Equational theories

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Chapter 1

Basic theory of magmas

Definition 1.1 (Magma)**.** A *magma* is a set G equipped with a binary operation $\circ : G \times G \to G$. A *homomorphism* $\varphi : G \to H$ between two magmas is a map such that $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in G$. An *isomorphism* is an invertible homomorphism.

Groups, semi-groups, and monoids are familiar examples of magmas. However, in general we do not expect magmas to have any associative properties.

A magma is called *empty* if it has cardinality zero, *singleton* if it has cardinality one, and *non-trivial* otherwise.

The number of magma structures on a set G of cardinality n is of course n^{n^2}, which is [1]

$$1, 1, 16, 19683, 4294967296, 298023223876953125, ...$$

(https://oeis.org/A002489). Up to isomorphism, the number of finite magmas of cardinality n up to isomorphism is the slightly slower growing sequence

$$1, 1, 10, 3330, 178981952, 2483527537094825, 14325590003318891522275680, ...$$

(https://oeis.org/A001329).

Definition 1.2 (Free Magma)**.** The *free magma* M_X generated by a set X (which we call an *alphabet*) is the set of all finite formal expressions built from elements of X and the operation $\circ$. An element of M_X will be called a *word* with alphabet X. The *order* of a word is the number of $\circ$ symbols needed to generate the word. Thus for instance X is precisely the set of words of order 1 in M_X.

For sake of concreteness, we will take the alphabet X to default to the natural numbers $\mathbb{N}$ if not otherwise specified.

For instance, if $X = \{0, 1\}$, then M_X would consist of the following words:

- 0, 1 (the words of order 0);

- $0 \circ 0$, $0 \circ 1$, $1 \circ 0$, $1 \circ 1$ (the words of order 1);

- $0 \circ (0 \circ 0)$, $0 \circ (0 \circ 1)$, $0 \circ (1 \circ 0)$, $0 \circ (1 \circ 1)$, $1 \circ (0 \circ 0)$, $1 \circ (0 \circ 1)$, $1 \circ (1 \circ 0)$, $1 \circ (1 \circ 1)$, $(0 \circ 0) \circ 0$, $(0 \circ 0) \circ 1$, $(0 \circ 1) \circ 0$, $(0 \circ 1) \circ 1$, $(1 \circ 0) \circ 0$, $(1 \circ 0) \circ 1$, $(1 \circ 1) \circ 0$, $(1 \circ 1) \circ 1$ (the words of order 2);

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[1]All sequences start from $n = 0$ unless otherwise specified.

- etc.

Lemma 1.3. *For a finite alphabet X, the number of words of order n is $C_n |X|^{n+1}$, where C_n is the nth Catalan number and X is the cardinality of X.*

Proof. Follows from standard properties of Catalan numbers. □

The first few Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, \dots$$

(https://oeis.org/A000108).

Definition 1.4 (Induced homomorphism)**.** Given a function $f : X \to G$ from an alphabet X to a magma G, the *induced homomorphism* $\varphi_f : M_X \to G$ is the unique extension of f to a magma homomorphism. Similarly, if $\pi : X \to Y$ is a function, we write $\pi_* : M_X \to M_Y$ for the unique extension of π to a magma homomorphism.

For instance, if $f : \{0, 1\} \to G$ maps $0, 1$ to x, y respectively, then

$$\varphi_f(0 \circ 1) = x \circ y$$

$$\varphi_f(1 \circ (0 \circ 1)) = y \circ (x \circ y)$$

and so forth. If $\pi \colon \mathbb{N} \to \mathbb{N}$ is the map $\pi(n) := n + 1$, then

$$\pi_*(0 \circ 1) = 1 \circ 2$$

$$\pi_*(1 \circ (0 \circ 1)) = 2 \circ (1 \circ 2)$$

and so forth.

Definition 1.5 (Law)**.** Let X be a set. A *law* with alphabet X is a formal expression of the form $w \simeq w'$, where $w, w' \in M_X$ are words with alphabet X (thus one can identify laws with alphabet X with elements of $M_X \times M_X$). A magma G *satisfies* the law $w \simeq w'$ if we have $\varphi_f(w) = \varphi_f(w')$ for all $f : X \to G$, in which case we write $G \vDash w \simeq w'$.

