On reducing the Erdös-Szekeres problem into a constraint unsatisfiability problem regarding certain multisets

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Abstract

We introduce the theory of *div point set*, which aims to provide a framework to study the combinatoric nature of any set of points in general position on an Euclidean plane. We then show that the Erdös-Szekeres conjecture can be proven through proving the unsatisfiability of some first-order logic formulae concerning some sets of 5-cardinal multisets over boolean variables under certain constraints.

1 Introduction

More than half a century ago Erdös and Szekeres [1] proved that for all $n \geq 3$, there exists an integer N such that among any N points in general position on an Euclidean plane, there always exists n points forming a convex polygon, and conjectured that the smallest number for N is determined by the function $g(n) = 2^{n-2} + 1$. This was known as the Erdös-Szekeres conjecture (and the problem of determining such N was often referred to the *Happy End Problem*, as it led to the marriage of Szekeres and Klein, who first proposed the question). 25 years after the initial paper, Erdös and Szekeres [2] showed that $g(x)$ cannot be less than 2^{n-2}. Currently the best known bounds for $g(x)$ are

$$2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 1$$

Many improvements for the upper bound have been made throughout the decades. The current upper bound was obtained by Tóth and Valtr [3] in 1998 as an improvement to the previous upper bound by Kleitman and Pachter [4] in the same year.

There are also attempts to verify individual instances of n. In 2002 Szekeres and Peters [5] showed using an exhaustive computer search that the conjecture holds for $n = 6$. Even to this day it remains the best known result. Rather than describing a computer Proof for $n \geq 7$ or improving the upper bound, our aim in this article is to demonstrate that solving some instances of a certain multiset unsatisfiability problem would prove the Erdös and Szekeres conjecture, through the theory of *div point set*.

1.1 preliminary

Throughout the article we would assume Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). The term "class" would be used to denote a collection of sets satisfying some predicate ϕ. A general form of Kuratowski definition would be used to define tuples. 2-tuples would be referred to as ordered pairs. A set of n-tuples would be referred to as relation (and described as binary relation when it is a set of ordered pairs). It would not matter as how natural numbers are defined as long as they satisfy Peano axioms. $\mathbb{N}_{\geq c}$ would be used to refer to the set of natural numbers greater or equal to some $c \in \mathbb{N}$ (e.g. $0 \notin \mathbb{N}_{\geq 1}$). For any 2 natural numbers a, b, $\binom{a}{b}$ denotes the binomial coefficient *a choose b*. Everything would be formulated under first order logic ($\wedge$,$\vee$,$\neg$,$\Rightarrow$ and $\Leftrightarrow$ would mean *and*, *or*, *not*, *imply* and *iff* respectively). We write $A := B$ if A is defined to be equivalent to B. $\forall x_1 \in A \forall x_2 \in A \forall x_3 \in A ... \forall x_n \in A$ would be abbreviated as

$$\forall x_1, x_2, x_3 ... x_n \in A$$

and $\exists x_1 \in A \exists x_2 \in A \exists x_3 \in A ... \exists x_n \in A$ as

$$\exists x_1, x_2, x_3 ... x_n \in A$$

For any set V, $|V|$ would be used to denote its cardinality, and $\mathscr{P}(V)$ be used to denote its power set:

$$\mathscr{P}(V) = \{v : v \subseteq V\}$$

We say a set V is totally ordered over certain binary relation $\geq$ iff for all a, b and c in V,

$$(a \geq b \wedge b \geq a) \Leftrightarrow (a = b)$$
$$(a \geq b \wedge c \geq b) \Leftrightarrow (a \geq c)$$
$$(a \geq b) \vee (b \geq a)$$

The subscript of a set union or set intersection may be omitted to indicate that union or intersection is applied to each element in the set:

For any set, A,

$$\bigcup A = \bigcup_{a \in A} a = a_1 \cup a_2 \cup ..a_n$$

$$\bigcap A = \bigcap_{a \in A} a = a_1 \cap a_2 \cap ..a_n$$

where $|A| = n$ and $a_1, a_2, ..a_n$ are all n distinct elements in A

For any k-tuple T, $\pi_i(T)$ would be used to denote the i-th element of T where $i, k \in \mathbb{N}$ and $i \leq k$; $\pi_\cup(T)$ would be used the denote the union of 1st, 2nd ... k-th elements of a k-tuple;

and $\pi_{\cap}(T)$ would be used to denote intersection in such fashion:

$$\text{For any } k\text{-tuple, } T, \quad \begin{aligned} \pi_{\cup}(T) &= \bigcup_{i=1}^{k} \pi_i(T) \\ \pi_{\cap}(T) &= \bigcap_{i=1}^{k} \pi_i(T) \end{aligned}$$

A single-argument function is any binary relation, f, satisfying

$$\begin{aligned}
&\forall x \in X \\
&\qquad \exists r \in f = \pi_1(r) \\
&\forall r \in f \\
&\qquad \pi_1(r) \in X \\
&\qquad \pi_2(r) \in Y \\
&\forall r_1, r_2 \in f \\
&\qquad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2)
\end{aligned}$$

for some none-empty sets X (often referred to as domain) and Y (referred to as co-domain). We often express the relationship between f, X, and Y as:

$$f : X \longrightarrow Y$$

We write $f(x) = y$ iff there exists an ordered pair (x, y) in f. A function is always assumed to be single-argument, unless otherwise stated. A function f is injective iff

$$\begin{aligned}
&\forall r_1, r_2 \in f \\
&\qquad r_1 = r_2 \Leftrightarrow \pi_2(r_1) = \pi_2(r_2)
\end{aligned}$$

It is subjective iff

$$\begin{aligned}
&\forall y \in Y \\
&\qquad \exists r \in f \quad y = \pi_2(r)
\end{aligned}$$

It is bijective iff it is both injective and surjective, in which case $\xrightarrow{1:1}$ would be used to denote such property. To avoid ambiguity, for any function $f : X \longrightarrow Y$, we would use $f^{members}$ to denote a new function, from $\mathscr{P}(X)$ to $\mathscr{P}(Y)$ such that

$$f^{members}(x) := \bigcup_{a \in x} \{f(a)\}$$

Here is a generalization of it, $f^{members^n}$, defined recursively:

$$f^{members^n}(x) := \bigcup_{a \in x} \{f^{members^{n-1}}(a)\} \text{ where } n \in \mathbb{N}_{\geq 2}$$
$$f^{members^1}(x) := f^{members}(x)$$

A multiset is a generalization of set, where the same element can occur multiple times, making a difference. Two multisets are equal iff (1) both multisets contain the same distinct elements and (2) for each distinct element, it occurs the same number of times in both multisets. A multiset is defined as an ordered pair (A, m_{m}) where $m_{\mathrm{m}} : A \longrightarrow \mathbb{N}_{\geq 1}$ is a function that describes the number of occurrences of some element in the multiset, and A is a set of all distinct elements in the multiset. The cardinality of a multiset (A, m_{m}) is defined as the sum of all $m_{\mathrm{m}}(x)$ for $x \in A$. Multisets are expressed using square brackets, $[\,]$, as compared to sets which use curly brackets, $\{\}$. Here is an example:

$$[f(x) : x \in \mathbb{N}_{\geq 1} : x \leq 3] = [1, 1, 1] = (\{1\}, \{(1, 3)\})$$
$$\text{where } f(x) = 1$$

A hypergraph is a generalization of graph, where an edge can contain any number of vertices. It is defined as an ordered pair (V, E) where E is a subset of $\mathscr{P}(V) \setminus \varnothing$. Elements in V are referred to as vertices while elements in E are referred to as edges or hyperedges. A hypergraph is k-uniformed when

$$\forall e \in E \quad |e| = k$$

where $k \in \mathbb{N}_{\geq 1}$. A full vertex coloring on some graph or hypergraph, (V, E), is defined as a function, $C : V \longrightarrow cDom$, such that

$$|C| = |V|$$
$$\forall c \in C \quad \pi_1(c) \in V \wedge \pi_2(c) \in cDom$$
$$\forall c_1, c_2 \in C \quad c_1 = c_2 \Leftrightarrow \pi_1(c_1) = \pi_1(c_2)$$

where $cDom \subset \mathbb{N}$, and it is often referred to as the set of colors. When $|Dom| = 2$, we say the coloring is monochromatic. We would use $FullCol(G, cDom)$ to denote the set of all possible full vertex colorings on a graph G of the set of colors $cDom$. For any graph G of n vertices, and any non-empty $cDom$,

$$|FullCol(G, cDom)| = n^{|cDom|}$$

2 *Div point set* as a representation for any set of points in general position

We start off by introducing an object which we would be referring to as *div point set*.

Definition 1. A *div point set* is any order-pair (P, Θ_P) satisfying

$$|\Theta_P| = \binom{|P|}{2} \wedge P \neq \varnothing \tag{2.1}$$

$$\forall D_n \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ \bigcup \delta_n = P \setminus d_n \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.2}$$

$$\forall D_n, D_m \in \Theta_P \quad \left| \begin{array}{l} (d_n, \delta_n) \coloneqq D_n \\ (d_m, \delta_m) \coloneqq D_m \\ d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.3}$$

We would be using $\mathscr{DPS}^*$ to denote the class of all ordered pairs satisfying (2.1), (2.2) and (2.3). Thus $\mathscr{X}$ is a *div point set* iff $\mathscr{X} \in \mathscr{DPS}^*$.

For any n points in general position, where $n \geq 2$, we can always select any 2 arbitrary points and draw a line across them, dividing the rest of the points into 2 disjoint sets. So long as the points are in general position, we can be sure that no 3 points forms a line, and thus each of the remaining $n-2$ points would always be in one of these sets. Let's refer to these 2 disjoint sets as *divs* produced by a *divider* made up of 2 distinct points, and the points in the *divs* as *TBD points* of the *divider* (*TBD* is short for *to-be-distributed-among-divs*). The process of selecting 2 distinct points from a set of point P, creating a *divider*, and producing 2 *divs* can be repeated $\binom{|P|}{2}$ times until all sets of 2 points in P are selected.

Any set of points P in general position on an Euclidean plane where $|P| \geq 2$ can be represented by some *div point set* (P, Θ_P). Each member of $D_n \in \Theta_P$ would be referred to as a *dividon*, to be interpreted as follows:

$$(d_n, \delta_n) \coloneqq D_n \quad \left| \begin{array}{l} \{a, b\} \coloneqq d_n \\ \quad a \text{ and } b \text{ represent the 2 points which make up the } \textit{divider} \\ \{div_1, div_2\} \coloneqq \delta_n \\ \quad div_1 \text{ and } div_2 \text{ represent the 2 } \textit{divs} \text{ produced by the } \textit{divider} \\ \quad \bigcup \delta_n \text{ thus represents the set of } \textit{TBD points} \text{ of the } \textit{divider} \end{array} \right. \tag{2.4}$$

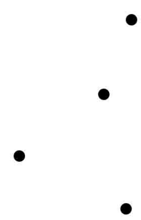

Figure I

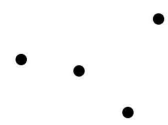

Figure II

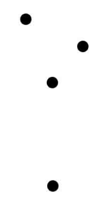

Figure III

The sets of points in *Figures I, II* and *III* can be represented by any *div point set* (A, Θ_A) as long as A is a set of 4 arbitrary elements a, b, c, d and

$$\begin{aligned}\Theta_A = \{&(\{a,b\}, \{(\{c\}, \{d\}\}),\\ &(\{a,c\}, \{(\{b\}, \{d\}\}),\\ &(\{a,d\}, \{(\{b\}, \{c\}\}),\\ &(\{b,c\}, \{(\{a,d\}, \varnothing\}),\\ &(\{b,d\}, \{(\{a,c\}, \varnothing\}),\\ &(\{c,d\}, \{(\{a,b\}, \varnothing\})\})\end{aligned}$$

To make sense of the *div point set* representation, we label the third point from the bottom in *Figure I* and the second point from the bottom in *Figures II* and *III* as a (note that each of these is the point surrounded by the remaining 3 points in the figure). For the rest of the points in each figure we simply label them arbitrarily as b, c, and d.

Only a handful of *div point sets* can be used to represent points in general position in $\mathbb{E}^2$. For majority of $\mathscr{X} \in \mathscr{DPS}^*$, let $(P, \Theta_P) \coloneqq \mathscr{X}$, there exists no meaningful interpretation for P as some sets of points in $\mathbb{E}^2$ such that each $D \in \Theta_P$ is a *dividon* that describes how *TBD points* are distributed between the 2 *divs* produced by each *divider*. A classical example would be (Q, Θ_Q) where Q is a set of 4 arbitrary elements a, b, c, d and

$$\begin{aligned}\Theta_Q = \{&(\{a,b\}, \{(\{c,d\}, \varnothing\}),\\ &(\{a,c\}, \{(\{b,d\}, \varnothing\}),\\ &(\{a,d\}, \{(\{b,c\}, \varnothing\}),\\ &(\{b,c\}, \{(\{a,d\}, \varnothing\}),\\ &(\{b,d\}, \{(\{a,c\}, \varnothing\}),\\ &(\{c,d\}, \{(\{a,b\}, \varnothing\})\})\end{aligned}$$

For a *div point set* (P, Θ_P) to have a meaningful interpretation for P as some set of points in $\mathbb{E}^2$, it has to satisfy certain conditions. For any 3 distinct points, x, y, and z in general position in $\mathbb{E}^2$, let $\langle x, y\rangle^z$ denote the *div* containing z produced by the *divider* made up of the point x and y, and $\langle x, y\rangle^{\text{-}z}$ denote the *div* not containing z produced by the *divider*. After some experimentation with points in $\mathbb{E}^2$, we would make the observation that the following formulas always hold true for any distinct points a, b, c, d in $\mathbb{E}^2$. (2.5) is trivially true, while (2.6), (2.7) and (2.8) are demonstrated in *Figures IV*, *V* and *VI* respectively.

