Equational theories

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Chapter 1

Basic theory of magmas

Definition 1.1 (Magma)**.** A *magma* is a set G equipped with a binary operation $\circ : G \times G \to G$. A *homomorphism* $\varphi : G \to H$ between two magmas is a map such that $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in G$. An *isomorphism* is an invertible homomorphism.

Groups, semi-groups, and monoids are familiar examples of magmas. However, in general we do not expect magmas to have any associative properties. In some literature, magmas are also known as groupoids, although this term is also used for a slightly different object (a category with inverses).

A magma is called *empty* if it has cardinality zero, *singleton* if it has cardinality one, and *non-trivial* otherwise.

The number of magma structures on a set G of cardinality n is of course n^{n^2}, which is [1]

$$1, 1, 16, 19683, 4294967296, 298023223876953125, \ldots$$

(OEIS A002489). Up to isomorphism, the number of finite magmas of cardinality n up to isomorphism is the slightly slower growing sequence

$$1, 1, 10, 3330, 178981952, 2483527537094825, 14325590003318891522275680, \ldots$$

(OEIS A001329).

Definition 1.2 (Free Magma)**.** The *free magma* M_X generated by a set X (which we call an *alphabet*) is the set of all finite formal expressions built from elements of X and the operation $\circ$. An element of M_X will be called a *word* with alphabet X. The *order* of a word is the number of $\circ$ symbols needed to generate the word. Thus for instance X is precisely the set of words of order 0 in M_X.

For sake of concreteness, we will take the alphabet X to default to the natural numbers $\mathbb{N}$ if not otherwise specified.

For instance, if $X = \{0, 1\}$, then M_X would consist of the following words:

- 0, 1 (the words of order 0);

- $0 \circ 0$, $0 \circ 1$, $1 \circ 0$, $1 \circ 1$ (the words of order 1);

- $0 \circ (0 \circ 0)$, $0 \circ (0 \circ 1)$, $0 \circ (1 \circ 0)$, $0 \circ (1 \circ 1)$, $1 \circ (0 \circ 0)$, $1 \circ (0 \circ 1)$, $1 \circ (1 \circ 0)$, $1 \circ (1 \circ 1)$, $(0 \circ 0) \circ 0$, $(0 \circ 0) \circ 1$, $(0 \circ 1) \circ 0$, $(0 \circ 1) \circ 1$, $(1 \circ 0) \circ 0$, $(1 \circ 0) \circ 1$, $(1 \circ 1) \circ 0$, $(1 \circ 1) \circ 1$ (the words of order 2);

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[1] All sequences start from $n = 0$ unless otherwise specified.

- etc.

Lemma 1.3. *For a finite alphabet X, the number of words of order n is $C_n|X|^{n+1}$, where C_n is the nth Catalan number and X is the cardinality of X.*

Proof. Follows from standard properties of Catalan numbers. ∎

The first few Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, ...$$

(OEIS A000108).

Definition 1.4 (Induced homomorphism)**.** Given a function $f : X \to G$ from an alphabet X to a magma G, the *induced homomorphism* $\varphi_f : M_X \to G$ is the unique extension of f to a magma homomorphism. Similarly, if $\pi \colon X \to Y$ is a function, we write $\pi_* \colon M_X \to M_Y$ for the unique extension of π to a magma homomorphism.

For instance, if $f : \{0, 1\} \to G$ maps $0, 1$ to x, y respectively, then

$$\varphi_f(0 \circ 1) = x \circ y$$

$$\varphi_f(1 \circ (0 \circ 1)) = y \circ (x \circ y)$$

and so forth. If $\pi \colon \mathbb{N} \to \mathbb{N}$ is the map $\pi(n) := n + 1$, then

$$\pi_*(0 \circ 1) = 1 \circ 2$$

$$\pi_*(1 \circ (0 \circ 1)) = 2 \circ (1 \circ 2)$$

and so forth.

Definition 1.5 (Law)**.** Let X be a set. A *law* with alphabet X is a formal expression of the form $w \simeq w'$, where $w, w' \in M_X$ are words with alphabet X (thus one can identify laws with alphabet X with elements of $M_X \times M_X$). A magma G *satisfies* the law $w \simeq w'$ if we have $\varphi_f(w) = \varphi_f(w')$ for all $f : X \to G$, in which case we write $G \vDash w \simeq w'$.

Thus, for instance, the commutative law

$$0 \circ 1 \simeq 1 \circ 0 \tag{1.1}$$

is satisfied by a magma G if and only if

$$x \circ y = y \circ x \tag{1.2}$$

for all $x, y \in G$. We refer to (1.2) as the *equation* associated to the law (1.1). One can think of equations as the "semantic" interpretation of a "syntactic" law. However, we shall often abuse notation and a law with its associated equation thus we shall (somewhat carelessly) also refer to (1.2) as "the commutative law" (rather than "the commutative equation").