Thus, for instance, the commutative law

$$0 \circ 1 \simeq 1 \circ 0 \tag{1.1}$$

is satisfied by a magma G if and only if

$$x \circ y = y \circ x \tag{1.2}$$

for all $x, y \in G$. We refer to (1.2) as the *equation* associated to the law (1.1). One can think of equations as the "semantic" intrepretation of a "syntactic" law. However, we shall often abuse notation and a law with its associated equation thus we shall (somewhat carelessly) also refer to (1.2) as "the commutative law" (rather than "the commutative equation").

Lemma 1.6 (Pushforward)**.** *Let $w \simeq w'$ be a law with some alphabet X, G be a magma, and $\pi : X \to Y$ be a function. If $G \vDash w \simeq w'$, then $G \vDash \pi_*(w) \simeq \pi_*(w')$. In particular, if π is a bijection, the statements If $G \vDash w \simeq w'$, then $G \vDash \pi_*(w) \simeq \pi_*(w')$ are equivalent.*

If π is a bijection, we will call $\pi_*(w) \simeq \pi_*(w')$ a *relabeling* of the law $w \simeq w'$. Thus for instance

$$5 \circ 7 \simeq 7 \circ 5$$

is a relabeling of the commutative law (1.1). By the above lemma, relabeling does not affect whether a given magna satisfies a given law.

Proof. Trivial. $\qquad\square$

Lemma 1.7 (Equivalence). *Let G be a magma and X be an alphabet. Then the relation $G \vDash w \simeq w'$ is an equivalence relation on M_X.*

Proof. Trivial. $\qquad\square$

Define the total order of a law $w \simeq w'$ to be the sum of the orders of w and w'.

Lemma 1.8 (Counting laws up to relabeling). *Up to relabeling, the number of laws $w \simeq w'$ of total order n is $C_{n+1}B_{n+2}$.*

Proof. Follows from the properties of Catalan and Bell numbers. $\qquad\square$

The first few Bell numbers (starting from $n = 0$) are

$$1, 1, 2, 5, 15, 52, 203, ...$$

(https://oeis.org/A000110).
The sequence in Lemma 1.8 is

$$2, 10, 75, 728, 8526, 115764, ...$$

(https://oeis.org/A289679).
Now we would also like to count laws up to relabeling and symmetry.

Lemma 1.9 (Counting laws up to relabeling and symmetry). *Up to relabeling and symmetry, the number of laws $w \simeq w'$ of total order n is*

$$C_{n+1}B_{n+2}/2$$

when n is odd, and

$$(C_{n+1}B_{n+2} + C_{n/2}(2D_{n+2} - B_{n+2}))/2$$

when n is even, where D_n is the number of partitions of $[n]$ up to reflection.

Proof. Elementary counting. $\qquad\square$

The sequence D_n is (starting from $n = 0$)

$$1, 1, 2, 4, 11, 32, 117, ...$$

(https://oeis.org/A103293), and the sequence in Lemma 1.9 is (starting from $n = 0$)

$$2, 5, 41, 364, 4294, 57882, 888440, ...$$

check this matches the formula
We can also identify all laws of the form $w \simeq w$ with the trivial law $0 \simeq 0$. The number of such laws of total order n is zero if n is odd, and $C_{n/2}B_{n/2+1}$ if n is even. We conclude:

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Lemma 1.10 (Counting laws up to relabeling, symmetry, and triviality)**.** *Up to relabeling, symmetry, and triviality, the number of laws of total order n is*

$$C_{n+1}B_{n+2}/2$$

if n is odd, 2 if $n = 0$, and

$$(C_{n+1}B_{n+2} + C_{n/2}(2D_{n+2} - B_{n+2}))/2 - C_{n/2}B_{n/2+1}$$

if $n \geq 2$ is even.

Proof. Routine counting. $\qquad\qquad\square$

This sequence is

$$2, 5, 39, 364, 4284, 57882, 888365, ...$$

check this matches the formula.

In particular, up to relabeling, symmetry, and triviality, there are exactly 4694 laws of total order at most 4. A list can be found here. A script for generating them may be found here. The list is sorted by the total number of operations, then by the number of operations on the LHS. Within each such class we define an order on expressions by variable $<$ operation, and lexical order on variables.