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad a \in \langle b, c\rangle^d \Leftrightarrow d \in \langle b, c\rangle^a\\ &\qquad a \in \langle b, c\rangle^{\text{-}d} \Leftrightarrow d \in \langle b, c\rangle^{\text{-}a}\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad c \in \langle a, b\rangle^{\text{-}d}\\ &\qquad \Leftrightarrow ((a \in \langle b, c\rangle^d \wedge a \in \langle b, d\rangle^c)\\ &\qquad\qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{\text{-}c}))\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad c \in \langle a, b\rangle^{d}\\ &\qquad \Leftrightarrow ((a \in \langle b, c\rangle^d \wedge a \in \langle b, d\rangle^{\text{-}c})\\ &\qquad\qquad \vee (a \in \langle b, c\rangle^{\text{-}d} \wedge a \in \langle b, d\rangle^{c}))\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\forall a, b, c, d\\ &\qquad a \in \langle b, c\rangle^{-d} \wedge a \in \langle b, d\rangle^{-c} \Rightarrow a \in \langle c, d\rangle^b\end{aligned} \tag{2.8}$$

In the context of *div point sets*, (2.5) is always true by (2.2) (recall $\bigcap \delta = \varnothing$), while (2.6), (2.7) and (2.8) can be rewritten as constraints on the *dividons* of a *div point set* as shown in (2.10), (2.11), and (2.12), using a function, ϕ, for determining if two arbitrary points belong to the same *div* in some δ of a *dividon*:

$$\phi(\delta, w) = \begin{cases} 1 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 = div_2 \\ 0 & \text{if} \quad (a \in div_1 \wedge b \in div_2) \Leftrightarrow div_1 \neq div_2 \end{cases} \text{ where } \left| \begin{array}{l} \delta = \{div_1, div_2\} \\ w = \{a, b\} \end{array} \right. \tag{2.9}$$

For any *div point set* (P, Θ_P),

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\quad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad |R| = 4 \\
&\quad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\} \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 1 \\
&\qquad\quad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\quad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad |R| = 4 \\
&\quad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcup_{n=1}^{3} \pi_1(D_n) = R \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\} \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = 0 \\
&\qquad\quad \Leftrightarrow \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) \neq \phi(\pi_2(D_3), R \setminus \pi_1(D_3))\ )
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\quad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\quad |R| = 4 \\
&\quad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad \bigcap_{n=1}^{2} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{2} \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3) \\
&\qquad \Rightarrow (\ \phi(\pi_2(D_1), R \setminus \pi_1(D_1)) = \phi(\pi_2(D_2), R \setminus \pi_1(D_2)) = 0 \\
&\qquad\quad \Rightarrow \phi(\pi_2(D_3), R \setminus \pi_1(D_3)) = 1\ )
\end{aligned}
\tag{2.12}
$$

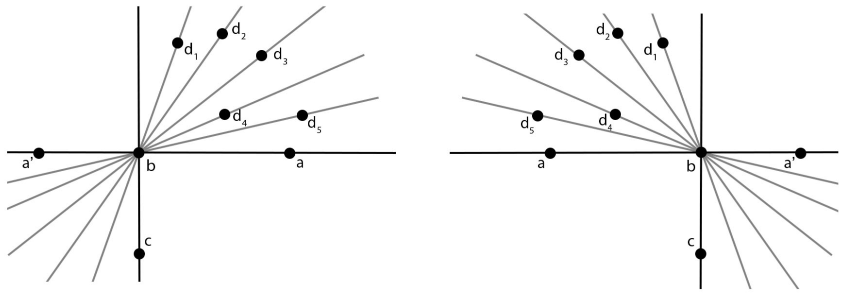

Figure IV

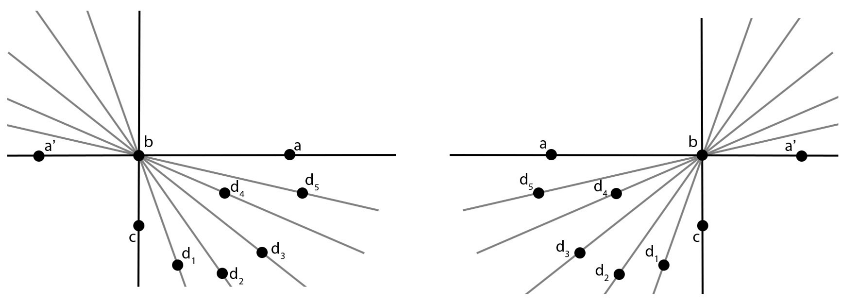

Figure V

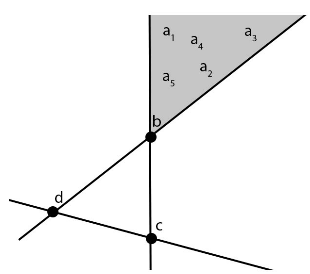

Figure VI

Axiom 1. A *div point sets* (P, Θ_P) has an interpretation for P as some set of points in $\mathbb{E}^2$ such that $D \in \Theta_P$ each describes the relative positions of the points (in terms of how the *TBD points* of each *divider* is distributed between 2 *divs* it produced) iff it is in $\mathscr{DPS}^+$, the class of *div point sets* satisfying (2.10), (2.11), and (2.12).

Remark. For *div point sets* of 3 or less points, it is vacuously true that they satisfy (2.10), (2.11), and (2.12) and thus they are by default in the class $\mathscr{DPS}^+$. This is consistent with Euclidean geometry: any set of 3 points in general position can be represented by any *div point set* of 3 points, and the same goes to any set of 2 points, and any set of 1 point.

Definition 2. We say that two *div point sets* (A, Θ_A) and (B, Θ_B) are isomorphic iff there exists a bijective function $f : A \xrightarrow{1:1} B$ which preseves the structure of the *divions*. Notationally,

$$
\begin{aligned}
(A, \Theta_A) \cong (B, \Theta_B) \Leftrightarrow \exists f : A \xrightarrow{1:1} B \\
&\forall D_A \in \Theta_A \\
&\quad \exists D_B \in \Theta_B \\
&\qquad (d_a, \delta_a) := D_A \\
&\qquad (d_b, \delta_b) := D_B \\
&\qquad f^{members}(d_a) = d_b \Leftrightarrow f^{members^2}(\delta_a) = \delta_b
\end{aligned}
\tag{2.13}
$$

in which case f would be referred to as the isomorphism between the two sets.

Remark. It is trivially true that all div point sets (P, Θ_P) in $\mathscr{DPS}^*$ where $|P| \leq 3$ are isomorphic to any div point sets (Q, Θ_Q) in $\mathscr{DPS}^*$ where $|Q| = |P|$.

Theorem 1. $\neg(\mathfrak{X} \cong Conc_4^1) \Leftrightarrow (\mathfrak{X} \cong Conv_4)$ for all $\mathfrak{X} \in \mathscr{DPS}_4^+$ where $\mathscr{DPS}_4^+$ denotes the *div point sets* of 4 points in $\mathscr{DPS}^+$ and

$$
\begin{aligned}
Conc_4^1 =& (Cc_4^1, \Theta_{Cc_4^1}) & Conv_4 =& (Cv_4, \Theta_{Cv_4}) \\
Cc_4^1 =& \{1, 2, 3, 4\} & Cv_4 =& \{1, 2, 3, 4\} \\
\Theta_{Cc_4^1} =& \{(\{1, 2\}, \{\{3\}, \{4\}\}), & \Theta_{Cv_4} =& \{(\{1, 2\}, \{\{3, 4\}, \varnothing\}), \\
& (\{1, 3\}, \{\{2\}, \{4\}\}), && (\{1, 3\}, \{\{2\}, \{4\}\}), \\
& (\{1, 4\}, \{\{2\}, \{3\}\}), && (\{1, 4\}, \{\{2, 3\}, \varnothing\}), \\
& (\{2, 3\}, \{\{1, 4\}, \varnothing\}), && (\{2, 3\}, \{\{1, 4\}, \varnothing\}), \\
& (\{2, 4\}, \{\{1, 3\}, \varnothing\}), && (\{2, 4\}, \{\{1\}, \{3\}), \\
& (\{3, 4\}, \{\{1, 2\}, \varnothing\})\} && (\{3, 4\}, \{\{1, 2\}, \varnothing\})\}
\end{aligned}
\tag{2.14}
$$

Proof for Theorem 1.

Summary. In Part 1 of the proof we would define a function ψ that returns 0 or 1 based on the *divs* of a *dividon* of some *div point set* in $\mathscr{DPS}_4^+$. In Part 2 we would define $\mathscr{DPS}_4^{\mathbb{N}}$ and a function Col that uses ψ, and show that for every $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, there exists a unique full vertex monochromatic coloring $Col(\pi_2(\mathscr{X}))$ on some hypergraph H, where the vertices of H are the dividers of the *div points sets* in $\mathscr{DPS}_4^{\mathbb{N}}$. In Part 3 we would define the edges of H in such a manner that the coloring $Col(\pi_2(\mathscr{X}))$ on H satisfies some conditions iff $\mathscr{X}$ satisfies (2.10) and (2.11). In Part 4 we demonstrate that for the coloring to satisfy the conditions, there exists only 3 $\mathscr{Scenarios}$, and colorings in $\mathscr{Scenario}$ 2 and 3 are isomorphic to $Col(\pi_2(Conc_4^1))$ and $Col(\pi_2(Conv_4))$, and $Conc_4^1$ and $Conv_4$ satisfy (2.12), but not the other *div point set* the coloring in $\mathscr{Scenario}$ 1 is based on, and thus proving Theorem 1.

Part 1. For any div point set (P, Θ_P) in $\mathscr{DPS}_4^+$, since $|P| = 4$, we can be certain that

$$\begin{aligned}
&\forall D \in \Theta_P \\
&\qquad \pi_2(D) \in \{type_0, type_1\} \\
&\qquad \{a, b\} = P \setminus \pi_1(D) \\
&\qquad type_0 = \{\{a\}, \{b\}\} \\
&\qquad type_1 = \{\{a, b\}, \varnothing\}
\end{aligned} \tag{2.15}$$

Recall that in (2.9), we define a function ϕ that takes in some $\pi_2(D)$ and a set of 2 *TBD points*, and returns 1 if the *TBD points* belong to the same *div* in $\pi_2(D)$, or 0 if they belong to different *divs* in $\pi_2(D)$. For $\mathscr{X} \in \mathscr{DPS}_4^+$, we can define a new function ψ, a simpler version of ϕ that does basically the same thing by exploiting (2.15), namely the fact every $\pi_2(D)$ is either $type_0$ or $type_1$ (since that there are only 2 *TBD points* for each divider):

$$\psi(\delta) = \begin{cases} 1 & \text{if } \exists div \in \delta \quad |div| = 2 \\ 0 & \text{if } \forall div \in \delta \quad |div| = 1 \end{cases} \tag{2.16}$$

For every *dividon* D of any $\mathscr{X} \in \mathscr{DPS}_4^+$, we have

$$\phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D)) \tag{2.17}$$

Part 2. Let's define $\mathscr{DPS}_4^{\mathbb{N}}$ to be the set of all *div point sets* (P, Θ_P) for which $P = \{1, 2, 3, 4\}$. All $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ would have the same *dividers* (Recall the set of *dividers* is just the set of elements in $\mathscr{P}(P)$ whose cardinality is 2.) Now let $H = (V, E)$ be a hypergraph whose vertices are the *dividers* of *div point sets* in $\mathscr{DPS}_4^{\mathbb{N}}$. Using ψ, we can define a bijective function, Col, that transforms the set of *dividons* of a *div point set* in $\mathscr{DPS}_4^{\mathbb{R}}$ into some full vertex monochromatic coloring for H.

$$\begin{aligned}
Col : \{\pi_2(\mathscr{X}) : \mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}\} &\longrightarrow FullCol(H, \{0, 1\}) \\
Col(\Omega_P) &= \{(\pi_1(D), \psi(\pi_2(D))) : D \in \Omega_P\}
\end{aligned} \tag{2.18}$$

It is bijective because

$$\begin{aligned}&\forall \mathscr{X}_1, \mathscr{X}_2 \in \mathscr{DPS}^{\mathbb{R}}\\ &\quad Col(\pi_2(\mathscr{X}_1)) = Col(\pi_2(\mathscr{X}_2)) \Leftrightarrow \mathscr{X}_1 = \mathscr{X}_2\end{aligned} \tag{2.19}$$

due to the fact that ψ is bijective for every dividon of any *div point set* of 4 points.

Part 3. Now let's define any set of three *dividers* containing 1 element in common to be an edge of H (recall that the vertices are the *dividers*), notationally,

$$E = \{e \in \mathscr{P}(V) : |e| = 3 \wedge |\bigcap e| = 1\} \tag{2.20}$$

H is a 3-uniform hypergraph with 4 hyperedges. For $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$ to satisfy (2.10) and (2.11) is equivalent to having $Col(\pi_2(\mathscr{X})) \in FullCol(H, \{0,1\})$ to satisfy the following:

I. If a vertex, V, is colored 0, the other 2 vertices belonging to the same edge as V must have the same coloring.

II. If a vertex, V, is colored 1, the other 2 vertices belonging to the same edge as V must have different colorings.

This is due to the fact, for any $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, (2.10) and (2.11) can be rewritten as having the colors on the vertices of each edge to satisfy some formulae, namely the following:

$$\begin{aligned}&\forall e \in E\\ &\quad \forall d_1, d_2, d_3 \in e\\ &\quad\quad d_1 \neq d_2 \neq d_3\\ &\quad\quad \Leftrightarrow Col(\pi_2(\mathscr{X}))(d_1) = 1 \Leftrightarrow Col(\pi_2(\mathscr{X}))(d_2) = Col(\pi_2(\mathscr{X}))(d_3))\end{aligned} \tag{2.21}$$

$$\begin{aligned}&\forall e \in E\\ &\quad \forall d_1, d_2, d_3 \in e\\ &\quad\quad d_1 \neq d_2 \neq d_3\\ &\quad\quad \Leftrightarrow (Col(\pi_2(\mathscr{X}))(d_1) = 0 \Leftrightarrow Col(\pi_2(\mathscr{X}))(d_2) \neq Col(\pi_2(\mathscr{X}))(d_3))\end{aligned} \tag{2.22}$$

(recall that Col is the function to transform some $\pi_2(\mathscr{X})$ into a coloring, while $Col(\pi_2(\mathscr{X}))$ is the actual coloring, which is defined as a function in *Preliminary*) This is a result of

$$\begin{aligned}&\forall p_1, p_2, p_3, p_4 \in P\\ &\quad R := \bigcup_{n=1}^{4} \{p_n\}\\ &\quad \forall D \in \Theta_P\\ &\quad\quad \phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D), P \setminus \pi_1(D)) = \psi(\pi_2(D))\end{aligned} \tag{2.23}$$

for any div point set (P, Θ_P) for which $|P| = 4$ (recall (2.15)), and any *dividons* D_1, D_2 and D_3 where

$$|\bigcap_{n=1}^{3} \pi_1(D_n)| = 1 \wedge |\bigcup_{n=1}^{3} \pi_1(D_n)| = 4 \tag{2.24}$$

would always have the *dividers* d_1, d_2 and d_3 respectively, where

$$|\bigcap_{n=1}^{3} d_n| = 1 \wedge d_1 \neq d_2 \neq d_3 \tag{2.25}$$

which are precisely what makes up an edge in E (recall (2.20)). Therefore a *div point set* of 4 points, $\mathscr{X}$, satisfies (2.10) and (2.11) iff $Col(\pi_2(\mathscr{X}))$ satisfies I and II.

Part 4. To satisfy I and II, 3 vertices belonging to the same edge must either be colored $[0, 0, 0]$ or $[0, 1, 1]$.

Suppose we start off by giving some vertices belonging to the same edge the coloring of $[0, 0, 0]$, by I this would indicate that the rest of the vertices need to have the same colors (recall that each vertex belongs to 2 different edges). We can either end up with H having all vertices colored 0 (let's call it $\mathscr{Scenario}$ 1), or 3 vertices colored 0 and 3 vertices colored 1 (let's call it $\mathscr{Scenario}$ 2).

Now suppose we start off by giving some vertices belonging to the same edge the coloring of $[0, 1, 1]$, by I this would indicate that the remaining 2 vertices of another edge, which the vertex colored 0 belongs to, must have the same colors. If we give them the coloring of $[0, 0]$, we would have an edge with vertices colored $[0, 0, 0]$, and the last uncolored vertex must then be colored 1, so we end up in $\mathscr{Scenario}$ 2 again. If we give them the coloring of $[1, 1]$, we would end up with 1 vertex colored 0 and 4 vertices colored 1, in which case the last uncolored vertex would need to be colored 0, since it belongs to 2 edges both with 2 vertices colored 1. Let's name this $\mathscr{Scenario}$ 3, where 2 vertices are colored 0 and 4 vertices are colored 1.