Definition 1.6 (Models)**.** Given an arbitrary set Γ of laws, a magma G is a *model* of Γ with the (overloaded) notation $G \vDash \Gamma$ if $G \vDash w \simeq w'$ for every $w \simeq w'$ in Γ; we also say that G *obeys* Γ. Given a law E, we write $\Gamma \vDash E$ if every magma G that models Γ, also models E.

Definition 1.7 (Derivation)**.** Given a set Γ of laws and a law $w \simeq w'$ over a fixed alphabet X, we say that Γ *derives* $w \simeq w'$, and write $\Gamma \vdash w \simeq w'$, if the law can be obtained using a finite number of applications of the following rules:

1. if $w \simeq w' \in \Gamma$, then $\Gamma \vdash w \simeq w'$.

2. $\Gamma \vdash w \simeq w$ for any word w.

3. if $\Gamma \vdash w \simeq w'$ then $\Gamma \vdash w' \simeq w$.

4. if $\Gamma \vdash w \simeq w'$ and $\Gamma \vdash w' \simeq w''$ then $\Gamma \vdash w \simeq w''$.

5. if $\Gamma \vdash w \simeq w'$ then $\Gamma \vdash w\sigma \simeq w'\sigma$, where σ is an arbitrary map from X to words in M_X and $w\sigma$ replaces each occurrence of an element of X with it's image by σ in w.

6. if $\Gamma \vdash w_1 \simeq w_2$ and $\Gamma \vdash w_3 \simeq w_4$ then $\Gamma \vdash w_1 \circ w_3 \simeq w2 \circ w_4$

This definition is useful because of the following theorem:

Theorem 1.8 (Birkhoff's completeness theorem)**.** *For any set of laws Γ and words w, w' over a fixed alphabet*

$$\Gamma \vdash w \simeq w' \text{ iff } \Gamma \vDash w \simeq w'.$$

Proof. (Sketch) The 'only if' component is soundness, and follows from verifying that the rules of inference in Definition 1.7 holds for $\vDash$. The 'if' part is completeness, and is proven by constructing the magma of words, quotiented out by the relation $\Gamma \vdash w \simeq w'$, which is easily seen to be an equivalence relation respecting the magma operation. $\square$

Corollary 1.9 (Compactness theorem)**.** *Let Γ be a collection of laws, and let E be a law. Then $\Gamma \vDash E$ if and only if there exists a finite subset Γ' of Γ such that $\Gamma' \vDash E$.*

Proof. The claim is obvious for $\vdash$, and the claim then follows from Theorem 1.8. $\square$

Lemma 1.10 (Pushforward)**.** *Let $w \simeq w'$ be a law with some alphabet X, G be a magma, and $\pi : X \to Y$ be a function. If $G \vDash w \simeq w'$, then $G \vDash \pi_*(w) \simeq \pi_*(w')$. In particular, if π is a bijection, the statements If $G \vDash w \simeq w'$, then $G \vDash \pi_*(w) \simeq \pi_*(w')$ are equivalent.*

Proof. Trivial. $\square$

If π is a bijection, we will call $\pi_*(w) \simeq \pi_*(w')$ a *relabeling* of the law $w \simeq w'$. Thus for instance

$$5 \circ 7 \simeq 7 \circ 5$$

is a relabeling of the commutative law (1.1). By the above lemma, relabeling does not affect whether a given magna satisfies a given law.

Lemma 1.11 (Equivalence)**.** *Let G be a magma and X be an alphabet. Then the relation $G \vDash w \simeq w'$ is an equivalence relation on M_X.*

Proof. Trivial. $\square$

Define the total order of a law $w \simeq w'$ to be the sum of the orders of w and w'.

Lemma 1.12 (Counting laws up to relabeling)**.** *Up to relabeling, the number of laws $w \simeq w'$ of total order n is $C_{n+1}B_{n+2}$.*

Proof. Follows from the properties of Catalan and Bell numbers. $\square$

The first few Bell numbers are

$$1, 1, 2, 5, 15, 52, 203, ...$$

(OEIS A000110).

The sequence in Lemma 1.12 is

$$2, 10, 75, 728, 8526, 115764, ...$$

(OEIS A289679).

Now we would also like to count laws up to relabeling and symmetry.

Lemma 1.13 (Counting laws up to relabeling and symmetry). *Up to relabeling and symmetry, the number of laws $w \simeq w'$ of total order n is*

$$C_{n+1}B_{n+2}/2$$

when n is odd, and

$$(C_{n+1}B_{n+2} + C_{n/2}(2D_{n+2} - B_{n+2}))/2$$

when n is even, where D_n is the number of partitions of $[n]$ up to reflection.

Proof. Elementary counting. $\qquad\square$

The sequence D_n is

$$1, 1, 2, 4, 11, 32, 117, ...$$

(OEIS A103293), and the sequence in Lemma 1.13 is

$$2, 5, 41, 364, 4294, 57882, 888440, ...$$

(OEIS A376620).

