 107\}, \{66, 67, 69, 88, 108\}, \{66, 68, 70, 89, 109\}, \{67, 68, 71, 90, 110\}, \{69, 70, 71, 91, 111\}, \\
& \{72, 73, 76, 82, 112\}, \{72, 74, 77, 83, 113\}, \{72, 75, 78, 84, 114\}, \{73, 74, 79, 85, 115\}, \{73, 75, 80, 86, 116\}, \{74, 75, 81, 87, 117\}, \\
& \{76, 77, 79, 88, 118\}, \{76, 78, 80, 89, 119\}, \{77, 78, 81, 90, 120\}, \{79, 80, 81, 91, 121\}, \{82, 83, 85, 88, 122\}, \{82, 84, 86, 89, 123\}, \\
& \{83, 84, 87, 90, 124\}, \{85, 86, 87, 91, 125\}, \{88, 89, 90, 91, 126\}, \{92, 93, 96, 102, 112\}, \{92, 94, 97, 103, 113\}, \{92, 95, 98, 104, 114\}, \\
& \{93, 94, 99, 105, 115\}, \{93, 95, 100, 106, 116\}, \{94, 95, 101, 107, 117\}, \{96, 97, 99, 108, 118\}, \{96, 98, 100, 109, 119\}, \\
& \{97, 98, 101, 110, 120\}, \{99, 100, 101, 111, 121\}, \{102, 103, 105, 108, 122\}, \{102, 104, 106, 109, 123\}, \{103, 104, 107, 110, 124\}, \\
& \{105, 106, 107, 111, 125\}, \{108, 109, 110, 111, 126\}, \{112, 113, 115, 118, 122\}, \{112, 114, 116, 119, 123\}, \{113, 114, 117, 120, 124\}, \\
& \{115, 116, 117, 121, 125\}, \{118, 119, 120, 121, 126\}, \{122, 123, 124, 125, 126\}\}
\end{aligned} \tag{3.6}$$



which shows how *sub div point sets* of 4 points of  $\mathfrak{X}_9$  (each represented by a distinct natural number above) would be disturbed among *sub div point sets* of 5 points of  $\mathfrak{X}_9$ . Notice how it satisfies (3.3) (e.g. for every *sub div point set* of 5 points,  $\mathfrak{X}_5$ , there exists a set in  $\mathcal{P}_n(\mathcal{P}_n(\mathcal{Sdp}_{\circ\ell}(\mathfrak{X}_9, 5), 5), 5)$  where its elements are different supersets of  $\mathfrak{X}_5$ , in which the *div point sets* of 5 points all have a *sub div point set* of 4 points in common i.e.  $|\bigcap_{\ell \in \mathfrak{S}} \mathcal{Sdp}_{\circ\ell}(\ell, 4)| = \binom{9-4-1}{4} = 1$  where  $s \in S$  and  $S$  is the set containing such supersets i.e.  $S \in \mathcal{P}_n(\mathcal{P}_n(\mathcal{Sdp}_{\circ\ell}(\mathfrak{X}_9, 5), 5), 5)$ ).

We suspect that (3.2) is nothing more than an elegant result of what happens when we have a structure whose sub-structures possess the combinatorial characteristics described above and the same time having to abide by some simple rule, which, in this case, is what described in (3.7) below.

### 3.2 the problem $UNSAT_{multiset}^{\mathcal{DPS}^+}$

**Definition 8.**  $UNSAT_{multiset}$  is the decision problem of determining if there exists no value-assignment for all variables in  $V$ , distributed in a certain manner among the multisets in  $M$ , such that it satisfies the FOL formulae in  $C$ , where the value-assignment is defined to be a function,  $Z$ : for all  $v$  in  $V$ ,  $Z(v) = x$  for some  $x \in D$  where  $D$ , the set of values a variable can be assigned to, is often referred to as the domain. An instance of  $UNSAT_{multiset}$  can thus be represented as a 4-tuple  $(V, D, M, C)$ .

