Freudenthal-Hopf

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Introduction

The goal of this project is to formalise the Freudenthal-Hopf theorem from Geometric Group Theory, which states that the Cayley graph of a finitely-generated group has zero, one, two, or infinitely many ends.

Chapter 1

Ends

1.1 Introduction

This chapter outlines an approach to defining *ends* of a graph. For concreteness, we restrict the definition of ends to just graphs, although the concept of ends can also be formulated for arbitrary topological spaces.

Intuitively, the *ends* of a space are meant to capture the distinct ways in which the space approaches infinity.

1.2 Defining *Ends*

This idea is often formally captured in the following way:

Definition 1.1 (Ends)**.** *Let G be a graph on a vertex set V.*

An end *of G is defined as a function assigning to each finite subset of V a connected component in its complement, subject to a consistency condition that the component assigned to any subset of a finite set K must contain the component assigned to K.*

Remark 1.2. *If K and L are two subsets of the vertex set of a graph with $K \subset L$, any non-empty connected component in the complement of L determines a unique connected component in the complement of K that contains it.*

This follows from the fact that any non-empty connected set of a graph is contained in a unique connected component.

For the purposes of formalisation, a Category theoretic description of 1.1 turns out to be more suitable, as several properties of ends can then be deduced from highly general results in Category theory that are present in Lean's mathematics library `mathlib`.

Definition 1.3. *For a type V, let* FinIncl *be the directed system of finite subsets of V considered a category under inclusion.*

Definition 1.4 (Ends functor)**.** *For a graph G on a vertex set V, let* ComplComp *be the contravariant functor from 1.3 to the category of types assigning to each finite set the set of connected components in its complement. The morphisms are mapped according to 1.2.*

Remark 1.5. *When the graph G is infinite, there is at least one connected component outside each finite set of vertices K.*

When the graph is locally finite, the number of connected components outside each finite set K is finite, since the connected components cover the boundary *of K, which is finite when the graph is locally finite.*

Together, these conditions ensure that 1.4 assigns to each object in 1.3 a non-empty and finite type.

It follows from general category theory that the limit of this inverse system exists and is non-empty.

Definition 1.6 (Sections of a Functor)**.** *The* sections *of a functor F from a category J to the category of types are the choices of a point $u(j) \in F(j)$ for each $j \in J$ such that $F(f)(u(j)) = u(j')$, for each $f : j \to j'$.*

Remark 1.7. *The sections of a functor 1.6 are closely related to limits in Category theory. The limit of a digram always exists in the category of types.*

Suppose L is the limit of a diagram J in the category of types. For any $l \in L$, one can construct a section of J by assigning to each $j \in J$ the image of l under the projection to j. Conversely, every section determines a consistent assignment of objects in the diagram, and hence an element of the limit L.

The following definition of ends is due to Kyle Miller:

Definition 1.8 (Ends)**.** *The* Ends *of a graph G are the sections of the functor 1.4 corresponding to G.*

Chapter 2

Functoriality

2.1 Introduction

Sufficiently nice maps between graphs can induce maps between their corresponding sets of ends. This chapter describes an approach to formalising this functoriality in terms of a notion of *reachability* of points in a graph.

Chapter 3

The Freudenthal-Hopf theorem

This chapter outlines the final proof of the *Freudenthal-Hopf theorem*.