A pictorial description of the colorings is shown in Figure VII.

Scenario 1 describes a coloring isomorphic to $Col(\pi_2(\mathscr{X}_\varnothing))$ where $\mathscr{X}_\varnothing \in \mathscr{DPS}_4^{\mathbb{N}}$ and

$$\begin{aligned}\pi_2(\mathscr{X}_\varnothing) =\{&(\{1,2\}, \{(\{3,4\}, \varnothing\}),\\ &(\{1,3\}, \{(\{2,4\}, \varnothing\}),\\ &(\{1,4\}, \{(\{2,3\}, \varnothing\}),\\ &(\{2,3\}, \{(\{1,4\}, \varnothing\}),\\ &(\{2,4\}, \{(\{1,3\}, \varnothing\}),\\ &(\{3,4\}, \{(\{1,2\}, \varnothing\})\})\end{aligned}$$

while Scenario 2 describes a coloring isomorphic to $Col(\pi_2(Conc_4^1))$ and scenario 3 descrbies a coloring isomorphic to $Col(\pi_2(Conv_4))$. $Conc_4^1$ and $Conv_4$ both satisfy (2.12), and $\mathscr{X}_\varnothing$

does not. Since any div point set of 4 points is isomorphic to some $\mathscr{X} \in \mathscr{DPS}_4^{\mathbb{N}}$, and only $Conc_4^1$ and $Conv_4$ satisfy all (2.10), (2.11), and (2.12), we conclude that

$$\forall X \in \mathscr{DPS}_4^+ \quad \exists a \in \{Conc_4^1, Conv_4\} \quad X \cong a$$

$\square$

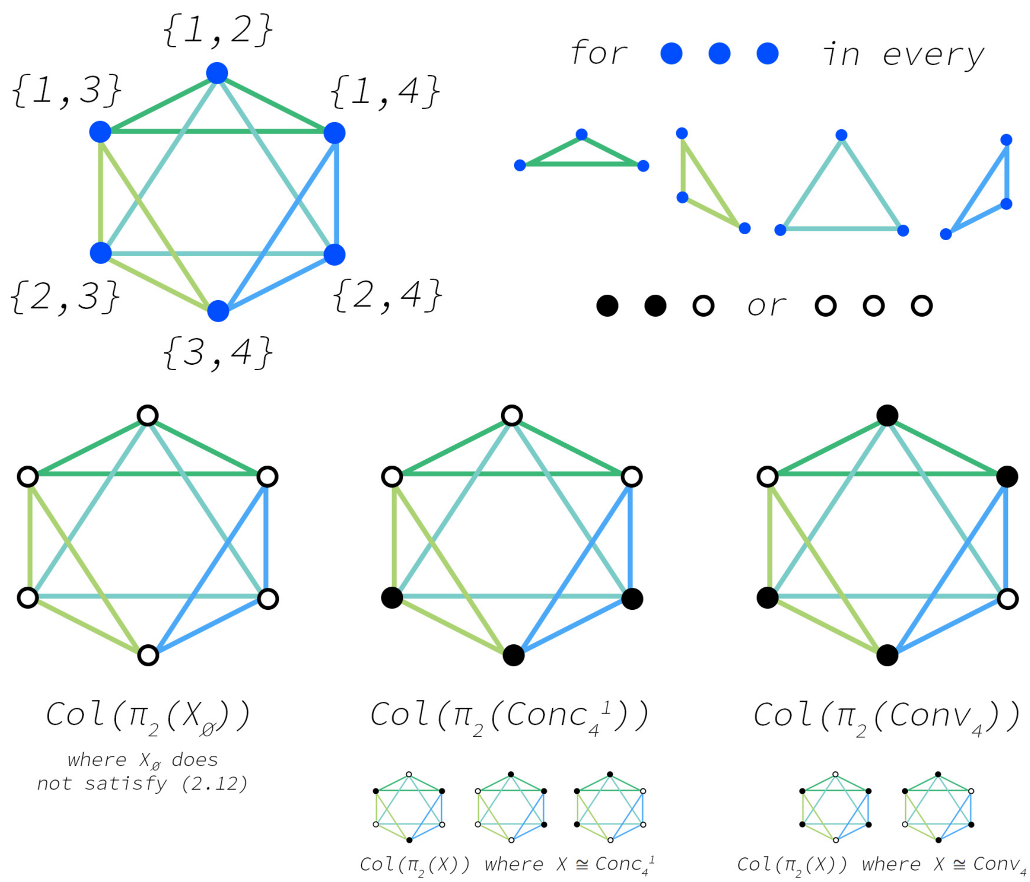

Figure VII

Remark. In Euclidean geometry, Theorem 1 can be interpreted as stating that for any set of 4 distinct points in general positions, it is either the case that it forms a structure where 1 point is inside a triangle formed by connecting the rest of 3 points, or the case that a convex polygon can be created by connecting the 4 points in a certain manner, which can be verified rather easily by a human child with a pen, a piece of paper and a love for Euclidean geometry.

2.1 *unit div point set* and *sub div point set*

For *div point sets* of 5 or more points, the function ψ defined in (2.16) would not be really useful since there would be 3 or more *TBD points* in each *dividon*. That means we cannot apply to same technique above to derive *div point sets* of 5 or more points satisfying (2.10), (2.11) and (2.12). With that in mind, we introduce the object *unit div point set* which makes use of *unit dividons*.

Definition 3. A *unit div point set* is any order-pair (P, Ω_P) satisfying (2.26), (2.27) and (2.28).

$$|\Omega_P| = \binom{|P|}{2}\binom{|P|-2}{2} \wedge P \neq \varnothing \tag{2.26}$$

$$\forall D_n \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ |d_n| = 2 \\ d_n \in \mathscr{P}(P) \\ |\delta_n| = 2 \\ |\bigcup \delta_n| = 2 \\ \bigcup \delta_n \in \mathscr{P}(P \setminus d_n) \\ \bigcap \delta_n = \varnothing \end{array} \right. \tag{2.27}$$

$$\forall D_n, D_m \in \Omega_P \quad \left| \begin{array}{l} (d_n, \delta_n) := D_n \\ (d_m, \delta_m) := D_m \\ d_n \cup \bigcup \delta_n = d_m \cup \bigcup \delta_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.28}$$

We would be using $\mathscr{UDPS}^*$ to denote the class of all *unit div point set*.

Remark. Similar to how *div point sets* of 4 points always satisfy (2.15), a *unit div point set* always satisfies (2.29).

$$\begin{array}{l} \forall \mathfrak{X} \in \mathscr{UDPS}^* \\ \quad (P, \Omega_P) := \mathfrak{X} \\ \quad \forall D \in \Omega_P \\ \quad\quad \pi_2(D) \in \{type_0, type_1\} \\ \quad\quad \{a, b\} \subseteq P \setminus \pi_1(D) \\ \quad\quad\quad type_0 = \{\{a\}, \{b\}\} \\ \quad\quad\quad type_1 = \{\{a, b\}, \varnothing\} \end{array} \tag{2.29}$$

For any *unit div point set*, (P, Ω_P), we can use ψ (defined in (2.16)) to map every $\pi_2(D) \in \Omega_P$ to some $k \in \{0, 1\}$.

Remark. One may immediately notice that any *div point sets* of 4 points also satisfy (2.26), (2.27) and (2.28), and any *unit div point set* of 4 points also satisfy (2.1), (2.2) and (2.3), and that is the say

$$\{\mathcal{X}_{udps} \in \mathcal{UDPS}^* : |\pi_1(X)| = 4\} = \{\mathcal{X}_{dps} \in \mathcal{DPS}^* : |\pi_1(X)| = 4\} \tag{2.30}$$

by virtue of the fact that

$$\binom{|4|}{2}\binom{|4-2|}{2} = \binom{|4|}{2} \tag{2.31}$$

and

$$\forall \mathcal{X} \in \mathcal{UDPS}^* \quad \left| \begin{array}{l} (P, \Omega_P) \coloneqq \mathcal{X} \\ |P| = 4 \\ \forall D_n \in \Omega_P \\ \qquad \delta_n = P \setminus d_n \\ \forall D_n, D_m \in \Omega_P \\ \qquad d_n = d_m \Leftrightarrow D_n = D_m \end{array} \right. \tag{2.32}$$

As we can see, the difference between a *div point set* and a *unit div point set* lies in that the former relies on a single *dividon* to describe the distribution of $|P| - 2$ *TBD points* between the 2 *divs* produced by a *divider*, while the later relies on $\binom{|P|-2}{2}$ *unit dividons* for that (since each *unit dividon* only describes the distribution of 2 *TBD points*). For every $\mathcal{X}_{dps} \in \mathcal{DPS}^*$ there exists a unique $\mathcal{X}_{udps} \in \mathcal{UDPS}^*$ which $\mathcal{X}_{dps}$ can be transformed into, by breaking down each *dividon* into $\binom{|P|-2}{2}$ *unit dividons*, achievable using the function $\mathcal{bd}$ defined below.

$$\begin{aligned} \mathcal{bd}(D, P) &= \{(\pi_1(D), \mathcal{d}_u(x, \pi_2(D)) : x \in \mathcal{P}(P \setminus \pi_1(D)) : |x| = 2\} \\ \mathcal{d}_u(x, divs) &= \begin{cases} \{x, \varnothing\}, & \text{if } x \subseteq divs \\ \{\{a\}, \{b\}\}, & \text{if } a \in div_1 \wedge b \in div_2 \\ & \text{where } x = \{a, b\} \text{ and } divs = \{div_1, div_2\} \end{cases} \end{aligned} \tag{2.33}$$

Definition 4. The function $\mathcal{F}^{\mathcal{DPS}}_{\rightarrow udps}$ transforms a *div point set* into a *unit div point set.*

$$\mathcal{F}^{\mathcal{DPS}}_{\rightarrow udps}(\mathcal{X}_{dps}) = (\pi_1(\mathcal{X}_{dps}), \bigcup \{\mathcal{bd}(D, \pi_1(\mathcal{X}_{dps})) : D \in \pi_2(\mathcal{X}_{dps})\} \tag{2.34}$$

$\mathcal{F}^{\mathcal{DPS}}_{\rightarrow udps}$ can be implemented in Haskell as follows:

```
import Control.Monad
import Data.List ((\\))
```

```haskell
powerList = filterM (const [True, False])

f:: ([Int],[([Int],[[Int]])]) -> ([Int],[([Int],[[Int]])])
f (points,dividons) = (points,unit_dividons)
  where
    unit_dividons = foldl (++) [] $ map get_unit_dividons dividons
    get_unit_dividons (d,(delta1:_)) = [(d,(\(a:b:_)->
      if a `in_same_div_as_b` b
        then [[a,b],[]]
        else [[a],[b]])
      x ) |
      x <- powerList (points \\ d), length x == 2,
      let (in_same_div_as_b) a b = (a `elem` delta1) == (b `elem` delta1)]
```

Remark. If we apply $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ on *div point sets* of 4 points we would immediately realize that $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ returns the same ordered pair, since for *div point sets* of 4 points, $\Omega_{sub} \subset \Omega_P$ in (2.34) would contain only one element and the element is some $D_\Theta \in \Theta_P$. For *div point sets* of 5 or more points Ω_{sub} would contain 3 or more elements, thus

$$\forall \mathscr{X} \in \mathscr{DPS}^* \qquad \mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}(\mathscr{X}) = \mathscr{X} \Leftrightarrow |\pi_1(\mathscr{X})| = 4 \tag{2.35}$$

On the other hand, applying $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ on *div point sets* with 3 or less points would result in $(P, \varnothing)$ since $\binom{n-2}{2} = 0$ for $n < 4$ and that is not going to be useful. So it is more sensible to define $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ over *div point sets* of 4 or more points.

$$\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}} : \mathscr{DPS}^*_{\geq 4} \longrightarrow \mathscr{UDPS}^* \tag{2.36}$$

Lemma 1. $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}} : \mathscr{DPS}^*_{\geq 4} \longrightarrow \mathscr{UDPS}^*$ is injective but not surjective. If the codomain is defined to be $\mathscr{UDPS}^\Theta$, the set of *unit div point sets* of 4 or more points satisfying (2.37), $\mathscr{F}_{\rightarrow udps}^{\mathscr{DPS}}$ is then bijective.

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\qquad (D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3) \\
&\qquad \Rightarrow (\psi(\pi_2(D_1)) = 1 \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))) \\
&\qquad\qquad \wedge (\psi(\pi_2(D_1)) = 0 \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3)))
\end{aligned} \tag{2.37}$$

Proof for Lemma 1. It is injective because in (2.34) Ω_{sub} differs depending on $D \in \Theta_P$ as a result of d_u in $d\mathfrak{b}$ being injective. It is not surjective onto the co-domain $\mathscr{UDPS}^*$, but surjective onto the co-domain $\mathscr{UDPS}^\Theta$, as a consequence of

I. $|\delta_n| = 2$ in (2.2): *Unit div point sets* with *unit dividons* such as

$$\{(a,b),(\{c\},\{d\})\},\{(a,b),(\{c\},\{e\})\},\{(a,b),(\{e\},\{d\})\}$$

can only be transformed from a *div point set* where $|\delta_n| = 3$ for some *dividion*, in this case:

$$\{(a,b),(\{c\},\{d\},\{e\})\}$$

Thus we have

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\quad D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3 \\
&\quad \Leftrightarrow \neg((\psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = \psi(\pi_2(D_3)) = 0)
\end{aligned} \tag{2.38}$$

II. Associativity: if a and b are in the same *div*, and b and c are in the same *div*, a and c must be in the same *div*. So unit *div* points set with *unit dividons* such as

$$\{(a,b),(\{c,d\},\varnothing)\},\{(a,b),(\{c,e\},\varnothing)\},\{(a,b),(\{e\},\{d\})\}$$

can not be transformed from any *div point set*. Thus we have

$$\begin{aligned}
&\forall D_1, D_2, D_3 \in \Omega_P \\
&\quad D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3 \\
&\quad \Leftrightarrow \neg(\psi(\pi_2(D_1)) = \psi(\pi_2(D_2) = 1 \wedge \psi(\pi_2(D_3)) = 0)
\end{aligned} \tag{2.39}$$