Chapter 2

Subgraph laws

In this project we study the 4694 laws (up to symmetry and relabeling) of total order at most 4.

Selected laws of interest are listed below, as well as in this file. Laws in this list will be referred to as "subgraph equations", as we shall inspect the subgraph of the implication subgraph induced by these equations.

Definition 2.1 (Equation 1)**.** Equation 1 is the law $0 == 0$ (or the equation $x = x$).

This is the trivial law, satisfied by all magmas. It is self-dual.

Definition 2.2 (Equation 2)**.** Equation 2 is the law $0 == 1$ (or the equation $x = y$).

This is the singleton law, satisfied only by the empty and singleton magmas. It is self-dual.

Definition 2.3 (Equation 3)**.** Equation 3 is the law $0 == 0 \circ 0$ (or the equation $x = x \circ x$).

This is the idempotence law. It is self-dual.

Definition 2.4 (Equation 4)**.** Equation 4 is the law $0 == 0 \circ 1$ (or the equation $x = x \circ y$).

This is the left absorption law.

Definition 2.5 (Equation 5)**.** Equation 5 is the law $0 == 1 \circ 0$ (or the equation $x = y \circ x$).

This is the right absorption law (the dual of Definition 2.4).

Definition 2.6 (Equation 6)**.** Equation 6 is the law $0 == 1 \circ 1$ (or the equation $x = y \circ y$).

This law is equivalent to the singleton law.

Definition 2.7 (Equation 7)**.** Equation 7 is the law $0 == 1 \circ 2$ (or the equation $x = y \circ z$).

This law is equivalent to the singleton law.

Definition 2.8 (Equation 8)**.** Equation 8 is the law $0 == 0 \circ (0 \circ 0)$ (or the equation $x = x \circ (x \circ x)$).

Definition 2.9 (Equation 23)**.** Equation 23 is the law $0 == (0 \circ 0) \circ 0$ (or the equation $x = (x \circ x) \circ x$).

This is the dual of Definition 2.8.

Definition 2.10 (Equation 38)**.** Equation 38 is the law $0 \circ 0 == 0 \circ 1$ (or the equation $x \circ x = x \circ y$).

This law asserts that the magma operation is independent of the second argument.

Definition 2.11 (Equation 39)**.** Equation 39 is the law $0 \circ 0 == 1 \circ 0$ (or the equation $x \circ x = y \circ x$).

This law asserts that the magma operation is independent of the first argument (the dual of Definition 2.10).

Definition 2.12 (Equation 40)**.** Equation 40 is the law $0 \circ 0 == 1 \circ 1$ (or the equation $x \circ x = y \circ y$).

This law asserts that all squares are constant. It is self-dual.

Definition 2.13 (Equation 41)**.** Equation 41 is the law $0 \circ 0 == 1 \circ 2$ (or the equation $x \circ x = y \circ z$).

This law is equivalent to the constant law, Definition 2.17.

Definition 2.14 (Equation 42)**.** Equation 42 is the law $0 \circ 1 == 0 \circ 2$ (or the equation $x \circ y = x \circ z$).

Equivalent to Definition 2.10.

Definition 2.15 (Equation 43)**.** Equation 43 is the law $0 \circ 1 == 1 \circ 0$ (or the equation $x \circ y = y \circ x$).

The commutative law. It is self-dual.

Definition 2.16 (Equation 45)**.** Equation 45 is the law $0 \circ 1 == 2 \circ 1$ (or the equation $x \circ y = z \circ y$).

This is the dual of Definition 2.14.

Definition 2.17 (Equation 46)**.** Equation 46 is the law $0 \circ 1 == 2 \circ 3$ (or the equation $x \circ y = z \circ w$).

The constant law: all products are constant. It is self-dual.

Definition 2.18 (Equation 168)**.** Equation 168 is the law $0 == (1 \circ 0) \circ (0 \circ 2)$ (or the equation $x = (y \circ x) \circ (x \circ z)$).

The law of a central groupoid. It is self-dual.