We shall now present the problem  $UNSAT_{multiset}^{\mathcal{DPS}^+}$ , a special case of  $UNSAT_{multiset}$ , of which, if an instance is solved, we can use the result that it yields *True* (i.e. it is not satisfiable) to prove that for a specific  $n \in \mathbb{N}_{\geq 5}$ , there exists no *div point set* of  $2^{n-2} + 1$  points,  $\mathfrak{X}$ , where it satisfies

$$\begin{aligned} \forall \mathfrak{X}_5 \in \mathcal{Sdp}_{\circ\ell}(\mathfrak{X}, 5) \\ [Assign(\mathfrak{X}_4) : \mathfrak{X}_4 \in \mathcal{Sdp}_{\circ\ell}(\mathfrak{X}_5, 4)] \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0], [0, 0, 0, 0, 0]\} \end{aligned} \quad (3.7)$$

but does not satisfy

$$\begin{aligned} \exists \mathfrak{A}_3 \in \mathcal{Sdp}_{\circ\ell}(\mathfrak{X}, n) \\ \forall \mathfrak{A}_{33} \in \mathcal{Sdp}_{\circ\ell}(\mathfrak{X}, 4) \quad Assign(\mathfrak{A}_{33}) = 0 \end{aligned} \quad (3.8)$$

where

$$Assign(\mathfrak{X}) = \begin{cases} 1 & \text{if } \mathfrak{X} \cong Conc_4^1 \\ 0 & \text{if } \mathfrak{X} \cong Conv_4 \end{cases} \quad (3.9)$$

Since for any *div point sets* in  $\mathcal{DPS}^+$ , its *sub div point sets* of 5 points satisfy (3.7) (recall Theorem 2), this would thus prove that *div point set* in  $\mathcal{DPS}^+$  must too satisfy (3.8), and

consequently proving (3.2) which can be rewritten as follows

$$\begin{aligned}
& \forall n \in \mathbb{N}_{\geq 3} \\
& \quad \forall \mathcal{A} \in \mathcal{DPS}^+ \\
& \quad |\pi_1(A)| > 2^{n-2} \\
& \quad \Leftrightarrow \exists \mathcal{A}_3 \in \mathcal{SDPS}_{of}(\mathcal{A}, n) \\
& \quad \quad \forall \mathcal{A}_{33} \in \mathcal{SDPS}_{of}(\mathcal{A}_3, 4) \quad Assign(\mathcal{A}_{33}) = 0
\end{aligned} \tag{3.10}$$

(recall that for all  $k \in \mathbb{N}_{\geq 3}$  and  $n \geq k$ , any *sub div point set* of  $k$  points of any  $Conv_n$  is isomorphic to some  $Conv_k$ ).

**Definition 9.**  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  is a special case of  $UNSAT_{multiset}$  where some instance,  $(V, D, M, C)$ , of  $UNSAT_{multiset}$  is an instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  iff for some  $n \geq 5$ ,

$$\begin{aligned}
|V| &= \binom{2^{n-2} + 1}{4} \\
D &= \{0, 1\} \\
M &= A \cup B
\end{aligned} \tag{3.11}$$

where  $A$  is a set of 5-cardinal multisets, and  $B$  is a set of  $n$ -cardinal multisets. For some *div point set* of  $n$  points,  $\mathcal{X}$ , the variables in  $V$  are distributed in  $m \in A$  the same way as how elements in  $\mathcal{SDPS}_{of}(\mathcal{X}, 4)$  are distributed in  $\mathcal{X}_{\mathcal{SDPS}} \in \mathcal{SDPS}_{of}(\mathcal{X}, 5)$ , while the variables are distributed in  $m \in B$  the same way as how elements in  $\mathcal{SDPS}_{of}(\mathcal{X}, 4)$  are distributed in  $\mathcal{X}_{\mathcal{SDPS}} \in \mathcal{SDPS}_{of}(\mathcal{X}, n)$ . And  $C$  is the set of the formulae (3.12) and (3.13).

$$\forall a \in A \quad a \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0], [0, 0, 0, 0, 0]\} \tag{3.12}$$

$$\forall b \in B \quad b \neq \underbrace{[0, 0, 0, \dots, 0, 0]}_{\binom{n}{4} \text{ 0's}} \tag{3.13}$$