We can also identify all laws of the form $w \simeq w$ with the trivial law $0 \simeq 0$. The number of such laws of total order n is zero if n is odd, and $C_{n/2}B_{n/2+1}$ if n is even. We conclude:

Lemma 1.14 (Counting laws up to relabeling, symmetry, and triviality). *Up to relabeling, symmetry, and triviality, the number of laws of total order n is*

$$C_{n+1}B_{n+2}/2$$

if n is odd, 2 if $n = 0$, and

$$(C_{n+1}B_{n+2} + C_{n/2}(2D_{n+2} - B_{n+2}))/2 - C_{n/2}B_{n/2+1}$$

if $n \geq 2$ is even.

Proof. Routine counting. $\qquad\square$

This sequence is

$$2, 5, 39, 364, 4284, 57882, 888365, ...$$

(OEIS A376640).

In particular, up to relabeling, symmetry, and triviality, there are exactly 4694 laws of total order at most 4. A list can be found here. A script for generating them may be found here. The list is sorted first by the total number of operations, then by the number of operations on the LHS. Within each such class we define an order on expressions by lexical order on variables (ordered x, y, z, w, u, v). The equation are arranged to be minimal with respect to this sorting order, thus the LHS will be shorter than or equal than the RHS, and earlier in the lexical order if the LHS and RHS are of equal length.

Chapter 2

Selected laws

In this project we study the 4694 laws (up to symmetry and relabeling) of total order at most 4. Selected laws of interest are listed below, as well as in this file.

Definition 2.1 (Equation 1)**.** Equation 1 is the law $0 \simeq 0$ (or the equation $x = x$).

This is the trivial law, satisfied by all magmas. It is self-dual.

Definition 2.2 (Equation 2)**.** Equation 2 is the law $0 \simeq 1$ (or the equation $x = y$).

This is the singleton law, satisfied only by the empty and singleton magmas. It is self-dual.

Definition 2.3 (Equation 3)**.** Equation 3 is the law $0 \simeq 0 \circ 0$ (or the equation $x = x \circ x$).

This is the idempotence law. It is self-dual.

Definition 2.4 (Equation 4)**.** Equation 4 is the law $0 \simeq 0 \circ 1$ (or the equation $x = x \circ y$).

This is the left absorption law.

Definition 2.5 (Equation 5)**.** Equation 5 is the law $0 \simeq 1 \circ 0$ (or the equation $x = y \circ x$).

This is the right absorption law (the dual of Definition 2.4).

Definition 2.6 (Equation 6)**.** Equation 6 is the law $0 \simeq 1 \circ 1$ (or the equation $x = y \circ y$).

This law is equivalent to the singleton law.

Definition 2.7 (Equation 7)**.** Equation 7 is the law $0 \simeq 1 \circ 2$ (or the equation $x = y \circ z$).

This law is equivalent to the singleton law.

Definition 2.8 (Equation 8)**.** Equation 8 is the law $0 \simeq 0 \circ (0 \circ 0)$ (or the equation $x = x \circ (x \circ x)$).

Definition 2.9 (Equation 14)**.** Equation 14 is the law $0 \simeq 1 \circ (0 \circ 1)$ (or the equation $x = y \circ (x \circ y)$).

Appears in Problem A1 from Putnam 2001.

Definition 2.10 (Equation 23)**.** Equation 23 is the law $0 \simeq (0 \circ 0) \circ 0$ (or the equation $x = (x \circ x) \circ x$).

This is the dual of Definition 2.8.

Definition 2.11 (Equation 29)**.** Equation 29 is the law $0 \simeq (1 \circ 0) \circ 1$ (or the equation $x = (y \circ x) \circ y$).

Appears in Problem A1 from Putnam 2001. Dual to Definition 2.9.

Definition 2.12 (Equation 38)**.** Equation 38 is the law $0 \circ 0 \simeq 0 \circ 1$ (or the equation $x \circ x = x \circ y$).

This law asserts that the magma operation is independent of the second argument.

Definition 2.13 (Equation 39)**.** Equation 39 is the law $0 \circ 0 \simeq 1 \circ 0$ (or the equation $x \circ x = y \circ x$).

This law asserts that the magma operation is independent of the first argument (the dual of Definition 2.12).

Definition 2.14 (Equation 40)**.** Equation 40 is the law $0 \circ 0 \simeq 1 \circ 1$ (or the equation $x \circ x = y \circ y$).

This law asserts that all squares are constant. It is self-dual.

Definition 2.15 (Equation 41)**.** Equation 41 is the law $0 \circ 0 \simeq 1 \circ 2$ (or the equation $x \circ x = y \circ z$).

This law is equivalent to the constant law, Definition 2.19.

Definition 2.16 (Equation 42)**.** Equation 42 is the law $0 \circ 1 \simeq 0 \circ 2$ (or the equation $x \circ y = x \circ z$).

Equivalent to Definition 2.12.

