Equational theories

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Chapter 1

Introduction

Definition 1.1. A Magma is a set G equipped with a binary operation $\circ : G \times G \to G$.

A *law* is an equation involving a finite number of indeterminate variables and the operation $\circ$. A magma G then obeys that law if the equation holds for all possible choices of indeterminate variables in G. For instance, the commutative law

$$x \circ y = y \circ x$$

holds in a magma G if and only if that magma is abelian.

We will be interested in seeing which laws imply which other laws, in the sense that magmas obeying the former law automatically obey the latter. We will also be interested in *anti-implications* showing that one law does *not* imply another, by producing examples of magmas that obey the former law but not the latter.

The number of finite magmas of length $n = 0, 1, 2, \ldots$, up to isomorphism, is

$$1, 1, 10, 3330, 178981952, 2483527537094825, 14325590003318891522275680, \ldots$$

(https://oeis.org/A001329).

The singleton or empty magma obeys all equational laws. One can ask whether an equational law admits nontrivial finite or infinite models. The following result was established in [1]:

Theorem 1.2. *The equational law*

$$(((y \circ y) \circ y) \circ x) \circ ((y \circ y) \circ z) = x \tag{1.1}$$

has an infinite model, but no non-trivial finite model.

Proof. Suppose for contradiction that we have a non-trivial model of (1.1). Write $y^2 := y \circ y$ and $y^3 := y^2 \circ y$. For any y, z, introduce the functions $f_y : x \mapsto y^3 \circ x$ and $g_{yz} : x \mapsto x \circ (y^2 \circ z)$. The law (1.1) says that g_{yz} is a left-inverse of f_y, hence by finiteness these are inverses and g_{yz} is independent of z. In particular

$$f(y^3) = g_{yy}(y^3) = g_{yz}(y^3) = f(y^2 \circ z)$$

and hence $y^2 \circ z$ is independent of z. Thus

$$f_y(x) = (y^2 \circ y) \circ x = (y^2 \circ y^2) \circ x$$

is independent of x. As f_y is invertible, this forces the magma to be trivial, a contradiction.

To construct an infinite magma, take the positive integers $\mathbb{Z}^+$ with the operation $x \circ y$ defined as

- 2^x if $y = x$;
- 3^y if $x = 1 \neq y$;
- $\min(j, 1)$ if $x = 3^j$ and $y \neq x$; and
- 1 otherwise.

Then $y^2 = 2^y$, $y^3 = 1$, and $y^2 \circ z$ a power of two for all y, z, and $(1 \circ x) \circ w = x$ for all x whenever w is a power of two, so (1.1) is satisfied. $\square$

Chapter 2

Subgraph equations

In this project we study the 4694 equational laws (up to symmetry and relabeling) that involve at most four applications of the binary operation $\circ$. The full list of such laws may be found here, and a script for generating them may be found here.

Selected equations of interest are listed below, as well as in this file. Equations in this list will be referred to as “subgraph equations”, as we shall inspect the subgraph of the implication subgraph induced by these equations.

Definition 2.1 (Equation 1)**.** Equation 1 is the law $x = x$.

This is the trivial law, satisfied by all magmas.

Definition 2.2 (Equation 2)**.** Equation 2 is the law $x = y$.

This is the singleton law, satisfied only by the empty and singleton magmas.

Definition 2.3 (Equation 3)**.** Equation 3 is the law $x = x \circ x$.

This is the idempotence law.

Definition 2.4 (Equation 4)**.** Equation 4 is the law $x = x \circ y$.

This is the left absorption law.

Definition 2.5 (Equation 5)**.** Equation 5 is the law $x = y \circ x$.

This is the right absorption law (the dual of Definition 2.4).

Definition 2.6 (Equation 6)**.** Equation 6 is the law $x = y \circ y$.

This law is equivalent to the singleton law.

Definition 2.7 (Equation 7)**.** Equation 7 is the law $x = y \circ z$.

This law is equivalent to the singleton law.

Definition 2.8 (Equation 8)**.** Equation 8 is the law $x = x \circ (x \circ x)$.

Definition 2.9 (Equation 38)**.** Equation 38 is the law $x \circ x = x \circ y$.

This law asserts that the magma operation is independent of the second argument.