The distribution of variables in  $A$  and  $B$  can be implement in Haskell as follows:

---

```

import Data.List
import Data.Maybe
type Multiset = [Integer]

merge (a:x) (b:y) = (a,b) : merge x y
merge [] _ = []

choose :: Integer -> Integer -> Integer
n 'choose' k
  | k < 0    = 0
  | k > n    = 0

```

```

    | otherwise = factorial n `div` (factorial k * factorial (n-k))

factorial :: Integer -> Integer
factorial n = foldl (*) 1 [1..n]

combine :: Integer -> [Integer] -> [[Integer]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                  , ys <- combine (n-1) xs' ]

number_of_points = (\n->(2^(n-2)+1))

n_setOf_m_Multisets :: Integer -> Integer -> [Multiset]
n_setOf_m_Multisets m n = [ map fromJust $ map ((flip lookup) encoding)
    (combine 4 m_points) | m_points <- combine n [1..m] ]
  where
    encoding = merge (combine 4 [1..m]) [1..(m `choose` 4)]

setA :: Integer -> [Multiset]
setA n = n_setOf_m_Multisets (number_of_points n) 5

setB :: Integer -> [Multiset]
setB n = [ x | x <- n_setOf_m_Multisets (number_of_points n) n, 2 `elem` x ]

```

---

**Remark.** A different implementation may result in a different  $M$  for the same  $n$ . Nonetheless, the different  $M$  obtained from a different implementation would be isomorphic to the  $M$  obtained from this implementation, in which case we would consider that distribution to be the same. Thus as far as unsatisfiability is concerned, for every  $n \in \mathbb{N}_{\geq 5}$ , there exists exactly one instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$ .

Here is the simplest instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  (when  $n = 5$ ): since  $A = B$ , we have  $|M| = |A| = |B| = \binom{2^{5-2}+1}{5} = 126$  multisets, and  $|V| = \binom{2^{5-2}+1}{4} = 126$  variables as well

(with each denoted by  $v_n$  below), distributed among the multisets in  $M$  as follows:

