Equational theories

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September 26, 2024

Chapter 1

Introduction

Definition 1.1. A Magma is a set G equipped with a binary operation $\circ : G \times G \to G$.

A *law* is an equation involving a finite number of indeterminate variables and the operation $\circ$. A magma G then obeys that law if the equation holds for all possible choices of indeterminate variables in G. For instance, the commutative law

$$x \circ y = y \circ x$$

holds in a magma G if and only if that magma is abelian.

We will be interested in seeing which laws imply which other laws, in the sense that magmas obeying the former law automatically obey the latter. We will also be interested in *anti-implications* showing that one law does *not* imply another, by producing examples of magmas that obey the former law but not the latter.

Chapter 2

Equations

Definition 2.1 (Equation 1)**.** Equation 1 is the law $x = x$.

Definition 2.2 (Equation 2)**.** Equation 2 is the law $x = y$.

Definition 2.3 (Equation 3)**.** Equation 3 is the law $x = x \circ x$.

Definition 2.4 (Equation 4)**.** Equation 4 is the law $x = x \circ y$.

Definition 2.5 (Equation 5)**.** Equation 5 is the law $x = y \circ x$.

Definition 2.6 (Equation 6)**.** Equation 6 is the law $x = y \circ y$.

Definition 2.7 (Equation 7)**.** Equation 7 is the law $x = y \circ z$.

Definition 2.8 (Equation 8)**.** Equation 8 is the law $x = x \circ (x \circ x)$.

Definition 2.9 (Equation 38)**.** Equation 38 is the law $x \circ x = x \circ y$.

Definition 2.10 (Equation 39)**.** Equation 39 is the law $x \circ x = y \circ x$.

Definition 2.11 (Equation 40)**.** Equation 40 is the law $x \circ x = y \circ y$.

Definition 2.12 (Equation 41)**.** Equation 41 is the law $x \circ x = y \circ z$.

Definition 2.13 (Equation 42)**.** Equation 42 is the law $x \circ y = x \circ z$.

Definition 2.14 (Equation 43)**.** Equation 43 is the law $x \circ y = y \circ x$.

Definition 2.15 (Equation 46)**.** Equation 46 is the law $x \circ y = z \circ w$.

Definition 2.16 (Equation 168)**.** Equation 168 is the law $x = (y \circ x) \circ (x \circ z)$.

Definition 2.17 (Equation 387)**.** Equation 387 is the law $x \circ y = (y \circ y) \circ x$.

Definition 2.18 (Equation 4512)**.** Equation 4512 is the law $x \circ (y \circ z) = (x \circ y) \circ z$.

Definition 2.19 (Equation 4513)**.** Equation 4513 is the law $x \circ (y \circ z) = (x \circ y) \circ w$.

Definition 2.20 (Equation 4522)**.** Equation 4522 is the law $x \circ (y \circ z) = (x \circ w) \circ u$.

Definition 2.21 (Equation 4582)**.** Equation 4582 is the law $x \circ (y \circ z) = (w \circ u) \circ v$.

Chapter 3

Implications

To reduce clutter, trivial or very easy implications will not be displayed here.

Theorem 3.1 (387 implies 43)**.** *Definition 2.17 implies Definition 2.14.*

Proof. (From MathOverflow). By Definition 2.17, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{3.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (2.17) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{3.2}$$

Now, replacing x by $x \circ x$ in (3.1) and then using (3.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{3.3}$$

Also, from two applications of (3.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (3.3) simplifies to $x \circ y = y \circ x$, which is Definition 2.14. □

Theorem 3.2 (7 equivalent to 2)**.** *Definition 2.7 is equivalent to Definition 2.2.*

Proof. When $x = y * z$, obviously $x = y$ because $x = x * x$ and $y = x * x$. the other way around is trivial. □

Theorem 3.3 (6 equivalent to 2)**.** *Definition 2.6 is equivalent to Definition 2.2.*

Proof. Similar to the previous argument. □

More generally, any equation of the form $x = f(y, z, w, u, v)$ is equivalent to Equation 2 (eventually once we have enough API for Magma relations, we could formalize this general claim rather than establish it on a case by case basis). It is also trivial that Equation 2 implies every other equation.

Chapter 4

Counterexamples

Theorem 4.1 (46 does not imply 4)**.** *Definition 2.15 does not imply Definition 2.4.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := 0$. $\square$

Theorem 4.2 (4 does not imply 4582)**.** *Definition 2.4 does not imply Definition 2.21.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. $\square$

Theorem 4.3 (4 does not imply 43)**.** *Definition 2.4 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. $\square$

Theorem 4.4 (4582 does not imply 42)**.** *Definition 2.21 does not imply Definition 2.13.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = y = 0$ and 2 otherwise. $\square$

Theorem 4.5 (4582 does not imply 43)**.** *Definition 2.21 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 3 if $x = 1$ and $y = 2$ and 4 otherwise. $\square$

Theorem 4.6 (42 does not imply 43)**.** *Definition 2.13 does not imply Definition 2.14.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x$. $\square$

Theorem 4.7 (42 does not imply 4512)**.** *Definition 2.13 does not imply Definition 2.18.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + 1$. $\square$

Theorem 4.8 (43 does not imply 42)**.** *Definition 2.14 does not imply Definition 2.13.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. $\square$

Theorem 4.9 (43 does not imply 4512)**.** *Definition 2.14 does not imply Definition 2.18.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x \cdot y + 1$. $\square$

Theorem 4.10 (4513 does not imply 4522)**.** *Definition 2.19 does not imply Definition 2.20.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y$ equal to 1 if $x = 0$ and $y \leq 2$, 2 if $x = 0$ and $y > 2$, and x otherwise. $\square$

Theorem 4.11 (4512 does not imply 4513)**.** *Definition 2.18 does not imply Definition 2.19.*

Proof. Use the natural numbers $\mathbb{N}$ with operation $x \circ y := x + y$. $\square$

Theorem 4.12 (387 does not imply 42)**.** *Definition 2.17 does not imply Definition 2.13.*

Proof. Use the boolean type Bool with $x \circ y := x||y$. $\square$

Theorem 4.13 (43 does not imply 387)**.** *Definition 2.14 does not imply Definition 2.17.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. $\square$

Theorem 4.14 (387 does not imply 4512)**.** *Definition 2.17 does not imply Definition 2.18.*

Proof. Use the reals $\mathbb{R}$ with $x \circ y := (x + y)/2$. $\square$

Theorem 4.15 (3 does not imply 42)**.** *Definition 2.3 does not imply Definition 2.13.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := y$. $\square$

Theorem 4.16 (3 does not imply 4512)**.** *Definition 2.3 does not imply Definition 2.18.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y$ equal to x when $x = y$ and $x + 1$ otherwise. $\square$

Theorem 4.17 (46 does not imply 3)**.** *Definition 2.15 does not imply Definition 2.3.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := 0$. $\square$

Theorem 4.18 (43 does not imply 3)**.** *Definition 2.14 does not imply Definition 2.3.*

Proof. Use the natural numbers $\mathbb{N}$ with $x \circ y := x + y$. $\square$