Definition 2.17 (Equation 43)**.** Equation 43 is the law $0 \circ 1 \simeq 1 \circ 0$ (or the equation $x \circ y = y \circ x$).

The commutative law. It is self-dual.

Definition 2.18 (Equation 45)**.** Equation 45 is the law $0 \circ 1 \simeq 2 \circ 1$ (or the equation $x \circ y = z \circ y$).

This is the dual of Definition 2.16.

Definition 2.19 (Equation 46)**.** Equation 46 is the law $0 \circ 1 \simeq 2 \circ 3$ (or the equation $x \circ y = z \circ w$).

The constant law: all products are constant. It is self-dual.

Definition 2.20 (Equation 168)**.** Equation 168 is the law $0 \simeq (1 \circ 0) \circ (0 \circ 2)$ (or the equation $x = (y \circ x) \circ (x \circ z)$).

The law of a central groupoid. It is self-dual.

Definition 2.21 (Equation 381)**.** Equation 381 is the law $0 \circ 1 \simeq (0 \circ 2) \circ 1$ (or the equation $x \circ y = (x \circ z) \circ y$).

Appears in Putnam 1978, Problem A4, part (b).

Definition 2.22 (Equation 387)**.** Equation 387 is the law $0 \circ 1 \simeq (1 \circ 1) \circ 0$ (or the equation $x \circ y = (y \circ y) \circ x$).

Introduced in MathOverflow.

Definition 2.23 (Equation 1689)**.** Equation 1689 is the law $0 \simeq (1 \circ 0) \circ ((0 \circ 2) \circ 2)$ (or the equation $x = (y \circ x) \circ ((x \circ z) \circ z)$).

Mentioned in [2].

Definition 2.24 (Equation 3722)**.** Equation 3722 is the law $0 \circ 1 \simeq (0 \circ 1) \circ (0 \circ 1)$ (or the equation $x \circ y = (x \circ y) \circ (x \circ y)$).

Appears in Putnam 1978, Problem A4, part (a). It is self-dual.

Definition 2.25 (Equation 3744)**.** Equation 3744 is the law $0 \circ 1 \simeq (0 \circ 2) \circ (3 \circ 1)$ (or the equation $x \circ y = (x \circ z) \circ (w \circ y)$).

This law is called a "bypass operation" in Putnam 1978, Problem A4. It is self-dual.

Definition 2.26 (Equation 4512)**.** Equation 4512 is the law $0 \circ (1 \circ 2) \simeq (0 \circ 1) \circ 2$ (or the equation $x \circ (y \circ z) = (x \circ y) \circ z$).

The associative law. It is self-dual.

Definition 2.27 (Equation 4513)**.** Equation 4513 is the law $0 \circ (1 \circ 2) \simeq (0 \circ 1) \circ 3$ (or the equation $x \circ (y \circ z) = (x \circ y) \circ w$).

Definition 2.28 (Equation 4522)**.** Equation 4522 is the law $0 \circ (1 \circ 2) \simeq (0 \circ 3) \circ 4$ (or the equation $x \circ (y \circ z) = (x \circ w) \circ u$).

Dual to Definition 2.30.

Definition 2.29 (Equation 4564)**.** Equation 4564 is the law $0 \circ (1 \circ 2) \simeq (3 \circ 1) \circ 2$ (or the equation $x \circ (y \circ z) = (w \circ y) \circ z$).

Dual to Definition 2.27.

Definition 2.30 (Equation 4579)**.** Equation 4579 is the law $0 \circ (1 \circ 2) \simeq (3 \circ 4) \circ 2$ (or the equation $x \circ (y \circ z) = (w \circ u) \circ z$).

Dual to Definition 2.28.

Definition 2.31 (Equation 4582)**.** Equation 4582 is the law $0 \circ (1 \circ 2) \simeq (3 \circ 4) \circ 5$ (or the equation $x \circ (y \circ z) = (w \circ u) \circ v$).

This law asserts that all triple constants (regardless of bracketing) are constant.
We also note some selected laws of order more than 5.

Definition 2.32 (Equation 5105)**.** Equation 5105 is the law $0 \simeq 1 \circ (1 \circ (1 \circ (0 \circ (2 \circ 1))))$ (or the equation $x = y \circ (y \circ (y \circ (x \circ (z \circ y))))$).

This law of order 5 was mentioned in [2].

Definition 2.33 (Equation 28393)**.** Equation 28393 is the law $0 \simeq (((0 \circ 0) \circ 0) \circ 1) \circ (0 \circ 2)$ (or the equation $x = (((x \circ x) \circ x) \circ y) \circ (x \circ z)$).

This law of order 5 was introduced by Kisielewicz [3].

Definition 2.34 (Equation 374794)**.** Equation 374794 is the law $0 \simeq (((1 \circ 1) \circ 1) \circ 0) \circ ((1 \circ 1) \circ 2)$ (or the equation $x = (((y \circ y) \circ y) \circ x) \circ ((y \circ y) \circ z)$).