{[v1, v2, v7, v22, v57], [v1, v3, v8, v23, v58], [v1, v4, v9, v24, v59], [v1, v5, v10, v25, v60], [v1, v6, v11, v26, v61], [v2, v3, v12, v27, v62], [v2, v4, v13, v28, v63], [v2, v5, v14, v29, v64], [v2, v6, v15, v30, v65], [v3, v4, v16, v31, v66], [v3, v5, v17, v32, v67], [v3, v6, v18, v33, v68], [v4, v5, v19, v34, v69], [v4, v6, v20, v35, v70], [v5, v6, v21, v36, v71], [v7, v8, v12, v37, v72], [v7, v9, v13, v38, v73], [v7, v10, v14, v39, v74], [v7, v11, v15, v40, v75], [v8, v9, v16, v41, v76], [v8, v10, v17, v42, v77], [v8, v11, v18, v43, v78], [v9, v10, v19, v44, v79], [v9, v11, v20, v45, v80], [v10, v11, v21, v46, v81], [v12, v13, v16, v47, v82], [v12, v14, v17, v48, v83], [v12, v15, v18, v49, v84], [v13, v14, v19, v50, v85], [v13, v15, v20, v51, v86], [v14, v15, v21, v52, v87], [v16, v17, v19, v53, v88], [v16, v18, v20, v54, v89], [v17, v18, v21, v55, v90], [v19, v20, v21, v56, v91], [v22, v23, v27, v37, v92], [v22, v24, v28, v38, v93], [v22, v25, v29, v39, v94], [v22, v26, v30, v40, v95], [v23, v24, v31, v41, v96], [v23, v25, v32, v42, v97], [v23, v26, v33, v43, v98], [v24, v25, v34, v44, v99], [v24, v26, v35, v45, v100], [v25, v26, v36, v46, v101], [v27, v28, v31, v47, v102], [v27, v29, v32, v48, v103], [v27, v30, v33, v49, v104], [v28, v29, v34, v50, v105], [v28, v30, v35, v51, v106], [v29, v30, v36, v52, v107], [v31, v32, v34, v53, v108], [v31, v33, v35, v54, v109], [v32, v33, v36, v55, v110], [v34, v35, v36, v56, v111], [v37, v38, v41, v47, v112], [v37, v39, v42, v48, v113], [v37, v40, v43, v49, v114], [v38, v39, v44, v50, v115], [v38, v40, v45, v51, v116], [v39, v40, v46, v52, v117], [v41, v42, v44, v53, v118], [v41, v43, v45, v54, v119], [v42, v43, v46, v55, v120], [v44, v45, v46, v56, v121], [v47, v48, v50, v53, v122], [v47, v49, v51, v54, v123], [v48, v49, v52, v55, v124], [v50, v51, v52, v56, v125], [v53, v54, v55, v56, v126], [v57, v58, v62, v72, v92], [v57, v59, v63, v73, v93], [v57, v60, v64, v74, v94], [v57, v61, v65, v75, v95], [v58, v59, v66, v76, v96], [v58, v60, v67, v77, v97], [v58, v61, v68, v78, v98], [v59, v60, v69, v79, v99], [v59, v61, v70, v80, v100], [v60, v61, v71, v81, v101], [v62, v63, v66, v82, v102], [v62, v64, v67, v83, v103], [v62, v65, v68, v84, v104], [v63, v64, v69, v85, v105], [v63, v65, v70, v86, v106], [v64, v65, v71, v87, v107], [v66, v67, v69, v88, v108], [v66, v68, v70, v89, v109], [v67, v68, v71, v90, v110], [v69, v70, v71, v91, v111], [v72, v73, v76, v82, v112], [v72, v74, v77, v83, v113], [v72, v75, v78, v84, v114], [v73, v74, v79, v85, v115], [v73, v75, v80, v86, v116], [v74, v75, v81, v87, v117], [v76, v77, v79, v88, v118], [v76, v78, v80, v89, v119], [v77, v78, v81, v90, v120], [v79, v80, v81, v91, v121], [v82, v83, v85, v88, v122], [v82, v84, v86, v89, v123], [v83, v84, v87, v90, v124], [v85, v86, v87, v91, v125], [v88, v89, v90, v91, v126], [v92, v93, v96, v102, v112], [v92, v94, v97, v103, v113], [v92, v95, v98, v104, v114], [v93, v94, v99, v105, v115], [v93, v95, v100, v106, v116], [v94, v95, v101, v107, v117], [v96, v97, v99, v108, v118], [v96, v98, v100, v109, v119], [v97, v98, v101, v110, v120], [v99, v100, v101, v111, v121], [v102, v103, v105, v108, v122], [v102, v104, v106, v109, v123], [v103, v104, v107, v110, v124], [v105, v106, v107, v111, v125], [v108, v109, v110, v111, v126], [v112, v113, v115, v118, v122], [v112, v114, v116, v119, v123], [v113, v114, v117, v120, v124], [v115, v116, v117, v121, v125], [v118, v119, v120, v121, v126], [v122, v123, v124, v125, v126]}

It is no surprise that the distribution of variables in  $m \in M$  above is exactly that of *sub div point sets* of 4 points in  $\mathcal{X}_5 \in \mathcal{SDps}_{of}(\mathcal{X}_9, 5)$  where  $\mathcal{X}_9$  is any *div point set* of 9 points (as shown in (3.6)). Each variable in  $V$  represents  $Assign(\mathcal{X}_4)$  for a particular element  $\mathcal{X}_4 \in \mathcal{SDps}_{of}(\mathcal{X}, 4)$  where  $\mathcal{X}$  is a *div point set* of  $2^{n-2} + 1$  points for some  $n \in \mathbb{N}_{\geq 5}$ . If there exists no value-assignment  $Z$  satisfying formulae in  $C$ , we can be certain that there exists no *div point set* of  $2^{n-2} + 1$  points,  $\mathcal{X}$ , where it satisfies (3.7) but does not satisfy, (3.8) as mentioned above, and, consequently, proving the the  $n$ -instance of conjecture.