Definition 2.19 (Equation 387)**.** Equation 387 is the law $0 \circ 1 == (1 \circ 1) \circ 0$ (or the equation $x \circ y = (y \circ y) \circ x$).

Definition 2.20 (Equation 4512)**.** Equation 4512 is the law $0 \circ (1 \circ 2) == (0 \circ 1) \circ 2$ (or the equation $x \circ (y \circ z) = (x \circ y) \circ z$).

The associative law. It is self-dual.

Definition 2.21 (Equation 4513)**.** Equation 4513 is the law $0 \circ (1 \circ 2) == (0 \circ 1) \circ 3$ (or the equation $x \circ (y \circ z) = (x \circ y) \circ w$).

Definition 2.22 (Equation 4522)**.** Equation 4522 is the law $0 \circ (1 \circ 2) == (0 \circ 3) \circ 4$ (or the equation $x \circ (y \circ z) = (x \circ w) \circ u$).

Dual to Definition 2.24.

Definition 2.23 (Equation 4564)**.** Equation 4564 is the law $0 \circ (1 \circ 2) == (3 \circ 1) \circ 2$ (or the equation $x \circ (y \circ z) = (w \circ y) \circ z$).

Dual to Definition 2.21.

Definition 2.24 (Equation 4579)**.** Equation 4579 is the law $0 \circ (1 \circ 2) == (3 \circ 4) \circ 2$ (or the equation $x \circ (y \circ z) = (w \circ u) \circ z$).

Dual to Definition 2.22.

Definition 2.25 (Equation 4582). Equation 4582 is the law $0 \circ (1 \circ 2) == (3 \circ 4) \circ 5$ (or the equation $x \circ (y \circ z) = (w \circ u) \circ v$).

This law asserts that all triple constants (regardless of bracketing) are constant.

Here is a more complicated law, introduced by Kisielewicz [1]:

Definition 2.26 (Equation 374794). Equation 374794 is the law $0 == (((1 \circ 1) \circ 1) \circ 0) \circ ((1 \circ 1) \circ 2)$ (or the equation $x = (((y \circ y) \circ y) \circ x) \circ ((y \circ y) \circ z))$.

We will be interested in seeing which laws imply which other laws, in the sense that magmas obeying the former law automatically obey the latter. We will also be interested in *anti-implications* showing that one law does *not* imply another, by producing examples of magmas that obey the former law but not the latter.

The singleton or empty magma obeys all equational laws. One can ask whether an equational law admits nontrivial finite or infinite models. The following result was established in [1]:

Theorem 2.27 (Kisielewicz theorem). *Definition 2.26 has an infinite model, but no non-trivial finite model.*

Proof. Suppose for contradiction that we have a non-trivial model of Definition 2.26. Write $y^2 := y \circ y$ and $y^3 := y^2 \circ y$. For any y, z, introduce the functions $f_y : x \mapsto y^3 \circ x$ and $g_{yz} : x \mapsto x \circ (y^2 \circ z)$. Definition 2.26 says that g_{yz} is a left-inverse of f_y, hence by finiteness these are inverses and g_{yz} is independent of z. In particular

$$f(y^3) = g_{yy}(y^3) = g_{yz}(y^3) = f(y^2 \circ z)$$

and hence $y^2 \circ z$ is independent of z. Thus

$$f_y(x) = (y^2 \circ y) \circ x = (y^2 \circ y^2) \circ x$$

is independent of x. As f_y is invertible, this forces the magma to be trivial, a contradiction.

To construct an infinite magma, take the positive integers $\mathbb{Z}^+$ with the operation $x \circ y$ defined as

- 2^x if $y = x$;

- 3^y if $x = 1 \neq y$;

- $\min(j, 1)$ if $x = 3^j$ and $y \neq x$; and

- 1 otherwise.

Then $y^2 = 2^y$, $y^3 = 1$, and $y^2 \circ z$ a power of two for all y, z, and $(1 \circ x) \circ w = x$ for all x whenever w is a power of two, so Definition 2.26 is satisfied. $\square$

Chapter 3

General implications

Definition 3.1 (Implication). A law $w == w'$ is said to *imply* another law $w'' == w'''$, if every magma G that satisfies the former, satisfies the latter:

$$G \vDash w == w' \implies G \vDash w'' == w'''.$$

Two laws are said to be *equivalent* if they imply each other.