This law of order 6 was introduced by Kisielewicz [3].
The singleton or empty magma obeys all equational laws. One can ask whether an equational law admits nontrivial finite or infinite models. An *Austin law* is a law which admits infinite models, but no nontrivial finite models. Austin [1] established the first such law, namely the order 9 law

$$(((1 \circ 1) \circ 1) \circ 0) \circ (((1 \circ 1) \circ ((1 \circ 1) \circ 1)) \circ 2) \simeq 0.$$

A shorter Austin law of order 6 was established in [3]:

Theorem 2.35 (Kisielewicz's first Austin law)**.** *Definition 2.34 is an Austin law.*

Proof. Suppose for contradiction that we have a non-trivial model of Definition 2.34. Write $y^2 :=$ $y \circ y$ and $y^3 := y^2 \circ y$. For any y, z, introduce the functions $f_y : x \mapsto y^3 \circ x$ and $g_{yz} : x \mapsto x \circ (y^2 \circ z)$. Definition 2.34 says that g_{yz} is a left-inverse of f_y, hence by finiteness these are inverses and g_{yz} is independent of z. In particular

$$f(y^3) = g_{yy}(y^3) = g_{yz}(y^3) = f(y^2 \circ z)$$

and hence $y^2 \circ z$ is independent of z. Thus

$$f_y(x) = (y^2 \circ y) \circ x = (y^2 \circ y^2) \circ x$$

is independent of x. As f_y is invertible, this forces the magma to be trivial, a contradiction.

To construct an infinite magma, take the positive integers $\mathbb{Z}^+$ with the operation $x \circ y$ defined as

- 2^x if $y = x$;

- 3^y if $x = 1 \neq y$;

- $\min(j, 1)$ if $x = 3^j$ and $y \neq x$; and

- 1 otherwise.

Then $y^2 = 2^y$, $y^3 = 1$, and $y^2 \circ z$ a power of two for all y, z, and $(1 \circ x) \circ w = x$ for all x whenever w is a power of two, so Definition 2.34 is satisfied. $\qquad\square$

An even shorter law (order 5) was obtained by the same author in a followup paper [2]:

Theorem 2.36 (Kisielewicz theoremII)**.** *Definition 2.33 is an Austin law.*

Proof. Using the y^2 and y^3 notation as before, the law reads

$$x = (y^3 \circ x) \circ (y \circ z). \tag{2.1}$$

In particular, for any y, the map $T_y : x \mapsto y^3 \circ x$ is injective, hence bijective in a finite model G. In particular we can find a function $f : G \to G$ such that $T_y f(y) = y^3$ for all y Applying (2.1) with $x = f(y)$, we conclude

$$T_y(y \circ z) = y^3 \circ (y \circ z) = f(y)$$

and thus $y \circ z$ is independent of z by injectivity of T_y. Thus, the left-hand side of (2.1) does not depend on x, and so the model is trivial. This shows there are no non-trivial finite models.

To establish an infinite model, use $\mathbb{N}$ with $x \circ y$ defined by requiring

$$y \circ y = 2^y; \quad 2^y \circ y = 3^y$$

and

$$3^y \circ x = 3^y 5^x$$

for $x \neq 3^y$, and

$$(3^y 5^x) \circ z = x$$

for $z \neq 3^y 5^x$. Finally set

$$2^{3^y} \circ z = 3^y$$

for $z \neq 3^y, 2^{3^y}$. All other assignments of $\circ$ may be made arbitrarily. It is then a routine matter to establish (2.1). $\qquad\square$

In that paper a computer search was also used to show that no law of order four or less is an Austin law.

An open question is whether Definition 2.32 is an Austin law. We have the following partial result from [2]:

Theorem 2.37 (Equation 5105 has no non-trivial finite models)**.** *Definition 2.32 has no non-trivial finite models.*

Proof. From Definition 2.32 we see that the map $w \mapsto y \circ w$ is onto; in particular, we may substitute $u := z \circ y$ and conclude that

$$x = y \circ (y \circ (y \circ (x \circ u)))$$

for all x, y, u. In particular $x \mapsto x \circ u$ is injective. If the model is finite, then both maps must be invertible, hence the magma is cancellative. Since

$$y \circ (y \circ (y \circ (x \circ (z \circ y)))) = x = y \circ (y \circ (y \circ (x \circ (z' \circ y))))$$

we conclude on inverting many left and right magma operations that $z = z'$, giving triviality. $\square$

Chapter 3

General implications

We will be interested in seeing which laws imply which other laws, in the sense that magmas obeying the former law automatically obey the latter. We will also be interested in *anti-implications* showing that one law does *not* imply another, by producing examples of magmas that obey the former law but not the latter. Here is a formal definition.

Definition 3.1 (Implication)**.** A law E is said to *imply* another law E' if $\{E\} \vDash E'$, or equivalently:

$$G \vDash w \simeq w' \implies G \vDash w'' \simeq w''' \text{ for all magmas } G$$

Two laws are said to be *equivalent* if they imply each other.