Combining (2.39) and (2.38) gives (2.37). □

Lemma 2. A *unit div point set* (P, Ω_P) has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$ such that $D \in \Omega_P$ each describes the relative positions of the points (in terms of how 2 *TBD points* of each *divider* is distributed between *divs* it produced) iff it is in $\mathcal{UDPS}^+$, the class of *unit div point sets* of 4 ore more points satisfying (2.37), (2.41), (2.42), and (2.43), in which ξ is a function that returns the union of the *divider* and the *TBD points* in a *unit dividon*, D, notationally,

$$\xi(D) = \pi_1(D) \cup \bigcup \pi_2(D) \tag{2.40}$$

For any *unit div point set* (P, Ω_P),

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Omega_P \\
&\qquad\qquad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad\qquad \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = 1 \\
&\qquad\qquad\qquad \Leftrightarrow \psi(\pi_2(D_2)) = \psi(\pi_2(D_3))\ )
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad\qquad \wedge \bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = 0 \\
&\qquad\qquad\qquad \Leftrightarrow \psi(\pi_2(D_2)) \neq \psi(\pi_2(D_3))\ )
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
&\forall p_1, p_2, p_3, p_4 \in P \\
&\qquad R := \bigcup_{n=1}^{4} \{p_n\} \\
&\qquad |R| = 4 \\
&\qquad \Leftrightarrow \forall D_1, D_2, D_3 \in \Theta_P \\
&\qquad\qquad (\ \xi(D_1) = \xi(D_2) = \xi(D_3) = R \wedge D_1 \neq D_2 \neq D_3 \\
&\qquad\qquad \wedge \bigcap_{n=1}^{2} \pi_1(D_n) = \{p_4\} \wedge \bigcup_{n=1}^{2} \pi_1(D_n) \setminus \{p_4\} = \pi_1(D_3)\ ) \\
&\qquad\qquad \Rightarrow (\ \psi(\pi_2(D_1)) = \psi(\pi_2(D_2)) = 0 \\
&\qquad\qquad\qquad \Rightarrow \psi(\pi_2(D_3)) = 1\ )
\end{aligned}
\tag{2.43}
$$

Proof for Lemma 2. A *div point set* $\mathscr{X}_{dps}$ satisfying (2.10), (2.11), and (2.12), iff the *unit div point set* $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_{dps})$ satisfies (2.41), (2.42), and (2.43). This can be demonstrated in a similar way as (2.23): for any *unit divdion* D_u of some *unit div point set*, $\mathscr{A}_{udps}$, and its corresponding *divdion* D of the *div point set* $\mathscr{A}_{dps}$ where $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{A}_{dps}) = \mathscr{A}_{udps}$ - corresponding in the sense that $D_u \in \mathscr{bd}(D, \pi_2(\mathscr{A}_{dps}))$ and so $\pi_1(D_u) = \pi_1(D)$ - let $R := \xi(D_u)$, we would have

$$\phi(\pi_2(D), R \setminus \pi_1(D)) = \phi(\pi_2(D_u), \bigcup \pi_2(D_u)) = \psi(\pi_2(D_u)) \tag{2.44}$$

By restricting the *unit dividons* into $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$, we can then replace the predicate $\bigcap_{n=1}^{3} \pi_1(D_n) = \{p_4\}$ with $D_1 \neq D_2 \neq D_3$, and every occurrence of $\phi(\pi_2(D_n), R \setminus \pi_1(D_n))$ with $\psi(\pi_2(D_n))$ (for $n \in \{1, 2, 3\}$) in (2.10), (2.11), and (2.12). This would gives (2.41), (2.42), and (2.43): they are basically a different way of expressing (2.10), (2.11), and (2.12) in the case of *unit div point sets*. Thus any *unit div point set* (P, Ω_P) in $\mathscr{UDPS}^+$ has an interpretation for P as some set of 4 or more points in $\mathbb{E}^2$, similar to how any *div point set* (P, Θ_P) in $\mathscr{DPS}^+$ has an interpretation for P by *Axiom 1.* □

Lemma 3. A *unit div point set* is in $\mathscr{UDPS}^+$ iff it is isomorphic to some *unit div point set* (P, Ω_P) in $\mathscr{UDPS}^{\mathbb{N}}$ where $Col_{udps}(\Omega_P)$, a full vertex monochromatic coloring on H_{udps}, satisfies (2.48) and (2.49), while $\mathscr{UDPS}^{\mathbb{N}}$ is the class of all *unit div point sets* (P, Θ_P) where $P \subset \mathbb{N}$ and $|P| \geq 4$, and Col_{udps} is a function similar to Col in (2.18),

$$Col_{udps}(\Omega_P) = \{((\pi_1(D), \bigcup \pi_2(D)), \psi(\pi_2(D)) : D \in \Omega_P\} \tag{2.45}$$

and H_{udps} is a 3-and-6-uniform hypergraph with 2 sets of hyperedges, E_1 and E_2, defined as a 3-tuple $H_{udps} = (V_{udps}, E_1, E_2)$, constructed based on P:

$$\begin{aligned} V_{udps} &= \bigcup \{V_{of}(d) : d \in \mathscr{P}(P) : |d| = 2\} \\ E_1 &= \{e \in \mathscr{P}(V) : |e| = 6 \wedge \forall v_1, v_2 \in e \; \pi_{\cup}(v_1) = \pi_{\cup}(v_2)\} \\ E_2 &= \{e \in \mathscr{P}(V) : |e| = 3 \wedge \forall v_1, v_2 \in e \; \pi_1(v_1) = \pi_2(v_2) \wedge |\bigcup_{v \in e} \pi_2(v)| = 3\} \end{aligned} \tag{2.46}$$

with $V_{of}(d)$ being a function that returns a set of ordered pair consists of *divider* and *TBD points* of *unit dividons* of the same *divider*,

$$V_{of}(d) = \{(d, P_{TBD}) : P_{TBD} \in \mathscr{P}(P \setminus x) : |P_{TBD}| = 2\} \tag{2.47}$$

and, finally, here are the conditions that the coloring needs to satisfy:

$$
\begin{aligned}
&\forall e \in E_1 \\
&\quad \exists v_1, v_2 \in e \quad \left| \begin{array}{l} \pi_1(v_1) = \pi_2(v_2) \\ \pi_1(v_2) = \pi_2(v_1) \\ C(v_1) = C(v_2) = 0 \\ C^{members}(e \setminus \{v_1, v_2\}) = \{1\} \end{array} \right. \\
&\quad \Leftrightarrow \neg\exists v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} |\pi_1(v_1) \cap \pi_1(v_2) \cap \pi_1(v_3)| = 1 \\ C(v_1) = C(v_2) = C(v_3) = 0 \\ C^{members}(e \setminus \{v_1, v_2, v_3\}) = \{1\} \end{array} \right.
\end{aligned}
\tag{2.48}
$$

$$
\begin{aligned}
&\forall e \in E_2 \\
&\quad \forall v_1, v_2, v_3 \in e \quad \left| \begin{array}{l} v_1 \neq v_2 \neq v_3 \\ \Rightarrow (C(v_1) = 1 \Leftrightarrow C(v_2) = C(v_3)) \\ \qquad \wedge (C(v_1) = 0 \Leftrightarrow C(v_2) \neq C(v_3)) \end{array} \right.
\end{aligned}
\tag{2.49}
$$

wherein C is the coloring based on the set of *unit divdons* of some $\mathscr{X}_{\mathscr{updś}} \in \mathscr{UDPS}^{\mathbb{N}}$, i.e. $C = Col_{\mathscr{udps}}(\pi_2(\mathscr{X}_{\mathscr{upds}}))$.

Remark. One may notice that the construction of $H_{\mathscr{udps}}$ depends solely on $\pi_1(\mathscr{X}_{\mathscr{upds}})$ (i.e. the points of a *unit div point set*), as different from the coloring, which depends solely on $\pi_2(\mathscr{X}_{\mathscr{upds}})$ (i.e. the set of *unit dividons*). This is similar to how the 3-uniform hypergraph H and the coloring on its vertices are defined back in the Proof for Theorem 1.

However, each vertex of $H_{\mathscr{udps}}$ is an ordered pair, structurally different from each vertex of H which is a set with cardinality of 2. Such definition for the vertices of $H_{\mathscr{udps}}$ in terms of not only the *divider* of a *unit dividon* but also its *TBD points* is necessary. This is because for any *unit div point set*, (P, Ω_P), there exists $\binom{|P|-2}{2}$ distinct *unit dividons* sharing a common *divider*, where $\binom{|P|-2}{2} > 1$ when $|P| \geq 5$. In order to distinguish *unit dividons* from one another in a *unit div point set* of 5 or more points, we would need to know both its *divider* and its *TBD points*.

Remark. For any *unit div point set*, (P, Ω_P) where $|P| = 4$, E_2 of $H_{\mathscr{upds}}$ constructed based on P is an empty set, and thus (2.49) is trivially true for any coloring on such $H_{\mathscr{upds}}$. On the other hand, there would only be 1 edge in E_1 and the coloring $C_{\mathscr{udps}}(\Omega_P)$ satisfies (2.48) iff (P, Ω_P) is isomorphic to $\mathscr{X} \in \{Conv^4, Conc^4_1\}$: in (2.48), the first-order predicate before the logical connective $\Leftrightarrow$ is true iff $\mathscr{X}$ is isomorphic to $Conv^4$, while the first-order predicate after $\Leftrightarrow$ is true iff $\mathscr{X}$ is isomorphic to $Conc^4_1$.

Proof for Lemma 3. Every *unit div point set* of 4 or more points is isomorphic to some *unit div point set* in $\mathscr{UDPS}^{\mathbb{N}}$. For a *unit div point set* to be in $\mathscr{UDPS}^{+}$, it has to satisfy (2.37), (2.41), (2.42), and (2.43). It is clear that a *unit div point set*, (P, Ω_P), satisfies (2.37) iff $Col(\Omega_P)$ on the H_{udps} constructed based on P satisfies (2.49): (2.49) is simply a different way of writing (2.37) by first defining the order pairs $(d_n, \bigcup \delta_n)$ of some *unit dividons* $D_n = (d_n, \delta_n)$ that satisfy the necessary conditions (namely $(D_1 \neq D_2 \neq D_3 \wedge \pi_1(D_1) = \pi_1(D_2) = \pi_1(D_3) \wedge |\bigcup_{n=1}^{3} \pi_2(D_n)| = 3)$) to be vertices of an edge in E_2 (recall (2.46)). On the other hand (P, Ω_P) would satisfy (2.41), (2.42), and (2.43) iff $Col(\Omega_P)$ on H_{udps} constructed based on P satisfies (2.48).

(2.41), (2.42), and (2.43) can be summarized as formulae with universal quantification of 4 points in P, where if these points are distinct, some conditional proposition regarding certain distinct *unit dividons* in Ω_P must be true. The common characteristic of the conditional proposition in all 3 formulae is that $\xi(D_1) = \xi(D_2) = \xi(D_3) = R$ is a part of the conjunction that makes up the antecedent. For any unit div point sets of 4 or more points, there are a total of 6 *unit dividons* D where $\xi(D) = R$ for any $R \subseteq P$ where $|R| = 4$, obtainable using $\mathscr{UDs}$, a function which takes in a set of 4 points, R, and returns a set of such ordered pair:

$$\begin{aligned} \mathscr{UDs}(R) &= \{\mathscr{ud}(d) : d \in \mathscr{P}(R) : |d| = 2\} \\ \mathscr{ud}(d) &= (d,\ R \setminus d) \end{aligned} \tag{2.50}$$

One may notice that $\mathscr{UDs}(R)$ always has a cardinality of $\binom{4}{2} = 6$ and that E_1 of the H_{udps} constructed based on some P can be expressed in terms of $\mathscr{UDs}$.

$$E_1 = \{\mathscr{UDs}(R) : R \in \mathscr{P}(P) : |R| = 4\} \tag{2.51}$$

By *Theorem 1*, a *unit div point set* of 4 points (recall that *div point sets* of 4 points are their own *unit div point sets*) satisfies (2.41), (2.42), and (2.43) iff it is isomorphic to either $Conc_4^1$ or $Conc_4$. More fundamentally, this means that any *unit div point set*, (P, Ω_P), satisfies (2.41), (2.42), and (2.43) iff each set of 6 *unit dividons*, $\Omega' \subseteq \Omega_P$, where for all $D \in \Omega'$, $\xi(D)$ is equivalent to a subset of 4 cardinality of P, is isomorphic[1] to either $\pi_2(Conc_4^1)$ or $\pi_2(Conv_4)$. The set of all such Ω' for any $\mathscr{X} \in \mathscr{UDPS}^*$, can be expressed as a function $All_{\Omega'}$ where:

$$\begin{aligned} All_{\Omega'}(\mathscr{X}) &= \{\Omega'_{basedOn}(R, \pi_2(\mathscr{X})) : R \in \mathscr{P}(\pi_1(\mathscr{X})) : |R| = 4\} \\ \Omega'_{basedOn}(R, \Omega_P) &= \{D : D \in \Omega_P : \xi(D) = R\} \end{aligned} \tag{2.52}$$

One may then realize that the following equation holds true for E_1 of any H_{upds} constructed based on $\pi_1(\mathscr{X})$:

$$E_1 = \{\{(\pi_1(D), \bigcup \pi_2(D)) : D \in \Omega'\} : \Omega' \in All_{\Omega'}(\mathscr{X})\} \tag{2.53}$$

Therefore any *unit div point set* of n points in $\mathscr{UDPS}^{\mathbb{N}}$, (P, Ω_P), satisfies (2.41), (2.42), and (2.43) iff for all 4-cardinal $R \subseteq P$, a subset C' of $Col_{\mathscr{udps}}(\Omega_P)$, the monochromatic vertex coloring on $H_{\mathscr{upds}}$ constructed based on P, where for all $c \in C'$, $\xi(\pi_1(c)) = R$, is isomorphic[2] to either the coloring $Col(\pi_2(Conc_4^1))$ or $Col(\pi_2(Conv_4))$. Notationally,

$$\begin{aligned}
&\forall R \in \{P' \in \mathscr{P}(P) : |P'| = 4\} \\
&\quad C' := C'_{of}(R, Col_{\mathscr{udps}}(\Omega_P)) \\
&\quad C' \cong Col(\pi_2(Conc_4^1)) \Leftrightarrow \neg(C' \cong Col(\pi_2(Conv_4))) \\
&\text{where } C'_{of}(R, C) = \{c \in C : \xi(\pi_1(c)) = R\}
\end{aligned} \tag{2.54}$$

which is what is expressed in (2.48). □

Note. *isomorphic*[1]: The definition of isomorphism in (2.56) is that of *div point sets*, but the isomorphism we are talking about here is that of sets of *unit dividons*, which can be defined as follows:

$$\begin{aligned}
\Omega_1 \cong^1 \Omega_2 \Leftrightarrow\ & |\Omega_1| = |\Omega_2| \\
& \wedge \exists f_\Omega : \textstyle\bigcup_{D \in \Omega_1} \pi_1(D) \xrightarrow{1:1} \bigcup_{D \in \Omega_2} \pi_1(D) \\
& \quad \forall D_1 \in \Omega_1 \\
& \qquad \exists D_2 \in \Omega_2 \\
& \qquad\quad (d_1, \delta_1) := D_1 \\
& \qquad\quad (d_2, \delta_2) := D_2 \\
& \qquad\quad f^{members}(d_1) = d_2 \Leftrightarrow f^{members^2}(\delta_1) = \delta_2
\end{aligned} \tag{2.55}$$

It is necessary to specify Ω_1 and Ω_2 to have the same cardinality, since it is possible for f_Ω, a bijective function satisfying the condition, to exist in the case when $|\Omega_1| \neq |\Omega_2|$.

isomorphic[2]: The isomorphism we are talking about here is that of colors, which can be defined as follows:

$$\begin{aligned}
C_1 \cong^2 C_2 \Leftrightarrow\ & \exists f_C : \{\pi_1(c) : c \in C_1\} \xrightarrow{1:1} \{\pi_1(c) : c \in C_2\} \\
& \quad \forall c_1 \in C_1 \\
& \qquad \exists c_2 \in C_2 \\
& \qquad\quad (v_1, color_1) := c_1 \\
& \qquad\quad (v_2, color_2) := c_2 \\
& \qquad\quad f(v_1) = v_2 \Rightarrow color_1 = color_2
\end{aligned} \tag{2.56}$$

Definition 5. We say that $\mathscr{X}_1 \in \mathscr{DPS}^*$ is a *sub div point set* of $\mathscr{X}_2 \in \mathscr{DPS}^*$ (denoted by $\leq$) iff the set of *unit divdion* of the corresponding *unit div point set* of $\mathscr{X}_1$ is a subset of that of $\mathcal{X}_2$. Notationally,

$$\begin{aligned} &\forall \mathcal{X}_1, \mathcal{X}_2 \in \mathcal{DPS}^* \\ &\quad (A, \Omega_A) := \mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{X}_1) \\ &\quad (B, \Omega_B) := \mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{X}_2) \\ &\quad \mathcal{X}_1 \leq \mathcal{X}_2 \Leftrightarrow \Omega_A \subseteq \Omega_B \end{aligned} \tag{2.57}$$

For clarification, 2 *sub div point sets* of some *div point set*, (S_1, Ω_{S_1}) and (S_2, Ω_{S_2}), are distinct *sub div point sets* if $S_1 \neq S_2$, which is to say, distinctness here is not defined in terms of isomorphism, but equality (i.e. by the axiom of extensionality in ZFC).