Lemma 3.2 (Pre-order). *Implication is a pre-order on the set of laws, and equivalence is an equivalence relation.*

Proof. Trivial. □

Implications between these laws are depicted in Figure 3.1.

Lemma 3.3 (Minimal element). *The law $0 == 0$ is the minimal element in this pre-order.*

Proof. Trivial. □

Lemma 3.4 (Maximal element). *The law $0 == 0$ is the minimal element in this pre-order.*
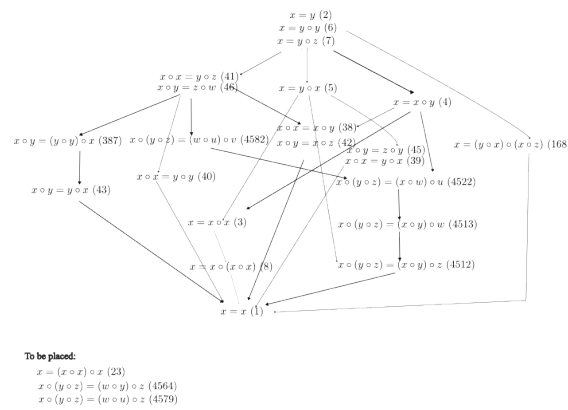
Proof. Trivial. □

Figure 3.1: Implications between the above equations, displayed as a Hasse diagram.

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Every magma G has a *reversal* G^{op}, formed by by replacing the magma operation $\circ$ with its opposite $\circ^{\mathrm{op}} : (x, y) \mapsto y \circ x$. There is a natural isomorphism between these magmas, which induces an involution $w \mapsto w^{\mathrm{op}}$ on words $w \in M_X$. Every law $w == w'$ then has a *dual* $w^{\mathrm{op}} == (w')^{\mathrm{op}}$.

For instance, the dual of the law $0 \circ 1 = 0 \circ 2$ is $1 \circ 0 = 2 \circ 0$, which after relabeling is $0 \circ 1 = 2 \circ 1$. A list of equations and their duals can be found here. Of the 4694 equations under consideration, 84 are self-dual, leaving 2305 pairs of dual equations.

The pre-ordering on laws has a duality symmetry:

Lemma 3.5 (Duality of laws). *If $w == w'$ implies $w'' == w'''$, then $w^{\mathrm{op}} == (w')^{\mathrm{op}}$ implies $w''^{\mathrm{op}} == (w''')^{\mathrm{op}}$.*

Proof. This follows from the fact that a magma G satisfies a law $w == w'$ if and only if G^{op} satisfies $w^{\mathrm{op}} == (w')^{\mathrm{op}}$. $\square$

Some equational laws can be "diagonalized":

Theorem 3.6 (Diagonalization). *An equational law of the form*

$$F(x_1, \dots, x_n) = G(y_1, \dots, y_m), \tag{3.1}$$

where $x_1, \dots, x_n$ and $y_1, \dots, y_m$ are distinct elements of the alphabet, implies the diagonalized law

$$F(x_1, \dots, x_n) = F(x_1', \dots, x_n').$$

where $x_1', \dots, x_n'$ are distinct from $x_1, \dots, x_n$ In particular, if $G(y_1, \dots, y_m)$ can be viewed as a specialization of $F(x_1', \dots, x_n')$, then these two laws are equivalent.

Proof. From two applications of (3.1) one has

$$F(x_1, \dots, x_n) = G(y_1, \dots, y_m)$$

and

$$F(x_1', \dots, x_n') = G(y_1, \dots, y_m)$$

whence the claim. $\square$

Thus for instance, Definition 2.7 is equivalent to Definition 2.2.

Chapter 4

Subgraph implications

Interesting implications between the subgraph equations in Chapter 2. To reduce clutter, trivial or very easy implications will not be displayed here.

Theorem 4.1 (387 implies 43). *Definition 2.19 implies Definition 2.15.*

Proof. (From MathOverflow). By Definition 2.19, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{4.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (2.19) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{4.2}$$

Now, replacing x by $x \circ x$ in (4.1) and then using (4.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{4.3}$$

Also, from two applications of (4.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (4.3) simplifies to $x \circ y = y \circ x$, which is Definition 2.15. $\qquad\square$

Chapter 5

Subgraph counterexamples

Some counterexamples for the anti-implications between the subgraph equations in Chapter 2.