Definition 2.10 (Equation 39)**.** Equation 39 is the law $x \circ x = y \circ x$.

This law asserts that the magma operation is independent of the first argument (the dual of Definition 2.9).

Definition 2.11 (Equation 40)**.** Equation 40 is the law $x \circ x = y \circ y$.

This law asserts that all squares are constant.

Definition 2.12 (Equation 41)**.** Equation 41 is the law $x \circ x = y \circ z$.

This law is equivalent to the constant law, Definition 2.15.

Definition 2.13 (Equation 42)**.** Equation 42 is the law $x \circ y = x \circ z$.

Equivalent to Definition 2.9.

Definition 2.14 (Equation 43)**.** Equation 43 is the law $x \circ y = y \circ x$.

The commutative law.

Definition 2.15 (Equation 46)**.** Equation 46 is the law $x \circ y = z \circ w$.

The constant law: all products are constant.

Definition 2.16 (Equation 168)**.** Equation 168 is the law $x = (y \circ x) \circ (x \circ z)$.

The law of a central groupoid.

Definition 2.17 (Equation 387)**.** Equation 387 is the law $x \circ y = (y \circ y) \circ x$.

Definition 2.18 (Equation 4512)**.** Equation 4512 is the law $x \circ (y \circ z) = (x \circ y) \circ z$.

The associative law.

Definition 2.19 (Equation 4513)**.** Equation 4513 is the law $x \circ (y \circ z) = (x \circ y) \circ w$.

Definition 2.20 (Equation 4522)**.** Equation 4522 is the law $x \circ (y \circ z) = (x \circ w) \circ u$.

Definition 2.21 (Equation 4582)**.** Equation 4582 is the law $x \circ (y \circ z) = (w \circ u) \circ v$.

This law asserts that all triple constants (regardless of bracketing) are constant.
Implications between these laws are depicted in Figure 2.1.

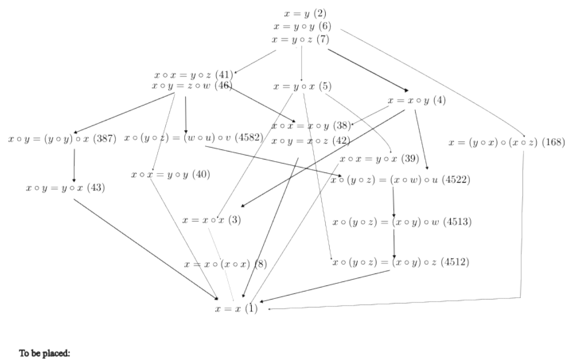

Figure 2.1: Implications between the above equations.

Chapter 3

General implications

In this chapter we record some general implications between equational laws.

Theorem 3.1 (Singleton law implies all other laws)**.** *The singleton law (Definition 2.2) implies all other laws.*

Proof. This is clear from substitution. □

Theorem 3.2 (All laws imply the trivial law)**.** *All laws imply the trivial law (Definition 2.1).*

Proof. Trivial. □

Every law E has a *dual* E^{op}, formed by replacing the magma operation $\circ$ with its opposite $\circ^{\mathrm{op}} : (x, y) \mapsto y \circ x$. For instance, the opposite of the law $x \circ y = x \circ z$ is $y \circ x = z \circ x$.

The implication graph has a duality symmetry:

Theorem 3.3 (Duality)**.** *If E, F are equational laws, then E implies F if and only if E^{op} implies F^{op}.*

Proof. This is because a magma M obeys a law E if and only if the opposite magma M^{op} obeys E^{op}. □

Some equational laws can be "diagonalized":

Theorem 3.4 (Diagonalization)**.** *An equational law of the form*

$$F(x_1, \dots, x_n) = G(y_1, \dots, y_m), \tag{3.1}$$

where $x_1, \dots, x_n$ and $y_1, \dots, y_m$ are distinct indeterminates, implies the diagonalized law

$$F(x_1, \dots, x_n) = F(x'_1, \dots, x'_n).$$

In particular, if $G(y_1, \dots, y_m)$ can be viewed as a specialization of $F(x'_1, \dots, x'_n)$, then these two laws are equivalent.