Definition 6. $\mathcal{S}dps_{of}$ is a function that returns the set of all *sub div point sets* of m points for some div point set where $m \in \mathbb{N}_{\geq 4}$.

$$\mathcal{S}dps_{of}(\mathcal{X}_{dps}, m) = \{\mathcal{S}dps(\mathcal{X}_{dps}, P_s) : P_s \in \pi_1(\mathcal{X}_{dps})(P) : |P_s| = m\} \tag{2.58}$$

where $\mathcal{S}dps$ is a function that returns the *sub div point set* of a set of points, P_s, of a *div point set* of $\mathcal{X}_{dps}$:

$$\begin{aligned} &\mathcal{S}dps(\mathcal{X}_{dps}, P_s) = \mathcal{F}_{\vec{d}ps}^{\mathcal{UDPS}}((P_s, \{D : D \in \pi_2(\mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{X}_{dps})) : \xi(D) \subseteq P_s\})) \\ &\text{where } \mathcal{F}_{\vec{d}ps}^{\mathcal{UDPS}} \text{ is the inverse of } \mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}} \end{aligned} \tag{2.59}$$

Since a *div point set* of n points always has $\binom{n}{m}$ distinct *sub div point sets* of m points, where $m \leq n$ and $m \geq 4$, $\mathcal{S}dps_{of}(\mathcal{X}_{dps}, m)$ has the cardinality of $\binom{|\pi_1(\mathcal{X}_{dps})|}{m}$.

Lemma 4. For any *div point set*, $\mathcal{X}$, and any natural number m greater or equal to 4, let $\mathcal{A}$ and $\mathcal{B}$ to be any 2 distinct *sub div point sets* of m points of $\mathcal{X}$, and k be the number of points $\mathcal{A}$ and $\mathcal{B}$ have in common, $\mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{A})$ and $\mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{B})$ always have $6\binom{k}{4}$ *unit dividons* in common. Notationally,

$$\begin{aligned} &\forall \mathcal{X} \in \mathcal{DPS}^* \\ &\quad \forall m \in \mathbb{N}_{\geq 1} \\ &\quad\quad \forall \mathcal{A}, \mathcal{B} \in \mathcal{S}dps_{of}(\mathcal{X}, m) \\ &\quad\quad\quad |\pi_2(\mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{A})) \cap \pi_2(\mathcal{F}_{\vec{u}dps}^{\mathcal{DPS}}(\mathcal{B}))| = 6\binom{|\pi_1(\mathcal{A}) \cap \pi_1(\mathcal{B})|}{4} \end{aligned} \tag{2.60}$$

Proof for Lemma 4. For any $m > |\pi_1(\mathcal{X})|$, the proposition on all elements $\mathcal{A}, \mathcal{B} \in \mathcal{S}dps_{of}(\mathcal{X}, m)$ is vacuously true. For $m = |\pi_1(\mathcal{X})|$, it is obvious that the proposition is true: since every *dividon* of a *div point set* can be broken down into $\binom{|P|-2}{2}$ *unit dividon*, any unit div point set of n points would have $\binom{n-2}{2}\binom{n}{2} = 6\binom{n}{4}$ *unit dividons* in total. For $m < 4$, the proposition is trivially true because $\binom{m}{4} = 0$ and *unit div point sets* of 3 or less points have no *unit dividons* (recall (2.26)). For any $m < |\pi_1(\mathscr{X})|$ but greater than 3, the proposition can be proven true by first observing that $\mathscr{UD}s(R) \cap \mathscr{UD}s(R') = \varnothing \Leftrightarrow R \neq R'$ (recall (2.50)) for any sets R and R' with cardinality of 4, which indicates no 2 distinct *unit div point set* of 4 points have a *unit dividon* in common. Notationally,

$$\begin{gathered}\forall \mathscr{A}, \mathscr{B} \in \{\mathscr{X} : \mathscr{X} \in \mathscr{UDPS}^* : |\pi_1(\mathscr{X})| = 4\} \\ \pi_2(\mathscr{A}) \cap \pi_2(\mathscr{B}) = \varnothing \Leftrightarrow \pi_1(\mathscr{A}) \neq \pi_1(\mathscr{B})\end{gathered} \tag{2.61}$$

However, for any 2 *unit div point sets* of 5 or more points, $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$, if they have 4 points in common, let the set of such 4 points be R (i.e. $R \in \mathscr{P}(\pi_1(\mathscr{A}_{udps}) \cap \pi_2(\mathscr{A}_{udps})) \wedge |R| = 4$) each D' in $\mathscr{UD}s(R)$ would be equivalent to $\xi(D_a)$ and $\xi(D_b)$ where D_a and D_b are respectively *unit dividon* of $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$ having the same *divider* and *TBD points*. In the case when $\mathscr{A}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{A}_{dps})$ and $\mathscr{B}_{udps} = \mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{B}_{dps})$ for some $\mathscr{A}_{dps}$ and $\mathscr{B}_{dps}$ that are both *sub div point sets* of a certain $\mathscr{X} \in \mathscr{DPS}^*$, $D_a = D_b$ for every such respective *unit dividons* of $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$. This implies that for every such distinct R, $\mathscr{A}_{udps}$ and $\mathscr{B}_{dps}$ have 6 *unit dividons* in common. Let k be the number of points $\mathscr{A}_{udps}$ and $\mathscr{B}_{udps}$ have in common, the number of such distinct R is precisely k *chooses* 4 i.e. $\binom{\pi_1(\mathscr{A}_{udps}) \cap \pi_1(\mathscr{B}_{udps})}{4}$. $\square$

Theorem 2. Let $\mathscr{DPS}^+_5$ denote the class of all *div point sets* of 5 points in $\mathscr{DPS}^+$. All $\mathscr{X} \in \mathscr{DPS}^+_5$ either have 4, 2 or 0 distinct *sub div point set* of 4 points isomorphic to $Conc^1_4$ (with the remaining *sub div point sets* of 4 points isomorphic to $Conv_4$).

Proof for Theorem 2.

Summary. In *Part 1* we prove that there exists no $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ has precisely 1, 3 or 5 elements isomorphic to $Conc^1_4$. In *Part 2* we prove that there exists $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ has precisely 0, 2 or 4 elements isomorphic to $Conc^1_4$.

Part 1. A *div point set* $\mathscr{X}_{dps}$ is in $\mathscr{DPS}^+$ iff $\mathscr{F}^{\mathscr{DPS}}_{\to udps}(\mathscr{X}_{dps})$ is in $\mathscr{UDPS}^+$. Any *unit div point set* of 5 points in $\mathscr{UDPS}^+$ always has an even number of *unit dividons* D where $\psi(\pi_2(D)) = 0$, since it is isomorphic to some (P, Ω_P) in $\mathscr{UDPS}^{\mathbb{N}}$ where the coloring $Col_{udps}(\Omega_P)$ satisfies (2.49). For $Col_{udps}(\Omega_P)$ to satisfy (2.49), every e in E_2 must has its vertices colored $[1,0,0]$ or $[1,1,1]$. Since in any *unit div point set* of 5 points, there exists only $\binom{5-2}{2} = 3$ distinct *unit dividon* with the same *divider*, edges in E_2 are disjoint (recall (2.46)), and therefore any coloring satisfying (2.49) would have an even number of vertices colored 0. $Conc^1_4$ has an odd number of *unit dividons* D where $\psi(\pi_2(D)) = 0$, while $Conv_4$ has an even number for such *unit dividons*. Therefore there exists no *unit div point sets* of 5 points, $\mathscr{X}_{udps}$, in $\mathscr{UDPS}^+$ such that $All_{\Omega'}(\mathscr{X}_{udps})$ (defined (2.52)) contains an odd number of elements isomorphic[1] to $\pi_2(Conc^1_4)$. We thereby conclude that there exists no $\mathscr{X} \in \mathscr{DPS}^+_5$ where $\mathscr{S}dps_{of}(\mathscr{X}, 4)$ has precisely 1, 3 or 5 elements isomorphic to $Conc^1_4$.

Part 2. There exists *unit div point sets* of 5 points in $\mathscr{UDPS}^+$ with precisely 4, 2, or 0 *sub div point sets* of 4 points isomorphic to $Conc^1_4$. We would prove it by demonstrating that it is possible to construct *unit div point sets* of 5 points, $\mathscr{X}_{\mathfrak{updp}}$, isomorphic to some (P, Ω_P) in $\mathscr{UDPS}^{\mathbb{N}}$ where the coloring $Col_{\mathfrak{udps}}(\Omega_P)$ satisfies (2.48) and (2.49) and there are precisely 4, 2, or 0 distinct $\Omega' \in All_{\Omega'}(\mathscr{X}_{\mathfrak{udps}})$ isomorphic[1] to $\pi_2(Conc^1_4)$, (with the remaining Ω' isomorphic to $Conv_4$).

I. To construct such *unit div point sets* $\mathscr{X}_{\mathfrak{udps}}$ where $All_{\Omega'}(\mathscr{X}_{\mathfrak{udps}})$ contains 0 elements isomorphic to $Conc^1_4$ and 5 elements isomorphic to $\pi_2(Conv_4)$, we would need to make sure there are only 2 *unit dividons* $D \in \Omega'$ where $\phi(D) = 0$ for all Ω' in $All_{\Omega'}(\mathscr{X}_{\mathfrak{udps}})$. Let's denote the set of all such *unit dividons* as D^*, the 5 elements in $All_{\Omega'}(\mathscr{X}_{\mathfrak{udps}})$ as Ω'_1, Ω'_2, Ω'_3, Ω'_4, Ω'_5 and each 2 such unit divdions in Ω'_n as D^1_n and D^2_n for $n \in \{1, 2, 3, 4, 5\}$, i.e. $\{D^1_n, D^2_n\} = \Omega'_n \cap D^*$. For the coloring to satisfy (2.48) and (2.49), we simply let any 2 *unit dividons* D^x_n, D^x_m where $x \in \{1, 2\}$ and $n \neq m$ to have a common *divider*, while avoiding to have 3 distinct *unit dividon* in D^* to a common *divider* , and the same time ensuring that

$$\begin{aligned} \pi_1(D^1_n) &= \bigcup \pi_2(D^2_n) \\ \pi_1(D^2_n) &= \bigcup \pi_2(D^1_n) \end{aligned} \tag{2.62}$$

(recall (2.48)). That is to say, for some subsets of 2 cardinality, A, B, C, D, E, F of $\pi_1(X_{udps})$, we have

$$\begin{aligned} \pi_1(D^1_1) = \bigcup \pi_2(D^2_1) = \pi_1(D^1_2) = \bigcup \pi_2(D^2_2) = A \\ \pi_1(D^2_1) = \bigcup \pi_2(D^1_1) = \pi_1(D^2_3) = \bigcup \pi_2(D^1_3) = B \\ \pi_1(D^2_2) = \bigcup \pi_2(D^1_2) = \pi_1(D^2_4) = \bigcup \pi_2(D^1_4) = C \\ \pi_1(D^1_4) = \bigcup \pi_2(D^2_4) = \pi_1(D^1_5) = \bigcup \pi_2(D^2_5) = D \\ \pi_1(D^2_5) = \bigcup \pi_2(D^1_5) = \pi_1(D^2_6) = \bigcup \pi_2(D^1_6) = E \\ \pi_1(D^1_3) = \bigcup \pi_2(D^2_3) = \pi_1(D^1_6) = \bigcup \pi_2(D^2_6) = F \end{aligned} \tag{2.63}$$

where

$$A \neq B \neq C \neq D \neq E \neq F$$
$$(A \cap B) = (A \cap C) = (B \cap F) = (C \cap D) = (D \cap E) = (E \cap F) = \varnothing$$

Suppose $\mathscr{X}$ is a *div point set* where $\mathscr{F}^{\mathscr{DPS}}_{\to \mathfrak{udps}}(\mathscr{X})$ is isomorphic to the *unit div point set* described above: $\mathscr{X}$ would be isomorphic to $Conv_5$ defined in (2.69) below.

II. To construct such *unit div point sets* $\mathscr{X}_{\mathfrak{udps}}$ where $All_{\Omega'}(\mathscr{X}_{\mathfrak{udps}})$ contains 2 elements isomorphic to $Conc^1_4$ and 3 elements isomorphic to $\pi_2(Conv_4)$ - let's use the same notations in I - we would need to make sure that

$$\begin{aligned} \forall n \in \{1,2,3\} \qquad & \{D_n^1, D_n^2\} = \Omega'_n \cap D^* \\ \forall n \in \{4,5\} \qquad & \{D_n^1, D_n^2, D_n^3\} = \Omega'_n \cap D^* \end{aligned} \tag{2.64}$$

The *divider* of *unit dividons* D_n^1, D_n^2, D_n^3 for $n \in \{4,5\}$ need to have 1 element in common:

$$|\pi_1(D_n^1) \cap \pi_1(D_n^1) \cap \pi_1(D_n^1)| = 1 \tag{2.65}$$

(recall (2.48)), while (2.62) continues to apply to D_n^1, D_n^2 for $n \in \{1,2,3\}$. For the coloring to satisfy (2.48) and (2.49), we can have D_4^x to share the same *divider* as D_5^x for all $x \in \{1,2\}$, while letting the remaining *unit dividons* in D_4 and D_5, namely D_4^3 and D_5^3, to respectively share the same *divider* as D_1^1 and D_2^1, and the remaining *unit dividons* in D_1 and D_2, namely D_1^2 and D_2^2, to respectively share the same *dividers* as the two *dividons* in D_3. That is to say, for the distinct points $a, b, c, d, e \in \pi_1(X_{udps})$, we have

$$\begin{aligned} \pi_1(D_4^1) = \pi_1(D_5^1) &= \{a,b\} \\ \pi_1(D_4^2) = \pi_1(D_5^2) &= \{a,c\} \\ \pi_1(D_4^3) = \pi_1(D_1^1) = \bigcup \pi_2(D_1^2) &= \{a,d\} \\ \pi_1(D_5^3) = \pi_1(D_2^1) = \bigcup \pi_2(D_2^2) &= \{a,e\} \\ \pi_1(D_1^2) = \bigcup \pi_2(D_1^1) = \pi_1(D_3^1) = \bigcup \pi_2(D_3^2) &\subset P \setminus \{a,d\} \\ \pi_1(D_2^2) = \bigcup \pi_2(D_2^1) = \pi_1(D_3^2) = \bigcup \pi_2(D_3^1) &\subset P \setminus \{a,e\} \end{aligned} \tag{2.66}$$

Suppose $\mathscr{X}$ is a *div point set* where $\mathscr{F}^{\mathscr{DPS}}_{\rightarrow udps}(\mathscr{X})$ is isomorphic to the *unit div point set* described above: $\mathscr{X}$ would be isomorphic to $Conc_5^1$ defined in (2.69) below.