Theorem 5.1 (46 does not imply 4). *Definition 2.17 does not imply Definition 2.4.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := 0$. □

Theorem 5.2 (4 does not imply 4582). *Definition 2.4 does not imply Definition 2.25.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 5.3 (4 does not imply 43). *Definition 2.4 does not imply Definition 2.15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 5.4 (4582 does not imply 42). *Definition 2.25 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = y = 0$ and 2 otherwise. □

Theorem 5.5 (4582 does not imply 43). *Definition 2.25 does not imply Definition 2.15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 3 if $x = 1$ and $y = 2$ and 4 otherwise. □

Theorem 5.6 (42 does not imply 43). *Definition 2.14 does not imply Definition 2.15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. □

Theorem 5.7 (42 does not imply 4512). *Definition 2.14 does not imply Definition 2.20.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + 1$. □

Theorem 5.8 (43 does not imply 42). *Definition 2.15 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. □

Theorem 5.9 (43 does not imply 4512). *Definition 2.15 does not imply Definition 2.20.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x \cdot y + 1$. □

Theorem 5.10 (4513 does not imply 4522). *Definition 2.21 does not imply Definition 2.22.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = 0$ and $y \leq 2$, 2 if $x = 0$ and $y > 2$, and x otherwise. $\square$

Theorem 5.11 (4512 does not imply 4513). *Definition 2.20 does not imply Definition 2.21.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. $\square$

Theorem 5.12 (387 does not imply 42). *Definition 2.19 does not imply Definition 2.14.*

Proof. Use the boolean type Bool with $x \circ y := x || y$. $\square$

Theorem 5.13 (43 does not imply 387). *Definition 2.15 does not imply Definition 2.19.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. $\square$

Theorem 5.14 (387 does not imply 4512). *Definition 2.19 does not imply Definition 2.20.*

Proof. Use the reals $\mathbb{R}$ with $x \circ y := (x + y)/2$. $\square$

Theorem 5.15 (3 does not imply 42). *Definition 2.3 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := y$. $\square$

Theorem 5.16 (3 does not imply 4512). *Definition 2.3 does not imply Definition 2.20.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y$ equal to x when $x = y$ and $x + 1$ otherwise. $\square$

Theorem 5.17 (46 does not imply 3). *Definition 2.17 does not imply Definition 2.3.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := 0$. $\square$

Theorem 5.18 (43 does not imply 3). *Definition 2.15 does not imply Definition 2.3.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. $\square$

Chapter 6

Equivalence with the constant and singleton laws

85 laws have been shown to be equivalent to the constant law (Definition 2.17), and 815 laws have been shown to be equivalent to the singleton law (Definition 2.2).

These are the laws up to 4 operations that follow from diagonalization of 2.2 and of 2.17.

In order to formalize these in Lean, a search was run on the list of equations to discover diagonalizations of these two specific laws: equations of the form $x = R$ where R doesn't include x, and equations of the form $x \circ y = R$ where R doesn't include x or y.

The proofs themselves all look alike, and correspond exactly to the two steps described in the proof of 3.6. The Lean proofs were generated semi-manually, using search-and-replace starting from the output of `grep` that found the diagonalized laws.

In the case of the constant law, equation 2.13 ($x \circ x = y \circ z$) wasn't detected using this method. It was added manually to the file with the existing proof from the sub-graph project.

Chapter 7

Simple rewrites

53,905 implications were automatically generated by simple rewrites.
describe the process of automatically generating these implications here.

Chapter 8

Trivial auto-generated theorems

4.2m implications proven by a transitive reduction of 15k theorems were proven using simple rewrite proof scripts.

include more details of the methodology, and any comparisons with other generated implication data sets.

Bibliography

[1] Andrzej Kisielewicz. Varieties of algebras with no nontrivial finite members. In *Lattices, semigroups, and universal algebra (Lisbon, 1988)*, pages 129–136. Plenum, New York, 1990.