Equational theories

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Chapter 1

Introduction

Definition 1. A Magma is a set G equipped with a binary operation $\circ : G \times G \to G$.

A *law* is an equation involving a finite number of indeterminate variables and the operation $\circ$. A magma G then obeys that law if the equation holds for all possible choices of indeterminate variables in G. For instance, the commutative law

$$x \circ y = y \circ x$$

holds in a magma G if and only if that magma is abelian.

We will be interested in seeing which laws imply which other laws, in the sense that magmas obeying the former law automatically obey the latter. We will also be interested in *anti-implications* showing that one law does *not* imply another, by producing examples of magmas that obey the former law but not the latter.

Chapter 2

Equations

Definition 2 (Equation 1)**.** Equation 1 is the law $x = x$.

Definition 3 (Equation 2)**.** Equation 2 is the law $x = y$.

Definition 4 (Equation 3)**.** Equation 3 is the law $x = x \circ x$.

Definition 5 (Equation 4)**.** Equation 4 is the law $x = x \circ y$.

Definition 6 (Equation 5)**.** Equation 5 is the law $x = y \circ x$.

Definition 7 (Equation 6)**.** Equation 6 is the law $x = y \circ y$.

Definition 8 (Equation 7)**.** Equation 7 is the law $x = y \circ z$.

Definition 9 (Equation 8)**.** Equation 8 is the law $x = x \circ (x \circ x)$.

Definition 10 (Equation 38)**.** Equation 38 is the law $x \circ x = x \circ y$.

Definition 11 (Equation 39)**.** Equation 39 is the law $x \circ x = y \circ x$.

Definition 12 (Equation 40)**.** Equation 40 is the law $x \circ x = y \circ y$.

Definition 13 (Equation 41)**.** Equation 41 is the law $x \circ x = y \circ z$.

Definition 14 (Equation 42)**.** Equation 42 is the law $x \circ y = x \circ z$.

Definition 15 (Equation 43)**.** Equation 43 is the law $x \circ y = y \circ x$.

Definition 16 (Equation 46)**.** Equation 46 is the law $x \circ y = z \circ w$.

Definition 17 (Equation 168)**.** Equation 168 is the law $x = (y \circ x) \circ (x \circ z)$.

Definition 18 (Equation 387)**.** Equation 387 is the law $x \circ y = (y \circ y) \circ x$.

Definition 19 (Equation 4512)**.** Equation 4512 is the law $x \circ (y \circ z) = (x \circ y) \circ z$.

Definition 20 (Equation 4513)**.** Equation 4513 is the law $x \circ (y \circ z) = (x \circ y) \circ w$.

Definition 21 (Equation 4552)**.** Equation 4552 is the law $x \circ (y \circ z) = (x \circ w) \circ u$.

Definition 22 (Equation 4582)**.** Equation 4582 is the law $x \circ (y \circ z) = (w \circ u) \circ v$.

Chapter 3

Implications

To reduce clutter, trivial or very easy implications will not be displayed here.

Theorem 23 (387 implies 43)**.** *Definition 18 implies Definition 15.*

Proof. (From MathOverflow). By Definition 18, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{3.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (18) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{3.2}$$

Now, replacing x by $x \circ x$ in (3.1) and then using (3.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{3.3}$$

Also, from two applications of (3.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3.3) simplifies to $x \circ y = y \circ x$, which is Definition 15. □

Theorem 24 (7 equivalent to 2)**.** *Definition 8 is equivalent to Definition 3.*

Proof. When $x = y * z$, obviously $x = y$ because $x = x * x$ and $y = x * x$. the other way around is trivial. □

Theorem 25 (6 equivalent to 2)**.** *Definition 7 is equivalent to Definition 3.*

Proof. Similar to the previous argument. □

More generally, any equation of the form $x = f(y, z, w, u, v)$ is equivalent to Equation 2 (eventually once we have enough API for Magma relations, we could formalize this general claim rather than establish it on a case by case basis). It is also trivial that Equation 2 implies every other equation.

Chapter 4

Counterexamples

Theorem 26 (46 does not imply 4)**.** *Definition 16 does not imply Definition 5.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := 0$. $\square$

Theorem 27 (4 does not imply 4582)**.** *Definition 5 does not imply Definition 22.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. $\square$

Theorem 28 (4 does not imply 43)**.** *Definition 5 does not imply Definition 15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. $\square$

Theorem 29 (4582 does not imply 42)**.** *Definition 22 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = y = 0$ and 2 otherwise. $\square$

Theorem 30 (4582 does not imply 43)**.** *Definition 22 does not imply Definition 15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 3 if $x = 1$ and $y = 2$ and 4 otherwise. $\square$

Theorem 31 (42 does not imply 43)**.** *Definition 14 does not imply Definition 15.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. $\square$

Theorem 32 (42 does not imply 4512)**.** *Definition 14 does not imply Definition 19.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + 1$. $\square$

Theorem 33 (43 does not imply 42)**.** *Definition 15 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. $\square$

Theorem 34 (43 does not imply 4512)**.** *Definition 15 does not imply Definition 19.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x \cdot y + 1$. $\square$

Theorem 35 (4513 does not imply 4552)**.** *Definition 20 does not imply Definition 21.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = 0$ and $y \leq 2$, 2 if $x = 0$ and $y > 2$, and x otherwise. $\square$

Theorem 36 (4512 does not imply 4513)**.** *Definition 19 does not imply Definition 20.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. □

Theorem 37 (387 does not imply 42)**.** *Definition 18 does not imply Definition 14.*

Proof. Use the boolean type Bool with $x \circ y := x||y$. □

Theorem 38 (43 does not imply 387)**.** *Definition 15 does not imply Definition 18.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. □

Theorem 39 (387 does not imply 4512)**.** *Definition 18 does not imply Definition 19.*

Proof. Use the reals $\mathbb{R}$ with $x \circ y := (x + y)/2$. □

Theorem 40 (3 does not imply 42)**.** *Definition 4 does not imply Definition 14.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := y$. □

Theorem 41 (3 does not imply 4512)**.** *Definition 4 does not imply Definition 19.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y$ equal to x when $x = y$ and $x + 1$ otherwise. □

Theorem 42 (46 does not imply 3)**.** *Definition 16 does not imply Definition 4.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := 0$. □

Theorem 43 (43 does not imply 3)**.** *Definition 15 does not imply Definition 4.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. □