III. To construct such *unit div point sets* $\mathscr{X}_{udps}$ where $All_{\Omega'}(\mathscr{X}_{udps})$ contains 4 elements isomorphic to $Conc_4^1$ and 1 element isomorphic to $\pi_2(Conv_4)$ - let's use the same notations in II - this time we would need to make sure that

$$\begin{aligned} \forall n \in \{1\} \qquad & \{D_n^1, D_n^2\} = \Omega'_n \cap D^* \\ \forall n \in \{2,3,4,5\} \qquad & \{D_n^1, D_n^2, D_n^3\} = \Omega'_n \cap D^* \end{aligned} \tag{2.67}$$

where (2.65) applies to D_n^1, D_n^2 and D_n^3 for $n \in \{2,3,4,5\}$ and (2.62) continues to apply to D_n^1, D_n^2 for $n \in \{1\}$. For the coloring to satisfy (2.48) and (2.49), we can let D_4^x to share the same *divider* as D_5^x,and D_2^x to share the same *divider* as D_3^x, for $x \in \{1,2\}$. And then we let the remaining *unit dividons* in D_4 and D_5, namely D_4^3 and D_5^3, to respectively share the same *divider* as D_3^3 and D_1^1, while the remaining *unit dividons* in D_2, namely D_2^3 to share the same *divider* as D_1^2. That is to say, for the distinct points $a, b, c, d, e \in \pi_1(X_{udps})$, we have

$$\begin{aligned}\pi_1(D_4^1) &= \pi_2(D_5^1) = \{a, b\} \\ \pi_1(D_4^2) &= \pi_2(D_5^2) = \{a, c\} \\ \pi_1(D_4^3) &= \pi_1(D_3^3) = \{a, d\} \\ \pi_1(D_3^1) &= \pi_1(D_2^1) = \{e, d\} \\ \pi_1(D_3^2) &= \pi_1(D_2^2) = \{c, d\} \\ \pi_1(D_5^3) = \pi_1(D_1^1) &= \bigcup \pi_2(D_1^2) = \{a, e\} \\ \pi_1(D_2^3) = \pi_1(D_1^2) &= \bigcup \pi_2(D_1^1) = \{b, d\}\end{aligned} \tag{2.68}$$

Suppose $\mathcal{X}$ is a *div point set* where $\mathcal{F}_{\to udps}^{\mathcal{DPS}}(\mathcal{X})$ is isomorphic to the *unit div point set* described above: $\mathcal{X}$ would be isomorphic to $Conc_5^2$ defined in (2.69) below.

□

Remark. A stronger version of *Theorem 2* would state that for all $\mathcal{X}_{dps} \in \mathcal{DPS}_5^+$, $\mathcal{X}_{dps}$ is either isomorphic to $Conv_5$, $Conc_5^1$ or $Conc_5^2$, where

$$\begin{aligned}Conv_5 =&(Cv_5, \Theta_{Cv_5}) \\ Cv_5 =&\{1,2,3,4,5\} \\ \Theta_{Cv_5} =&\{(\{1,2\},\{(\{3,4\},\varnothing\}), \\ &(\{1,3\},\{\{2\},\{4,5\}\}), \\ &(\{1,4\},\{\{2,3\},\{5\}\}), \\ &(\{1,5\},\{\{2,3,4\},\varnothing\}), \\ &(\{2,3\},\{\{1,4,5\},\varnothing\}), \\ &(\{2,4\},\{\{1,5\},\{3\}\}), \\ &(\{2,5\},\{\{1\},\{3,4\}\}), \\ &(\{3,4\},\{\{1,2,5\},\varnothing\}) \\ &(\{3,5\},\{\{1,2\},\{4\}\}) \\ &(\{3,4\},\{\{1,2,5\},\varnothing\})\end{aligned}
\quad
\begin{aligned}Conc_5^1 =&(Cc_5^1, \Theta_{Cc_5^1}) \\ Cc_5^1 =&\{1,2,3,4,5\} \\ \Theta_{Cc_5^1} =&\{(\{1,2\},\{(\{3,4,5\},\varnothing\}), \\ &(\{1,3\},\{\{2,5\},\{4\}\}), \\ &(\{1,4\},\{\{2,3,5\},\varnothing\}), \\ &(\{1,5\},\{\{2\},\{3,4\}\}), \\ &(\{2,3\},\{\{1,4,5\},\varnothing\}), \\ &(\{2,4\},\{\{1\},\{3,5\}\}), \\ &(\{2,5\},\{\{1,4\},\{3\}\}), \\ &(\{3,4\},\{\{1,2,5\},\varnothing\}) \\ &(\{3,5\},\{\{1,2\},\{4\}\}) \\ &(\{4,5\},\{\{1,2,5\},\varnothing\})\end{aligned}
\quad
\begin{aligned}Conc_5^2 =&(Cc_5^2, \Theta_{Cc_5^2}) \\ Cc_5^2 =&\{1,2,3,4,5\} \\ \Theta_{Cc_5^2} =&\{(\{1,2\},\{(\{3,4,5\},\varnothing\}), \\ &(\{1,3\},\{\{2,4,5\},\varnothing\}), \\ &(\{1,4\},\{\{2\},\{3,5\}\}), \\ &(\{1,5\},\{\{2,4\},\{3\}\}), \\ &(\{2,3\},\{\{1,4,5\},\varnothing\}), \\ &(\{2,4\},\{\{1\},\{3,5\}\}), \\ &(\{2,5\},\{\{1,4\},\{3\}\}), \\ &(\{3,4\},\{\{1\},\{2,5\}\}) \\ &(\{3,5\},\{\{1,4\},\{2\}\}) \\ &(\{4,5\},\{\{1,3\},\{2\}\})\end{aligned} \tag{2.69}$$

To prove this version of *Theorem 2* we would need to prove that there exists no *div point sets* in $\mathcal{DPS}_5^+$ not isomorphic to $Conv_5$, $Conc_5^1$ or $Conc_5^2$.

Remark. Let $All_{of\Omega'}(\mathcal{X}, n)$ be a generalization of $All_{\Omega'}(\mathcal{X})$ such that

$$All_{of\Omega'}(\mathcal{X}, n) = \{\Omega'_{basedOn}(R, \pi_2(\mathcal{X})) : R \in \mathcal{P}(\pi_1(\mathcal{X})) : |R| = n\} \tag{2.70}$$

Using *Theorem 2*, we can prove that the proposition below about *unit div point set* is false:

A *unit div point set* of 5 or more points, $\mathcal{X}_{udps}$, is in $\mathcal{UDPS}^+$ iff all elements in $All_{of\Omega'}(\mathcal{X}_{udps}, n)$ are isomorphic to the set of *unit dividons* of some *unit div point set* in $\mathcal{UDPS}^+$, for any $n \in \mathbb{N}_{\geq 4}$ less than $|\pi_1(\mathcal{X}_{udps})|$.

However, the following weaker version of it still holds true.

If $\mathcal{X}_{udps}$ is in $\mathcal{UDPS}^+$, all members of $All_{of\Omega'}(\mathcal{X}_{udps}, n)$ are also in $\mathcal{UDPS}^+$ for any $n \in \mathbb{N}_{\geq 4}$ less than $|\pi_1(\mathcal{X}_{udps})|$.

This is equivalent as stating that for any *unit div point set*, $\mathcal{X}_s$, whose *unit dividons* is a subset of that of some $\mathcal{X} \in \mathcal{UDPS}^+$, there certainly exists an interpretation for $\pi_1(\mathcal{X}_s)$ as some set of points in $\mathbb{E}^2$, which is just a subset of the interpretation for $\pi_1(\mathcal{X})$ as some set of points in $\mathbb{E}^2$, since $\pi_1(\mathcal{X}_s) \subseteq \pi_1(\mathcal{X})$.

There is undoubtedly some similarity between the false proposition above, and the following proposition which is too false:

A *div point set* of 4 or more points, $\mathcal{X}_{dps}$, is in $\mathcal{DPS}^+$ iff all elements in $\mathcal{S}dps_{of}(\mathcal{X}_{dps}, n)$ are also in $\mathcal{DPS}^+$, for any $n \in \mathbb{N}_{\geq 3}$ less than $|\pi_1(X_{dps})|$. If this is true, it would imply that $\mathcal{DPS}^* = \mathcal{DPS}^+$, which is obviously false.

However, if we closely examine this proposition, we would realize that it would be true if not for the case when $n = 3$: since it is vacuously true that any *div point sets* of 3 points satisfy (2.10), (2.11), and (2.12), we cannot conclude that a certain *div point set* satisfies (2.10), (2.11), and (2.12), even if all its *sub div point sets* of 3 points satisfy them. Now recall *Lemma 3* where E_2 of the hypergraph based on P is an empty set in the case when $|P| = 4$ and as a result, it is vacuously true that such E_2 always satisfies (2.49). This is why the proposition regarding *unit div point sets* above is false: we cannot conclude that a certain *unit div point set* is isomorphic to some *unit div point set*, (P, Ω_P), in $\mathcal{UDPS}^{\mathbb{N}}$ where $Col_{udps}(\Omega_P)$ satisfies (2.48) and (2.49), even if all elements in $All_{of\Omega'}(\mathcal{X}_{udps}, 4)$ are isomorphic to some *unit div point set*, (P, Ω_P), in $\mathcal{UDPS}^{\mathbb{N}}$ where $Col_{udps}(\Omega_P)$ satisfies them.

It can be proven that in the case when $n \in \mathbb{N}_{\geq 5}$, the proposition regarding *unit div point sets* above is true.

2.2 *convexity*

The notion that there exists n points forming a convex polygon among some set of points in $\mathbb{E}^2$ can be expressed through *convexity* in the context of *div point sets*.

Definition 7. A *div point set* (P, Θ_P) has a *convexity* of n iff there exists (Q, Θ_Q) such that $(Q, \Theta_Q) \leq (P, \Theta_P)$ and (Q, Θ_Q) is isomorphic to $Conv_n$, defined as follow

$$Conv_n = (P, \{(d, \delta_{conv}(d, P)) : d \in \mathscr{P}(P) : |d| = 2\})$$
$$\text{where } \begin{cases} P := \{x \in \mathbb{N}_{\geq 1} : x \leq n\} \\ \delta_{conv}(d, P) = \{\{p : p \in P : inside(p, d)\}, \{p : p \in P : outside(p, d)\}\} \\ inside(p, d) = (p > min(d) \wedge p < max(d)) \\ outside(p, d) = (p < min(d) \vee p > max(d)) \\ min(d) \text{ returns the smallest number in } d \\ max(d) \text{ returns the biggest number in } d. \end{cases} \tag{2.71}$$

where n is a natural number ≥ 3. Here is an implementation of it as a function in Haskell:

```
import Data.List

combine :: Int -> [a] -> [[a]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                      , ys <- combine (n-1) xs']

convex:: Int -> ([Int],[([Int],[[Int]])])
convex n = (points, dividons)
  where
    points = [1..n]
    dividers = combine 2 points
    dividons = [(divider,[div1,div2])
      | divider@(a:b:_) <- dividers,
      let divs = points \\ divider,
      let div1 = [ x | x <- divs, x > a, x < b ],
      let div2 = divs \\ div1 ]
```

Axiom 2. For any $\mathscr{X}$ in $\mathscr{DPS}^+$, $\mathscr{X}$ has an interpretation for $\pi_1(\mathscr{X})$ as some set of points in E^2 among which there exists n points forming a convex polygon, iff $\mathscr{X}$ has a convexity of n. More precisely, there exists an interpretation for $P' \subseteq \pi_1(\mathscr{X})$ as some set of n points in E^2 forming a convex polygon iff $Sdps(\mathscr{X}, P')$ is isomorphic to $Conv_n$, for all $n \geq 3$.

Remark. One may notice that for any $n \geq 4$, all *sub div point sets* of $n-1$ points of $Conv_n$ are isomorphic to $Conv_{n-1}$. By *Axiom 2*, that is equivalent to the following proposition: for any $n \geq 4$, after removing any one point from a set of n points that are the vertices of a convex polygon on an Euclidean plane, the remaining points too forms a convex polygon, which is trivially true.

Remark. By *Axiom 2*, we can conclude from *Theorem 2* that for any 5 points in general position on an Euclidean plane, there always exists 4 points forming a convex polygon.

3 *A reduction to a multiset unsatisfiability problem*

The Erdös-Szekeres conjecture can be expressed as a conjunction of (3.1) and (3.2) in the theory of *div point set*.

$$\begin{aligned}&\forall n \in \mathbb{N}_{\geq 3}\\ &\qquad \exists \mathscr{A} \in \mathscr{DPS}^{+} \quad |\pi_1(A)| = 2^{n-2} \wedge \exists \mathscr{A}_{\jmath} \leq \mathscr{A} \quad \mathscr{A}_{\jmath} \ncong Conv_n\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\forall n \in \mathbb{N}_{\geq 3}\\ &\qquad \forall \mathscr{A} \in \mathscr{DPS}^{+} \quad |\pi_1(A)| > 2^{n-2} \Leftrightarrow \exists \mathscr{A}_{\jmath} \leq \mathscr{A} \quad \mathscr{A}_{\jmath} \cong Conv_n\end{aligned} \tag{3.2}$$

Since the lower bound has been proven to be $2^{n-2}+1$, all is left is to prove (3.2) and the conjecture would be proven.

