Equational theories

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September 26, 2024

Chapter 1

Introduction

Definition 1. A Magma is a set G equipped with a binary operation $\circ : G \times G \to G$.

Chapter 2

Equations

Definition 2 (Equation 1)**.** Equation 1 is the law $x = x$.

Definition 3 (Equation 2)**.** Equation 2 is the law $x = y$.

Definition 4 (Equation 3)**.** Equation 3 is the law $x = x \circ x$.

Definition 5 (Equation 4)**.** Equation 4 is the law $x = x \circ y$.

Definition 6 (Equation 5)**.** Equation 5 is the law $x = y \circ x$.

Definition 7 (Equation 6)**.** Equation 6 is the law $x = y \circ y$.

Definition 8 (Equation 7)**.** Equation 7 is the law $x = y \circ z$.

Definition 9 (Equation 8)**.** Equation 8 is the law $x = x \circ (x \circ x)$.

Definition 10 (Equation 42)**.** Equation 42 is the law $x \circ y = x \circ z$.

Definition 11 (Equation 43)**.** Equation 43 is the law $x \circ y = y \circ x$.

Definition 12 (Equation 46)**.** Equation 46 is the law $x \circ y = z \circ w$.

Definition 13 (Equation 387)**.** Equation 387 is the law $x \circ y = (y \circ y) \circ x$.

Definition 14 (Equation 4512)**.** Equation 4512 is the law $x \circ (y \circ z) = (x \circ y) \circ z$.

Definition 15 (Equation 4513)**.** Equation 4513 is the law $x \circ (y \circ z) = (x \circ y) \circ w$.

Definition 16 (Equation 4552)**.** Equation 4552 is the law $x \circ (y \circ z) = (x \circ w) \circ u$.

Definition 17 (Equation 4582)**.** Equation 4582 is the law $x \circ (y \circ z) = (w \circ u) \circ v$.

Chapter 3

Implications

To reduce clutter, trivial or very easy implications will not be displayed here.

Theorem 18 (387 implies 43)**.** *Definition 13 implies Definition 11.*

Proof. (From `MathOverflow`https://mathoverflow.net/a/450905/766). By Definition 13, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{3.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (13) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{3.2}$$

Now, replacing x by $x \circ x$ in (3.1) and then using (3.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{3.3}$$

Also, from two applications of (3.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3.3) simplifies to $x \circ y = y \circ x$, which is Definition 11. $\square$

Chapter 4

Counterexamples

Theorem 19 (46 does not imply 3)**.** *Definition 12 does not imply Definition 4.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := 0$. □

Theorem 20 (3 does not imply 4582)**.** *Definition 4 does not imply Definition 17.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 21 (3 does not imply 43)**.** *Definition 4 does not imply Definition 11.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 22 (Equation 4582 does not imply Equation 42)**.** *Definition 17 does not imply Definition 10.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = y = 0$ and 2 otherwise. □

Theorem 23 (Equation 4582 does not imply Equation 43)**.** *Definition 17 does not imply Definition 11.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 3 if $x = 1$ and $y = 2$ and 4 otherwise. □

Theorem 24 (Equation 42 does not imply Equation 43)**.** *Definition 10 does not imply Definition 11.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 25 (Equation 42 does not imply Equation 4512)**.** *Definition 10 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + 1$. □

Theorem 26 (Equation 43 does not imply Equation 42)**.** *Definition 11 does not imply Definition 10.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. □

Theorem 27 (Equation 43 does not imply Equation 4512)**.** *Definition 11 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x \cdot y + 1$. $\square$

Theorem 28 (Equation 4513 does not imply Equation 4552)**.** *Definition 15 does not imply Definition 16.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = 0$ and $y \leq 2$, 2 if $x = 0$ and $y > 2$, and x otherwise. $\square$

Theorem 29 (Equation 4512 does not imply Equation 4513)**.** *Definition 14 does not imply Definition 15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. $\square$