**Remark.**  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  can be reduced into the Boolean Unsatisfiability Problem, the complement of  $SAT$ , by first converting each multiset in  $A$  into the DNF formula below:

$$\bigvee_{v_0 \in \mathcal{A}} (\neg v_0 \wedge \bigwedge_{v_1 \in \mathcal{A} \setminus \{v_0\}} v_1) \vee \bigvee_{\mathcal{A}_{|3|} \in \mathcal{A}_{|3|}^*} (\bigwedge_{v_0 \in \mathcal{A}_3} \neg v_0 \wedge \bigwedge_{v_1 \in V \setminus \mathcal{A}_{|3|}} v_1) \vee (\bigwedge_{v_0 \in \mathcal{A}} \neg v_0) \quad (3.14)$$

where  $\mathcal{A}_{|3|}^* = \{\mathcal{A}_{|3|} \in \mathbb{P}(\mathcal{A}) : |\mathcal{A}_{|3|}| = 3\}$  and  $\mathcal{A}$  denotes the set of variables in each multiset, and converting each multiset in  $B$  into the DNF formula below:

$$\bigvee_{v \in \mathcal{B}} v \quad (3.15)$$

where  $\mathcal{B}$  denotes the set of variables in each multiset, then joining all these DNF formulae conjunctively. One may realize that, in the case when  $\mathcal{B} = \mathcal{A}$ , the conjunction of  $\bigvee_{u \in \mathcal{B}} u$  and  $\bigwedge_{v_0 \in \mathcal{A}} \neg v$  gives a tautology, and thus for the instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  where  $n = 5$ , we would have a simpler propositional formula. The same observation can be made in the

set of FOL formulae of such instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  where in order to satisfy (3.13), we would need to restrict (3.12) to  $\forall a \in A \ a \in \{[1, 1, 1, 1, 0], [1, 1, 0, 0, 0]\}$ .

We thereby conclude that a plausible approach to proving the upper-bound of the Erdős-Szekeres conjecture through  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  is by induction i.e. we first solve for the instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  where  $n = 5$  - apparently accomplishable with a modern SAT solver running on a high performance computer - and then we prove the inductive hypothesis that  $\forall m \in \mathbb{N}_{\geq 5} \ UNSAT(m) \Rightarrow UNSAT(m+1)$  where  $UNSAT(k)$  denotes the unsatisfiability of the instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  in which  $n = k$ .

**Remark.** The Erdős-Szekeres conjecture would not be disproven even if a certain instance of  $UNSAT_{multiset}^{\mathcal{DPS}^+}$  turns out to yield *False* (i.e. it is satisfiable), since satisfying the constraints only implies that there exists a *div point set* of  $2^{n-2} + 1$  points for a particular  $n \in \mathbb{N}_{\geq 5}$  where

- I. none of its *sub div point sets* of  $n$  points is isomorphic to  $Conv_n$
- II. each of its *sub div point sets* of 5 points has 4, 2 or 0 distinct *sub div point sets* of 4 points isomorphic to  $Conc_4^1$

from which we cannot conclude that such *div point set* is in  $\mathcal{DPS}^+$ , unless it too satisfies the stronger version of *Theorem 2* i.e. unless proven so, we should not rule out the possibility for a *div point set* of 5 points to not be in  $\mathcal{DPS}^+$  despite having 4, 2 or 0 distinct *sub div point sets* of 4 points isomorphic to  $Conc_4^1$  (with the remaining isomorphic to  $Conv_4$ ).

To disprove the Erdős-Szekeres conjecture, not only do we need to show that (3.2) is false, we need to demonstrate that there exists no other constraints besides (2.10), (2.11), and (2.12) which  $\mathcal{X} \in \mathcal{DPS}^*$  has to satisfy such that there exists an interpretation for  $\pi_1(\mathcal{X})$  as some set of points in  $\mathbb{E}^2$  i.e. *Axiom 1*'s consistency with Euclidean geometry.

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