3.1 a combinatorial characteristics of sub div point sets

As we examine *div point sets* of v points for $v > 5$, we would notice this rather interesting fact about *sub div point sets*: for any natural number $a > 1$, let $\mathscr{SDPSS}$ be the set of all *sub div point set* of $v-a$ points of any *div point sets* of v points, for all $\mathscr{X}_{\mathscr{SDPS}}$ in $\mathscr{SDPSS}$, we can always select $v-a$ distinct sets of $a+1$ div point sets in $\mathscr{X}_{\mathscr{SDPS}}$ such that for each of these sets, it contains $\mathscr{X}_{\mathscr{SDPS}}$, and all div points sets in it have $\binom{v-a-b}{t}$ sub div point sets of t points in common, for all $t \in \mathbb{N}_{\geq 1}$. Notationally,

$$\begin{aligned}
&\forall \mathscr{X} \in \mathscr{DPS}^{+}\\
&\quad v \coloneqq |\pi_1(\mathscr{X})|\\
&\quad \forall a \in \mathbb{N}_{\geq 1}\\
&\qquad \mathscr{SDPSS} \coloneqq \mathscr{S}dps_{of}(\mathscr{X}, v-a)\\
&\qquad \forall \mathscr{X}_{\mathscr{SDPS}} \in \mathscr{SDPSS}\\
&\qquad\quad \forall b \in \{x : x \in \mathbb{N}_{\geq 1} : x < v-a\}\\
&\qquad\qquad \exists \mathscr{S} \in \mathscr{P}_n(\mathscr{P}_n(\mathscr{SDPSS}, a+b), v-a)\\
&\qquad\qquad\quad \forall \jmath \in \mathscr{S}\\
&\qquad\qquad\qquad \mathscr{X}_{\mathscr{SDPS}} \in \jmath\\
&\qquad\qquad\qquad \forall t \in \mathbb{N}_{\geq 1}\\
&\qquad\qquad\qquad\quad |\bigcap_{\ell \in \jmath} \mathscr{S}dps_{of}(\ell, t)| = \binom{v-a-b}{t}
\end{aligned} \tag{3.3}$$

where

$$\mathscr{P}_n(S, n) = \{x : x \in \mathscr{P}(S) : |x| = n\} \tag{3.4}$$

To understand why such combinatorial characteristics exists, consider this: any 2 *sub div point sets*, $\mathcal{S}_1$ and $\mathcal{S}_2$ of a certain div point set is distinct as long as they are of distinct points i.e. $\mathcal{S}_1 \neq \mathcal{S}_2 \Leftrightarrow \pi_1(\mathcal{S}_1) \neq \pi_1(\mathcal{S}_2)$ and are the same as long as they are of the same points i.e. $\mathcal{S}_1 = \mathcal{S}_2 \Leftrightarrow \pi_1(\mathcal{S}_1) = \pi_1(\mathcal{S}_2)$, thus (3.3) is equivalent as stating: for any set $\mathcal{N}$ with the same cardinality as $\mathbb{N}$,

$$
\begin{aligned}
&\forall X \in \mathscr{P}(\mathcal{N}) \\
&\quad v \coloneqq |X| \\
&\quad \forall a \in \mathbb{N}_{\geq 1} \\
&\quad\quad PS \coloneqq \mathscr{P}_n(X, v-a) \\
&\quad\quad \forall z \in PS \\
&\quad\quad\quad \forall b \in \{x : x \in \mathbb{N}_{\geq 1} : x < v - a\} \\
&\quad\quad\quad\quad \exists S \in \mathscr{P}_n(\mathscr{P}_n(PS, a+b), v-a) \\
&\quad\quad\quad\quad\quad \forall s \in S \\
&\quad\quad\quad\quad\quad\quad z \in s \\
&\quad\quad\quad\quad\quad\quad \forall n \in \mathbb{N}_{\geq 1} \\
&\quad\quad\quad\quad\quad\quad\quad |\bigcap_{l \in s} \mathscr{P}_n(l, n)| = \binom{v-a-b}{n}
\end{aligned}
\tag{3.5}
$$

For the purpose of illustration, suppose we have some div point set $\mathcal{X}_9$ of 9 points, let $isom$ be an bijective function from $\mathcal{Sdps}_{of}(\mathcal{X}_9, 4)$ to a set of natural numbers, N, where $N = \{n : n \in \mathbb{N}_{\geq 1} : n \leq |\mathcal{Sdps}_{of}(\mathcal{X}_9, 4)|$, $\{\{iso(\mathcal{X}_4) : \mathcal{X}_4 \in \mathcal{X}_5\} : \mathcal{X}_5 \in \mathcal{Sdps}_{of}(\mathcal{X}_9, 5)\}$ would be isomorphic to:

$$
\begin{aligned}
&\{\{1,2,7,22,57\},\{1,3,8,23,58\},\{1,4,9,24,59\},\{1,5,10,25,60\},\{1,6,11,26,61\},\{2,3,12,27,62\},\\
&\{2,4,13,28,63\},\{2,5,14,29,64\},\{2,6,15,30,65\},\{3,4,16,31,66\},\{3,5,17,32,67\},\{3,6,18,33,68\},\\
&\{4,5,19,34,69\},\{4,6,20,35,70\},\{5,6,21,36,71\},\{7,8,12,37,72\},\{7,9,13,38,73\},\{7,10,14,39,74\},\\
&\{7,11,15,40,75\},\{8,9,16,41,76\},\{8,10,17,42,77\},\{8,11,18,43,78\},\{9,10,19,44,79\},\{9,11,20,45,80\},\\
&\{10,11,21,46,81\},\{12,13,16,47,82\},\{12,14,17,48,83\},\{12,15,18,49,84\},\{13,14,19,50,85\},\{13,15,20,51,86\},\\
&\{14,15,21,52,87\},\{16,17,19,53,88\},\{16,18,20,54,89\},\{17,18,21,55,90\},\{19,20,21,56,91\},\{22,23,27,37,92\},\\
&\{22,24,28,38,93\},\{22,25,29,39,94\},\{22,26,30,40,95\},\{23,24,31,41,96\},\{23,25,32,42,97\},\{23,26,33,43,98\},\\
&\{24,25,34,44,99\},\{24,26,35,45,100\},\{25,26,36,46,101\},\{27,28,31,47,102\},\{27,29,32,48,103\},\{27,30,33,49,104\},\\
&\{28,29,34,50,105\},\{28,30,35,51,106\},\{29,30,36,52,107\},\{31,32,34,53,108\},\{31,33,35,54,109\},\{32,33,36,55,110\},\\
&\{34,35,36,56,111\},\{37,38,41,47,112\},\{37,39,42,48,113\},\{37,40,43,49,114\},\{38,39,44,50,115\},\{38,40,45,51,116\},\\
&\{39,40,46,52,117\},\{41,42,44,53,118\},\{41,43,45,54,119\},\{42,43,46,55,120\},\{44,45,46,56,121\},\{47,48,50,53,122\},\\
&\{47,49,51,54,123\},\{48,49,52,55,124\},\{50,51,52,56,125\},\{53,54,55,56,126\},\{57,58,62,72,92\},\{57,59,63,73,93\},\\
&\{57,60,64,74,94\},\{57,61,65,75,95\},\{58,59,66,76,96\},\{58,60,67,77,97\},\{58,61,68,78,98\},\{59,60,69,79,99\},\\
&\{59,61,70,80,100\},\{60,61,71,81,101\},\{62,63,66,82,102\},\{62,64,67,83,103\},\{62,65,68,84,104\},\{63,64,69,85,105\},\\
&\{63,65,70,86,106\},\{64,65,71,87,107\},\{66,67,69,88,108\},\{66,68,70,89,109\},\{67,68,71,90,110\},\{69,70,71,91,111\},\\
&\{72,73,76,82,112\},\{72,74,77,83,113\},\{72,75,78,84,114\},\{73,74,79,85,115\},\{73,75,80,86,116\},\{74,75,81,87,117\},\\
&\{76,77,79,88,118\},\{76,78,80,89,119\},\{77,78,81,90,120\},\{79,80,81,91,121\},\{82,83,85,88,122\},\{82,84,86,89,123\},\\
&\{83,84,87,90,124\},\{85,86,87,91,125\},\{88,89,90,91,126\},\{92,93,96,102,112\},\{92,94,97,103,113\},\{92,95,98,104,114\},\\
&\{93,94,99,105,115\},\{93,95,100,106,116\},\{94,95,101,107,117\},\{96,97,99,108,118\},\{96,98,100,109,119\},\\
&\{97,98,101,110,120\},\{99,100,101,111,121\},\{102,103,105,108,122\},\{102,104,106,109,123\},\{103,104,107,110,124\},\\
&\{105,106,107,111,125\},\{108,109,110,111,126\},\{112,113,115,118,122\},\{112,114,116,119,123\},\{113,114,117,120,124\},\\
&\{115,116,117,121,125\},\{118,119,120,121,126\},\{122,123,124,125,126\}\}
\end{aligned}
\tag{3.6}
$$

which shows how *sub div point sets* of 4 points of $\mathcal{X}_9$ (each represented by a distinct natural number above) would be disturbed among *sub div point sets* of 5 points of $\mathcal{X}_9$. Notice how it satisfies (3.3) (e.g. for every *sub div point set* of 5 points, $\mathcal{X}_5$, there exists a set in $\mathcal{P}_n(\mathcal{P}_n(\mathcal{S}dps_{of}(\mathcal{X}_9, 5), 5), 5)$ where its elements are different supersets of $\mathcal{X}_5$, in which the *div point sets* of 5 points all have a *sub div point set* of 4 points in common i.e. $|\bigcap_{\ell \in s} \mathcal{S}dps_{of}(\ell, 4)| = \binom{9-4-1}{4} = 1$ where $s \in S$ and S is the set containing such supersets i.e. $S \in \mathcal{P}_n(\mathcal{P}_n(\mathcal{S}dps_{of}(\mathcal{X}_9, 5), 5), 5)$).

We suspect that (3.2) is nothing more than an outcome of what happens when we have a structure whose sub-structures possess the combinatorial characteristics described above and the same time having to abide by some simple rule, which in this case is what described in (3.7) below.

3.2 the problem $UNSAT^{\mathcal{DPS}^+}_{multiset}$

Definition 8. $UNSAT_{multiset}$ is the decision problem of determining if there exists no value-assignment for all variables in V, distributed in a certain manner among the multisets in M, such that it satisfies the FOL formulae in C, where the value-assignment is defined to be a function, Z: for all v in V, $Z(v) = x$ for some $x \in D$ where D, the set of values a variable can be assigned to, is often referred to as the domain. An instance of $UNSAT_{multiset}$ can thus be represented as a 4-tuple (V, D, M, C).

We shall now present the problem $UNSAT^{\mathcal{DPS}^+}_{multiset}$, a special case of $UNSAT_{multiset}$, for which, if an instance of it is solved, it would prove that for some $n \in \mathbb{N}_{\geq 5}$, there exists no *div point set* of $2^{n-2}+1$ points, $\mathcal{X}$, where $\mathcal{X}_5 \in \mathcal{S}dps_{of}(\mathcal{X}, 5)$ satisfies

$$[Assign(\mathcal{X}_4) : \mathcal{X}_4 \in \mathcal{S}dps_{of}(\mathcal{X}_5, 4)] \in \{[1,1,1,1,0],[1,1,0,0,0],[0,0,0,0,0]\} \tag{3.7}$$

but $\mathcal{X}$ does not satisfy

$$\begin{aligned} &\exists \mathcal{A}_s \in \mathcal{S}dps_{of}(\mathcal{X}, n) \\ &\qquad\qquad \forall \mathcal{A}_{ss} \in \mathcal{S}dps_{of}(\mathcal{X}, 4) \quad Assign(\mathcal{A}_{ss}) = 0 \end{aligned} \tag{3.8}$$

where

$$Assign(\mathcal{X}) = \begin{cases} 1 & \text{if} \quad \mathcal{X} \cong Conc^1_4 \\ 0 & \text{if} \quad \mathcal{X} \cong Conv_4 \end{cases} \tag{3.9}$$

Since for any *div point sets* in $\mathcal{DPS}^+$, its *sub div point sets* of 5 points satisfy (3.7) (recall Theorem 2), this would thus prove that *div point set* in $\mathcal{DPS}^+$ must too satisfy (3.8), and consequently proving (3.2) which can be rewritten as follows

$$\begin{aligned}
&\forall n \in \mathbb{N}_{\geq 3} \\
&\quad \forall \mathscr{A} \in \mathscr{DPS}^{+} \\
&\qquad |\pi_1(A)| > 2^{n-2} \\
&\qquad \Leftrightarrow \exists \mathscr{A}_s \in \mathscr{Sdps}_{of}(\mathscr{A}, n) \\
&\qquad\quad \forall \mathscr{A}_{ss} \in \mathscr{Sdps}_{of}(\mathscr{A}_s, 4) \quad Assign(\mathscr{A}_{ss}) = 0
\end{aligned} \tag{3.10}$$

(recall that for all $k \in \mathbb{N}_{\geq 3}$ and $n \geq k$, any *sub div point set* of k points of any $Conv_n$ is isomorphic to some $Conv_k$).

Definition 9. $UNSAT^{\mathscr{DPS}^+}_{multiset}$ is a special case of $UNSAT_{multiset}$ where some instance, (V, D, M, C), of $UNSAT_{multiset}$ is an instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ iff for some $n \geq 5$,

$$\begin{aligned}
|V| &= \binom{2^{n-2}+1}{4} \\
D &= \{0, 1\} \\
M &= A \cup B
\end{aligned} \tag{3.11}$$

where A is a set of 5-cardinal multisets, and B is a set of n-cardinal multisets. For some *div point set* of n points, $\mathscr{X}$, the variables in V are distributed in $m \in A$ the same way as how elements in $\mathscr{Sdps}_{of}(\mathscr{X}, 4)$ are distributed in $\mathscr{X}_{\mathscr{SPDPS}} \in \mathscr{Sdps}_{of}(\mathscr{X}, 5)$, while the variables are distributed in $m \in B$ the same way as how elements in $\mathscr{Sdps}_{of}(\mathscr{X}, 4)$ are distributed in $\mathscr{X}_{\mathscr{SPDPS}} \in \mathscr{Sdps}_{of}(\mathscr{X}, n)$. And C is the set of the formulae (3.12) and (3.13).

$$\forall a \in A \qquad a \in \{[1,1,1,1,0], [1,1,0,0,0], [0,0,0,0,0]\} \tag{3.12}$$

$$\forall b \in B \qquad b \neq \underbrace{[0,0,0,...,0,0]}_{\binom{n}{4}\ 0's} \tag{3.13}$$

The distribution of variables in A and B can be implement in Haskell as follows:

```
import Data.List
import Data.Maybe
type Multiset = [Integer]

merge (a:x) (b:y) = (a,b) : merge x y
merge [] _ = []

choose :: Integer -> Integer -> Integer
n `choose` k
    | k < 0     = 0
    | k > n     = 0
```

```
  | otherwise = factorial n `div` (factorial k * factorial (n-k))

factorial :: Integer -> Integer
factorial n = foldl (*) 1 [1..n]

combine :: Integer -> [Integer] -> [[Integer]]
combine 0 _ = [[]]
combine n xs = [ y:ys | y:xs' <- tails xs
                       , ys <- combine (n-1) xs']

number_of_points = (\n->(2^(n-2)+1))

n_setOf_m_Multisets:: Integer -> Integer -> [Multiset]
n_setOf_m_Multisets m n = [ map fromJust $ map ((flip lookup) encoding)
    (combine 4 m_points) | m_points <- combine n [1..m] ]
    where
        encoding = merge (combine 4 [1..m]) [1..(m `choose` 4)]

setA :: Integer -> [Multiset]
setA n = n_setOf_m_Multisets (number_of_points n) 5

setB :: Integer -> [Multiset]
setB n = [ x | x <- n_setOf_m_Multisets (number_of_points n) n, 2 `elem` x ]
```

Remark. A different implementation may result in a different M for the same n. Nonetheless, the different M obtained from a different implementation would be isomorphic to the M obtained from this implementation, in which case we would consider that distribution to be the same. Thus as far as unsatisfiability is concerned, for every $n \in \mathbb{N}_{\geq 5}$, there exists exactly one instance of $UNSAT^{\mathcal{DPS}^+}_{multiset}$.