Lemma 3.2 (Pre-order)**.** *If we define $E \leq E'$ if E implies E', then this is a pre-order on the set of laws, and equivalence is an equivalence relation.*

Note that we view the stronger law as less than or equal to the weaker law. This is because the class of magmas that obey the stronger law is a subset of the class of magmas that obey the weaker law. It is also consistent with the conventions of Lean's Mathlib.

Proof. Trivial. $\qquad\square$

Implications between the laws from Chapter 2 are depicted in Figure 3.1.

Lemma 3.3 (Maximal element)**.** *The law $0 \simeq 0$ is the maximal element in this pre-order.*

Proof. Trivial. $\qquad\square$

Lemma 3.4 (Minimal element)**.** *The law $0 \simeq 1$ is the minimal element in this pre-order.*

Proof. Trivial. $\qquad\square$

Every magma G has a *reversal* G^{op}, formed by by replacing the magma operation $\circ$ with its opposite $\circ^{\mathrm{op}} : (x, y) \mapsto y \circ x$. There is a natural isomorphism between these magmas, which induces an involution $w \mapsto w^{\mathrm{op}}$ on words $w \in M_X$. Every law $w \simeq w'$ then has a *dual* $w^{\mathrm{op}} \simeq (w')^{\mathrm{op}}$.

For instance, the dual of the law $0 \circ 1 = 0 \circ 2$ is $1 \circ 0 = 2 \circ 0$, which after relabeling is $0 \circ 1 = 2 \circ 1$. A list of equations and their duals can be found here. Of the 4694 equations under consideration, 84 are self-dual, leaving 2305 pairs of dual equations.
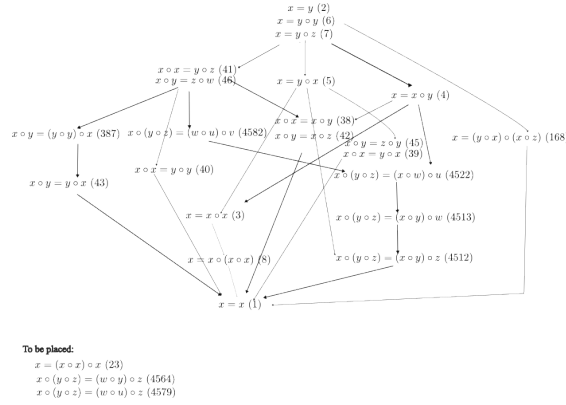
The pre-ordering on laws has a duality symmetry:

Figure 3.1: Implications between the above equations, displayed as a Hasse diagram.

Lemma 3.5 (Duality of laws). *If $w \simeq w'$ implies $w'' \simeq w'''$, then $w^{\mathrm{op}} \simeq (w')^{\mathrm{op}}$ implies $w''^{\mathrm{op}} \simeq (w''')^{\mathrm{op}}$.*

Proof. This follows from the fact that a magma G satisfies a law $w \simeq w'$ if and only if G^{op} satisfies $w^{\mathrm{op}} \simeq (w')^{\mathrm{op}}$. $\qquad\square$

Some equational laws can be "diagonalized":

Theorem 3.6 (Diagonalization). *An equational law of the form*

$$F(x_1, \dots, x_n) = G(y_1, \dots, y_m), \tag{3.1}$$

where $x_1, \dots, x_n$ and $y_1, \dots, y_m$ are distinct elements of the alphabet, implies the diagonalized law

$$F(x_1, \dots, x_n) = F(x_1', \dots, x_n').$$

where $x_1', \dots, x_n'$ are distinct from $x_1, \dots, x_n$ In particular, if $G(y_1, \dots, y_m)$ can be viewed as a specialization of $F(x_1', \dots, x_n')$, then these two laws are equivalent.

Proof. From two applications of (3.1) one has

$$F(x_1, \dots, x_n) = G(y_1, \dots, y_m)$$

and

$$F(x_1', \dots, x_n') = G(y_1, \dots, y_m)$$

whence the claim. $\qquad\square$

Thus for instance, Definition 2.7 is equivalent to Definition 2.2.

Theorem 3.7 (Laws implied by the constant law). *If w, w' each have order at least one, then the law $w \simeq w'$ is implied by the constant law (Definition 2.19). If exactly one of w, w' has order zero, and the law $w \simeq w'$ is not implied by the constant law.*

Proof. Routine. $\qquad\square$

Theorem 3.8 (Criterion for implication). *If $w \simeq w'$ is such that every variable appears the same number of times in both w and w', and $w \simeq w'$ implies another law $w'' \simeq w'''$, then every variable appears the same number of times in both w'' and w'''.*

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Proof. Consider the magma $\mathbb{R}$ with the addition law $+$. By hypothesis, this magma obeys $w \simeq w'$, and hence $w'' \simeq w'''$, giving the claim by comparing coefficients of the linear forms associated to w'' and w''' in this magma. $\qquad\square$

Chapter 4

Implications between selected laws

We collect here some notable implications between the the selected laws in Chapter 2. By Theorem 1.8, every implication can basically be established by a finite number of rewrites. In most cases, the sequence of rewrites is quite straightforward, and the implication is very easy, but we record some less obvious examples.

