Equational theories

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Chapter 1

Introduction

Definition 1. A Magma is a set G equipped with a binary operation $\circ : G \times G \to G$.

A *law* is an equation involving a finite number of indeterminate variables and the operation $\circ$. A magma G then obeys that law if the equation holds for all possible choices of indeterminate variables in G. For instance, the commutative law

$$x \circ y = y \circ x$$

holds in a magma G if and only if that magma is abelian.

We will be interested in seeing which laws imply which other laws, in the sense that magmas obeying the former law automatically obey the latter. We will also be interested in *anti-implications* showing that one law does *not* imply another, by producing examples of magmas that obey the former law but not the latter.

Chapter 2

Equations

Definition 2 (Equation 1)**.** Equation 1 is the law $x = x$.

Definition 3 (Equation 2)**.** Equation 2 is the law $x = y$.

Definition 4 (Equation 3)**.** Equation 3 is the law $x = x \circ x$.

Definition 5 (Equation 4)**.** Equation 4 is the law $x = x \circ y$.

Definition 6 (Equation 5)**.** Equation 5 is the law $x = y \circ x$.

Definition 7 (Equation 6)**.** Equation 6 is the law $x = y \circ y$.

Definition 8 (Equation 7)**.** Equation 7 is the law $x = y \circ z$.

Definition 9 (Equation 8)**.** Equation 8 is the law $x = x \circ (x \circ x)$.

Definition 10 (Equation 42)**.** Equation 42 is the law $x \circ y = x \circ z$.

Definition 11 (Equation 43)**.** Equation 43 is the law $x \circ y = y \circ x$.

Definition 12 (Equation 46)**.** Equation 46 is the law $x \circ y = z \circ w$.

Definition 13 (Equation 387)**.** Equation 387 is the law $x \circ y = (y \circ y) \circ x$.

Definition 14 (Equation 4512)**.** Equation 4512 is the law $x \circ (y \circ z) = (x \circ y) \circ z$.

Definition 15 (Equation 4513)**.** Equation 4513 is the law $x \circ (y \circ z) = (x \circ y) \circ w$.

Definition 16 (Equation 4552)**.** Equation 4552 is the law $x \circ (y \circ z) = (x \circ w) \circ u$.

Definition 17 (Equation 4582)**.** Equation 4582 is the law $x \circ (y \circ z) = (w \circ u) \circ v$.

Chapter 3

Implications

To reduce clutter, trivial or very easy implications will not be displayed here.

Theorem 18 (387 implies 43)**.** *Definition 13 implies Definition 11.*

Proof. (From MathOverflow). By Definition 13, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{3.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (13) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{3.2}$$

Now, replacing x by $x \circ x$ in (3.1) and then using (3.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{3.3}$$

Also, from two applications of (3.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3.3) simplifies to $x \circ y = y \circ x$, which is Definition 11. $\square$

Chapter 4

Counterexamples

Theorem 19 (46 does not imply 4)**.** *Definition 12 does not imply Definition 5.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := 0$. □

Theorem 20 (4 does not imply 4582)**.** *Definition 5 does not imply Definition 17.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 21 (4 does not imply 43)**.** *Definition 5 does not imply Definition 11.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 22 (Equation 4582 does not imply Equation 42)**.** *Definition 17 does not imply Definition 10.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = y = 0$ and 2 otherwise. □

Theorem 23 (Equation 4582 does not imply Equation 43)**.** *Definition 17 does not imply Definition 11.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 3 if $x = 1$ and $y = 2$ and 4 otherwise. □

Theorem 24 (Equation 42 does not imply Equation 43)**.** *Definition 10 does not imply Definition 11.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 25 (Equation 42 does not imply Equation 4512)**.** *Definition 10 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + 1$. □

Theorem 26 (Equation 43 does not imply Equation 42)**.** *Definition 11 does not imply Definition 10.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. □

Theorem 27 (Equation 43 does not imply Equation 4512)**.** *Definition 11 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x \cdot y + 1$. $\square$

Theorem 28 (Equation 4513 does not imply Equation 4552)**.** *Definition 15 does not imply Definition 16.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = 0$ and $y \leq 2$, 2 if $x = 0$ and $y > 2$, and x otherwise. $\square$

Theorem 29 (Equation 4512 does not imply Equation 4513)**.** *Definition 14 does not imply Definition 15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. $\square$

Theorem 30 (Equation 387 does not imply Equation 42)**.** *Definition 13 does not imply Definition 10.*

Proof. Use the boolean type Bool with $x \circ y := x||y$. $\square$

Theorem 31 (Equation 43 does not imply Equation 387)**.** *Definition 11 does not imply Definition 13.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. $\square$

Theorem 32 (Equation 387 does not imply Equation 4512)**.** *Definition 13 does not imply Definition 14.*

Proof. Use the reals $\mathbb{R}$ with $x \circ y := (x + y)/2$. $\square$