Here is the simplest instance of $UNSAT^{\mathcal{DPS}^+}_{multiset}$ (when $n = 5$): since $A = B$, we have $|M| = |A| = |B| = \binom{2^{5-2}+1}{5} = 126$ multisets, and $|V| = \binom{2^{5-2}+1}{4} = 126$ variables as well (with each denoted by v_n below), distributed among the multisets in M as follows:

$\{[v_1, v_2, v_7, v_{22}, v_{57}], [v_1, v_3, v_8, v_{23}, v_{58}], [v_1, v_4, v_9, v_{24}, v_{59}], [v_1, v_5, v_{10}, v_{25}, v_{60}], [v_1, v_6, v_{11}, v_{26}, v_{61}], [v_2, v_3, v_{12}, v_{27}, v_{62}],$
$[v_2, v_4, v_{13}, v_{28}, v_{63}], [v_2, v_5, v_{14}, v_{29}, v_{64}], [v_2, v_6, v_{15}, v_{30}, v_{65}], [v_3, v_4, v_{16}, v_{31}, v_{66}], [v_3, v_5, v_{17}, v_{32}, v_{67}], [v_3, v_6, v_{18}, v_{33}, v_{68}],$
$[v_4, v_5, v_{19}, v_{34}, v_{69}], [v_4, v_6, v_{20}, v_{35}, v_{70}], [v_5, v_6, v_{21}, v_{36}, v_{71}], [v_7, v_8, v_{12}, v_{37}, v_{72}], [v_7, v_9, v_{13}, v_{38}, v_{73}], [v_7, v_{10}, v_{14}, v_{39}, v_{74}],$
$[v_7, v_{11}, v_{15}, v_{40}, v_{75}], [v_8, v_9, v_{16}, v_{41}, v_{76}], [v_8, v_{10}, v_{17}, v_{42}, v_{77}], [v_8, v_{11}, v_{18}, v_{43}, v_{78}], [v_9, v_{10}, v_{19}, v_{44}, v_{79}], [v_9, v_{11}, v_{20}, v_{45}, v_{80}],$
$[v_{10}, v_{11}, v_{21}, v_{46}, v_{81}], [v_{12}, v_{13}, v_{16}, v_{47}, v_{82}], [v_{12}, v_{14}, v_{17}, v_{48}, v_{83}], [v_{12}, v_{15}, v_{18}, v_{49}, v_{84}], [v_{13}, v_{14}, v_{19}, v_{50}, v_{85}], [v_{13}, v_{15}, v_{20}, v_{51}, v_{86}],$
$[v_{14}, v_{15}, v_{21}, v_{52}, v_{87}], [v_{16}, v_{17}, v_{19}, v_{53}, v_{88}], [v_{16}, v_{18}, v_{20}, v_{54}, v_{89}], [v_{17}, v_{18}, v_{21}, v_{55}, v_{90}], [v_{19}, v_{20}, v_{21}, v_{56}, v_{91}], [v_{22}, v_{23}, v_{27}, v_{37}, v_{92}],$
$[v_{22}, v_{24}, v_{28}, v_{38}, v_{93}], [v_{22}, v_{25}, v_{29}, v_{39}, v_{94}], [v_{22}, v_{26}, v_{30}, v_{40}, v_{95}], [v_{23}, v_{24}, v_{31}, v_{41}, v_{96}], [v_{23}, v_{25}, v_{32}, v_{42}, v_{97}], [v_{23}, v_{26}, v_{33}, v_{43}, v_{98}],$
$[v_{24}, v_{25}, v_{34}, v_{44}, v_{99}], [v_{24}, v_{26}, v_{35}, v_{45}, v_{100}], [v_{25}, v_{26}, v_{36}, v_{46}, v_{101}], [v_{27}, v_{28}, v_{31}, v_{47}, v_{102}], [v_{27}, v_{29}, v_{32}, v_{48}, v_{103}], [v_{27}, v_{30}, v_{33}, v_{49}, v_{104}],$
$[v_{28}, v_{29}, v_{34}, v_{50}, v_{105}], [v_{28}, v_{30}, v_{35}, v_{51}, v_{106}], [v_{29}, v_{30}, v_{36}, v_{52}, v_{107}], [v_{31}, v_{32}, v_{34}, v_{53}, v_{108}], [v_{31}, v_{33}, v_{35}, v_{54}, v_{109}], [v_{32}, v_{33}, v_{36}, v_{55}, v_{110}],$
$[v_{34}, v_{35}, v_{36}, v_{56}, v_{111}], [v_{37}, v_{38}, v_{41}, v_{47}, v_{112}], [v_{37}, v_{39}, v_{42}, v_{48}, v_{113}], [v_{37}, v_{40}, v_{43}, v_{49}, v_{114}], [v_{38}, v_{39}, v_{44}, v_{50}, v_{115}], [v_{38}, v_{40}, v_{45}, v_{51}, v_{116}],$
$[v_{39}, v_{40}, v_{46}, v_{52}, v_{117}], [v_{41}, v_{42}, v_{44}, v_{53}, v_{118}], [v_{41}, v_{43}, v_{45}, v_{54}, v_{119}], [v_{42}, v_{43}, v_{46}, v_{55}, v_{120}], [v_{44}, v_{45}, v_{46}, v_{56}, v_{121}], [v_{47}, v_{48}, v_{50}, v_{53}, v_{122}],$
$[v_{47}, v_{49}, v_{51}, v_{54}, v_{123}], [v_{48}, v_{49}, v_{52}, v_{55}, v_{124}], [v_{50}, v_{51}, v_{52}, v_{56}, v_{125}], [v_{53}, v_{54}, v_{55}, v_{56}, v_{126}], [v_{57}, v_{58}, v_{62}, v_{72}, v_{92}], [v_{57}, v_{59}, v_{63}, v_{73}, v_{93}],$
$[v_{57}, v_{60}, v_{64}, v_{74}, v_{94}], [v_{57}, v_{61}, v_{65}, v_{75}, v_{95}], [v_{58}, v_{59}, v_{66}, v_{76}, v_{96}], [v_{58}, v_{60}, v_{67}, v_{77}, v_{97}], [v_{58}, v_{61}, v_{68}, v_{78}, v_{98}], [v_{59}, v_{60}, v_{69}, v_{79}, v_{99}],$
$[v_{59}, v_{61}, v_{70}, v_{80}, v_{100}], [v_{60}, v_{61}, v_{71}, v_{81}, v_{101}], [v_{62}, v_{63}, v_{66}, v_{82}, v_{102}], [v_{62}, v_{64}, v_{67}, v_{83}, v_{103}], [v_{62}, v_{65}, v_{68}, v_{84}, v_{104}], [v_{63}, v_{64}, v_{69}, v_{85}, v_{105}],$
$[v_{63}, v_{65}, v_{70}, v_{86}, v_{106}], [v_{64}, v_{65}, v_{71}, v_{87}, v_{107}], [v_{66}, v_{67}, v_{69}, v_{88}, v_{108}], [v_{66}, v_{68}, v_{70}, v_{89}, v_{109}], [v_{67}, v_{68}, v_{71}, v_{90}, v_{110}], [v_{69}, v_{70}, v_{71}, v_{91}, v_{111}],$
$[v_{72}, v_{73}, v_{76}, v_{82}, v_{112}], [v_{72}, v_{74}, v_{77}, v_{83}, v_{113}], [v_{72}, v_{75}, v_{78}, v_{84}, v_{114}], [v_{73}, v_{74}, v_{79}, v_{85}, v_{115}], [v_{73}, v_{75}, v_{80}, v_{86}, v_{116}], [v_{74}, v_{75}, v_{81}, v_{87}, v_{117}],$
$[v_{76}, v_{77}, v_{79}, v_{88}, v_{118}], [v_{76}, v_{78}, v_{80}, v_{89}, v_{119}], [v_{77}, v_{78}, v_{81}, v_{90}, v_{120}], [v_{79}, v_{80}, v_{81}, v_{91}, v_{121}], [v_{82}, v_{83}, v_{85}, v_{88}, v_{122}], [v_{82}, v_{84}, v_{86}, v_{89}, v_{123}],$
$[v_{83}, v_{84}, v_{87}, v_{90}, v_{124}], [v_{85}, v_{86}, v_{87}, v_{91}, v_{125}], [v_{88}, v_{89}, v_{90}, v_{91}, v_{126}], [v_{92}, v_{93}, v_{96}, v_{102}, v_{112}], [v_{92}, v_{94}, v_{97}, v_{103}, v_{113}], [v_{92}, v_{95}, v_{98}, v_{104}, v_{114}],$
$[v_{93}, v_{94}, v_{99}, v_{105}, v_{115}], [v_{93}, v_{95}, v_{100}, v_{106}, v_{116}], [v_{94}, v_{95}, v_{101}, v_{107}, v_{117}], [v_{96}, v_{97}, v_{99}, v_{108}, v_{118}], [v_{96}, v_{98}, v_{100}, v_{109}, v_{119}],$
$[v_{97}, v_{98}, v_{101}, v_{110}, v_{120}], [v_{99}, v_{100}, v_{101}, v_{111}, v_{121}], [v_{102}, v_{103}, v_{105}, v_{108}, v_{122}], [v_{102}, v_{104}, v_{106}, v_{109}, v_{123}], [v_{103}, v_{104}, v_{107}, v_{110}, v_{124}],$
$[v_{105}, v_{106}, v_{107}, v_{111}, v_{125}], [v_{108}, v_{109}, v_{110}, v_{111}, v_{126}], [v_{112}, v_{113}, v_{115}, v_{118}, v_{122}], [v_{112}, v_{114}, v_{116}, v_{119}, v_{123}], [v_{113}, v_{114}, v_{117}, v_{120}, v_{124}],$
$[v_{115}, v_{116}, v_{117}, v_{121}, v_{125}], [v_{118}, v_{119}, v_{120}, v_{121}, v_{126}], [v_{122}, v_{123}, v_{124}, v_{125}, v_{126}]\}$

It is no surprise that the distribution of variables in $m \in M$ above is exactly that of *sub div point sets* of 4 points in $\mathscr{X}_5 \in \mathscr{S}dps_{of}(\mathscr{X}_9, 5)$ where $\mathscr{X}_9$ is any *div point set* of 9 points (as shown in (3.6)). Each variable in V represents $Assign(\mathscr{X}_4)$ for a particular element $\mathscr{X}_4 \in \mathscr{S}dps_{of}(\mathscr{X}, 4)$ where $\mathscr{X}$ is a *div point set* of $2^{n-2}+1$ points for some $n \in \mathbb{N}_{\geq 5}$. If there exists no value-assignment Z satisfying formulae in C, we can be certain that there exists no *div point set* of $2^{n-2}+1$ points, $\mathscr{X}$, where every $\mathscr{X}_5 \in \mathscr{S}dps_{of}(\mathscr{X}, 5)$ satisfies (3.7) but $\mathscr{X}$ does not satisfy (3.8) as mentioned above.

Remark. $UNSAT^{\mathscr{DPS}+}_{multiset}$ can be reduced into the Boolean Unsatisfiability Problem, the complement of SAT, by first converting each multiset in A into the DNF formula below:

$$\bigvee_{v_0 \in \mathscr{A}} (\neg v_0 \wedge \bigwedge_{v_1 \in \mathscr{A} \setminus \{v_0\}} v_1) \vee \bigvee_{\mathscr{A}_{|3|} \in \mathscr{A}^*_{|3|}} (\bigwedge_{v_0 \in \mathscr{A}_3} \neg v_0 \wedge \bigwedge_{v_1 \in V \setminus \mathscr{A}_{|3|}} v_1) \vee (\bigwedge_{v_0 \in \mathscr{A}} \neg v_0) \tag{3.14}$$

where $\mathscr{A}^*_{|3|} = \{\mathscr{A}_{|3|} \in \mathbb{P}(\mathscr{A}) : |\mathscr{A}_{|3|}| = 3\}$ and $\mathscr{A}$ denotes the set of variables in each multiset, and converting each multiset in B into the DNF formula below:

$$\bigvee_{v \in \mathscr{B}} v \tag{3.15}$$

where $\mathscr{B}$ denotes the set of variables in each multiset, then joining all these DNF formulae conjunctively. One may realize that, in the case when $\mathscr{B} = \mathscr{A}$,the conjunction of $\bigvee_{u \in \mathscr{B}} u$ and $\bigwedge_{v_0 \in \mathscr{A}} \neg v$ gives a tautology, and thus for the instance of $UNSAT^{\mathscr{DPS}+}_{multiset}$ where $n = 5$, we would have a simpler propositional formula. The same observation can be made in the set of FOL formulae of such instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ where in order to satisfy (3.13), we would need to restrict (3.12) to $\forall a \in A \;\; a \in \{[1,1,1,1,0],[1,1,0,0,0]\}$.

We thereby conclude that a plausible approach to proving the upper-bound of the Erdös-Szekeres conjecture through $UNSAT^{\mathscr{DPS}^+}_{multiset}$ is by induction i.e. we first solve for the instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ where $n = 5$ - apparently accomplishable with a modern SAT solver running on a high performance computer - and then we prove the inductive hypothesis that for all $m \in \mathbb{N}_{\geq 5}$,

$$\mathscr{UNSAT}(m) \Rightarrow \mathscr{UNSAT}(m+1) \tag{3.16}$$

where $\mathscr{UNSAT}(k)$ denotes the unsatisfiability of the instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ in which $n = k$.

Remark. The Erdös-Szekeres conjecture would not be disproven even if a certain instance of $UNSAT^{\mathscr{DPS}^+}_{multiset}$ turns out to be satisfiable, since satisfying the constraints only implies that there exists a *div point set* of $2^{n-2}+1$ points for a particular $n \in \mathbb{N}_{\geq 5}$ where

I. none of its *sub div point sets* of n points is isomorphic to $Conv_n$

II. each of its *sub div point sets* of 5 points has 4, 2 or 0 distinct *sub div point sets* of 4 points isomorphic to $Conc^1_4$

from which we cannot conclude that such *div point set* is in $\mathscr{DPS}^+$, unless it too satisfies the stronger version of *Theorem 2* i.e. unless proven so, we should not rule out the possibility for a *div point set* of 5 points to not be in $\mathscr{DPS}^+$ despite having 4, 2 or 0 distinct *sub div point sets* of 4 points isomorphic to $Conc^1_4$ (with the remaining isomorphic to $Conv_4$).

To disprove the Erdös-Szekeres conjecture, not only do we need to show that (3.2) is false, we need to demonstrate that there exists no other constraints besides (2.10), (2.11), and (2.12) which $\mathscr{X} \in \mathscr{DPS}^*$ has to satisfy such that there exists an interpretation for $\pi_1(\mathscr{X})$ as some set of points in $\mathbb{E}^2$ i.e. *Axiom 1*'s consistency with Euclidean geometry.

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