Theorem 4.1 (387 implies 43)**.** *Definition 2.22 implies Definition 2.17.*

Proof. (From MathOverflow). By Definition 2.22, one has the law

$$(x \circ x) \circ y = y \circ x. \tag{4.1}$$

Specializing to $y = x \circ x$, we conclude

$$(x \circ x) \circ (x \circ x) = (x \circ x) \circ x$$

and hence by another application of (2.22) we see that $x \circ x$ is idempotent:

$$(x \circ x) \circ (x \circ x) = x \circ x. \tag{4.2}$$

Now, replacing x by $x \circ x$ in (4.1) and then using (4.2) we see that

$$(x \circ x) \circ y = y \circ (x \circ x)$$

so in particular $x \circ x$ commutes with $y \circ y$:

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ (x \circ x). \tag{4.3}$$

Also, from two applications of (4.1) one has

$$(x \circ x) \circ (y \circ y) = (y \circ y) \circ x = x \circ y.$$

Thus (4.3) simplifies to $x \circ y = y \circ x$, which is Definition 2.17. $\qquad\square$

Theorem 4.2 (29 equivalent to 14)**.** *Definition 2.11 is equivalent to Definition 2.9.*

This result was posed as Problem A1 from Putnam 2001.

Proof. By Lemma 3.5 it suffices to show that Definition 2.11 implies Definition 2.9. From Definition 2.11 one has

$$x = ((x \circ y) \circ x) \circ (x \circ y)$$

and also

$$y = (x \circ y) \circ x$$

giving $x = y \circ (x \circ y)$, which is Definition 2.9. $\qquad\square$

Theorem 4.3 (14 implies 29)**.** *Definition 2.9 implies Definition 2.11.*

This result was posed as Problem A1 from Putnam 2001.

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following result was problem A4 on Putnam 1978.

Theorem 4.4 (3744 implies 3722, 381)**.** *Definition 2.25 implies Definition 2.24 and Definition 2.21.*

Proof. By hypothesis, one has

$$x \circ y = (x \circ z) \circ (w \circ y)$$

for all x, y, z, w. Various specializations of this give

$$x \circ y = (x \circ z) \circ (y \circ y) \tag{4.4}$$
$$x \circ z = (x \circ z) \circ (x \circ z) \tag{4.5}$$
$$(x \circ z) \circ y = ((x \circ z) \circ (x \circ z)) \circ (y \circ y). \tag{4.6}$$

The equation (4.5) gives Definition 2.24, while (4.4), (4.5), (4.6) gives

$$x \circ y = (x \circ z) \circ y$$

which is Definition 2.21. $\qquad\square$

Theorem 4.5 (1689 is equivalent to 2)**.** *Definition 2.23 is equivalent to Definition 2.2.*

Proof. The implication of Definition 2.23 from Definition 2.2 is trivial. The converse is a surprisingly long chain of implications; see pages 326–327 of [2]. The initial law

$$x = (y \circ x) \circ ((x \circ z) \circ z)$$

is used to obtain, in turn,

$$x \circ ((((x \circ y) \circ y) \circ z) \circ z) = (x \circ y) \circ y,$$

$$(x \circ (y \circ z)) \circ (z \circ ((z \circ w) \circ w)) = y \circ z,$$

$$x \circ (y \circ ((y \circ z) \circ z)) = (x \circ y) \circ y,$$

$$((x \circ (y \circ z)) \circ z) \circ z = y \circ z,$$

$$(x \circ (y \circ (z \circ w))) \circ (z \circ w) = y \circ (z \circ w),$$

$$(x \circ (y \circ z)) \circ (y \circ z) = x \circ (y \circ z),$$

$$((x \circ y) \circ ((y \circ z) \circ z)) \circ ((y \circ z) \circ z) = y,$$

$$((x \circ y) \circ ((y \circ z) \circ z)) \circ ((y \circ z) \circ z) = ((x \circ ((x \circ y) \circ ((y \circ z) \circ z))) \circ ((y \circ z) \circ z)) \circ ((y \circ z) \circ z),$$

$$x \circ ((x \circ y) \circ y) = x,$$

$$x \circ (x \circ (y \circ z)) = x,$$

$$(x \circ y) \circ y = x \circ y,$$

$$(x \circ x) \circ x = x,$$

$$(x \circ y) \circ y = y,$$

$$x \circ y = y.$$

$\square$

Chapter 5

Selected magmas

Each magma can be used to establish anti-implications: if Γ is the set of all laws obeyed by a magma G, then we have $E \not\leq E'$ whenever $E \in \Gamma$ and $E' \notin \Gamma$. Large numbers of implications can already be obtained from

- All magmas of order at most 4, up to isomorphism (of which there are $178,985,294$);

- All commutative magmas of order 5, up to isomorphism **determine their count**;

- Cyclic groups $\mathbb{Z}/N\mathbb{Z}$ with $2 \leq N \leq 12$ and $x \circ y = ax^2 + bxy + cy^2 + dx + ey$ for randomly chosen a, b, c, d, e.