Proof. From two applications of (3.1) one has

$$F(x_1, \dots, x_n) = G(y_1, \dots, y_m)$$

and

$$F(x'_1, \dots, x'_n) = G(y_1, \dots, y_m)$$

whence the claim. □

Thus for instance, Definition 2.7 is equivalent to Definition 2.2.

Chapter 4

Subgraph implications

Interesting implications between the subgraph equations in Chapter 2. To reduce clutter, trivial or very easy implications will not be displayed here.

Theorem 4.1 (387 implies 43)**.** *Definition 2.17 implies Definition 2.14.*

Proof. (From MathOverflow). By Definition 2.17, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{4.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (2.17) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{4.2}$$

Now, replacing x by $x \circ x$ in (4.1) and then using (4.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{4.3}$$

Also, from two applications of (4.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (4.3) simplifies to $x \circ y = y \circ x$, which is Definition 2.14. □

Chapter 5

Subgraph counterexamples

Some counterexamples for the anti-implications between the subgraph equations in Chapter 2.

Theorem 5.1 (46 does not imply 4)**.** *Definition 2.15 does not imply Definition 2.4.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := 0$. ☐

Theorem 5.2 (4 does not imply 4582)**.** *Definition 2.4 does not imply Definition 2.21.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. ☐

Theorem 5.3 (4 does not imply 43)**.** *Definition 2.4 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. ☐

Theorem 5.4 (4582 does not imply 42)**.** *Definition 2.21 does not imply Definition 2.13.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = y = 0$ and 2 otherwise. ☐

Theorem 5.5 (4582 does not imply 43)**.** *Definition 2.21 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 3 if $x = 1$ and $y = 2$ and 4 otherwise. ☐

Theorem 5.6 (42 does not imply 43)**.** *Definition 2.13 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. ☐

Theorem 5.7 (42 does not imply 4512)**.** *Definition 2.13 does not imply Definition 2.18.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + 1$. ☐

Theorem 5.8 (43 does not imply 42)**.** *Definition 2.14 does not imply Definition 2.13.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. ☐

Theorem 5.9 (43 does not imply 4512)**.** *Definition 2.14 does not imply Definition 2.18.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x \cdot y + 1$. ☐

Theorem 5.10 (4513 does not imply 4522)**.** *Definition 2.19 does not imply Definition 2.20.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = 0$ and $y \leq 2$, 2 if $x = 0$ and $y > 2$, and x otherwise. □

Theorem 5.11 (4512 does not imply 4513)**.** *Definition 2.18 does not imply Definition 2.19.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. □

Theorem 5.12 (387 does not imply 42)**.** *Definition 2.17 does not imply Definition 2.13.*

Proof. Use the boolean type Bool with $x \circ y := x||y$. □

Theorem 5.13 (43 does not imply 387)**.** *Definition 2.14 does not imply Definition 2.17.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. □

Theorem 5.14 (387 does not imply 4512)**.** *Definition 2.17 does not imply Definition 2.18.*

Proof. Use the reals $\mathbb{R}$ with $x \circ y := (x + y)/2$. □

Theorem 5.15 (3 does not imply 42)**.** *Definition 2.3 does not imply Definition 2.13.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := y$. □

Theorem 5.16 (3 does not imply 4512)**.** *Definition 2.3 does not imply Definition 2.18.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y$ equal to x when $x = y$ and $x + 1$ otherwise. □

Theorem 5.17 (46 does not imply 3)**.** *Definition 2.15 does not imply Definition 2.3.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := 0$. □

Theorem 5.18 (43 does not imply 3)**.** *Definition 2.14 does not imply Definition 2.3.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. □

Chapter 6

Equivalence with the constant and singleton laws

85 laws have been shown to be equivalent to the constant law (Definition 2.15), and 815 laws have been shown to be equivalent to the singleton law (Definition 2.2).

describe the process of automatically generating these implications here.

Chapter 7

Simple rewrites

53,905 implications were automatically generated by simple rewrites.

describe the process of automatically generating these implications here.

Bibliography

[1] Andrzej Kisielewicz. Varieties of algebras with no nontrivial finite members. In *Lattices, semigroups, and universal algebra (Lisbon, 1988)*, pages 129–136. Plenum, New York, 1990.