Some other magmas have been used to establish counterexamples:

- The cyclic group $\mathbb{Z}/6\mathbb{Z}$ with the addition law.

- The natural numbers with law $x \circ y = x + 1$.

- The natural numbers with law $x \circ y = xy + 1$.

- The reals with $x \circ y = (x + y)/2$.

- The natural numbers with $x \circ x$ equal to x when $x = y$ and $x + 1$ otherwise.

Chapter 6

Equivalence with the constant and singleton laws

85 laws have been shown to be equivalent to the constant law (Definition 2.19), and 815 laws have been shown to be equivalent to the singleton law (Definition 2.2).

These are the laws up to 4 operations that follow from diagonalization of 2.2 and of 2.19.

In order to formalize these in Lean, a search was run on the list of equations to discover diagonalizations of these two specific laws: equations of the form $x = R$ where R doesn't include x, and equations of the form $x \circ y = R$ where R doesn't include x or y.

The proofs themselves all look alike, and correspond exactly to the two steps described in the proof of 3.6. The Lean proofs were generated semi-manually, using search-and-replace starting from the output of `grep` that found the diagonalized laws.

In the case of the constant law, equation 2.15 ($x \circ x = y \circ z$) wasn't detected using this method. It was added manually to the file with the existing proof from the sub-graph project.

Chapter 7

Simple rewrites

53,905 implications were automatically generated by simple rewrites.
   describe the process of automatically generating these implications here.

Chapter 8

Trivial auto-generated theorems

Approximately 4.5m transitive implications were proven by a transitive reduction of about 15k theorems. Most of these implications were derived from being the first automated run to connect the largest equivalence classes, hence creating a large set of transitively closed implications.

Scripts generated theorems to try simple combinations of equation rewrites to reach the desired goal for every unknown implication. The generated proof scripts were run with lean and the successful theorems were extracted. An example of the types of generated rewrites that were tested:

```
repeat intro
apply

repeat intro
try { rw [<-h] }
try { rw [<-h, <-h] }
try { rw [<-h, <-h, <-h] }
try { rw [<-h, <-h, <-h, <-h] }
try { rw [<-h, <-h, <-h, <-h, <-h] }
repeat rw [h]

repeat intro
try {
  nth_rewrite 1 [h]
  try { rw [h] }
  try { rw [<-h] }
}
try {
  nth_rewrite 2 [h]
  try { rw [h] }
  try { rw [<-h] }
}
try {
  nth_rewrite 3 [h]
  try { rw [h] }
  try { rw [<-h] }
}
```

```
try {
  nth_rewrite 4 [h]
  try { rw [h] }
  try { rw [<-h] }
}
try {
  nth_rewrite 1 [h]
  nth_rewrite 1 [h]
  try { rw [h] }
  try { rw [<-h] }
}
...
```

Chapter 9

Enumerating Small Finite Magmas

describe the process of automatically generating these implications here.

Chapter 10

Equation Search

After previous brute force had derived many implications expressible as a small number of rewrites, this equation search leveraged the implication graph to search further. It began at a hypothesis, and then attempted to perform substitutions by equations it transitively implied. Due to it's naive implementation, it can not perform all forms of substutions, it is limited to substutions that match the exact text of another equation. This search method benefits from being able to start from a hypothesis and can search for any number of goals, rather than having to search the combined space of all possible hypotheses and goals, and can 'reach' farther positions in the search graph than simple rewriting alone.

An example proof illustrates the logic it uses:

```
have eq3315 (x y : G) : x   y = x   (y   (x   x)) := by
  apply Apply.Equation12_implies_Equation11 at h
  apply RewriteHypothesis.Equation11_implies_Equation3323 at h
  apply Apply.Equation3323_implies_Equation3315 at h
  apply h
have eq52 (x y : G) : x = x   (y   (x   x)) := by
  apply Apply.Equation12_implies_Equation61 at h
  apply Apply.Equation61_implies_Equation54 at h
  apply Apply.Equation54_implies_Equation52 at h
  apply h
repeat intro
nth_rewrite 1 [eq3315]
nth_rewrite 1 [← eq52]
apply h
repeat assumption
```

Bibliography

[1] A. K. Austin. A note on models of identities. *Proc. Amer. Math. Soc.*, 16:522–523, 1965.

[2] A. Kisielewicz. Austin identities. *Algebra Universalis*, 38(3):324–328, 1997.

[3] Andrzej Kisielewicz. Varieties of algebras with no nontrivial finite members. In *Lattices, semigroups, and universal algebra (Lisbon, 1988)*, pages 129–136. Plenum, New York, 1990.